DERIVED ℓ-ADIC ZETA FUNCTIONS

JONATHAN CAMPBELL, JESSE WOLFSON, AND INNA ZAKHAREVICH

ABSTRACT. Let $K_0(\mathcal{V}_k)$ be the Grothendieck ring of varieties. Motivic measures often arise as group homomorphisms $K_0(\mathcal{V}_k) \longrightarrow K_0(\mathcal{E})$, where $\mathcal{E}$ is an exact category. In this paper we give a recipe for lifting such homomorphisms to maps of spectra $K(\mathcal{V}_k) \longrightarrow K(\mathcal{E})$, where $K(\mathcal{V}_k)$ is the Grothendieck spectrum of varieties constructed by Campbell and Zakharevich. We consider two special cases: the classical local zeta function, thought of as a homomorphism $K_0(\mathcal{V}_{\mathbf{F}_q}) \longrightarrow K_0(\mathrm{End}(\mathbf{Q}_\ell))$, and the compactly-supported Euler characteristic, thought of as a homomorphism $K_0(\mathcal{V}_{\mathbf{C}}) \longrightarrow K_0(\mathbf{Q})$. We use these to prove that the Grothendieck spectrum of varieties contains nontrivial geometric information in its higher homotopy groups by showing that the map $\mathbb{S} \longrightarrow K(\mathcal{V}_k)$ is nontrivial in higher dimensions when k is finite or $\mathbf{C}$, and, moreover, that when k is finite this map is not surjective on higher homotopy groups.

1. INTRODUCTION

Let k be a field. In this paper, by "variety over k" we mean a reduced, separated k-scheme of finite type. The Grothendieck ring of varieties over k, denoted $K_0(\mathcal{V}_k)$, is the abelian group generated by isomorphism classes $[X]$ of k-varieties, with the relation $[X] = [Z] + [X - Z]$ for $Z \hookrightarrow X$ a closed inclusion. Thus $K_0(\mathcal{V}_k)$ is the *universal additive invariant*; any function $f\colon \{\text{varieties}\} \longrightarrow A$ (where A is an abelian group) satisfying $f(X) = f(X - Z) + f(Z)$ (usually called a *motivic measure*) factors through $K_0(\mathcal{V}_k)$. Many important invariants of varieties induce motivic measures; for example point counts (for k finite) or Hodge numbers (for $k \subset \mathbf{C}$) produce such homomorphisms.

The first and third authors (see [Cam, Zak17a, CWZ]) have constructed a higher Grothendieck ring of varieties, namely, a spectrum $K(\mathcal{V}_k)$ such that $\pi_0 K(\mathcal{V}_k) \cong K_0(\mathcal{V}_k)$. Two natural questions arise:

(1) Do classical motivic measures lift to maps of spectra?
(2) What arithmetic or geometric information do the higher homotopy groups $K_i(\mathcal{V}_k)$ encode?

One of the most important motivic measures of a variety is its zeta function. When $k = \mathbf{F}_q$ and $A = (1 + \mathbf{Z}[\![t]\!], \times)$, this can be written as

$$X \mapsto Z(X, t) = \exp \sum_{n=1}^{\infty} \frac{|X(\mathbf{F}_{q^n})|}{n} t^n.$$

The zeta function satisfies the defining relation $Z(X,t) = Z(Y,t)Z(X-Y,t)$ for closed subvarieties $Y \hookrightarrow X$, and thus gives a well-defined invariant on the ring $K_0(\mathcal{V}_k)$. If we think of the $\mathbf{F}_{q^n}$-points of X as the fixed points of the n-th power of Frobenius, then the zeta function is uniquely determined by the data of Frobenius acting on the $\overline{\mathbf{F}}_q$-points of k. By the Grothendieck-Lefschetz fixed point theorem, the number fixed points of Frobenius is equal to the trace of Frobenius acting on the ℓ-adic cohomology; thus we can instead write the zeta function as **REFERENCE**

$$X \mapsto \prod_{i=0}^{\infty} \det(1 - tFrob_q | H^i_c(X, \mathbf{Q}_\ell))^{(-1)^{i+1}}.$$

Note that this now depends only on the ℓ-adic compactly supported cohomology of X and the action of Frobenius. (For more detail on this, together with a beautiful discussion of other formulations of the zeta function, see [Ram15].)

If we instead consider the case when X is not finite, but rather a subfield of $\mathbf{C}$, we can construct a similar invariant by considering the compactly supported Euler characteristic of X. We can write

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim H^i_c(X(\mathbf{C}); \mathbf{Q}).$$

This, again, is a motivic measure that only depends on the compactly supported cohomology of X.

These two examples bring us to the statement of the main theorem.

Theorem 1.1.

(1) *Let k be a field, k^s a separable closure of k, and $\ell \neq \mathrm{char}(k)$ a prime. Denote by $\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell)$ the exact category of finitely generated continuous representations of $\mathrm{Gal}(k^s/k)$ over $\mathbf{Z}_\ell$. The function from k-varieties to the Grothendieck ring of continuous $\mathrm{Gal}(k^s/k)$ representations*

$$X \mapsto \sum_i (-1)^i [H^i_{et,c}(X \times_k k^s; \mathbf{Z}_\ell)]$$

lifts to a map of K-theory spectra

$$\boldsymbol{\zeta} \colon K(\mathcal{V}_k) \longrightarrow K(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell)).$$

(2) *Let k be a subfield of $\mathbf{C}$, and let R be any commutative ring. The function from k-varieties to the Grothendieck ring of finitely generated R-modules*

$$X \mapsto \sum_i (-1)^i [H^i(X(\mathbf{C}); R)]$$

lifts to a map of K-theory spectra

$$\mathbf{E} \colon K(\mathcal{V}_k) \longrightarrow K(R).$$

We refer to $\boldsymbol{\zeta}$ as the *derived ℓ-adic zeta function* and to $\mathbf{E}$ as the *derived R-Euler characteristic*. These are homotopical enrichments of the standard zeta function and R-Euler characteristic. When k is a finite field, $\boldsymbol{\zeta}$ is exactly the lift of the zeta function.

Corollary 1.2. *Let $k = \mathbf{F}_q$ be a finite field and $\ell \nmid q$ a prime. Write $W(R)$ for the ring of (big) Witt vectors of the ring R. The classical zeta function of $\mathbf{F}_q$-varieties lifts to a map of spectra*

$$\boldsymbol{\zeta} \colon K(\mathcal{V}_{\mathbf{F}_q}) \longrightarrow K(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q); \mathbf{Z}_\ell)),$$

which fits into a commuting square

$$\begin{array}{ccc} K_0(\mathcal{V}_{\mathbf{F}_q}) & \xrightarrow{\pi_0 \boldsymbol{\zeta}} & K_0(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q); \mathbf{Z}_\ell)) \\ \downarrow{\scriptstyle Z(-,t)} & & \downarrow{\scriptstyle \det(1 - Frob_q^* t)} \\ W(\mathbf{Z}) & \longrightarrow & W(\mathbf{Z}_\ell) \end{array}$$

after applying π_0.

This answers the question (1) by lifting the classical zeta function to a derived invariant; for more details and the proof, see the end of Section 4.

For the question (2), one might begin by asking: are there nontrivial elements in $K_i(\mathcal{V}_k)$ for $i > 0$? One simple example of an SW-category is **FinSet**, the category of finite sets; by

REFERENCE: BARRATT–PRIDDY–QUILLEN the K-theory of this is equivalent to $\mathbb{S}$, the sphere spectrum. Note that for any variety X we can define a map

$$\sigma_X \colon \mathbb{S} \longrightarrow K(\mathcal{V}_k)$$

by thinking of $\mathbb{S}$ as $K(\mathbf{FinSet})$ and sending the finite set F to $\coprod_F X$. When k is a subfield of $\mathbf{C}$ this gives enough information to detect some of these nontrivial elements.

Theorem 1.3. *Let k be a subfield of $\mathbf{C}$. Then there are arbitrarily high non-trivial homotopy groups of $K(\mathcal{V}_k)$.*

The proof proceeds by tracing through the map $\pi_*\sigma_X$ and showing that in degrees $4s-1$ it is non-trivial. Thus in particular $K_{4s-1}(\mathcal{V}_k)$ is non-trivial. Using an elaboration of this proof could produce many more non-trivial homotopy groups. For example, from the multiplicative structure of $K(\mathcal{V}_k)$ and the inclusion $i\colon \mathrm{Aut}(X) \longrightarrow K_1(\mathcal{V}_k)$, we have, for any automorphism f of X, the composite

$$\pi_*\mathbb{S} \longrightarrow K_*(\mathcal{V}_k) \xrightarrow{\cdot i(f)} K_{*+1}(\mathcal{V}_k) \longrightarrow K_{*+1}(\mathbf{Z}).$$

We are hopeful that this gives a method for detecting other non-trivial homotopy groups of $K_*(\mathcal{V}_k)$, but leave it to future work. We note that this is an approach very similar to the one Bökstedt and Waldhausen [BW87] use to detect non-trivial homotopy groups in the algebraic K-theory of spaces, $A(*)$.

When k is finite we can obtain more refined information. For a finite field $k = \mathbf{F}_q$, the map $X \mapsto X(\mathbf{F}_q)$ defines a map $K(\mathcal{V}_{\mathbf{F}_q}) \longrightarrow \mathbb{S}$ which is a cosection of $\sigma_{\mathrm{Spec}\, k}$, yielding $K(\mathcal{V}_{\mathbf{F}_q}) \simeq \mathbb{S} \vee \widetilde{K}(\mathcal{V}_{\mathbf{F}_q})$. Thus an analogous statement to that of Theorem 1.3 can simply state that when k is finite, $\pi_*\mathbb{S}$ is a summand of $K_*(\mathcal{V}_k)$. Thus, a much more subtle question is:

Do there exist nontrivial elements in $\widetilde{K}_i(\mathcal{V}_k)$ for $i > 0$?

Note that this statement makes sense for any k, by defining $\widetilde{K}(\mathcal{V}_k) = \mathrm{hocofib}\, \sigma_{\mathrm{Spec}\, k}$.

We use the derived zeta function to answer this question affirmatively.

Theorem 1.4. *The group $\widetilde{K}_1(\mathcal{V}_k)$ is nontrivial whenever k is a subfield of $\mathbf{R}$, a finite field with $|k| \equiv 3 \pmod 4$, or a global or local field with a place of cardinality $3 \pmod 4$.*

To prove this theorem we use the derived 2-adic zeta function. For any category $\mathcal{C}$, let $\mathrm{Aut}(\mathcal{C})$ be the category of pairs (P, f), where $P \in \mathcal{C}$ and $f \in \mathrm{Aut}(P)$. Morphisms $(P, f) \longrightarrow (Q, g)$ are morphisms $h\colon P \longrightarrow Q$ such that $hf = gh$. When $\mathcal{C}$ is exact it induces an exact structure on $\mathrm{Aut}(\mathcal{C})$. Specializing from a representation of $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ to its value on $Frob_q$ gives a functor $\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q); \mathbf{Z}_\ell) \longrightarrow \mathrm{Aut}(\mathbf{Q}_\ell)$. Composing the derived zeta function with the K-theory of this functor and applying π_1, gives a homomorphism $K_1(\mathcal{V}_k) \longrightarrow K_1(\mathrm{Aut}(\mathbf{Q}_2))$. The group $K_1(\mathrm{Aut}(\mathbf{Q}_2))$ is relatively well-understood. In [Gra79], Grayson constructs a homomorphism $\sigma_2\colon K_1(\mathrm{Aut}(\mathbf{Q}_2)) \longrightarrow K_2(\mathbf{Q}_2)$. By Moore's Theorem (see e.g. [Mil71, Appendix]), the 2-adic Hilbert symbol induces a (split) surjection $(-,-)_2\colon K_2(\mathbf{Q}_2) \twoheadrightarrow \mathbf{Z}/2\mathbf{Z}$, and by composing these maps, we produce a map

$$h_2\colon K_1(\mathcal{V}_k) \longrightarrow K_1(\mathrm{Aut}(\mathbf{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbf{Q}_2) \xrightarrow{(-,-)_2} \mathbf{Z}/2\mathbf{Z}.$$

We then show $h_2 \circ \pi_1\sigma_{\mathrm{Spec}\, k}$ is trivial, but $h_2 \circ \pi_1\sigma_{\mathbb{P}^1}$ is surjective.

Corollary 1.2 and Theorem 1.4 are both applications of Theorem 1.1, in which $\boldsymbol{\zeta}$ is constructed. In order to construct $\boldsymbol{\zeta}$, we use a K-theory machinery first created by the first author in [Cam]. The usual categories one wants to work with as inputs for a K-theory machine are Waldhausen categories [Wal85]. Unfortunately, these do not work to produce $K(\mathcal{V}_k)$, which is why the first and third authors introduced their formalisms. In [Cam], the difficulty is circumvented by defining a modification of Waldhausen categories called SW-categories (the S is for "scissors") where one can define algebraic K-theory for $\mathcal{V}_k$ in much the same way one does for Waldhausen categories. However, in order to get maps $K(\mathcal{C}) \longrightarrow K(\mathcal{W})$ where $\mathcal{C}$ is an SW-category and $\mathcal{W}$ is a Waldhausen category, one needs the notion of a "W-exact functor" introduced in [Cam]. It needs to satisfy certain variance conditions reminiscent of push-pull formulae (see Section 2 for details). To construct the derived ℓ-adic zeta function, we take the SW-category $\mathcal{V}_k$ and the Waldhausen category $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell))$ of homologically finite and bounded chain complexes of Galois representations. To go from one to the other we need to use compactly-supported étale cohomology. Unfortunately, this is not sufficiently functorial; to resolve this, we construct a helper category which decorates each variety with a compactification. Once the compactification is chosen, compactly-supported étale cohomology is functorial and satisfies the axioms of a W-exact functor. Applying K-theory, we obtain the derived ℓ-adic zeta function.

We view this work as part of a larger program to lift motivic measures to the spectral/homotopical level. For example, the outline we follow should adapt to give lifts for other cohomologically defined motivic measures, e.g. p-adic zeta functions, Serre polynomials, and the Gillet–Soulé measure [GS96]. One might similarly ask for lifts of Kapranov's motivic zeta function, or of the motivic measure used by Larsen and Lunts [LL03] to show that motivic zeta function is *not* rational as a map out of $K_0(\mathcal{V}_{\mathbf{C}})$. See Section 7 for a more detailed discussion.

This paper is organized as follows. In Section 2, we quickly review Waldhausen K-theory and introduce SW-categories. In Section 3, we review the background and necessary results from ℓ-adic cohomology and Galois representations. In Section 4 we review our general approach to constructing derived motivic measures, and provide the the construction of the motivic measure arising from singular cohomology. Section 5 contains the full construction of the derived ℓ-adic zeta function. In Section 6 we use the results of the previous section to construct nontrivial elements in the higher K-theory of varieties over both $\mathbf{C}$ and finite fields. We close, in Section 7, by discussing questions for future work.

Acknowledgments. We thank Bhargav Bhatt, Denis-Charles Cisinski, Sean Howe, Keerthi Madapusi Pera and Nick Rozenblyum for helpful correspondence. We thank Oliver Braunling, Kiran Kedlaya, Dan Petersen, Ravi Vakil, Chuck Weibel, Kirsten Wickelgren and Ilya Zakharevich for many helpful questions and comments on an earlier draft. J.W. was supported in part by NSF Grant No. DMS-1400349. I.Z. was supported in part by an NSF MSPRF grant and NSF Grant No. DMS-1654522.

Notation 1.5. Throughout, when dealing with schemes or varieties, we let $Z \hookrightarrow Y$ denote a closed inclusion and $X \overset{\circ}{\to} Y$ denote an open inclusion.

2. SW-categories and K-Theory

In [Zak17a], the third author defines a spectrum $K(\mathcal{V}_k)$ whose zeroth homotopy group is the Grothendieck ring of varieties over k. In [Cam], the first author gives an alternate construction of this spectrum. In this paper, we use the latter construction to produce maps out of $K(\mathcal{V}_k)$, so we review the structure necessary to produce this spectrum.

Most definitions of K-theory work with categories where a suitable notion of quotient exists, for example Quillen's exact categories [Qui73] or Waldhausen's categories [Wal85]. These notions of quotient are then used to define the exact sequences that K-theory is defined to "split." When dealing with the category of varieties, we have no such quotients. Instead, our "exact sequences" are sequences of the form $Z \hookrightarrow X \overset{\circ}{\leftarrow} (X - Z)$ where the first map is a closed inclusion and the second is an open inclusion. The notion of an SW-category is meant to modify Waldhausen's definition of categories with cofibrations and weak equivalences to allow the use of such "exact sequences." For ease of reading, we review Waldhausen's construction before recalling the first author's construction.

Definition 2.1 (Waldhausen category, [Wal85, Section 1.2])**.** A *Waldhausen category*[1] is a category $\mathcal{C}$ equipped with two distinguished subcategories: cofibrations and weak equivalences, denoted $\mathbf{co}(\mathcal{C})$ and $\mathbf{w}(\mathcal{C})$. The arrows in $\mathbf{co}(\mathcal{C})$ are denoted by hooked arrows $\hookrightarrow$. Arrows representing weak equivalences are decorated with $\sim$. These categories satisfy the following axioms:

1. $\mathcal{C}$ has a zero object 0.
2. All isomorphisms are contained in $\mathbf{co}(\mathcal{C})$ and $\mathbf{w}(\mathcal{C})$.
3. For all objects A of $\mathcal{C}$, the morphism $0 \longrightarrow A$ is a cofibration.
4. (**pushouts**) For any diagram
$$C \longleftarrow A \hookrightarrow B$$
where $A \hookrightarrow B$ is a cofibration, the pushout exists and the morphism $C \hookrightarrow B \cup_A C$ is a cofibration.
5. (**gluing**) For any diagram
$$\begin{array}{ccccc} C & \longleftarrow & A & \hookrightarrow & B \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ C' & \longleftarrow & A' & \hookrightarrow & B' \end{array}$$
where the vertical morphisms are weak equivalences the induced morphism
$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$
is also a weak equivalence.

We will be using a slightly more general definition of the K-theory of a Waldhausen category, as we need this flexibility for our main results. For a more standard description of Waldhausen's K-theory, see for example [Rog92, Section 1].

We begin with some preliminary definitions.

Definition 2.2. Let $\mathcal{C}$ be a category. The category $\operatorname{Ar}\mathcal{C}$ has, as its objects, the morphisms of $\mathcal{C}$. The morphisms in $\operatorname{Ar}\mathcal{C}$ from a morphism $f\colon A \longrightarrow B$ to a morphism $g\colon C \longrightarrow D$ are commutative squares
$$\begin{array}{ccc} A & \longrightarrow & C \\ f\downarrow & & \downarrow g \\ B & \longrightarrow & D \end{array} .$$
If $\mathcal{C}$ is equipped with a subcategory of weak equivalences then a morphism in $\operatorname{Ar}\mathcal{C}$ is considered a weak equivalence if both horizontal morphisms in the diagram above are weak equivalences.

Let $[n]$ be the ordered set $\{0 < \ldots < n\}$ considered as a category. Then $\operatorname{Ar}[n]$ can be considered to be the set of pairs $(i,j) \in [n] \times [n]$ with $i \leq j$. We are often considering functors $X\colon \operatorname{Ar}[n] \longrightarrow \mathcal{C}$; in this case, we write $X_{i,j}$ for $X(i,j)$.

Definition 2.3. Let $\mathcal{C}$ be a Waldhausen category. A morphism in $\operatorname{Ar}\mathcal{C}$ is a *weak cofibration* if it is weakly equivalent (via a zigzag of weak equivalences) in $\operatorname{Ar}\mathcal{C}$ to a cofibration. A square in $\mathcal{C}$ is *homotopy cocartesian* if it is weakly equivalent (by a zigzag) to a pushout square where either both horizontal or both vertical morphisms are cofibrations.

Definition 2.4 ($S'_\bullet$-construction, see [BM11, Definition 2.3])**.** Let $\mathcal{C}$ be a Waldausen category. We define $S'_n\mathcal{C}$ to be the category of functors
$$X\colon \operatorname{Ar}[n] \longrightarrow \mathcal{C}$$
with morphisms natural transformations, subject to the conditions

[1]Referred to as a "category with cofibrations and weak equivalences" by Waldhausen.

- the initial map $0 \longrightarrow A_{i,i}$ is a weak equivalence for all i.
- When $i \le j \le k$, $X_{i,j} \longrightarrow X_{i,k}$ is a weak cofibration.
- For any $i \le j \le k$ the square

$$\begin{array}{ccc} X_{i,j} & \longrightarrow & X_{i,k} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{j,k} \end{array}$$

is a homotopy cocartesian square.

A map $A \longrightarrow B$ in $S'_n\mathcal{C}$ is a weak equivalence when each component map $A_{i,j} \longrightarrow B_{i,j}$ is a weak equivalence; it is a cofibration when each component map $A_{i,j} \longrightarrow B_{i,j}$ is a cofibration and the map $A_{i,k} \cup_{A_{i,j}} B_{i,j} \longrightarrow B_{i,k}$ is a weak cofibration.

Remark 2.5. The $S'_n\mathcal{C}$ assemble to form a simplicial category (i.e. a simplicial object in the category of small categories). For more detail on this, see [BM11, Section 2].

We now define the algebraic K-theory spectrum of a Waldhausen category. Unfortunately, the $S'_\bullet$-construction does not work correctly for all Waldhausen categories, but only those satisfying a condition Blumberg–Mandell call *functorial factorization of weak cofibrations (FFWC)*. All examples that we are concerned with satisfy this condition, but a discussion of the condition is not illuminating for the sake of the current discussion; we prove that this is the case in Appendix A and restrict our attention to categories satisfying FFWC.

Definition 2.6. Let $\mathcal{C}$ be a Waldhausen category satisfying FFWC. Let $wS'_n\mathcal{C}$ denote the subcategory of weak equivalences of $S'_n\mathcal{C}$ and let $N_\bullet wS'_n\mathcal{C}$ denote the nerve of that category. The topological space $K^t(\mathcal{C})$ is defined by

$$K^t(\mathcal{C}) = \Omega|N_\bullet wS'_\bullet\mathcal{C}|$$

where $|-|$ denotes the geometric realization of a bisimplicial set. The spectrum $K(\mathcal{C})$ is defined by taking a (functorial) fibrant-cofibrant replacement in the category of spectra of the spectrum whose m-th space is

$$|N_\bullet w \underbrace{S'_\bullet \cdots S'_\bullet}_{m \text{ times}} \mathcal{C}|.$$

The most important example of a Waldhausen category for the purposes of this paper is the following:

Example 2.7. Let $\mathcal{E}$ be any exact category. If we define the admissible monomorphisms to be the cofibrations and the isomorphisms to be the weak equivalences then the Waldhausen K-theory of $\mathcal{E}$ and Quillen's K-theory of $\mathcal{E}$ are equivalent. Let $\mathrm{Ch}^{\mathrm{str}}(\mathcal{E})$ be the category of bounded chain complexes in $\mathcal{E}$; by [TT90, Theorem 1.11.7], the incluson $\mathcal{E} \longrightarrow \mathrm{Ch}^{\mathrm{str}}(\mathcal{E})$ given by mapping $\mathcal{E}$ to the chain complexes concentrated at 0 is an equivalence on K-theory. The inverse map on K_0 is exactly the Euler characteristic.

However, we need a stronger version of this. Let R be a ring, and suppose that $\mathcal{E}$ is ${\mathrm{Mod}^{\mathrm{fg}}}_R$, the category of finitely generated R-modules. Let $\mathrm{Ch}^{\mathrm{b}}(R)$ be the category of chain complexes of (possibly infinitely generated) R-modules whose cohomology is bounded and in ${\mathrm{Mod}^{\mathrm{fg}}}_R$. We claim that the induced inclusion $K({\mathrm{Mod}^{\mathrm{fg}}}_R) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(R))$ is an equivalence.

We prove this by breaking the map up into three compositions:

$$K({\mathrm{Mod}^{\mathrm{fg}}}_R) \longrightarrow K(\mathrm{Ch}^{\mathrm{str}}(R)) \longrightarrow K(\mathrm{Ch}^{\mathrm{fb}}(R)) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(R)).$$

Here, $\mathrm{Ch}^{\mathrm{fb}}(R)$ is the category of chain complexes of finitely generated R-modules which are homologically bounded. The first of these is an equivalence by [TT90, Theorem 1.11.7]. The second is an equivalence by [Wei13, Section V.2.7.1]. Thus it remains to consider the third. We prove that this is an equivalence by using Waldhausen's Approximation Theorem, [Wal85, Theorem 1.6.7].

To apply the theorem we must show that for any map $f\colon A \longrightarrow B$, where $A \in \mathrm{Ch}^{\mathrm{fb}}(R)$ and $B \in \mathrm{Ch}^{\mathrm{b}}(R)$ there exists a cofibration $g\colon A \hookrightarrow A'$ in $\mathrm{Ch}^{\mathrm{fb}}(R)$ and a weak equivalence $f'\colon A' \longrightarrow B$ such that $f = f'g$. Note that (by possibly first factoring f as a cofibration followed by a weak equivalence) it suffices to check this when f is itself a cofibration; in particular f is levelwise injective. Suppose that the cohomology of B is nonzero below dimension k. We define $A'_m = 0$ if $m > k$. For $m \leq k$ we define A'_m to be the submodule of B_m generated by

- the image of A_m,
- a choice of generators for $H^m(B)$, and
- a choice of generators for the relations between the generators for $H^{m+1}(B)$.

(Since the cohomology is bounded we can just construct these starting at $m = k$ and working downwards.) Since each of these only involves a finite number of generators, A'_m is finitely generated. Thus A' is levelwise a subcomplex of B which has the same cohomology and satisfies the desired conditions. Thus the map $K(\mathrm{Ch}^{\mathrm{fb}}(R)) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(R))$ is a weak equivalence.

It is not necessarily the case that for all exact categories, $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})$ satisfies FFWC. In Appendix A we show that the specific cases we are interested in in this paper do; the results of this appendix also suggest that for almost all cases of $\mathcal{E}$ that are of interest FFWC holds.

We now turn to defining SW-categories. As much of the intuition necessary for working with these comes from Waldhausen's $S_\bullet$-construction we describe the structures in parallel language. However, we omit the "weakness" hypotheses, since we need to work inside SW-categories more strictly than in Waldhausen categories.

Definition 2.8 (SW-category [Cam, Definition 3.23])**.** An *SW-category* is a category $\mathcal{C}$ equipped with three distinguished subcategories: cofibrations, complement maps, and weak equivalences, denoted $\mathbf{co}(\mathcal{C})$, $\mathbf{comp}(\mathcal{C})$ and $\mathbf{w}(\mathcal{C})$. The arrows in $\mathbf{co}(\mathcal{C})$ are denoted by hooked arrows $\hookrightarrow$ and the arrows in $\mathbf{comp}(\mathcal{C})$ are denoted by $\overset{\circ}{\rightarrow}$. Arrows representing weak equivalences are decorated with $\sim$. The category $\mathcal{C}$ is further equipped with a collection of *subtraction sequences* $\{Z \hookrightarrow X \overset{\circ}{\leftarrow} U\}$.

All of this data is required to satisfy the following axioms.

(1) $\mathcal{C}$ has an initial object, denoted $\emptyset$. Furthermore, $\mathcal{C}$ has coproducts.
(2) For any $C, D \in \mathcal{C}$, the canonical map $C \longrightarrow C \amalg D$ is both a cofibration and a complement map.
(3) All isomorphisms are contained in $\mathbf{comp}(\mathcal{C})$, $\mathbf{co}(\mathcal{C})$, and $\mathbf{w}(\mathcal{C})$.
(4) The pullback of a complement map (resp. cofibration) along any map exists and is a complement map (resp. cofibration).
(5) Subtraction satisfies the following axioms:
    (a) Every cofibration $Z \hookrightarrow X$ participates in a subtraction sequence $Z \hookrightarrow X \overset{\circ}{\leftarrow} Y$ such that Y and $Y \longrightarrow X$ are unique up to unique isomorphism. We informally write $Y = X - Z$ in this case.
    (b) The pullback of a subtraction sequence

$$Z \hookrightarrow X \overset{\circ}{\leftarrow} U$$

along a map $W \longrightarrow X$ is again a subtraction sequence.

(c) Functoriality in cartesian squares: Given a cartesian diagram where every map is a cofibration

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

there is a unique map $X - W \longrightarrow Z - Y$.

(6) (**pushouts**) Given a diagram

$$\begin{tikzcd} W \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & X \arrow[d, dotted] \\ Y \arrow[r, dotted] & Z \end{tikzcd}$$

with both f and g cofibrations, a pushout Z exists and both dotted arrows are cofibrations. Furthermore, every diagram of this form is required to be a pullback.

(7) (**pushout products**) Given a pullback diagram

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

where all arrows are cofibrations, the map $X \amalg_W Y \longrightarrow Z$ is a cofibration.

(8) (**subtraction and pushouts**) Given a diagram

$$\begin{tikzcd} X' \arrow[d, hook] & W' \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, hook] \\ X & W \arrow[l, hook'] \arrow[r, hook] & Y \\ X'' \arrow[u, "\circ"] & W'' \arrow[l, hook'] \arrow[r, hook] \arrow[u, "\circ"] & Y'' \arrow[u, "\circ"] \end{tikzcd}$$

where the columns are subtraction sequences and upper left and upper right squares are pullback squares, then the pushouts along the rows form a subtraction sequence:

$$X' \amalg_{W'} Y \hookrightarrow X \amalg_W Y \overset{\circ}{\leftarrow} X'' \amalg_{W''} Y''$$

(9) (**gluing**) Given the diagram

$$\begin{tikzcd} Y \arrow[d, "\sim"] & X \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\sim"] & Z \arrow[d, "\sim"] \\ Y' & X' \arrow[l, hook'] \arrow[r, hook] & Z' \end{tikzcd}$$

where the vertical arrows are weak equivalences, there is an induced weak equivalence

$$Y \amalg_X Z \xrightarrow{\sim} Y' \amalg_{X'} Z'$$

(10) (**subtraction and weak equivalences**) Given a commuting square

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\sim"'] & Y \arrow[d, "\sim"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

there is a weak equivalence

$$Y - X \xrightarrow{\sim} Y' - X'.$$

making the diagram

$$\begin{array}{ccccc} X & \hookrightarrow & Y & \xleftarrow{\circ} & Y - X \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \hookrightarrow & Y' & \xleftarrow{\circ} & Y' - X' \end{array}$$

commute.

This may look like quite a bit of data. The first five axioms are meant to codify the notion of "subtraction" in an arbitrary category with cofibrations and complement maps standing in for closed and open embeddings in topological spaces. Axioms 6-10 are essentially technical in nature, providing the exact conditions under which K-theory may be defined.

There are many examples of these kinds of categories. The following is the motivating example.

Example 2.9. The category $\mathcal{V}_k$ of varieties over a field k, is an SW-category, where cofibrations are closed immersions, complements are open immersions, and the weak equivalences are isomorphisms. (This is proven in detail in the results leading up to [Cam, Prop. 3.28]) The subtraction sequences are defined as follows. Given a closed inclusion $i\colon Z \longrightarrow X$, i determines a homeomorphism of Z onto a closed set $i(Z)$. We consider the open set $X - i(Z)$ and give it a scheme structure by restricting the structure sheaf on X. Thus

$$X - Z = (X - i(Z), \mathcal{O}_X|_{X-Z}).$$

The definition of an SW-category is designed to provide exactly the structure needed to carry out a Waldhausen-style $S_\bullet$-construction when we have subtraction instead of quotients. However, we need one auxiliary definition.

Definition 2.10. We define $\widetilde{\mathrm{Ar}}[n]$ to be the full subcategory of $[n]^{\mathrm{op}} \times [n]$ consisting of pairs (i, j) with $i \leq j$.

Definition 2.11 ($\widetilde{S}_\bullet$-construction)**.** Let $\mathcal{C}$ be an SW-category. We define $\widetilde{S}_n\mathcal{C}$ to be the category with objects functors

$$X\colon \widetilde{\mathrm{Ar}}[n] \longrightarrow \mathcal{C}$$

with morphisms natural transformations, subject to the conditions

- $X_{i,i} = \emptyset$, the initial object
- When $j < k$, $X_{i,j} \longrightarrow X_{i,k}$ is a cofibration.
- The subdiagram

$$X_{i,j} \longrightarrow X_{i,k} \longleftarrow X_{j,k}$$

  is a subtraction sequence for all $i < j < k$.

Remark 2.12. The $\widetilde{S}_n\mathcal{C}$ assemble to form a simplicial category (i.e. a simplicial object in the category of small categories). Each of these is itself an SW-category, so this construction can be iterated. For details, see [Cam, **REFERENCE**].

We may finally define the algebraic K-theory spectrum of an SW-category.

Definition 2.13. Let $\mathcal{C}$ be an SW-category. Let $w\widetilde{S}_n\mathcal{C}$ denote the subcategory of weak equivalence of $\widetilde{S}_n\mathcal{C}$ and let $N_\bullet w\widetilde{S}_n\mathcal{C}$ denote the nerve of that category. The topological space $K^t(\mathcal{C})$ is defined by

$$K^t(\mathcal{C}) = \Omega|N_\bullet w\widetilde{S}_\bullet\mathcal{C}|$$

where $|-|$ denotes the geometric realization of a bisimplicial set. The spectrum $K(\mathcal{C})$ is defined by taking a (functorial) fibrant-cofibrant replacement in the category of spectra of the spectrum whose m-th space is

$$|N_\bullet w \underbrace{\widetilde{S}_\bullet \cdots \widetilde{S}_\bullet}_{m \text{ times}} \mathcal{C}|.$$

There is a notion of exact functors for SW-categories:

Definition 2.14. Let $\mathcal{C}, \mathcal{D}$ be SW-categories. A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ are called *exact* if

1. (1) F preserves the initial object: $F(\emptyset) = \emptyset$.
2. (2) F preserves subtraction sequences
3. (3) F preserves pushout diagrams.

Proposition 2.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be SW-categories and let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Then F descends to a map of spectra $K(\mathcal{C}) \longrightarrow K(\mathcal{D})$.*

Proof. This follows directly from the definition of exact functor. □

Most of the maps in which we are interested do not have SW-categories as codomains; instead, we wish to be able to construct a functor from an SW-category to a Waldhausen category. This requires we use a different definition in order to define the map of K-theories, since we cannot just hit the source and target with the $\widetilde{S}_\bullet$ construction or $S'_\bullet$-construction. In fact, because of the change in variance, the proper notion is not a functor at all — instead it is a pair of functors, one covariant and one contravariant. One should keep in mind here the dual of compactly supported cohomology, which is covariant on closed inclusions and contravariant on open. There is an additional functor that needed to keep track of weak equivalences, so, in all it is a triple of functors.

Definition 2.16 (Based on [Cam, Defn. 5.2])**.** Let $\mathcal{C}$ be an SW-category and $\mathcal{D}$ a Waldhausen category. A *weakly W-exact functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple of functors $(F_!, F^!, F^w)$ such that

1. (1) $F_!$ is a functor $F_!: \mathbf{co}(\mathcal{C}) \longrightarrow \mathcal{D}$ from the category of cofibrations in $\mathcal{C}$ to $\mathcal{D}$. For a map i we abbreviate $F_!(i)$ to $i_!$.
2. (2) $F^!$ a contravariant functor $F^!: \mathbf{comp}(\mathcal{C})^{\text{op}} \longrightarrow \mathcal{D}$, from the category of complement maps in $\mathcal{C}$ to $\mathcal{D}$. For a map j we abbreviate $F^!(j)$ to $j^!$.
3. (3) F^w is a functor $\mathbf{w}(\mathcal{C}) \longrightarrow \mathbf{w}(\mathcal{D})$.
4. (4) For objects $X \in \mathcal{C}$, $F^!(X) = F_!(X) = F^w(X)$ and we denote all three by $F(X)$.
5. (5) For every *cartesian* diagram in $\mathcal{C}$

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Y & \underset{j'}{\hookrightarrow} & W \end{array}$$

where the horizontal maps are cofibrations and the vertical maps are complements, we obtain a commuting diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{j_!} & F(Z) \\ {\scriptstyle i^!}\uparrow & & \uparrow{\scriptstyle (i')^!} \\ F(Y) & \xrightarrow{j'_!} & F(W) \end{array} .$$

(6) For a subtraction sequence in $\mathcal{C}$

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X - Z$$

the square

$$\begin{array}{ccc} F(X) & \xrightarrow{i_!} & F(Y) \\ \downarrow & & \downarrow j^! \\ F(0) & \longrightarrow & F(Y-X) \end{array}$$

is a weakly cocartesian square in $\mathcal{D}$.

(7) For any commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & Y \\ i_X \downarrow & & \downarrow i_Y \\ X' & \underset{f'}{\hookrightarrow} & Y' \end{array}$$

where the horizontal morphisms are cofibrations and the vertical morphisms are weak equivalences, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{f_!} & F(Y) \\ F^w(i_X) \downarrow & & \downarrow F^w(i_Y) \\ F(X') & \xrightarrow[(f')_!]{} & F(Y') \end{array}$$

A similar statement holds for complement maps.

Remark 2.17. For notational ease, we denote W-exact functors by $(F_!, F^!, F^w): \mathcal{C} \longrightarrow \mathcal{D}$ or even $F: \mathcal{C} \longrightarrow \mathcal{D}$, when no confusion can arise.

Having defined this, one can prove the following.

Proposition 2.18 (Based on [Cam, Prop. 5.3])**.** *Let $\mathcal{C}$ be an SW-category and $\mathcal{D}$ a Waldhausen category with FFWC and let $(F_!, F^!, F^w): \mathcal{C} \longrightarrow \mathcal{D}$ be a weakly W-exact functor. Then $(F_!, F^!, F^w)$ determines a spectrum map*

$$K(\mathcal{C}) \xrightarrow{F} K(\mathcal{D}).$$

Proof. It suffices to prove that this map exists before taking fibrant-cofibrant replacement, since our replacement is functorial. But this follows exactly from the definitions of $K(\mathcal{C})$ and $K(\mathcal{D})$, since a weakly W-exact functor takes a simplex in the spaces defining $K(\mathcal{C})$ to a simplex in the spaces defining $K(\mathcal{D})$. □

As a consequence of the definition of K-theory, we obtain the following result, which can be used to pick out interesting elements of K_1 of an SW-category. The proof is just as for Waldhausen K-theory as in [Wal85, §1.5].

Proposition 2.19. *Let X be an object in an SW-category (resp. Waldhausen category) $\mathcal{C}$. There is a homomorphism $\xi_X: \mathrm{Aut}(X) \longrightarrow K_1(\mathcal{C})$, which is natural in $\mathcal{C}$ in the sense that for any exact (resp. weakly W-exact) functor $F: \mathcal{C} \longrightarrow \mathcal{D}$, $\xi_{F(X)} = \pi_1 F \circ \xi_X$.*

Proof. We begin by recalling how this statement works for Waldhausen categories (see [Wal85, p.341]). Given an automorphism $f\colon X\longrightarrow X$ there is a corresponding 1-simplex in $w\mathcal{C}$. Composing with the map $|w\mathcal{C}|\longrightarrow\Omega|wS_\bullet\mathcal{C}|$ gives the desired map $\mathrm{Aut}(X)\longrightarrow K_1\mathcal{C}$. This construction is visibly natural for exact functors $F\colon\mathcal{C}\longrightarrow\mathcal{D}$, and uses only that automorphisms include into $\pi_1|w\mathcal{C}|$ and that we have $|w\mathcal{C}|\longrightarrow|wS'_\bullet\mathcal{C}|$. As the same is true for the $S'_\bullet$-construction, an analogous statement holds.

For SW-categories $\mathcal{C}$, the construction works the same way since the simplicial set $w\widetilde{S}_\bullet\mathcal{C}$ is still reduced and isomorphisms are a subcategory of weak equivalences.

For a weakly W-exact functor, $F=(F_!,F^!,F^w)$ from an SW-category $\mathcal{C}$ to a Waldhausen category $\mathcal{D}$ we obtain a simplicial map $w\widetilde{S}_\bullet\mathcal{C}\longrightarrow w\widetilde{S}'_\bullet\mathcal{D}$. For $X\in\mathcal{C}$, there are maps $\mathrm{Aut}(X)\longrightarrow\mathrm{Aut}(F(X))$, and $|w\mathcal{C}|\longrightarrow|w\mathcal{D}|$ induced by F^w. It is then clear that the diagram

$$\begin{array}{ccccc}
\mathrm{Aut}(X) & \longrightarrow & \pi_1 w\mathcal{C} & \longrightarrow & \pi_1(\Omega|w\widetilde{S}_\bullet\mathcal{C}|)\\
\downarrow & & \downarrow & & \downarrow\\
\mathrm{Aut}\,F(X) & \longrightarrow & \pi_1 w\mathcal{D} & \longrightarrow & \pi_1(\Omega|w\widetilde{S}'_\bullet\mathcal{D}|)
\end{array}$$

commutes, which is the statement of the proposition. □

An important tool in the general method discussed in Section 4 is a lemma designed to identify when the K-theories of two SW-categories are equivalent. Although the conditions of this lemma look complicated, in the geometric cases we are interested in they are generally very natural. For a detailed example of an application, see Example 2.23.

Lemma 2.20. *Let $U\colon\mathcal{A}\longrightarrow\mathcal{C}$ be an exact functor of SW-categories. Suppose that the following extra conditions hold:*

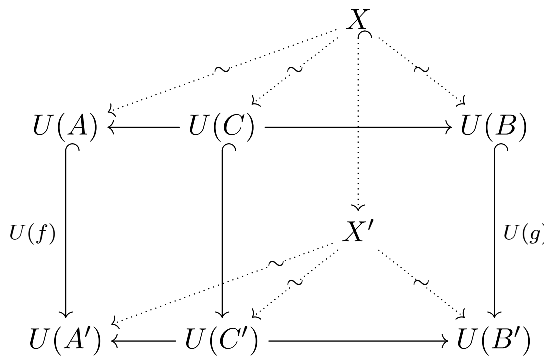
(1) *U is surjective on objects. Moreover, one of the following holds:*
   (a) *For all cofibrations $f\colon X\hookrightarrow X'$, if $U(A')=X'$ then there exists a cofibration $A\hookrightarrow A'$ whose image under U is f.*
   (b) *For all cofibrations $f\colon X\hookrightarrow X'$, if $U(A)=X$ then there exists a cofibration $A\hookrightarrow A'$ whose image under U is f.*
(2) *If for $f,g\colon A\longrightarrow B$ in $\widetilde{S}_n\mathcal{A}$, there exists a weak equivalence $h\colon X\longrightarrow U(A)$ in $\widetilde{S}_n\mathcal{C}$ such that $U(f)h=U(g)h$, then there exists a weak equivalence $\widetilde{h}\colon Z\longrightarrow A$ in $\widetilde{S}_n\mathcal{A}$ such that h factors through $U(\widetilde{h})$ and such that $f\widetilde{h}=g\widetilde{h}$.*
(3) *Given any diagram*

$$\begin{array}{ccccc}
U(A) & \xleftarrow{\sim} & X & \xrightarrow{\sim} & U(B)\\
{\scriptstyle U(f)}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle U(g)}\\
U(A') & \xleftarrow{\sim} & X' & \xrightarrow{\sim} & U(B')
\end{array}$$

there exists a diagram

$$\begin{array}{ccccc}
A & \longleftarrow & C & \longrightarrow & B\\
{\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle g}\\
A' & \longleftarrow & C' & \longrightarrow & B'
\end{array}$$

in $\mathcal{A}$ making the diagram



commute.

Then $K(U)$ is an equivalence.

Proof. It suffices to prove that, for $n \geq 0$, the map $w\widetilde{S}_n(\mathcal{A}) \longrightarrow w\widetilde{S}_n(\mathcal{C})$ induces a weak equivalence on geometric realizations. By Quillen's Theorem A [Qui73, Theorem A], it suffices to show that for any $\alpha \in w\widetilde{S}_n(\mathcal{C})$, the undercategory $\alpha/w\widetilde{S}_n(\mathcal{A})$ is cofiltering. Recall that α is represented by a diagram

$$X_1 \hookrightarrow \cdots \hookrightarrow X_n \ .$$

For this, we first observe that by condition (1) the category $\alpha/w\widetilde{S}_n(\mathcal{A})$ is non-empty. We use the second version of (1); the proof for the first works analogously. Since U is surjective on cofibrations by (1), there exists a cofibration $A_1 \longrightarrow A_2$ that maps to $X_1 \longrightarrow X_2$. Now, since $U(A_2) = X_2$, there exists a cofibration $A_2 \longrightarrow A_3$ whose image under U is $X_2 \longrightarrow X_3$. Working inductively, we see that there is a sequence $A_1 \longrightarrow \cdots \longrightarrow A_n$ of cofibrations whose image under U is α. To show that $\alpha/w\widetilde{S}_n(\mathcal{A})$ is nonempty we take this sequence and the identity map from α.

We now show that $\alpha/w\widetilde{S}_n(\mathcal{A})$ is co-filtering. We start by showing that any two parallel arrows are equalized by a third. An object $A \in \alpha/w\widetilde{S}_n(\mathcal{A})$ is a diagram

$$\begin{array}{ccc} X_1 & \hookrightarrow \cdots \hookrightarrow & X_n \\ \sim\downarrow & & \downarrow\sim \\ U(A_1) & \hookrightarrow \cdots \hookrightarrow & U(A_n) \end{array} \ .$$

Given two such objects A and B, and two morphisms $f, g\colon A \longrightarrow B$, we must show that there exists a map $h\colon Z \longrightarrow A$ in $\alpha/w\widetilde{S}_n(\mathcal{A})$ such that $fh = gh$. This is exactly guaranteed by condition (2).

It remains to show that for every pair of objects in $\alpha/w\widetilde{S}_n(\mathcal{A})$, there exists a third object which maps into each of them. This is guaranteed by condition (3). □

Our main example is the SW-category of varieties together with a choice of compactification; we show that forgetting the choices induces an equivalence on K-theory.

Definition 2.21. Let k be a field. We define the SW-category $\mathcal{V}_k^{cptd}$ as follows. Objects of $\mathcal{V}_k^{cptd}$ are open embeddings $X \overset{\circ}{\to} \overline{X}$ where X is a k-variety and $\overline{X}$ is a proper k-variety. Morphisms

$(X \overset{\circ}{\to} \overline{X}) \longrightarrow (Y \overset{\circ}{\to} \overline{Y})$ are commuting squares

$$\begin{array}{ccc} X & \xrightarrow{\circ} & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xrightarrow{\circ} & \overline{Y} \end{array} .$$

A morphism $(X \overset{\circ}{\to} \overline{X}) \longrightarrow (Y \overset{\circ}{\to} \overline{Y})$ is a

cofibration: if f and $\overline{f}$ are closed embeddings,
complement: if f is an open embedding and $\overline{f}$ is a closed embedding, and
weak equivalence: if f is an isomorphism.

A sequence $(Z \overset{\circ}{\to} \overline{Z}) \longrightarrow (X \overset{\circ}{\to} \overline{X}) \overset{\circ}{\leftarrow} (U \overset{\circ}{\to} \overline{U})$ is a subtraction sequence if the left map is a cofibration, the right map is a complement, $Z \hookrightarrow X \overset{\circ}{\leftarrow} U$ is a subtraction sequence in $\mathcal{V}_k$, and the closed embedding $\overline{U} \longrightarrow \overline{X}$ has set-theoretic image equal to the closure of $\overline{X} - \overline{Z}$ in $\overline{X}$.

Lemma 2.22. *The category $\mathcal{V}_k^{cptd}$ with cofibrations, complements, weak equivalences, and subtraction defined as above satisfies the axioms of an SW-category.*

Proof. We verify the axioms in turn. Note that limits and colimits are computed pointwise in $\mathcal{V}_k^{cptd}$. As a result, Axiom 1 is automatically satisfied. Axiom 2 follows immediately because for any varieties X, Y, the embedding $X \longrightarrow X \coprod Y$ is both open and closed. Axiom 3 is also immediate, because isomorphisms in $\mathcal{V}_k^{cptd}$ are pointwise, and any isomorphism of varieties is simultaneously an open and closed embedding. Axiom 4 holds for the same reason as in $\mathcal{V}_k$. We now verify the remaining axioms in turn.

(5a) Let $(Z \overset{\circ}{\to} \overline{Z}) \hookrightarrow (X \overset{\circ}{\to} \overline{X})$ be a cofibration. Then $Z \hookrightarrow X$ determines the open embedding $X - Z \hookrightarrow X$ up to unique isomorphism, and similarly, $\overline{Z} \hookrightarrow \overline{X}$ determines the closed embedding $\overline{\overline{X} - \overline{Z}} \longrightarrow \overline{X}$ up to unique isomorphism.

(5b) Given a diagram

$$\begin{array}{ccccc} & & (W, \overline{W}) & & \\ & & \downarrow & & \\ (Z, \overline{Z}) & \hookrightarrow & (X, \overline{X}) & \overset{\circ}{\longleftarrow} & (U, \overline{U}) \end{array}$$

where the bottom row is a subtraction sequence, we have a subtraction sequence

$$W \times_X Z \hookrightarrow W \overset{\circ}{\longleftarrow} W \times_X Y$$

in $\mathcal{V}_k$. Moreover, because taking closure commutes with pullback, we have

$$\begin{aligned} \overline{W} \times_{\overline{X}} \overline{U} &\cong \overline{W} \times_{\overline{X}} \overline{\overline{X} - \overline{Z}} \\ &= \overline{\overline{W} \times_{\overline{X}} (\overline{X} - \overline{Z})} \end{aligned}$$

where the outer $\overline{(-)}$ denotes closure in $\overline{W}$. Further, because the pullback of the complement of a subvariety is the complement of its pullback, this is isomorphic to

$$= \overline{\overline{W} - \overline{W} \times_{\overline{X}} \overline{Z}}.$$

Therefore, $(W \times_X Z, \overline{W} \times_{\overline{X}} \overline{Z}) \hookrightarrow (W, \overline{W}) \overset{\circ}{\longleftarrow} (W \times_X U, \overline{W} \times_{\overline{X}} \overline{U})$ is a subtraction sequence.

(5c) Given a Cartesian square

$$\begin{array}{ccc} (W,\overline{W}) & \hookrightarrow & (X,\overline{X}) \\ \downarrow & & \downarrow \\ (Y,\overline{Y}) & \hookrightarrow & (Z,\overline{Z}) \end{array}$$

in which all maps are cofibrations, the unique map $X-W \hookrightarrow Z-Y$ is an open embedding. Further, there is a unique map $\overline{X}-\overline{W}\longrightarrow\overline{Z}-\overline{Y}$ and, because $\overline{X}\longrightarrow\overline{Z}$ is a closed emebdding, this extends uniquely to a closed embedding

$$\overline{\overline{X}-\overline{W}}\longrightarrow\overline{\overline{Z}-\overline{Y}}.$$

Axioms (6), (7), and (8) hold for the same reason they hold in $\mathcal{V}_k$, namely [Sch05, Corollary 3.9] and the remarks following. Axioms (9) and (10) follow immediately from the definition of weak equivalences in $\mathcal{V}_k^{cptd}$ along with Axioms 6 and 5a respectively. □

Example 2.23. Let $\mathcal{C} = \mathcal{V}_k$ and let $\mathcal{A} = \mathcal{V}_k^{cptd}$, with U being the forgetful functor. We claim that the conditions of Lemma 2.20 hold. We check the conditions in turn:

(1) By Nagata [Nag62, Theorem 4.3][2] every k-variety X admits an open embedding $X \overset{\circ}{\to} \overline{X}$ into a proper k-variety $\overline{X}$. Thus U is surjective on objects. Moreover, for any cofibration $X \hookrightarrow X'$, if $\overline{X}'$ is a compactification of X' then the closure of X in $\overline{X}'$ is a compactification of X. Thus condition (a) holds.

(2) Weak equivalences in $\mathcal{V}_k$ are isomorphisms. In this case this condition says that given two parallel maps f and g in $\widetilde{S}_n\mathcal{V}_k^{cptd}$

$$\begin{array}{ccccc} (X_1,\overline{X}_1) & \hookrightarrow \cdots \hookrightarrow & (X_n,\overline{X}_n) \\ \downarrow & & \downarrow \\ (Y_1,\overline{Y}_1) & \hookrightarrow \cdots \hookrightarrow & (Y_n,\overline{Y}_n) \end{array}$$

such that f and g are equal when restricted to X_i, then there exists a weak equivalence h in $\widetilde{S}_n\mathcal{V}_k^{cptd}$

$$\begin{array}{ccccc} (X_1,\overline{\overline{X}}_1) & \hookrightarrow \cdots \hookrightarrow & (X_n,\overline{\overline{X}}_n) \\ \downarrow & & \downarrow \\ (X_1,\overline{X}_1) & \hookrightarrow \cdots \hookrightarrow & (X_n,\overline{X}_n) \end{array}$$

such that $fh = gh$. But for this, we can take $\overline{\overline{X}}_i$ to be the closure of X_i in $\overline{X}_i$, and h to be given by the canonical inclusions. Then, for each i, the ith components of fh and gh agree on a dense set in $\overline{\overline{X}}_i$, and thus are equal. Therefore $fh = gh$.

(3) This property states that given a closed embedding of varieties $X \hookrightarrow X'$ together with induced embeddings of two choices of compactification $(X,\overline{X}) \hookrightarrow (X',\overline{X'})$ and $(X,\overline{\overline{X}}) \hookrightarrow (X',\overline{\overline{X'}})$, there exists a third choice of embeddings of compactifications that dominates both of these. Note that if we can find a compactification of X' that dominates both $\overline{X'}$ and $\overline{\overline{X'}}$ then we can take the closure of X in it to produce the desired embedding. Thus all that we must show is that for any two choices of compactification, there is a third that

[2] For a modern treatment of Nagata's Theorem, see [Con07], esp. Theorem 4.1.

dominates them. For example, we can take the closure of X inside $\overline{X} \times \overline{X}'$, as in [Del77, §IV-10.5]).

3. Preliminaries on ℓ-adic Cohomology

In this section we recall standard facts that we need about ℓ-adic cohomology with its continuous Galois action. We take [Del77] and [FK80] as standard references.

3.1. **Continuous Galois Representations.** Let k be a field, let k^s be a separable closure, and let $\ell \neq \mathrm{char}(k)$ be a prime. The separable Galois group $\mathrm{Gal}(k^s/k)$ is a profinite group, and canonically carries the profinite topology

$$\mathrm{Gal}(k^s/k) = \varprojlim_{L/k \text{ fin.,sep.}} \mathrm{Gal}(L/k)$$

where the finite groups $\mathrm{Gal}(L/k)$ are discrete, and the limit is in the category of topological groups. Let R be a ring. Recall that for a (discrete) R-module A, a *continuous representation* $\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Aut}_R(A)$ is one which factors through a finite subgroup $\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Gal}(L/k) \longrightarrow \mathrm{Aut}_R(A)$ for some separable L/k. Denote by $\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); R)$ the category of finitely generated continuous representations of $\mathrm{Gal}(k^s/k)$ over R. Recall the following (cf. [Del77, Arcata II.4.4]).

Proposition 3.1. *Let k be a field, and k^s a separable closure. Let R be a ring. Denote by* $\mathrm{Sh}(\mathrm{Spec}(k); R)$ *the category of étale sheaves of (discrete) finitely generated R-modules on* $\mathrm{Spec}(k)$*. The functor*

$$\mathrm{Sh}(\mathrm{Spec}(k); R) \longrightarrow \mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); R)$$
$$F \mapsto F|_{\mathrm{Spec}(k^s)}$$

is an equivalence of categories.

We are especially interested in continuous ℓ-adic representations. Recall the following reformulation of the category of finitely generated $\mathbf{Z}_\ell$-modules (we follow the presentation of [FK80, Ch I.12]). We consider projective diagrams

$$F \colon \mathbf{Z}_{\geq} \longrightarrow \mathrm{Mod}(\mathbf{Z}_\ell),$$

where $\mathbf{Z}_{\geq}$ is the category whose objects are integers and where there is a unique morphism $m \longrightarrow n$ whenever $m \geq n$. The category of diagrams is an abelian category, and in particular has images,kernels, cokernels, etc. defined pointwise. For $r \in \mathbf{Z}$, denote by $F[r]$ the shifted diagram, i.e. with $F[r]_m \coloneqq F_{r+m}$.

Definition 3.2. A projective diagram $F \colon \mathbf{Z}_{\geq} \longrightarrow \mathrm{Mod}(\mathbf{Z}_\ell)$ satisfies:

(1) the *Mittag–Leffler* (*ML*) condition if for every n, there exists $t \geq n$ such that for all $m \geq t$,
$$\mathrm{Image}(F_m \longrightarrow F_n) = \mathrm{Image}(F_t \longrightarrow F_n),$$
(2) the *Mittag–Leffler–Artin–Rees* (*MLAR*) condition if there exists some $t \geq 0$ such that for all $r \geq t$,
$$\mathrm{Image}(F[r] \longrightarrow F) = \mathrm{Image}(F[t] \longrightarrow F).$$

Definition 3.3. Define $\mathrm{Pro}_{MLAR}(\mathbf{Z}_\ell)$ to be the procategory of MLAR projective systems in which each F_n is torsion. Concretely, for two such systems F and G,

$$\hom_{\mathrm{Pro}_{MLAR}(\mathbf{Z}_\ell)}(F, G) \coloneqq \varinjlim_{r \geq 0} \hom(F[r], G).$$

The key purpose of Mittag–Leffler systems is that on such systems, the inverse limit is an exact functor. The ML property is frequently satisfied, for instance, if all the modules F_n are finite length, then F is an ML system.

Definition 3.4. An *ℓ-adic* system is a projective diagram F such that $F_n = 0$ for $n < 0$ and for all n

(1) F_n is a module of finite length,
(2) $\ell^{n+1} F_n = 0$, and
(3) the map $F_{n+1} \longrightarrow F_n$ induces an isomorphism

$$F_{n+1}/\ell^{n+1} F_{n+1} \cong F_n.$$

An *A–R ℓ-adic* system is any object of $\mathrm{Pro}_{MLAR}(\mathbf{Z}_\ell)$ which is isomorphic to an ℓ-adic system. Denote by $\mathrm{AR}(\ell) \subset \mathrm{Pro}_{MLAR}(\mathbf{Z}_\ell)$ the full sub-category of A–R-ℓ-adic systems.

We can now give the promised reformulation of the category $\mathrm{Mod}_f(\mathbf{Z}_\ell)$ of finitely generated $\mathbf{Z}_\ell$-modules (cf. e.g. [FK80, Proposition I.12.4])

Proposition 3.5. *The inverse limit*

$$\varprojlim\colon \mathrm{AR}(\ell) \longrightarrow \mathrm{Mod}_f(\mathbf{Z}_\ell)$$

is an equivalence of exact categories.

We can use the proposition to give a similar reformulation of the category $\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell)$. Recall that $\mathbf{Z}_\ell$ is a profinite ring, with the profinite (equivalently "adic" topology). Similarly, for any finitely generated $\mathbf{Z}_\ell$-module A, the group $\mathrm{Aut}_{\mathbf{Z}_\ell}(A)$ is canonically a topological group, with the profinite topology. Mutatis mutandis, we obtain from Definitions 3.3 and 3.4 a notion of ℓ-adic systems of continuous $\mathrm{Gal}(k^s/k)$ representations, and a category $\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell)$ of such. Concretely, objects are given by projective systems

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots$$

where for each n, F_n is a continuous representation of $\mathrm{Gal}(k^s/k)$ in finitely generated $\mathbf{Z}/\ell^n\mathbf{Z}$-modules and the analogous conditions to those of Definition 3.4 hold. Analogously to Proposition 3.5, we have the following.

Proposition 3.6. *The inverse limit*

$$\varprojlim\colon \mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell) \longrightarrow \mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell)$$

is an equivalence of exact categories.

3.2. **ℓ-adic Sheaves.** We now extend the above to sheaves. We consider schemes X for which $\ell \in \mathcal{O}(X)$ is invertible. Mutatis mutandis, we obtain from Definition 3.3 a definition of MLAR projective systems of ℓ-torsion étale sheaves, and the category $\mathrm{Pro}_{MLAR}(\mathrm{Sh}(X), \mathbf{Z}_\ell)$ of such.

Definition 3.7. Let X be a scheme and ℓ a prime invertible on X. An *ℓ-adic sheaf* on X is projective system $F\colon \mathbf{Z}_{\geq} \longrightarrow \mathrm{Sh}(X; \mathbf{Z})$ of étale sheaves of abelian groups on X such that

(1) the sheaves F_n are constructible for all n,
(2) $F(n) = 0$ for $n < 0$, and
(3) the map $F_{n+1} \longrightarrow F_n$ induces isomorphisms

$$F_{n+1} \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong F_n.$$

An *A–R-ℓ-adic sheaf* is any object of $\mathrm{Pro}_{MLAR}(\mathrm{Sh}(X), \mathbf{Z}_\ell)$ which is isomorphic to an ℓ-adic sheaf. We denote the category of A–R-ℓ-adic sheaves by $\mathrm{Sh}(X; \mathbf{Z}_\ell)$.

The following is the key theorem we use for ℓ-adic sheaves (cf. [FK80, Theorem I.12.15]).

Theorem 3.8 (Finiteness Theorem for ℓ-adic Sheaves)**.** *Let $f\colon X \longrightarrow S$ be a compactifiable mapping, ℓ a prime invertible on X, and $n \mapsto F_n$ an A–R-ℓ-adic sheaf on X. Then for all $\nu \geq 0$, the system $n \mapsto R^\nu f_!(F_n)$ is an A–R-ℓ-adic sheaf on S.*

Using the smooth base change theorem (cf. e.g. [FK80, Theorem I.7.3]) and Propositions 3.1 and 3.5 above, we immediately deduce the following.

Corollary 3.9. *Let k be a field, k^s a separable closure, and let X be a variety of finite type over k. Then for any A–R-ℓ-adic sheaf F on X, a choice of compactification $j\colon X \hookrightarrow \overline{X}$ and a choice of functorial flasque resolution $Q_{\overline{X}}$ on $\mathrm{Sh}(\overline{X};\mathbf{Z}_\ell)$ determine an extension of the assignment*

$$F \mapsto H^*_c(X_{/k^s}; F)$$

to a functor

$$\begin{aligned}\mathrm{Sh}(X;\mathbf{Z}_\ell) &\longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}_\ell))\\ F &\mapsto \Gamma_{\overline{X}} Q_{\overline{X}}(j_!F)_{|_{k^s}}\end{aligned}$$

4. The general approach

In this section we discuss our general approach for constructing derived motivic measures. The main difficulty in constructing a motivic measure directly is that there are *choices* involved in the construction. For example, consider the measure

$$\chi_{et}\colon X \longmapsto \sum (-1)^i [H^i_c(X,\mathbf{Q}_\ell)] \in K_0(\mathbf{Q}_\ell).$$

When taking ℓ-adic cohomology with compact supports, it is necessary to choose a compactification of X. While this choice does not affect the resulting cohomology, it does mean that the construction is not *functorial*, and thus does not lift directly to K-theory.

The general method for getting around this problem is to use an auxillary category $\mathcal{A}$, which includes all necessary choices, and then construct the measure out of that. To resolve the alternting sum (which is clearly not a well-defined construct in the category of $\mathbf{Q}_\ell$-modules) we take chain complexes. As discussed in Example 2.7, the inclusion of an exact category as the chain complexes concentrated at 0 is an equivalence, with the "Euler characteristic" as the inverse. Thus we produce a diagram

$$\mathcal{V}_k \xleftarrow{U} \mathcal{A} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathbf{Q}_\ell)^{\mathrm{op}} \longleftarrow \mathbf{Q}_\ell\textbf{-Mod}^{\mathrm{op}}.$$

Applying K-theory to this diagram produces a diagram in which all of the backwards-facing maps are weak equivalences, and thus invertible (up to homotopy). Inverting these gives the desired map

$$K(\mathcal{V}_k) \longrightarrow K(\mathbf{Q}_\ell).$$

Remark 4.1. We had two notational questions in this paper pulling us in opposite directions for the question of variance. Exact categories are by definition symmetric, but Waldhausen categories and SW-categories are not. However, the categories of chain complexes of modules are biWaldhausen, since they can be equipped with model category structures with quasiisomorphisms as weak equivalences; thus we could choose either variance for the right-hand side of our functor. However, on the left-hand side of the functor we had a difficulty.

On one hand, the standard terminology for exact functors is covariant on cofibrations and admissible monics. Thus, since cohomology with compact supports is contravariant on closed embeddings, to be consistent we had to declare the codomain to be $\mathrm{Ch}(R)^{\mathrm{op}}$.

On the other hand, cohomology is by definition contravariant. Thus we could have instead defined W-exact functors to be contravariant on cofibrations, instead of complement maps, and thus mapped into $\mathrm{Ch}(R)$.

Given that we wanted the development of SW-categories to fit nicely into the current literature on K-theory, we chose the first option. However, the second option is equally valid, and could have been achieved relatively easily via a small change in the axioms of a W-exact functor. Neither choice, of course, changes the underlying reality: compactly supported cohomology is covariant on open embeddings and contravariant on closed embeddings.

The main example that we use in this paper is $\mathcal{A} = \mathcal{V}_k^{cptd}$, defined in Definition 2.21. The functor U simply forgets the choice of compactification.

We can use $\mathcal{V}_k^{cptd}$ to construct two different motivic measures:

Theorem 4.2.

(1) *Let k be a field and k^s a separable closure of k. Let $\ell \neq \mathrm{char}(k)$ be a prime, and denote by $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell))$ the category of homologically bounded chain complexes of continuous representations of $\mathrm{Gal}(k^s/k)$ over $\mathbf{Z}_\ell$ with finitely generated cohomology. There exists a weakly W-exact functor*

$$F\colon \mathcal{V}_k^{cptd} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}_\ell))^{op}$$

such that for any object $(X, \overline{X})$ of $\mathcal{V}_k^{cptd}$,

$$H^*(F(X, \overline{X})) \cong \mathrm{Gal}(k^s/k) \circlearrowleft H^*_{et,c}(X \times_k k^s; \mathbf{Z}_\ell).$$

Here, the Galois group acts naturally on k^s, and the map induced by a (cofibration or complement) map $(X, \overline{X}) \longrightarrow (Y, \overline{Y})$ is the natural map

$$H^*_{et,c}(X \times_k k^s; \mathbf{Z}_\ell) \longleftrightarrow H^*_{et,c}(Y \times_k k^s; \mathbf{Z}_\ell)$$

(with the direction for the map chosen to have the correct variance depending on whether $(X, \overline{X}) \longrightarrow (Y, \overline{Y})$ was a cofibration or complement map).

(2) *Let k be a subfield of $\mathbf{C}$, and R a commutative ring. Let $\mathrm{Ch}^{\mathrm{b}}(R)$ be the category of homologically bounded chain complexes of R-modules with finitely generated cohomology. There exists a weakly W-exact functor*

$$G\colon \mathcal{V}_k^{cptd} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(R)^{op}$$

such that for any object $(X, \overline{X})$ of $\mathcal{V}_k^{cptd}$,

$$H^*(G(X, \overline{X})) \cong H^*_c(X(\mathbf{C}); R).$$

Proof. The proof of part (1) is long and technical, so we defer it to Section 5. We now prove part (2).

In the interest of conciseness, for the rest of this proof instead of writing $X(\mathbf{C})$ we abuse notation and write X. We also suppress the coefficients on cohomology, which are always be taken to be R.

We define the functor G as follows. Let

$$G(X, \overline{X}) = C^*_{\mathrm{sing}}(\overline{X}, \overline{X} - X),$$

be the chain complex of singular cochains with coefficients in R on the topological space $\overline{X}$ which are zero outside X. The functor $G^!\colon \mathbf{co}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(R)^{\mathrm{op}}$ sends a closed embedding $(Z, \overline{Z}) \longrightarrow (X, \overline{X})$ to the usual pullback on cohomology. Note that this is well-defined, since (as Z is closed in X) $\overline{Z} - Z \subseteq \overline{X} - X$. The functor $G_!\colon \mathbf{comp}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(R)^{\mathrm{op}}$ sends an open embedding $(U, \overline{U}) \longrightarrow (X, \overline{X})$ to the map which extends a cochain on $\overline{U}$ to a cochain on $\overline{X}$ by defining it to be 0 on all cochains which are not contained in $\overline{U}$. The functor $G^w\colon \mathbf{w}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(R)^{\mathrm{op}}$ is defined similarly, with the analogous observation that a weak equivalence $(X, \overline{X}) \longrightarrow (X, \overline{\overline{X}})$ produces a map of pairs $(\overline{X}, \overline{X} - X) \longrightarrow (\overline{\overline{X}}, \overline{\overline{X}} - X)$.

If G is a weakly W-exact functor then it satisfies the conditions in the statement of the theorem, so it remains to prove that it is actually weakly W-exact. Conditions (1)-(4) hold by definition. We check the others in turn.

(5) By the definitions of the maps, it suffices to check that the diagram commutes on the compactification components. Thus the statement we need to check is that for any cartesian diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{j} & \overline{Z} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i'} \\ \overline{Y} & \xrightarrow{j'} & \overline{W} \end{array}$$

where all maps are closed embeddings, $(i')^! \circ j'_! = j_! \circ i^!$. In this case, both of the compositions around the diagram take a cochain $\alpha\colon C_*(\overline{Y}, \overline{Y} - Y) \longrightarrow R$ to the cochain $\alpha'\colon C_*(\overline{Z}, \overline{Z} - Z) \longrightarrow R$ which takes a chain $\sigma\colon \Delta^n \longrightarrow \overline{Z}$ to 0 if σ doesn't factor through $\overline{X}$, and to $\alpha(i \circ \sigma)$ if it does. Thus the diagram commutes.

(6) Suppose that we are given a subtraction sequence

$$(Z,\overline{Z}) \xhookrightarrow{\ i\ } (X,\overline{X}) \xleftarrow{\ j\ } (U,\overline{U}) \ .$$

Note that we have the following commutative diagram of pairs of spaces:

$$\begin{array}{ccccc} (\overline{U},\overline{U}-U) & & & & \\ \big\downarrow & & & & \\ (\overline{X},\overline{X}-U) & \longleftarrow & (\overline{X},\overline{X}-X) & \longleftarrow & (\overline{X}-U,\overline{X}-X) \\ & & & \nwarrow & \big\uparrow \\ & & & & (\overline{Z},\overline{Z}-Z) \end{array}$$

Applying C^* gives an exact sequence of cochain complexes across the middle, as it is the sequence associated to the triple $(\overline{X}, \overline{X}-U, \overline{X}-X)$. By excision, all vertical isomorphisms become quasi-isomorphisms on cochains. Thus we have the diagram

$$\begin{array}{ccccc} C^*(\overline{U},\overline{U}-U) & & & & \\ \sim\big\downarrow\big\uparrow & \searrow^{i_!} & & & \\ C^*(\overline{X},\overline{X}-U) & \longrightarrow & C^*(\overline{X},\overline{X}-X) & \longrightarrow & C^*(\overline{X}-U,\overline{X}-X) \\ & & & \searrow_{j^!} & \big\downarrow\sim \\ & & & & (\overline{Z},\overline{Z}-Z) \end{array}$$

where the dotted arrow is the result of applying C^*, and the curved arrow is the quasi-inverse that extends chains by 0. Both of the solid arrow triangles commute. Thus the sequence

$$G(U,\overline{U}) \xrightarrow{i_!} G(X,\overline{X}) \xrightarrow{j^!} G(Z,\overline{Z})$$

is weakly equivalent to an exact sequence, and thus is weakly exact.

(7) The condition for complement maps holds because G^w and $G^!$ are given by the same functor. The condition for cofibrations holds because in both cases we extend a cochain on $\overline{X'}$ to a cochain on $\overline{Y}$ which is zero outside of X'; the only difference is that applying the maps in one direction extends the cochain to $\overline{Y'}$ first, and then to $\overline{Y}$, and in the other direction it extends to $\overline{X}$ first and then to $\overline{Y}$.

□

Assuming Theorem 4.2(1), we can now prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We prove the first part of the theorem, which follows from Theorem 4.2(1). The second part of the theorem follows analogously from Theorem 4.2(2).

Consider the following diagram, where the two categories on the left are the SW-categories of varieties and varieties with compactifications, and the two categories on the right are the Waldhausen categories of continuous Galois representations and homologically finite and bounded chain complexes thereof:

$$\begin{array}{ccc}
 & & \mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}_\ell)^{\mathrm{op}} \\
 & & \downarrow{\scriptstyle \cdot[0]} \\
\mathcal{V}_k^{cptd} & \xrightarrow{F} & \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}_\ell))^{\mathrm{op}} \\
\downarrow{\scriptstyle U} & & \\
\mathcal{V}_k & &
\end{array}$$

Here, the functor F is the functor constructed in Theorem 4.2(1). By Example 2.23 and Example 2.7, after applying K-theory the two vertical functors become equivalences. Thus after applying K-theory we get the following diagram of spectra:

$$\begin{array}{ccc}
 & & K(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}_\ell)) \\
 & & {\scriptstyle \cdot[0]}\downarrow\uparrow{\scriptstyle \chi} \\
K(\mathcal{V}_k^{cptd}) & \xrightarrow{K(F)} & K(\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}_\ell))) \\
{\scriptstyle K(U)}\downarrow\uparrow{\scriptstyle h} & & \\
K(\mathcal{V}_k) & &
\end{array}$$

Here, χ is the inverse equivalence to $\cdot[0]$ corresponding to the Euler characteristic and h is the inverse equivalence to $K(U)$. Both of these exist and are well-defined and unique up to a contractible space of choices since all of the spectra are fibrant and cofibrant. We define

$$\boldsymbol{\zeta} := \chi \circ K(F) \circ h.$$

A similar argument gives

$$\mathbf{E} := \chi \circ K(G) \circ h$$

where now χ is the inverse to the weak equivalence $K(\mathrm{Mod}^{\mathrm{fg}}{}_R) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(R))$. □

We refer to $\boldsymbol{\zeta}$ as the *derived ℓ-adic zeta function* and $\mathbf{E}$ as the *derived R-Euler characteristic* Almkvist [Alm78] and Grayson [Gra79] imply the following corollary.

Corollary 4.3. *Let k be a field, k^s a separable closure, $\ell \neq \mathrm{char}(k)$ a prime, and $g \in \mathrm{Gal}(k^s/k)$ any element. Then the assignment*

$$X \mapsto g \circlearrowleft H^*_{et,c}(X \times_k k^s;\mathbf{Z}_\ell)$$

lifts to a map of K-theory spectra

$$g_* \circ \boldsymbol{\zeta} \colon K(\mathcal{V}_k) \longrightarrow K(\mathrm{Aut}(\mathbf{Z}_\ell)).$$

Passing to K_0 and taking the characteristic polynomial, we recover the "g-Zeta-function"

$$\begin{aligned} Z_g \colon K_0(\mathcal{V}_k) &\longrightarrow W(\mathbf{Z}_\ell) \\ [X] &\mapsto \det(1 - g^*t; H^*_{et,c}(X \times_k k^s;\mathbf{Z}_\ell)). \end{aligned}$$

Taking $k = \mathbf{F}_q$ and g to be Frobenius, the "g-Zeta-function" is precisely the classical zeta function (cf. e.g. the discussion on p. 171-174 of [FK80]); in this special case Corollary 4.3 is exactly Corollary 1.2.

5. Proof of Theorem 4.2(1)

By Proposition 3.6, it suffices to construct a W-exact functor

$$F\colon \mathcal{V}_k^{cptd} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}^{\mathrm{AR}}(\mathrm{Gal}(k^s/k);\ell))^{\mathrm{op}} \tag{5.1}$$

such that the cohomology of the chain complex produces compactly supported ℓ-adic cohomology (with the Galois action induced by the action on k^s)

Remark 5.2.

(1) We prove in this section that the functor which assigns compactly supported cochains, defined via the Godement resolution, to the constant ℓ-adic sheaf $\mathbf{Z}_\ell$ takes subtraction sequences of k-varieties to exact sequences of homologically bounded chain complexes of sheaves on $\mathrm{Spec}(k)$.

(2) Our proof applies to the constant sheaf $\mathbf{Z}_\ell$, considered as a uniform system of sheaves on all k-varieties. The key property we use is that for $f\colon Y \longrightarrow X$, $f^*\mathbf{Z}_{\ell,X} \cong \mathbf{Z}_{\ell,Y}$. Our proof does not apply to any collection of sheaves on k-varieties for which this identity ever fails. In particular, we do **not** prove that there is a chain level realization of any cohomology functor which is exact on the category of *all* sheaves.

(3) Our proof exploits a key difference between sheaf cohomology and singular cohomology of topological spaces. Namely, given a disjoint decomposition of a space $X = Z \cup (X - Z)$, it is not the case that a singular chain is either contained in Z or in $X - Z$. By contrast, every point of X is contained in one piece of the decomposition or the other. The Godement resolution is defined purely in terms of the points of the space; this is what allows our construction of compactly supported cochains to give an exact functor.

We fix once and for all the Godement resolution as our functorial flasque resolution for étale sheaves over a variety X,

$$Q_X\colon \mathcal{F} \longmapsto \mathcal{G}^\bullet\mathcal{F}.$$

Notation 5.3. In this section, we make the following shorthand definitions.

A^n_X**:** denotes, for a k-variety X, the sheaf $(\mathbf{Z}/\ell^n\mathbf{Z})_{X|_{k^s}}$.

S_X**:** denotes the functor $\Gamma_X Q_X$. Note that this functor lands in the category of chain complexes, *not* in the derived category, which is why we avoid the standard notation $R\Gamma$.

Following the discussion in Section 2, we proceed by defining functors $F^w_n, F^n_!$ and $F^!_n$ taking values in $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}/\ell^n\mathbf{Z}))$. We then show these fit together to define an A–R-ℓ-adic system of continuous Galois representations. As all of the $\mathrm{Gal}(k^s/k)$-actions follow from the action on k^s, we omit these from the notation, but the implication should be that all functors record this data as well.

We begin by stating a technical lemma whose results we use throughout this section.

Lemma 5.4. *Let X be a k-variety, let $\iota\colon U \hookrightarrow X$ be the inclusion of an open subvariety. Let $j\colon X \longrightarrow Y$ be a proper map for which $j \circ \iota$ is an embedding. Then for all sheaves $\mathcal{F}$ on U, there is an induced isomorphism (of chain complexes of sheaves on* $\mathrm{Spec}(k)$*)*

$$j_\Gamma\colon S_X(\iota_!\mathcal{F}) \xrightarrow{\cong} S_Y(j_!\iota_!\mathcal{F}).$$

This isomorphism is natural in the following two senses:

(1) *For any map of sheaves $g\colon \mathcal{F} \longrightarrow \mathcal{G}$ on U, the diagram*

$$\begin{array}{ccc} S_X(\iota_!\mathcal{F}) & \xrightarrow{j_\Gamma} & S_Y(j_!\iota_!\mathcal{F}) \\ \downarrow{\scriptstyle S_X\iota_!(g)} & & \downarrow{\scriptstyle S_Y j_!\iota_!(g)} \\ S_X(\iota_!\mathcal{G}) & \xrightarrow{j_\Gamma} & S_Y(j_!\iota_!\mathcal{G}) \end{array}$$

commutes.

(2) *Given a commuting diagram*

$$\begin{array}{ccccc} U & \xrightarrow{(j_1)|_U} & V & & \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \\ X & \xrightarrow{j_1} & Y & \xrightarrow{j_2} & Z \end{array}$$

where ι_i is an open embedding and j_i is a proper map such that $j_i \circ \iota_i$ is an embedding for $i = 1, 2$, and $j_1(U) \subset V$, then for any sheaf $\mathcal{F}$ on U,

$$(j_2)_\Gamma \circ (j_1)_\Gamma = (j_2 \circ j_1)_\Gamma.$$

Proof. Since j is proper, $j_! = j_*$, so it suffices to prove the statement of the lemma with j_* substituted for $j_!$.

Recall that the Godement resolution $\mathcal{G}^\bullet(\mathcal{F})$ of an étale sheaf $\mathcal{F}$ on a k-variety X is defined inductively as follows (cf. e.g. [FK80, p. 129]). Fix algebraic closures $\Omega_0 := \overline{k}$ and $\Omega_n := \overline{k(t_1, \ldots, t_n)}$ for all n. Let M_X be the set of geometric points

$$x\colon \operatorname{Spec}(\Omega_n) \longrightarrow X \qquad (n \in \mathbf{N})$$

Note that every point in X is associated to at least one geometric point in M_X (this is what allows us to define the Godement resolution with respect to this set of points of the étale site of X). Define

$$\mathcal{G}^0_X(\mathcal{F}) := \prod_{x \in M_X} x_* x^* \mathcal{F}$$

Equivalently, $\mathcal{G}^0_X(\mathcal{F}) = \delta_{X*}\delta_X^*\mathcal{F}$ where $\delta_X\colon \coprod_{M_X} x \longrightarrow X$ is the inclusion of points.

The functor $\mathcal{G}^0_X(-)$ is exact (it is exact on stalks), and comes with a diagonal inclusion

$$\mathcal{F} \longrightarrow \mathcal{G}^0_X(\mathcal{F})$$

given by mapping a section to its stalks. Define

$$\mathcal{G}^1_X(\mathcal{F}) := \mathcal{G}^0_X(\operatorname{coker}(\mathcal{F} \longrightarrow \mathcal{G}^0_X(\mathcal{F}))).$$

In general, define

$$\mathcal{G}^{i+1}_X(\mathcal{F}) := \mathcal{G}^0_X(\operatorname{coker}(\mathcal{G}^{i-1}_X(\mathcal{F}) \longrightarrow \mathcal{G}^i_X(\mathcal{F})))$$

Then the assignment $\mathcal{F} \mapsto \mathcal{G}^\bullet_X(\mathcal{F})$ has the following properties:

(1) it defines an exact functor $Q_X\colon \operatorname{Sh}(X) \longrightarrow \operatorname{Ch}^{\mathrm{b}}(\operatorname{Sh}(X))$,
(2) the sheaves $\mathcal{G}^i(\mathcal{F})$ are flabby for all i,
(3) the map $\mathcal{F} \longrightarrow \mathcal{G}^\bullet(\mathcal{F})$ is a resolution.

In particular, the functor $S_X := \Gamma_X \circ Q_X\colon \operatorname{Sh}(X) \longrightarrow \operatorname{Ch}^{\mathrm{b}}(\operatorname{Sh}(\operatorname{Spec}(k)))$ is exact.

Now let $\iota\colon U \longrightarrow X$ be an open embedding, let $\mathcal{F}$ be a sheaf on U, and let $j\colon X \longrightarrow Y$ be a proper map such that $j\iota\colon U \longrightarrow Y$ is an embedding. We claim that there is a canonical isomorphism (of chain complexes in $\operatorname{Sh}(Y)$)

$$j_*\mathcal{G}^\bullet_X(\iota_!\mathcal{F}) \xrightarrow{\cong} \mathcal{G}^\bullet_Y(j_*\iota_!\mathcal{F}).$$

Granting the claim, we obtain j_Γ by applying Γ_Y to this isomorphism and pre-composing with the natural isomorphism $\Gamma_X(-) \cong \Gamma_Y \circ j_*(-)$.

The key observation underpinning the claim is that any map of k-varieties $f\colon W \longrightarrow Z$ determines a map of sets $f\colon M_W \longrightarrow M_Z$ (just by post-composing f with each map $x\colon \operatorname{Spec}(\Omega_n) \longrightarrow W$, and thus a commuting diagram (of non-finite type varieties)

$$\begin{array}{ccc} \coprod_{M_W} w & \xrightarrow{\delta_f} & \coprod_{M_Z} z \\ \downarrow{\scriptstyle \delta_W} & & \downarrow{\scriptstyle \delta_Z} \\ W & \xrightarrow{f} & Z \end{array}$$

If f is an open embedding, then by the definition of $f_!$, we have a canonical "base change" isomorphism

$$\delta_Z^* f_! \mathcal{F} \cong \delta_{f*} \delta_W^* \mathcal{F}.$$

Since f is an embedding, δ_f is an open embedding as well, and is also closed because the varieties in question are discrete. Therefore $\delta_{f!} = \delta_{f*}$.

We are now ready to prove the claim by induction. For the base of the induction,

$$\begin{aligned} j_* \mathcal{G}_X^0(\iota_! \mathcal{F}) &:= j_* \delta_{X*} \delta_X^* \iota_! \mathcal{F} \\ &\cong j_* \delta_{X*} \delta_{\iota *} \delta_U^* \mathcal{F} && \text{(because } \iota \text{ is an embedding)} \\ &\cong \delta_{Y*} \delta_{j*} \delta_{\iota *} \delta_U^* \mathcal{F} && \text{(by the functoriality of } (-)_*) \end{aligned}$$

Finally, because $j \circ \iota$ is an embedding, by inspection of the definitions, we see that there is a canonical isomorphism

$$\delta_{j*} \delta_{\iota *} \delta_U^* \mathcal{F} \cong \delta_Y^* j_* \iota_! \mathcal{F}$$

(i.e. $y_* y^* j_* \iota_! \mathcal{F} = y_* y^* \mathcal{F}$ for $y \in M_U \subset M_Y$ and $y_* y^* j_* \iota_! \mathcal{F} = 0$ otherwise.) We conclude that

$$j_* \mathcal{G}_X^0(\iota_! \mathcal{F}) \cong \delta_{Y*} \delta_Y^* j_* \iota_! \mathcal{F} =: \mathcal{G}_Y^0(j_* \iota_! \mathcal{F}).$$

This settles the base of the induction. But, for the induction step, note that in the argument above, we have

$$j_* \mathcal{G}_X^0(\iota_! \mathcal{F}) \cong j_* \delta_{X*} \delta_{\iota *} \delta_U^* \mathcal{F} \cong j_* \iota_! \delta_{U*} \delta_U^* \mathcal{F}$$

where the second isomorphism follows by inspection from the definition of $\iota_!$. This implies that

$$\mathcal{G}_X^1(\iota_! \mathcal{F}) := \operatorname{coker}(\iota_! \mathcal{F} \longrightarrow \iota_! \delta_{U*} \delta_U^* \mathcal{F}) \cong \iota_! \mathcal{G}_U^1(\mathcal{F}).$$

In particular, $\mathcal{G}_X^1(\iota_! \mathcal{F})$ is again an extension by 0 of a sheaf on U. Because j is proper, j_* preserves colimits, so

$$j_* \mathcal{G}_X^1(\iota_! \mathcal{F}) \cong \mathcal{G}_Y^1(j_* \iota_! \mathcal{F})$$

and we can now apply the same argument as above to conclude that there exists a canonical isomorphism

$$j_* \mathcal{G}_X^i(\iota_! \mathcal{F}) \cong \mathcal{G}_Y^i(j_* \iota_! \mathcal{F})$$

for all i, thus proving the claim.

It remains to show the two naturality properties. The naturality with respect to morphisms of sheaves $g\colon \mathcal{F} \longrightarrow \mathcal{G}$ follows immediately from the functoriality of the constructions above.

For the second naturality property, a direct inspection of the construction above shows that the naturality property follows from the properness of the maps j_i and the universal properties of cokernels and products (using that $\delta_{X*} \delta_X^* \mathcal{F} = \prod_{x \in M_X} \mathcal{F}_x$). Concretely, the naturality follows by writing out the explicit definitions of the maps and sheaves on $\operatorname{Spec}(k)$ and then repeatedly using that if $I \xrightarrow{\iota} J \xrightarrow{j} K$ are maps of sets such that $\iota\colon I \longrightarrow J$ and $j\iota\colon I \longrightarrow K$ are injective, then for any sheaf of abelian groups A on the discrete space I, there are canonical isomorphisms

$$\begin{aligned}\prod_I A_i \cong \prod_I A_i \times \prod_{J-I} 0 \cong \prod_J (\iota_! A)_j \\ \cong \prod_I A_i \times \prod_{K-I} 0 \cong \prod_K ((j\iota)_! A)_k.\end{aligned}$$

□

We turn our attention to constructing the functor

$$F_n^!\colon \mathbf{comp}(\mathcal{V}_k^{cptd})^{\mathrm{op}} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \mathbf{Z}/\ell^n\mathbf{Z}))^{\mathrm{op}}.$$

On objects, it is given by

$$F_n^!(\gamma_X\colon X \overset{\circ}{\to} \overline{X}) \coloneqq S_{\overline{X}}\gamma_{X!}A_X^n. \tag{5.5}$$

By definition, a complement map j in $\mathcal{V}_k^{cptd}$ consists of a commuting square

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ {\scriptstyle \gamma_U}\downarrow & & \downarrow{\scriptstyle \gamma_X} \\ \overline{U} & \xrightarrow{\overline{j}} & \overline{X} \end{array}$$

where j is an open embedding and $\overline{j}$ is a closed embedding. Given such, we obtain a map

$$F_n^! j\colon S_{\overline{U}}\gamma_{U!}A_U^n \longrightarrow S_{\overline{X}}\gamma_{X!}A_X^n$$

via the following composition:

$$\begin{aligned} S_{\overline{U}}\gamma_{U!}A_U^n \xrightarrow{\overline{j}_\Gamma} S_{\overline{X}}\overline{j}_!\gamma_{U!}A_U^n \xrightarrow{\cong} S_{\overline{X}}\gamma_{X!}j_!A_U^n \\ \xrightarrow{\cong} S_{\overline{X}}\gamma_{X!}j_!j^*A_X^n \xrightarrow{S_{\overline{X}}\gamma_{X!}(\epsilon)} S_{\overline{X}}\gamma_{X!}A_X^n. \end{aligned}$$

The first map exists by Lemma 5.4. For the last morphism, ϵ is the counit of the adjunction $j_! \dashv j^*$ (which exists because j is an open embedding). The two isomorphisms come from the canonical identification $\overline{j}_!\gamma_{U!} = \gamma_{X!}j_!$ and $A_U^n \cong j^*A_X^n$.

Lemma 5.6. *The assignment $j \mapsto F_n^! j$ is functorial on* $\mathbf{comp}(\mathcal{V}_k^{cptd})^{op}$.

Proof. The definition immediately implies that identities are mapped to identities, so we only need to check that composition is respected. For this, note that a composable pair of cofibrations in $\mathcal{V}_k^{cptd}$ consists of a commuting diagram

$$\begin{array}{ccccc} U & \xrightarrow{j_1} & X & \xrightarrow{j_2} & Y \\ {\scriptstyle \gamma_U}\downarrow & & \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma_Y} \\ \overline{U} & \xrightarrow{\overline{j}_1} & \overline{X} & \xrightarrow{\overline{j}_2} & \overline{Y} \end{array}$$

in which the maps j_i are open embeddings and the maps $\overline{j_i}$ are closed embeddings. The above diagram yields the following commutative diagram:

$$\begin{array}{ccccc}
S_{\overline{U}}\gamma_{U!}A^n_U & \searrow^{(\bar{j}_2\bar{j}_1)_\Gamma} & & & \\
\downarrow{\scriptstyle \bar{j}_{1\Gamma}} & & & & \\
S_{\overline{X}}(\bar{j}_{1!}\gamma_{U!}A^n_U & \xrightarrow{\bar{j}_{2\Gamma}} & S_{\overline{Y}}(\bar{j}_2\bar{j}_1)_!\gamma_{U!}A^n_U & \xrightarrow{\cong} & S_{\overline{Y}}(\bar{j}_2\bar{j}_1)_!\gamma_{U!}(j_2j_1)^*A^n_Y \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
S_{\overline{X}}\gamma_{X!}j_{1!}j_1^*A^n_X & \xrightarrow{\bar{j}_{2\Gamma}} & S_{\overline{Y}}\bar{j}_{2!}\gamma_{X!}j_{1!}j_1^*A^n_X & & S_{\overline{Y}}\gamma_{Y!}(j_2j_1)_!(j_2j_1)^*A^n_Y \\
\downarrow{\scriptstyle S_{\overline{X}}\gamma_{X!}\epsilon_1} & & \downarrow{\scriptstyle S_{\overline{Y}}\bar{j}_{2!}\gamma_{X!}\epsilon_1} & & \downarrow{\scriptstyle S_{\overline{Y}}\gamma_{Y!}(\epsilon_{21})} \\
S_{\overline{X}}\gamma_{X!}A^n_X & \xrightarrow{\bar{j}_{2\Gamma}} & S_{\overline{Y}}\bar{j}_{2!}\gamma_{X!}A^n_X & & \downarrow \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow \\
 & & S_{\overline{Y}}\gamma_{Y!}j_{2!}j_2^*A^n_Y & \xrightarrow{S_{\overline{Y}}\gamma_{Y!}\epsilon_2} & S_{\overline{Y}}\gamma_{Y!}A^n_Y
\end{array}$$

The left-hand side of the diagram commutes by the naturality properties of $\bar{j}_{2\Gamma}$ (by Lemma 5.4); the right-hand side of the diagram commutes because it commutes in $\mathrm{Sh}(\overline{Y}\times_k k^s)$ before applying $S_{\overline{Y}}$. The composition around the bottom is the map $F_n^!(j_2)\circ F_n^!(j_1)$, and the composition around the top is the map $F_n^!(j_2\circ j_1)$. Since the diagram commutes, $F_n^!$ is functorial. ☐

We now define the functor

$$F^n_!\colon \mathbf{co}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\mathbf{Z}/\ell^n\mathbf{Z}))^{\mathrm{op}}.$$

On objects, it is equal to $F_n^!$ (see (5.5)). By definition, a cofibration i in $\mathcal{V}_k^{cptd}$ consists of a commuting square

$$\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
{\scriptstyle\gamma_Z}\downarrow & & \downarrow{\scriptstyle\gamma_X} \\
\overline{Z} & \xrightarrow{\bar{i}} & \overline{X}
\end{array}$$

where the horizontal maps are closed embeddings. Given such, we obtain a map

$$F^n_! i\colon S_{\overline{X}}\gamma_{X!}A^n_X \longrightarrow S_{\overline{Z}}\gamma_{Z!}A^n_Z$$

using the composition of morphisms

$$\begin{aligned}
S_{\overline{X}}\gamma_{X!}A^n_X \xrightarrow{S_{\overline{X}}\gamma_{X!}(\eta)} S_{\overline{X}}\gamma_{X!}i_*i^*A^n_X \xrightarrow{\cong} S_{\overline{X}}\gamma_{X!}i_!A^n_Z \\
\xrightarrow{\cong} S_{\overline{X}}\bar{i}_!\gamma_{Z!}A^n_Z \xrightarrow{\bar{i}_\Gamma^{-1}} S_{\overline{Z}}\gamma_{Z!}A^n_Z.
\end{aligned}$$

Here, η is the unit of the adjunction $i^*\dashv i_*$, and the last two isomorphisms come from the canonical identifications $i_! = i_*$ (because i is proper), $\gamma_{X!}i_! = \bar{i}_!\gamma_{Z!}$ (by functoriality of !) and $A^n_Z \cong i^*A^n_X$. The last map is the inverse of the isomorphism of Lemma 5.4.

Lemma 5.7. *The functor $F^n_!$ is well-defined.*

Proof. The definition immediately implies that identities are mapped to identities, so we only need to check that composition is respected. A composable pair of complement maps in $\mathcal{V}_k^{cptd}$

consists of a commuting diagram

$$\begin{array}{ccccc} W & \xrightarrow{i_2} & Z & \xrightarrow{i_1} & X \\ {\scriptstyle \gamma_W}\downarrow & & {\scriptstyle \gamma_Z}\downarrow & & \downarrow{\scriptstyle \gamma_X} \\ \overline{W} & \xrightarrow{\bar{i}_2} & \overline{Z} & \xrightarrow{\bar{i}_1} & \overline{X} \end{array}$$

in which the horizontal maps are closed embeddings. Given this diagram, we have the following diagram:

$$\begin{array}{ccccc}
S_{\overline{X}}\gamma_{X!}A^n_X & \xrightarrow{\;S_{\overline{X}}\gamma_{X!}\eta_{12}\;}\!\!\searrow & & & \\
{\scriptstyle S_{\overline{X}}\gamma_{X!}\eta_1}\downarrow & & & & \\
S_{\overline{X}}\gamma_{X!}i_{1*}i_1^*A^n_X & \xrightarrow{S_{\overline{X}}\gamma_{X!}i_{1*}\eta_2} & S_{\overline{X}}\gamma_{X!}(i_1i_2)_*(i_1i_2)^*A^n_X & \xrightarrow{\cong} & S_{\overline{X}}\gamma_{X!}(i_1i_2)_!A^n_W \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} & & \\
S_{\overline{X}}\bar{i}_{1!}\gamma_{Z!}A^n_Z & \xrightarrow{S_{\overline{X}}\bar{i}_{1!}\gamma_{Z!}\eta_2} & S_{\overline{X}}\bar{i}_{1!}\gamma_{Z!}i_{2*}i_2^*A^n_Z & & \downarrow{\scriptstyle \cong} \\
{\scriptstyle \bar{i}_{1\Gamma}}\downarrow & & \downarrow{\scriptstyle \cong} & & \\
S_{\overline{Z}}\gamma_{Z!}A^n_Z & & S_{\overline{X}}\bar{i}_{1!}\gamma_{Z!}i_{2!}A^n_W & \xrightarrow{\cong} & S_{\overline{X}}(\bar{i}_1\bar{i}_2)_!\gamma_{W!}A^n_W \\
{\scriptstyle S_{\overline{Z}}\gamma_{Z!}\eta_2}\downarrow & & {\scriptstyle \bar{i}_{1\Gamma}^{-1}}\downarrow & & \downarrow{\scriptstyle (\bar{i}_1\bar{i}_2)_\Gamma^{-1}} \\
S_{\overline{Z}}\gamma_{Z!}i_{2*}i_2^*A^n_Z & \xrightarrow{\cong} & S_{\overline{Z}}\bar{i}_{2!}\gamma_{W!}A^n_W & \xrightarrow{\bar{i}_{2\Gamma}^{-1}} & S_{\overline{W}}\gamma_{W!}A^n_W
\end{array}$$

Here, η_a is the unit of the adjunction $i_a^* \dashv (i_a)_*$ for $a = 1, 2$ and η_{12} is the unit for the adjuction $(i_1i_2)^* \dashv (i_1i_2)_*$. The composition around the top is $F^n_!(i_1i_2)$; the composition around the bottom is $F^n_!(i_2) \circ F^n_!(i_1)$. The diagram commutes by the naturality of η_1,η_2,η_{12} and by Lemma 5.4. Thus $F^n_!$ is a functor. □

It now remains to construct F^w_n. We define it using the same formula as for $F^!_n$. Note that as the proof of Lemma 5.7 only used the results of Lemma 5.4, the proof works analogously to show that F^w_n is well-defined.

Lemma 5.8. *The collection of functors* $\{(F^!_n, F^n_!, F^w_n)\}$ *defines a* W*-exact functor*

$$F \colon \mathcal{V}_k^{cptd} \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell))^{op}.$$

Proof. The maps $F^!_{n+1} \longrightarrow F^!_{n+1} \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \longrightarrow F^!_n$ and similarly for $F^n_!$ and F^w_n endow the collections $\{F^!_n\}$, $\{F^n_!\}$ and $\{F^w_n\}$ with the structure of projective systems. From the construction, for each fixed X, $\{F^!_n(X)\}$, $\{F^n_!(X)\}$ and $\{F^w_n\}$ are obtained by applying $S_{\overline{X}}\gamma_{X!}$ to A–R-ℓ-adic systems of sheaves on $X \times_k k^s$. Therefore, by [FK80, Theorem I.12.15], $\{F^!_n(X)\}$, $\{F^n_!(X)\}$ and $\{F^w_n(X)\}$ are A–R-ℓ-adic complexes of sheaves on k^s, i.e. A–R-ℓ-adic complexes of continuous $\mathrm{Gal}(k^s/k)$-modules. So, we indeed have functors

$$\begin{aligned} F^! &\colon \mathbf{comp}(\mathcal{V}_k^{cptd})^{\mathrm{op}} \longrightarrow (\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell))^{\mathrm{op}})^{\mathrm{op}} \\ F_! &\colon \mathbf{co}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell))^{\mathrm{op}} \\ F^w &\colon \mathbf{w}(\mathcal{V}_k^{cptd}) \longrightarrow \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k); \ell))^{\mathrm{op}}. \end{aligned}$$

It remains to verify that F is W-exact. Axioms (1)-(4) hold by definition, so we check the remainder in turn.

First, consider Axiom (5). It suffices to prove it for F_n. This is a large but straightforward diagram chase using the definitions of $F_n^!$ and $F_!^n$. For those who would like to see the details, we present them in Appendix B.

Now we check Axiom (6): that F takes subtraction sequences to exact sequences. Again, it suffices to show it for F_n. Given a subtraction sequence in $\mathcal{V}_k^{cptd}$

$$(Z,\overline{Z}) \xhookrightarrow{\;j\;} (X,\overline{X}) \xleftarrow{\;i\;} (U,\overline{U})$$

we obtain a sequence in $\mathrm{Sh}(X \times_k k^s)$

$$0 \longrightarrow j_! A_U^n \longrightarrow A_X^n \longrightarrow i^* A_Z^n \longrightarrow 0$$

and we see that this is exact by direct inspection (i.e. by verifying exactness on stalks). Applying $S_{\overline{X}}\gamma_{X!}$, we obtain an exact sequence in $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}^{\mathrm{AR}}_{\mathrm{cts}}(\mathrm{Gal}(k^s/k);\ell))$ (using that the Godement resolution is an exact functorial flabby resolution, cf. e.g. [FK80, p. 129]). Consider the following diagram:

$$\begin{array}{ccccc}
S_{\overline{U}}\gamma_{U!}A_U^n & & & & \\
\downarrow{\scriptstyle \bar{j}_\Gamma} & \searrow^{F_n^!(j)} & & & \\
S_{\overline{X}}\gamma_{X!}j_!A_U^n & \longrightarrow & S_{\overline{X}}\gamma_{X!}A_X^n & \longrightarrow & S_{\overline{X}}\gamma_{X!}i^*A_Z^n \\
 & & & \searrow_{F_!^n(i)} & \downarrow{\scriptstyle \bar{i}_\Gamma^{-1}} \\
 & & & & S_{\overline{Z}}\gamma_{Z!}A_Z^n
\end{array}$$

The exact sequence across the middle is isomorphic to the diagonal sequence, which is thus also exact. Therefore F_n takes subtraction sequences to exact sequences, as desired.

It remains to verify Axiom (7); as before, it suffices to prove that it holds for F_n. The proof of (7) for complement maps is identical to the proof of Lemma 5.7, since it is simply checking that the transformation defined in Lemma 5.4 respects composition. The proof of (7) for cofibrations is identical to the proof of (5), since any such commutative diagram is automatically a pullback square, and F_n^w is defined identically to $F_!^n$. ☐

6. Nontrivial Classes in the Higher K-Theory of Varieties

Let **FinSet** be the category of finite sets. Consider the map $\mathbb{S} \longrightarrow K(\mathcal{V}_k)$ induced by the exact functor $\textbf{FinSet} \longrightarrow \mathcal{V}_k$ given by $A \mapsto \coprod_A \mathrm{Spec}(k)$. This induces a homomorphism

$$K_*(\mathbb{S}) \longrightarrow K_*(\mathcal{V}_k).$$

The stable homotopy groups of spheres have a rich higher structure, but it is not clear that this structure is not annihilated when passing to varieties. In this section we explore this map in the cases where k is a subfield of $\mathbf{C}$ and when k is a finite field. In the case when k is a subfield of $\mathbf{C}$ we show that the image of this map is nontrivial even above the 0-th homotopy group; in the case when k is finite we show that $K_*(\mathbb{S})$ is a direct summand of $K_*(\mathcal{V}_k)$ and prove that this map is not surjective when $* = 1$. Moreover, we are able to extend this result to local and global fields which contain a place of cardinality congruent to 3 mod 4, as well as to subfields of $\mathbf{R}$.

Our analysis considers a particularly simple family of maps involving permutations of varieties. In order to construct these maps, it is useful to have a model of the sphere spectrum $\mathbb{S}$ within SW-categories. The following lemmas furnish this.

Lemma 6.1. *The category of finite sets* **FinSet** *is an SW-category where cofibrations are monomorphisms and subtraction sequences are diagrams* $[m] \hookrightarrow [n] \hookleftarrow [n-m]$ *in which the images of the two maps are disjoint.*

Proposition 6.2. *Let K^W denote the K-theory of a Waldhausen category, and K^{SW} the K-theory of an SW-category. Let* **FinSet** *be the category of finite sets, considered as an SW-category, and let* $\mathbf{FinSet}_+$ *be the category of finite pointed sets, considered as a Waldhausen category (both with injections as the cofibrations). Then*

$$K^W(\mathbf{FinSet}_+) \simeq K^{SW}(\mathbf{FinSet})$$

Proof. We prove this by exhibiting an isomorphism of simplicial sets

$$S_\bullet \mathbf{FinSet}_+ \cong \widetilde{S}_\bullet \mathbf{FinSet}.$$

Given a cofibration sequence $[m]_+ \hookrightarrow [n]_+ \longrightarrow [n-m]_+$, we can define a corresponding subtraction sequence $[m] \hookrightarrow [n] \hookleftarrow [n-m]$. The map $[m] \hookrightarrow [n]$ is exactly the same as $[m]_+ \hookrightarrow [n]_+$, but missing the basepoint. The map $[n-m] \hookrightarrow [n]$ is obtained by observing that $[n]_+ \longrightarrow [n-m]_+$ is a monomorphism off of $+$. The $[n-m] \hookrightarrow [n]$ is then the inverse of this monomorphism. ☐

We now define the family of maps we use.

Definition 6.3. For each $X \in \mathcal{V}_k$, define

$$\sigma_X \colon \mathbf{FinSet} \longrightarrow \mathcal{V}_k$$

to be the exact functor of SW-categories $\mathbf{FinSet} \longrightarrow \mathcal{V}_k$ induced by

$$[n] \longmapsto \underbrace{X \amalg \cdots \amalg X}_{n \text{ times}}.$$

Thus we get a family of maps

$$\pi_* \sigma_X \colon \pi_*(\mathbb{S}) \longrightarrow K_*(\mathcal{V}_k).$$

6.1. **Subfields of C.** We appeal to some facts from Adams [Ada66] and Quillen's letter to Milnor [Qui76]. Recall that there is a homomorphism from the stable homotopy of the stable orthogonal group to the stable homotopy groups of spheres: $J \colon \pi_*^s \mathbf{O} \longrightarrow \pi_* \mathbb{S}$. This is constructed as follows. Since every element of $O(n)$ defines a map $\mathbf{R}^n \longrightarrow \mathbf{R}^n$, by one point compactification, it defines a point map $S^n \longrightarrow S^n$. Thus, there is a map of spaces $O(n) \longrightarrow \mathrm{Map}_*(S^n, S^n) \cong \Omega^n S^n$. Taking homotopy groups and stabilizing yields the J-homomorphism. The following is due to Adams

Theorem 6.4. [Ada66, Thm. 1.5] *The map $J \colon \pi_{4s-1}\mathbf{O} \longrightarrow \pi_{4s-1}\mathbb{S}$ exhibits $\pi_{4s-1}\mathbf{O}$ as a direct summand of $\pi_{4s-1}\mathbb{S}$ and the image is*

$$J(\pi_{4s-1}(\mathbf{O})) \cong C_{d_s} \qquad d_s = \text{denominator}\left(\frac{B_s}{4s}\right)$$

where B_s is the sth Bernoulli number, and C_{d_s} denotes the cyclic group of order d_s.

There is also a map $p_i \colon \pi_i \mathbb{S} \longrightarrow K_i(\mathbf{Z})$ induced by the inclusion $B\Sigma_\infty \longrightarrow BGL(\mathbf{Z})$. For our purposes, it is more useful to regard it as the map induced by the exact functor of Waldhausen categories

$$P \colon \mathbf{FinSet}_+ \longrightarrow \mathrm{Mod}^{\mathrm{fg}}(\mathbf{Z}) \qquad [n]_+ \mapsto \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$$

Quillen's letter to Milnor gives the following.

Theorem 6.5. [Qui76] *The composite map*

$$\pi_{4s-1}(\mathbf{O}) \xrightarrow{J} \pi_{4s-1}(\mathbb{S}) \xrightarrow{p_{4s-1}} K_{4s-1}(\mathbf{Z})$$

is injective.

We can use these results to identify nontrivial elements in $K_{2s-1}(\mathcal{V}_k)$.

Theorem 6.6. *Let k be a subfield of* **C**. *There are infinitely many non-trivial homotopy groups of $K_*(\mathcal{V}_k)$. In particular, for all $s > 0$, $K_{4s-1}(\mathcal{V}_k)$ is non-trivial.*

Proof. Fix $s > 0$, and let $X \in \mathcal{V}_k$ be a projective variety such that its compactly-generated Euler characteristic $\chi_c(X)$ is relatively prime to d_s. (Note that this is always possible, as for example $\chi_c(\mathbf{C}P^n) = n$ for all n.) Let $C_c^*(X)$ be the compactly-supported singular chains on the complex points of X. Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{V}_k & \xleftarrow{U} & \mathcal{V}_k^{cptd} & \dashrightarrow & \mathrm{Ch}^{\mathrm{b}}(\mathbf{Z}) & \xleftarrow{\cdot[0]} & \mathrm{Mod}^{\mathrm{fg}}(\mathbf{Z}) \\
 & \nwarrow^{\sigma_X} & \uparrow{\scriptstyle \sigma_{(X,X)}} & & & \nwarrow^{\otimes C_c^*(X)} & \\
 & & \mathbf{FinSet} & \dashrightarrow & \mathbf{FinSet}_+ & \xrightarrow{P} & \mathrm{Mod}^{\mathrm{fg}}(\mathbf{Z})
\end{array}$$

where the left half of the diagram consists of SW-categories and the right half consists of Waldhausen categories; both dashed arrows are W-exact functors. The bottom dashed arrow takes a finite set S to S_+, and takes cofibrations to themselves; a complement map is taken to the map which takes each element in the image of the complement map to its preimage, and each element not in the image to the basepoint. Applying π_0 and noting that the two solid arrows across the top become isomorphisms, we recover the motivic measure constructed in Theorem 4.2(2). Upon applying K-theory we get a diagram

$$\begin{array}{ccccccc}
K_{4s-1}(\mathcal{V}_k) & \xleftarrow{\cong} & K_{4s-1}(\mathcal{V}_k^{cptd}) & \longrightarrow & K_{4s-1}(\mathrm{Ch}^{\mathrm{b}}(\mathbf{Z})) & \xrightarrow{\chi} & K_{4s-1}(\mathbf{Z}) \\
 & \nwarrow^{\sigma_X} & \uparrow{\scriptstyle \sigma_{(X,X)}} & & & \nwarrow & \uparrow{\scriptstyle \cdot\chi_c(X)} \\
\pi_{4s-1}(\mathbf{O}) & \xrightarrow{J} & \pi_{4s-1}\mathbb{S} & \xrightarrow{\cong} & \pi_{4s-1}\mathbb{S} & \xrightarrow{p_{4s-1}} & K_{4s-1}(\mathbf{Z})
\end{array}$$

By Theorems 6.4 and 6.5, the composition across the bottom $\pi_{4s-1}(\mathbf{O}) \longrightarrow K_{4s-1}(\mathbf{Z})$ is injective; since $\chi_c(X)$ is relatively prime to d_s, the composition $\pi_{4s-1}(\mathbf{O}) \longrightarrow K_{4s-1}(\mathbf{Z})$ across the bottom and to the upper-right is also injective. Since this factors through $K_{4s-1}(\mathcal{V}_k)$ the theorem follows. □

Remark 6.7. In the above proof we are not really using anything interesting about X; in particular, we could take X to be $\operatorname{Spec} k$ and the proof still goes through. This makes sense, as the image of J sees nontrivial elements of K-theory which can be described in terms of permutations, which are automorphisms of 0-dimensional varieties.

However, we believe that the more complex proof is useful due to the possibility of generalization. It should be possible to use the same construction in the proof to show that there are more interesting nontrivial elements in higher homotopy groups of $K(\mathcal{V}_k)$ by exploiting the structure of the cohomology of $C_c^*(X)$. For example, if the measure can be enriched to land in the K-theory of mixed Hodge structures then by selecting an X with a nontrivial mixed Hodge structure the above proof would detect an element which is *not* in the image of the map $K_*(\S) \longrightarrow K_*(\mathcal{V})$ induced by the inclusion of 0-dimensional varieties.

6.2. **Finite Fields.** When k is finite the functor

$$\begin{array}{ccc}
\mathcal{V}_k & \longrightarrow & \mathbf{FinSet} \\
X & \longmapsto & X(k)
\end{array}$$

gives a splitting of the K-theory spectrum $K(\mathcal{V}_k)$ as $K(\mathcal{V}_k) \simeq \mathbb{S} \vee \widetilde{K}(\mathcal{V}_k)$ (and thus of each homotopy group). A priori it may be the case that all higher homotopy groups of $K(\mathcal{V}_k)$ are in the image of the homotopy groups of $\mathbb{S}$.

We show that this is not the case by using $\boldsymbol{\zeta}$ to identify a nonzero element in $\widetilde{K}_1(\mathcal{V}_k)$ for $k = \mathbf{F}_q$ (with $q \equiv 3 \bmod 4$). For such k, we construct a surjective homomorphism $h_2 \colon K_1(\mathcal{V}_k) \longrightarrow \mathbf{Z}/2$ such that the composition $\pi_1(\mathbb{S}) \longrightarrow K_1(\mathcal{V}_k) \xrightarrow{h_2} \mathbf{Z}/2$ is trivial. The map h_2 is defined as follows.

Let $\star$ be the operation defined in [Mil71, §8]. Grayson [Gra79] has shown that given a pair of commuting automorphisms f, g on a $\mathbf{Q}_\ell$-vector space V, the map

$$(f, g) \mapsto f^{-1} \star g,$$

induces a homomorphism $s_\ell\colon K_1(\mathrm{Aut}(\mathbf{Q}_\ell)) \longrightarrow K_2(\mathbf{Q}_\ell)$. Moore's Theorem (see [Mil71, Appendix], or [Wei05, Theorem 57] and its proof) shows that the mod-ℓ Hilbert symbol gives a split surjection $(-,-)_\ell\colon K_2(\mathbf{Q}_\ell) \twoheadrightarrow \mu(\mathbf{Q}_\ell)$ onto the roots of unity in $\mathbf{Q}_\ell$ with kernel an uncountable uniquely divisible abelian group $U_2(\mathbf{Q}_\ell)$. For q odd, we then define h_2 to be the composition

$$K_1(\mathcal{V}_{\mathbf{F}_q}) \xrightarrow{\pi_1(Frob_*\circ\boldsymbol{\zeta})} K_1(\mathrm{Aut}(\mathbf{Q}_2)) \xrightarrow{s_2} K_2(\mathbf{Q}_2) \xrightarrow{(-,-)_2} \mathbf{Z}/2,$$

where $Frob_*$ is the map defined in Corollary 4.3 for $g = Frob$ and $(-,-)_2$ is the 2-adic Hilbert symbol.

Fix a variety X, and consider the following diagram:

$$\begin{array}{ccccc} & \overset{\pi_1\sigma_{(X,X)}}{\dashrightarrow} & K_1(\mathcal{V}_k^{cptd}) & \xrightarrow{h_2} & \mathbf{Z}/2 \\ & & \downarrow{\scriptstyle K_1(U)} & & \\ \pi_1\mathbb{S} & \xrightarrow{\pi_1\sigma_X} & K_1(\mathcal{V}_k) & & \end{array} \tag{6.8}$$

If X is proper then $\sigma_{(X,X)}$ exists, and the diagram commutes with the dotted arrow added.

Theorem 6.9. *When $k = \mathbf{F}_q$ with $q \equiv 3 \pmod 4$ the map h_2 defined above detects a nontrivial class in $\widetilde{K}_1(\mathcal{V}_k)$. In particular, the spectrum $\widetilde{K}(\mathcal{V}_k)$ is not an Eilenberg–MacLane spectrum.*

Proof. Let $\eta \in \pi_1(\mathbb{S})$ be the nonzero element. We show that h_2 is nonzero but contains $(\pi_*\sigma_{\mathrm{Spec}\,k})(\eta)$ in its kernel. This shows that $\widetilde{K}_1(\mathcal{V}_k) \neq 0$ and therefore that $\widetilde{K}(\mathcal{V}_k)$ is not an Eilenberg–MacLane spectrum.

Let $\tau\colon \{1,2\} \longrightarrow \{1,2\}$ be the transposition of two points.

Tracing through the definition, $h_2(\pi_1\sigma_{\mathrm{Spec}\,k}(\tau)) = (-1,1)_2 = 1$ in $\mathbf{Z}/2$.

Now let $X = \mathbb{P}^1$, and consider the automorphism $\alpha\colon x \mapsto 1/x$ acting on $\mathbb{P}^1$. For $k = \mathbf{F}_q$, we can write $\alpha \circlearrowleft (H^*_{et,c}(\mathbb{P}^1|_{\overline{\mathbf{F}}_q}; \mathbf{Q}_2), Frob_q^{\mathbb{P}^1})$ as a direct sum

$$(1 \circlearrowleft \mathbf{Q}_2(0)) \oplus (-1 \circlearrowleft \mathbf{Q}_2(-1)).$$

The map s_2 sends everything with the identity acting on it to the unit, so the image of this under h_2 is $(-1, q)_2$, which, when $q \equiv 3 \bmod 4$, is -1. This gives the desired element in $\widetilde{K}_1$. □

6.3. **Elements in K_1 for other fields.** Suppose that k is a global or local fields with a place of cardinality equivalent to 3 mod 4, or $k \subset \mathbf{R}$. The results from the previous section hold with almost identical proofs, with slightly different definitions.

For k a global or local field with a place of cardinality equivalent to 3 mod 4, we pick a Frobenius element ϕ for this place, and define h_2 to be the composition

$$K_1(\mathcal{V}_k) \xrightarrow{\pi_1(\phi_*\circ\boldsymbol{\zeta})} K_1(\mathrm{Aut}(\mathbf{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbf{Q}_2) \xrightarrow{(-,-)_2} \mathbf{Z}/2,$$

Similarly for $k \subset \mathbf{R}$, we define h_2 to be the composition

$$K_1(\mathcal{V}_k) \xrightarrow{\pi_1((\bar{\cdot})_*\circ\boldsymbol{\zeta})} K_1(\mathrm{Aut}(\mathbf{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbf{Q}_2) \xrightarrow{(-,-)_2} \mathbf{Z}/2,$$

where $\bar{\cdot}$ denotes complex conjugation and $(\bar{\cdot})_*$ is the map defined in Corollary 4.3 for $g = \bar{\cdot}$.

Theorem 6.10. *For k a global or local field with a place of cardinality equivalent to 3 mod 4, or for $k \subset \mathbf{R}$, $\widetilde{K}(\mathcal{V}_k)$ is not an Eilenberg–MacLane spectrum.*

Remark 6.11. We expect that $\widetilde{K}(\mathcal{V}_k)$ is not an Eilenberg–MacLane spectrum in general, however the particular class that we use h_2 to detect requires the assumptions on k. Using the methods of this paper, it should be possible to find a different example that gives a nontrivial class for all odd q, all global and local fields with a place of odd cardinality, and all non-algebraically closed subfields of $\mathbf{C}$. For even q, one would need the p-adic (rather than the ℓ-adic) analogue of the derived zeta function to employ the present approach.

Proof of Theorem 6.10. This proof works the same way as the proof of Theorem 6.9.

If k is a global or local field with a place of cardinality 3 mod 4, then, for any Frobenius element ϕ for this place, the action of ϕ on the étale cohomology of $\mathbb{P}^1$ factors through the action on the special fiber. In particular, the above computation similarly shows that the map h_2 takes the class $\alpha \circlearrowleft \mathbb{P}^1$ to -1, and thus gives the desired element in $\widetilde{K}_1(\mathcal{V}_k)$.

For $k \subset \mathbf{R}$, we can consider the same X. We write $\pi_1\alpha \circlearrowleft (H^*_{et,c}(X|_{\mathbf{C}}; \mathbf{Q}_2), \bar{\cdot})$ as a direct sum

$$(1 \circlearrowleft (\mathbf{Q}_2, 1)) \oplus (-1 \circlearrowleft (\mathbf{Q}_2, -1)).$$

The map s_2 sends everything with the identity acting on it to the unit, so the image of this under h_2 is $(-1,-1)_2 = -1$. This gives the desired element in $\widetilde{K}_1$. □

Theorem 1.4 is a direct consequence of Theorems 6.9 and 6.10.

Remark 6.12. It should be possible to do a more powerful analysis on K_1 by exploiting the rich structure of automorphism groups of varieties and applying Proposition 2.19.

7. Questions for Future Work

Indecomposable Elements in $K(\mathcal{V}_k)$. Theorems 6.9 and 6.10 establish that there are nontrivial classes in the higher K-theory of varieties that do not come from the sphere spectrum. However, one could ask a more refined question: since $K(\mathcal{V}_k)$ is an E_∞-ring spectrum, its homotopy groups $K_*(\mathcal{V}_k)$ form a ring. We therefore have a ready supply of elements of $K_*(\mathcal{V}_k)$: those in the image of the multiplication

$$\beta\colon K_0(\mathcal{V}_k) \otimes \pi_*(\mathbb{S}) \xrightarrow{1\otimes\sigma_{\mathrm{Spec}\,k}} K_0(\mathcal{V}_k) \otimes K_*(\mathcal{V}_k) \longrightarrow K_*(\mathcal{V}_k).$$

We call such elements *decomposable*. A priori, it may be the case that this map is surjective, and that therefore all higher homotopy groups of $K(\mathcal{V}_k)$ are decomposable. The example constructed in Section 6 is decomposable, since this can just be written as $\eta \cdot [\mathbb{P}^1]$.

Question 7.1. **Indecomposable elements.** Do there exist indecomposable elements in $K_*(\mathcal{V}_k)$?

As we explain in Remark 7.2 below, we do not expect the map h_2 to be able to distinguish decomposable from non-decomposable elements. Instead, we hope that by expanding the collection of derived motivic measures and employing Proposition 2.19 judiciously, a suitable invariant could be found.

Remark 7.2. We have shown that the derived ℓ-adic zeta function factors through integral compactly supported motivic cohomology. This suggests that the invariant $h_2\colon K_1(\mathcal{V}_k) \longrightarrow K_2(\mathbf{Q}_2)$ in Section 6 factors through the composition

$$K_1(\mathrm{Aut}(\mathbf{Z})) \longrightarrow K_2(\mathbf{Z}) \longrightarrow K_2(\mathbf{Q}) \longrightarrow K_2(\mathbf{Q}_2).$$

We highlight three implications of this expected factoring:

(1) It underscores the importance of the 2-adic Hilbert symbol, as opposed to the ℓ-adic Hilbert symbol for $\ell \neq 2$. Indeed, by Tate's computation of $K_2(\mathbf{Q})$ (see e.g. [Mil71, Theorem 11.6]), the map $K_2(\mathbf{Z}) \longrightarrow K_2(\mathbf{Q})$ is split injective, with the splitting given by the 2-adic Hilbert symbol. In particular, the Hilbert symbols $(-,-)_\ell$ for $\ell \neq 2$ identically

vanish on $K_2(\mathbf{Z})$. Further, no classes outside the summand $\mu(\mathbf{Q}_2) \subset K_2(\mathbf{Q}_2)$ are in the image of $K_2(\mathbf{Z})$.

(2) It suggests that we should not expect the map h_2 to be able to distinguish indecomposable elements in $\widetilde{K}_1(\mathcal{V}_k)$. Indeed, Milnor's computation [Mil71, Corollary 10.2] shows that the map $K_1(\mathrm{Aut}(\mathbf{Z})) \longrightarrow K_2(\mathbf{Z})$ is surjective and the nontrivial class in $K_2(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ is mapped onto by decomposable classes.

(3) It suggests that the higher invariants of the derived zeta functions should be in some sense independent of ℓ. It would be fruitful to understand this more precisely!

Other Derived Motivic Measures. The recipe in this paper should work to construct derived motivic measures for other cohomological invariants. We took ℓ-adic cohomology as the basis for our derived zeta function. One would like analogous maps for the other Weil cohomology theories.

Problem 7.3. **Derived p-adic zeta functions.** Let k be a perfect field of characteristic p with Witt vectors $W(k)$. Construct a map of K-theory spectra

$$K(\mathcal{V}_k) \longrightarrow K(\mathrm{Aut}(W(k)))$$

which lifts the function sending a variety X to $H^*_{rig,c}(X/W(k))$ to its compactly supported rigid cohomology (with constant coefficients) acted on by the Frobenius automorphism.

We expect that the construction should parallel that in Section 4, with the category $\mathcal{V}^{cptd}$ replaced by a category of varieties X equipped with a choice of compactification $X \hookrightarrow \overline{X}$, and a choice of map of admissible triples $(X, Y, \mathcal{Y}) \longrightarrow (\overline{X}, \overline{Y}, \overline{\mathcal{Y}})$ extending $X \hookrightarrow \overline{X}$ as in [Ber86, Section 3], and with rigid cohomology replacing the ℓ-adic constructions. Note that, Tsuzuki's finiteness theorem [Tsu03, Theorem 5.1.1] plays an essential role in defining the W-exact functor.

Problem 7.4. **Derived Serre Polynomial.** Let k be a field of characteristic 0. Construct a map of K-theory spectra taking values in the K-theory of integral mixed Hodge structures

$$K(\mathcal{V}_k) \longrightarrow K(MHS_{\mathbf{Z}})$$

which lifts the function sending a variety X to $H^*_c(X(\mathbf{C}); \mathbf{Z})$ with its canonical mixed Hodge structure.

We expect that the construction should parallel that in Section 4, with the category $\mathcal{V}^{cptd}$ replaced by a category of varieties X equipped with a choice of compactification $X \hookrightarrow \widetilde{X}$, and a choice of cubical hyperresolution $\widetilde{X}_\bullet \longrightarrow \widetilde{X}$ of the pair $(\widetilde{X}, \widetilde{X} - X)$ (see e.g. [PS08, Chapter 5]), and with logarithmic differential forms in lieu of the ℓ-adic constructions.

The framework of motives suggests that the derived ℓ-adic zeta function, along with the two maps described above, should factor through a derived motivic measure built from motivic cohomology.

Problem 7.5. **Derived Gillet–Soulé.** Let k be a field admitting resolution of singularities. Construct a map of K-theory spectra taking values in the K-theory of integral Chow motives over k

$$K(\mathcal{V}_k) \longrightarrow K(\mathbf{M}_k)$$

which lifts the motivic measure of Gillet–Soulé [GS96] sending a k-variety X to its compactly supported integral Chow motive. Prove that the derived ℓ-adic zeta function and the derived Serre polynomial factor through this map.

We expect that the replacement of $\mathcal{V}^{cptd}$ should be the same as for the Serre polynomial.

For general fields k, one might expect to have a motivic measure based on integral Voevodsky motives through which all of the above maps factor.

Moving further away from cohomological invariants, one of the richest motivic measures is Kapranov's motivic zeta function

$$K_0(\mathcal{V}_k) \longrightarrow W(K_0(\mathcal{V}_k))$$
$$[X] \mapsto \sum_{i=0}^{\infty} [\mathrm{Sym}^i(X)] t^i.$$

Question 7.6. **Derived motivic zeta function.** Does Kapranov's motivic zeta function lift to a map of K-theory spectra?

For motivation, recall that Weil's realization that the classical zeta function of a variety over a finite field can be obtained cohomologically provided a robust strategy for proving that the zeta function of such varieties is rational. Similarly, a lift of Kapranov's motivic zeta function to a map of K-theory spectra might be expected to go a long way toward proving that the motivic zeta function is rational, in an appropriate sense.

Recall, however, that purely as a map out of $K_0(\mathcal{V}_{\mathbf{C}})$, Kapranov's motivic zeta function is *not* rational. This was proven by Larsen and Lunts [LL03], and the key tool in their proof was a motivic measure

$$\mu_{LL} \colon K_0(\mathcal{V}_{\mathbf{C}}) \longrightarrow \mathbf{Z}[SB_{\mathbf{C}}]$$

taking values in the free abelian group on stable birational equivalences classes of complex varieties. Note that, since $\mathbb{P}^1 \sim_{SB} *$, Larsen and Lunts' measure takes the class of the affine line to 0. In particular, it still may be the case that Kapranov's motivic zeta is rational after inverting the affine line, or performing some other modification of $K_0(\mathcal{V}_k)$ (cf. [LL04]). This underpins the "in the appropriate sense" above.

Question 7.7. **Derived Larsen–Lunts.** Does the Larsen–Lunts measure lift to a map of K-theory spectra? Using the formalism of assemblers, the third-named author was able to accomplish this [Zak17b]. However, it would be desirable to have a direct construction of this motivic measure. For this, one would need an SW-category which naturally encodes stable birational equivalence of projective varieties.

Appendix A. Functorial factorization of weak cofibrations

In this appendix, we define and verify in the cases of interest the technical condition "functorial factorization of weak cofibrations" defined in [BM08]. The condition is meant to be a weakening of Waldhausen's cylinder functors [Wal85, Sect. 1.6].

Definition A.1. Let [1] denote the ordered set $0 < 1$ considered as a category, and let [2] the ordered set $0 < 1 < 2$, also considered as a category. A *functorial factorization* is a functor $\varphi\colon \mathrm{Fun}([1], \mathcal{C}) \longrightarrow \mathrm{Fun}([2], \mathcal{C})$ such that $(d^1)^* \circ \varphi = 1_{\mathrm{Fun}([1],\mathcal{C})}$. Here, $d^1\colon [1] \longrightarrow [2]$ takes 0 to 0 and 1 to 2.

Definition A.2. Let $\mathcal{C}$ be a Waldhausen category. A *weak equivalence* between morphisms $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ is a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \sim\downarrow & & \downarrow\sim \\ C & \xrightarrow{g} & D \end{array}$$

A *weak cofibration* is a map $A \longrightarrow B$ that admits a zig-zig of weak equivalences to a cofibration $A' \hookrightarrow B'$.

Definition A.3. [BM08, Definition 2.2] Let $\mathcal{C}$ be a Waldhausen category. Write $\mathrm{Fun}^{wc}([1],\mathcal{C})$ for the full subcategory of $\mathrm{Fun}([1],\mathcal{C})$ consisting of the functors whose image is a weak cofibration. Let $\mathrm{Fun}^{c,w}([2],\mathcal{C})$ be the full subcategory of $\mathrm{Fun}([2],\mathcal{C})$ consisting of those diagrams which are a cofibration followed by a weak equivalence. A *functorial factorization of weak cofibrations* is a functor $\varphi\colon \mathrm{Fun}^{wc}([1],\mathcal{C}) \longrightarrow \mathrm{Fun}^{c,w}([2],\mathcal{C})$ such that $(f^1)^* \circ \varphi = 1_{\mathrm{Fun}^{wc}([1],\mathcal{C})}$.

Let $\mathcal{C}$ be a cofibrantly generated model category. We thus have a functorial factorization

$$\mathrm{Fun}([1],\mathcal{C}) \longrightarrow \mathrm{Fun}^{c,w}([2],\mathcal{C}),$$

given by the functorial factorization of morphisms into a cofibration followed by an acyclic fibration. If a Waldhausen category arises as a subcategory of a model category, we can often leverage this factorization to obtain functorial factorizations inside the Waldhausen category. The main problem is that objects in a Waldhausen category need to be small (in some sense), whereas functorial factorizations often produce very large objects. However, in Example 2.7 we showed that being *homologically* small is sufficient; when weak equivalences are quasi-isomorphisms this is therefore sufficient.

To produce the functorial factorizations that we need we appeal to a theorem of Hovey [Hov01] which produces model category structures on categories of chain complexes.

Theorem A.4. [Hov01, Theorem 2.2] *Let $\mathcal{A}$ be a Grothendieck abelian category. Then* $\mathrm{Ch}(\mathcal{A})$ *admits a cofibrantly generated model structure where*

- *weak equivalences are quasi-isomorphisms*
- *cofibrations are injections*
- *fibrations have the right lifting property with respect to trivial cofibrations*

This theorem is also proved in [Bek00, Proposition 3.13] with a convenient characterization of Grothendieck abelian categories [Bek00, Proposition 3.10].

Theorem A.5. *The category* $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}(G_k);\ell)$ *satisfies FFWC.*

Proof. As noted in Proposition 3.1, the category $\mathrm{Rep}_{\mathrm{cts}\,cts}(G_k;\ell)$ is equivalent to the category $\mathrm{Sh}^{et}(\mathrm{Spec}(k);\ell)$. This is the category of sheaves of an abelian group on a ringed site, as such, it is a Grothendieck category. By Hovey's Theorem, $\mathrm{Ch}(\mathrm{Rep}_{\mathrm{cts}\,cts}(G_k;\ell)$ admits a cofibrantly generated structure. Thus, all morphisms in $\mathrm{Ch}(\mathrm{Rep}_{\mathrm{cts}\,cts}(G_k;\ell)$ have functorial factorizations. Given a weak cofibration $A \longrightarrow B$ factor it as $A \hookrightarrow C \longrightarrow B$; it remains to show that $C \in \mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}\,cts}(G_k;\ell))$. However, since $C \longrightarrow B$ is a weak equivalence and B is homologically bounded, C must be as well. □

Theorem A.6. *The category* $\mathrm{Ch}^{\mathrm{b}}(R)$ *satisfies FFWC.*

Proof. The category Mod_R is a Grothendieck abelian category. As such, $\mathrm{Ch}(\mathrm{Mod}_R)$ has a cofibrantly generated injective model structure. Thus, the category of all maps $X \longrightarrow Y$ in $\mathrm{Ch}(\mathrm{Mod}_R)$ has a functorial factorization of the required form. Restricting to $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Mod}_R)$, as in the previous proof, we see that $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Mod}_R)$ does as well. □

Appendix B. The proof of Axiom (5)

Axiom (5) states the following. Suppose that we are given a cartesian diagram

$$\begin{array}{ccc} (X,\overline{X}) & \overset{i}{\Vdash\!\!\longrightarrow} & (Z,\overline{Z}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (Y,\overline{Y}) & \overset{j}{\Vdash\!\!\longrightarrow} & (W,\overline{W}). \end{array}$$

We must show that

$$\begin{array}{ccc} F_n(X,\overline{X}) & \xrightarrow{F_n^!(i)} & F_n(Z,\overline{Z}) \\ {\scriptstyle F_n^!(f)}\big\uparrow & & \big\uparrow{\scriptstyle F_!^n(g)} \\ F_n(Y,\overline{Y}) & \xrightarrow{F_n^!(j)} & F_n(W,\overline{W}) \end{array}$$

commutes. Note that since f and g are both closed, we know that on sheaves $f_* = f_!$ and $g_* = g_!$.

First, note that the following diagram (in $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Rep}_{\mathrm{cts}}^{\mathrm{AR}}(\mathrm{Gal}(k^s/k));\ell)$) commutes.

$$\begin{array}{ccccccc}
S_{\overline{Y}}\gamma_{Y!}A_Y^n & \xrightarrow{\bar{j}_\Gamma} & S_{\overline{W}}\gamma_{W!}j_!A_Y^n & \xrightarrow{\cong} & S_{\overline{W}}\gamma_{W!}j_!j^*A_W^n & & \\
{\scriptstyle S_{\overline{Y}}\gamma_{Y!}\eta}\big\downarrow & & \big\downarrow{\scriptstyle S_{\overline{W}}\gamma_{W!}j_!\eta} & & \big\downarrow{\scriptstyle S_{\overline{W}}\gamma_{W!}j_!\eta_{j^*}} & & \\
S_{\overline{Y}}\gamma_{Y!}f_*f^*A_Y^n & \xrightarrow{\bar{j}_\Gamma} & S_{\overline{W}}\gamma_{W!}j_!f_*f^*A_Y^n & \xrightarrow{\cong} & S_{\overline{W}}\gamma_{W!}j_!f_*f^*j^*A_W^n & & \\
 & & & & \big\downarrow{\scriptstyle\cong} & & \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} & & S_{\overline{W}}\gamma_{W!}j_!f_*i^*g^*A_W^n & & \\
 & & & & \big\downarrow{\scriptstyle\cong} & & \\
S_{\overline{Y}}\gamma_{Y!}f_!A_X^n & \xrightarrow{\bar{j}_\Gamma} & S_{\overline{W}}\gamma_{W!}j_!f_!A_X^n & \xrightarrow{\cong} & S_{\overline{W}}\gamma_{W!}j_!f_!i^*A_Z^n & & \\
 & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} & & \\
{\scriptstyle \bar{f}_\Gamma^{-1}}\big\downarrow & * & S_{\overline{W}}\gamma_{W!}g_!i_!A_X^n & \xrightarrow{\cong} & S_{\overline{W}}\gamma_{W!}g_!i_!i^*A_Z^n & \xrightarrow{S_{\overline{W}}\gamma_{W!}g_!\epsilon} & S_{\overline{W}}\gamma_{W!}g_!A_Z^n \\
 & & \big\downarrow{\scriptstyle \bar{g}_\Gamma^{-1}} & & \big\downarrow{\scriptstyle \bar{g}_\Gamma^{-1}} & & \big\downarrow{\scriptstyle \bar{g}_\Gamma^{-1}} \\
S_{\overline{X}}\gamma_{X!}A_X^n & \xrightarrow{\bar{i}_\Gamma} & S_{\overline{Z}}\gamma_{Z!}i_!A_X^n & \xrightarrow{\cong} & S_{\overline{Z}}\gamma_{Z!}i_!i^*A_Z^n & \xrightarrow{S_{\overline{Z}}\gamma_{Z!}\epsilon} & S_{\overline{Z}}\gamma_{Z!}A_Z^n
\end{array}$$

The pentagon marked $*$ commutes because of the naturality of $(-)_\Gamma$. The composition around the bottom is $F_n^!(i) \circ F_!^n(f)$.

Now consider the following diagram in $\mathrm{Sh}(W \times_k k^s)$:

$$\begin{array}{ccccc}
j_!j^*A_W^n & \xrightarrow{=} & j_!j^*A_W^n & \xrightarrow{\epsilon} & A_W^n \\
{\scriptstyle j_!\eta_{j^*}}\big\downarrow & \star & \big\downarrow{\scriptstyle j_!j^*\eta} & & \big\downarrow{\scriptstyle \eta} \\
j_!f_*f^*j^*A_W^n & & j_!j^*g_*g^*A_W^n & \xrightarrow{\epsilon} & g_*g^*A_W^n \\
{\scriptstyle\cong}\big\downarrow & \nearrow{\scriptstyle j_!\alpha_{g^*}} & & & \\
j_!f_*i^*g^*A_W^n & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} \\
{\scriptstyle\cong}\big\downarrow & & & & \\
j_!f_!i^*A_Z^n & \xrightarrow{j_!\alpha} & j_!j^*g_!A_Z^n & \xrightarrow{\epsilon} & g_!A_Z^n \\
{\scriptstyle\cong}\big\downarrow & & \dagger & & \big\downarrow{\scriptstyle\cong} \\
g_!i_!i^*A_Z^n & & \xrightarrow{\qquad g_!\epsilon \qquad} & & g_!A_Z^n
\end{array}$$

Here, it is important to keep in mind that since the original diagram of varieties is Cartesian (and since $f_* = f_!$ and $g_* = g_!$), there is a natural isomorphism

$$\alpha\colon f_!i^* \cong f_*i^* \Longrightarrow j^*g_* \cong j^*g_!.$$

The only two parts of this diagram which do not commute by definition are the two pentagons marked $\star$ and $\dagger$. To see that these commute, it suffices to check that the two diagrams

$$\begin{array}{ccc} f_*f^*j^*A^n_W & \xleftarrow{\eta(f)_{j^*}} & j^*A^n_W \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle j^*\eta(g)} \\ f_*i^*g^*A^n_W & \xrightarrow{\alpha_{g^*}} & j^*g_*g^*A^n_W \end{array}$$

and

$$\begin{array}{ccc} j_!f_!i^*A^n_Z & \xrightarrow{j_!\alpha} & j_!j^*g_!A^n_Z \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle \epsilon(j)} \\ g_!i_!i^*A^n_Z & \xrightarrow{\epsilon(i)} & g_!A^n_Z \end{array}$$

commute. That these commute follows directly from the definition of the base change homomorphism (see e.g. [FK80, p. 60]). More conceptually, base change (and proper push forward) preserves the base change isomorphism.

After applying $S_{\overline{W}}\gamma_{W!}$ to this diagram, it fits into the rectangle in the upper-right in the above diagram; then the composition around the top is $F^n_!(g) \circ F^!_n(j)$. This proves Axiom (5).

References

[Ada66] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71. MR 0198470

[Alm78] Gert Almkvist, *K-theory of endomorphisms*, J. Algebra **55** (1978), no. 2, 308–340. MR 523461 (80i:18018)

[Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475. MR 1780498

[Ber86] P. Berthelot, *Géométrie rigide et cohomologie de variétés algébriques de caractéristique p*, Mémoires de la S.M.F. **23** (1986), 7–32.

[BM08] Andrew J. Blumberg and Michael A. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math. **200** (2008), no. 2, 155–179. MR 2413133

[BM11] ———, *Algebraic K-theory and abstract homotopy theory*, Adv. Math. **226** (2011), no. 4, 3760–3812. MR 2764905

[BW87] Marcel Bökstedt and Friedhelm Waldhausen, *The map $BSG \longrightarrow A(*) \longrightarrow QS^0$*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 418–431. MR 921487

[Cam] J. Campbell, *The K-theory spectrum of varieties*, arXiv:1505.03136.

[Con07] B. Conrad, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257.

[CWZ] J. Campbell, J. Wolfson, and I. Zakharevich, *A comparison of constructions of the Grothendieck spectrum of varieties*, In preparation.

[Del77] P. Deligne, *Cohomologie Étale (SGA 4 1/2)*, Springer, 1977.

[FK80] E. Freitag and R. Kiehl, *Étale Cohomology and the Weil Conjectures*, Springer-Verlag, 1980.

[Gra79] D. R. Grayson, *K_2 and the K-theory of automorphisms*, J. Algebra **58** (1979), no. 1, 12–30.

[GS96] H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. reine angew. Math. **478** (1996), 127–176.

[Hov01] Mark Hovey, *Model category structures on chain complexes of sheaves*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2441–2457. MR 1814077

[LL03] M. Larsen and V. Lunts, *Motivic measures and stable birational geometry*, Mosc. Math. J. **3** (2003), no. 1, 85–95.

[LL04] ———, *Rationality criteria for motivic zeta functions*, Compos. Math. **140** (2004), no. 6, 1537–1560.

[Mil71] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971, Annals of Mathematics Studies, No. 72. MR 0349811

[Nag62] M. Nagata, *Imbeddings of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), no. 1, 1–10.

[PS08] C.A.M. Peters and J.H.M. Steenbrink, *Mixed Hodge structures*, Springer, 2008.

[Qui73] D. Quillen, *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, ch. Higher algebraic K-theory: I, pp. 85–147, Springer, 1973.

[Qui76] ———, *Letter from Quillen to Milnor on* $\mathrm{Im}(\pi_i O \to \pi_i^{\mathrm{s}} \to K_i\mathbf{Z})$, 182–188. Lecture Notes in Math., Vol. 551. MR 0482758

[Ram15] N. Ramachandran, *Zeta functions, Grothendieck groups, and the Witt ring*, Bull. Sci. Math. **139** (2015), no. 6, 599–627. MR 3395874

[Rog92] J. Rognes, *A spectrum level rank filtration in algebraic K-theory*, Topology **31** (1992), no. 4, 813–845. MR 1191383

[Sch05] K. Schwede, *Gluing schemes and a scheme without closed points*, Recent progress in arithmetic and algebraic geometry, Contemp. Math., vol. 386, Amer. Math. Soc., 2005, pp. 157–172.

[Tsu03] N. Tsuzuki, *Cohomological descent of rigid cohomology for proper coverings*, Invent. Math. **151** (2003), 101–133.

[TT90] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, vol. III, Birkhäuser Boston, 1990, pp. 247–435.

[Wal85] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic Geometry and Topology (New Brunswick, N.J, 1983), Lecture Notes in Mathematics, vol. 1126, Springer, 1985.

[Wei05] C. A. Weibel, *Algebraic K-theory of rings of integers in local and global fields*, Handbook of K-Theory, vol. 1, Springer-Verlag, 2005, pp. 139–190.

[Wei13] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory. MR 3076731

[Zak17a] I. Zakharevich, *The K-theory of assemblers*, Adv. Math. **304** (2017), 1176–1218. MR 3558230

[Zak17b] Inna Zakharevich, *The annihilator of the Lefschetz motive*, Duke Math. J. **166** (2017), no. 11, 1989–2022. MR 3694563

Department of Mathematics, Vanderbilt University
E-mail address: `j.campbell@vanderbilt.edu`

Department of Mathematics, University of California, Irvine
E-mail address: `wolfson@uci.edu`

Department of Mathematics, Cornell University
E-mail address: `zakh@math.cornell.edu`