Hypergraphs not containing a tight tree with a bounded trunk II: 3-trees with a trunk of size 2

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Abstract

A *tight* r*-tree* T is an r-uniform hypergraph that has an edge-ordering $e_1, e_2, \dots, e_t$ such that for each $i \geq 2$, e_i has a vertex v_i that does not belong to any previous edge and $e_i - v_i$ is contained in e_j for some $j < i$. Kalai conjectured in 1984 that every n-vertex r-uniform hypergraph with more than $\frac{t-1}{r}\binom{n}{r-1}$ edges contains every tight r-tree T with t edges.

A *trunk* T' of a tight r-tree T is a tight subtree T' of T such that vertices in $V(T) \setminus V(T')$ are leaves in T. Kalai's Conjecture was proved in 1987 for tight r-trees that have a trunk of size one. In a previous paper we proved an asymptotic version of Kalai's Conjecture for all tight r-trees that have a trunk of bounded size. In this paper we continue that work to establish the exact form of Kalai's Conjecture for all tight 3-trees on at least 20 edges that have a trunk of size two.

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1 Introduction. Trees, trunks, and Kalai's conjecture

For an r-uniform hypergraph (r*-graph*, for short) H, the *Turán number* $\mathrm{ex}_r(n, H)$ is the largest m such that there exists an n-vertex r-graph G with m edges that does not contain H. Estimating $\mathrm{ex}_r(n, H)$ is a difficult problem even for r-graphs with a simple structure. Here we consider Turán-type problems for so called tight r-trees. A *tight* r*-tree* ($r \geq 2$) is an r-graph whose edges can be ordered so that each edge e apart from the first one contains a vertex v_e that does not belong to any preceding edge but the set $e - v_e$ is contained in some preceding edge. Such an ordering is called a *proper ordering* of the edges. A usual graph tree is a tight 2-tree.

A vertex v in a tight r-tree T is a *leaf* if it has degree one in T. A *trunk* T' of a tight r-tree T is a tight subtree of T such that in some proper ordering of the edges of T the edges of T' are listed first

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and vertices in $V(T)\setminus V(T')$ are leaves in T. Hence, each $e\in E(T)\setminus E(T')$ contains an $(r-1)$-subset of some $e'\in E(T')$ and a leaf in T (that lies outside $V(T')$). In the case of $r=2$ each $e\in E(T)\setminus E(T')$ is a pendant edge. Every tight tree T with at least two edges has a trunk (for example, T minus the last edge in a proper ordering is a trunk). Let $c(T)$ denote the minimum size of a trunk of T. We write $e(H)$ for the number of edges in H.

In this paper we consider the following classical conjecture.

Conjecture 1.1 (Kalai 1984, see in [1]). *Let T be a tight r-tree with t edges. Then* $\mathrm{ex}_r(n,T)\le \frac{t-1}{r}\binom{n}{r-1}$.

The coefficient $(t-1)/r$ in this conjecture, if it is true, is optimal as one can see using constructions obtained from partial Steiner systems due to Rödl [4]. The conjecture turns out to be difficult even for very special cases of tight trees, in fact for $r=2$ it is the famous Erdős-Sós conjecture. The following partial result on Kalai's conjecture was proved in 1987.

Theorem 1.2 ([1]). *Let T be a tight r-tree with t edges and $c(T)=1$. Suppose that G is an n-vertex r-graph with $e(G)>\frac{t-1}{r}\binom{n}{r-1}$. Then G contains a copy of T.*

In a previous paper [2], we showed that Conjecture 1.1 holds *asymptotically* for tight r-trees with a trunk of a bounded size. Our result is as follows. Define $a(r,c):=(r^r+1-\frac{1}{r})(c-1)$.

Theorem 1.3 ([2]). *Let T be a tight r-tree with t edges and $c(T)\le c$. Then*

$$\mathrm{ex}_r(n,T)\le\left(\frac{t-1}{r}+a(r,c)\right)\binom{n}{r-1}.$$

The goal of this paper is to prove the conjecture in *exact* form for infinitely many 3-trees.

Theorem 1.4. *Let T be a tight 3-tree with t edges and $c(T)\le 2$. If $t\ge 20$ then*

$$\mathrm{ex}_3(n,T)\le\frac{t-1}{3}\binom{n}{2}.$$

Beside ideas and observations from [2], discharging is quite helpful here.

2 Notation and preliminaries. Shadows and default weights

In this section, we introduce some notation and list a couple of simple observations from [2]. For the sake of self-containment, we present their simple proofs as well.

The *shadow* of an r-graph G is $\partial(G):=\{S:|S|=r-1,\ \textit{and } S\subseteq e \textit{ for some } e\in e(G)\}$.

The *link* of a set $D\subseteq V(G)$ in an r-graph G is defined as $L_G(D):=\{e\setminus D: e\in E(G), D\subseteq e\}$.

The *degree* of D, $d_G(D)$, is the number the edges of G containing D. If G is an r-graph and $|D|=r-1$, the elements of $L_G(D)$ are vertices. In this case, we also use $N_G(D)$ to denote $L_G(D)$. Many times we drop the subscript G. For $1\le p\le r-1$, the *minimum p-degree* of G is

$$\delta_p(G):=\min\{d_G(D):|D|=p,\ \textit{and}\ D\subseteq e\ \textit{ for some }\ e\in E(G)\}.$$

For an r-graph G and $D \in \partial(G)$, let $w(D) := \frac{1}{d_G(D)}$. For each $e \in E(G)$, let

$$w(e) := \sum_{D \in \binom{e}{r-1}} w(D) = \sum_{D \in \binom{e}{r-1}} \frac{1}{d_G(D)}.$$

We call w the *default weight function* on $E(G)$ and $\partial(G)$. Frankl and Füredi [1] (and later some others) used the following simple property of this function.

Proposition 2.1. *Let G be an r-graph. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then*

$$\sum_{e \in E(G)} w(e) = |\partial(G)|.$$

Proof. By definition,

$$\sum_{e \in E(G)} w(e) = \sum_{e \in E(G)} \left(\sum_{D \in \binom{e}{r-1}} \frac{1}{d_G(D)} \right) = \sum_{D \in \partial(G)} \left(\sum_{e \in E(G), D \subseteq e} \frac{1}{d_G(D)} \right) = \sum_{D \in \partial(G)} 1 = |\partial(G)|. \quad \square$$

An *embedding* of an r-graph H into an r-graph G is an injection $f : V(H) \to V(G)$ such that for each $e \in E(H)$, $f(e) \in E(G)$. The following proposition is folklore.

Proposition 2.2. *Let G be an r-graph with $e(G) > q|\partial(G)|$. Then G contains a subgraph G' with $\delta_{r-1}(G') \geq \lfloor q \rfloor + 1$.*

Proof. Starting from G, if there exists $D \in \partial(G)$ of degree at most $\lfloor q \rfloor$ in the current r-graph, we remove the edges of this r-graph containing D. Let G' be the final r-graph. Since we have deleted at most $q|\partial(G)| < e(G)$ edges, G' is nonempty. By the stopping rule, $\delta_{r-1}(G') \geq \lfloor q \rfloor + 1$. $\square$

3 Lemmas for Theorem 1.4

The idea behind the proof of Theorem 1.4 is to find in the host 3-graph G a special pair of edges with good properties where we plan to map the trunk of size 2 of T. We use the weight argument together with discharging to find such special pairs in the next two lemmas.

Given edges $e = abc$ and $f = adc$ in a 3-graph G sharing pair ac, for a pair $\{x, y\} \subset \{a, b, c, d\}$, let $d'_{e,f}(x, y)$ denote the number of $z \in V(G) \setminus \{a, b, c, d\}$ such that $xyz \in G$. By definition

$$d'_{e,f}(x, y) \geq d(x, y) - 2 \qquad \textit{for every } \{x, y\} \subset \{a, b, c, d\}. \tag{1}$$

Lemma 3.1. *Let $m \geq 20$ be a positive integer and let G be a 3-graph satisfying $e(G) > \frac{m}{3}|\partial(G)|$ and $\delta_2(G) > \frac{m}{3}$. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then there exist edges $e = abc$ and $f = adc$ in G satisfying*
(a) $w(e) < \frac{3}{m}$ *and* $w(ac) < \frac{1}{m}$,
(b) $\min\{d'_{e,f}(a, b), d'_{e,f}(c, b)\} \geq \lfloor \frac{m}{3} \rfloor$,

(c) $\max\{d'_{e,f}(a,b), d'_{e,f}(c,b)\} \geq \lfloor \frac{2m}{3} \rfloor$, *and*
(d) *either* $3(w(f) - \frac{3}{m}) < (\frac{3}{m} - w(e))$ *or* $\max\{d'_{e,f}(a,d), d'_{e,f}(c,d)\} \geq m-1$.

Proof. For convenience, let $w_0 = \frac{3}{m}$. By Proposition 2.1, $\sum_{e \in G} w(e) = |\partial(G)|$. So,

$$\frac{1}{e(G)} \sum_{e \in G} w(e) = \frac{|\partial(G)|}{e(G)} < \frac{1}{m/3} = w_0. \tag{2}$$

Hence the average weight of an edge in G is less than w_0. We call an edge $e \in E(G)$ *light* if $w(e) < w_0$ and *heavy* otherwise. A pair $\{x, y\}$ of vertices in G is *good*, if $d(xy) \geq m+1$.

To find the desired pair of edges e, f we first do some marking of edges. For every light edge e, fix an ordering, say a, b, c, of its vertices so that $d(ab) \leq d(bc) \leq d(ac)$. We call ab, bc, ac the *low, medium, high* sides of e, respectively.

Since e is light, $w(e) = \frac{1}{d(ab)} + \frac{1}{d(bc)} + \frac{1}{d(ac)} < w_0 = \frac{3}{m}$, it follows that

$$d(ac) > m, \quad d(bc) > \frac{3m}{2}, \quad d(ab) > \frac{m}{3}. \tag{3}$$

In particular, ac is good. We define markings involving e based on three cases.

Case M1*:* $d(ab) \geq \lfloor m/3 \rfloor + 2$ and $d(bc) \geq \lfloor 2m/3 \rfloor + 2$. In this case, we let e *mark* every edge containing ac apart from itself.

Case M2*:* $d(ab) \leq \lfloor m/3 \rfloor + 1$. By (3), $d(ab) = \lfloor m/3 \rfloor + 1$, and since e is light,

$$d(ac) \geq d(bc) > \frac{1}{\frac{3}{m} - \frac{3}{m+3}} = \frac{m(m+3)}{9}. \tag{4}$$

We let e mark all the edges $acx \neq e$ containing ac such that abx is not an edge in G. By (4), in this case

$$e \text{ marks at least } \quad \frac{m(m+3)}{9} - \frac{m+3}{3} = \frac{(m+3)(m-3)}{9} \quad \text{edges.} \tag{5}$$

Case M3*:* $d(bc) \leq \lfloor 2m/3 \rfloor + 1$. By (3), $d(bc) = \lfloor 2m/3 \rfloor + 1$. Let e mark all the edges $acx \neq e$ containing ac such that bcx is not an edge in G. Since e is light,

$$d(ac) > \frac{1}{\frac{3}{m} - 2\frac{3}{2m+3}} = \frac{m(2m+3)}{9}. \tag{6}$$

Similarly to (5), in this case

$$e \text{ marks at least } \quad \frac{m(2m+3)}{9} - \frac{2m+3}{3} = \frac{(2m+3)(m-3)}{9} \quad \text{edges.} \tag{7}$$

We perform the above marking procedure for each light edge e.

Claim 1. If e is a light edge and f is an edge marked by e then (a)-(c) hold. Further, if f is light, then the lemma holds for (e, f).

Proof of Claim 1. Suppose $e = abc$, where a, b, c are ordered as described earlier and suppose $f = acd$. Then (a) holds by e being light and by (3). (b) holds, since either $d(ab) \geq \lfloor m/3 \rfloor + 2$ or $d(ab) = \lfloor m/3 \rfloor + 1$ and $d'_{e,f}(a,b) = d(ab) - 1$ (because $abd \notin G$). Similarly, (c) holds, since either $d(bc) \geq \lfloor 2m/3 \rfloor + 2$ or $d(bc) = \lfloor 2m/3 \rfloor + 1$ and $d'_{e,f}(b,c) = d(bc) - 1$ (because $bcd \notin G$). Now, if f is also a light edge then (d) holds since $w(f) - \frac{3}{m} < 0 < \frac{3}{m} - w(e)$. □

By Claim 1, we may henceforth assume that every marked edge is heavy. We will now use a discharging procedure to find our pair (e, f). Let the initial charge $ch(e)$ of every edge e in G equal to $w(e)$. Then $\sum_{e \in G} ch(e) = \sum_{e \in G} w(e) = |\partial(G)|$. We will redistribute charges among the edges of G so that the total sum of charges does not change and the resulting charge of each heavy edge remains at least w_0.

The discharging rule is as follows. Suppose a heavy edge f was marked by exactly q light edges. If $q = 0$, then let the new charge $ch^*(f)$ equal $ch(f)$. Otherwise, let f transfer to each light edge e that marks it a charge of $(ch(f) - w_0)/q$ so that $ch^*(f) = w_0$. It is easy to see that the total charge does not change in this discharging process. Hence, by (2), there is an edge e with $ch^*(e) < w_0$. By our discharging rule, e must be a light edge. Suppose e marked p edges. In each of Cases M1,M2, M3, e marks at least one edge. So $p > 0$. Among all edges e marked, let f be one that gave the least charge to e. By definition, f gave e a charge of at most $(ch^*(e) - ch(e))/p < (w_0 - ch(e))/p$. We claim that the pair (e, f) satisfies the lemma. Suppose $e = abc$, where a, b, c are ordered as before, and suppose $f = acd$. By Claim 1, (a), (b), and (c) hold. It remains to prove (d). If all three pairs in f are good, then $w(f) < \frac{3}{m}$, contradicting f being heavy. So, at most two of the pairs in f are good. By our earlier discussion, ac is good. If one of ad and cd is also good, then the second part of (d) holds. So we may assume that ac is the only good pair in f. Let q be the number of the light edges that marked f. By the marking process, a light edge only marks edges containing its high side and the high side is a good pair. Since ac is the only good pair in f, each of the q light edges that marked f contains ac and has ac as its high side.

First, suppose that Case M1 was applied to e. Then all the edges containing ac other than e were marked, which by our assumption must be heavy. In particular, this implies that $q = 1$. By our rule, f gave e a charge of $ch(f) - w_0$. By our choice of f, each of the $d(ac) - 1 \geq m$ edges of G containing ac (other than e) gave e a charge of at least $ch(f) - w_0$. Hence, $w_0 > ch^*(e) \geq ch(e) + m(ch(f) - w_0)$, from which the first part of (d) follows.

Next, suppose that Case M2 was applied to e. Then $d(ab) \leq \lfloor m/3 \rfloor + 1$. If $q > \lfloor m/3 \rfloor + 1$, then one of light edges containing ac, say acx, satisfies that $abx \notin G$. By rule, e marked acx, contradicting our assumption that no light edge was marked. So $q \leq \lfloor m/3 \rfloor + 1$. Similarly if Case 3 was applied to e then $q \leq \lfloor 2m/3 \rfloor + 1$. In both of these cases, e marked at least $\frac{(m+3)(m-3)}{9}$ edges, and by the choice of f, each of these edges gave to e charge at least $(ch(f) - w_0)/q$. Since $ch^*(e) < w_0$, we conclude

$$w_0 - ch(e) > \frac{(m+3)(m-3)}{9} \frac{ch(f) - w_0}{q} \geq \frac{(m+3)(m-3)}{3(2m+3)}(ch(f) - w_0).$$

Since $m \geq 20$, this means

$$\frac{ch(f) - w_0}{w_0 - ch(e)} < \frac{3(2m+3)}{(m+3)(m-3)} \leq \frac{3 \cdot 45}{24 \cdot 18} = \frac{5}{16} < \frac{1}{3}.$$

So, the first part of (d) holds. □

For an edge e, by $d_{\min}(e)$ we denote the *minimum codegree* over all three pairs of vertices in e.

Lemma 3.2. *Let G be a 3-graph satisfying $e(G) > \gamma|\partial(G)|$. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then there exists a pair of edges e, f with $|e \cap f| = 2$ such that*

1. $w(e) < \frac{1}{\gamma}$,
2. $|e \cap f| = d_{\min}(e)$,
3. $w(f) < \frac{1}{\gamma} + \frac{3}{d_{\min}(e)-1}(\frac{1}{\gamma} - w(e))$.

Proof. For convenience, let $w_0 = \frac{1}{\gamma}$. As in the proof of Lemma 3.1, call an edge e with $w(e) < w_0$ *light* and an edge e with $w(e) \geq w_0$ *heavy.* As before, the average average of $w(e)$ over all e is $|\partial(G)|/e(G) < w_0$. For each light edge e, let us mark a pair of vertices in it of codegree $d_{\min}(e)$. If e is a light edge with a marked pair xy and f is another light edge containing xy, then our statements already hold. So we assume that no marked pair of any light edge lies in another light edge. Let us initially assign a charge of $w(e)$ to each edge e in G. Then the average charge of an edge in G is less than w_0. We now apply the following discharging rule. For each heavy edge f, transfer $\frac{1}{3}(w(f) - w_0)$ of the charge to each light edge e whose marked pair is contained in f. Note that for each f there are at most 3 such e. In particular, each heavy edge still has charge at least w_0 after the discharging.

Since discharging does not change the total charge, there exists some edge e with charge less than w_0. By the previous sentence, e is a light edge in G . Let xy be its marked pair. There are $d_{\min}(e) - 1$ other edges containing it, each of which is heavy. Each such edge f has given a charge of $\frac{1}{3}(w(f) - w_0)$ to w_0. For e to still have a charge less than w_0, one of these edges f satisfies $\frac{1}{3}(w(f) - w_0) < \frac{w_0 - w(e)}{d_{\min}(e)-1}$. Hence $w(f) < \frac{1}{\gamma} + \frac{3}{d_{\min}(e)-1}(\frac{1}{\gamma} - w(e))$. □

Our third lemma proves a special case of Theorem 1.4.

Lemma 3.3. *Let T be a tight 3-tree with $t \geq 5$ edges. Suppose T has a trunk $\{e_1, e_2\}$ of size 2 such that $d_T(e_1 \cap e_2) \geq \lfloor \frac{t-1}{3} \rfloor + 2$. Let G be an n-vertex 3-graph that does not contain T. Then $e(G) \leq \frac{t-1}{3}|\partial(G)|$.*

Proof. For convenience, let $m = t - 1$. Let G be a 3-graph with $e(G) > \frac{m}{3}|\partial(G)|$. Then G contains a subgraph G' such that $e(G') > \frac{m}{3}|\partial(G')|$ and $\delta_2(G') > \frac{m}{3}$. For convenience, we assume G itself satisfies these two conditions. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then G satisfies the conditions of Lemma 3.1. Let the edges $e = abc$ and $f = adc$ satisfy the claim of that lemma, where a, b, c are ordered as in Lemma 3.1. In particular, by (a), e is light and ac is good, i.e. $d(ac) \geq m + 1$. By our assumptions, $d(ab) \leq d(bc)$. By parts (b) and (c),

$$d'_{e,f}(a, b) \geq \left\lfloor \frac{m}{3} \right\rfloor \quad \textit{and} \quad d'_{e,f}(c, b) \geq \left\lfloor \frac{2m}{3} \right\rfloor. \tag{8}$$

We rename pairs $\{a, d\}$ and $\{c, d\}$ as D_1 and D_2 so that $d'_{e,f}(D_1) = \min\{d'_{e,f}(a, d), d'_{e,f}(c, d)\}$ and $d'_{e,f}(D_2) = \max\{d'_{e,f}(a, d), d'_{e,f}(c, d)\}$. We claim that in these terms,

$$d'_1 := d'_{e,f}(D_1) \geq \left\lfloor \frac{m}{3} \right\rfloor - 1 \quad \textit{and} \quad d'_2 := d'_{e,f}(D_2) \geq \left\lfloor \frac{m}{3} \right\rfloor. \tag{9}$$

By (1) and the fact that $\delta_2(G) > \frac{m}{3}$, $d'_1, d'_2 \geq \lfloor \frac{m}{3} \rfloor - 1$. We will use part (d) of Lemma 3.1 to show that $d'_2 \geq \lfloor \frac{m}{3} \rfloor$. If the second part of (d) holds, then $d'_2 \geq m-1$ and we are done. So suppose the first part of Lemma 3.1 (d) holds instead, i.e. $3(w(f) - w_0) < (w_0 - w(e))$. Then $w(f) < \frac{4}{3}w_0 = \frac{4}{m}$. If $d'_1 = d'_2 = \lfloor \frac{m}{3} \rfloor - 1$, then $d(D_1) = d(D_2) = \lfloor \frac{m}{3} \rfloor + 1$ and hence

$$w(f) > \frac{2}{\lfloor \frac{m}{3} \rfloor + 1} \geq \frac{6}{m+3} \geq \frac{4}{m}$$

when $m > 9$, a contradiction. Thus, $d'_2 \geq \lfloor \frac{m}{3} \rfloor$ and (9) holds.

By our assumption, T has a trunk $\{e_1, e_2\}$ with $d_T(e_1 \cap e_2) \geq \lfloor \frac{m}{3} \rfloor + 2$. Suppose $e_1 = xyu$ and $e_2 = xyv$ so that $e_1 \cap e_2 = xy$. By our assumption, each edge in $E(T) \setminus \{e_1, e_2\}$ contains a pair in e_1 or e_2 and a vertex outside $e_1 \cup e_2$. For each pair B contained in e_1 or e_2, let $N'_T(B) = N_T(B) \setminus \{x, y, u, v\}$ and $\mu(B) = |N'_T(B)|$. Then $\mu(xy) = d_T(xy) - 2$, and $\mu(B) = d_T(B) - 1$ for each $B \in \{xu, xv, yu, yv\}$,. By definition,

$$\mu(xy) + \mu(xu) + \mu(xv) + \mu(yu) + \mu(yv) = t - 2 = m - 1. \tag{10}$$

Since $\mu(xy) = d_T(xy) - 2 \geq \lfloor \frac{m}{3} \rfloor > \frac{m}{3} - 1$, we have

$$\mu(xu) + \mu(xv) + \mu(yu) + \mu(yv) < \frac{2m}{3}. \tag{11}$$

We consider three cases, and in each case we find an embedding of T into G.

Case 1. $d'_{e,f}(a, b) \geq \lfloor \frac{2m}{3} \rfloor$. Recall that by (8), $d'_{e,f}(c, b) \geq \lfloor \frac{2m}{3} \rfloor$. By symmetry we may assume that $\mu(xu) + \mu(yu) \geq \mu(xv) + \mu(yv)$ and that $\mu(xv) \geq \mu(yv)$. Then by (11) $\mu(xv) + \mu(yv) \leq \lfloor \frac{m}{3} \rfloor$, so we construct an embedding ϕ of T into G as follows.

First, let $\phi(u) = b$ and $\phi(v) = d$. Then choose distinct $\phi(x), \phi(y) \in \{a, c\}$ so that $\phi(\{y, v\}) = D_1$ and $\phi(\{x, v\}) = D_2$. This maps e_1 to e and e_2 to f. Since $\mu(yv) < \frac{1}{4}\frac{2m}{3} = \frac{m}{6}$, by (9) we can next map $N'_T(yv)$ into $N'_G(D_1)$. Now, since $\mu(yv) + \mu(xv) < \frac{1}{2}\frac{2m}{3} = \frac{m}{3}$, again by (9) we can map $N'_T(xv)$ into $N'_G(D_2) \setminus \phi(N'_T(yv))$. If $\phi(x) = a, \phi(y) = c$, then by the condition of Case 1 and (11), we can map $N'_T(yu)$ into $N'_G(bc) \setminus \phi(N'_T(yv) \cup N'_T(xv))$ and $N'_T(xu)$ into $N'_G(ac) \setminus \phi(N'_T(yv) \cup N'_T(xv))$. The case $\phi(x) = c, \phi(y) = a$ is similar. Finally, embed $N'_T(xy)$ into $N'_G(ac)$.

Case 2. $\lfloor \frac{m}{3} \rfloor \leq d'_{e,f}(a, b) \leq \lfloor \frac{2m}{3} \rfloor - 1$ and $d'_1 \geq \lfloor \frac{m}{3} \rfloor$. Then we can strengthen the second part of (9) to

$$d'_2 \geq \left\lfloor \frac{m}{2} \right\rfloor. \tag{12}$$

Indeed, (9) holds immediately if the second part of (d) holds in Lemma 3.1; so we may assume $3(w(f) - w_0) < (w_0 - w(e))$. By the condition of Case 2,

$$w_0 - w(e) \leq \frac{3}{m} - \frac{3}{2m+3} = \frac{3(m+3)}{m(2m+3)}.$$

From this, we get

$$w(f) < \frac{3}{m} + \frac{(m+3)}{m(2m+3)} = \frac{7m+12}{m(2m+3)}.$$

If $d'_2 \le \lfloor \frac{m}{2} \rfloor - 1$, then

$$w(f) > \frac{2}{d'_2 + 2} \ge 2\frac{2}{m+2},$$

which is larger than $\frac{7m+12}{m(2m+3)}$ for $m \ge 24$. This contradiction proves (12).

For convenience, suppose $D_1 = cd$ (the case $D_1 = ad$ is similar). By symmetry, we may assume that $\mu(xu) + \mu(yv) \le \mu(yu) + \mu(xv)$ and that $\mu(yu) \ge \mu(xv)$. Then by (11),

$$\mu(xu) + \mu(yv) \le \left\lfloor \frac{m}{3} \right\rfloor, \quad \mu(xu) + \mu(yv) + \mu(xv) \le \left\lfloor \frac{m}{2} \right\rfloor. \tag{13}$$

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively and embedding in order $N'_T(yv)$ into $N'_G(cd)$, $N'_T(xu)$ into $N'_G(ab)$, $N'_T(xv)$ into $N'_G(ad)$, $N'_T(yu)$ into $N'_G(bc)$, and $N'_T(xy)$ into $N'_G(ac)$ greedily. Conditions (10), (11), (12) and (13) ensure that such an embedding exists.

Case 3. $\lfloor \frac{m}{3} \rfloor \le d'_{e,f}(a,b) \le \lfloor \frac{2m}{3} \rfloor - 1$ and $d'_1 = \lfloor \frac{m}{3} \rfloor - 1$. We now strengthen (12) to

$$d'_2 \ge \left\lfloor \frac{2m}{3} \right\rfloor. \tag{14}$$

Indeed, exactly as in the proof of (12), we derive that $w(f) < \frac{7m+12}{m(2m+3)}$. If $d'_2 \le \lfloor \frac{2m}{3} \rfloor - 1$, then

$$\frac{3}{2m+3} \le \frac{1}{d'_2 + 2} < \frac{7m+12}{m(2m+3)} - \frac{1}{d'_1 + 2} \le \frac{7m+12}{m(2m+3)} - \frac{3}{m+3},$$

which is not true for $m \ge 20$. This proves (14).

As in Case 2, suppose $D_1 = cd$ (the case $D_1 = ad$ is similar). By symmetry, we may assume that $\mu(xu) + \mu(yv) \le \mu(yu) + \mu(xv)$ and that $\mu(xu) \ge \mu(yv)$. Then by (11),

$$\mu(xu) + \mu(yv) \le \left\lfloor \frac{m}{3} \right\rfloor, \quad \mu(yv) \le \left\lfloor \frac{m}{6} \right\rfloor. \tag{15}$$

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively and embedding in order $N'_T(yv)$ into $N'_G(cd)$, $N'_T(xu)$ into $N'_G(ab)$, $N'_T(xv)$ into $N'_G(ad)$, $N'_T(yu)$ into $N'_G(bc)$, and $N'_T(xy)$ into $N'_G(ac)$ greedily. Conditions (10), (11), (14) and (15) ensure that such an embedding exists. □

4 Proof of Theorem 1.4

We prove the shadow version of Theorem 1.4, which immediately implies Theorem 1.4.

Theorem 1.4′. *Let $t \ge 20$ be an integer. Let T be a tight 3-tree with t edges and $c(T) \le 2$. If G is an r-graph that does not contain T then $e(G) \le \frac{t-1}{3}|\partial(G)|$.*

Proof. First, let us point that in this proof, we exploit Lemma 3.2 and will not need Lemma 3.1 in an explicit way. Let T be a tight 3-tree with $t \ge 20$ edges that contains a trunk $\{e_1, e_2\}$ of size 2. For convenience, let $m = t - 1$. Let G be a 3-graph with $e(G) > \frac{m}{3}|\partial(G)|$. We prove that G contains T. As before we may assume that $\delta_2(G) > \frac{m}{3}$. Let w be the default weight function on $E(G)$ and $\partial(G)$.

By Lemma 3.2, there exist edges e and f in G such that $d(e \cap f) = d_{\min}(e)$, $w(e) < \frac{3}{m}$, and (using $m \geq 19$)

$$\text{if } d(e\cap f) > \tfrac{m}{2}, \text{ then } w(f) < \frac{3}{m} + \frac{3}{(m+1)/2-1}\left(\frac{3}{m} - w(e)\right) \leq \frac{3}{m} + \frac{1}{3}\left(\frac{3}{m} - w(e)\right) \leq \frac{4}{m}, \quad (16)$$

and

$$\text{if } d(e\cap f) \leq \tfrac{m}{2}, \text{ then } w(f) < \frac{3}{m} + \frac{3}{\lceil m/3 \rceil - 1}\left(\frac{3}{m} - w(e)\right) \leq \frac{3}{m} + \frac{1}{2}\left(\frac{3}{m} - \frac{2}{m}\right) = \frac{4}{m}. \quad (17)$$

Suppose $e = acb$ and $f = acd$, so that $e \cap f = ac$. For each pair D contained in e or f, let $N'_G(D) = N_G(D) \setminus \{a, b, c, d\}$ and $d'_G(D) = |N'_G(D)|$. Then $d'_G(D) \geq d_G(D) - 2$. Consider T. Suppose $e_1 = xyu$ and $e_2 = xyv$, so that $e_1 \cap e_2 = xy$. If $d_T(xy) \geq \lfloor \frac{m}{3} \rfloor + 2$, then we apply Lemma 3.3 and are done. Hence we may assume that

$$d_T(xy) \leq \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

For each pair B contained in e_1 or e_2, let $N'_T(B) = N_T(B) \setminus \{x, y, u, v\}$ and let $\mu(B) = |N'_T(B)|$. Then $\mu(xy) = d_T(xy) - 2$ and $\mu(B) = d_T(B) - 1$ for the other pairs. Also, we have

$$\mu(xu) + \mu(yu) + \mu(xv) + \mu(yv) + \mu(xy) = m - 1. \quad (18)$$

Since $\mu(xy) = d_T(xy) - 2 \leq \frac{m}{3} - 1$,

$$\mu(xy) + \frac{i}{4}(m - 1 - \mu(xy)) \leq \frac{m}{3} + \frac{im}{6} - 1 \qquad \forall i \in [4]. \quad (19)$$

Let us view e, f as glued together at ac with e on the left and f on the right. Let

$$L_{max} = \max\{d_G(ab), d_G(bc)\}, \qquad L_{min} = \min\{d_G(ab), d_G(bc)\},$$
$$R_{max} = \max\{d_G(ad), d_G(cd)\}, \qquad R_{min} = \min\{d_G(ad), d_G(cd)\}.$$

Since $d(ac) = d_{\min}(e)$, $L_{max} \geq L_{min} \geq d_G(ac)$. Since $w(e) < \frac{3}{m}$, we have

$$L_{max} > m. \quad (20)$$

We consider two cases. In each case, we find an embedding of T into G.

Case 1. $L_{min} > m$. This implies $d'_G(ab), d'_G(bc) \geq m - 1$. By symmetry, we may assume that $d_G(ad) \geq d_G(cd)$ so that $d_G(ad) = R_{max}$ and $d_G(cd) = R_{min}$. Now, consider T. By symmetry, we may assume that $\mu(xu) + \mu(yu) \geq \mu(xv) + \mu(yv)$ and that $\mu(xv) \geq \mu(yv)$. Then $\mu(yv) \leq \frac{1}{4}(m - 1 - \mu(xy))$ and $\mu(xv) + \mu(yv) \leq \frac{1}{2}(m - 1 - \mu(xy))$. This, together with (19) implies

$$\mu(yv) \leq \left\lfloor \tfrac{m}{4} \right\rfloor, \qquad \mu(xv) + \mu(yv) \leq \left\lfloor \frac{m}{2} \right\rfloor - 1,$$
$$\mu(yv) + \mu(xy) \leq \left\lfloor \tfrac{m}{2} \right\rfloor - 1, \qquad \mu(xv) + \mu(yv) + \mu(xy) \leq \left\lfloor \frac{2m}{3} \right\rfloor - 1. \quad (21)$$

Case 1.1. $d_G(ac) > \frac{2m}{3}$. By (16), $\frac{1}{R_{max}} + \frac{1}{R_{min}} < w(f) < \frac{4}{m}$, so $R_{max} > \frac{m}{2}$. Since $\delta_2(G) > \frac{m}{3}$, we have $R_{min} > \frac{m}{3}$. Hence

$$d'_G(ab), d'_G(bc) \geq m-1, \quad d'_G(ac) \geq \left\lfloor \frac{2m}{3} \right\rfloor - 1, \quad d'_G(ad) \geq \left\lfloor \frac{m}{2} \right\rfloor - 1, \quad d'_G(cd) \geq \left\lfloor \frac{m}{3} \right\rfloor - 1. \tag{22}$$

Now we can embed T into G as follows. First, we map x, y, u, v to a, b, c, d respectively. This maps e_1 to e and e_2 to f. Then we map $N'_T(yv)$ into $N'_G(cd)$ followed by $N'_T(xv)$ into $N'_G(ad)$. Next, we map $N'_T(xy)$ into $N'_G(ac)$, $N'_T(yu)$ into $N'_G(bc)$, and $N'_T(xu)$ into $N'_G(ab)$ in that order. Conditions (21) and (22) ensure that such an embedding exists.

Case 1.2. $d_G(ac) \leq \frac{2m}{3}$. Then $w(e) \geq \frac{3}{2m}$. If $d_G(ac) > \frac{m}{2}$, then by (16), $w(f) < \frac{3}{m} + \frac{1}{3}(\frac{3}{m} - \frac{3}{2m}) = \frac{7}{2m}$. On the other hand, if $d_G(ac) \leq \frac{m}{2}$, then $w(e) \geq \frac{2}{m}$ and by (17), $w(f) < \frac{3}{m} + \frac{1}{2}(\frac{3}{m} - \frac{2}{m}) = \frac{7}{2m}$. So in any case,

$$\frac{1}{R_{max}} + \frac{1}{R_{min}} < w(f) - w(ac) < \frac{7}{2m} - \frac{3}{2m} = \frac{2}{m}.$$

Then $R_{max} > m$ and $R_{min} > \frac{m}{2}$. Also, since $\delta_2(G) > \frac{m}{3}$, we have $d_G(ac) > \frac{m}{3}$. Hence,

$$d'_G(ab), d'_G(bc) \geq m-1, \quad d'_G(ac) \geq \left\lfloor \frac{m}{3} \right\rfloor - 1, \quad d'_G(ad) \geq m-1, \quad d'_G(cd) \geq \left\lfloor \frac{m}{2} \right\rfloor - 1. \tag{23}$$

Now we can embed T into G as follows. First, we map x, y, u, v to a, b, c, d respectively. This maps e_1 to e and e_2 to f. Then we map $N'_T(xy)$ into $N'_G(ac)$. This is doable since $d'_T(xy) = d_T(xy) - 2 \leq \lfloor \frac{m}{3} \rfloor - 1$ while $d'_G(ac) \geq \lfloor \frac{m}{3} \rfloor - 1$. Then we map $N'_T(yv)$ into $N'_G(cd)$ followed by $N'_T(xv)$ into $N'_G(ad)$. Next, we map $N'_T(yu)$ into $N'_G(bc)$, and $N'_T(xu)$ into $N'_G(ab)$ in that order. Conditions (21) and (23) ensure that such an embedding exists.

Case 2. $L_{min} \leq m$. By symmetry, we may assume that $d_G(ab) \geq d_G(bc)$ so that $d_G(ab) = L_{max}$ and $d_G(bc) = L_{min}$. We have $\frac{1}{L_{min}} + \frac{1}{d_G(ac)} < w(e) < \frac{3}{m}$. Since $d(ac) = d_{\min}(e)$, $d_G(ac) \leq L_{min} \leq m$. This yields $L_{min} > \frac{2m}{3}$, $\frac{m}{2} < d_G(ac) \leq m$, and $w(e) > \frac{2}{m}$. By (20), $L_{max} > m$. Thus,

$$d'_G(ab) \geq m-1, \quad d'_G(bc) \geq \left\lfloor \frac{2m}{3} \right\rfloor - 1, \quad d'_G(ac) \geq \left\lfloor \frac{m}{2} \right\rfloor - 1. \tag{24}$$

Since $d_G(ac) > m/2$, by (16),

$$w(f) < \frac{3}{m} + \frac{1}{3}\frac{1}{m} = \frac{10}{3m} \text{ and } \frac{1}{R_{max}} + \frac{1}{R_{min}} \leq w(f) - \frac{1}{d_G(ac)} < \frac{10}{3m} - \frac{1}{m} = \frac{7}{3m}. \tag{25}$$

Case 2.1 $R_{max} > m$. By our assumption and (25),

$$R_{max} > m, \quad R_{min} > \frac{3m}{7}.$$

First suppose that $d_G(ad) \geq d_G(cd)$. Then

$$d'_G(ad) \geq m-1, \quad d'_G(cd) \geq \left\lfloor \frac{3m}{7} \right\rfloor - 1. \tag{26}$$

By symmetry, we may assume that $\mu(xu)+\mu(xv) \geq \mu(yu)+\mu(yv)$ and that $\mu(yu) \geq \mu(yv)$. Then by these assumptions and (19), we have

$$\mu(yv) \leq \left\lfloor \frac{m}{4} \right\rfloor - 1, \quad \mu(yv)+\mu(xy) \leq \left\lfloor \frac{m}{2} \right\rfloor - 1, \quad \mu(yv)+\mu(xy)+\mu(yu) \leq \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{27}$$

Now we can embed T into G as follows. First, we map x, y, u, v to a, b, c, d respectively. This maps e_1 to e and e_2 to f. Then we map $N'_T(yv)$ into $N'_G(cd)$ followed by $N'_T(xy)$ into $N'_G(ac)$. Next, we map $N'_T(yu)$ into $N'_G(bc)$, $N'_T(xv)$ into $N'_G(ad)$, and $N'_T(xu)$ into $N'_G(ab)$ in that order. Conditions (24), (26) and (27) ensure that such an embedding exists.

Next, suppose that $d_G(cd) \geq d_G(ad)$. Then

$$d'_G(ad) \geq \left\lfloor \frac{3m}{7} \right\rfloor - 1, \quad d'_G(cd) \geq m-1. \tag{28}$$

By symmetry, we may assume that $\mu(xu)+\mu(yv) \geq \mu(xv)+\mu(yu)$ and that $\mu(yu) \geq \mu(xv)$. By these assumptions and (19), we have

$$\mu(xv) \leq \left\lfloor \frac{m}{4} \right\rfloor - 1, \quad \mu(xv)+\mu(xy) \leq \left\lfloor \frac{m}{2} \right\rfloor - 1, \quad \mu(xv)+\mu(xy)+\mu(yu) \leq \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{29}$$

Now we can embed T into G as follows. First, we map x, y, u, v to a, b, c, d respectively. This maps e_1 to e and e_2 to f. Then we map $N'_T(xv)$ into $N'_G(ad)$ followed by $N'_T(xy)$ into $N'_G(ac)$. Next, we map $N'_T(yu)$ into $N'_G(bc)$, $N'_T(yv)$ into $N'_G(cd)$, and $N'_T(xu)$ into $N'_G(ab)$ in that order. Conditions (24), (28) and (29) ensure that such an embedding exists.

Case 2.2 $R_{max} \leq m$. Since $R_{min} \leq R_{max} \leq m$, by (25), we again have $R_{max} > \frac{6m}{7}$, and

$$\frac{1}{R_{min}} < \frac{7}{3m} - \frac{1}{m} = \frac{4}{3m}; \quad \text{so} \quad R_{min} > \frac{3m}{4}.$$

By (25), $w(f) < \frac{10}{3m}$. Also, $\frac{1}{L_{min}} \geq \frac{1}{L_{max}} \geq \frac{1}{m}$. Hence,

$$w(ac) < \frac{10}{3m} - \frac{2}{m} = \frac{4}{3m} \quad \textit{and hence} \quad d'_G(ac) \geq \left\lfloor \frac{3m}{4} \right\rfloor - 1. \tag{30}$$

First, suppose that $d_G(ad) \geq d_G(cd)$. Then

$$d'(ad) \geq \left\lfloor \frac{6m}{7} \right\rfloor - 1, \quad d'(cd) \geq \left\lfloor \frac{3m}{4} \right\rfloor - 1. \tag{31}$$

By symmetry, we may assume that $\mu(xu)+\mu(xv) \geq \mu(yu)+\mu(yv)$ and that $\mu(xu) \geq \mu(xv)$. In particular,

$$\mu(xu) \geq \frac{1}{4}\left(m-1-\mu(xy)\right) \geq \frac{1}{4}\left(m-1-\frac{m}{3}+1\right) = \frac{m}{6}. \tag{32}$$

By (19), (24), (31), and (32), we can greedily embed T into G by mapping x, y, u, v to a, c, b, d, respectively and mapping in order $N'_T(yv)$ into $N'_G(cd)$, $N'_T(xy)$ into $N'_G(ac)$, $N'_T(yu)$ into $N'_G(bc)$,

$N'_T(xv)$ into $N'_G(ad)$, and $N'_T(xu)$ into $N'_G(ab)$.

Next, suppose that $d_G(cd) \geq d_G(ad)$. Then $d'(ad) \geq \lfloor \frac{3m}{4} \rfloor - 1$ and $d'(cd) \geq \lfloor \frac{6m}{7} \rfloor - 1$. By symmetry, we may assume that $\mu(xu) + \mu(yv) \geq \mu(xv) + \mu(yu)$ and that $\mu(xu) \geq \mu(yv)$. Again, (32) holds. We can greedily embed T into G by mapping x, y, u, v to a, c, b, d, respectively and mapping in order $N'_T(yu)$ into $N'_G(bc)$, $N'_T(xy)$ into $N'_G(ac)$, $N'_T(xv)$ into $N'_G(ad)$, $N'_T(yv)$ into $N'_G(cd)$, and $N'_T(xu)$ into $N'_G(ab)$. □

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