HOCHSCHILD CONIVEAU SPECTRAL SEQUENCE AND THE BEILINSON RESIDUE

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Abstract. Higher-dimensional residues can be constructed either following Grothendieck–Hartshorne and local cohomology, or following Tate–Beilinson using Lie algebra homology. We show that there is a natural link: we develop the Hochschild analogue of the coniveau spectral sequence. The rows of our spectral sequence look a lot like the Cousin complexes in *Residues & Duality*. Note that these are for coherent cohomology. We prove that they agree by an 'HKR isomorphism with supports'. Using the close ties of Hochschild homology to Lie algebra homology, this yields a direct comparison.

The coniveau spectral sequence and the Gersten complex originally arose in algebraic K-theory. But if one replaces in its construction K-theory by Hochschild homology, everything goes through. Still, it appears that this analogue has not really been studied much so far (if at all). We discuss it in this paper, building on the work of B. Keller [Kel98b] and P. Balmer [Bal09].

Let X/k be a Noetherian separated scheme. Write HH^x for the Hochschild spectrum with support in a point x. Our Hochschild–Cousin complex will take the form

$$\cdots \longrightarrow \coprod_{x\in X^0} HH^x_{-q}(\mathscr{O}_{X,x}) \longrightarrow \coprod_{x\in X^1} HH^x_{-q-1}(\mathscr{O}_{X,x}) \longrightarrow \cdots \longrightarrow \coprod_{x\in X^n} HH^x_{-q-n}(\mathscr{O}_{X,x}) \longrightarrow \cdots$$

and appears as the rows in the E_1-page of a corresponding Hochschild coniveau spectral sequence,

$$^{HH}E_1^{p,q} := \coprod_{x\in X^p} HH^x_{-p-q}(\mathscr{O}_{X,x}) \Rightarrow HH_{-p-q}(X). \tag{0.1}$$

Some things are different from K-theory: As Hochschild homology does *not* satisfy dévissage, one cannot replace the HH^x by Hochschild homology of the residue field.

There is a similar, but much older complex: The *coherent cohomology* Cousin complex from *Residues & Duality* [Har66]. It has the form

$$\cdots \longrightarrow \coprod_{x\in X^0} H^q_x(X,\mathscr{F}) \longrightarrow \coprod_{x\in X^1} H^{q+1}_x(X,\mathscr{F}) \longrightarrow \cdots \longrightarrow \coprod_{x\in X^n} H^{q+n}_x(X,\mathscr{F}) \longrightarrow \cdots, \tag{0.2}$$

where $H^\bullet_x$ denotes (coherent) local cohomology of a coherent sheaf $\mathscr{F}$ with support in a point x. It also arises as a row in the E_1-page of a spectral sequence $^{Cous}E_1^{p,q}(\mathscr{F}) \Rightarrow H^{p+q}(X,\mathscr{F})$. We prove that if X/k is smooth and $\mathscr{F} := \Omega^*_{X/k}$, this complex is canonically isomorphic to our Hochschild–Cousin complex. We do this by a Hochschild–Kostant–Rosenberg (HKR) isomorphism with supports:

Theorem (HKR with supports)**.** *Let k be a field, R a smooth k-algebra and $t_1,\ldots,t_n$ a regular sequence. Then there is a canonical isomorphism*

$$H^n_{(t_1,\ldots,t_n)}(R,\Omega^{n+i}) \xrightarrow{\sim} HH_i^{(t_1,\ldots,t_n)}(R).$$

For $n=0$, this becomes the classical HKR isomorphism. On the left, H^*_I refers to (coherent) local cohomology. On the right hand side, HH^I_* refers to the Hochschild homology of the category of I-supported perfect complexes.

Although not being spelled out in this form, this is a consequence of the much more general theory due to Keller [Kel98b]. We shall need this explicit formulation, and give a very elementary proof. This will be

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Proposition 2.0.1. We feel that this straight-forward extension of the standard HKR isomorphism would deserve to be much more widely known.

Theorem. *Let k be a field and X/k a smooth separated scheme. For every integer q, the q-th row on the E_1-page of the Hochschild coniveau spectral sequence is isomorphic to the zero-th row of the E_1-page of the coherent cohomology Cousin coniveau spectral sequence of Ω^{-q}. That is: For every integer q, there is a canonical isomorphism of chain complexes*

$$^{HH}E_1^{\bullet,q} \xrightarrow{\sim} {}^{Cous}E_1^{\bullet,0}(\Omega^{-q}) .$$

Entry-wise, this isomorphism is induced from the HKR isomorphism with supports.

This will be Theorem 2.1.3. As all the other rows on the right-hand side turn out to vanish, one can re-package this claim as follows:

$$^{HH}E_1^{\bullet,\bullet} \xrightarrow{\sim} {}^{Cous}E_1^{\bullet,\bullet}(\tilde{\Omega}) \qquad \text{with} \qquad \tilde{\Omega} := \bigoplus_m \Omega^m[m].$$

Theorem. *Let k be a field of characteristic zero and X/k a smooth separated scheme. Then the Hochschild coniveau spectral sequence of line 0.1 degenerates on the E_2-page.*

This kind of behaviour is exactly the same as for a spectral sequence due to C. Weibel [Wei97], which also converges to the Hochschild homology of X, starting from $^{Weibel}E_2^{p,q} = H^p(X, \mathscr{H}\mathscr{H}_{-q})$, where $\mathscr{H}\mathscr{H}_{-q}$ is Zariski-sheafified Hochschild homology. However, his spectral sequence is constructed in a different fashion (hypercohomology spectral sequence) and does not come with a description of the E_1-page as we give it in Equation 0.1.

The Chern character from algebraic K-theory, $K \to HH$, then induces a morphism of coniveau spectral sequences, and by the above comparison to *Residues & Duality*, to (coherent) local cohomology. On the E_1-page, these maps are induced by pointwise maps:

Definition (Chern character with supports)**.** *If X/k is smooth and $x \in X$ any scheme point, then we construct a map (§2.1.2)*

$$K_m(\kappa(x)) \longrightarrow H^p_x(X, \Omega^{p+m})$$

with $p := \operatorname{codim}_X \overline{\{x\}}$, inducing maps $^KE_1^{p,q} \longrightarrow {}^{Cous}E_1^{p,0}(\Omega^{-q})$, where $^KE_1^{p,q}$ is the usual coniveau spectral sequence for algebraic K-theory, as in Quillen [Qui73].

See §2.1.2 for the actual definition.

So far, we work in analogy to algebraic K-theory. In the second part of the paper, we focus on a completely different issue. The coherent Cousin complex, line 0.2, appears in *Residues & Duality* [Har66] as an injective resolution – and is usually looked at from a quite different perspective: If X/k is a smooth proper variety of pure dimension n with $f : X \to \operatorname{Spec} k$ the structure map, the shriek pullback is known concretely: $f^!\mathscr{O}_k \cong \Omega^n_{X/k}[n]$. Grothendieck duality then stems from the adjunction $f_* \leftrightarrows f^!$ and the co-unit map $\operatorname{Tr}_f : f_*f^!\mathscr{O}_k \to \mathscr{O}_k$, which induces

$$\operatorname{Tr}_f : H^n(X, \Omega^n_{X/k}) \longrightarrow k.$$

The coherent Cousin complex then provides an injective resolution of $\Omega^n_{X/k}$; even more than that it is a so-called dualizing complex. Although we will not explain this here, the map Tr_f can be unravelled explicitly in terms of (higher) residues. Tate [Tat68] and Beilinson [Beĭ80] have proposed an approach to residues based on higher adèles of a scheme. Adèles provide a further resolution of the sheaf $\Omega^n_{X/k}$ and give rise to a certain Lie homology map $H^{\mathrm{Lie}}_{n+1}(-,-) \to k$ which turns out to give an explicit description of these residue maps. The duality theory aspect of the adèles (in dimension > 1) is due to Yekutieli [Yek92], [Yek98].

In [Bra18] it was shown that this approach to the residue can also be re-phrased in terms of the Hochschild homology of certain (non-commutative) algebras defined from the adèles. Along with the first part of the paper, it seems more than tempting to believe that this should allow us to phrase the Tate–Beilinson residue in terms of differentials in the Hochschild–Cousin complex. We show that this is indeed the case:

Theorem (Main Comparison Theorem)**.** *The Tate–Beilinson residue in the Lie homology of adèles* [Tat68], [Beĭ80] *can be expressed in terms of the differentials of our Hochschild–Cousin complex: Specifically, the Tate–Beilinson Lie homology residue symbol*

$$\Omega^n_{\mathrm{Frac}\,L_n/k} \longrightarrow H^{\mathrm{Lie}}_{n+1}((A_n)_{Lie}, k) \overset{\phi_{Beil}}{\longrightarrow} k$$

(as defined in [Beĭ80, §1, Lemma, (b)]*) also agrees with*

$$\Omega^n_{\mathrm{Frac}\,L_n/k} \longrightarrow HH^{\eta_0}_n(L_n) \longrightarrow HH^{\eta_0}_n(C_0) \longrightarrow HH_n(A_n) \overset{\phi_{HH}}{\longrightarrow} k.$$

(see Theorem 5.2.2 for details and notation)

This statement is intentionally vague since we do not want to introduce the necessary notation and background on adèles of schemes in this introduction. This result will be stated in precise form in §5.2, along with a review of the adèle theory. In coarse strokes, we paint the following picture:

$$\begin{array}{ccccc}
\text{coherent Cousin complex} & \longleftrightarrow & \text{Hochschild Cousin complex} & \longleftrightarrow & \text{adèles of } \Omega^n \\
\uparrow & & \uparrow & & \uparrow \\
\text{local coherent cohomology} & & \text{Hochschild homology with supports} & & \text{non-commutative Hochschild homology} \\
 & & & & \downarrow \\
 & & & & \text{Lie homology}
\end{array}$$

It should be said that J. Lipman had the idea to use Hochschild homology for residues already in 1987 [Lip87]. However, his construction is very different from ours. He constructs the residue manually, while we use Keller's analogue of the localization sequence in K-theory [Kel99], which only became available in 1999, along with the particularly flexible technique for coniveau due to Balmer [Bal09], which is even more recent.

Outline. We proceed as follows: In §1 we recall the necessary material on Hochschild homology. In §2 we prove the HKR isomorphism with supports. In §3 we give an independent treatment of the Hochschild residue à la [Bra18]. In §4 we provide the necessary material on Tate categories. These categories provide the crucial bridge to transport Hochschild homology from a classical geometric to an adèle perspective. In §4.4 we develop a '*relative Morita theory*'. If an exact category $\mathscr{C}$ happens to be equivalent to a projective module category, say $\mathscr{C} \overset{\sim}{\to} P_f(E)$ for an algebra E, we will need to understand how such a presentation changes if we consider a fully exact sub-category $\mathscr{C}' \hookrightarrow \mathscr{C}$, or a quotient exact category $\mathscr{C}/\mathscr{C}'$, provided $\mathscr{C}'$ is left or right s-filtering. This might be of independent interest. In §5 we combine all these tools to establish a commutative square relating the Beilinson–Tate residue with boundary maps in Keller's localization sequence for Hochschild homology.

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1. The many definitions of Hochschild homology

Let us quickly survey what we understand as Hochschild homology. There are a large number of definitions which apply in greater or smaller generality. We will quickly sketch the transition from the classical definition of Hochschild up to the definition for dg categories of Keller.

For k a commutative ring and A a flat k-algebra one classically defines a complex $(C_\bullet, b)$ by $C_i(A) = A^{\otimes i+1}$,

$$b(a_0 \otimes \cdots \otimes a_i) := \sum_{j=0}^{i-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i + (-1)^i a_i a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \tag{1.1}$$

and its homology is the *Hochschild homology* of A. Philosophically, this is conveniently viewed as a concrete complex quasi-isomorphic to a certain derived tensor product, namely

$$C_\bullet \sim A \otimes^{\mathbf{L}}_{A \otimes_k A^{op}} A, \tag{1.2}$$

but it is the former definition which led to Mitchell's generalization to categories [Mit72]: For $\mathscr{A}$ a k-linear small category such that all $\mathrm{Hom}_k(-,-)$ are flat k-modules, one defines

$$C_i(\mathscr{A}) := \coprod \mathrm{Hom}(X_i, X_0) \otimes_k \mathrm{Hom}(X_{i-1}, X_i) \otimes_k \cdots \otimes_k \mathrm{Hom}(X_0, X_1), \tag{1.3}$$

where the coproduct runs over all $(i+1)$ tuples of objects in $\mathscr{A}$. A differential b can be defined by the same formula as before, this time instead of multiplying elements of A, one composes the respective morphisms. In order to stress the analogy with Equation 1.1 the reader might at first sight prefer to use an indexing starting with $\mathrm{Hom}(X_0, X_1) \otimes \ldots$, but this comes with the disadvantage that in composing $X_0 \overset{a_0}{\to} X_1$ and $X_1 \overset{a_1}{\to} X_2$ the composition is $a_1 a_0$ and not $a_0 a_1$. Thus, in order to use the same formula for b, one has to use a reversed numbering.

Remark 1.0.1. If we regard a ring A as a category $\mathbf{A}$ with one object "A" and $\mathrm{Hom}_{\mathbf{A}}(A, A) := A$, the classical definition of Equation 1.1 literally becomes a special case of Mitchell's categorical definition.

However, both constructions are only the 'correct' ones in very special cases. For example, for A a general k-algebra, i.e. not necessarily flat over k, one works instead with $C_i(A) := A_\bullet \otimes_k \cdots \otimes_k A_\bullet$, where $A_\bullet \to A$ is a flat resolution of A and adapts the definition of the differential to deal with dg algebras, as done by Keller in [Kel98a]. In a similar spirit, for $\mathscr{A}$ a k-linear dg category, one replaces the definition of Equation 1.3 by a version where the complexes $\mathrm{Hom}(-,-)$ get replaced by flat resolutions. This leads to the definition of Keller that we shall also use in the present paper; the flat case suffices for our purposes:

Definition 1.0.2. *Let k be a commutative ring and $\mathscr{A}$ a small k-linear dg-flat dg category. In particular, all*

- $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ *are k-flat dg k-modules, and*
- *the composition*
$$\mathrm{Hom}_{\mathscr{A}}(Y, Z) \otimes_k \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X, Z)$$
is a morphism of dg k-modules.

Then define for homogeneous morphisms $(a_i, \ldots, a_0) \in C_i(\mathscr{A})$ (as in Equation 1.3)

$$b(a_i, \ldots, a_0) := \sum_{j=0}^{i-1} (-1)^j (a_i, \ldots, a_i \circ a_{i-1}, \ldots, a_0) + (-1)^{n+\sigma} (a_0 \circ a_i, a_{i-1}, \ldots, a_1),$$

where $\sigma = (\deg a_0)(\deg a_1 + \cdots + \deg a_{i-1})$.

See for example [Kel98b, §3.2] or [Kel99, §1.3]. The general version without flatness assumption is constructed in [Kel99, §3.9]. We also remind the reader that for an exact category the category of complexes itself does not reflect any datum of the exact structure, so that the derived category of an exact category $\mathscr{E}$ has to be defined as the Verdier quotient $D^b\mathscr{E} := \mathscr{K}^b(\mathscr{E})/\mathscr{A}c^b(\mathscr{E})$, where $\mathscr{K}^b(\mathscr{E})$ is the triangulated category of bounded complexes in $\mathscr{E}$ modulo chain homotopies and $\mathscr{A}c^b(\mathscr{E})$ the subcategory of acyclic complexes, subtly depending on the exact structure. See [Büh10, §10] for a very detailed excellent review.

As is suggested from the derived category of an exact category, the Hochschild homology of $\mathscr{E}$ is then defined as follows:

Definition 1.0.3 (Keller, [Kel99, §1.4])**.** *Let $\mathscr{E}$ be a flat k-linear exact category. Then its* Hochschild homology *is*

$$HH(\mathscr{E}) := Cone\left(C_\bullet \mathscr{A}c^b(\mathscr{E}) \longrightarrow C_\bullet \mathscr{C}^b(\mathscr{E})\right),$$

where $\mathscr{C}^b(\mathscr{E})$ is the dg category of bounded complexes in $\mathscr{E}$ and $\mathscr{A}c^b(\mathscr{E})$ its dg subcategory of acyclic complexes.

In the present paper we will mostly be interested in the Hochschild homology of perfect complexes with support. Let us briefly recall the concept of perfect complexes, following Thomason-Trobaugh [TT90, §2]: Let X be a scheme. A complex $\mathscr{F}_\bullet$ of $\mathscr{O}_X$-module sheaves is called *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles. This definition goes back to [SGA, Exp. I].

For $f : X \to Y$ a morphism, the total left derived functor $\mathbf{L}f^* : D^-(\mathscr{M}od_{\mathscr{O}_Y}) \to D^-(\mathscr{M}od_{\mathscr{O}_X})$ of the pullback f^* preserves perfection so that there is a pullback functor $f^* : \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$. If f is flat, this functor is literally just the entrywise pullback of a perfect complex (see [TT90, §2.5.1]).

For $f : X \to Y$ a proper and perfect[1] morphism of Noetherian schemes, the total right derived functor $\mathbf{R}f_* : D^-(\mathscr{M}od_{\mathscr{O}_X}) \to D^-(\mathscr{M}od_{\mathscr{O}_Y})$ preserves perfection so that there is a pushforward functor $f_* : \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$, see [TT90, Thm. 2.5.4].

Let Z be a closed subset. We write $\mathrm{Perf}_Z(X)$ for the category of perfect complexes on X whose pullback to the open $U := X - Z$ is acyclic. The homological support $\mathrm{supph}\,(\mathscr{F})$ of a perfect complex $\mathscr{F}_\bullet$ is the support of the total homology of $\mathscr{F}_\bullet$, i.e. it is the union of the supports of the sheaves $\bigcup_i \mathrm{supp}\,\mathscr{H}^i(\mathscr{F}_\bullet)$.

The category $\mathrm{Perf}_Z(X)$ sadly does not fit into the framework of exact categories. According to taste, the reader may view $\mathrm{Perf}_Z(X)$ (and then $\mathrm{Perf}(X) = \mathrm{Perf}_X(X)$ as well) as a stable ∞-category. Alternatively, it can also be modelled as a Waldhausen category, i.e. a classical 1-category with quasi-isomorphisms of complexes as weak equivalences. Using either formalism there is an associated homotopy category, $D_Z(X) := \mathrm{Ho}(\mathrm{Perf}_Z(X))$, a triangulated category. The dg perspective leads directly to a very similar definition as before:

Definition 1.0.4 (Keller, [Kel98b, §4.3])**.** *Let X be a scheme and Z a closed subset. Define the* Hochschild homology of X with support in Z *by*

$$HH^Z(X) := Cone\left(C_\bullet(\mathscr{A}c\,\mathrm{Perf}\,X) \longrightarrow C_\bullet(\mathrm{Perf}_Z\,X)\right),$$

where $\mathrm{Perf}_Z\,X$ is the category of perfect complexes on X acyclic on $X - Z$, and $\mathscr{A}c\,\mathrm{Perf}\,X$ is the category of all acyclic perfect complexes. We write $HH(X) := HH^X(X)$ for the variant without support condition.[2]

Instead of $\mathscr{A}c\,\mathrm{Perf}\,X$ we could also write $\mathrm{Perf}_\varnothing\,X$ of course; these are literally the same categories. Finally, we should also discuss a sheaf perspective [Wei96], [Swa96, §2, end of page 59]: Let k be a field now. For X a k-scheme one can consider the presheaf of complexes of k-modules

$$U \mapsto C_\bullet(\mathscr{O}_X(U)) \tag{1.4}$$

and let $\mathscr{C}_\bullet$ be its Zariski sheafification (Weibel denotes it as $\mathscr{C}^h_*$ in [Wei96, §1]). Note that $\mathscr{O}_X(U)$ is a flat k-algebra, so for $C_\bullet$ one can use the classical definition as in Equation 1.1. Unfortunately, $\mathscr{C}_\bullet$ is not a quasi-coherent sheaf. However, its homology turns out to be quasi-coherent.

Theorem 1.0.5 (Geller, Weibel)**.** *Let X be a k-scheme.*

1. *The homology sheaves $\mathscr{H}\mathscr{H}_i := H_i(\mathscr{C}_\bullet)$ are quasi-coherent.*
2. *The Zariski sheafification of $U \mapsto HH_i(\mathscr{O}_X(U))$ agrees with the sheaf $\mathscr{H}\mathscr{H}_i$.*
3. *On each affine open $U \subseteq X$, one has canonical isomorphisms*

$$H^p(U, \mathscr{H}\mathscr{H}_i) \cong \begin{cases} 0 & \text{for } p \neq 0 \\ HH_i(\mathscr{O}_X(U)) & \text{for } p = 0. \end{cases}$$

4. *$\mathscr{H}\mathscr{H}_i$ also makes sense as an étale sheaf and $H^p(X_{\acute{e}t}, \mathscr{H}\mathscr{H}_i) \cong H^p(X_{Zar}, \mathscr{H}\mathscr{H}_i)$.*

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[1]e.g. smooth morphisms or regular closed immersions. A general closed immersion need not be perfect, in particular a general finite morphism need not be perfect. Any finite type morphism to a smooth scheme is perfect.

[2]One might be tempted to prefer writing "HH_Z" for the theory with support in Z, but it leads to the impractical notation $HH_{Z,i}$. Also, for homology with closed support (Borel–Moore), the superscript notation H^c_i or H^{cl}_i is widespread. Ultimately, it remains a matter of taste, of course.

See [Wei96, Proposition 1.2] for a discussion and [WG91, Cor. 0.4] for the proof. This is all we need for the present paper, but much more is known, e.g. cdh descent for X smooth [CHW08], [CHSW08].

Example 1.0.6. *The Zariski descent and the Hochschild-Kostant-Rosenberg isomorphism imply that on a smooth k-scheme the sheaves $\mathscr{H}\mathscr{H}_i$ and Ω^i $(:= \Omega^i_{X/k})$ are isomorphic.*

Weibel and Swan now define a version of Hochschild homology of a scheme via

$$HH_i^{Weibel}(X) := H^{-i}(X_{Zar}, \mathscr{C}_\bullet),$$

where H^* refers to the sheaf (hyper)cohomology of the sheaf of complexes [Wei96, Eq. 1.1]. Geller and Weibel show that for X affine this agrees with the classical definition in terms of rings, $HH_i^{Weibel}(X) \cong HH_i(\mathscr{O}_X(X))$, see [WG91, Thm. 4.1]. More generally, Keller established a beautiful theorem linking this sheaf perspective with the categorical viewpoint.

Theorem 1.0.7 (Keller)**.** *Let k be a field and X a Noetherian separated k-scheme. Then there is a canonical isomorphism $HH_i^{Weibel}(X) \overset{\sim}{\to} HH_i(X)$, where $HH_i(X)$ refers to the Hochschild homology of perfect complexes as in Definition 1.0.4.*

This is [Kel98b, Thm. 5.2][3]. Keller's paper also provides details on the switch between two slightly different definitions of the sheaf hypercohomology underlying Weibel's definition. Besides all this, Equation 1.2 suggests an entirely different definition of Hochschild homology of a scheme, proposed by Swan [Swa96]. However, it turns out to agree with the previous definition:

Theorem 1.0.8 (Yekutieli)**.** *Let k be a field and X a finite type k-scheme. Then there is a quasi-isomorphism of complexes of sheaves $\mathscr{C}_\bullet\mathscr{O}_X \overset{\sim}{\to} \mathscr{O}_X \otimes^{\mathbf{L}}_{\mathscr{O}_{X\times X}} \mathscr{O}_X$.*

See [Yek02, Proposition 3.3]. In the same paper Yekutieli also constructed an alternative complex $\widehat{\mathscr{C}_\bullet}$ of completed Hochschild chains, which is itself quasi-coherent, suitably interpreted, and not just quasi-coherent after taking homology. One can also define Hochschild homology on the derived level, following Căldăraru and Willerton [CW10]. Their paper also shows equivalence to Weibel's approach.

Finally, we also need a completely different direction of generalization of Hochschild homology: the case of rings *without units*. Formulations in terms of modules or perfect complexes over non-unital rings appear to be very subtle (but see work of Quillen [Qui96] and Mahanta [Mah11]), so we will not enter into the matter of setting up a categorical viewpoint, and just stick to algebras.

Conventions. We shall reserve the word *ring* for a commutative, unital associative algebra. A ring morphism will always preserve the unit of multiplication. This leaves us with the word *algebra* whenever we need to work with more general structures. For us an associative algebra A will *not* be assumed to be unital. Likewise, we do *not* require morphisms of algebras to preserve a unit, even if it exists. As an example, note that this implies that any one-sided or two-sided ideal $I \subseteq A$ is itself an associative algebra and the inclusion $I \hookrightarrow A$ a morphism of algebras.

Definition 1.0.9. *The algebra A is called*

(1) locally left unital *(resp.* locally right unital*) if for every finite subset $S \subseteq A$ there exists an element $e_S \in S$ such that $e_S a = a$ (resp. $ae_S = a$) for all $a \in S$;*
(2) locally bi-unital *if it is both locally left unital and locally right unital.*

Remark 1.0.10. Locally bi-unital does not imply that we can find e_S such that $e_S a = a = ae_S$ holds for all a in any finite subset $S \subseteq A$. It makes no statement about the mutual relation of left- and right-units.

If A is a non-unital associative k-algebra, the definition of the complex as in Equation 1.1 still makes perfect sense. Nonetheless, it turns out that this is not quite the right thing to do, a 'correction term' is required, as was greatly clarified and resolved by M. Wodzicki [Wod89]: One defines the so-called bar complex $B_\bullet$ and with the cyclic permutation operator t, and one forms the bi-complex $C_i^{\mathrm{corr}}(A) := [C_i(A) \xrightarrow{1-t} B_{i+1}(A)]$ (we will not define $B_\bullet$ or t here, all details can be found in [Wod89, §2, especially

[3]Keller proves the result on the level of mixed complexes. But Hochschild homology can be defined in terms of mixed complexes, leading directly to the present formulation.

page 598 l. 5]). This complex turns out to model a well-behaved theory of Hochschild homology even if A is non-unital. If A is unital, $B_\bullet$ turns out to be acyclic so that we recover the previous definition, but this works even more generally:

Proposition 1.0.11 (Wodzicki)**.** *If A is locally left unital (or locally right unital), $B_\bullet$ is acyclic, so that the complex $C_\bullet(A)$ models Hochschild homology.*

This is [Wod89, Cor. 4.5]. Let us rephrase this: As long as we only work with locally left or right unital associative algebras, we may just work with the complex in line 1.1 as the definition of Hochschild homology. And this will be precisely the situation in this paper, so the reader may feel free to ignore $B_\bullet$ and $C_\bullet^{\mathrm{corr}}$ entirely.

We need one more ingredient: Suppose A is a (possibly non-unital) associative algebra and I a two-sided ideal. Then we get an exact sequence of associative algebras

$$I \hookrightarrow A \twoheadrightarrow A/I. \tag{1.5}$$

Theorem 1.0.12 (Wodzicki)**.** *Suppose we are given an exact sequence as in Equation 1.5. If A and I are locally left unital (or locally right unital), then there is a fiber sequence*

$$HH(I) \longrightarrow HH(A) \longrightarrow HH(A/I) \longrightarrow +1.$$

Proof. This is just a special case of a more general formalism, we refer to the paper [Wod89] for the entire story, or the book [Lod92, §1.4]. In the case at hand we can proceed as follows: One gets a fiber sequence

$$HH(A, I) \longrightarrow HH(A) \longrightarrow HH(A/I) \longrightarrow +1,$$

where $HH(A, I)$ refers to relative Hochschild homology, which is just defined as the cone, so the existence of this sequence is tautological. By Wodzicki's Excision Theorem [Wod89, Theorem 3.1], we get an equivalence $HH(I) \xrightarrow{\sim} HH(A, I)$. This is only true when I and A are H-unital (in the sense of loc. cit.), as is guaranteed by our assumptions and [Wod89, Cor. 4.5]. □

1.1. **Coniveau filtration.**

1.1.1. *Coherent cohomology with supports.* Let us first recall the construction of the coniveau spectral sequence in sheaf cohomology. We briefly summarize some facts about local cohomology that we shall need. A detailed presentation has been given by Hartshorne [Har67], in a different format also in [Har66, Ch. IV] or [Con00, §3.1].

Let X be a topological space. A subset Z is called *locally closed* if it can be written as the intersection of an open and a closed subset. Equivalently, a closed subset $Z \subseteq_{cl} V$ of $V \subseteq_{op} X$ an open subset. For a sheaf $\mathscr{F}$ one defines a new sheaf

$$\underline{\Gamma_Z}\mathscr{F}(U) := \{s \in \mathscr{F}(U) \mid \operatorname{supp} s \subseteq Z\},$$

the sheaf of sections with support in Z. Note that if $j : Z \hookrightarrow X$ is an open subset, $\underline{\Gamma_Z}\mathscr{F} = j_* j^{-1}\mathscr{F}$. Moreover, $\underline{\Gamma_Z}$ is a left exact functor from the category of abelian group sheaves on X to itself. Right derived functors exist, are denoted by $\mathscr{H}^p_Z\mathscr{F} := \mathbf{R}^p\underline{\Gamma_Z}\mathscr{F}$, and called *local cohomology* sheaves [Har67, §1]. We write $H^p_Z(X, \mathscr{F})$ for the right derived functors of the functor $\mathscr{F} \mapsto H^0(X, \underline{\Gamma_Z}\mathscr{F})$. There is also a product

$$\underline{\Gamma_{Z_1}}\mathscr{F} \otimes \underline{\Gamma_{Z_2}}\mathscr{G} \longrightarrow \underline{\Gamma_{Z_1 \cap Z_2}}(\mathscr{F} \otimes \mathscr{G}) \tag{1.6}$$

for sheaves $\mathscr{F}, \mathscr{G}$ and locally closed subsets Z_1, Z_2. We shall mainly need the following property: If Z is a locally closed subset, $Z' \subseteq Z$ a closed subset, then $Z - Z'$ is also a locally closed subset in X and there is a distinguished triangle

$$\mathbf{R}\underline{\Gamma_{Z'}}\mathscr{F} \longrightarrow \mathbf{R}\underline{\Gamma_Z}\mathscr{F} \longrightarrow \mathbf{R}\underline{\Gamma_{Z-Z'}}\mathscr{F} \longrightarrow +1 \tag{1.7}$$

(see [Har67, Lemma 1.8 or Proposition 1.9], also [Har66, Ch. IV, "Variation 2", p. 219]). For $Z := X$ and $Z' \subseteq X$ a closed subset, this specializes to $\mathbf{R}\underline{\Gamma_{Z'}}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow j_* j^{-1}\mathscr{F} \longrightarrow +1$, where $j : U \hookrightarrow X$ denotes the open immersion of the open complement $U := X \setminus Z'$ (see [Har66, Ch. IV, "Variation 3", p. 220]). If X is a scheme, one can say quite a bit more:

Lemma 1.1.1. *Suppose X is a Noetherian scheme and $\mathscr{F}$ a quasi-coherent sheaf.*

(1) *([Har67, Proposition 2.1]) Then the* $\mathscr{H}^p_Z\mathscr{F}$ *are also quasi-coherent sheaves.*
(2) *([Har67, Thm. 2.8]) If* Z *is a closed subscheme with ideal sheaf* $\mathscr{I}_Z$*, there is a canonical isomorphism of quasi-coherent sheaves, functorial in* $\mathscr{F}$*,*

$$\underrightarrow{\operatorname{colim}}_{\ell}\, \mathscr{E}xt^p_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^{\ell}_Z, \mathscr{F}) \xrightarrow{\sim} \mathscr{H}^p_Z\mathscr{F}. \tag{1.8}$$

Lemma 1.1.2 (dual "dimension axiom")**.** *If* R *is a ring and* $I = (f_1, \ldots, f_r)$*. Then for every* R*-module* M*, we have* $H^p_I(R, M) = 0$ *for* $p > r$*.*

See for example [ILL$^+$07, Cor. 7.14]. We will frequently use the following property, often allowing us to reduce to local rings:

Lemma 1.1.3 (Excision, [Har67, Proposition 1.3])**.** *Let* Z *be locally closed in* X*,* $V \subseteq X$ *open so that* $Z \subseteq V \subseteq X$*. Then there is a canonical isomorphism* $H^p_Z(X, \mathscr{F}) \xrightarrow{\sim} H^p_Z(V, \mathscr{F}\mid_V)$*, induced by the pullback of sections along the open immersion.*

Lemma 1.1.4 ([Har67, Proposition 5.9])**.** *Let* R *be a Noetherian ring,* $I' \subseteq I$ *ideals and* M *a finitely generated* R*-module. Then there are canonical isomorphisms* $H^p_I(\operatorname{Spec} R, M) \xrightarrow{\sim} H^p_{\hat{I}}(\operatorname{Spec} \widehat{R}, \widehat{M})\mid_R$*, where* $\widehat{(-)}$ *in both cases refers to the* I'*-adic completion, so that* $H^p_{\hat{I}}(\widehat{M})$ *is an* $\widehat{R}$*-module.*

If x is a (not necessarily closed) point of the scheme X, we write

$$H^i_x(X, \mathscr{F}) := \operatorname*{colim}_{Z} H^i_Z(X, \mathscr{F}),$$

where Z runs through all locally closed subsets of X which contain the point x, directed towards smaller sets. Equivalently, let Z run through all locally closed subsets of X such that

$$x \in Z \subseteq \overline{\{x\}},$$

or equivalently running through all open neighbourhoods of x inside (and with respect to the subspace topology of) the closed subscheme $\overline{\{x\}}$.

Next, one builds the Cousin complex in *coherent cohomology*. Let Z^p denote a closed subset of X with $\operatorname{codim}_X Z^p \geq p$. One can read the colimit (under inclusion) under all such,

$$F_p H^i(X, \mathscr{F}) := \underrightarrow{\operatorname*{colim}}_{Z^p} \operatorname{im}(H^i_{Z^p}(X, \mathscr{F}) \to H^i(X, \mathscr{F})), \tag{1.9}$$

as a filtration of the cohomology of the sheaf $\mathscr{F}$. Taking the underlying filtered complex spectral sequence, one arrives at the "Cousin coniveau spectral sequence", due to Grothendieck:

Proposition 1.1.5 ([Har66, Ch. IV])**.** *This filtration induces a convergent spectral sequence with*

$$^{Cous}E^{p,q}_1(\mathscr{F}) := \coprod_{x \in X^p} H^{p+q}_x(X, \mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

The rows of the E_1-page read

$$0 \longrightarrow \coprod_{x \in X^0} H^q_x(X, \mathscr{F}) \xrightarrow{d} \coprod_{x \in X^1} H^{q+1}_x(X, \mathscr{F}) \xrightarrow{d} \cdots \xrightarrow{d} \coprod_{x \in X^n} H^{q+n}_x(X, \mathscr{F}) \longrightarrow 0 \tag{1.10}$$

(this is the q-th row, concentrated columnwise in the range $0 \leq p \leq n$ for $n := \dim X$). The differential d agrees with the upward arrow in the following Diagram 1.11: We get a long exact sequence from Equation 1.7 and replicating suitable excerpts twice, we get the rows of the diagram

$$\begin{tikzcd}
H^{i+1}_{Z^{p+2}}(X,\mathscr{F}) \arrow[r] & H^{i+1}_{Z^{p+1}}(X,\mathscr{F}) \arrow[r] & \coprod_{x\in X^{p+1}} H^{i+1}_x(X,\mathscr{F}) \arrow[r, "\partial"] & H^{i+2}_{Z^{p+2}}(X,\mathscr{F}) \\
H^{i}_{Z^{p+1}}(X,\mathscr{F}) \arrow[r] & H^{i}_{Z^{p}}(X,\mathscr{F}) \arrow[r] & \coprod_{x\in X^{p}} H^{i}_x(X,\mathscr{F}) \arrow[r, "\partial"] \arrow[u] & H^{i+1}_{Z^{p+1}}(X,\mathscr{F}) \arrow[ull]
\end{tikzcd} \tag{1.11}$$

The leftward diagonal arrow is just the identity morphism. Define the upward arrow to be the composition. It is precisely the map d. We refer to [Har66] for more details. For *local* Cohen-Macaulay schemes we are in the pleasant situation that the complex in Equation 1.10 is exact. More precisely:

Proposition 1.1.6. *Suppose $X = \operatorname{Spec} R$ for R a Noetherian Cohen-Macaulay local ring. Then the sequence in Equation 1.10 is exact. In particular, if we read its entries as sheaves, i.e.*

$$U \mapsto \coprod_{x \in U^p} H_x^{p+q}(X, \mathscr{O}_X) \qquad \text{(for } U \subseteq X \text{ Zariski open)},$$

the complex in Equation 1.10 provides a flasque resolution of the sheaf $U \mapsto \mathscr{H}^q(U, \mathscr{O}_X)$.

This is a special case of [Har66, Ch. IV, Proposition 2.6], in a mild variation of [Har66, Ch. IV, Example on p. 239].

Corollary 1.1.7. *For X Noetherian and Cohen-Macaulay, suppose $\mathscr{F}$ is a coherent sheaf. Then there is a flasque resolution of the sheaf $\mathscr{F}$, namely*

$$0 \longrightarrow \mathscr{F} \longrightarrow \underline{\coprod_{x \in X^0} H_x^0(X, \mathscr{F})} \longrightarrow \cdots \longrightarrow \underline{\coprod_{x \in X^n} H_x^n(X, \mathscr{F})} \longrightarrow 0.$$

This is known as the Cousin resolution *of $\mathscr{F}$. If X is Gorenstein and $\mathscr{F}$ locally free, this is an injective resolution of $\mathscr{F}$.*

Since we shall mostly work with smooth schemes, the weaker Gorenstein and Cohen-Macaulay conditions are usually implied. In the literature one often allows more general filtrations than those by codimension, but we have no use for this increase in flexibility, see however [Con00, Ch. III] or [Bal09].

1.1.2. *Hochschild homology with supports.* We will now repeat the story of §1.1.1 for the Hochschild homology of categories of perfect complexes with support, see §1 for the definition. We assume that the scheme X is Noetherian and separated.

In principle, we shall do precisely the same constructions, but the inner machinery is of quite a different type. This has not so much to do with perfect complexes, but rather with a very different homological perspective. Whereas we based the last section on the distinguished triangle

$$\mathbf{R}\underline{\Gamma_{Z'}}\mathscr{F} \longrightarrow \mathbf{R}\underline{\Gamma_{Z}}\mathscr{F} \longrightarrow \mathbf{R}\underline{\Gamma_{Z-Z'}}\mathscr{F} \longrightarrow +1, \tag{1.12}$$

regarding just the cohomology with supports of coherent sheaves, we will now replace this by the localization sequence of categories

$$\operatorname{Perf}_{Z'} X \longrightarrow \operatorname{Perf}_Z X \longrightarrow \operatorname{Perf}_{Z-Z'} X \longrightarrow +1.$$

Just as the above sequence induces a long exact sequence in cohomology, the latter induces a long exact sequence in the Hochschild homology of these categories (and in fact, just as well for their algebraic K-theory, cyclic homology, etc.).

So far HH^Z was only defined for Z closed. Using the above sequence, we can define HH^Z for Z *locally closed* in X by writing it as $Z = Z_2 - Z_1$ with Z_1, Z_2 closed in X. If x is a (not necessarily closed) point of the scheme X, we write

$$HH^x(X) := \operatorname*{colim}_Z HH^Z(X),$$

where Z runs through all locally closed subsets of X which contain the point x, directed towards smaller sets.

The constructions of this section play a fundamental and classical rôle in algebraic K-theory and originate essentially from Quillen [Qui73]. We adapt them to Hochschild homology, however. In order to do so, we use a particularly strong variant of the construction due to Balmer [Bal09].

Define the full subcategory of perfect complexes of homological support $\geq p$ by

$$\operatorname{Perf}_{Z^p}(X) := \{\mathscr{F}_\bullet \in \operatorname{Perf}(X) \mid \operatorname{codim}_X(\operatorname{supph} \mathscr{F}) \geq p\}.$$

As before, this can either be viewed as a stable ∞-category or a Waldhausen category with quasi-isomorphisms as weak equivalences. The analogue of the filtration in Equation 1.9 is now the filtration

$$\cdots \hookrightarrow \operatorname{Perf}_{Z^2}(X) \hookrightarrow \operatorname{Perf}_{Z^1}(X) \hookrightarrow \operatorname{Perf}_{Z^0}(X) = \operatorname{Perf}(X). \tag{1.13}$$

Then

$$\operatorname{Perf}_Z(X) \longrightarrow \operatorname{Perf}(X) \longrightarrow \operatorname{Perf}(U) \tag{1.14}$$

is an exact sequence of stable ∞-categories, known as the *localization sequence*. There is also a product

$$\begin{aligned} \operatorname{Perf}_{Z_1}(X) \times \operatorname{Perf}_{Z_2}(X) &\longrightarrow \operatorname{Perf}_{Z_1 \cap Z_2}(X) \\ \mathscr{F}_{\bullet} \otimes \mathscr{G}_{\bullet} &\longmapsto (\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G})_{\bullet} \end{aligned} \tag{1.15}$$

sending bounded complexes of perfect sheaves to their derived tensor product. By $\operatorname{supp}(\mathscr{F} \otimes \mathscr{G}) = \operatorname{supp} \mathscr{F} \cap \operatorname{supp} \mathscr{G}$, the tensor product of perfect complexes with supports in Z_i (for $i = 1, 2$) is a bi-exact functor to perfect complexes with support in $Z_1 \cap Z_2$. As before, one can construct a convergent spectral sequence, essentially due to P. Balmer [Bal09], namely we obtain the following:

Proposition 1.1.8. *The filtration in Equation 1.13 gives rise to a convergent spectral sequence with*

$${}^{HH}E_1^{p,q} := \coprod_{x \in X^p} HH^x_{-p-q}(\mathscr{O}_{X,x}) \Rightarrow HH_{-p-q}(X).$$

The rows of the E_1-page read

$$\cdots \longrightarrow \coprod_{x \in X^0} HH^x_{-q}(\mathscr{O}_{X,x}) \xrightarrow{d} \coprod_{x \in X^1} HH^x_{-q-1}(\mathscr{O}_{X,x}) \xrightarrow{d} \cdots \xrightarrow{d} \coprod_{x \in X^n} HH^x_{-q-n}(\mathscr{O}_{X,x}) \longrightarrow \cdots \tag{1.16}$$

(this is the q-th row, concentrated columnwise in the range $0 \leq p \leq n$ for $n := \dim X$). The differential d agrees with the upward arrow in the following Diagram 1.17: Replicating copies of the long exact sequence in Hochschild homology associated to localizations as in Equation 1.14, but adapted to the filtration $\operatorname{Perf}_{Z^p}(X)$, we arrive at the diagram

$$\begin{tikzcd} HH_{i-1}^{Z^{p+2}}(X) \arrow[r] & HH_{i-1}^{Z^{p+1}}(X) \arrow[r] & \coprod_{x \in X^{p+1}} HH^x_{i-1}(\mathscr{O}_{X,x}) \arrow[r, "\partial"] & HH_{i-2}^{Z^{p+2}}(X) \\ HH_i^{Z^{p+1}}(X) \arrow[r] & HH_i^{Z^p}(X) \arrow[r] & \coprod_{x \in X^p} HH^x_i(\mathscr{O}_{X,x}) \arrow[r, "\partial"] \arrow[u] & HH_{i-1}^{Z^{p+1}}(X) \arrow[ull] \end{tikzcd} \tag{1.17}$$

imitating Diagram 1.11 that we had constructed before.

Remark 1.1.9 (Failure of $\mathbf{A}^1$-invariance). The complex in line 1.16 is the analogue of the Gersten complex in algebraic K-theory. In the K-theory of coherent sheaves, one can replace the analogous K-theory groups with support by the K-theory of the residue field by dévissage. This is why the K-theory Gersten complex is usually written down in the simpler fashion which does not mention any conditions on support. One of the starting points of this paper was: How can one formulate a Gersten complex for Hochschild homology? There is a general device producing Gersten complexes for $\mathbf{A}^1$-invariant Zariski sheaves with transfers [Voe00, Thm. 4.37] as well as a coniveau spectral sequence [MVW06, Remark 24.12]. However, Hochschild homology is *not* $\mathbf{A}^1$-invariant, so these tools do not apply in our context. One could still use the technology of [CTHK97], which does not depend on $\mathbf{A}^1$-invariance in any form. Instead, we use Balmer's triangulated technique, which also does not hinge on $\mathbf{A}^1$-invariance [Bal09, Thm. 2].

Proof. We leave it to the reader to fill in the details of the construction as described. Alternatively, the reader can just follow the argument of Balmer [Bal09, Thm. 2] and replace K-theory everywhere with Hochschild homology: Namely, from the filtration of Equation 1.13 we get an exact sequence of dg categories

$$\operatorname{Perf}_{Z^{p+1}}(X) \longrightarrow \operatorname{Perf}_{Z^p}(X) \longrightarrow \operatorname{Perf}_{Z^p}(X)/\operatorname{Perf}_{Z^{p+1}}(X)$$

and thanks to a strikingly general result of Balmer [Bal07, Thm. 3.24] the idempotent completion of the right-most category can be identified as

$$(\operatorname{Perf}_{Z^p}(X)/\operatorname{Perf}_{Z^{p+1}}(X))^{ic} \xrightarrow{\sim} \coprod_{x \in X^p} \operatorname{Perf}_x(\mathscr{O}_{X,x}). \tag{1.18}$$

In Balmer's paper [Bal09, Thm. 2] this argument is spelled out as an exact sequence of triangulated categories with Waldhausen models, whereas we have spelled it out as an exact sequence of dg categories. The exactness of either viewpoint is equivalent to the other, see [BGT13, Proposition 5.15]. The rôle of Schlichting's localization theorem is taken by Keller's localization theorem [Kel99] (a very clean and brief statement is also found in [Kel98b, §5.5, Theorem]). The convergence of the spectral sequence follows readily from the fact that its horizontal support is bounded since $X^p = \varnothing$ for $p \notin [0, \dim X]$. □

Remark 1.1.10. For later reference, let us make the functor in line 1.18 more precise: For each point $x \in X^p$, this is the pullback along the flat morphism $j_x : \operatorname{Spec} \mathscr{O}_{X,x} \hookrightarrow X$. Balmer shows that this induces the relevant equivalence [Bal07, §4.1]. In fact, he shows more: Perfect complexes can be regarded as a tensor triangulated category and under fairly weak assumptions the points of its Balmer spectrum (i.e. the prime $\otimes$-ideals, see [Bal05]) correspond canonically to the points of the scheme X. If $\mathscr{P}(x)$ denotes the prime $\otimes$-ideal of this point, one gets an exact sequence of triangulated categories

$$\mathscr{P}(x) \longrightarrow \operatorname{Perf}(X) \xrightarrow{j^*} \operatorname{Perf}(\mathscr{O}_{X,x}).$$

See [Bal07].

1.2. **Hochschild homology of different categories.** As for K-theory, one could consider the Hochschild homology not just of perfect complexes (which is the standard choice, because it is best-behaved for most applications), but also of coherent sheaves $\operatorname{Coh}_Z(X)$ with support. Both viewpoints are related by the following standard fact:

Proposition 1.2.1. *If X is a regular finite-dimensional Noetherian separated scheme, there are triangulated equivalences*

$$\operatorname{Perf}(X) \xrightarrow{\sim} D^b_{coh}(\operatorname{Mod}(\mathscr{O}_X)) \xleftarrow{\sim} D^b(\operatorname{Coh}(X)),$$

where the middle term is the bounded derived category of $\mathscr{O}_X$-module sheaves whose cohomology are coherent sheaves.

This was proven in [SGA, Exposé I], see also [TT90, §3]. The converse is also true: If the first arrow is a triangle equivalence, X must have been regular [LS16, Proposition 2.1]. In analogy to Equation 1.14 we have a localization sequence in Hochschild homology, but for coherent sheaves, induced from the exact sequence of abelian categories

$$\operatorname{Coh}_Z(X) \longrightarrow \operatorname{Coh}(X) \longrightarrow \operatorname{Coh}(U),$$

inducing an exact sequence of stable ∞-categories. If X is regular, so is U, and since this exact sequence determines the left-hand side term $\operatorname{Coh}_Z(X)$, it follows that $\operatorname{Coh}_Z(X) \simeq \operatorname{Perf}_Z(X)$. In general, [TT90, §3] is an excellent reference for this type of material.

Corollary 1.2.2. *If X is a regular Noetherian scheme, it does not make a difference whether we carry out the constructions of §1.1.2 for perfect complexes with support, or coherent sheaves with support. The results are canonically isomorphic.*

Remark 1.2.3. As algebraic K-theory satisfies dévissage, one obtains an equivalence $K(\operatorname{Coh}(Z)) \xrightarrow{\sim} K(\operatorname{Coh}_Z(X))$ for X Noetherian. The Hochschild analogue

$$HH(\operatorname{Coh}(Z)) \xrightarrow{?} HH(\operatorname{Coh}_Z(X))$$

is *false*. The issue is not on the level of categories, but rather that Hochschild homology does not satisfy dévissage. The failure of dévissage was originally discovered by Keller [Kel99, §1.10].

Proposition 1.2.4 (Thomason)**.** *There is a fully faithful triangular functor $D^b \operatorname{VB}(X) \to \operatorname{Perf}(X)$ from the bounded derived category of vector bundles on X to perfect complexes. If X has an ample family of line bundles, this is an equivalence of triangulated categories.*

See [TT90], Section 3.

1.3. **Flat pullback functoriality.** Next, we want to study the functoriality of the coniveau spectral sequences under flat morphisms. There is the standard pullback of differential forms and moreover the pullback of perfect complexes $f^* : \operatorname{Perf}(Y) \to \operatorname{Perf}(X)$, defined as the total left derived functor of the pullback of complexes. If f is flat, this literally sends a strictly perfect complex $\mathscr{F}_\bullet$ to $f^*\mathscr{F}_\bullet = f^{-1}\mathscr{F}_\bullet \otimes_{\mathscr{O}_Y} \mathscr{O}_X$.

Lemma 1.3.1. *Suppose k is a field. Let X, Y be smooth k-schemes and $f : X \to Y$ any morphism.*

(1) *If X, Y are affine, the induced pullbacks induce a commutative square*

$$\begin{tikzcd} \Omega^i_Y \arrow[r] \arrow[d, "f^*"'] & HH_i\mathrm{Perf}(Y) \arrow[d, "f^*"] \\ \Omega^i_X \arrow[r] & HH_i\mathrm{Perf}(X) \end{tikzcd}$$

with the horizontal arrows the Hochschild-Kostant-Rosenberg ("HKR") isomorphisms; cf. §2 for a reminder on the HKR isomorphism.

(2) *For X, Y not necessarily affine, the pullbacks of the Zariski sheafifications*

$$\begin{tikzcd} \Gamma(Y, \Omega^i) \arrow[r] \arrow[d, "f^*"'] & \Gamma(Y, \mathscr{HH}_i) \arrow[d, "f^*"] \\ \Gamma(X, \Omega^i) \arrow[r] & \Gamma(X, \mathscr{HH}_i) \end{tikzcd}$$

induce a commutative square.

In the case of a flat morphism, we can describe the induced morphisms on the respective spectral sequences in an explicit fashion.

Proposition 1.3.2 (Flat Pullbacks)**.** *Let $f : X \to Y$ be a flat morphism between Noetherian schemes.*

(1) *Then the pullback of differential forms induces a morphism of spectral sequences,*

$$f^* : {}^{Cous}E_r^{p,q}(Y, \Omega^n) \longrightarrow {}^{Cous}E_r^{p,q}(X, \Omega^n) \; .$$

On the E_1-page this map unwinds as follows: Given $x \in X^p$, $y \in Y^p$ the map between the respective summands is zero if $f(x) \neq y$ and the canonical map $H_y^{p+q}(\mathscr{O}_{Y,y}, \Omega^n) \to H_x^{p+q}(\mathscr{O}_{X,x}, \Omega^n)$ otherwise.

(2) *Then the pullback $f^* : \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$ is an exact functor and induces a morphism of spectral sequences, which we shall also denote by*

$$f^* : {}^{HH}E_r^{p,q}(Y) \longrightarrow {}^{HH}E_r^{p,q}(X) \; .$$

On the E_1-page this map unwinds as follows: Given $x \in X^p$, $y \in Y^p$ the map between the respective summands is zero if $f(x) \neq y$ and the canonical map $HH^y_{-p-q}(\mathscr{O}_{Y,y}) \to HH^x_{-p-q}(\mathscr{O}_{X,x})$ otherwise.

(3) *In either case for a given $y \in Y^p$ we have $x \in X^p \cap f^{-1}(y)$ exactly if x is the generic point of an irreducible component of the scheme-theoretic fibre $f^{-1}(y)$. In particular, for any given $y \in Y^p$ there are only finitely many such.*

Proof. Follow [She79, Proposition 1.2], which is written for K-theory, but can easily be adapted. □

2. Hochschild-Kostant-Rosenberg isomorphism with supports

This section will be devoted to a crucial comparison result: We will show that a certain excerpt of the long exact sequence in relative local homology, i.e. coming from Equation 1.7, is canonically isomorphic to a matching excerpt of the localization sequence in Hochschild homology. This is heavily inspired by Keller's beautiful paper [Kel98b]. The main consequence is that the boundary maps of these two sequences, even though they originate from quite different sources, actually agree.

Let us briefly recall that if R is a smooth k-algebra, the Hochschild-Kostant-Rosenberg map

$$\begin{aligned} \phi_{*,0} : \Omega^*_{R/k} &\longrightarrow HH_*(R) \\ f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &\longmapsto \sum_{\pi \in S_{n+1}} \mathrm{sgn}(\pi) f_{\pi(0)} \otimes \cdots \otimes f_{\pi(n)} \end{aligned} \tag{2.1}$$

induces an isomorphism of graded algebras − this is the classical Hochschild-Kostant-Rosenberg isomorphism ([Lod92, Thm. 3.4.4]). We obtain the following isomorphisms as a trivial consequence:

$$H^0(R,\Omega^i) \underset{\simeq}{\xrightarrow{\psi_{i,0}}} \Omega^i_{R/k} \underset{\simeq}{\xrightarrow{\phi_{i,0}}} HH_i(R).$$

Here $\psi_{i,0}$ denotes the tautological identification (we are in the affine situation). The first part of the following proposition can be seen as a generalization of this fact to Hochschild homology with support in a regularly embedded closed subscheme.

Proposition 2.0.1 (H-K-R with support)**.** *Let k be a field, R a smooth k-algebra and $t_1,\dots,t_n$ a regular sequence.*

(1) (Isomorphisms) *There are canonical isomorphisms*

$$\phi_{i,n}\circ\psi_{i,n} : H^n_{(t_1,\dots,t_n)}(R,\Omega^{n+i}) \longrightarrow HH_i^{(t_1,\dots,t_n)}(R), \tag{2.2}$$

functorial in k-algebra morphisms $R\to R'$ sending $t_1,\dots,t_n$ to a regular sequence.

(2) (Boundary maps) *The following diagram*

$$\begin{tikzcd}
0 \arrow[r] & HH_i^{(t_1,\dots,t_n)}(R) \arrow[r] & HH_i^{(t_1,\dots,t_n)}(R[t_{n+1}^{-1}]) \arrow[r, "\partial"] & HH_{i-1}^{(t_1,\dots,t_{n+1})}(R) \arrow[r] & 0 \\
0 \arrow[r] & H^n_{(t_1,\dots,t_n)}(R,\Omega^{n+i}) \arrow[r] \arrow[u, "\cong"'] & H^n_{(t_1,\dots,t_n)}(R[t_{n+1}^{-1}],\Omega^{n+i}) \arrow[r, "\partial"] \arrow[u, "\cong"'] & H^{n+1}_{(t_1,\dots,t_{n+1})}(R,\Omega^{n+i}) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd} \tag{2.3}$$

commutes, where the top row is an excerpt of the localization sequence for Hochschild homology, the bottom row of the long exact relative local homology sequence coming from Equation 1.7 and the upward arrows are the isomorphisms $\phi_{i,n}\circ\psi_{i,n}$. In particular, these excerpts of the long exact sequences are short exact.

(3) (Products) *Suppose $t_1,\dots,t_n$ and $t'_1,\dots,t'_m$ are regular sequences such that their concatenation is also a regular sequence (this is also known as "*transversally intersecting*"). The isomorphisms in (1) respect the natural product structures, i.e. of Equation 1.6 and Equation 1.15, so that the diagram*

$$\begin{tikzcd}
H^n_{(t_1,\dots,t_n)}(R,\Omega^{n+i})\otimes H^m_{(t'_1,\dots,t'_m)}(R,\Omega^{m+j}) \arrow[r] \arrow[d, "\cong"'] & H^{n+m}_{(t_1,\dots,t_n,t'_1,\dots,t'_m)}(R,\Omega^{n+m+i+j}) \arrow[d, "\cong"] \\
HH_i^{(t_1,\dots,t_n)}(R)\otimes HH_j^{(t'_1,\dots,t'_m)}(R) \arrow[r] & HH_{i+j}^{(t_1,\dots,t_n,t'_1,\dots,t'_m)}(R)
\end{tikzcd}$$

commutes.

(4) *Part (2) remains true if one replaces the usual (perfect complex) localization sequence for Hochschild homology by the localization sequence based on coherent sheaves with support, cf. §1.2.*

Proof. This is, albeit not explicitly, a consequence of Keller [Kel98b, §4-§5] (loc. cit. phrases it for mixed complexes à la Kassel, but this clearly implies the Hochschild case). One can, however, also prove the above claims rather directly by an induction on codimension, and we will give this alternative proof: It naturally splits into two parts, establishing first the isomorphisms ψ (this is classical, we just unwind it explicitly to be sure that all maps agree), and then the isomorphisms ϕ later, so we really want to establish the isomorphisms

$$H^n_{(t_1,\dots,t_n)}(R,\Omega^{n+i}) \xrightarrow{\psi_{i,n}} \frac{\Omega^{n+i}_{R[t_1^{-1},\dots,t_n^{-1}]/k}}{\sum_{j=1}^n \Omega^{n+i}_{R[t_1^{-1},\dots,\widehat{t_j^{-1}},\dots,t_n^{-1}]/k}} \xrightarrow{\phi_{i,n}} HH_i^{(t_1,\dots,t_n)}(R), \tag{2.4}$$

which is a little more detailed than the Equation 2.2. Hence, we first focus entirely on establishing for all i, n, the commutative diagrams:

$$
\begin{tikzcd}
\frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_n^{-1}]/k}}{\sum_{j=1}^{n}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_n^{-1}]/k}} \arrow[r] & \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_n^{-1},t_{n+1}^{-1}]/k}}{\sum_{j=1}^{n}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_n^{-1},t_{n+1}^{-1}]/k}} \arrow[r, "\mathrm{quot}"] & \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_{n+1}^{-1}]/k}}{\sum_{j=1}^{n+1}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_{n+1}^{-1}]/k}} \\
H^n_{(t_1,\ldots,t_n)}(R,\Omega^{n+i}) \arrow[r] \arrow[u, "\cong"'] & H^n_{(t_1,\ldots,t_n)}(R[t_{n+1}^{-1}],\Omega^{n+i}) \arrow[r, "\partial"] \arrow[u, "\cong"'] & H^{n+1}_{(t_1,\ldots,t_{n+1})}(R,\Omega^{n+i}) \arrow[u, "\cong"']
\end{tikzcd}
\tag{2.5}
$$

We do this by induction on n. We handle the case $n = 0$ first: We have $H^0(R, \Omega^i) \cong \Omega^i_{R/k}$, which settles the entire left square (the denominator sum over $j = 1, \ldots, n$ is void). The long exact sequence from Equation 1.7 tells us that

$$H^0_{(t_1)}(R,\Omega^i) \to H^0(R,\Omega^i) \to H^0(R[t_1^{-1}],\Omega^i) \to H^1_{(t_1)}(R,\Omega^i) \to H^1(R,\Omega^i)$$

is exact. Evaluation of the individual terms yields the exact sequence

$$0 \to \Omega^i_{R/k} \to \Omega^i_{R[t_1^{-1}]/k} \to H^1_{(t_1)}(R,\Omega^i) \to 0,$$

since $H^0_{(t_1)}(R,\Omega^i) = 0$ as Ω^i has no sections annihilated by t_1 (it is a free module), and $H^1(R,\Omega^i) = 0$ since $\operatorname{Spec} R$ is affine, so there is no higher coherent cohomology. This settles the bottom row of the diagram. The upper-right term fits by direct inspection.

Suppose the case n is settled. The long exact sequence from Equation 1.7 tells us that

$$\cdots \to H^n_{(t_1,\ldots,t_{n+1})}(R,\Omega^{n+i}) \overset{(*)}{\to} H^n_{(t_1,\ldots,t_n)}(R,\Omega^{n+i}) \to H^n_{(t_1,\ldots,t_n)}(R[t_{n+1}^{-1}],\Omega^{n+i}) \to H^{n+1}_{(t_1,\ldots,t_{n+1})}(R,\Omega^{n+i}) \cdots \tag{2.6}$$

is exact. The two middle terms by induction hypothesis identify with

$$\frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_n^{-1}]/k}}{\sum_{j=1}^{n}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_n^{-1}]/k}} \overset{\alpha}{\longrightarrow} \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_{n+1}^{-1}]/k}}{\sum_{j=1}^{n}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_{n+1}^{-1}]/k}}, \tag{2.7}$$

which is injective since we invert only nonzerodivisors (and the module of differential forms is free). Thus, the map $(*)$ must be the zero map. The next term on the right in line 2.6 would be $H^{n+1}_{(t_1,\ldots,t_n)}(R,\Omega^{n+i})$, which is zero since the ideal is generated by just n elements (Lemma 1.1.2). This proves that the bottom row in Equation 2.3 is short exact, and as a result its third term is just the quotient of the map in Equation 2.7, thus establishing Diagram 2.5. Now take the upward isomorphism on the right as the definition for

$$\psi_{i-1,n+1} : H^{n+1}_{(t_1,\ldots,t_{n+1})}(R,\Omega^{n+i}) \longrightarrow \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_{n+1}^{-1}]/k}}{\sum_{j=1}^{n+1}\Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_{n+1}^{-1}]/k}},$$

establishing the isomorphism $\psi_{i-1,n+1}$ in the first part of the claim (note that i was arbitrary all along, so it is no problem that we constructed $\psi_{i-1,n+1}$ on the basis of $\psi_{i,n}$). From now on we can assume to have all $\psi_{-,-}$ and Diagrams 2.5 available (for all i and n).

Next, we employ the localization sequence for the corresponding categories of perfect complexes with support, giving the long exact sequence

$$\cdots \to HH_i^{(t_1,\ldots,t_{n+1})}(R) \to HH_i^{(t_1,\ldots,t_n)}(R) \to HH_i^{(t_1,\ldots,t_n)}(R[t_{n+1}^{-1}]) \overset{\partial}{\to} HH_{i-1}^{(t_1,\ldots,t_{n+1})}(R) \to \cdots. \tag{2.8}$$

We start a new induction, again along n. For $n = 0$ this sequence reads

$$\cdots \to HH_i^{(t_1)}(R) \to HH_i(R) \to HH_i(R[t_1^{-1}]) \overset{\partial}{\to} HH_{i-1}^{(t_1)}(R) \to \cdots.$$

and via the Hochschild-Kostant-Rosenberg isomorphism identifies with

$$\cdots \to HH_i^{(t_1)}(R) \overset{\beta}{\to} \Omega^i_{R/k} \overset{\alpha}{\to} \Omega^i_{R[t_1^{-1}]/k} \overset{\partial}{\to} HH_{i-1}^{(t_1)}(R) \overset{\beta}{\to} \Omega^{i-1}_{R/k} \overset{\alpha}{\to} \Omega^{i-1}_{R[t_1^{-1}]/k} \overset{\partial}{\to} \cdots.$$

The maps denoted by α in the localization sequence are induced from the pullback of a perfect complex to the open along $\operatorname{Spec} R[t_1^{-1}] \hookrightarrow \operatorname{Spec} R$, and is known to correspond on differential forms to the same –

the pullback to the open. Thus, the morphisms α are injective and thus the maps denoted by β must be zero maps. This settles the exactness of the top row in diagram 2.3 for $n = 0$. In fact, by direct inspection of the maps, it establishes the commutativity of the entire diagram. Now suppose the case n is settled. Using the induction hypothesis we can identify the middle bit of Equation 2.8 with the map of Equation 2.7. This yields the identification

$$\cdots \to HH_i^{(t_1,\ldots,t_{n+1})}(R) \xrightarrow{\beta} \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_n^{-1}]/k}}{\sum_{j=1}^n \Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_n^{-1}]/k}}$$

$$\cdots \xrightarrow{\alpha} \frac{\Omega^{n+i}_{R[t_1^{-1},\ldots,t_n^{-1},t_{n+1}^{-1}]/k}}{\sum_{j=1}^n \Omega^{n+i}_{R[t_1^{-1},\ldots,\widehat{t_j^{-1}},\ldots,t_n^{-1},t_{n+1}^{-1}]/k}} \xrightarrow{\partial} HH_{i-1}^{(t_1,\ldots,t_{n+1})}(R) \xrightarrow{\beta} \cdots$$

and again the injectivity of α (it is the same map as in Equation 2.7) implies that the maps β must be zero. This settles the exactness of the top row and the commutativity of diagram 2.3 in general. Patching it to the diagrams 2.5 of the first part of the proof finishes the argument.

It remains to prove (3). The product is induced in local cohomology from Equation 1.6, composed with the product of the exterior algebra on 1-forms, i.e.

$$\underline{\Gamma_{Z_1}}\Omega^i \otimes \underline{\Gamma_{Z_2}}\Omega^j \longrightarrow \underline{\Gamma_{Z_1\cap Z_2}}(\Omega^i \otimes \Omega^j) \longrightarrow \underline{\Gamma_{Z_1\cap Z_2}}(\Omega^{i+j});$$

and in Hochschild homology from the bi-exact tensor functor

$$\mathrm{Perf}_{Z_1} X \times \mathrm{Perf}_{Z_2} X \longrightarrow \mathrm{Perf}_{Z_1\cap Z_2} X.$$

The compatibility of products for $n = m = 0$ follows directly from the classical HKR isomorphism in Equation 2.1. Consider the commutative diagram of Equation 2.3 for $n = 0$. We see that both upward arrows respect the product, and the horizontal arrows (i.e. pullback to an open subscheme) respect the product as well. We see that a product with a term in H^1 can be computed by lifting it along ∂ to H^0 and computing the product there and mapping it back to H^1 (e.g. in the middle term of Equation 2.4). This deduces the claim for all products $H^i \otimes H^j$ with $i, j \leq 1$ from the H^0-case. With the same argument lift elements along ∂ from H^n to H^{n-1}, compute products there, to inductively prove the claim for all products $H^i \otimes H^j$ with $i, j \leq n$ once it is proven for all $i, j \leq n-1$. For (4) it suffices to invoke Cor. 1.2.2; everything carries over verbatim. ☐

2.1. **The E_1-pages.** We can use the results of the previous section in order to compare the different coniveau spectral sequences from §1.1. We need some basic facts regarding the vanishing of Hochschild or local cohomology groups for local rings:

Proposition 2.1.1. *Let k be a field and $(R, \mathfrak{m})$ an essentially smooth local k-algebra of dimension n. Then*

$$HH_i^{\mathfrak{m}}(R) = \begin{cases} 0 & \text{for } i > 0 \\ H^n_{\mathfrak{m}}(R, \Omega^{n+i}) & \text{for } -n \leq i \leq 0 \\ 0 & \text{for } i < -n \end{cases}$$

and if M is a finitely generated R-module,

$$H^p_{\mathfrak{m}}(R, M) = \begin{cases} \mathrm{Hom}_R(M, \Omega^n)^\vee & \text{for } p = n \\ 0 & \text{for } p \neq n, \end{cases}$$

where $(-)^\vee := \mathrm{Hom}_R(-, E)$ denotes the Matlis dual (for E some injective hull of $\kappa(\mathfrak{p}) = R/\mathfrak{m}$ as an R-module).

Proof. (see [ILL+07] for background) Let $t_1, \ldots, t_n$ be a regular sequence, so that $\mathfrak{m} = (t_1, \ldots, t_n)$. By Proposition 2.0.1 and because Ω^1 is free of rank n, we have

$$HH_i^{\mathfrak{m}}(R) \cong H^n_{\mathfrak{m}}(R, \Omega^{n+i}) \cong H^n_{\mathfrak{m}}(R, R) \otimes \Omega^{n+i}.$$

Since Ω^{n+i} is zero for $i > 0$, and similarly for $i < -n$, we immediately get the first claim. Next, by (the simplest form of) Local Duality we have

$$H^p_{(t_1,\ldots,t_n)}(R, M) \cong \begin{cases} \mathrm{Hom}_R(M, \omega_R)^\vee & \text{for } p = n \\ 0 & \text{for } p \neq n, \end{cases}$$

where M is an arbitrary finitely generated R-module and ω_R a canonical module over k. Since R is a smooth k-algebra, $\omega_R := \Omega^n$ is a canonical module, and so we get the second claim. $\square$

Let us compare the E_1-pages of the two different spectral sequences. They are fairly different. For the coherent Cousin coniveau spectral sequence, it is supported in the first quadrant and has the following shape:

$$\begin{array}{c|ccccccc}
q & \vdots & & & & & & \\
2 & \coprod_{x\in X^0} H^2_x(\Omega^n) & & \ddots & & & & \\
1 & \coprod_{x\in X^0} H^1_x(\Omega^n) & \to & \coprod_{x\in X^1} H^2_x(\Omega^n) & \to & \coprod_{x\in X^2} H^3_x(\Omega^n) & & \\
0 & \coprod_{x\in X^0} H^0_x(\Omega^n) & \to & \coprod_{x\in X^1} H^1_x(\Omega^n) & \to & \coprod_{x\in X^2} H^2_x(\Omega^n) & \to\cdots & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}$$

We have $H^p_x(X,\Omega^n) = H^p_x(\mathscr{O}_{X,x},\Omega^n)$ by excision, Lemma 1.1.3, and since $\dim(\mathscr{O}_{X,x}) = \operatorname{codim}_X(x)$, Proposition 2.1.1 implies that the groups on this E_1-page vanish unless the cohomological degree matches the codimension of the point in question. However, this is only the case for the $q = 0$ row. We are left with the following E_1-page:

$$\begin{array}{c|ccccccc}
q & \vdots & & & & & & \\
1 & 0 & \to & 0 & & \ddots & & \\
0 & \coprod_{x\in X^0} H^0_x(\Omega^n) & \to & \coprod_{x\in X^1} H^1_x(\Omega^n) & \to & \coprod_{x\in X^2} H^2_x(\Omega^n) & \to\cdots & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}$$

Thus, it collapses to a single row already on the E_1-page. As it converges to $H^{p+q}(X,\Omega^n)$, we re-obtain a special case of Cor. 1.1.7:

Corollary 2.1.2. *If X/k is smooth separated, there are canonical isomorphisms*

$$H^p(X,\Omega^n) \cong H^p(X, Cous_\bullet(\Omega^n)),$$

coming from the E_1-page degeneration of the coherent Cousin coniveau spectral sequence.

Now let us compare these results to the Hochschild coniveau spectral sequence. It is supported in the first and fourth quadrant.

$$\begin{array}{c|ccccccc}
q & \vdots & & & & & & \\
1 & \coprod_{x\in X^0} HH^x_{-1}(X) & & & & & & \\
0 & \coprod_{x\in X^0} HH^x_0(X) & \to & \coprod_{x\in X^1} HH^x_{-1}(X) & \to & \ddots & & \\
-1 & \coprod_{x\in X^0} HH^x_1(X) & \to & \coprod_{x\in X^1} HH^x_0(X) & \to & \coprod_{x\in X^2} HH^x_{-1}(X) & & \\
-2 & \coprod_{x\in X^0} HH^x_2(X) & \to & \coprod_{x\in X^1} HH^x_1(X) & \to & \coprod_{x\in X^2} HH^x_0(X) & \to\cdots & \\
 & \vdots & & \vdots & & \vdots & & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}$$

If we make use of our HKR theorem with supports, this can be rephrased in terms of local cohomology groups.

$$
\begin{array}{c|ccccccc}
q & \vdots & & \vdots & & & & \\
1 & 0 & & 0 & & 0 & & \\
0 & \coprod_{x\in X^0} H^0_x(\Omega^0) & \to & \coprod_{x\in X^1} H^1_x(\Omega^0) & \to & \ddots & & \\
-1 & \coprod_{x\in X^0} H^0_x(\Omega^1) & \to & \coprod_{x\in X^1} H^1_x(\Omega^1) & \to & \coprod_{x\in X^2} H^2_x(\Omega^1) & & \\
-2 & \coprod_{x\in X^0} H^0_x(\Omega^2) & \to & \coprod_{x\in X^1} H^1_x(\Omega^2) & \to & \coprod_{x\in X^2} H^2_x(\Omega^2) & \to \cdots & \\
 & \vdots & & \vdots & & \vdots & & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}
$$

In particular, this interpretation reveals that we are actually facing a spectral sequence which is supported exclusively in the fourth quadrant. So far, this leaves open how the HKR isomorphism with supports interacts with the rightward arrows. We will rectify this now.

Theorem 2.1.3 (Row-by-row Comparison)**.** *Let k be a field and X/k a smooth separated scheme. The $(-q)$-th row on the E_1-page of the Hochschild coniveau spectral sequence is isomorphic to the zero-th row of the E_1-page of the coherent Cousin coniveau spectral sequence of Ω^q. That is: For every integer q, there is a canonical isomorphism of chain complexes*

$$ {}^{HH}E_1^{\bullet,-q} \xrightarrow{\sim} {}^{Cous}E_1^{\bullet,0}(\Omega^q)\ . $$

Entry-wise, this isomorphism is induced from the HKR isomorphism with supports.

Proof. The idea is the following: From Proposition 2.0.1 we already know that if U is a smooth affine k-scheme and Z a closed subscheme, defined by a regular element $f \in \mathscr{O}(U)$, the boundary maps in Hochschild homology with supports resp. local cohomology are compatible. Thus, in order to prove this claim, we need to show that the evaluation of the differential d on the respective E_1-pages can be reduced to evaluating such boundary maps.

To carry this out, recall that the equivalence in line Equation 1.18 is induced from the pullback $j_x : \operatorname{Spec} \mathscr{O}_{X,x} \hookrightarrow X$ (Remark 1.1.10). For every open subscheme neighbourhood $U \subseteq X$ containing $x \in X$, we get a canonical factorization of j^* as

$$ \operatorname{Perf}(X) \longrightarrow \operatorname{Perf}(U) \longrightarrow \operatorname{Perf}(\mathscr{O}_{X,x}). \tag{2.9} $$

Each perfect complex on $\operatorname{Spec} \mathscr{O}_{X,x}$ comes from all sufficiently small open subschemes $U \ni x$, and they become isomorphic if and only if this already happens on a sufficiently small open U. See [Bal07, §4.1, especially p. 1247] for a discussion of this. Suppose $Z^{[0]} \supseteq Z^{[1]} \supseteq \dots$ are closed subsets of X such that $\operatorname{codim}_X(Z^{[p]}) \geq p$. Then we get a filtration

$$ \cdots \hookrightarrow \operatorname{Perf}_{Z^{[2]}}(X) \hookrightarrow \operatorname{Perf}_{Z^{[1]}}(X) \hookrightarrow \operatorname{Perf}_{Z^{[0]}}(X) = \operatorname{Perf}(X), $$

analogous to the one in line 1.13. There is a partial order on the set of all such filtrations $Z^{[0]} \supseteq Z^{[1]} \supseteq \dots$, where $Z' \geq Z$ if and only if $Z'^{[p]} \supseteq Z^{[p]}$ holds for all p. We may form a spectral sequence based on this filtration, as above. We will not have an analogue of Equation 1.18 available in this context, but we still get a convergent spectral sequence converging to $HH_{-p-q}(X)$. Taking the colimit of this spectral sequence over all filtrations $\{Z^{[\cdot]}\}$, we obtain the above spectral sequence. The advantage of a spectral sequence of a filtration $\{Z^{[\cdot]}\}$ is that the $Z^{[i]}$ are reduced closed subschemes with open subschemes as complements so that the boundary maps ∂ of this spectral sequence correspond to a localization sequence for a true open-closed complement. By the above colimit argument, for any element $\alpha \in HH^x_i(\mathscr{O}_{X,x})$ for $x \in X^p$, we may compute the differential d on the E_1-page by performing the computation on the E_1-page of a concrete filtration $\{Z^{[\cdot]}\}$. In particular, we may choose this filtration sufficiently fine such that (1) we can work with an *affine* neighbourhood $U \ni x$ in line 2.9, and (2) such that there exists some $f \in \mathscr{O}_X(U)$ so that the codimension ≥ 1 closed subset in $\mathscr{O}_{X,x}$ is cut out by f, i.e. the stalk of the ideal sheaf $\mathscr{I}_{Z^{[p+1]},x} \subseteq \mathscr{O}_{X,x}$ is generated by f, and (3) the class α is pulled back from some $\tilde{\alpha} \in HH^x_i(U)$. If one finds a U such that (1) holds, one may need to shrink it further to ensure (2) holds as well, and then even smaller to ensure (3). Then, inspecting Diagram 1.17, we may compute the differential d on the ${}^{HH}E_1$-page by

$$\underset{\ni\alpha}{\coprod_{x\in X^p} HH_i^x(\mathscr{O}_{X,x})} \leftarrow \underset{\ni\tilde{\alpha}}{\coprod_{x\in X^p} HH_i^x(U)} \xrightarrow{\partial} HH_{i-1}^{\tilde{Z}}(U) \longrightarrow \coprod_{x\in X^{p+1}} HH_{i-1}^x(\mathscr{O}_{X,x}).$$

As we assume that X is smooth, this means that d reduces to evaluating the boundary map ∂ in the localization sequence corresponding to cutting out a regular element f from $\operatorname{Spec}\mathscr{O}_X(U)$, a smooth affine k-scheme. On the other hand, the ${}^{Cous}E_1$-differential d is given by Diagram 1.11, and we can factor this analogously (with the same open subset U) as

$$\underset{\ni HKR(\alpha)}{\coprod_{x\in X^p} H_x^p(\mathscr{O}_{X,x},\Omega^*)} \leftarrow \underset{\ni HKR(\tilde{\alpha})}{\coprod_{x\in X^p} H_x^p(U,\Omega^*)} \xrightarrow{\partial} H_{\tilde{Z}}^{p+1}(U,\Omega^*) \longrightarrow \coprod_{x\in X^{p+1}} H_x^{p+1}(\mathscr{O}_{X,x},\Omega^*),$$

By Proposition 2.0.1 the boundary map ∂ in local cohomology here is compatible with the corresponding boundary map in Hochschild homology with supports. In other words: As the differential d of the coherent Cousin coniveau spectral sequence can be reduced in a completely analogous way to the same affine open U, it follows that the HKR isomorphism commutes with computing the boundary map in the respective row of the ${}^{Cous}E_1$-page. Our claim follows. $\square$

We know from Corollary 2.1.2 that the cohomology of the q-th row agrees with the sheaf cohomology of Ω^{-q}. Thus, the E_2-page of the Hochschild coniveau spectral sequence reads

$$\begin{array}{r|cccc}
q & \vdots & \vdots & & \\
1 & 0 & 0 & 0 & \\
0 & H^0(X,\Omega^0) & H^1(X,\Omega^0) & \ddots & \\
-1 & H^0(X,\Omega^1) & H^1(X,\Omega^1) & H^2(X,\Omega^1) & \\
-2 & H^0(X,\Omega^2) & H^1(X,\Omega^2) & H^2(X,\Omega^2) & \cdots \\
 & \vdots & \vdots & \vdots & \\
\hline
 & 0 & 1 & 2 & p
\end{array} \tag{2.10}$$

Remark 2.1.4. If X is affine, say $X := \operatorname{Spec} A$, the higher sheaf cohomology groups vanish, i.e. $H^p(X,\Omega^{-q}) = 0$ for all $p \neq 0$. Thus, the E_2-page has collapsed to a single column, and the convergence of the spectral sequence just becomes the statement that

$$H^0(X,\Omega^{-q}) \cong HH_{-q}(X) \qquad \text{for all } q \in \mathbf{Z}$$

and since the left-hand side agrees with $\Omega^{-q}_{A/k}$, we recover the ordinary HKR isomorphism.

At least if the base field k has characteristic zero, this E_2-page degenerates in general, even if X is not affine. This follows from incompatible Hodge degrees, as we explain in the following sub-section:

2.1.1. *Interplay with Hodge degrees.* Suppose k is a field of characteristic zero. Then the Hochschild homology of commutative k-algebras comes with a filtration, known either as Hodge or λ-filtration. It was introduced by Gerstenhaber and Schack [GS87] and Loday [Lod89]. Weibel has extended this filtration to separated Noetherian k-schemes[4] in his paper [Wei97]. One obtains a canonical and functorial direct sum decomposition

$$HH_p(X) = \bigoplus_j HH_p(X)^{(j)}. \tag{2.11}$$

See [Wei97, Proposition 1.3]. He also proved that $HH_p(X)^{(j)} = H^{j-p}(X,\Omega^j)$ holds for smooth k-schemes X, providing a very explicit relation to the usual Hodge decomposition [Wei97, Cor. 1.4]. Based on this, we can define a Hodge decomposition on Hochschild homology with supports as well.

[4] Actually to an even broader class of schemes.

If the base field k has characteristic zero, we may define

$$HH_Z(X)^{(j)} := \mathrm{hofib}\left(HH\,(X)^{(j)} \longrightarrow HH\,(X-Z)^{(j)}\right).$$

Since the usual Hochschild homology just splits into direct summands functorially, as in Equation 2.11, the spectral sequence constructed in §1.1.2 splits into a direct sum of spectral sequences. The same happens to our HKR isomorphism with supports, Proposition 2.0.1:

Theorem 2.1.5 (Compatibility with Hodge degrees)**.** *Let k be a field of characteristic zero.*

(1) *Suppose R is a smooth k-algebra and $t_1, \ldots, t_n$ a regular sequence. Then the Hodge decomposition refines the isomorphism of Proposition 2.0.1 in the following fashion:*

$$HH_i^{(t_1,\ldots,t_n)}(R)^{(j)} = \begin{cases} H^n_{(t_1,\ldots,t_n)}(R, \Omega^{n+i}) & \text{if } n+i=j \\ 0 & \text{if } n+i \neq j. \end{cases}$$

(2) *Suppose X is a smooth separated k-scheme. Then the spectral sequence of §1.1.2 splits as a direct sum of spectral sequences*

$$\left({}^{HH}E_1^{p,q}\right)^{(j)} := \coprod_{x \in X^p} HH^x_{-p-q}(\mathscr{O}_{X,x})^{(j)} \Rightarrow HH_{-p-q}(X)^{(j)}.$$

(3) *Suppose X is a smooth separated k-scheme. The spectral sequence ${}^{HH}E$ degenerates on the E_2-page, i.e. all differentials in Figure 2.10 are zero.*

Remark 2.1.6. The results (2) and (3) are very close to well-known older results of Weibel. For example, the spectral sequence in (2) has a large formal resemblance to the one constructed in Weibel [Wei97, Proposition 1.2]. However, he uses a quite different construction to set up his spectral sequence. He uses the hypercohomology spectral sequence of his sheaf approach to the Hochschild homology of a scheme, as in line 1.4. He also obtains an E_2-degeneration statement with essentially the same proof as ours, [Wei97, Cor. 1.4], for his spectral sequence.

Proof. (1) The proof is exactly the same as we have given for Proposition 2.0.1. By functoriality the Hodge decomposition can be dragged through the entire proof systematically. Only the first step of the proof changes, where one has to use that the ordinary HKR isomorphism is supported entirely in the n-Hodge part:

$$\Omega^n_{R/k} \xrightarrow{\sim} HH_n(R)^{(n)} \qquad \text{and} \qquad HH_n(R)^{(j)} = 0 \quad (\text{for } j \neq n).$$

This is [Lod89, 3.7. Théorème] or [Lod92, Theorem 4.5.12], for example.
(2) Immediate.
(3) The E_1-page of the Hodge degree j graded part takes the shape

$$\begin{array}{r|ccccccc}
q & \vdots & & & & & & \\
1 & \coprod_{x\in X^0} HH^x_{-1}(X)^{(j)} & & & & & & \\
0 & \coprod_{x\in X^0} HH^x_{0}(X)^{(j)} & \to & \coprod_{x\in X^1} HH^x_{-1}(X)^{(j)} & \to & \ddots & & \\
-1 & \coprod_{x\in X^0} HH^x_{1}(X)^{(j)} & \to & \coprod_{x\in X^1} HH^x_{0}(X)^{(j)} & \to & \coprod_{x\in X^2} HH^x_{-1}(X)^{(j)} & & \\
-2 & \coprod_{x\in X^0} HH^x_{2}(X)^{(j)} & \to & \coprod_{x\in X^1} HH^x_{1}(X)^{(j)} & \to & \coprod_{x\in X^2} HH^x_{0}(X)^{(j)} & \to \cdots & \\
 & \vdots & & \vdots & & \vdots & & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}$$

and applying the refined HKR isomorphism with supports to these entries, part (1) of our claim implies that all rows vanish except for the row with $q=-j$. As a result, it follows that the spectral sequence degenerates. As our original spectral sequence ${}^{HH}E^{\bullet,\bullet}$ is just a direct sum of these $\left({}^{HH}E_1^{p,q}\right)^{(j)}$, it follows that all differentials of the ${}^{HH}E_2$-page must be zero (because the differentials then also are direct sums of the differentials of the individual $\left({}^{HH}E_1^{p,q}\right)^{(j)}$, so they cannot map between different Hodge graded parts). $\square$

This also leads to a version of the 'Gersten resolution', which differs from the classical coherent Cousin resolution in the way it is constructed, but not in its output. For an abelian group A, we write $(i_x)_*A$ to denote the constant sheaf A on the scheme point x and extended by zero elsewhere.

Corollary 2.1.7 ("Hochschild–Cousin resolution"). *Suppose X/k is a smooth separated scheme over a field k. Then*

$$\mathscr{H}\mathscr{H}_n \xrightarrow{\sim} \left[\coprod_{x \in X^0} (i_x)_* HH_n^x(\mathscr{O}_{X,x}) \longrightarrow \coprod_{x \in X^1} (i_x)_* HH_{n-1}^x(\mathscr{O}_{X,x}) \longrightarrow \cdots \right]_{0,n} \tag{2.12}$$

is a quasi-isomorphism of complexes of sheaves, and yields a flasque resolution of the Zariski sheaf $\mathscr{H}\mathscr{H}_n \cong \Omega^n$ for any n.

It will also be possible to prove this using a transfer-based method à la [CTHK97], [Wei05].

Proof. We give two proofs: (1) We can use Theorem 2.1.3, proving that the complex of sheaves on the right is canonically isomorphic to the coherent Cousin resolution of Corollary 1.1.7. The latter is a resolution even under far less restrictive assumptions than smoothness, relying on the tools of [Har66]. (2) Alternatively, suppose k is of characteristic zero. We may consider the Hochschild coniveau spectral sequence ${}^{HH}E^{\bullet,\bullet}$ of U for any open immersion $U \hookrightarrow X$. We obtain a presheaf of spectral sequences, which we sheafify in the Zariski topology. We denote it by ${}^{HH}\mathscr{E}^{p,q}$. As this process also sheafifies the limit of the spectral sequence, we get a spectral sequence of sheaves

$${}^{HH}\mathscr{E}_1^{p,q} := \coprod_{x \in X^p} (i_x)_* HH_{-p-q}^x(\mathscr{O}_{X,x}) \Rightarrow \mathscr{H}\mathscr{H}_{-p-q}(X).$$

The direct sum decomposition of Theorem 2.1.5 is functorial in pullback along opens, so ${}^{HH}\mathscr{E}^{p,q}$ degenerates on the second page. Restrict to the direct summand of the $\mathscr{H}\mathscr{H}_n$ which we are interested in. This leaves only one non-zero entry on the E_2-page. The sheaves in Equation 2.12 are clearly flasque and since the E_2-page has just one entry, the resolution property follows easily (it implies the exactness in all higher degrees). $\square$

2.1.2. *Chern character with supports.* Let X/k be a smooth scheme and $x \in X$ a scheme point of codimension $\operatorname{codim}_X \overline{\{x\}} = p$. We will define a *Chern character with supports*,

$$\mathscr{T}(x) : K_m(\kappa(x)) \longrightarrow H_x^p(X, \Omega^{p+m}).$$

The definition is simple: As X/k is smooth, dévissage and excision for K-theory yield a canonical isomorphism $K(\kappa(x)) \cong K^x(\mathscr{O}_{X,x}) \cong K^x(X)$, where K^x denotes K-theory with support in $\{x\}$. The spectrum-level Chern character $K \to HH$ (à la McCarthy [McC94], Definition 4.4.1, in the version of Keller [Kel99], Section 0.1) induces a map $K^x(X) \to HH^x(X)$. Excision and the HKR isomorphism with supports for Hochschild homology then yield $HH_m^x(X) \cong HH_m^x(\mathscr{O}_{X,x}) \cong H_x^p(\mathscr{O}_{X,x}, \Omega^{p+m})$. We call the composition of these maps $\mathscr{T}(x)$.

Example 2.1.8. *If X/k is an integral smooth scheme with generic point η, the map $\mathscr{T}(\eta) : K_*(k(X)) \to \Omega^*_{k(X)/k}$ is just the trace map $K \to HH$, applied to the rational function field of X.*

Proposition 2.1.9. *Let X/k be a Noetherian scheme over a field k.*

(1) *Then the Chern character (a.k.a. trace map)*

$$K(X) \longrightarrow HH(X)$$

induces a morphism of spectral sequences ${}^{K}E^{\bullet,\bullet} \to {}^{HH}E^{\bullet,\bullet}$, where ${}^{K}E^{\bullet,\bullet}$ denotes Balmer's coniveau spectral sequence of [Bal09].

(2) *If X is smooth over k, we may compose it with the comparison map to the coherent Cousin spectral sequence, and then the map between the E_1-pages is the Chern character for supports:*

$$\mathscr{T}(x) : {}^{K}E_1^{p,q} \longrightarrow {}^{Cous}E_1^{p,0}(\Omega^{-q})$$
$$K_{-p-q}(\kappa(x)) \longrightarrow H_x^p(X, \Omega^{-q})$$

Here we have used that Balmer's coniveau spectral sequence agrees with Quillen's from [Qui73] *thanks to the smoothness assumption.*

Proof. (1) This is true by functoriality. We have constructed ${}^{HH}E^{\bullet,\bullet}$ based on the same filtration that Balmer uses for K-theory (cf. Proposition 1.1.8). The Chern character $K \to HH$ is compatible with the respective localization sequences, and thus the trace functorially induces a morphism of spectral sequences. On the E_1-page, this morphism induces morphisms

$$T_{p,q} : \coprod_{x\in X^p} K^x_{-p-q}(\mathscr{O}_{X,x}) \longrightarrow \coprod_{x\in X^p} HH^x_{-p-q}(\mathscr{O}_{X,x})$$

and by comparing supports, these morphisms are Cartesian in the sense that the direct summand of $x \in X^p$ on the left maps exclusively to the direct summand belonging to the same x on the right-hand side. That is, $T_{p,q} = \sum T^x_{p,q}$ with

$$T^x_{p,q} : K^x_{-p-q}(\mathscr{O}_{X,x}) \longrightarrow HH^x_{-p-q}(\mathscr{O}_{X,x}).$$

(2) Now, assume that X/k is smooth and separated. We may then, equivalently, use the K-theory of coherent sheaves on the left, and then dévissage. So, using the dévissage isomorphism on the left-hand side in the above equation, and the HKR isomorphism with supports on the right-hand side, we obtain

$$T^{x\prime}_{p,q} : K_{-p-q}(\kappa(x)) \cong K^x_{-p-q}(\mathscr{O}_{X,x}) \longrightarrow HH^x_{-p-q}(\mathscr{O}_{X,x}) \cong H^p_x(\mathscr{O}_{X,x}, \Omega^{-q}).$$

Using excision of the right-hand side, this transforms into the definition of $\mathscr{T}(x)$. □

3. Cubical algebras and their residue symbol

3.1. **Introduction to the comparison problem.** The next sections will be devoted to relating our Hochschild–Cousin complex to the residue theory of [Tat68], [Beĭ80]. Let us briefly sketch the story in dimension one, in order to motivate how we shall proceed.

3.1.1. *Residue à la Tate.* In [Tat68] Tate defines the residue of a rational 1-form on an integral curve X/k at a closed point $x \in X$ as follows: Let $\widehat{\mathscr{K}_{X,x}} := \operatorname{Frac} \widehat{\mathscr{O}}_{X,x}$ denote the local field at the point x. It can also be constructed by completing the function field with respect to the metric of the valuation associated to x. Now $\widehat{\mathscr{K}_{X,x}}$ is a locally linearly compact topological k-vector space.[5] It has infinite dimension. Any rational functions $f, g \in k(X)$ act on it by continuous k-linear endomorphisms, i.e. we could also read them as elements $f, g \in \operatorname{End}^{cts}_k(\widehat{\mathscr{K}_{X,x}})$. If P^+ denotes any projector splitting the inclusion $\widehat{\mathscr{O}}_{X,x} \hookrightarrow \widehat{\mathscr{K}_{X,x}}$, Tate shows that the commutator $[P^+ f, g]$ has sufficiently small image to define a trace on it, and defines a map

$$\Omega^1_{\widehat{\mathscr{K}_{X,x}}/k} \longrightarrow k, \qquad f\,\mathrm{d}g \longmapsto \operatorname{Tr}[P^+ f, g].$$

He shows that this agrees with the usual residue of the 1-form $\omega := f\,\mathrm{d}g$ at x. In [Beĭ80], [BFM91], [BBE02] this construction gets interpreted in terms of a central extension of Lie algebras, giving a Lie algebra cohomology class

$$\phi_{Tate} \in H^2_{\mathrm{Lie}}((\widehat{\mathscr{K}_{X,x}})_{Lie}, k). \tag{3.1}$$

3.1.2. *Residue via localization.* A completely different approach to think about the residue might be to use the boundary map ∂ in Keller's localization sequence,

$$HH_1(\widehat{\mathscr{O}}_{X,x}) \longrightarrow HH_1(\widehat{\mathscr{K}_{X,x}}) \xrightarrow{\partial} HH^x_0(\widehat{\mathscr{O}}_{X,x}) \xrightarrow{\operatorname{Tr}} k. \tag{3.2}$$

Via the HKR comparison map, $\Omega^1_{\widehat{\mathscr{K}_{X,x}}/k} \to HH_1(\widehat{\mathscr{K}_{X,x}})$, this also produces a map $\Omega^1_{\widehat{\mathscr{K}_{X,x}}/k} \to k$, which *should* be the residue.

---

[5] In Tate's set-up (cf. p. 3 of *loc. cit*), given a subspace $A \subset V$, the collection of linear subspaces $\{A' \mid A' < A\}$ satisfies the axioms for the closed sets in a (linear) topology, and Tate's algebra E is just the ring of operators on V which are continuous with respect to this linear topology.

3.1.3. *Comparison.* We sketch the comparison for $n = 1$. This can serve as a guide through the entire proof. We begin with the boundary map in the Hochschild localization sequence, that is

$$HH_1(\widehat{\mathscr{K}_{X,x}}) \xrightarrow{\partial} HH_0^x(\widehat{\mathscr{O}}_{X,x}) \tag{3.3}$$

from Equation 3.2. This is the boundary map coming from the localization sequence of

$$\mathrm{Coh}_{\overline{\{x\}}}\,\widehat{\mathscr{O}}_{X,x} \longrightarrow \mathrm{Coh}\,\widehat{\mathscr{O}}_{X,x} \longrightarrow \mathrm{Coh}\,\widehat{\mathscr{K}_{X,x}},$$

where we have taken the freedom to drop writing "Spec" for denoting the affine schemes attached to these rings. The first idea is to map this sequence to a sequence built from Pro-categories. We will review these categories below; for the moment just imagine a category whose objects are projective systems.

(3.4)

$$\begin{tikzcd}
\mathrm{Coh}_{\overline{\{x\}}}(\widehat{\mathscr{O}}_{X,x}) \arrow[r] \arrow[d, "1"] & \mathrm{Coh}(\widehat{\mathscr{O}}_{X,x}) \arrow[r] \arrow[d] & \mathrm{Coh}(\widehat{\mathscr{K}_{X,x}}) \arrow[d] \\
\mathrm{Coh}_{\overline{\{x\}}}(\widehat{\mathscr{O}}_{X,x}) \arrow[r] & \mathsf{Pro}^a_{\aleph_0}(\mathrm{Coh}_{\overline{\{x\}}}(\widehat{\mathscr{O}}_{X,x})) \arrow[r] & \mathsf{Pro}^a_{\aleph_0}(\mathrm{Coh}_{\overline{\{x\}}}(\widehat{\mathscr{O}}_{X,x}))/\mathrm{Coh}_{\overline{\{x\}}}(\widehat{\mathscr{O}}_{X,x})
\end{tikzcd}$$

The left downward arrow is the identity functor, while the middle downward functor "unravels" $\widehat{\mathscr{O}}_{X,x}$-modules as a formal projective limit of modules with support in $\overline{\{x\}}$. For example, $\mathscr{O}_{X,x}$ itself would be sent to

$$\varprojlim_i \left(\mathscr{O}_{X,x}/\mathfrak{m}_x^i\right),$$

but *as this projective system*, and not in terms of evaluating the limit in some category. The right downward arrow is the one which makes the diagram commute. We get a commutative square of boundary maps: Instead of working with ∂, we can compatibly work with the boundary map coming from the bottom sequence. The quotient category on the right can essentially be replaced by a Tate category – another concept which we shall recall below. The reader may imagine a category which mixes Ind- and Pro-limits. The idea is that it models local linear compactness abstractly. One may visualize the downward right arrow, after this replacement, as a functor

$$\mathrm{Coh}\,\widehat{\mathscr{K}_{X,x}} \longrightarrow \mathsf{Tate}_{\aleph_0}(\widehat{\mathscr{O}}_{X,x})$$

sending a $\widehat{\mathscr{K}_{X,x}}$-module to its realization as a formal inductive-projective system along

$$\underset{s\in\mathscr{O}_{X,x}\setminus\{0\}}{\underrightarrow{\mathrm{colim}}}\ \varprojlim_i \left(\frac{1}{s}\mathscr{O}_{X,x}/\mathfrak{m}_x^i\right).$$

The next idea is to use the philosophy that the Pro-categories or Tate categories in question are module categories over certain non-commutative algebras. Using a relative Morita theory (a formalism we shall develop below in §4.4; it extends Morita theory to quotient categories), the lower sequence now turns out to be compatible with the Hochschild sequence attached to an exact sequence of non-unital non-commutative algebras.

This leads to two facts: Firstly, as the Tate categories and Pro-categories were modelled to imitate locally linearly compact vector spaces, one finds that the non-commutative algebra appearing here (defined without topology, using Tate categories) is actually isomorphic to the one which appears in Tate's work,

$$A := \mathrm{End}_k^{cts}(\widehat{\mathscr{K}_{X,x}}) \cong \mathrm{End}_{\mathsf{Tate}_{\aleph_0}(k)}(\widehat{\mathscr{K}_{X,x}}). \tag{3.5}$$

Secondly, it remains to identify the boundary map between these non-commutative algebras (starting from A in the case at hand). These have a special structure; they are so-called Beilinson cubical algebras (we recall this concept below). Once we reach this point, we have shown that the boundary map ∂ of line 3.3 is compatible with a boundary map in the Hochschild homology of such cubical algebras. Finally, in the paper [Bra18] it was shown that the latter is compatible to Tate and Beilinson's map in Lie homology.

Concretely, this is the "Lie-to-Hochschild comparison theorem" of the Introduction, loc. cit:

$$\begin{tikzcd}
 & H_n^{\mathrm{Lie}}(\mathfrak{g},\mathfrak{g}) \arrow[dl] \arrow[dr] & \\
H_{n+1}^{\mathrm{Lie}}(\mathfrak{g},k) \arrow[dr] \arrow[ddr, "\phi_{Beil}"'] & & HH_n(A) \arrow[dl] \arrow[ddl, "\phi_{HH}"] \\
 & HH_n(A) \arrow[d, "\phi_C"'] & \\
 & k &
\end{tikzcd}$$

In the cited paper, this is stated also for cyclic homology HC; both is possible. Here A is the cubical algebra in question. In our toy case $n = 1$ the A is precisely the one of Equation 3.5, $\mathfrak{g}$ is its Lie algebra. Next, ϕ_{Beil} is the same as Tate's map (since Beilinson's residue is the higher-dimensional generalization of Tate's one-dimensional theory), i.e. $\phi_{Beil} = \phi_{Tate}$ of Equation 3.1, just write the Lie *co*homology class as a functional on Lie *ho*mology. Finally, ϕ_C is the trace of the boundary map of algebras discussed above, so *the* input which the above discussion had let us to. This closes the circle. The switch to Lie homology involves a degree jump. This explains why the Hochschild boundary map

$$HH_1(\widehat{\mathscr{K}}_{X,x}) \xrightarrow{\partial} HH_0^x(\widehat{\mathscr{O}}_{X,x}) \xrightarrow{\mathrm{Tr}} k$$

of Equation 3.2 suddenly becomes a map

$$\phi_{Tate} : H_2^{\mathrm{Lie}}((\widehat{\mathscr{K}}_{X,x,Lie}, k) \xrightarrow{\mathrm{Tr}} k$$

originating from degree 2 Lie homology (and for general n from degree $n+1$ Lie homology to k).

How does this picture generalize to $n > 1$? Firstly, instead of working with a single Ind-Pro system, we now have to iterate this n times. This complicates the notation. Instead of a single boundary map, we need to take n boundary maps. However, on each individual level things remain essentially as in the one-dimensional case. On the Lie algebra cohomology side, this do not admit such a reduction as an inductive step, see [Bra18]. However, in Hochschild homology we can use a non-commutative form of excision, which circumvents this issue, see loc. cit.

3.2. Definition of the abstract symbol.

Definition 3.2.1 ([Beĭ80])**.** *Let k be a field. A* Beilinson n-fold cubical algebra *is*

1. *an associative k-algebra A;*
2. *two-sided ideals I_i^+, I_i^- such that $I_i^+ + I_i^- = A$ for $i = 1, \dots, n$;*
3. *define $I_i^0 = I_i^+ \cap I_i^-$ and call $I_{tr} := \bigcap_{i=1,\dots,n} I_i^0$ the* trace-class *operators of A.*

A trace *on an n-fold cubical algebra is a morphism $\tau : I_{tr}/[I_{tr}, A] \longrightarrow k$.*

Although the following property is stronger than necessary to develop the formalism, it will be handy to single out a particularly friendly type of such algebras:

Definition 3.2.2. *We say that $(A, (I_i^{\pm}))$ is* good *if for every $c = 1, \dots, n$ the intersection $I_1^0 \cap \cdots \cap I_c^0$ is locally bi-unital (in the sense of Definition 1.0.9).*

This assumption will be made for the following reasons: Firstly, the simplifications due to Wodzicki's Proposition 1.0.11 apply, and secondly we can easily define a whole hierarchy of further cubical algebras, which we may imagine as going down dimension by dimension.

Lemma 3.2.3. *Let $(A, (I_i^{\pm})_{i=1,\dots,n})$ be a good n-fold cubical algebra. Define*

$$A' := I_1^0 \qquad \text{and} \qquad J_{i-1}^{\pm} := I_i^{\pm} \cap I_1^0$$

for $i = 2, \dots, n$.

1. *Then $(A', (J_i)_{i=1,\dots,n-1})$ is a good $(n-1)$-fold cubical algebra and both algebras have the same trace-class operators.*
2. *The natural homomorphism $I_{tr}/[I_{tr}, I_{tr}] \xrightarrow{\sim} I_{tr}/[I_{tr}, A]$ is an isomorphism.*

The proof of the second claim is very easy, but based on a trick which might not be particularly obvious if one only looks at the claim.

Proof. *(Step 1)* Clearly $A' = I_1^0$ is a (non-unital) associative k-algebra. For $i = 1, \ldots, n-1$ we compute

$$J_i^+ + J_i^- = (I_{i+1}^+ \cap I_1^0) + (I_{i+1}^- \cap I_1^0) \subseteq A \cap I_1^0 = I_1^0.$$

For the converse inclusion, let $x \in I_1^0$ be given. Since $(A, (I_i^\pm))$ is assumed good, there is a local left unit for the singleton finite set $\{x\}$ in I_1^0, say $x = ex$ with $e \in I_1^0$. By $I_{i+1}^+ + I_{i+1}^- = A$, write $x = x^+ + x^-$ with $x^\pm \in I_{i+1}^\pm$. Thus, $x = ex = e(x^+ + x^-) = ex^+ + ex^-$ and since I_1^0 is a two-sided ideal, $ex^s \in I_1^0 \cap I_{i+1}^s$ for $s \in \{+, -\}$. As this works for all $x \in I_1^0$, we get $I_1^0 \subseteq J_i^+ + J_i^-$. Thus, A' is an $(n-1)$-fold cubical algebra. Note that this argument would work just as well with local right units. The trace-class operators are

$$I_{tr}(A') = \bigcap\nolimits_{i=1,\ldots,n-1} J_i^+ \cap J_i^- = \bigcap\nolimits_{i=1,\ldots,n-1} I_{i+1}^+ \cap I_{i+1}^- \cap I_1^0 = \bigcap\nolimits_{i=1,\ldots,n} I_i^+ \cap I_i^- = I_{tr}(A).$$

(Step 2) We need to check that $(A', (J_i^\pm)_{i=1,\ldots,n-1})$ is good, i.e. the local bi-unitality of

$$J_1^0 \cap \cdots \cap J_c^0 = (I_1^0 \cap I_{i+1}^0) \cap \cdots \cap (I_1^0 \cap I_{c+1}^0) = I_1^0 \cap \cdots \cap I_{c+1}^0$$

for any $c = 1, \ldots, n-1$. And these are locally bi-unital since $(A, (I_i^\pm))$ is good. This completes the proof of the first claim.
(Step 3) It remains to prove the second claim. In fact, this is true as soon as A is any associative algebra and I any locally right unital two-sided ideal: It is clear that $[I, I] \subseteq [A, I]$ and we shall show the reverse inclusion: Let any $t \in I$ and $a \in A$ be given. Let e be a local right unit for the singleton set $\{t\} \subset I$. By the ideal property, $ea \in I$ and $at \in I$. Thus, the left-hand side of the following equation lies in $[I, A]$, namely $[ea, t] - [e, at] = eat - tea - eat + ate = ate - tea \overset{(*)}{=} at - ta = [a, t]$, where we have used the local right unit property for $(*)$. Without right unitality, there would have been no chance for this kind of argument. $\square$

Suppose A is a good n-fold cubical algebra with a trace τ. In the paper [Bra18] a canonical functional $\phi_{HH} : HH_n(A) \to k$ was constructed, functorial in morphisms of cubical algebras. We will give a self-contained exposition of this construction:

Let $(A_n, (I_i^\pm), \tau)$ be a good n-fold cubical algebra over k with a trace τ. We define

$$A_{n-1} := I_1^0 \qquad \text{and} \qquad J_{i-1}^\pm := I_i^\pm \cap I_1^0, \tag{3.6}$$

where $i = 0, \ldots, n-1$. By Lemma 3.2.3 this is again a good cubical algebra over k. Define

$$\Lambda : A_n \longrightarrow A_n / A_{n-1}, \qquad x \longmapsto x^+ \tag{3.7}$$

where $x = x^+ + x^-$ is any decomposition with $x^\pm \in I_1^\pm$.

Remark 3.2.4. This map is *not* the natural quotient map!

Lemma 3.2.5. *The map Λ is well-defined.*

Proof. By the axiom $I_1^+ + I_1^- = A_n$ of a cubical algebra, such an element x^+ always exists. It is not unique, but if x^* is another choice, by the exactness of $I_1^0 \to I_1^+ \oplus I_1^- \to A_n \to 0$ we have $x^+ - x^* \in I_1^+ \cap I_1^- = I_1^0 = A_{n-1}$. $\square$

As A_{n-1} is a two-sided ideal in A_n, we get an exact sequence of associative algebras

$$0 \longrightarrow A_{n-1} \longrightarrow A_n \overset{\text{quot}}{\longrightarrow} A_n / A_{n-1} \longrightarrow 0. \tag{3.8}$$

This sequence induces a long exact sequence in Hochschild homology via Theorem 1.0.12, and we shall denote the boundary map by δ.

Definition 3.2.6 ([Bra18])**.** *Define*

$$d : HH(A_n) \xrightarrow{\Lambda} HH(A_n / A_{n-1}) \xrightarrow{\delta} \Sigma HH(A_{n-1}). \tag{3.9}$$

We can repeat this construction and obtain a morphism:

Definition 3.2.7 ([Bra18]). *Suppose $(A, (I_i^{\pm}))$ is good n-fold cubical algebra over k with a trace τ. Define*

$$\phi_{HH} : HH_n(A) \longrightarrow HH_0(I_{tr}) \longrightarrow k, \qquad \gamma \mapsto \tau \underbrace{d \circ \cdots \circ d}_{n\ times} \gamma. \tag{3.10}$$

We call this the abstract Hochschild symbol *of A.*

4. Relation with Tate categories

4.1. **Exact categories.** We will give a brief, almost self-contained review of the formalism of Tate categories. Let $\mathscr{C}$ be an exact category [Büh10]. In particular, among its morphisms, we reserve the symbols "$\hookrightarrow$" resp. "$\twoheadrightarrow$" for admissible monics resp. admissible epics. An admissible sub-object refers to a sub-object such that the inclusion is an admissible monic. We write $\mathscr{C}^{ic}$ to denote the idempotent completion of $\mathscr{C}$.

Lemma 4.1.1. *Let $\mathscr{C}$ be an exact category.*

(1) *The idempotent completion $\mathscr{C}^{ic}$ has a canonical exact structure such that $\mathscr{C} \hookrightarrow \mathscr{C}^{ic}$ is an exact functor reflecting exactness.*
(2) *If $\mathscr{C}$ is split exact, so is $\mathscr{C}^{ic}$.*

Proof. (1) [Büh10, Proposition 6.13], (2) [BGW16c, Proposition 5.23]. □

Next, recall that every exact category can 2-universally be embedded into a Grothendieck abelian category, called $\mathsf{Lex}(\mathscr{C})$, such that it becomes an extension-closed full sub-category and a kernel-cokernel pair is exact if and only if this is so in $\mathsf{Lex}(\mathscr{C})$, in the classical sense of exactness. This is known as the *Quillen embedding* $\mathscr{C} \hookrightarrow \mathsf{Lex}(\mathscr{C})$. See [TT90, §A.7], [Sch04, §1.2] or [Büh10, Appendix A] for a detailed treatment.

4.2. **Ind- and Pro-categories, Tate categories.** Let κ be an infinite cardinal. An *admissible Ind-diagram of cardinality* κ is a functor $X : I \to \mathscr{C}$ with I a directed poset of cardinality at most κ which maps the arrows of I to admissible monics in $\mathscr{C}$. Since $\mathsf{Lex}(\mathscr{C})$ is co-complete, any such diagram has a colimit in this category. Thus, the following definition makes sense:

Definition 4.2.1. *Let $\mathscr{C}$ be an exact category and κ an infinite cardinal.*

(1) *The essential image of all admissible Ind-diagrams of cardinality κ in $\mathsf{Lex}(\mathscr{C})$ is the category of* admissible Ind-objects*, and is denoted by $\mathsf{Ind}^a_\kappa(\mathscr{C})$.*
(2) *Define $\mathsf{Pro}^a_\kappa(\mathscr{C}) := \mathsf{Ind}^a_\kappa(\mathscr{C}^{op})^{op}$, the exact category of admissible Pro-objects.*

See also [BGW16c, §4] for a different perspective on Pro-objects. Definition 4.2.1 is due to Bernhard Keller for $\kappa = \aleph_0$ [Kel90]. See [BGW16c, §3] for a detailed treatment of the general case. One shows that $\mathsf{Ind}^a_\kappa(\mathscr{C})$ is extension-closed inside $\mathsf{Lex}(\mathscr{C})$ and therefore carries a canonical exact structure induced from $\mathsf{Lex}(\mathscr{C})$ [Büh10, Lemma 10.20], [BGW16c, Theorem 3.7], and the functor $\mathscr{C} \hookrightarrow \mathsf{Ind}^a_\kappa(\mathscr{C})$, sending objects to the constant diagram, is exact. We write $\mathsf{Ind}^a(\mathscr{C})$, $\mathsf{Pro}^a(\mathscr{C})$ etc. without a qualifier κ if we do not wish to impose any restriction on the cardinality.

For the sake of legibility, we shall henceforth mostly drop κ from the notation, but all these results would also be valid for the variants constrained by an infinite cardinal κ bound. Precise information about such variations can always be found in the cited sources.

Definition 4.2.2. *Consider the commutative square of exact categories and exact functors,*

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathsf{Ind}^a\mathscr{C} \arrow[d] \\ \mathsf{Pro}^a\mathscr{C} \arrow[r] & \mathsf{Ind}^a\mathsf{Pro}^a\mathscr{C}. \end{tikzcd} \tag{4.1}$$

A lattice *in an object $X \in \mathsf{Ind}^a\mathsf{Pro}^a(\mathscr{C})$ is an admissible sub-object $L \hookrightarrow X$ such that $L \in \mathsf{Pro}^a(\mathscr{C})$ and $X/L \in \mathsf{Ind}^a(\mathscr{C})$.* [BGW16c, §5]

(1) *The category of elementary Tate objects, denoted by* $\mathsf{Tate}^{el}(\mathscr{C})$ *or* $1\text{-}\mathsf{Tate}^{el}(\mathscr{C})$*, is the full sub-category of* $\mathsf{Ind}^a\mathsf{Pro}^a(\mathscr{C})$ *of objects having a lattice (this lattice is not part of the data). The category of Tate objects, denoted* $\mathsf{Tate}(\mathscr{C})$ *or* $1\text{-}\mathsf{Tate}(\mathscr{C})$*, is the idempotent completion* $\mathsf{Tate}^{el}(\mathscr{C})^{ic}$. [BGW16c, §5, Theorem 5.6]
(2) *More generally, define* $n\text{-}\mathsf{Tate}^{el}(\mathscr{C}) := \mathsf{Tate}^{el}\left((n-1)\text{-}\mathsf{Tate}(\mathscr{C}) \right)$ *and* $n\text{-}\mathsf{Tate}(\mathscr{C}) := n\text{-}\mathsf{Tate}^{el}(\mathscr{C})^{ic}$ *as its idempotent completion. We will refer to the objects of these categories as* n-Tate objects.
(3) *Once we fix an elementary Tate object* $X \in \mathsf{Tate}^{el}(\mathscr{C})$*, the lattices form a poset, called the* Sato Grassmannian $Gr(X)$*, by defining* $L' \leq L$ *whenever* $L' \hookrightarrow L$ *is an admissible monic.*

We refer to [BGW16c] for a detailed treatment. See [AK10], [Pre11] for earlier work on iterating Tate categories. The following facts are of essential importance:

Theorem 4.2.3. *Let* $\mathscr{C}$ *be an exact category.*
(1) *If* $L' \hookrightarrow L$ *are lattices in an object* $X \in \mathsf{Tate}^{el}(\mathscr{C})$*, then* $L/L' \in \mathscr{C}$.
(2) *Suppose* $\mathscr{C}$ *is idempotent complete. Then the poset* $Gr(V)$ *is directed and co-directed, i.e. any finite set of lattices has a common sub-lattice and a common over-lattice.*
(3) *If* $X \in \mathsf{Tate}^{el}(\mathscr{C})$ *lies in the sub-categories of Pro-objects and Ind-objects simultaneously, we have* $X \in \mathscr{C}$*. If* $\mathscr{C}$ *is idempotent complete, the same holds true for* $X \in \mathsf{Tate}(\mathscr{C})$.

Proof. (1) [BGW16c, Proposition 6.6], (2) [BGW16c, Theorem 6.7], (3) [BGW16c, Proposition 5.9], [BGW16c, Proposition 5.28]. ☐

There are also some basic factorizations for in- and out-going morphisms under the inclusions of categories in Diagram 4.1 and lattices:

Proposition 4.2.4. *Let* $\mathscr{C}$ *be an exact category.*
(1) *Every morphism* $Y \xrightarrow{a} X$ *in* $\mathsf{Tate}^{el}(\mathscr{C})$ *with* $Y \in \mathsf{Pro}^a(\mathscr{C})$ *can be factored as* $Y \xrightarrow{\tilde{a}} L \hookrightarrow X$ *with* L *a lattice in* X.
(2) *Every morphism* $X \xrightarrow{a} Y$ *in* $\mathsf{Tate}^{el}(\mathscr{C})$ *with* $Y \in \mathsf{Ind}^a(\mathscr{C})$ *can be factored as* $X \twoheadrightarrow X/L \xrightarrow{\tilde{a}} Y$ *with* L *a lattice in* X.

Proof. A complete proof is given in [BGW17, Proposition 2.7]. ☐

4.3. **Quotient exact categories.** If $\mathscr{C} \hookrightarrow \mathscr{D}$ is an exact sub-category, this does not yet suffice to define a quotient exact category "$\mathscr{D}/\mathscr{C}$"with the expected properties. However, as was shown by Schlichting, a sufficient condition for such a category to exist is that $\mathscr{C} \hookrightarrow \mathscr{D}$ is "left or right s-filtering".[6] This is a technical notion and we refer to the original paper [Sch04], or for a quick review to [BGW16c, §2]. Ultimately, $\mathscr{D}/\mathscr{C}$ arises as the localization $\mathscr{D}[\Sigma^{-1}]$, where Σ is the smallest class of morphisms encompassing (1) admissible epics with kernels in $\mathscr{C}$, (2) admissible monics with cokernels in $\mathscr{C}$, (3) and which is closed under composition. The left/right s-filtering conditions imply the existence of a calculus of left/right fractions. As was observed by T. Bühler, in the left s-filtering case, these conditions also imply that inverting admissible epics with kernels in $\mathscr{C}$ is sufficient (see [BGW16c, Proposition 2.19] for a careful formulation of the latter). Let us summarize a number of fully exact sub-categories which have these particular properties:

Proposition 4.3.1. *Let* $\mathscr{C}$ *be an exact category.*
(1) $\mathscr{C} \hookrightarrow \mathsf{Ind}^a(\mathscr{C})$ *is left* s*-filtering.*
(2) $\mathscr{C} \hookrightarrow \mathsf{Pro}^a(\mathscr{C})$ *is right* s*-filtering.*
(3) $\mathsf{Pro}^a(\mathscr{C}) \hookrightarrow \mathsf{Tate}^{el}(\mathscr{C})$ *is left* s*-filtering.*
(4) $\mathsf{Ind}^a(\mathscr{C}) \hookrightarrow \mathsf{Tate}^{el}(\mathscr{C})$ *is right* s*-filtering if* $\mathscr{C}$ *is idempotent complete.*
(5) $\mathsf{Ind}^a(\mathscr{C}) \cap \mathsf{Pro}^a(\mathscr{C}) = \mathscr{C}$*, viewed as full sub-categories of* $\mathsf{Tate}^{el}(\mathscr{C})$.

[6] For a simple example of why this is needed, let $\mathscr{D} = \mathsf{Vect}_f(k)$ be the category of finite dimensional vector spaces over a field k, and let $\mathscr{C} \subset \mathscr{D}$ be the full subcategory of even dimensional vector spaces. Then $\mathscr{C}$ is left special in $\mathscr{D}$ but not left filtering (consider the map $k^{\oplus 2} \twoheadrightarrow k$). Because $\mathscr{D}$ is generated by a single object (namely the line k), any exact category quotient $\mathscr{D}/\mathscr{C}$ would have all objects isomorphic to 0.

Proof. (1) [BGW16c, Proposition 3.10], (2) [BGW16c, Theorem 4.2], (3) [BGW16c, Proposition 5.8], (4) [BGW16c, Remark 5.35], [BGHW18, Cor. 2.4], (5) [BGW16c, Proposition 5.9]. ☐

The construction of this type of quotient category is compatible with the formation of derived categories in the following sense:

Proposition 4.3.2 (Schlichting). *Let $\mathscr{C}$ be an idempotent complete exact category and $\mathscr{C} \hookrightarrow \mathscr{D}$ a right (or left) s-filtering inclusion as a full sub-category of an exact category $\mathscr{D}$. Then*

$$D^b(\mathscr{C}) \hookrightarrow D^b(\mathscr{D}) \twoheadrightarrow D^b(\mathscr{D}/\mathscr{C})$$

is an exact sequence of triangulated categories.

This is [Sch04, Proposition 2.6]. The construction of the derived category of an exact category is explained in Keller [Kel96, §11] or Bühler [Büh10, §10].

Theorem 4.3.3 (Keller's Localization Theorem, [Kel99, §1.5, Theorem]). *Let $\mathscr{C}$ be an exact category and $\mathscr{C} \hookrightarrow \mathscr{D}$ a right (or left) s-filtering inclusion as a full sub-category of an exact category $\mathscr{D}$. Then*

$$HH(\mathscr{C}) \longrightarrow HH(\mathscr{D}) \longrightarrow HH(\mathscr{D}/\mathscr{C}) \longrightarrow +1$$

is a fiber sequence in Hochschild homology.

This is due to Keller [Kel99, §1.5, Theorem]. The following result was first proven for countable cardinality by Sho Saito [Sai15]:

Proposition 4.3.4. *For any infinite cardinal κ and exact category $\mathscr{C}$, there is an exact equivalence of exact categories*

$$\mathsf{Tate}^{el}_{\kappa}(\mathscr{C})/\mathsf{Pro}^{a}_{\kappa}(\mathscr{C}) \xrightarrow{\sim} \mathsf{Ind}^{a}_{\kappa}(\mathscr{C})/\mathscr{C} \ .$$

See [BGW16c, Proposition 5.32] for a detailed proof. It turns out that this result admits a symmetric dual statement, which will be more useful for the purposes of this paper.

Proposition 4.3.5 ([BGW16c, Proposition 5.34]). *Let $\mathscr{C}$ be an idempotent complete exact category. There is an exact equivalence of exact categories*

$$\mathsf{Tate}^{el}(\mathscr{C})/\mathsf{Ind}^{a}(\mathscr{C}) \xrightarrow{\sim} \mathsf{Pro}^{a}(\mathscr{C})/\mathscr{C} \ ,$$

sending an object $X \in \mathsf{Tate}^{el}(\mathscr{C})$ to L, where L is any lattice $L \hookrightarrow X$, and morphisms $f : X \to X'$ to a suitable restriction $f\mid_L : L \to L'$ with $L' \hookrightarrow X'$ a suitable lattice (which exists by Proposition 4.2.4(1)). This defines a well-defined functor. The inverse equivalence is induced from the inclusion of categories $\mathsf{Pro}^{a}(\mathscr{C}) \hookrightarrow \mathsf{Tate}^{el}(\mathscr{C})$.

4.4. **Relative Morita theory.** In this section we develop a series of results aiming at the comparison of n-Tate categories with projective module categories. The following lemma is the starting point for this type of consideration. Recall that we write $P_f(R)$ to denote the category of finitely generated projective right R-modules.

Definition 4.4.1. *Let $\mathscr{C}$ be an exact category. We say that $S \in \mathscr{C}$ is a* generator *if every object $X \in \mathscr{C}$ is a direct summand of $S^{\oplus n}$ for n sufficiently large.*

Lemma 4.4.2. *Let $\mathscr{C}$ be an idempotent complete and split exact category with generator S. Then the functor*

$$\mathscr{C} \longrightarrow P_f(\mathrm{End}_{\mathscr{C}}(S)), \qquad Z \longmapsto \mathrm{Hom}_{\mathscr{C}}(S, Z)$$

is an exact equivalence of categories.

Proof. This is for example spelled out in [BGW16b, Lemma 20]. ☐

While such comparison results have been known for decades, there seems to be very little literature studying the 2-functoriality of them. The rest of the section will work out explicit descriptions of the relevant maps in all the cases relevant for the paper. A number of these results might be of independent interest.

4.4.1. *Sub-categories.*

Lemma 4.4.3. *Let $\mathscr{D}$ be a split exact category. Suppose $\mathscr{C} \hookrightarrow \mathscr{D}$ is a fully exact sub-category. Then $\mathscr{C}$ is also split exact. Suppose S is a generator for $\mathscr{C}$ and $\tilde{S} \in \mathscr{D}$ a generator for $\mathscr{D}$. Suppose*

$$\tilde{S} = S \oplus S'$$

for some $S' \in \mathscr{D}$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathscr{C}^{ic} \arrow[r] \arrow[d, "\sim"'] & \mathscr{D}^{ic} \arrow[d, "\sim"] \\
P_f(\mathrm{End}_{\mathscr{C}}(S)) \arrow[r] & P_f(\mathrm{End}_{\mathscr{D}}(\tilde{S}))
\end{tikzcd}$$

whose downward arrows are exact equivalences and the bottom rightward arrow is

$$M \longmapsto M \otimes_{\mathrm{End}_{\mathscr{D}}(S)} \mathrm{Hom}_{\mathscr{D}}(\tilde{S}, S),$$

and equivalently this functor is induced by the (non-unital) algebra homomorphism

$$\mathrm{End}_{\mathscr{C}}(S) \longrightarrow \mathrm{End}_{\mathscr{D}}(S \oplus S'), \qquad f \longmapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

Although it feels like this should be standard, we have not been able to locate a source in the literature.

Proof. (Step 1) Since $\mathscr{C} \hookrightarrow \mathscr{D}$ reflects exactness, $\mathscr{C}$ is also split exact. Thus, its idempotent completion $\mathscr{C}^{ic}$ is also split exact by Lemma 4.1.1. Moreover, if S is a generator for $\mathscr{C}$, then it is also a generator for $\mathscr{C}^{ic}$ since every object in $\mathscr{C}^{ic}$ is a direct summand of an object in $\mathscr{C}$, and these are in turn direct summands of $S^{\oplus n}$ for some n. The same argument works for $\mathscr{D}^{ic}$. Then the 2-universal property of idempotent completion [Büh10, Proposition 6.10] promotes $\mathscr{C} \hookrightarrow \mathscr{D}$ to the top row in the following diagram:

$$\begin{tikzcd}
\mathscr{C}^{ic} \arrow[r] \arrow[d, "\sim"'] & \mathscr{D}^{ic} \arrow[d, "\sim"] \\
P_f(\mathrm{End}_{\mathscr{C}}(S)) \arrow[r, "?"'] & P_f(\mathrm{End}_{\mathscr{D}}(\tilde{S}))
\end{tikzcd} \tag{4.3}$$

As both $\mathscr{C}^{ic}$ and $\mathscr{D}^{ic}$ are split exact and idempotent complete and possess generators, Lemma 4.4.2 induces exact equivalences, given by the downward arrows. Note that an object $Z \in \mathscr{C}$ is sent to

$$Z \mapsto \mathrm{Hom}_{\mathscr{C}}(S, Z) \qquad \text{resp.} \qquad Z \mapsto \mathrm{Hom}_{\mathscr{D}}(\tilde{S}, Z),$$

depending on which path we follow in the above diagram. Since $\mathscr{C}$ is a full sub-category of $\mathscr{D}$, the first functor agrees with $Z \mapsto \mathrm{Hom}_{\mathscr{D}}(S, Z)$. We claim that Diagram 4.3 can be completed to a commutative square of exact functors by adding the following arrow as the bottom row:

$$P_f(\mathrm{End}_{\mathscr{C}}(S)) \longrightarrow P_f(\mathrm{End}_{\mathscr{D}}(\tilde{S}))$$
$$M \longmapsto M \otimes_{\mathrm{End}_{\mathscr{D}}(S)} \mathrm{Hom}_{\mathscr{D}}(\tilde{S}, S).$$

This claim is immediate when plugging in $Z := S$, but since every object in $\mathscr{C}$ is a direct summand of $S^{\oplus n}$ and this formula preserves direct summands, this implies the claim for all objects in $\mathscr{C}$. Since the categories are split exact, checking exactness of the functor reduces to checking additivity, which is immediate.

(Step 2) By $\tilde{S} = S \oplus S'$ we get the non-unital homomorphism of associative algebras in Equation 4.2. If $M \in P_f(\mathrm{End}_{\mathscr{C}}(S))$ and $f \in \mathrm{End}_{\mathscr{C}}(S)$ this means, just by matrix multiplication, that the equation

$$m \cdot f \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = m \otimes \begin{pmatrix} fa & fb \\ 0 & 0 \end{pmatrix}, \qquad m \in M$$

holds in $M \otimes_{\mathrm{End}_{\mathscr{C}}(S)} \mathrm{End}_{\mathscr{D}}(S \oplus S')$. Thus, we find that the map

$$M \otimes_{\mathrm{End}_{\mathscr{C}}(S)} \mathrm{End}_{\mathscr{D}}(S \oplus S') \longrightarrow M \otimes_{\mathrm{End}_{\mathscr{D}}(S)} \mathrm{Hom}_{\mathscr{D}}(\tilde{S}, S)$$
$$m \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto m \otimes \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is an isomorphism of right $\mathrm{End}_{\mathscr{D}}(S \oplus S')$-modules. As a result, the functor can also be described by tensoring along the non-unital algebra homomorphism. □

4.4.2. *Quotient categories.* While the previous result covers the case of a fully exact sub-category, we now want to address the same problem in the situation of a quotient category. Suppose $\mathscr{C} \hookrightarrow \mathscr{D}$ is a fully exact sub-category. We need stronger hypotheses to ensure the existence of the quotient category, namely those of §4.3. We then write

$$\langle X \to C \to X \text{ with } C \in \mathscr{C} \rangle$$

to denote the two-sided ideal of morphisms $X \to X$, for $X \in \mathscr{D}$, which factor over an object in $\mathscr{C}$. Note that this really produces a two-sided ideal, so it makes no difference whether we think of this as an ideal or as the ideal generated by such morphisms.

Lemma 4.4.4. *Let $\mathscr{D}$ be a split exact category with a generator S. Suppose $\mathscr{C} \hookrightarrow \mathscr{D}$ is a left (or right) s-filtering sub-category.*

1. *Then $\mathrm{End}_{\mathscr{D}/\mathscr{C}}(S) = \mathrm{End}_{\mathscr{D}}(S) \,/\, \langle S \to C \to S \text{ with } C \in \mathscr{C} \rangle$.*
2. *Moreover, the diagram*

$$\begin{tikzcd} \mathscr{D}^{ic} \arrow[r] \arrow[d, "\sim"'] & (\mathscr{D}/\mathscr{C})^{ic} \arrow[d, "\sim"] \\ P_f(\mathrm{End}_{\mathscr{D}}(S)) \arrow[r] & P_f(\mathrm{End}_{\mathscr{D}/\mathscr{C}}(S)) \end{tikzcd}$$

commutes, where the lower horizontal arrow is

$$M \longmapsto M \otimes_{\mathrm{End}_{\mathscr{D}}(S)} \mathrm{End}_{\mathscr{D}/\mathscr{C}}(S)\,.$$

Proof. We only prove the left s-filtering case: As $\mathscr{C} \hookrightarrow \mathscr{D}$ is left s-filtering, the quotient category $\mathscr{D}/\mathscr{C}$ exists. Idempotent completion has a suitable universal property as a 2-functor so that the resulting exact functor $\mathscr{D} \to \mathscr{D}/\mathscr{C}$ induces canonically a functor $\mathscr{D}^{ic} \to (\mathscr{D}/\mathscr{C})^{ic}$, justifying the top row [Büh10, Proposition 6.10].
(Step 1) By Schlichting's original construction [Sch04, Lemma 1.13] the quotient category $\mathscr{D}/\mathscr{C}$ arises as the localization $\mathscr{D}[\Sigma^{-1}]$ with a calculus of left fractions, where the class Σ is formed of (1) admissible monics with cokernel in $\mathscr{C}$, (2) admissible epics with kernels in $\mathscr{C}$, (3) and closed under composition. For this proof, we shall use that it suffices to localize at $\Sigma_e = \{\text{admissible epics with kernels in } \mathscr{C}\}$, i.e. $\mathscr{D}/\mathscr{C} := \mathscr{D}[\Sigma^{-1}] = \mathscr{D}[\Sigma_e^{-1}]$, by [BGW16c, Proposition 2.19]. This idea is due to T. Bühler. By loc. cit. this localization admits a calculus of left fractions[7]. This means that every morphism $X \to Y$ in $\mathscr{D}/\mathscr{C}$ is represented by a left roof

$$X \longrightarrow W \overset{\in \Sigma_e}{\longleftarrow} Y.$$

First of all, we shall show that all morphisms are equivalent to morphisms coming from $\mathscr{D}$, i.e. left roofs of the shape $X \longrightarrow Y \overset{1}{\longleftarrow} Y$: Suppose we are given an arbitrary left roof

$$\begin{tikzcd} & W & \\ X \arrow[ur, "f"] & & Y. \arrow[ul, "h"'] \end{tikzcd}$$

[7]The conventions of left and right fractions are as in Gabriel–Zisman [GZ67] or Bühler [Büh10]. This means that the meaning of left and right is opposite to the usage in [GM03], [KS06].

As $h \in \Sigma_e$ is a split epic, we may write $Y = W \oplus C$ for some object $C \in \mathscr{C}$ so that our roof takes the shape

$$\begin{tikzcd} & W & \\ X \arrow[ur, "f"] & & W \oplus C. \arrow[ul, "pr_W"'] \end{tikzcd}$$

Thanks to the commutative diagram

$$\begin{tikzcd} & W \arrow[d, "1"] & \\ X \arrow[ur, "f"] \arrow[dr, "f\oplus 0"'] & W & W \oplus C \arrow[ul, "pr_W"'] \arrow[dl, "1"] \\ & W \oplus C, \arrow[u, "pr_W"'] & \end{tikzcd}$$

we learn that this left roof is equivalent to the roof $X \longrightarrow W \oplus C \overset{1}{\longleftarrow} W \oplus C$ and rewriting this using $Y = W \oplus C$ in terms of Y, we have proven our claim. This means that $\mathrm{Hom}_{\mathscr{D}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}/\mathscr{C}}(X,Y)$ is surjective. It remains to determine the kernel. By the calculus of left fractions, two left roofs (which, as we had just proven, we may assume to come from genuine morphisms in $\mathscr{D}$) are equivalent if and only if there exists a commutative diagram of the shape

$$\begin{tikzcd} & Y \arrow[d, "\Sigma_e"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & H & Y \arrow[ul, "1"'] \arrow[dl, "1"] \\ & Y. \arrow[u, "\Sigma_e"'] & \end{tikzcd}$$

The existence of such a diagram is equivalent to the the equality of morphism

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\in \Sigma_e} H$$

and thus to $f - g$ mapping to zero in H. By the universal property of kernels, this is equivalent to the existence of a factorization $f - g : X \to \ker(Y \twoheadrightarrow H) \to Y$. Since $Y \twoheadrightarrow H$ lies in Σ_e, we have $\ker(Y \twoheadrightarrow H) \in \mathscr{C}$. The converse direction works the same way. Thus,

$$\mathrm{End}_{\mathscr{D}/\mathscr{C}}(S) = \mathrm{End}_{\mathscr{D}}(S) \, / \, \langle S \to C \to S \text{ with } C \in \mathscr{C} \rangle$$

and since the embedding $\mathscr{D}/\mathscr{C} \to (\mathscr{D}/\mathscr{C})^{ic}$ is fully faithful [Büh10, Remark 6.3], this description also applies to $(\mathscr{D}/\mathscr{C})^{ic}$.

(Step 2) Since $\mathscr{D}$ is split exact, $\mathscr{D}^{ic}$ is idempotent complete and still split exact by Lemma 4.1.1. Hence, Lemma 4.4.2 implies that the left-hand side downward arrow, $Z \mapsto \mathrm{Hom}_{\mathscr{D}^{ic}}(S, Z)$, is an equivalence of categories. The quotient category $\mathscr{D}/\mathscr{C}$ is an exact category where a kernel-cokernel sequence

$$A \longrightarrow B \longrightarrow C$$

is considered exact iff it is isomorphic to the image of an exact sequence in $\mathscr{D}$, [Sch04, Proposition 1.16]. This is the canonical exact structure on $\mathscr{D}/\mathscr{C}$, making $\mathscr{D} \to \mathscr{D}/\mathscr{C}$ an exact functor. Since $\mathscr{D}$ is split exact, this means B is isomorphic to the direct sum of the outer terms and this property stays true in $\mathscr{D}/\mathscr{C}$. Thus, $\mathscr{D}/\mathscr{C}$ also has the split exact structure (however there is no reason why it would have to be idempotent complete). The functor $\mathscr{D}/\mathscr{C} \to (\mathscr{D}/\mathscr{C})^{ic}$ to the idempotent completion is exact, and the idempotent completion $(\mathscr{D}/\mathscr{C})^{ic}$ must also be split exact by Lemma 4.1.1. If every object in $\mathscr{D}$ is a direct summand of $S^{\oplus n}$, this property stays true in $\mathscr{D}/\mathscr{C}$, and thus in $(\mathscr{D}/\mathscr{C})^{ic}$. Hence, Lemma 4.4.2 also applies to $(\mathscr{D}/\mathscr{C})^{ic}$, with the image of the same object, and thus there is an exact equivalence of categories via

$Z \mapsto \mathrm{Hom}_{(\mathscr{D}/\mathscr{C})^{ic}}(S,Z)$. This is a priori a right $\mathrm{End}_{(\mathscr{D}/\mathscr{C})^{ic}}(S)$-module, but by the full faithfulness of the idempotent completion this algebra agrees with $\mathrm{End}_{\mathscr{D}/\mathscr{C}}(S)$. Finally, we observe that in order to make the diagram commute the lower horizontal arrow must be

$$\mathrm{Hom}_{\mathscr{D}^{ic}}(S,Z) \mapsto \mathrm{Hom}_{(\mathscr{D}/\mathscr{C})^{ic}}(S,Z)$$

for all $Z \in \mathscr{D}$. But by Step 1 this is just quotienting out the ideal $\langle S \to C \to S \text{ with } C \in \mathscr{C}\rangle$ from the right, or equivalently tensoring with the corresponding quotient ring. This proves our claim. $\square$

Although this diverts a bit from our storyline and will not be used anywhere else in this paper, let us record an immediate application of this lemma:

Proposition 4.4.5. *Drinfeld's Calkin category* $\mathscr{C}_R^{\mathrm{Kar}}$*, as defined in* [Dri06, §3.3.1]*, is equivalent to the Calkin category* $\mathsf{Calk}(\mathscr{C}) := (\mathsf{Ind}^a(\mathrm{Mod}(R))/\mathrm{Mod}(R))^{ic}$ *of* [BGW16c, Def. 3.40].

Proof. The category $\mathrm{Mod}(R)$ is split exact with generator R. We obtain the claim by applying Lemma 4.4.4 to the left s-filtering inclusion $\mathrm{Mod}(R) \hookrightarrow \mathsf{Ind}^a(\mathrm{Mod}(R))$, and the description of this category given by the lemma agrees with the definition used by Drinfeld in [Dri06, §3.3.1]. $\square$

4.5. **Applications to Tate categories.**

4.5.1. *Iterated Morita calculus for n-Tate categories.* Now we can apply these results to Ind-, Pro- and Tate categories. For the sake of legibility we have divided the following arguments into several separate propositions. However, there will be a great overlap in notation so that it appears to be reasonable to introduce some overall notation for the length of this section.

Assume $\mathscr{C}$ is any split exact category with a generator $S \in \mathscr{C}$. Then we define objects

$$S[t^{-1}] := \textstyle\coprod_{\mathbf{N}} S \in \mathsf{Ind}^a(\mathscr{C}) \qquad \text{and} \qquad S[[t]] := \prod_{\mathbf{N}} S \in \mathsf{Pro}^a(\mathscr{C}),$$

where the (co)product is interpreted as the corresponding formal Ind- resp. Pro-limit object. In order to be absolutely precise, let us spell out what this means explicitly in terms of the actual definition of the respective exact categories, as in §4.2:

We define an admissible Ind-diagram and admissible Pro-diagram by

$$S[t^{-1}] : \mathbf{N} \longrightarrow \mathscr{C}, \quad n \mapsto \textstyle\coprod_{i=1}^{n} S, \qquad\qquad S[[t]] : \mathbf{N} \longrightarrow \mathscr{C}, \quad n \mapsto \prod_{i=1}^{n} S. \tag{4.4}$$

More specifically, we view the natural numbers $\mathbf{N}$ as a directed poset and define admissible diagrams by these formulae, where for $S[t^{-1}]$ a morphism $n \mapsto n+1$ in $\mathbf{N}$ is sent to the inclusion of $\coprod_{i=1}^{n} S \to \coprod_{i=1}^{n+1} S$, while for $S[[t]]$ we send it to the projection in the opposite direction. Finally, define

$$S((t)) := S[[t]] \oplus S[t^{-1}] \in \mathsf{Tate}^{el}(\mathscr{C}). \tag{4.5}$$

This is obviously an elementary Tate object since $S[[t]]$ is a lattice (see Definition 4.2.2). These objects completely characterize the Tate object categories in the following way:

Theorem 4.5.1 ([BGW16b])**.** *Let $\mathscr{C}$ be an idempotent complete split exact category with a generator*[8] *$S \in \mathscr{C}$.*

(1) *The category n-$\mathsf{Tate}_{\aleph_0}(\mathscr{C})$ is split exact and idempotent complete.*
(2) *The object $\tilde{S}_n := S((t_1))\cdots((t_n))$ is a generator of n-$\mathsf{Tate}^{el}(\mathscr{C})$ and n-$\mathsf{Tate}(\mathscr{C})$ and we have an exact equivalence of exact categories*

$$n\text{-}\mathsf{Tate}_{\aleph_0}(\mathscr{C}) \xrightarrow{\sim} P_f(A_n) \quad \text{with} \quad A_n := \mathrm{End}(\tilde{S}_n). \tag{4.6}$$

(3) *For every object $X \in n$-$\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C})$ its endomorphism algebra canonically carries the structure of a Beilinson n-fold cubical algebra. We refer to* [BGW16b] *for the construction.*

This theorem hinges crucially on our restriction to Tate objects of *countable* cardinality. See [BGW16c] for a detailed discussion and counter-examples due to J. Šťovíček and J. Trlifaj for strictly greater cardinalities. Also, if $\mathscr{C}$ is not split exact, there is no way to save the conclusions of this result. We refer to the introduction of [BGW16b] for an overview.

[8] Of course, if we instead have a finite system of generators, we can just take their direct sum as a single-object generator.

Proof. (Claim 1) By [BGW16c, Theorem 7.2] the category $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C})$ is split exact. Thus, its idempotent completion $n\text{-}\mathsf{Tate}_{\aleph_0}(\mathscr{C})$ fulfills the claim by Lemma 4.1.1.
(Claim 2) The statement about the generator for $n\text{-}\mathsf{Tate}^{el}(\mathscr{C})$ is proven in [BGW16c, Proposition 7.4]. Since $n\text{-}\mathsf{Tate}(\mathscr{C})$ is just the idempotent completion of this category, each of its objects is a direct summand of an object in $n\text{-}\mathsf{Tate}^{el}(\mathscr{C})$ and thus this generator also works for the idempotent completion. The equivalence of Equation 4.6 stems from Lemma 4.4.2. One has to check that the assumptions of the lemma hold true. See [BGW16b, Theorem 1] for the details.
(Claim 3) This is [BGW16b, Theorem 1]. We give a brief survey: Call a morphism $f : X \to Y$ of n-Tate objects *bounded* if it factors through a lattice in the target, say $X \to L \hookrightarrow Y$, and *discrete*, if it sends a lattice of the source to zero. For $X = Y$, one checks that these morphisms form two-sided ideals, $I_1^{\pm}$ and moreover $I_1^{+} + I_1^{-} = R$. For the latter, most assumptions are needed, especially $\mathscr{C}$ split exact and cardinality $\kappa := \aleph_0$. See [BGW16b] for counter-examples when these assumptions are not met. The ideals $I_2^{\pm}$ are defined inductively: For any nested pair of lattices $L' \hookrightarrow L \hookrightarrow X$, the quotient L/L' is an $(n-1)$-Tate object, and one defines I_2^{+} to be those morphisms such that for any factorization $\overline{f} : L_1/L_1' \to L_2'/L_2$ for $f\mid_{L_1}$, over suitable lattices L_1, L_1', L_2, L_2', the morphism $\overline{f}$ is bounded, as a morphism of $(n-1)$-Tate objects. Similarly for I_2^{-}. This pattern can be extended inductively to define $I_i^{\pm}$ for $i = 1, \ldots, n$. $\square$

Next, we need to check that the cubical algebra actually meets the well-behavedness criteria we are intending to use later.

Proposition 4.5.2. *The cubical algebra* $\operatorname{End}(\tilde{S}_n)$ *is good in the sense of Definition 3.2.2.*

Proof. Let us only treat the case of local left units. We prove this by induction in n, starting from $n = 1$. Suppose $M := \{f : X_1 \to X_2\}$ is a finite set of trace-class morphisms. In particular, each such f is a finite morphism (viewed as a 1-Tate object of $(n-1)$-Tate objects). Then, for each $f \in M$, being both bounded and discrete, we can find lattices $L_1' \hookrightarrow X_1$ and $L_2 \hookrightarrow X_2$ so that this f factors as

$$f : X_1 \twoheadrightarrow X_1/L_1' \longrightarrow L_2 \hookrightarrow X_2.$$

These being found, we find *one* L_1' resp. *one* L_2 having this property simultaneously for all f in the set by taking common sub- resp. over-lattices of the corresponding lattices for the individual f – this uses the (co-)directedness of the Sato Grassmannian, Theorem 4.2.3.

Fix any over-lattice L_1 of L_1'. Then for every sub-lattice L_2' of L_2 we get an induced morphism

$$f\mid_{L_1} : L_1/L_1' \longrightarrow L_2 \longrightarrow L_2/L_2'.$$

By assumption, each such $f\mid_{L_1}$ is a trace-class morphism of $(n-1)$-Tate objects, so we look at a finite set of trace-class morphisms and can find a local left unit, say e_1, by induction (if $n = 1$ arbitrary morphisms between objects in $\mathscr{C}$ are trace-class, so we can just use the identity morphism of $\mathscr{C}$. If $n \geq 2$ we argue by induction). It remains to lift these local left units to a map from X_1 to X_2.
(Step A) If we replace L_2' by a sub-lattice L_2'', we get a commutative diagram, depicted below on the left:

(4.7)
$$\begin{tikzcd} & L_1/L_1' \arrow[d] \arrow[dr] & \\ L_2'/L_2'' \arrow[r, hook] & L_2/L_2'' \arrow[r, two heads] & L_2/L_2' \end{tikzcd} \qquad \begin{tikzcd} & L_2 & \\ L_1/L_1' \arrow[ur] \arrow[r, hook] & L_1^{+}/L_1' \arrow[u] \arrow[r, two heads] & L_1^{+}/L_1 \end{tikzcd}$$

The downward arrow exists since we even have a map to L_2 without quotienting out anything. We get

$$f - \sigma_{L_2/L_2'} f : L_1/L_1' \longrightarrow L_2'/L_2'',$$

where σ is a section of the right-hand side epimorphism. Since f is trace-class and trace-class morphisms form an ideal, this morphism is also trace-class. Thus, we again face a trace-class morphism of $(n-1)$-Tate objects and again by induction, we find a local left unit, say e_2. Now the diagonal (2×2)-matrix $(e_1 \oplus e_2)$ is a local left unit on $L_2/L_2'' = L_2'/L_2'' \oplus L_2/L_2'$.
Since we work with a Tate object of countable cardinality, perform this inductively on an co-exhaustive

family of lattices L_2', going step-by-step to smaller sub-lattices. This produces a local left unit to the morphisms f, each restricted to L_1,

$$L_1/L_1' \longrightarrow L_2.$$

(Step B) Now, we proceed analogously and step-by-step replace L_1 by an over-lattice L_1^+. We get the commutative Diagram 4.7 (depicted on the right) above. This diagram commutes since our morphism was actually defined on X/L_1', so the restrictions to any lattices are necessarily compatible. Again, picking a left section σ in the top row, we get

$$f - f\sigma : L_1^+/L_1 \longrightarrow L_2$$

and since f is trace-class, so is this morphism. Now by Step A, we can find a local left unit e_2 for these morphisms (as f runs through our finite set of morphisms) and so the diagonal (2×2)-matrix $e_1 \oplus e_2$ is a local left unit on $L_1^+/L_1' = L_1/L_1' \oplus L_1^+/L_1$. For local right units an analogous argument works. This finishes the proof. $\square$

Based on the preceding theorem, we make the following section-wide definitions: Let $\mathscr{C}$ be a split exact and idempotent complete exact category with a generator S.

Let $n \geq 0$ be arbitrary. Define

$$\mathscr{C}_n := n\text{-}\mathsf{Tate}_{\aleph_0}(\mathscr{C}) \quad , \quad \tilde{S}_n := S((t_1))\cdots((t_n)) \quad , \quad A_n := \operatorname{End}(\tilde{S}_n) \tag{4.8}$$

By Theorem 4.5.1 the algebra A_n is a good n-fold cubical algebra and there is an exact equivalence of exact categories

$$\mathscr{C}_n \longrightarrow P_f(A_n), \qquad Z \longmapsto \operatorname{Hom}(\tilde{S}_n, Z). \tag{4.9}$$

In particular, all these exact categories are idempotent complete, split exact and come equipped with a convenient fixed generator. Since all A_n are cubical algebras, we shall freely write I_i^+, I_i^-, I_i^0 for the respective ideals of bounded, discrete or finite morphisms. See [BGW16b] for further background.

Below, we shall unravel step-by-step the nature of certain quotient and boundary homomorphisms coming from Theorem 4.5.1.

Proposition 4.5.3. *As always in this section, assume $\mathscr{C}$ is an idempotent complete split exact category with a generator $S \in \mathscr{C}$. Then the diagram*

$$\begin{tikzcd}
\mathsf{Tate}_{\aleph_0}(\mathscr{C}_n) \arrow[r] \arrow[d, "\sim"'] & (\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^{a}_{\aleph_0}(\mathscr{C}_n))^{ic} \arrow[d, "\sim"] \\
P_f(A_{n+1}) \arrow[r] & P_f(A_{n+1}/I_1^-)
\end{tikzcd}$$

commutes, where the top row rightward arrow is induced from the quotient functor of $\mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n) \hookrightarrow \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)$, and the bottom row rightward arrow is induced from the quotient morphism of the ideal inclusion $I_1^- \hookrightarrow A_{n+1}$. The downward arrows are exact equivalences.

Proof. Firstly, since $\mathscr{C}_n$ is split exact, the categories $\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$ and $\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)$ are also split exact categories [BGW16c, Theorem 4.2 (6)], [BGW16c, Proposition 5.23]. Moreover, $\mathscr{C}_n$ is idempotent complete and thus $\mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n) \hookrightarrow \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)$ is right s-filtering by Proposition 4.3.1. Furthermore, every object in $\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)$ is a direct summand of $\tilde{S} := \tilde{S}_{n+1}$. We use Lemma 4.4.4 in order to deduce that the diagram

$$\begin{tikzcd}
\mathsf{Tate}_{\aleph_0}(\mathscr{C}_n) \arrow[r] \arrow[d, "\sim"'] & (\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^{a}_{\aleph_0}(\mathscr{C}_n))^{ic} \arrow[d, "\sim"] \\
P_f(A_{n+1}) \arrow[r] & P_f(A_{n+1}/I^*)
\end{tikzcd}$$

commutes, where we have used that $\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)^{ic} = \mathsf{Tate}_{\aleph_0}(\mathscr{C}_n)$ in the upper left corner and $A_{n+1} := \operatorname{End}_{(n+1)\text{-}\mathsf{Tate}_{\aleph_0}(\mathscr{C})}(\tilde{S})$, and where the ideal I^* is generated by morphisms admitting a factorization $\tilde{S} \to$

$I \to \tilde{S}$ with $I \in \mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n)$. We claim that $I^* = I_1^-$, where I_1^- refers to the structure of A_{n+1} as an $(n+1)$-fold cubical algebra: Suppose $f \in I^*$. Then f factors as $\tilde{S} \to I \to \tilde{S}$ with $I \in \mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n)$ and by Proposition 4.2.4 there exists a lattice $L \hookrightarrow \tilde{S}$ such that we obtain a further factorization $\tilde{S} \twoheadrightarrow \tilde{S}/L \to I \to \tilde{S}$. In particular, f sends the lattice L to zero so that $f \in I_1^-$. Conversely, suppose $f \in I_1^-$. Let $L \hookrightarrow \tilde{S}$ be a lattice which is sent to zero. Then f factors as $\tilde{S} \twoheadrightarrow \tilde{S}/L \to \tilde{S}$ just by the universal property of quotients. As L is a lattice, $\tilde{S}/L \in \mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n)$, proving $f \in I^*$. This finishes the proof of $I^* = I_1^-$. □

We shall also need the following variation of the same idea.

Proposition 4.5.4. *As always in this section, assume $\mathscr{C}$ is an idempotent complete split exact category with a generator $S \in \mathscr{C}$. Define*

$$\hat{S} := S((t_1)) \cdots ((t_n))[[t_{n+1}]] \in \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n),$$

as in Equation 4.4. Then $E := \mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)}(\hat{S})$ is an $(n+1)$-fold cubical algebra and we have a commutative diagram

$$\begin{tikzcd} \left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)\right]^{ic} \arrow[r] \arrow[d, "\sim"'] & \left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n\right]^{ic} \arrow[d, "\sim"] \\ P_f(E) \arrow[r] & P_f(E/I_1^0(\hat{S})), \end{tikzcd} \tag{4.10}$$

where the top row rightward morphism is the quotient functor induced from $\mathscr{C}_n \hookrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$, the bottom row rightward morphism stems from the ideal inclusion $I_1^0 \hookrightarrow E$, and the downward arrows are exact equivalences of exact categories.

Proof. The $(n+1)$-fold cubical algebra structure is immediate from Theorem 4.5.1, employing that $\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n) \hookrightarrow \mathscr{C}_{n+1}$ is a full sub-category, so it does not matter whether we consider endomorphisms in $\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$ or the $(n+1)$-Tate category $\mathscr{C}_{n+1}$. By Proposition 4.3.1 the inclusion $\mathscr{C}_n \hookrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$ is right s-filtering. This produces the top row of the following diagram:

$$\begin{tikzcd} \mathscr{C}_n \arrow[r, hook] & \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n) \arrow[r, two heads] \arrow[d] & \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n \arrow[d] \\ & \left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)\right]^{ic} \arrow[r, two heads] \arrow[d] & \left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n\right]^{ic} \arrow[d] \\ & P_f(E) \arrow[r] & P_f(E/I_1^0). \end{tikzcd} \tag{4.11}$$

We construct the second row from the first by taking the fully faithful embedding into the idempotent completion; the right-ward functor exists by the 2-universal property [Büh10, Proposition 6.10]. Next, construct the third row by Lemma 4.4.4. To this end, we employ the shorthands

$$\hat{S} := S((t_1)) \cdots ((t_n))[[t_{n+1}]] \qquad \text{and} \qquad E := \mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)}(\hat{S})$$

so that this Lemma literally yields the third row

$$P_f(\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)}(\hat{S})) \twoheadrightarrow P_f(\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n}(\hat{S})) \tag{4.12}$$

along with the description

$$\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n}(\hat{S}) = (\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)}\hat{S}) / \left\langle \hat{S} \to C \to \hat{S} \text{ with } C \in \mathscr{C}_n \right\rangle. \tag{4.13}$$

However, $\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$ is a full sub-category of $\mathscr{C}_{n+1}$, so in Equation 4.12 we could just as well compute the left-hand side endomorphism algebra in $\mathscr{C}_{n+1}$. By Theorem 4.5.1 the latter is canonically an $(n+1)$-fold cubical algebra, so this structure is also available for the endomorphism algebra on the left-hand side in Equation 4.12, and in particular we can speak of the two-sided ideal I_1^0. Next, we claim that

$$I_1^0 = \left\langle \hat{S} \to C \to \hat{S} \text{ with } C \in \mathscr{C}_n \right\rangle \tag{4.14}$$

as two-sided ideals in Equation 4.13. Suppose $f \in I_1^0$. Then $f : \hat{S} \to \hat{S}$ is discrete as a morphism of 1-Tate objects (with values in n-Tate objects). That is, there is a lattice $L \hookrightarrow \hat{S}$ that is sent to zero. Thus, f factors as $\hat{S} \twoheadrightarrow \hat{S}/L \to \hat{S}$, where $\hat{S}/L$ is and Ind-object (since L is a lattice), and simultaneously a Pro-object since it is an admissible quotient of the Pro-object $\hat{S}$. Thus, $\hat{S}/L \in \mathscr{C}_n$ by Theorem 4.2.3, and thus f lies in the right-hand side ideal in Equation 4.14. Conversely, suppose f lies in the right-hand side ideal in Equation 4.14. Consider the map $\hat{S} \to C$. Since $\mathscr{C} \hookrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathscr{C})$ is right filtering by Proposition 4.3.1, this arrow admits a factorization $\hat{S} \twoheadrightarrow \tilde{C} \to C$ with $\tilde{C} \in \mathscr{C}_n$. Here $\ker(\hat{S} \twoheadrightarrow \tilde{C})$ exists, it is a lattice (since it is a sub-object of a Pro-object and thus itself a Pro-object, and the quotient by it lies in $\mathscr{C}_n$, which can trivially be viewed as an Ind-object), and so $\hat{S} \to C$ sends a lattice to zero and therefore so does $f : \hat{S} \to C \to \hat{S}$. Hence, $f \in I_1^-$. As $\hat{S}$ is a Pro-object, we trivially have $f \in I_1^+$ and thus $f \in I_1^+ \cap I_1^- = I_1^0$. This finishes the proof of Equation 4.14. Thus, Equation 4.12 becomes

$$P_f(E) = P_f(\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)}(\hat{S})) \to P_f(\mathrm{End}_{\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n}(\hat{S})) = P_f(E/I_1^0).$$

This settles the last row in Diagram 4.11. Note that the explicit description of the middle arrow in Lemma 4.4.4 under these identifications also confirms that $P_f(E) \to P_f(E/I_1^0)$ just comes from the map $E \twoheadrightarrow E/I_1^0$. $\square$

4.5.2. *Relation to the abstract Hochschild symbol.* Next, we shall replace the associative algebra E in the previous proposition by a certain ideal: With the notation of the proposition, write

$$\tilde{S}_{n+1} = S((t_1)) \cdots ((t_{n+1})) = \hat{S} \oplus S((t_1)) \cdots ((t_n))[t_{n+1}^{-1}]. \tag{4.15}$$

Now there is an (non-unital) embedding of algebras

$$E \hookrightarrow \mathrm{End}(S((t_1)) \cdots ((t_{n+1}))) = A_{n+1} \qquad \text{by} \qquad f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.16}$$

acting only on $\hat{S}$. Clearly, with this interpretation, f is sent into the ideal $I_1^+(\tilde{S}_{n+1})$ since each of these morphisms factors through the lattice $\hat{S} \hookrightarrow \tilde{S}_{n+1}$ of $\tilde{S}_{n+1}$ as an $(n+1)$-Tate object. Analogously, if f lies in $I_1^0(\hat{S})$, this embedding maps it to I_1^0 of $\tilde{S}_{n+1}$.

Diagram 4.10 induces a commutative square in Hochschild homology, depicted below as the upper square.

$$\begin{tikzcd} HH\left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)\right]^{ic} \arrow[r] \arrow[d] & HH\left[\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n\right]^{ic} \arrow[d] \\ HH(E) \arrow[r] \arrow[d] & HH(E/I_1^0(\hat{S})) \arrow[d] \\ HH(I_1^+) \arrow[r] & HH(I_1^+/I_1^0). \end{tikzcd} \tag{4.17}$$

The lower square arises from the embedding morphism which we have just discussed, Equation 4.16. Note that I_1^+ and I_1^0 refer to the ideals of A_{n+1}!

Lemma 4.5.5 ([Bra18])**.** *The following diagram of A_{n+1}-bimodules*

$$\begin{tikzcd}
I_1^0 \arrow[r, "diag"] \arrow[d, "="'] & I_1^+ \oplus I_1^- \arrow[r, "diff"] \arrow[d, "pr_{I_1^+}"'] & A_{n+1} \arrow[d, "(1)"] \arrow[dd, bend left, "\Lambda"] \\
A_n \arrow[r] \arrow[d, "="'] & I_1^+ \arrow[r] \arrow[d, "incl"'] & A_{n+1}/I_1^- \arrow[d, "(2)"] \\
A_n \arrow[r] & A_{n+1} \arrow[r] & A_{n+1}/A_n
\end{tikzcd}$$

is commutative with exact rows. The second and third row are also exact sequences of associative algebras. See Equation 3.7 for the definition of the morphism Λ.

Proof. We will construct this diagram row by row. The exactness of the first row stems from $I_1^0 := I_1^+ \cap I_1^-$ and $I_1^+ + I_1^- = A_{n+1}$, i.e. it comes directly from the axioms of a cubical algebra. The second row is obtained from quotienting out I_1^-. The downward arrows, in particular (1), are just the quotient maps. Note that the exactness of the second row implies that the quotient on the right-hand side can be rewritten as $A_n \hookrightarrow I_1^+ \twoheadrightarrow I_1^+/A_n$. The inclusion $I_1^+ \hookrightarrow A_{n+1}$ thus induces the last exact row. Note that the arrow (2) is induced from $I_1^+ \hookrightarrow A_{n+1}$. As a result, the composition of (1) and (2), $A_{n+1} \to A_{n+1}/A_n$, is *not* the quotient map, but precisely the map Λ (by diagram chase). $\square$

Now we apply Hochschild homology to Diagram 4.10 of Proposition 4.5.4. We get a commutative diagram

$$\begin{tikzcd}
HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)) \arrow[r] \arrow[d] & HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n) \arrow[r] \arrow[d] & \Sigma HH(\mathscr{C}_n) \arrow[d] \\
HH(E) \arrow[r] & HH(E/I_1^0(\hat{S})) \arrow[r] & \Sigma HH(I_1^0(\hat{S})).
\end{tikzcd} \tag{4.18}$$

This is induced from a square: In the top row we could identify the homotopy fiber by Keller's localization sequence, which is applicable by the exactness of the induced sequence of derived categories (Proposition 4.3.2). In the bottom row we can identify the homotopy fiber by the long exact sequence in the Hochschild homology of algebras, Theorem 1.0.12, of the algebra extension $I_1^0(\hat{S}) \hookrightarrow E \twoheadrightarrow E/I_1^0(\hat{S})$, and the (one-sided) unitality of I_1^0 (this holds since our cubical algebras are good). In view of Diagram 4.17 we can replace the lower row of Diagram 4.18 by the following middle row:

$$\begin{tikzcd}
HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)) \arrow[r] \arrow[d] & HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n) \arrow[r] \arrow[d] & \Sigma HH(\mathscr{C}_n) \arrow[d] \\
HH(I_1^+) \arrow[r] \arrow[d] & HH(I_1^+/I_1^0) \arrow[r] \arrow[d] & \Sigma HH(I_1^0) \arrow[d] \\
HH(A_{n+1}) \arrow[r] & HH(A_{n+1}/A_n) \arrow[r] & \Sigma HH(A_n),
\end{tikzcd} \tag{4.19}$$

where I_1^+, I_1^0 refer to the ideals of A_{n+1} as usual. Correspondingly, the bottom rows are induced from the inclusion $I_1^+ \hookrightarrow A_{n+1}$ resp. $A_{n+1} \hookrightarrow A_n$, as described in Lemma 4.5.5.

Proposition 4.5.6. *As always in this section, assume $\mathscr{C}$ is an idempotent complete split exact category with a generator $S \in \mathscr{C}$. Then the following diagram commutes:*

$$\begin{tikzcd}
HH(\mathsf{Tate}_{\aleph_0}(\mathscr{C}_n)) \arrow[r, "quot"] \arrow[d, "\sim"] & HH((\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n))^{ic}) \arrow[r, "\sim"] & HH((\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n)^{ic}) \arrow[d] \\
HH(A_{n+1}) \arrow[rr, "\Lambda"'] & & HH(A_{n+1}/A_n)
\end{tikzcd} \tag{4.20}$$

Proof. Proposition 4.5.3 provides a commutative diagram of exact categories and exact functors and applying Hochschild homology gives us the commutative diagram

$$\begin{tikzcd} HH\mathsf{Tate}_{\aleph_0}(\mathscr{C}_n) \arrow[r] \arrow[d, "\sim"'] & HH((\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^{a}_{\aleph_0}(\mathscr{C}_n))^{ic}) \arrow[d, "\sim"] \\ HH(A_{n+1}) \arrow[r] & HH(A_{n+1}/I_1^-) \end{tikzcd}$$

whose downward arrows are isomorphisms. Secondly, Proposition 4.3.5 tells us that the inclusion

$$\mathsf{Pro}^{a}_{\aleph_0}(\mathscr{C}_n) \hookrightarrow \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n) \tag{4.21}$$

induces the exact equivalence of exact categories

$$\mathsf{Pro}^{a}_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n \xrightarrow{\sim} \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^{a}_{\aleph_0}(\mathscr{C}_n). \tag{4.22}$$

If we apply Lemma 4.4.3 to the fully exact sub-category of Equation 4.21 we obtain the commutative square

$$\begin{tikzcd} \mathsf{Pro}^{a}_{\aleph_0}(\mathscr{C}_n)^{ic} \arrow[r] \arrow[d] & \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)^{ic} \arrow[d] \\ P_f(\mathrm{End}_{\mathscr{C}}(\tilde{S}_n[[t_{n+1}]])) \arrow[r] & P_f(\mathrm{End}_{\mathscr{D}}(\tilde{S}_{n+1})) \end{tikzcd}$$

($\tilde{S}_n[[t_{n+1}]]$ was previously also called $\hat{S}$; cf. Proposition 4.5.4) where the bottom rightward arrow stems from the algebra homomorphism

$$\mathrm{End}(\tilde{S}_n[[t_{n+1}]]) \longrightarrow \mathrm{End}(\tilde{S}_n[[t_{n+1}]] \oplus \tilde{S}_n[t_{n+1}^{-1}]), \qquad f \longmapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.23}$$

By Proposition 4.5.3 the equivalence $\mathsf{Tate}(\mathscr{C}_n) \xrightarrow{\sim} P_f(A_{n+1})$ identifies the quotient functor $\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n) \to \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^{a}_{\aleph_0}(\mathscr{C}_n)$ just with quotienting out the ideal I_1^-. In view of Equation 4.23 the inverse of the equivalence in Equation 4.22 corresponds to finding an endomorphism of $\tilde{S}_n[[t_{n+1}]]$ which is mapped under the map of Equation 4.23 to a given element (alternatively this follows from the description of the inverse functor on morphisms as given by Proposition 4.3.5). But this is easy to achieve concretely: Given $f \in \mathrm{End}_{\mathscr{D}}(\tilde{S}_{n+1})$ we compose it with a projector to $\tilde{S}_n[[t_{n+1}]]$, i.e. the composition of both these steps is realized by

$$\begin{gathered} A_{n+1} \longrightarrow A_{n+1}/I_1^- \longrightarrow \mathrm{End}(\tilde{S}_n[[t_{n+1}]]) \xrightarrow{(*)} E/I_1^0(\hat{S}) \\ \text{(with } E/I_1^0(\hat{S}) \text{ and the arrow } (*) \text{ as in Diag. 4.10)} \\ f \longmapsto P^+ f \longmapsto \overline{P^+ f}, \end{gathered} \tag{4.24}$$

where P^+ is the idempotent projecting $\tilde{S}_{n+1}$ to $\tilde{S}_n[[t_{n+1}]]$. Finally, note that we already know what the map

$$HH((\mathsf{Pro}^{a}_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n)^{ic}) \longrightarrow HH(A_{n+1}/A_n)$$

does: It arises as the composition of a series of arrows in Diagrams 4.18, 4.19, namely

$$HH(\,\mathsf{Pro}^{a}_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}^n\,) \to HH(E/I_1^0(\hat{S})) \to HH(I_1^+/I_1^0) \to HH(A_{n+1}/A_n),$$

where the last two arrows are just induced from non-unital inclusions of associative algebras into each other. As our lift of Equation 4.24 already gives us a concrete representative for $HH(E/I_1^0(\hat{S}))$, we see that

$$f \longmapsto P^+ f \tag{4.25}$$

is a representative of the algebra homomorphism making Diagram 4.20 commutative (This is by the way indeed a ring homomorphism: The failure of P^+ to respect multiplication of $f, g \in A_{n+1}$ is

$$\mathfrak{d} := P^+(f \cdot g) - (P^+ f) \cdot (P^+ g) = P^+ f(1 - P^+) g.$$

Since the image of P^+ lies in the lattice $\tilde{S}_n[[t_{n+1}]]$ of $\tilde{S}_{n+1}$, we have $\mathfrak{d} \in I_1^+$, and since the kernel of $1 - P^+$ contains the lattice, we also have $\mathfrak{d} \in I_1^-$. Thus, $\mathfrak{d} \in I_1^+ \cap I_1^- = I_1^0$ and therefore $\mathfrak{d} \equiv 0$ in $A_{n+1}/A_n = A_{n+1}/I_1^0$.) Finally, observe that the map in Equation 4.25 is a concrete representative of the map Λ (see Equation 3.7). This finishes the proof. □

Theorem 4.5.7. *As always in this section, assume $\mathscr{C}$ is an idempotent complete split exact category with a generator $S \in \mathscr{C}$. Then the natural diagram*

$$\begin{tikzcd} HH(\mathsf{Tate}_{\aleph_0}(\mathscr{C}_n)) \arrow[r] \arrow[d, "\sim"'] & \Sigma HH(\mathscr{C}_n) \arrow[d, "\sim"] \\ HH(A_{n+1}) \arrow[r, "d"'] & \Sigma HH(A_n) \end{tikzcd}$$

commutes. Here the downward arrows are the exact equivalences of Equation 4.9, and d is the homomorphism of degree -1 defined in Equation 3.9 (or in the paper [Bra18, §6]*).*

Proof. The top row stems from the square of exact categories in line 4.1. By the crucial idea of Sho Saito's paper [Sai15] (his proof of the Kapranov-Previdi delooping conjecture), after taking algebraic K-theory, this diagram becomes homotopy Cartesian. However, the same idea works with Hochschild homology, and we get the homotopy commutative diagram

$$\begin{tikzcd} HH(\mathscr{C}_n) \arrow[r] \arrow[d] & HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)) \arrow[r] \arrow[d] & HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n) \arrow[d] \\ HH(\mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n)) \arrow[r] & HH(\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)) \arrow[r] & HH(\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)/\mathsf{Ind}^a_{\aleph_0}(\mathscr{C}_n)) \end{tikzcd} \tag{4.26}$$

whose rows are fiber sequences, by Keller's localization Theorem, see Theorem 4.3.3. In more detail: The rows stem from the fact that $\mathscr{C}_n \hookrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)$ is right s-filtering resp. $\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n) \hookrightarrow \mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)$ is left s-filtering. All of these constructions are functorial on the level of exact functors of exact categories and this induces the downward arrows. Following Saito's idea, since the right-hand side downward map stems from an exact equivalence, Proposition 4.3.5, it is an equivalence, and thus the square on the left-hand side is homotopy bi-Cartesian. We obtain the equivalence $HH(\mathscr{C}_n) \xrightarrow{\sim} \Sigma HH(\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n))$ and equivalences which allow us to phrase this equivalence as the boundary map of the fiber sequence induced from the top row in Diagram 4.26. Aside: Note that the underlying equivalence

$$HH(\mathsf{Tate}^{el}_{\aleph_0}(\mathscr{C}_n)) \sim HH(\mathsf{Pro}^a_{\aleph_0}(\mathscr{C}_n)/\mathscr{C}_n).$$

of Hochschild spectra *does not* come from an exact equivalence of exact categories. By Proposition 4.5.6 we know that under the identification of either side with the Hochschild homology of a category of projective modules and following the middle downward arrow of Diagram 4.19, this is induced from the algebra homomorphism $\Lambda : HH(A_{n+1}) \to HH(A_{n+1}/A_n)$. But from Diagram 4.19 we also see that the boundary map of the localization sequence for $\mathscr{C}_n \hookrightarrow \mathsf{Pro}^a(\mathscr{C}_n)$ (in the top row) commutes with the boundary map of the long exact sequence in Hochschild homology of the algebra extension

$$A_n \hookrightarrow A_{n+1} \twoheadrightarrow A_{n+1}/A_n$$

(in the bottom row). We had denoted the latter boundary map by δ in Equation 3.9. Thus, in conjunction with the identification with the boundary map of the Tate category variant (Equation 4.26), the map is $\delta \circ \Lambda$, which is precisely the definition of the map d in the statement of the theorem. This finishes the proof. □

5. The Beilinson residue

5.1. **Adèles of a scheme.**

5.1.1. *Definition.* Let us recall as much material about adèles of schemes as we need. The original source is Beilinson's article [Beĭ80]. Let k be a field. Suppose X is a Noetherian separated k-scheme. Given a scheme point $\eta \in X$, we shall write $\overline{\{\eta\}}$ for its Zariski closure, equipped with the reduced closed sub-scheme structure. Moreover, we also abuse notation and write η for its defining ideal sheaf.

When given points $\eta_0, \eta_1 \in X$, we write "$\eta_0 \geq \eta_1$" if $\overline{\{\eta_0\}} \ni \eta_1$, i.e. η_1 is a specialization of η_0. Write $S(X)_n := \{(\eta_0 > \cdots > \eta_n), \eta_i \in X\}$ for length $n+1$ sequences without repetitions. Suppose $K_n \subseteq S(X)_n$ is a subset, for some chosen $n \geq 0$. Following [Beĭ80], define ${}_\eta(K_n) := \{(\eta_1 > \cdots > \eta_n)$ such that $(\eta > \eta_1 > \cdots > \eta_n) \in K_n\}$, a subset of $S(X)_{n-1}$.

Definition 5.1.1. *Let X be a Noetherian k-scheme.*

(1) *Assume $\mathscr{F}$ is a coherent sheaf. Define inductively*

$$A(K_0, \mathscr{F}) := \prod\nolimits_{\eta \in K_0} \varprojlim_i \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\eta}/\eta^i$$

for $n = 0$, and

$$A(K_n, \mathscr{F}) := \prod\nolimits_{\eta \in X} \varprojlim_i A({}_\eta K_n\ , \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\eta}/\eta^i) \tag{5.1}$$

for $n \geq 1$.

(2) *For a quasi-coherent sheaf $\mathscr{F}$, define $A(K_n, \mathscr{F}) := \underrightarrow{\operatorname{colim}}_{\mathscr{F}_j} A(K_n, \mathscr{F}_j)$, where $\mathscr{F}_j$ runs through the coherent sub-sheaves of $\mathscr{F}$.*

Note that the arguments in Equation 5.1 are usually only quasi-coherent, so this additional definition is necessary to give $A(-, \mathscr{F})$ a meaning, even if $\mathscr{F}$ happens to be coherent.

Theorem 5.1.2 (A. Beilinson)**.** *Let X be a Noetherian scheme.*

(1) *For any $n \geq 0$ and subset $K_n \subseteq S(X)_n$, the above defines an exact functor*

$$A(K_n, -) : \mathrm{QCoh}(X) \longrightarrow \mathrm{Mod}(\mathscr{O}_X),$$

(2) *and for every quasi-coherent sheaf $\mathscr{F}$, this gives rise to a flasque resolution*

$$0 \to \mathscr{F} \to \mathbf{A}^0_{\mathscr{F}} \to \mathbf{A}^1_{\mathscr{F}} \to \mathbf{A}^2_{\mathscr{F}} \to \dots, \tag{5.2}$$

where $\mathbf{A}^i_{\mathscr{F}}(U) := A(S(U)_i\, , \mathscr{F})$ for any Zariski open $U \subseteq X$.

We will not go into further detail. This result is taken from Beilinson's paper [Beĭ80, §2]. See Huber [Hub91] for the proof.

5.1.2. *Local structure for a single flag.* We fix a flag $\triangle = (\eta_0 > \cdots > \eta_r)$ with $\operatorname{codim}_X \overline{\{\eta_i\}} = i$ throughout this subsection. We may evaluate the adèle group $A_X(\triangle, \mathscr{O}_X)$ of Definition 5.1.1 for this individual flag. Unravelling the definition, it consists alternatingly of localizations at a multiplicative set, and completions at ideals. For the sake of the following arguments, we will introduce a notation to keep these two steps conceptually separated – this notation will not appear anywhere else again. Namely:

Definition 5.1.3. *Set $L_r := \mathscr{O}_{\eta_r}$ and $C_r := \widehat{\mathscr{O}_{\eta_r}}$. Inductively for $j \leq r$ let*

- $L_{j-1} := C_j[(\mathscr{O}_{\eta_j} - \eta_{j-1})^{-1}]$ *("localization")*
- $C_{j-1} := \varprojlim_{i_{j-1}} L_{j-1}/\eta_{j-1}^{i_{j-1}}$ *("completion")*

This proceeds downward along j until we reach $A_X(\triangle, \mathscr{O}_X) = C_0$. So this is a step-by-step description of the formation of an adèle completion. A detailed verification of this is given in [BGW16a, §4], which uses notation largely compatible with ours, except for our $C_{(-)}$ being called $A_{(-)}$ in Definition 4.4 loc. cit. The localizations and completions are ring maps which, as affine schemes, lead to the following sequence of flat morphisms:

$$\operatorname{Spec} A_X(\triangle, \mathscr{F}) \to \cdots \to \operatorname{Spec} C_{r-1} \to \operatorname{Spec} L_{r-1} \to \operatorname{Spec} C_r \to \operatorname{Spec} L_r \to X. \tag{5.3}$$

The behaviour of the prime ideals under these maps is very carefully studied in [Yek92, §3] and [BGW16a], but we will not need more than the following:

Lemma 5.1.4 ([BGW16a, Lemma 4.5]). *For any $i = 0, \ldots, r$ we have*

(1) C_j *is a faithfully flat Noetherian* $\mathscr{O}_{\eta_j}$*-algebra.*
(2) *The maximal ideals of the ring* C_j *are precisely the primes minimal over* $\eta_j C_j$*.*
(3) *The ring* C_j *is a finite product of* j*-dimensional reduced local rings, each complete with respect to its maximal ideal.*

5.1.3. *Coherent Cousin complex.* For the sake of legibility, let us allow ourselves (just for this section) the shorthand

$$H^r_x(X) := H^r_x(X, \Omega^n),$$

where $x \in X$ is any scheme point, and n any fixed integer. If R is a ring, we shall also write $H^r_x(R) := H^r_x(\operatorname{Spec} R)$. We may now consider the coherent Cousin complex of the scheme X for the coherent sheaf Ω^n, i.e. with the above shorthand

$$Cous^{\bullet}(X) : \cdots \xrightarrow{d} \coprod_{x_{r-2} \in X^{r-2}} H^{r-2}_{x_{r-2}}(X) \xrightarrow{d} \coprod_{x_{r-1} \in X^{r-1}} H^{r-1}_{x_{r-1}}(X) \xrightarrow{d} \coprod_{x_r \in X^r} H^r_{x_r}(X) \longrightarrow \cdots. \tag{5.4}$$

We write d for its differential and d^*_* for the components of d among the individual direct summands, as in

$$d = \sum_{x_r, x_{r+1}} (d^{x_r}_{x_{r+1}} : H^r_{x_r}(X) \longrightarrow H^{r+1}_{x_{r+1}}(X)).$$

We proceed as follows: For a flat morphism $f : X \to Y$ of schemes we know by Proposition 1.3.2 that there is an induced pullback of coherent Cousin complexes $f^* : \Gamma(Y, Cous^{\bullet}(Y)) \to \Gamma(X, Cous^{\bullet}(X))$ and even better, we understand in a very precise way the induced morphisms between the individual direct summands appearing in Equation 5.4 (see again Proposition 1.3.2 for details). For the flag $\triangle$ with $\eta_i \in X^i$ that we had fixed, we may consider the diagram consisting only of the summands of Equation 5.4 for $x_r := \eta_r$ (and the morphisms between them instead of d being just the respective component $d^{\eta_r}_{\eta_{r+1}}$). This yields a diagram, call it Q_X,

$$Q_X : \cdots \longrightarrow H^{r-2}_{\eta_{r-2}}(X) \longrightarrow H^{r-1}_{\eta_{r-1}}(X) \longrightarrow H^r_{\eta_r}(X) \longrightarrow 0.$$

(Of course this will *not* be a complex anymore; there is no reason the composition of individual d^*_* should be zero). Since f^* commutes with the differential d of the Cousin complex, the components $d^{\eta_r}_{\eta_{r+1}}$ individually also commute with f^*. Therefore f^* induces also a flat pullback between the diagrams of shape very much like Q_X, namely

$$\begin{array}{ccccccccc}
Q' : & \cdots \longrightarrow & \coprod_{x_{r-2}} H^{r-2}_{x_{r-2}}(Y) & \longrightarrow & \coprod_{x_{r-1}} H^{r-1}_{x_{r-1}}(Y) & \longrightarrow & \coprod_{x_r} H^r_{x_r}(Y) & \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
Q_Y : & \cdots \longrightarrow & H^{r-2}_{\eta_{r-2}}(Y) & \longrightarrow & H^{r-1}_{\eta_{r-1}}(Y) & \longrightarrow & H^r_{\eta_r}(Y) & \longrightarrow 0
\end{array}$$

where for each i the points x_i run through the finitely many irreducible components of the scheme-theoretic fiber $f^{-1}(\eta_i)$ (this is because by Proposition 1.3.2 the pullback f^* of the direct summands appearing in the lower row has non-zero image at most in these direct summands of the coherent Cousin complex of X). For example, if each of the points η_i has precisely one pre-image under f, the top row complex would literally have the shape of Q, but with the η_i each replaced by $f^{-1}(\eta_i)$.

Now consider the following commutative diagram (whose construction we will explain below):

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \\
 & & \| & & \| & & & & \\
4) & \to & \coprod H^{r-3}_{\eta_{r-3}}(C_{r-1}) & \to & \coprod H^{r-2}_{\eta_{r-2}}(C_{r-1}) & \to & \coprod H^{r-1}_{\eta_{r-1}}(C_{r-1}) & & \\
 & & \uparrow & & \uparrow & & \| & & \\
3) & \to & \coprod H^{r-3}_{\eta_{r-3}}(L_{r-1}) & \to & \coprod H^{r-2}_{\eta_{r-2}}(L_{r-1}) & \to & \coprod H^{r-1}_{\eta_{r-1}}(L_{r-1}) & & \\
 & & \| & & \| & & \| & & \\
2) & \to & \coprod H^{r-3}_{\eta_{r-3}}(C_{r}) & \to & \coprod H^{r-2}_{\eta_{r-2}}(C_{r}) & \to & \coprod H^{r-1}_{\eta_{r-1}}(C_{r}) & \to & H^{r}_{\eta_{r}}(C_{r}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \| \\
1) & \to & H^{r-3}_{\eta_{r-3}}(L_{r}) & \to & H^{r-2}_{\eta_{r-2}}(L_{r}) & \to & H^{r-1}_{\eta_{r-1}}(L_{r}) & \to & H^{r}_{\eta_{r}}(L_{r}) \\
 & & \| & & \| & & \| & & \| \\
0) & \to & H^{r-3}_{\eta_{r-3}}(X) & \to & H^{r-2}_{\eta_{r-2}}(X) & \to & H^{r-1}_{\eta_{r-1}}(X) & \to & H^{r}_{\eta_{r}}(X)
\end{array}
\tag{5.5}
$$

To construct this diagram, we begin with the bottom row and work upwards. The bottom row is a sequence of direct summands in the Cousin complex of X. The rows above now result inductively from applying the flat pullback along the respective morphisms in the chain of Equation 5.3. More precisely:

(Odd Rows) To obtain odd-indexed rows: This is the flat pullback of the row below along the localization

$$L_j := C_{j+1}[(\mathscr{O}_{\eta_{j+1}} - \eta_j)^{-1}].$$

The primes in such a localization correspond bijectively to those primes P of C_{j+1} with $P \cap (\mathscr{O}_{\eta_{j+1}} - \eta_j) = \varnothing$. Hence, the entire flag $\eta_0 > \cdots > \eta_j$ lies also in $\operatorname{Spec} L_j$. Since the local cohomology of the row below takes supports in $\eta_0, \ldots, \eta_j$ respectively, it follows that in each case excision (Lemma 1.1.3) guarantees that the flat pullback induces an isomorphism, explaining the equalities "$\|$". Note that under an open immersion a point has at most one pre-image, so direct summands do not fiber up into further direct summands when going upward.

(Even Rows) To obtain the even-indexed rows: This is the flat pullback of the row below along the completion

$$C_j := \varprojlim_{i_j} L_j / \eta_j^{i_j}.$$

Firstly, we note that in a completion a point can have several (finitely many) pre-images; therefore several summands may appear in the new row (as indicated in the above diagram); see Proposition 1.3.2 for a precise description of the map between these direct summands. Applying Lemma 1.1.4 to $I = I' := \eta_j$, we obtain that the pullback induces an isomorphism

$$H^p_{\eta_j}(\operatorname{Spec} L_j, M) \xrightarrow{\sim} H^p_{\eta_j}(\operatorname{Spec} C_j, \widehat{M}) \mid_{L_j} ,$$

producing the isomorphism of the right-most non-zero term with the corresonding term in the row below.

In the above diagram we find "downward staircase steps" on the right, of the shape

$$
\begin{array}{ccc}
\coprod H^{r-1}_{\eta_{r-1}}(C_{r-1}) & & \\
\| & & \\
\coprod H^{r-1}_{\eta_{r-1}}(L_{r-1}) & & \\
\| & & \\
\coprod H^{r-1}_{\eta_{r-1}}(C_{r}) & \to & H^{r}_{\eta_{r}}(C_{r}),
\end{array}
\tag{5.6}
$$

for varying r. The arrows "$\|$" are actually upward arrows coming from the flat pullback and we had seen above that these are isomorphisms in the situation at hand. So we may run them backwards, giving something that could be called an adèle enrichment of the usual boundary maps of the coherent Cousin complex. Let us give them a name:

Definition 5.1.5 (Adèle boundary maps, Cousin version)**.** *For a flag $\eta_0 > \cdots > \eta_n$ and a quasi-coherent sheaf $\mathscr{F}$ we call the morphisms*

$$\boldsymbol{\partial}^{\eta_r}_{\eta_{r+1}} : H^r_{\eta_r}(C_r) \longrightarrow H^{r+1}_{\eta_{r+1}}(C_{r+1}),$$

i.e. in self-contained notation,

$$\begin{aligned}&H^r_{\eta_r}(A(\eta_r > \cdots > \eta_n, \mathscr{O}_X),\, A(\eta_r > \cdots > \eta_n, \mathscr{F}))\\ &\qquad\longrightarrow H^{r+1}_{\eta_{r+1}}(A(\eta_{r+1} > \cdots > \eta_n, \mathscr{O}_X),\, A(\eta_{r+1} > \cdots > \eta_n, \mathscr{F})),\end{aligned}$$

the (Cousin) adèle boundary maps.

Using the HKR theorem with supports and its compatibility with boundary map on the local cohomology vs. Hochschild side, Proposition 2.0.1, there is also a Hochschild counterpart of the same map for $\mathscr{F} := \Omega^n$:

Definition 5.1.6 (Adèle boundary maps, Hochschild version)**.** *Suppose X is a smooth separated scheme of pure dimension n. For a flag $\eta_0 > \cdots > \eta_n$ with* $\operatorname{codim}_X \eta_i = i$ *we call the morphisms*

$$^{HH}\boldsymbol{\partial}^{\eta_r}_{\eta_{r+1}} : HH^{\eta_r}_{n-r}(C_r) \longrightarrow HH^{\eta_{r+1}}_{n-(r+1)}(C_{r+1})$$

the (Hochschild) adèle boundary maps.

5.1.4. *Tate realization.* As explained above, we have the concatenation of flat morphisms

$$\operatorname{Spec} A_X(\triangle, \mathscr{F}) \to \cdots \to \operatorname{Spec} C_{r-1} \to \operatorname{Spec} L_{r-1} \to \operatorname{Spec} C_r \to \operatorname{Spec} L_r \to X.$$

We will now construct exact functors originating from the module categories of the individual rings appearing along this composition, i.e. functors $\mathsf{Mod}_f(R) \to (\bigstar)$, where $\mathsf{Mod}_f(R)$ denotes the category of finitely generated R-modules and "$\bigstar$" will be suitably chosen exact categories built from Ind-, Pro- and Tate objects (as recalled in §4.2). The basic idea is that $A_X(\triangle, \mathscr{F})$ is a finite product of n-local fields [Yek92], [BGW16a] and can be presented as an n-Tate object in finite-dimensional k-vector spaces, say

$$A_X(\triangle, \mathscr{F}) = \underbrace{\varinjlim\varprojlim \cdots \varinjlim\varprojlim}_{\longleftarrow\alpha\longrightarrow} A_\alpha \qquad \text{with} \qquad A_\alpha \in \mathsf{Vect}_f(k),$$

and then there is an exact functor

$$\begin{aligned}\mathsf{Mod}_f(A_X(\triangle, \mathscr{F})) &\longrightarrow n\text{-}\mathsf{Tate}_{\aleph_0}(\mathsf{Vect}_f(k))\\ M &\longmapsto \underbrace{\varinjlim\varprojlim \cdots \varinjlim\varprojlim}_{\longleftarrow\alpha\longrightarrow} (M \otimes A_\alpha)\,.\end{aligned} \tag{5.7}$$

See [BGW16c, §7.2] for details. As the rings L_{n-r} arise from r alternating localizations and completions, and similarly for C_{n-r}, there are analogous exact functors taking values in r-Tate objects. At the risk of repeating ourselves, let us unravel a bit the structure of these analogues:

Each completion of a ring can be interpreted as a Pro-limit, given by a projective system (as depicted below on the left), and each localization as an Ind-limit, given by the inductive limit of finitely generated sub-modules inside the localization (as depicted below on the right):

$$\widehat{R} = \varprojlim_i R/I^i \qquad \text{and} \qquad \varinjlim_{t \in S} \frac{1}{t} R = R[S^{-1}].$$

We write $\frac{1}{t}R$ to denote the R-submodule of $R[S^{-1}]$ generated by the element $\frac{1}{t}$. Concretely, let X/k is an n-dimensional scheme. Then, by presenting C_{n-r} resp. L_{n-r} by alternating localizations and completions (as dictated by Definition 5.1.3), the analogue of the functor in line 5.7 yields exact functors

$$\begin{aligned}\mathsf{Mod}_f(C_n) &\longrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathsf{Vect}_f(k))\\ \mathsf{Mod}_f(L_{n-1}) &\longrightarrow \mathsf{Ind}^a_{\aleph_0}\mathsf{Pro}^a_{\aleph_0}(\mathsf{Vect}_f(k))\\ \mathsf{Mod}_f(C_{n-1}) &\longrightarrow \mathsf{Pro}^a_{\aleph_0}\mathsf{Ind}^a_{\aleph_0}\mathsf{Pro}^a_{\aleph_0}(\mathsf{Vect}_f(k))\\ &\;\;\vdots\end{aligned} \tag{5.8}$$

and in fact all the pairs of Ind-Pro-limits lie in the sub-category of Tate objects so that

$$\begin{array}{c} \vdots \\ \mathsf{Mod}_f(C_1) \longrightarrow \mathsf{Pro}^a_{\aleph_0}\left((n-1)\text{-}\mathsf{Tate}_{\aleph_0}\right)(\mathsf{Vect}_f(k)) \\ \mathsf{Mod}_f(L_0) \longrightarrow \left(n\text{-}\mathsf{Tate}_{\aleph_0}\right)(\mathsf{Vect}_f(k)) \\ \mathsf{Mod}_f(C_0) \longrightarrow \mathsf{Pro}^a_{\aleph_0}\left(n\text{-}\mathsf{Tate}_{\aleph_0}\right)(\mathsf{Vect}_f(k)) \end{array} \tag{5.9}$$

and $C_0 = A(\triangle, \mathscr{O}_X)$ still lies in $(n\text{-}\mathsf{Tate}_{\aleph_0})(\mathsf{Vect}_f(k))$ since the outermost Pro-limit is just taken over nil-thickenings of the irreducible components/minimal primes. These Pro-limits reduce to an eventually stationary projective system and thus already exist in the n-Tate category without having to take a further category of Pro-objects. As a result, $A(\triangle, \mathscr{O}_X)$-modules can naturally be sent to their associated n-Tate object in the category of finite-dimensional k-vector spaces.

Remark 5.1.7. The exactness of these functors can be shown step-by-step: For the inductive systems defining Ind-objects the exactness is immediately clear, and for the projective systems defining the Pro-objects one uses the Artin–Rees lemma. We refer to [BGW16c, §7.2]. A more detailed investigation of such functors $\mathbf{C}_Z : \mathrm{Coh}(X) \to \mathsf{Pro}^a_{\aleph_0}(\mathrm{Coh}_Z(X))$, $\mathscr{F} \mapsto [i \mapsto \mathscr{F}/\mathscr{J}^i_Z]$, where $\mathscr{J}_Z$ denotes the ideal sheaf of Z and $i \in \mathbf{Z}_{\geq 1}$, is given in [BGW17]. See [BGW17, Proposition 3.25].

Proposition 5.1.8. *We obtain a commutative diagram*

$$\begin{tikzcd} H^r_{\eta_r}(C_r) \arrow[r, "\sim"] \arrow[d] & H^r_{\eta_r}(L_r, \Omega^n) \arrow[r, "\sim"] \arrow[d] & HH^{\eta_r}_{n-r}(L_r) \arrow[r] \arrow[d] & HH_{n-r}((n-r)\text{-}\mathsf{Tate}_{\aleph_0}(\mathsf{Vect}_f(k))) \arrow[d] \\ H^{r+1}_{\eta_{r+1}}(C_{r+1}) \arrow[r, "\sim"] & H^{r+1}_{\eta_{r+1}}(L_{r+1}, \Omega^n) \arrow[r, "\sim"] & HH^{\eta_{r+1}}_{n-r-1}(L_{r+1}) \arrow[r] & HH_{n-r-1}((n-r-1)\text{-}\mathsf{Tate}_{\aleph_0}(\mathsf{Vect}_f(k))), \end{tikzcd}$$

where

1. *the first and second downward arrows the adèle boundary maps* $\boldsymbol{\partial}^{\eta_r}_{\eta_{r+1}}$ *of Definition 5.1.5,*
2. *the third downward arrow is the analogous adèle boundary map in Hochschild homology (i.e. the maps of Definition 5.1.6 up to the canonical isomorphism induced from swapping* L_r *with* C_r*),*
3. *the fourth downward arrow is induced from the delooping map*

$$HH(j\text{-}\mathsf{Tate}_{\aleph_0}(-)) \xrightarrow{\sim} \Sigma HH((j-1)\text{-}\mathsf{Tate}_{\aleph_0}(-)).$$

Proof. (Left square) In the left-most column we consider the adèle boundary map as constructed in Definition 5.1.5. The relevant local cohomology groups are invariant under the last completion (so this is Lemma 1.1.4, or see Diagram 5.6). This implies the commutativity of the left-most square.
(Middle square) We use the HKR isomorphism with supports both on the left and on the right and the fact that this transforms the boundary map in local cohomology into the boundary map of the Hochschild homology localization sequence, Proposition 2.0.1. As these maps are also differentials on the E_1-page of the coherent Cousin vs. coniveau spectral sequence, we may also directly cite Theorem 2.1.3, but unravelling its proof both results reduce to the same core.
(Right square) We use the realization functors with values in the relevant higher Tate categories as in lines 5.8-5.9. Thus, the commutativity of this square is equivalent to the fact that these realization functors transform the localization sequence boundary map into the delooping map of Hochschild homology. We discuss this at length in [BGW17], but see Appendix A for a quick overview. □

Proposition 5.1.9. *Pick $\mathscr{C} := \mathsf{Vect}_f(k)$ and let A_i denote the Beilinson cubical algebra as provided by Theorem 4.5.1 for this choice of $\mathscr{C}$. Then there is a canonical commutative square*

$$\begin{tikzcd} HH^{\eta_r}_{n-r}(L_r) \arrow[r] \arrow[d] & HH_{n-r}(A_{n-r}) \arrow[d, "d"] \\ HH^{\eta_{r+1}}_{n-r-1}(L_{r+1}) \arrow[r] & HH_{n-r-1}(A_{n-r-1}), \end{tikzcd}$$

where the left downward arrow is as in Proposition 5.1.8, the right downward arrow is the map d of Definition 3.2.6.

Proof. We pick $\mathscr{C} := \mathsf{Vect}_f(k)$, which is a split exact idempotent complete abelian category with the generator k (viewed as a one-dimensional k-vector space). Hence, we may use our version of Morita theory and apply Theorem 4.5.1. We obtain the cubical algebras A_{n-r} and A_{n-r-1}. By the cited theorem, there is an exact equivalence

$$n\text{-}\mathsf{Tate}_{\aleph_0}(\mathscr{C}) \xrightarrow{\sim} P_f(A_n), \tag{5.10}$$

inducing isomorphisms between the respective Hochschild homology groups. Next, we use Proposition 5.1.8 and using the isomorphisms of line 5.10, we may replace the objects in the right-most column by the Hochschild homology of the A_i. Thanks to Theorem 4.5.7 our diagram remains commutative if we replace the downward arrow between these objects by the map d. This results precisely in our claim. □

5.2. **Comparison with the Tate–Beilinson residue in Lie homology.** For every n-fold cubical algebra A over a field k, Beilinson constructs a canonical map

$$\phi_{Beil} : H^{\mathrm{Lie}}_{n+1}(A_{Lie}, k) \longrightarrow k,$$

where A_{Lie} denotes the Lie algebra of the associative algebra A, i.e. $[x, y] := xy - yx$. This is [Beĭ80, §1, Lemma, (a)]. For $n = 1$ this functional describes a class in $H^2(A_{Lie}, k)$, and thus a central extension known as Tate's central extension. Although not spelled out explicitly, it was originally constructed by Tate to have a coordinate-independent defnition of the residue on curves in his paper [Tat68]. See [Bra18] for a detailed review. To connect Lie homology with differential forms, use the square

$$\begin{tikzcd} H_n(A_{Lie}, A_{Lie}) \arrow[r, "\varepsilon"] \arrow[d, "I'"'] & HH_n(A) \arrow[d, "\phi_{HH}"] \\ H_{n+1}(A_{Lie}, k) \arrow[r, "\phi_{Beil}"'] & k, \end{tikzcd} \tag{5.11}$$

of [Bra18]. The map I' is a Lie analogue of the map I in the SBI sequence of Hochschild homology (also known as *Connes's periodicity sequence*, see [Lod92, Thm 2.2.1]). Any element coming from any commutative sub-algebra of A can be lifted to the upper left corner, showing that Beilinson's map and the abstract Hochschild symbol agree on such elements. See loc. cit. One slogan of the present paper might be:

We show that both Tate's and Beilinson's constructions essentially encode an iterated boundary map in Keller's localization sequence for Hochschild homology, after iteratively cutting out the divisors defining a saturated flag in the scheme.

The novelty here is the interpretation in terms, essentially, of differentials in the adèlic variant of the Hochschild–Cousin complex or equivalently coherent Cousin complex:

Theorem 5.2.1 (Adèle Cousin differentials via abstract Hochschild symbol)**.** *There is a commutative diagram*

$$\begin{tikzcd} H^0_{\eta_0}(C_0) \arrow[r, "\chi"] \arrow[d, "\rho"'] & HH_n(A_n) \arrow[d, "\phi_{HH}"] \\ H^n_{\eta_n}(C_n) \arrow[r, "\xi"'] & k, \end{tikzcd}$$

where

(1) χ *is the composition of all right-ward arrows in the top rows of Proposition 5.1.8 and then Proposition 5.1.9,*
(2) ρ *is the composition of the downward arrows in the same propositions, concatenated for* $n, n-1, \dots$ *down to* 0*,*
(3) ξ *is the trace of the local cohomology group of a closed point down to* k *(= literally the trace of an endomorphism of a finite-dimensional* k*-vector space)*
(4) ϕ_{HH} *is the abstract Hochschild symbol of Definition 3.2.7.*

Proof. The left downward arrow is a composition of adèle boundary maps in the Cousin version, Definition 5.1.5. Thanks to the HKR isomorphism with supports, in the concrete guise of Proposition 5.1.8, we may isomorphically work with Hochschild homology with supports instead, as on the left-hand side in the diagram in Proposition 5.1.9. Moreover, thanks to this proposition, we might again isomorphically replace these by maps d between the Hochschild homology groups of cubical algebras. Next, by the very definition of the abstract Hochschild symbol (Definition 3.2.7; or see [Bra18]) as a composition of all these maps d, we learn that by composing the isomorphisms that we have just discussed, the left downward arrow can be identified with the Hochschild symbol $HH_n(A_n) \to HH_0(A_0) \xrightarrow{\tau} k$. □

Theorem 5.2.2 (Agreement with Tate–Beilinson Lie map)**.** *Suppose* X/k *is a separated, finite type scheme of pure dimension* n*. Fix a flag* $\triangle = (\eta_0 > \cdots > \eta_n)$ *with* $\operatorname{codim}_X \overline{\{\eta_i\}} = i$*. The Tate–Beilinson Lie homology residue symbol*

$$\Omega^n_{\operatorname{Frac} L_n/k} \longrightarrow H_{n+1}((A_n)_{Lie}, k) \xrightarrow{\phi_{Beil}} k$$

(as defined [Beĭ80, §1, Lemma, (b)]*) also agrees with*

$$\Omega^n_{\operatorname{Frac} L_n/k} \longrightarrow HH^{\eta_0}_n(L_n) \longrightarrow HH^{\eta_0}_n(C_0) \longrightarrow HH_n(A_n) \xrightarrow{\phi_{HH}} k.$$

Here $L_{(-)}, C_{(-)}$ *are as in Definition 5.1.3, in particular they depend on* $\triangle$*.*

Proof. This is very easy now. As $R := \operatorname{Frac} L_n$ is commutative, any differential form $f_0 \, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ lifts to its symmetrization $\sum (-1)^\pi f_{\pi(0)} \otimes f_{\pi(1)} \wedge \cdots \wedge f_{\pi(n)}$ in the Chevalley–Eilenberg complex describing the Lie homology group $H_n(R_{Lie}, R_{Lie})$. One checks that the Chevalley–Eilenberg differential vanishes on commuting elements (this is trivial since the latter is a linear combination of terms each of which contains at least one commutator). Thus, by functoriality, even after mapping R into the non-commutative algebra A_n, we still have a Lie homology cycle. Thus, we have a lift to the upper left corner in Diagram 5.11 and along with Theorem 5.2.1 this implies the claim. □

Appendix A. Boundary map under localization

We recall the following basic construction:

Proposition A.0.1 (Pro Realization)**.** *Let* X *be a Noetherian scheme,* Z *a closed subset and* $U := X \setminus Z$ *the open complement. Define*

$$\mathbf{C}_Z : \operatorname{Coh}(X) \longrightarrow \mathsf{Pro}^a_{\aleph_0}(\operatorname{Coh}_Z(X)), \qquad \mathscr{F} \longmapsto \varprojlim_r j_{r,*} j_r^* \mathscr{F}, \tag{A.1}$$

where

- $\mathscr{F}$ *is an arbitrary coherent sheaf on* X*,*

- *$j_r : Z^{(r)} \hookrightarrow X$ the closed immersion of the r-th infinitesimal neighbourhood*[9] *of Z as a closed subscheme with the reduced subscheme structure, and*
- *the limit $\varprojlim_r$ is understood as the admissible Pro-diagram $\mathbf{N} \to j_{r,*}j_r^*\mathscr{F}$.*

Then this defines an exact functor and it sits in the commutative diagram of exact categories and exact functors:

$$\begin{tikzcd} \mathrm{Coh}_Z(X) \arrow[r] \arrow[d, "1"] & \mathrm{Coh}(X) \arrow[r] \arrow[d, "\mathbf{C}_Z"] & \mathrm{Coh}(U) \arrow[d] \\ \mathrm{Coh}_Z(X) \arrow[r] & \mathsf{Pro}^a_{\aleph_0}(\mathrm{Coh}_Z(X)) \arrow[r] & \mathsf{Pro}^a_{\aleph_0}(\mathrm{Coh}_Z(X))/\mathrm{Coh}_Z(X) \end{tikzcd} \tag{A.2}$$

The right downward arrow is induced from $\mathbf{C}_Z$ to the quotient categories, in view of the natural exact equivalence $\mathrm{Coh}(U) \cong \mathrm{Coh}(X)/\operatorname{Coh}_Z(X)$.

Proof. See [BGW17, Proposition 3.25] for a discussion of the functor $\mathbf{C}_Z$ (it is defined loc. cit. using the same notation). The left commutative square is immediate. For the right commutative square see Lemma 3.27 loc. cit. Alternatively, a number of statements of this type are discussed in [BGW14]. ☐

The rows in Diagram A.2 are exact sequences of exact categories, i.e. on the left-hand side we have fully exact sub-categories that are left- resp. right s-filtering in the middle exact categories and the right-hand side arrows are the quotient functors to the quotient exact categories. Thus, we obtain the commutative square

$$\begin{tikzcd} HH\,\mathrm{Coh}(U) \arrow[r] \arrow[d] & \Sigma HH\,\mathrm{Coh}_Z(X) \arrow[d, "1"] \\ HH\left(\mathsf{Pro}^a_{\aleph_0}\left(\mathrm{Coh}_Z(X)\right)/\operatorname{Coh}_Z(X)\right) \arrow[r] & \Sigma HH\,\mathrm{Coh}_Z(X) \end{tikzcd}$$

from Keller's localization sequence. Using the equivalence

$$HH\left(\mathsf{Pro}^a_{\aleph_0}\left(\mathrm{Coh}_Z(X)\right)/\operatorname{Coh}_Z(X)\right) \xrightarrow{\sim} HH\mathsf{Tate}_{\aleph_0}\left(\mathrm{Coh}_Z(X)\right)$$

this may be rephrased as

$$\begin{tikzcd} HH\,\mathrm{Coh}(U) \arrow[r] \arrow[d] & \Sigma HH\,\mathrm{Coh}_Z(X) \arrow[d, "1"] \\ HH\mathsf{Tate}_{\aleph_0}\left(\mathrm{Coh}_Z X\right) \arrow[r] & \Sigma HH\,\mathrm{Coh}_Z(X). \end{tikzcd}$$

As a result, the boundary map of the localization sequence for Hochschild homology of a closed-open complement $Z \hookrightarrow X \hookleftarrow U$ is compatible with a delooping boundary coming from the delooping property of the Tate category. From this fact, one also obtains that the differentials on the E_1-page of the Hochschild coniveau spectral sequence are compatible with the functor to a Tate category. This is the same argument as in the proof of Theorem 2.1.3, and simply based on the fact that these E_1-differentials can be realized as colimits of boundary maps of the ordinary localization sequence, see loc. cit.

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