Hypergraphs not containing a tight tree with a bounded trunk

Zoltán Füredi* Tao Jiang† Alexandr Kostochka‡
Dhruv Mubayi§ Jacques Verstraëte¶

December 10, 2017

Abstract

An r-uniform hypergraph is a *tight r-tree* if its edges can be ordered so that every edge e contains a vertex v that does not belong to any preceding edge and the set $e-v$ lies in some preceding edge. A conjecture of Kalai [13], generalizing the Erdős-Sós Conjecture for trees, asserts that if T is a tight r-tree with t edges and G is an n-vertex r-uniform hypergraph containing no copy of T, then G has at most $\frac{t-1}{r}\binom{n}{r-1}$ edges.

A *trunk* T' of a tight r-tree T is a tight subtree such that every edge of $T-T'$ has $r-1$ vertices in some edge of T' and a vertex outside T'. For $r\geq 3$, the only nontrivial family of tight r-trees for which this conjecture has been proved is the family of r-trees with trunk size one in [5] from 1987. Our main result is an asymptotic version of Kalai's conjecture for all tight trees T of bounded trunk size. This follows from our upper bound on the size of a T-free r-uniform hypergraph G in terms of the size of its shadow. We also give a short proof of Kalai's conjecture for tight r-trees with at most four edges. In particular, our result on the tight path of length 4 for 3-graphs implies the intersection shadow theorem of Katona [14].

1 Results and history of tight trees

Turán-type problems are among central in combinatorics. For integers $n\geq r\geq 2$ and an r-uniform hypergraph (r-*graph*, for short) H, the *Turán number* $\mathrm{ex}_r(n,H)$ is the largest m such that there exists an n-vertex r-graph G with m edges that does not contain H. One of well-known conjectures in extremal graph theory is the Erdős-Sós Conjecture (see [2]) that *every n-vertex graph G with more than $n(t-1)/2$ edges contains every tree with t edges as a subgraph*. In other words, they conjecture that $\mathrm{ex}_2(n,T)\leq n(t-1)/2$ for each tree with t edges. The conjecture, if true, would be best possible whenever t divides n, as seen by taking G to be the disjoint union of K_t's. There are many partial results on the conjecture. The most significant progress on the conjecture was made by Ajtai, Komlós, Simonovits, and Szemerédi [1], who solved the conjecture for all sufficiently large t.

*Research supported by grant K116769 from the National Research, Development and Innovation Office NKFIH, and by the Simons Foundation Collaboration grant #317487.

†Research partially supported by NSF award DMS-1400249.

‡Research of this author is supported in part by NSF grant DMS-1600592 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.

§Research partially supported by NSF award DMS-1300138.

¶Research supported by NSF award DMS-1556524.

In 1984, Kalai [13] made a more general conjecture for r-graphs. To describe the conjecture, we need the following notion of hypergraph trees. Let $r \geq 2$ be an integer. An r-graph T is called a tight *r-tree* if its edges can be ordered as $e_1, \ldots, e_t$ so that

$$\text{for each } i \geq 2, \text{ there are a vertex } v \in e_i \text{ and } 1 \leq s \leq i-1 \text{ such that } v \notin \bigcup_{j=1}^{i-1} e_j \text{ and } e_i - v \subset e_s. \tag{1}$$

Note that a graph tree is a tight 2-tree. We write $e(H)$ for the number of edges in H.

Conjecture 1.1 (Kalai 1984, see in [5])**.** *Let $r \geq 2$ and let T be a tight r-tree with $t \geq 2$ edges. Then* $\mathrm{ex}_r(n,T) \leq \frac{t-1}{r}\binom{n}{r-1}$.

Kalai observed that his conjecture, if true, is asymptotically optimal using constructions obtained from partial Steiner systems due to Rödl [20]. The recent work of Keevash [15] (see also [10]) on the existence of designs show that in fact for every $r \geq 2$ and t there are infinitely many n for which there is an n-vertex r-graph G with $e(G) = \frac{t-1}{r}\binom{n}{r-1}$ that contains none of the tight r-trees with t edges. For example, this bound can be achieved for all $n > n_0(r,t)$ when some divisibility properties hold, e.g., $n - r + 2$ is divisible by $(t+r-1)!$. This gives a lower bound $\frac{t-1}{r}\binom{n}{r-1} - O_{r,t}(n^{r-2})$ for all n.

A weaker upper bound

$$\mathrm{ex}_r(n,T) \leq (e(T)-1)\binom{n}{r-1} \qquad \text{for each tight } r\text{-tree } T \tag{2}$$

is implicit in several earlier works, and is explicit in [9] (see Proposition 5.4 there).

To prove Conjecture 1.1, we need to improve the bound in (2) by a factor of r. This turns out to be difficult even for very special cases of tight trees. It is only recently that the authors [8] were able to improve (2) in the case T is the tight r-uniform path with t edges by a factor of $1 - 1/r$. (For short paths, $t < (3/4)r$, Patkós [19] proved better coefficients).

So far, the only family of tight trees for which Kalai's conjecture is verified is the family of so-called *star-shaped* trees. A tight r-tree T is *star-shaped* if it contains an edge e_0 such that $|e \cap e_0| = r-1$ for each $e \in T \setminus \{e_0\}$.

Theorem 1.2 ([5])**.** *Let $n, r, t \geq 2$ be integers. Let G be an n-vertex r-graph with $e(G) > \frac{t-1}{r}\binom{n}{r-1}$. Then G contains every star-shaped tight r-tree with t edges.*

Given a tight r-tree T and a tight subtree T' of T, we say that T' is a *trunk* of T if there exists an edge-ordering of T satisfying (1) such that the edges of T' are listed first and for each $e \in E(T) \setminus E(T')$ there exists $e' \in E(T')$ such that $|e \cap e'| = r-1$. Let $c(T)$ be the minimum number of edges in a trunk of T. Hence, a star-shaped tight tree is a tight tree T with $c(T) = 1$, and Theorem 1.2 says that Kalai's Conjecture holds for tight r-trees T with $c(T) = 1$. Note from the definition above that for a tight tree T having $c(T) \leq c$ is equivalent to saying that all but at most c edges of T contain a vertex of degree 1.

The primary goal of this paper is to extend Theorem 1.2 to tight trees of bounded trunk size. Our main theorem says that for every fixed integers $r \geq 2$ and $c \geq 1$, Kalai's Conjecture holds asymptotically in $e(T)$ for tight r-trees T with $c(T) \leq c$.

Theorem 1.3. *Let n, r, t, c be positive integers, where $n \geq r \geq 2$ and $t \geq c \geq 1$. Let $a(r,c) = (r^r + 1 - \frac{1}{r})(c-1)$. Let T be a tight r-tree with t edges and $c(T) \leq c$. Then*

$$\mathrm{ex}_r(n,T) \leq \left(\frac{t-1}{r} + a(r,c)\right)\binom{n}{r-1}. \tag{3}$$

Note that Theorem 1.2 follows from Theorem 1.3 by setting $c = 1$. The main point of Theorem 1.3 is that the coefficient in front of $\binom{n}{r-1}$ is $(t-1)/r + O_{r,c}(1)$, while the coefficient in Kalai's conjecture is $(t-1)/r$.

We also give a (simple) proof of the fact that Kalai's Conjecture holds for tight r-trees with at most four edges.

Theorem 1.4. *Let $n \geq r \geq 2$ be integers and T be a tight r-tree with $t \leq 4$ edges. Then*

$$\mathrm{ex}_r(n,T) \leq \frac{t-1}{r}\binom{n}{r-1}.$$

The proofs of Theorems 1.3 and 1.4 are postponed to Sections 4 and 5.

2 Tight trees and shadows

An important notion in extremal set theory is that of *shadow*. Given an r-graph G, the *shadow* of G is

$$\partial(G) = \{S : |S| = r-1, \quad \textit{and} \quad S \subseteq e \quad \textit{for some} \quad e \in e(G)\}.$$

The result in [5] is more explicit than Theorem 1.2. It was shown that if T is any star-shaped tight r-tree with t edges and G is a T-free r-graph then $e(G) \leq \frac{t-1}{r}|\partial(G)|$, from which Theorem 1.2 immediately follows. There were several other results in the literature that bound the size of an H-free r-graph in terms of the size of its shadow. Katona [14] showed that if G is an intersecting r-graph then $e(G) \leq |\partial(G)|$. This is known as the *intersection shadow theorem*. More recently, Frankl [4] showed that if G is an r-graph that does not contain a matching of size $s+1$ then $e(G) \leq s|\partial(G)|$. Sometimes it is easier prove the bounds in terms of the shadow size than in terms of n using induction. Instead of Theorems 1.3–1.4 we will prove bounds on $e(G)$ in terms of $|\partial(G)|$, from which Theorems 1.3–1.4 will follow.

Based on our results, we propose the following conjecture, which we will show is equivalent to Kalai's conjecture.

Conjecture 2.1. *Let $r \geq 2, t \geq 1$ be integers. Let T be a tight r-tree with t edges. If G is an r-graph that does not contain T then $e(G) \leq \frac{t-1}{r}|\partial(G)|$.*

The lower bound constructions obtained from designs mentioned earlier show that the bound in Conjecture 2.1, if true, would be tight. Since for every r-graph G on n vertices one has $|\partial(G)| \leq \binom{n}{r-1}$ Conjecture 2.1 obviously implies Conjecture 1.1. We will show in Theorem 2.3 that Conjecture 1.1 also implies Conjecture 2.1.

Proposition 2.2. *Conjecture 2.1 is equivalent to Kalai's conjecture.*

Theorem 2.3. *If T is a tight tree then the limit*

$$\alpha(T) := \lim_{n\to\infty} \mathrm{ex}_r(n,T)/\binom{n}{r-1}$$

exists and is equal to its supremum. Moreover,

$$\alpha(T) = \sup\left\{\frac{e(G)}{|\partial(G)|} : G \text{ is a } T\text{-free } r\text{-graph}\right\}.$$

In particular for $\alpha := \alpha(T)$ we have $\mathrm{ex}(n,T) \le \alpha\binom{n}{r-1}$ and $e(G) \le \alpha|\partial(G)|$ for every n and for every T-free r-graph G.

Let H be a u-uniform hypergraph on v vertices ($v \ge u \ge 1$). An *almost disjoint induced packing* of H of size m on n vertices consists of v-subsets of $[n]$, $V_1,\dots,V_m$, and m copies of H on these sets, $H_1,\dots,H_m$, such that either $|V_i\cap V_j| < u$ or $|V_i\cap V_j| = u$, but in the latter case $V_i\cap V_j$ is not an edge of any of the H_k's. So V_k induces H_k in the union $\cup E(H_i)$. Obviously, $m \le \binom{n}{u}/e(H)$. For the proof of Theorem 2.3 we need a result from [6] about the existence of almost perfect induced packings of subhypergraphs with nearly disjoint vertex sets. We recall it in the form we need. Given H as $n\to\infty$ one has

$$\max m = (1+o(1))\binom{n}{u}/e(H). \tag{4}$$

In fact (4) is an application of the packing result of Frankl and Rödl [7] .

Lemma 2.4. *Let T be a tight r-tree and suppose that G is a T-free r-graph. Then for every $\varepsilon > 0$, there exists $n_0 = n_0(T,G,\varepsilon)$ such that for all $n > n_0$*

$$\mathrm{ex}(n,T) > \left(\frac{e(G)}{|\partial(G)|} - \varepsilon\right)\binom{n}{r-1}.$$

Proof of Lemma 2.4. To get a lower bound we need a construction F, a T-free r-graph on n vertices. Define $H = \partial(G)$ and apply (4) (with $u = r-1$) to obtain near optimal number of copies of $\partial(G)$, $H_1,\dots,H_m$ with vertex sets $V_1,\dots,V_m$. Put a copy of G, G_i, on each V_i such that $\partial(G_i) = H_i$. The resulting copies of G share no $(r-1)$-shadow and in particular are edge-disjoint. The union $F = \cup E(G_i)$ has $(1-o(1))(e(G)/|\partial(G)|)\binom{n}{r-1}$ edges and it is T-free. Indeed, F cannot contain a tight tree that moves from one copy of G_i to another. When we start to build the tree $T = \{e_1,\dots,e_t\}$ with $e_1 \in G_i$ then all other edges e_j must also belong to G_i so there is no such tree in F. □

Note that a similar proof idea was used by Huang and Ma [12] to disprove an Erdős-Sós/Verstraëte conjecture concerning tight cycles.

Remark 2.5. It follows from the proof of Lemma 2.4 that it still holds if T is replaced with any r-graph with a *connected* $(r-1)$-intersection graph, meaning that the auxiliary graph defined on $E(T)$ where $e, e' \in E(T)$ are adjacent if and only if $|e\cap e'| = r-1$ is connected.

Proof of Theorem 2.3. Define

$$\begin{aligned}\alpha(n,T) &:= \mathrm{ex}_r(n,T)/\binom{n}{r-1},\\ \beta(n,T) &:= \max\left\{\frac{e(G)}{|\partial G|}:\ G \text{ is a } T\text{-free } r\text{-graph on } n \text{ vertices}\right\}.\end{aligned}$$

Since $\beta(n,T) \le \beta(n+1,T)$ and $\beta(n,T) \le e(T)-1$ (by (2)) the limit $\beta = \beta(T) = \lim_{n\to\infty}\beta(n,T)$ exists, is positive, and is equal to its supremum. Since $\alpha(n,T) \le \beta(n,T)$ we have $\sup_n \alpha(n,T) \le \beta$. The proof of the existence of the limit α can be completed by Lemma 2.4 showing that for every $\varepsilon > 0$ taking a T-free r-graph G with $\frac{e(G)}{|\partial(G)|} > \beta - \varepsilon$ there exists an n_0 such that $\alpha(n,T) > \beta - 2\varepsilon$ for all $n > n_0$. □

3 Notation and preliminaries

Given an r-graph G and a subset $D \subseteq V(G)$, we define the *link* of D in G, denoted by $L_G(D)$, to be

$$L_G(D) = \{e \setminus D : e \in E(G), D \subseteq e\}.$$

The *degree* of D, denoted by $d_G(D)$, is defined to be $|L_G(D)|$; equivalently it is the number of edges of G that contain D. When G is r-uniform and $|D| = r-1$, elements of $L_G(D)$ are vertices. In this case, we also use $N_G(D)$ to denote $L_G(D)$ and call it the *co-neighborhood* of D in G. When the context is clear we will drop the subscripts in $L_G(D)$, $N_G(D)$ and $d_G(D)$. For each $1 \le p \le r-1$, we define the *minimum p-degree* of G to be

$$\delta_p(G) = \min\{d_G(D) : |D| = p, \quad \textit{and} \quad D \subseteq e \quad \textit{for some} \quad e \in E(G)\}.$$

Given an r-graph G, and $D \in \partial(G)$, let $w(D) = \frac{1}{d_G(D)}$. For each $e \in E(G)$, let

$$w(e) = \sum_{D \in \binom{e}{r-1}} w(D) = \sum_{D \in \binom{e}{r-1}} \frac{1}{d_G(D)}. \tag{5}$$

We call w the *default weight function* on $E(G)$ and $\partial(G)$. The following simple property of the default weight function is key to the weight method, employed in [5] and in various other works.

Proposition 3.1. *Let G be an r-graph. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then*

$$\sum_{e \in E(G)} w(e) = |\partial(G)|.$$

Proof. By definition,

$$\sum_{e\in E(G)} w(e) = \sum_{e\in E(G)} \sum_{D\in\binom{e}{r-1}} \frac{1}{d_G(D)} = \sum_{D\in\partial(G)} \sum_{e\in E(G), D\subseteq e} \frac{1}{d_G(D)} = \sum_{D\in\partial(G)} 1 = |\partial(G)|.$$

□

An r-graph G is called *r-partite* if $V(G)$ can be partitioned into r sets $A_1, \dots, A_r$ such that every edge of G contains one vertex from each A_i. We call $(A_1, \dots, A_r)$ an *r-partition* of G. Equivalently, we say that an r-graph G is *r-colorable* if G if there exists a vertex coloring of G with r colors such that each edge uses all r colors; we call such a coloring a *proper r-coloring* of G. The following proposition follows by induction on the number of edges in T.

Proposition 3.2. *Let $r \geq 2$. Every tight r-tree T has a unique r-partition.* □

Given r-graphs G and H, an *embedding* of H into G is an injection $f : V(H) \to V(G)$ such that for each $e \in E(H)$, $f(e) \in E(G)$.

Proposition 3.3. (Color-preserving embedding) *Let T be a tight r-tree with t edges. Let φ be a proper r-coloring of T. Let G be an r-partite graph with $\delta_{r-1}(G) \geq t$, where $(A_1, \cdots, A_r)$ is an r-partition of G. Then there exists an embedding f of T into G such that for each $u \in V(T)$ $f(u) \in A_{\varphi(u)}$.*

Proof. We use induction on t. The base step is trivial. Now, suppose $t \geq 2$. Let $e_1, \dots, e_t$ be an ordering of the edges of T that satisfies (1). Let $T' = T \setminus e_t$. Then T' is a tight r-tree with $t-1$ edges. By the induction hypothesis, there exists an embedding f of T' into G such that for each $u \in V(T')$, $f(u) \in A_{\varphi(u)}$. Let $D = e_t \cap e_{\alpha(t)}$ and let v be the unique vertex in $e_t \setminus e_{\alpha(t)} = V(T) \setminus V(T')$. Then $e_t = D \cup \{v\}$. Since $f(D)$ is an $(r-1)$-set contained in $f(e_{t-1})$ and $\delta_{r-1}(G) \geq t$, $d_G(f(D)) \geq t$. So there are at least t edges of G containing $f(D)$, at most $|V(T')| - (r-1) = t-1$ of which contain a vertex of $f(T')$. Hence there exists an edge e in G that contains $f(D)$ and a vertex z outside $f(T')$. We extend f by letting $f(v) = z$. Now f is an embedding of T into G.

It remains to show that $z \in A_{\varphi(v)}$. By permuting colors if needed, we may assume that $\varphi(v) = r$. Since $D \cup \{v\} \in E(T)$ and φ is proper, the colors used in D are $1, \dots, r-1$. By our assumption, vertices in $f(D)$ lie in $A_1, \dots, A_{r-1}$, respectively, which implies $z \in A_r$. □

The following proposition is folklore. We include a proof for completeness,

Proposition 3.4. *Let $r \geq 2$ and $q \geq 1$ be integers and let G be an r-graph with $e(G) > q|\partial(G)|$. Then G contains a subgraph G' with $\delta_{r-1}(G') \geq q+1$ and*

$$e(G') > q|\partial(G')|. \tag{6}$$

Proof. Among subgraphs G' of G satisfying (6), choose one with the fewest edges. We claim that $\delta_{r-1}(G') \geq q+1$. Indeed, if there is $D \in \partial(G')$ that is contained in at most q edges of G', then the r-graph G'' obtained from G' by deleting all edges containing D again satisfies (6), but has fewer edges than G', a contradiction. □

Another useful folklore fact is:

Proposition 3.5. *Let α be a positive real, $r \geq 3$ be an integer and G be an r-graph with $e(G) > \frac{\alpha}{r}|\partial(G)|$. Then there is $v \in V(G)$ such that the link $G_1 := L_G(\{v\})$ satisfies*

$$e(G_1) > \frac{\alpha}{r-1}|\partial(G_1)|.$$

Proof. Suppose that $|L_G(\{v\})| \le \frac{\alpha}{r-1}|\partial(L_G(\{v\})|$ for each $v \in V(G)$. Then

$$r \cdot e(G) = \sum_{v\in V(G)} d_G(v) = \sum_{v\in V(G)} |L_G(\{v\})| \le \frac{\alpha}{r-1} \sum_{v\in V(G)} |\partial(L_G(\{v\})|.$$

Since each edge $f \in \partial(G)$ contributes $r-1$ to $\sum_{v\in V(G)} |\partial(L_G(\{v\})|$ (1 to the link of each its vertex), this proves the proposition. □

We also need the following fact used in [5].

Proposition 3.6. *Let r be a positive integer. Let $d_1 \le d_2, \cdots \le d_r$ be positive reals. If $\sum_{i=1}^r \frac{1}{d_i} = s$, then for each $i \in [r]$, $d_i \ge \frac{i}{s}$.*

Proof. For each $i \in [r]$, since $\frac{1}{d_1} \ge \cdots \ge \frac{1}{d_i}$, we have $\frac{i}{d_i} \le \sum_{j=1}^i \frac{1}{d_j} \le s$. So, $d_i \ge \frac{i}{s}$. □

4 Proof of Theorem 1.3 on trees with bounded trunks

As discussed in the introduction, we prove the following stronger version of Theorem 1.3.

Theorem 1.3′. *Let n, r, t, c be positive integers, where $n \ge r \ge 2$ and $t \ge c \ge 1$. Let $a(r,c) = (r^r + 1 - \frac{1}{r})(c-1)$. Let T be a tight r-tree with t edges and $c(T) \le c$. If G is an r-graph that does not contain T then*

$$e(G) \le \left(\frac{t-1}{r} + a(r,c)\right)|\partial(G)|. \tag{7}$$

Proof of Theorem 1.3 ′. Suppose T is a tight r-tree with t edges and $c(T) = c$. Let G be an n-vertex r-graph with $e(G) > (\frac{t-1}{r} + a(r,c))|\partial(G)|$. We show that G contains T. For convenience, let

$$\gamma = \frac{t-1}{r} + a(r,c) - r^r(c-1) = \frac{t-1}{r} + (1 - \frac{1}{r})(c-1).$$

Then

$$e(G) > (\gamma + r^r(c-1))|\partial(G)|.$$

Let w be the default weight function on $E(G)$ and $\partial(G)$. By Proposition 3.1, $\sum_{e\in E(G)} w(e) = |\partial(G)|$. Let

$$H = \{e \in E(G) : w(e) \ge \frac{1}{\gamma}\} \text{ and } L = \{e \in E(G) : w(e) < \frac{1}{\gamma}\}.$$

By the definition of H,

$$\frac{1}{\gamma} e(H) \le \sum_{e\in H} w(e) \le \sum_{e\in G} w(e) = |\partial(G)|.$$

Hence $e(H) \le \gamma|\partial(G)|$. Since $e(G) > (\gamma + r^r(c-1))|\partial(G)|$, we have

$$e(L) > r^r(c-1)|\partial(G)|.$$

By averaging, L contains an r-partite subgraph L_1 with

$$e(L_1) \geq \frac{r!}{r^r} e(L) > \frac{r!}{r^r} r^r (c-1)|\partial(G)| \geq r!(c-1)|\partial(G)|. \tag{8}$$

Let $(A_1, \ldots, A_r)$ be an r-partition of L_1. Let $e \in E(L_1)$. Let σ be a permutation of $[r]$ such that

$$d_G(e \setminus A_{\sigma(1)}) \leq \cdots \leq d_G(e \setminus A_{\sigma(r)}).$$

We let $\pi(e) = (\sigma(1), \ldots, \sigma(r))$ and refer to it as the *pattern* of e. Since there are $r!$ different permutations of $[r]$, by the pigeonhole principle, some $\lceil e(L_1)/r! \rceil$ edges e of L_1 have the same pattern $\pi(e)$. Let L_2 be the subgraph of L_1 consisting of these edges. By (8),

$$e(L_2) \geq \frac{e(L_1)}{r!} > (c-1)|\partial(G)|.$$

By Lemma 3.4, L_2 contains a subgraph L_2^* such that

$$\delta_{r-1}(L_2^*) \geq c.$$

Recall that all edges in $L_2^* \subseteq L_1$ have the same pattern. By permuting indices if needed, we may assume that $\pi(e) = (1, 2, \ldots, r)$ for each $e \in L_2^*$. By our assumption,

$$d_G(e \setminus A_1) \leq \cdots \leq d_G(e \setminus A_r) \qquad \forall e \in L_2^*. \tag{9}$$

Also, by the definition of L,

$$w(e) = \sum_{i=1}^{r} \frac{1}{d_G(e \setminus A_i)} < \frac{1}{\gamma} \qquad \forall e \in L_2^* \subseteq L.$$

By Lemma 3.6 and (9), we have

$$d_G(e \setminus A_i) > i\gamma \qquad \forall e \in L_2^* \qquad \forall i \in [r]. \tag{10}$$

Now consider a trunk T' of T with c edges. By the definition of a trunk, if E' is any subset of $E(T) \setminus E(T')$ then $T' \cup E'$ is a tight tree with $c + |E'|$ edges. By Proposition 3.2, T' is r-partite. Let $(B_1, \ldots, B_r)$ be an r-partition of T'. For each $e \in E(T) \setminus E(T')$, by definition, there exists $\alpha(e) \in E(T')$ such that $|e \cap \alpha(e)| = r - 1$. Thus, $e \cap \alpha(e) = \alpha(e) \setminus B_i$ for some unique $i \in [r]$. For each $i \in [r]$, let

$$E_i = \{e \in E(T) \setminus E(T') : e \cap \alpha(e) = \alpha(e) \setminus B_i\}.$$

By permuting the subscripts in the r-partition $(B_1, \ldots, B_r)$ of T' if needed, we may assume that

$$|E_1| \leq \cdots \leq |E_r|.$$

Since $\sum_{i=1}^{r} |E_i| = t - c$, this implies

$$|E_1| + \cdots + |E_i| \leq \left\lfloor \frac{i(t-c)}{r} \right\rfloor \qquad \forall i \in [r]. \tag{11}$$

Since $e(T')=c$, $\delta_{r-1}(L_2^*)\ge c$, $(A_1,\dots,A_r)$ is an r-partition of L_2^* and $(B_1,\dots,B_r)$ is an r-partition of T', by Proposition 3.3, there exists an embedding h of T' into L_2^* such that for each $i\in[r]$ every vertex in B_i of T' is mapped into A_i. Now consider the edges in E_1. By the definition of E_1, for each $e\in E_1$ there is $\alpha(e)\in E(T')$ such that $e\cap\alpha(e)=\alpha(e)\setminus B_1$ and $h(\alpha(e\setminus B_1))=h(\alpha(e))\setminus A_1$. Since $h(\alpha(e))\in L_2^*$, by (10),

$$d_G(h(\alpha(e)\setminus A_1))\ge\lfloor\gamma\rfloor+1\qquad\forall e\in E_1. \tag{12}$$

Since $T'\cup E_1$ is a tight tree with

$$|E_1|+c\le\lfloor\frac{t-c}{r}\rfloor+c=\lfloor\frac{t-1}{r}+(1-\frac{1}{r})(c-1)\rfloor+1=\lfloor\gamma\rfloor+1$$

edges, and h is an embedding of T' into G, (12) ensures that we can greedily extend h to an embedding of $T'\cup E_1$ into G. In general, let $i\in[r]\setminus\{1\}$ and suppose that we have extended h to an embedding of $T'\cup E_1\cup\cdots\cup E_{i-1}$ into G. By the definition of E_i, for each $e\in E_i$ there is $\alpha(e)\in T'$ such that $e\cap\alpha(e)=\alpha(e)\setminus B_i$ and $h(e\cap\alpha(e))=h(\alpha(e))\setminus A_i$. By (10),

$$d_G(h(e\cap\alpha(e))\ge\lfloor i\gamma\rfloor+1\qquad\forall e\in E_i. \tag{13}$$

Since $T'\cup E_1\cup\cdots\cup E_i$ is a tight tree with

$$c+|E_1|+\cdots+|E_i|\le\lfloor\frac{i(t-c)}{r}\rfloor+c\le\lfloor i\gamma\rfloor+1$$

edges, and h is already an embedding of $T'\cup E_1\cup\cdots\cup E_{i-1}$ into G, (13) ensures that we can greedily extend h further to an embedding of $T'\cup E_1\cup\cdots\cup E_i$ into G. Hence we can find an embedding of T into G. □

5 Proof of Theorem 1.4 on trees with four edges

Again, we are proving the shadow version of the theorem:

Theorem 1.4′. *Let $n\ge r\ge 2$ be integers and T be a tight r-tree with $t\le 4$ edges. If G is an r-graph that does not contain T then $e(G)\le\frac{t-1}{r}|\partial(G)|$.*

We start from a partial case of such T, the 3-uniform tight path P_4^3 with 4 edges. The case of the path P_5^3 is still unsolved (to our knowledge).

Lemma 5.1. *Let $n\ge 5$ and G be an n-vertex 3-graph containing no tight path P_4^3 with four edges. Then $e(G)\le|\partial(G)|$.*

Observe that for 3-graphs Lemma 5.1 is stronger than Katona's intersecting shadow theorem, since an intersecting 3-graph must be P_4^3-free. There are many nearly extremal families with very different structures for Lemma 5.1 besides the ones obtained from Steiner systems $S(n,5,2)$. Here we mention just two. First, one can observe that the Erdős-Ko-Rado family $G:=\{g\in\binom{[n]}{3}:1\in g\}$ is P_4^3-free with

$$|\partial(G)|=\binom{n}{2}=\frac{n}{n-2}\binom{n-1}{2}=\frac{n}{n-2}e(G).$$

Second, for $n \equiv 0 \mod 3$ one can take a tournament $\overrightarrow{D}$ on $n/3$ vertices and a partition of $[n]$ into triples $V_1, V_2, \ldots, V_{n/3}$ and define the P_4^3-free triple system as

$$G := \left\{ g \in \binom{[n]}{3} : \text{ for some } \overrightarrow{ij} \in E(\overrightarrow{D}) \text{ one has } |V_i \cap g| = 2, |V_j \cap g| = 1 \right\}.$$

Then we have $|\partial(G)|/e(G) = \binom{n}{2}/9\binom{n/3}{2} = (n-1)/(n-3)$.

Proof of Lemma 5.1. Suppose G is an n-vertex 3-graph with the fewest edges such that

$$e(G) > |\partial(G)| \text{ and } G \text{ contains no } P_4^3. \tag{14}$$

By Proposition 3.4 and the minimality of G,

$$\delta_2(G) \geq 2. \tag{15}$$

Let w be the default weight function on G and $\partial(G)$. Since $\sum_{e\in G} w(e) = |\partial(G)| < e(G)$, by (14), G has an edge $e_0 = abc$ with

$$w(e_0) = \frac{1}{d(ab)} + \frac{1}{d(ac)} + \frac{1}{d(bc)} < 1. \tag{16}$$

We may assume $d(ab) \leq d(ac) \leq d(bc)$. Similarly to Proposition 3.6, in order (16) to hold, we need

$$d(ac) \geq 3 \quad \text{and} \quad d(bc) \geq 4. \tag{17}$$

By (15) and (16), we can greedily choose distinct $a', b', c' \in V(G) - \{a, b, c\}$ so that $abc', acb', bca' \in G$.

We claim that

$$ab'b, ac'c \in G. \tag{18}$$

Indeed, by (15) G has an edge $ab'x$ for some $x \neq c$. If $x \notin \{b, a'\}$, then G has a tight 4-path $a'bcab'x$, a contradiction to (14). So suppose $x = a'$. By (17), G has an edge bcy for some $y \notin \{a, a', b'\}$. Then G has a tight 4-path $ybcab'a'$, again a contradiction to (14). Thus $ab'b \in G$. Similarly, $ac'c \in G$, and (18) holds.

Next we similarly show that

$$a'ba, a'ca \in G. \tag{19}$$

Indeed, by (15) G has an edge $a'bx$ for some $x \neq c$. If $x \notin \{a, b'\}$, then G has a tight 4-path $b'acba'x$. Suppose $x = b'$. Then by (18), G has a tight 4-path $b'a'bcac'$, again a contradiction to (14). Thus $a'ba \in G$. Similarly, $a'ca \in G$, and (19) holds.

Together, (18) and (19) imply that $d_G(ab) \geq 4$ and $d_G(ac) \geq 4$. So, the proof of (18) yields similarly that $c'bc, b'cb \in G$. If the degree of each of $a'a, a'b, a'c$ is 2, then the 3-graph $G_2 = G \setminus \{a'ab, a'ac, a'bc\}$ has $|G| - 3$ edges and $|\partial(G_2)| = |\partial(G)| - 3$, a contradiction to the minimality of G. Thus we may assume that G has an edge $a'ax$, where $x \notin \{b, c\}$. By the symmetry between b' and c', we may assume $x \neq b'$. Then G has a tight 4-path $xa'abcb'$. □

Now we are ready to prove Theorem 1.4′.

Proof of Theorem 1.4 ′. We use induction on r.

Base Step. $r = 2$. In this case, $\partial(G) = V(G)$. For $t \le 3$ the statement is trivial. Let $t = 4$. There are three non-isomorphic (graph) trees with four edges: the path $P_4 = v_0v_1v_2v_3v_4$ with 4 edges, the star S_4 with center v_0 and leaves v_1, v_2, v_3, v_4, and the tree F_4 obtained from the star S_4 by replacing edge v_0v_4 with edge v_4v_3. So we want to show that for every graph G

$$\text{if } e(G) > \tfrac{3}{2}|V(G)| \text{ then } G \text{ contains each of } P_4, S_4 \text{ and } F_4. \tag{20}$$

The case of $T = P_4$ is a special case of the Erdős-Gallai Theorem [3]. The other two possibilities also are known from the literature, but we give a short proof. Consider a counterexample G to (20) with the fewest vertices. By Proposition 3.4, $\delta(G) \ge 2$. Since $\sum_{a\in V(G)} d(a) = 2e(G) > 3|V(G)|$, there is $a \in V(G)$ with $s \ge 4$ neighbors, say, $b_1, \dots, b_s$. In particular, G contains S_4 with center a. Since $\delta(G) \ge 2$, b_1 has a neighbor $b \ne a$. So we can embed F_4 into $G[\{a, b, b_1, \dots, b_4\}]$ by sending v_0 to a, v_3 to b_1, v_4 to b, and v_1 and v_2 to two vertices in $\{b_2, b_3, b_4\} - b$. Thus (20) holds.

Induction Step. Suppose $r \ge 3$, the theorem holds for all $r' < r$, T is a tight r-tree, and G is an r-graph with $e(G) > \frac{t-1}{r}|\partial(G)|$.

Case 1: T has a vertex v belonging to all edges. Let T_1 be the link $L_T(\{v\})$ of v. It is a tight $(r-1)$-tree with t edges. By Proposition 3.5, there is $a \in V(G)$ such that the link $G_1 := L_G(\{a\})$ satisfies $e(G_1) > \frac{t-1}{r-1}|\partial(G_1)|$. By the induction assumption, there is an embedding φ of T_1 into G_1. Then by letting $\varphi(v) = a$ we obtain an embedding of T into G.

Case 2: T has no vertex belonging to all edges. By the definition of a tight r-tree, this is possible only if $t = 4$, $r = 3$ and $T = P_4^3$. In this case, we are done by Lemma 5.1. □

6 Concluding remarks

- Theorem 2.3 shows that some shadow theorems in the literature are not really stronger than their nonshadow versions. In particular, this is the case whenever the forbidden r-graph T has a connected $(r-1)$-intersection graph (see Remark 2.5).

- It would be interesting to decide if Lemma 2.4 holds for other r-graphs besides tight trees and also for which r-graphs T $\lim_{n\to\infty} \mathrm{ex}_r(n, T)/\binom{n}{r-1}$ exist. In particular, we ask if $\lim_{n\to\infty} \mathrm{ex}_r(n, T)/\binom{n}{r-1}$ exists for each r-uniform tree T, where an r-graph is a *tree* if it is a subgraph of a tight tree. This question is not even solved when $r = 2$ and T is a graph forest, see, e.g., [18]. See [9] and [17] for recent results on the Turán numbers of some large families of r-uniform trees.

  Note that even for trees, if the limits $\alpha(T)$ and $\beta(T)$ exist they need not be equal. (See the proof of Theorem 2.3 for the definition of $\alpha(T)$ and $\beta(T)$.) Consider a linear path $P = P_4^r$ of length four, $E(P) := \{\{1, 2, \dots, r\}, \{r, r+1, \dots, 2r-1\}, \{2r-1, 2r, \dots, 3r-2\}, \{3r-2, 3r-1, \dots, 4r-3\}\}$. It is known [9, 17] that $\mathrm{ex}(n, T) = \binom{n-1}{r-1} + \binom{n-3}{r-2} + \varepsilon(n, r)$ for $n > n_0(r)$ and $r \ge 3$, where $\varepsilon(n, r) = 0$ except for $r = 3$, when it is 0, 1 or 2. So we have $\alpha(P) = 1$. On the other hand, the complete

r-graph G on $4r-4$ vertices avoids P_4^r and $e(G)/|\partial G| = \binom{4r-4}{r}/\binom{4r-4}{r-1} = (3r-3)/r \le \beta(P)$. Consequently, $0 < \alpha(P) < \beta(P)$ for $r \ge 3$. (Actually, a linear path of length 3 is also an appropriate example).

In the case $r = 2$ consider $T = kP_2$, a disjoint union of k paths of length 2 on $3k$ vertices. Gorgol [11] showed that $\alpha(kP_2) = k - 1/2$ while considering the complete graph on $3k-1$ vertices we get $\beta(kP_2) \ge (3k-2)/2$. Moreover, the Erdős-Gallai Theorem implies that here equality holds here.

- Recent substantial work by Keller and Lifshitz [16] studies the Turán number of some r-graphs F with small core. However their *junta method for hypergraphs* does not seem to apply here, since it seems to require that $r \gg |C|$ where C is the set of the vertices of F of degree at least 2.

- A direction we will continue to pursue is to reduce the error term $a(r,c)$ in the coefficient in Theorem 1.3. We have some nontrivial improvements. For example, in the first unsolved case, that is, when T is a 3-uniform tight tree with $c(T) = 2$, we have a proof that $a(3,2) \le 1/3$. Thus we have $\beta(T) \le t/3$ and $\mathrm{ex}_3(n,T) \le (t/3)\binom{n}{2}$.

Acknowledgements. This research was partly conducted during an American Institute of Mathematics Structured Quartet Research Ensembles workshop, and we gratefully acknowledge the support of AIM.

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Zoltán Füredi
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Reáltanoda utca 13-15
H-1053, Budapest, Hungary
E-mail: zfuredi@gmail.com.

Tao Jiang
Department of Mathematics
Miami University
Oxford, OH 45056, USA.
E-mail: jiangt@miamioh.edu.

Alexandr Kostochka
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
and Sobolev Institute of Mathematics
Novosibirsk 630090, Russia.
E-mail: kostochk@math.uiuc.edu.

Dhruv Mubayi
Department of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
Chicago, IL 60607, USA.
E-mail: mubayi@uic.edu.

Jacques Verstraëte
Department of Mathematics
University of California at San Diego
9500 Gilman Drive, La Jolla, California 92093-0112, USA.
E-mail: jverstra@math.ucsd.edu.