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# Shorting in Speculative Markets

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**ABSTRACT**

In models of trading with heterogeneous beliefs following Harrison-Kreps, short selling is prohibited and agents face constant marginal costs-of-carry. The resale option guarantees that prices exceed buy-and-hold prices and the difference is identified as a bubble. We propose a model where risk-neutral agents face asymmetric increasing marginal costs on long and short positions. Here, agents also value an option to delay, and a Hamilton-Jacobi-Bellman equation quantifies the influence of costs on prices. An unexpected decrease in shorting costs may deflate a bubble, linking financial innovations that facilitated shorting of mortgage-backed securities to the collapse of prices.

As Kindleberger and Aliber (2005) observe, many classical economists argued that the purchase of securities for resale rather than for investment income is what drives asset price bubbles. To explain such speculation in a dynamic equilibrium model, Harrison and Kreps (1978) study risk-neutral agents with fluctuating heterogeneous beliefs. In their model, long positions can be financed at a constant interest rate and short selling is ruled out. The buyer of an asset thus acquires both a stream of future dividends and an option to resell, which together with fluctuating beliefs guarantees that speculators are willing to pay more for an asset than they would pay if they were forced to hold the asset to maturity, that is, what risk-neutral investors are willing to pay to be able to speculate. Scheinkman and Xiong (2003) consider a model in which heterogeneous beliefs result from agents' overconfidence on different public signals and added trading costs. They show that these models generate a correlation between trading volume and overpricing,[1] a

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[1] See also Berestycki et al. (2019).

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characteristic associated with several major bubble episodes over the last three centuries.[2]

Another stylized fact is that bubble implosions often follow increases in supply. For instance, the implosion of the dotcom bubble was preceded by a massive increase in the float of Internet shares.[3] Similarly, while the South Sea bubble lasted less than one year, the amount of outstanding shares of the South Sea Company (SSC) more than doubled during that period and many other joint-stock companies were established.[4] However, the assumption of risk-neutral investors facing constant marginal costs in the earlier literature on disagreement and bubbles implies that supply is irrelevant for pricing.[5] Hong, Scheinkman, and Xiong (2006) analyze a two-period model with risk-averse investors in which unexpected increases in supply can deflate bubbles. The economics are straightforward—when agents are risk averse, their marginal valuation for a risky asset decreases with the amount they hold.

Short selling an asset can be seen as a source of additional supply. The collapse of prices for mortgage-backed securities (MBSs) in 2007 was preceded by a series of financial innovations that facilitated shorting: the creation of standardized credit default swaps (CDS) for MBS in 2005, the introduction of traded indexes for subprime mortgage-backed credit derivatives in 2006, and the use of CDS to construct synthetic collateralized debt obligations (CDOs) that allowed Wall Street to satisfy the global demand for U.S. AAA mortgage bonds without going through the relatively slow process of originating new mortgages.[6] The amounts shorted were substantial. Cordell, Huang, and Williams (2011) estimate that synthetic CDOs, issued mostly after 2005H2, more than doubled the amount of BBB Home Equity Bonds placed in CDOs between 1998 and 2007.[7] It is unlikely that this supply would have been absorbed without any price impact. In any case, starting in the second half of 2007, prices appear to exhibit substantial discounts relative to fundamentals.[8]

[2] See, for example, Carlos, Neal, and Wandschneider (2006) on the South Sea bubble, Hong and Stein (2007) on the Roaring Twenties, Ofek and Richardson (2003) and Cochrane (2002) on the Internet bubble, and Xiong and Yu (2011) on the Chinese warrant bubble.

[3] See Ofek and Richardson (2003).

[4] The directors of the SSC understood that bubble companies competed with the SSC's conversion scheme and could deflate its own bubble. Harris (1994) documents that the Bubble Act of 1720, which banned joint-stock companies unless if authorized by Royal Charter, was issued at the behest of the company to limit the competition for capital.

[5] Except for the assumption of positive net supply, questions concerning the supply of the asset subject to bubbles are also ignored in the rational bubbles literature (Santos and Woodford (1997)).

[6] See Scheinkman (2014) for a summary or Lewis (2015) for an excellent detailed account.

[7] BBB tranches of Home Equity Bonds were an important part of the CDO machine that transformed subprime mortgages into AAA-rated bonds.

[8] Beltran, Cordell, and Thomas (2017) provide a methodology to calculate the intrinsic value of a CDO and apply it to market data (see their Appendix A). They attribute the low prices to the increase in information asymmetry between buyers and sellers who followed the downgrades of MBS securities by rating agencies in summer 2007. Analyzing the pricing of index CDS postcrisis, Stanton and Wallace (2011) suggest that the pricing reflected a limited supply of insurance of asset-backed securities, presumably relative to demand.

In this paper, we propose a finite-horizon continuous-time model with $n$ types of investors who trade a single asset and aim to maximize expected cumulative net gains from trade. These investors are risk neutral, face a constant interest rate, and have fluctuating heterogeneous beliefs about the evolution of a Markov state that determines the asset's payoff. In contrast to previous literature, shorting is allowed. Investors pay costs that are proportional to the square of their positions but the constant of proportionality that defines the cost of going short may be larger than the corresponding constant for going long. This asymmetry between the costs of going short and long is a well-known feature of financial markets (see, e.g., D'Avolio (2002)). The assumptions in the earlier literature correspond to infinite costs for short positions and constant marginal costs for long positions. The costs in our model, by contrast, can be thought of as capturing the monetary costs of holding a position (in particular, increasing costs of capital), as well as risks that we do not explicitly model, such as market-wide liquidity shocks that would force agents to liquidate their positions at unfavorable prices or the recall risk faced by short sellers.

Since costs are quadratic, an agent's marginal valuation of an asset will decrease as their position increases, as would be the case for risk-averse agents. We therefore view our setup as an alternative to a much less tractable model with risk aversion, with many of the same forces present.[9] In particular, we show that an increase in the aggregate supply of the asset decreases equilibrium prices. Importantly, using the two cost coefficients as separate parameters allows us to impose asymmetric costs and study the impact of changes in relative costs on prices. By contrast, traditional models with risk aversion treat longs and shorts symmetrically or rule out shorts by imposing portfolio constraints.

We model the asset's equilibrium price as a function of time and the current state. Types that expect prices to increase on average over the next instant choose to go long, with the size of their position depending on the difference between their expected price changes and the marginal cost of carrying long positions. The other types choose to go short, by amounts that depend on their expected price changes and the cost of carrying short positions. Equilibrium requires that the longs absorb the shorts plus an exogenous supply. Theorem 1 below shows that there exists a unique equilibrium price function and that it can be characterized by a partial differential equation (PDE). This equation is of Hamilton-Jacobi-Bellman type with a novel form. In particular, the optimization runs over the ways to divide agents into two groups at any time and state; at the optimum, these are optimists (holding long positions in equilibrium) and pessimists (holding shorts). A noteworthy feature is that supply enters mathematically as a running cost (i.e., like intermediate consumption in Merton's problem). Theorem 1 also quantifies how these costs influence the effect of

[9] One difference from models of disagreement that use risk aversion to avoid no-shorting constraints is that the presence of holding costs allows an equilibrium to exist even when agents disagree about perceived arbitrage opportunities.

optimists' and pessimists' views. For instance, as shorting gets more expensive relative to being long, optimists have a larger effect on the asset's price.

We show that an increase in supply decreases the equilibrium price and that a decrease (increase) in the cost of long (short) positions increases the price. We also show that as the cost of long positions converges to zero, the equilibrium price function converges to a function that does not depend on the cost of holding a short position or the amount supplied. In contrast, as the cost of shorting becomes prohibitive, the equilibrium price converges to a function that depends on the cost of carrying long positions as well as the exogenous supply of the asset.

To discuss the impact of speculation, we first characterize the static equilibrium price, that is, the price that prevails when retrading is not allowed and agents are forced to use buy-and-hold strategies. Previous literature identifies the difference between the dynamic price (where retrading is possible) and the static price as the size of a bubble.

A buyer of the asset today may forecast that at some future date, she would be able to sell at a price that would exceed her own valuation of the asset at that date. Because of this *resale option*, she may be willing to pay more than what she believes is the discounted value of the payoff of an asset. In the classical models, this option leads equilibrium prices to exceed the price that would prevail if retrading were ruled out. In addition, there is an *option to delay,* an option that is not highlighted in the earlier literature on heterogeneous beliefs. A speculator may plan to buy additional units of the asset in future states of the world with a larger difference between the asset price and her marginal valuation. However, if the marginal cost of holding a long position is constant, this delay option has no impact in equilibrium. The intuition is that, since agents are risk neutral and the marginal cost-of-carry is constant, a buyer of a positive amount of the asset must be indifferent as to the amount of the asset she buys. Hence, the delay option has no value for this buyer and the dynamic equilibrium price cannot be smaller than the static price. We prove that this comparison holds even if shorting is allowed (see Proposition 7). We further show that in the presence of increasing marginal costs of going long, the option to delay may outweigh the resale option and cause the dynamic price to be *lower* than the static one, even when shorting is prohibited (see Example 1). Thus, the assumption in the earlier literature that delivers the result that speculative prices exceed static prices is not the prohibition of short sales, but rather the assumption of constant marginal costs for carrying long positions.

When shorting is allowed, the short party has corresponding options. An agent who acquires a short position today may forecast that at some future date, he would be able to repurchase the asset at a price below his own valuation at that date. This option to resell a short position, that is, the option to cover shorts, decreases the minimum amount that pessimists would be willing to receive for shorting the asset, and thus put downward pressure on prices. The short party also enjoys an option to delay. In Example 2, we show that when the cost of holding a short position is close to the cost of holding a long position, the equilibrium price may be less than the static price. We argue that this may occur because the long party values the resale option less than the short party values the repurchase option. Example 2 can also be used to illustrate that an unexpected decrease in the cost of shorting can lead to a collapse of an asset price bubble, thus rationalizing a link between the decrease in the cost of shorting in the MBS market from 2005 to 2007 and the collapse of CDO prices.

The equilibrium in our model is not first-best except in the limit case of homogeneous beliefs. We show that the equilibrium price and allocations obtain as solutions to the problem of a time-consistent planner who subsidizes and taxes the cost-of-carry to maximize the initial price. We use this planner's problem to explain the structure of the PDE that characterizes equilibrium prices.

Our paper connects to a number of other contributions in the literature. In a pioneering study that accounts for short-sales and risk aversion in a continuous-time setting of heterogeneous beliefs, Dumas, Kurshev, and Uppal (2009) consider a "complete markets" model with two classes of agents, one of which is overconfident about a public signal. Overconfident investors' reaction to the signal introduces a risk factor—sentiment risk—which carries a risk premium and causes stock prices to be excessively volatile. In their model, the delay option must be valuable and supply affects equilibrium prices, but these effects are not explicitly analyzed. Instead, Dumas, Kurshev, and Uppal (2009) focus on identifying the trading strategy that would allow a rational investor to take advantage of excessive stock price volatility and sentiment fluctuations. The authors show that rational investors choose a conservative portfolio that is sensitive to their predictions about future realizations of sentiment. A related paper by David (2008) assumes the existence of distinct processes for output and dividends of a stock in zero net supply. The drifts of these processes are given by an unobserved, finite-state, continuous-time Markov chain. Agents agree to disagree on the probabilities of transitions across states and use the zero-supply stock to speculate against each other—creating an additional source of risk. David (2008) focuses on the relationship between the equity premium and the time variation in agents' consumption. In these papers, the costs of going short and long are symmetric (the short party receives the equilibrium price and must pay the dividends that accumulate until the position is closed), and thus, one cannot examine the effect of changes in shorting costs, the main object of interest in our paper.

The literature on asset pricing with search frictions that follows Duffie, Gârleanu, and Pedersen (2005) assumes that agents have fluctuating private benefits from holding an asset and that opportunities for trading are randomly distributed. Strict concavity of private benefits implies that the supply of the asset affects equilibrium prices. Our assumption of quadratic costs of holding a position could be similarly motivated as private benefits (but in our case, the differences come from heterogeneity of beliefs rather than benefits). Fluctuating private benefits also generate options to resell and to delay trading, options that are discussed in Lagos and Rocheteau (2006), Feldhütter (2012), and Hugonnier, Lester, and Weill (2018). These authors point out that, depending on the curvature of private benefits, an increase in trading opportunities may increase or decrease the price of the asset. In particular, the comparison between prices that would prevail when retrading opportunities are more or less frequent is ambiguous. However, short selling or changes in shorting costs are not emphasized in this literature.

Duffie, Gârleanu, and Pedersen (2002)[10] highlight the mechanics of shorting that prevails in markets in which shorts pay a fee to borrow assets from longs.[11] Agents disagree on the expected final payoff of an asset but there is no fluctuation of beliefs and hence no speculative behavior—all purchases are buy-and-hold. The dynamics arise because pessimists must meet longs and borrow their shares and these meetings occur with an intensity of $\lambda$ per unit of time. In the model of Duffie, Gârleanu, and Pedersen (2002), an increase in supply decreases equilibrium prices (see Proposition 5). There are no explicit costs of shorting, but an increase in the intensity of meetings, $\lambda$, has an ambiguous effect on prices—it decreases prices because the supply of shorts increases but it increases prices because longs are more likely to lend their shares.

Cvitanić and Malamud (2011) also study heterogeneous agents in an equilibrium model, with a focus on survival and market impact. They find that long-run price and portfolio impact are equivalent to the survival of an agent under different measures. Again, there is no fluctuation of beliefs—agents are optimists or pessimists because they over- or underestimate the (constant) drift parameter of dividends. By contrast, in our model, the notion of optimism is endogenous and state-dependent.

Fostel and Geanakoplos (2012) study a collateral equilibrium[12] in a model of financial innovation with heterogeneous beliefs. The introduction of a CDS leads to a decrease in the price of the underlying security, with the decrease being more dramatic if tranching of the security is already present. The introduction of this new derivative affects equilibrium prices, but in the model of Fostel and Geanakoplos (2012), the initial beneficiaries of the new contract are the optimists, who in the language of Fostel and Geanakoplos (2012) benefit from "tranching cash." By contrast, in our model, the initial beneficiaries of this cost decrease are the pessimists. Lewis (2015) documents that starting in early 2005, a small group of traders who had pessimistic views on the housing market lobbied International Swaps and Derivatives Association (ISDA)—the trade organization for over-the-counter market participants—to create standardized CDS contracts on mortgage-market securities that facilitated shorting.[13]Oehmke and Zawadowski (2016) also examine the effect of introducing a CDS in a model in which traders differ on their horizons and beliefs. They postulate a per-unit cost for trading bonds that affects long and short positions equally, while CDS trading is free. Thus, the introduction of a CDS decreases the cost for both longs and shorts. It leads former bond buyers to switch to protection selling and former bond shorters to buy protection. Moreover long-horizon traders now hold a long position on the bond while buying protection.

[10] See also Vayanos and Weill (2008).

[11] Synthetic CDOs allowed pessimists to short CDOs without borrowing the underlying securities.

[12] See Geanakoplos and Zame (1997).

[13] See, for example, Lewis (2015, pp. 48-50) on the creation of standardized CDS.

The net effect may be an *increase* in bond prices. Although the mechanism described in Oehmke and Zawadowski (2016) may have played a role in the CDO market, the sharp drop in the prices of CDO tranches suggests that it was overwhelmed by the decrease in the cost of shorting.

The paper is organized as follows. Section I describes the problem and characterizes the equilibrium as the solution to a Hamilton-Jacobi-Bellman equation. Section II presents comparative statics and limiting results. Section III discusses the role of speculation, while Section IV addresses the planner's problem. Section V concludes. The appendices provide the proofs and several extensions of our model.

## I. Equilibrium Price

In this section, we describe our formal setup and show that it leads to a unique equilibrium. The equilibrium price is described by a PDE of Hamilton-Jacobi-Bellman type.

### *A. Definition of the Equilibrium Price*

We consider $n \geq 1$ types, each with a unit measure of agents, which trade a security over a finite time interval $[0, T]$. For brevity, we often refer to a type as an agent. The security has a single payoff $f(X(T))$ at horizon $T$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function and $X(\omega)$, $\omega \in \Omega$, is the $d$-dimensional state process.[14] While there is no ambiguity about $f$, agents agree to disagree on the evolution of the state process. The views of agent $i$ are represented by a probability measure $Q_i$ on $\Omega$ under which $X$ follows the stochastic differential equation (SDE)

$$dX(t) = b_i(t, X(t))\, dt + \sigma_i(t, X(t))\, dW_i(t), \quad X(0) = x, \tag{1}$$

where $W_i$ is a Brownian motion of dimension $d'$ and the functions

$$b_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d'}$$

are deterministic. We assume throughout that (the components of) $b_i$ and $\sigma_i$ are in $C_b^{1,2}$, the set of bounded continuous functions $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ whose partial derivatives $\partial_t g$, $\partial_{x_i} g$, and $\partial_{x_i x_j} g$ exist and are continuous and bounded on $[0, T) \times \mathbb{R}^d$.[15] Moreover, we assume that $\sigma_i^2$ is uniformly parabolic, that is, its eigenvalues are uniformly bounded away from zero.[16] These conditions imply that the SDE (1) has a unique (strong) solution.

[14] More precisely, we take $X$ to be the coordinate-mapping process on the space $\Omega = C([0, T], \mathbb{R}^d)$ of continuous $d$-dimensional paths, equipped with the canonical filtration and sigma field. In what follows all processes are assumed to be progressively measurable.

[15] These conditions could be relaxed considerably. The present form avoids issues of technical nature.

[16] Given a matrix $A$, we write $A^2$ for the product $AA^\top$ of $A$ with its transpose $A^\top$.

Notice that we allow the differences in beliefs to affect both drift and diffusion coefficients. As volatility is more amenable to statistical estimation than the rate of return, the differences on drifts will typically be more significant. While much of the literature on disagreement in asset markets focuses on constant volatility processes and thus naturally assumes perfect agreement on volatilities, ample evidence suggests that more complex processes involving stochastic and time-varying volatility are necessary to understand empirical features of asset prices. In this context, it is quite plausible, as Epstein and Ji (2013) argue, that agents also differ in their forecasts for volatility.[17] It therefore seems worthwhile to establish that our equilibrium is robust to such differences. We note, however, that all of our results have interest, and all of our examples remain valid, when agents disagree only about drifts.

Agents trade the security throughout the interval $[0, T]$ at a time $t$ price $P(t)$ to be determined in equilibrium. The agents are subject to an instantaneous cost-of-carry $c$ that differs across long and short positions,[18]

$$c(y) = \begin{cases} \frac{1}{2\alpha_+} y^2, & y \geq 0, \\ \frac{1}{2\alpha_-} y^2, & y \leq 0. \end{cases} \tag{2}$$

Here, the (inverse) cost coefficients $\alpha_\pm$ are given constants that satisfy[19]

$$0 < \alpha_- \leq \alpha_+,$$

which means that the cost of shorting is higher than the cost of going long. An *admissible* portfolio for an agent is a bounded process $\Phi$.[20] We write $\mathcal{A}$ for the collection of these portfolios. The value $\Phi(t)$ indicates the number of units the agent holds of the security at time $t$, where the number can be negative in

[17] Note that while the past trajectory of $\sigma(t, X(t))$ can be inferred from the observation of $X(t)$, agents may very well differ in their forecasts. This is obvious if $\sigma$ depends on time $t$, but even if not, the past observation of $\sigma(t, X(t))$ will typically reveal little of the function $\sigma$ when $X$ is nonrecurrent (e.g., of dimension larger than two). We will see in Theorem 1 below that the pricing of the security does indeed depend on the future volatility over the entire time interval. This is quite natural as the same would be true in standard risk-neutral pricing when $f$ is a derivative on a stock, for instance.

[18] The assumption that costs are proportional to the square of the position does not accommodate the fact that borrowers of the stock may pay a fee quoted as an annualized percentage of the value of the loaned securities (the rebate rate). This assumption is made to simplify the exposition and to allow us to concentrate on the effects of the size of a position on an agent's marginal valuation. In Appendix A, we discuss how our results can be generalized if an additional linear term is added to the costs-of-carry. In addition, for tractability, we assume that costs are a function of the size rather than the value of the position. See Appendix C for a discussion of costs as a function of position values.

[19] In Appendix B, we examine how the model changes if costs of going long and/or short vary across agents.

[20] Boundedness could be replaced by suitable integrability conditions without affecting our results.

the case of a short position. Given a (semimartingale) price process $P$, agent $i$ seeks to maximize the expected net payoff [21]

$$E_i\left[\int_0^T \Phi(t)\,dP(t) - \int_0^T c(\Phi(t))\,dt\right], \tag{3}$$

where the first integral represents the profit or loss from trading and the second integral is the cumulative cost-of-carry incurred. Criterion (3) can be rationalized by assuming that agents can borrow and lend at an interest rate of zero and that the cost function $c$ is measured in the unit of account but can also be taken as a primitive utility function. We take interest rates to be exogenous because most bubbles affect only part of the capital market and hence have little effect on risk-free rates. The assumption that this exogenous rate equals zero is made to simplify the notation. To account for this in our discussions, we refer, for instance, to the case where $c(y) = 0$ for $y \geq 0$ as a constant (rather than zero) marginal cost of being long. An admissible portfolio $\Phi_i$ is *optimal* for agents of type $i$ if it maximizes (3) over all $\Phi \in \mathcal{A}$. We examine symmetric equilibria in which agents of the same type choose the same portfolio.

As the final input of our model, we introduce a nonnegative *supply function* $s \in C_b^{1,2}$.[22] The supply $S(t) = s(t, X(t))$ is owned by third parties that supply the asset inelastically.[23] Notice that this formalism allows the payoff $f(X(T))$ to depend on $S(T)$. An *equilibrium price* is a process $P$ satisfying $P(T) = f(X(T))$ a.s. under all $Q_i$ for which there exist admissible portfolios $\Phi_i$, $i \in \{1, \ldots, n\}$, such that $\Phi_i$ is optimal for agent $i$ and the market-clearing condition

$$\sum_{i=1}^{n} \Phi_i(t) = S(t)$$

holds.[24] We are interested in Markovian equilibria, that is, equilibrium prices of the form $P(t) = v(t, X(t))$ for a function $v$, which we refer to as an *equilibrium price function*.

[21] To ensure that the expectation is well defined a priori, we set $E_i[Y] := -\infty$ whenever $E_i[\min\{0, Y\}] = -\infty$ for any random variable $Y$. For the processes $P$ that occur in our results below, (3) will be finite for any $\Phi \in \mathcal{A}$.

[22] Our results could be extended to a discontinuous supply shock as in Hong, Scheinkman, and Xiong (2006) using backward induction.

[23] Since the utility function in (3) is separable, the equilibrium price is invariant to endowments. Hence, we could have alternatively assumed an arbitrary ownership structure for the endowment across investor types—we opted for the simpler presentation.

[24] More precisely, $P$ is a continuous semimartingale, which ensures that the integrals of $\Phi_i$ are well defined. This is automatically satisfied for the processes considered below.

### B. Existence and PDE for the Equilibrium Price

The following notation will be useful to state our first result. Given $v \in C_b^{1,2}$, we define the function $\mathcal{L}^i v$ by

$$\mathcal{L}^i v(t,x) = \partial_t v(t,x) + b_i \partial_x v(t,x) + \frac{1}{2} \operatorname{Tr} \sigma_i^2 \partial_{xx} v(t,x). \tag{4}$$

Here, $\partial_x v$ denotes the gradient vector, $\partial_{xx} v$ the Hessian matrix, and $\operatorname{Tr} \sigma_i^2 \partial_{xx} v$ the trace of the matrix $\sigma_i^2 \partial_{xx} v$, that is, the sum of the entries on the diagonal. One can interpret $\mathcal{L}^i v(t,x)$ as the change in $v$ that agents of type $i$ expect over an infinitesimal time interval after $t$.

Before stating the general characterization of equilibria in Theorem 1 below, we develop the heuristics in two particular cases. We suppose that $X$ is one-dimensional and the coefficients $b_i$ and $\sigma_i$ are constant.

We first derive the first-order conditions for the portfolios. Suppose that we are in an equilibrium with price $P(t) = v(t, X(t))$. Itô's formula states that under $Q_i$,

$$\begin{aligned} dP(t) &= \partial_t v(t,x)\,dt + b_i \partial_x v(t,x)\,dt + \frac{1}{2}\sigma_i^2 \partial_{xx} v(t,x)\,dt + \sigma_i \partial_x v(t,x)\,dW_i(t) \\ &= \mathcal{L}^i v(t,x)\,dt + \sigma_i \partial_x v(t,x)\,dW_i(t). \end{aligned}$$

Thus, the expected final payoff (3) for a portfolio $\Phi$ is

$$E_i\left[\int_0^T \Phi(t)\,dP(t) - \int_0^T c(\Phi(t))\,dt\right] = E_i\left[\int_0^T \{\Phi(t)\mathcal{L}^i v(t, X(t)) - c(\Phi(t))\}\,dt\right],$$

where we have used the fact that the $dW$-integral has zero expectation. To optimize this quantity, we simply maximize the integrand with respect to $\Phi(t)$ at every $t$, that is, we set the marginal expected gain $\mathcal{L}^i v(t, X(t)) - c'(\Phi(t)) = 0$. The latter formula shows that the equilibrium *holding premium* for type $i$, which equals the expected price change $\mathcal{L}^i v$ since the interest rate is zero, is generated by the holding cost. Using the quadratic form (2) of $c$, we derive the optimal portfolio

$$\Phi_i(t) = \phi_i(t, X(t)), \quad \text{where} \quad \phi_i(t,x) = \alpha_{\operatorname{sign}(\mathcal{L}^i v(t,x))} \mathcal{L}^i v(t,x).$$

In particular, agents are myopic given the price function and its derivatives (whereas the price itself incorporates agents' expectations about the future of the state process $X$).

Next, we derive an equation for $v$ in two special cases. First, in the homogeneous case in which all agents have the same views: $b_i = b$ and $\sigma_i = \sigma$. Thus, $\mathcal{L}^i v(t,x) = \mathcal{L} v(t,x)$ is also independent of $i$ and the optimal positions are identical across agents; in particular, there is no short selling in equilibrium and the first-order condition becomes $\Phi_i(t) = \alpha_+ \mathcal{L}v(t, X(t))$. Market clearing requires that $\alpha_+ \mathcal{L}v(t, X(t)) = S(t)/n$, or

$$\partial_t v(t,x) + b\partial_x v(t,x) + \frac{1}{2}\sigma^2 \partial_{xx} v(t,x) - \frac{s(t,x)}{n\alpha_+} = 0.$$

This PDE is linear and supply enters as a running cost: the equilibrium price must compensate for the cost-of-carry.

Second, consider $n = 2$ types of agents who disagree on the drift coefficient $\mu_i$ but agree on the volatility $\sigma := \sigma_1 = \sigma_2$. To further simplify the derivation, consider the case of zero net supply. Market clearing requires $\phi_1 + \phi_2 = 0$, and thus one type must be long while the other must be short. Therefore, there are two possibilities at each $(t, x)$:

$$\mathcal{L}^1 v(t,x) \le 0 \text{ and } \mathcal{L}^2 v(t,x) \ge 0, \text{ thus } \alpha_- \mathcal{L}^1 v(t,x) + \alpha_+ \mathcal{L}^2 v(t,x) = 0, \text{ or}$$

$$\mathcal{L}^1 v(t,x) \ge 0 \text{ and } \mathcal{L}^2 v(t,x) \le 0, \text{ thus } \alpha_+ \mathcal{L}^1 v(t,x) + \alpha_- \mathcal{L}^2 v(t,x) = 0.$$

Recalling that $\alpha_- \le \alpha_+$, it follows that

$$\text{if } \mathcal{L}^1 v(t,x) \le 0 \text{ and } \mathcal{L}^2 v(t,x) \ge 0, \text{ then } \alpha_+ \mathcal{L}^1 v(t,x) + \alpha_- \mathcal{L}^2 v(t,x) \le 0;$$

$$\text{if } \mathcal{L}^1 v(t,x) \ge 0 \text{ and } \mathcal{L}^2 v(t,x) \le 0, \text{ then } \alpha_- \mathcal{L}^1 v(t,x) + \alpha_+ \mathcal{L}^2 v(t,x) \le 0.$$

Hence, in all cases, it holds that

$$\max\left\{\alpha_- \mathcal{L}^1 v(t,x) + \alpha_+ \mathcal{L}^2 v(t,x);\ \alpha_+ \mathcal{L}^1 v(t,x) + \alpha_- \mathcal{L}^2 v(t,x)\right\} = 0.$$

Next, divide the equation above by $\alpha_- + \alpha_+$ and plug in the definitions of $\mathcal{L}^1$ and $\mathcal{L}^2$. After rearranging terms, one obtains

$$\partial_t v(t,x) + \max_{(i,j)=(1,2),(2,1)} \left\{ \left( \frac{\alpha_-}{\alpha_- + \alpha_+} b_i + \frac{\alpha_+}{\alpha_- + \alpha_+} b_j \right) \partial_x v(t,x) \right\} + \frac{1}{2}\sigma^2 \partial_{xx} v(t,x) = 0.$$

Disagreement about drifts causes a nonlinearity in the first-order term. Similarly, disagreement about volatilities would cause a nonlinearity in the second-order term. The next theorem states that an analogous PDE uniquely characterizes the equilibrium price function in our model. In general, the maximization above over two possibilities is replaced by a maximization over "groups" $I \subseteq \{1, \ldots, n\}$ of agents; we denote by $|I|$ the number of agents in $I$ and by $I^c = \{1, \ldots, n\} \setminus I$ the complementary group.

THEOREM 1: *(i) There exists a unique equilibrium price function $v \in C_b^{1,2}$. The corresponding optimal portfolios are unique and given by $\Phi_i(t) = \phi_i(t, X(t))$,*[25] *where*

$$\phi_i(t,x) = \alpha_{\mathrm{sign}(\mathcal{L}^i v(t,x))} \mathcal{L}^i v(t,x). \tag{5}$$

[25] Uniqueness is understood up to $(Q_i \times dt)$-nullsets.

*(ii) The function $v \in C_b^{1,2}$ can be characterized as the unique solution of the PDE*

$$\partial_t v(t,x) + \sup_{I \subseteq \{1,\dots,n\}} \left( \mu_I(t,x)\partial_x v(t,x) + \frac{1}{2} \operatorname{Tr} \Sigma_I^2(t,x)\partial_{xx} v(t,x) - \kappa_I(t,x) \right) = 0 \tag{6}$$

*on $[0,T) \times \mathbb{R}^d$ with terminal condition $v(T,x) = f(x)$, where the supremum is taken over all subsets $I \subseteq \{1,\dots,n\}$ and the coefficients are defined as*

$$\mu_I(t,x) = \frac{\alpha_-}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i\in I} b_i(t,x) + \frac{\alpha_+}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i\in I^c} b_i(t,x), \tag{7}$$

$$\Sigma_I^2(t,x) = \frac{\alpha_-}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i\in I} \sigma_i^2(t,x) + \frac{\alpha_+}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i\in I^c} \sigma_i^2(t,x), \tag{8}$$

$$\kappa_I(t,x) = \frac{s(t,x)}{|I|\alpha_- + |I^c|\alpha_+}. \tag{9}$$

*Moreover, a maximizer for the supremum in (6) is given by*

$$I_*(t,x) = \{i \in \{1,\dots,n\} : \mathcal{L}^i v(t,x) < 0\}. \tag{10}$$

In equilibrium, group $I_*$ of (10) corresponds to the more pessimistic agents (those holding shorts), whereas $I_*^c$ corresponds to the optimists (those holding long positions). Formulas (7) and (8) for $\mu_I$ and $\Sigma_I$ can be seen as a weighted average of the agents' drift and volatility coefficients. The weights imply that when shorting is more expensive than being long (i.e., $\alpha_-$ is small relative to $\alpha_+$), optimists have a larger impact on the equilibrium price. In Section II, we show that when $\alpha_- \to 0$ or $\alpha_+ \to \infty$, the more pessimistic views are not reflected in the equilibrium price at all. The running cost $\kappa_I$ of (9) depends linearly on the exogenous supply $s$, which is divided by a weighted sum of the cost coefficients, the weights being the size of the set $I$ and its complement $I^c$, respectively. Since $\alpha_- \leq \alpha_+$, this cost increases with the number $|I|$ of types in group $I$.

We will see in Section IV that the precise form of the PDE (6) with a supremum can be explained through the problem of a planner with limited instruments. To obtain an initial intuition, note that using (4), the left-hand side of the PDE can be read as the difference between two quantities. The first is a weighted average over the instantaneous holding premia. The second quantity is related to the instantaneous marginal cost of carrying positions. The PDE sets this difference to zero when the weights correspond to the particular group $I = I_*$.

Mathematically, the PDE (6) is of Hamilton-Jacobi-Bellman type, which implies that $v$ can be represented as the value function of a stochastic optimal control problem. This is useful for our derivation of comparative statics and limiting results presented below but has no obvious economic interpretation. The control problem is presented in Appendix D.

REMARK 1: *The equilibrium price $v(t,x)$ is 0-homogeneous in $(\alpha_-, \alpha_+, s)$, which suggests that supply and costs are closely linked in our model. That is, if these parameters are replaced by $(\lambda\alpha_-, \lambda\alpha_+, \lambda s)$ for some $\lambda > 0$, the price does not change. This follows from Theorem 1 (ii) after observing that the coefficients $\mu_I$, $\Sigma_I$, and $\kappa_I$ are invariant under this substitution. In the special case $s = 0$, the homogeneity implies that the price depends on $(\alpha_-, \alpha_+)$ only via the ratio $\alpha_+/\alpha_-$.*

## II. Comparative Statics and Limiting Cases

In the first part of this section, we establish comparative statics with respect to the supply and cost parameters. In the second part, we analyze the limit $\alpha_+ \to \infty$ when there is no cost-of-carry for long positions, as well as the limit $\alpha_- \to 0$ when short positions are ruled out.

### *A. Comparative Statics*

We start with the dependence on supply.

PROPOSITION 1: *The equilibrium price function $v$ is monotone decreasing with respect to the supply function $s$: prices decrease with an increase in supply.*

Next, we turn to the cost parameters $\alpha_-$ and $\alpha_+$. The following proposition shows that the equilibrium price is decreasing with respect to the cost-of-carry for long positions and increasing with respect to the cost for short positions.

PROPOSITION 2: *The equilibrium price function $v$ is*

*(i) increasing with respect to $\alpha_+$,*
*(ii) decreasing with respect to $\alpha_-$, and*
*(iii) increasing with respect to the quotient $\alpha_+/\alpha_-$ if $s \equiv 0$.*

The proof uses our PDE characterization of the price and a comparison theorem from the theory of parabolic PDE.[26]

The following is a partial extension of (iii) to the case of nonzero supply, which is useful if $\alpha_-$ and $\alpha_+$ are varied simultaneously.

REMARK 2: *Let $\alpha_- \leq \alpha_+$ and $\alpha'_- \leq \alpha'_+$ be two pairs of cost coefficients and let $v$ and $v'$ be the corresponding equilibrium price functions. If the coefficients satisfy $\alpha_+/\alpha_- \leq \alpha'_+/\alpha'_-$ and $\alpha_- \leq \alpha'_-$, then $v \leq v'$.*

[26] Comparison theorems are useful to show that two functions satisfy an inequality on their domain if they are known to satisfy an (in)equality on the boundary. In our context, we may think of the equilibrium price function as satisfying the PDE $F(v, \beta) = 0$, where $\beta$ is a parameter. If $v_1$ and $v_2$ are price functions corresponding to different parameters $\beta_1$ and $\beta_2$, we know that they are equal at the boundary $t = T$ since they satisfy the same terminal condition $f$. If $v_2$ is a subsolution of the PDE for $v_1$, that is, $F(v_2, \beta_1) \geq 0$, the comparison theorem implies that $v_1 \geq v_2$. See, for example, Fleming and Soner (2006).

*For instance, it follows that if the costs-of-carry for long and short positions are increased by a common factor, then the price decreases.*

### *B. Limiting Models*

We discuss two limits for the cost coefficients that shed light on the relationship between our model and the earlier models discussed in the introduction. To make the dependence on the parameters explicit, we denote by $v^{\alpha_-,\alpha_+}(t,x)$ the equilibrium price function $v(t,x)$ for $\alpha_-, \alpha_+$.

#### *B.1. Zero Cost for Long Positions*

We first consider the limit $\alpha_+ \to \infty$ when the cost-of-carry for long positions tends to zero.

PROPOSITION 3: *As $\alpha_+ \to \infty$, the function $v^{\alpha_-,\alpha_+}$ converges to the unique solution $v^\infty \in C_b^{1,2}$ of the PDE*

$$\partial_t v + \sup_{i\in\{1,\dots,n\}} \left( b_i \partial_x v + \frac{1}{2} \operatorname{Tr} \sigma_i^2 \partial_{xx} v \right) = 0 \tag{11}$$

*with terminal condition $v(T,x) = f(x)$. In particular, $v^\infty$ is independent of $\alpha_-$ and $s$. The convergence is locally uniform in $(t,x)$ and is monotone increasing if $\alpha_+ \uparrow \infty$.*

We now discuss the limiting model that arises in Proposition 3, that is, with no cost-of-carry for long positions. We state these results without proofs since the arguments are very similar to the proof of Theorem 1.

The limiting model has an equilibrium price function $v := v^\infty$ that is unique and independent of the supply $s$ and the cost coefficient $\alpha_-$ for short positions. Thus, we obtain the results of previous models with risk-neutral agents in this limiting regime. The intuition for equation (11) is straightforward. In any equilibrium, if $j$ is one of the most optimistic types, we must have $\mathcal{L}^j v(t,x) \geq 0$. However, if the marginal cost of going long is zero, $\mathcal{L}^j v(t,x) = 0$ must hold. In particular, $j$ is indifferent with respect to nonnegative positions and equilibrium prices are independent of both the supply of the asset and the demand for shorting. However, in equilibrium, the optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ do depend on $s$ and $\alpha_-$. Given $(t,x)$, if $i$ is not a maximizer, $\mathcal{L}^i v(t,x) < 0$ and

$$\phi_i(t,x) = \alpha_- \mathcal{L}^i v(t,x)$$

as in (5); in particular, agent $i$ holds a short position. In equilibrium, the aggregate amount held by the most optimistic types is set by the market-clearing condition—they must hold the sum of the exogenous supply and all amounts shorted. If there is more than one maximizer $i$, then any distribution of the available amount (supply plus short positions) over these maximizers gives an optimal allocation.[27]

[27] See Muhle-Karbe and Nutz (2018) for an analysis of this case when shorting is constrained.

The properties described above for $\alpha_+ = \infty$ continue to hold in the limiting case $\alpha_- = 0$, that is, when there is no cost for long positions and short positions are prohibited. In particular, all but the most optimistic agents hold a flat position, and only the most optimistic characteristics play a role in determining the price. We therefore obtain the results of previous models with risk-neutral agents in this limiting regime.

REMARK 3: *The results for $\alpha_+ = \infty$ may be contrasted with the opposite extreme case in which the cost coefficients $\alpha_+$ and $\alpha_-$ are equal. Then, the drift and volatility coefficients*

$$\mu := \mu_I = \frac{1}{n}\sum_{i=1}^{n} b_i, \qquad \Sigma^2 := \Sigma_I^2 = \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2$$

*are independent of $I$ and equal to the arithmetic average of the coefficients in the agents' models, meaning that all agents contribute equally to the price. The running cost is $\kappa := \kappa_I = s/(n\alpha_+)$. Thus, (6) becomes the linear PDE*

$$\partial_t v + \frac{1}{n}\sum_{i=1}^{n} b_i \partial_x v(t,x) + \frac{1}{2n}\sum_{i=1}^{n} \operatorname{Tr} \sigma_i^2 \partial_{xx} v - \frac{s}{n\alpha_+} = 0,$$

*and by the Feynman-Kac formula, the equilibrium price is*

$$v(t,x) = E\left[ f(X^{t,x}(T)) - \int_t^T s(r, X^{t,x}(r))/(n\alpha_+)\, dr \right],$$

*where $X^{t,x}$ is a diffusion with drift $\mu$, volatility $\Sigma$, and initial condition $X^{t,x}(t) = x$ (see Appendix D). That is, the equilibrium price is simply the expected value of the security under the averaged coefficients of the agents, minus a cost term related to the supply.*

### *B.2. Infinite Cost for Short Positions*

We now discuss the limit $\alpha_- \to 0$, that is, the cost-of-carry for short positions tends to infinity.

PROPOSITION 4: *As $\alpha_- \to 0$, the function $v^{\alpha_-,\alpha_+}$ converges to the unique solution $v^{0,\alpha_+} \in C_b^{1,2}$ of the PDE*

$$\partial_t v + \sup_{\emptyset \neq J \subseteq \{1,\dots,n\}} \left( \frac{1}{|J|}\sum_{i\in J} b_i \partial_x v + \frac{1}{2}\operatorname{Tr} \frac{1}{|J|}\sum_{i\in J} \sigma_i^2 \partial_{xx} v - \frac{s}{|J|\alpha_+} \right) = 0 \qquad (12)$$

*with terminal condition $v(T,x) = f(x)$. In the special case $s = 0$, this PDE coincides with (11) and in particular the solution $v^{0,\alpha_+} = v^\infty$ is independent of $\alpha_+$. The convergence is locally uniform in $(t,x)$ and is monotone increasing if $\alpha_- \downarrow 0$.*

The limiting model that arises in Proposition 4 corresponds to a prohibition of shorting. This model has a unique equilibrium price function $v := v^{0,\alpha_+}$

that depends on the supply $s$ and the cost coefficient $\alpha_+$ for long positions. At each state $(t, x)$, we can think of the types as being divided into relatively optimistic agents, $J = \{i \in \{1, \dots, n\} : \mathcal{L}^i v(t, x) \geq 0\}$, and pessimists, $J^c$. We have $J \neq \emptyset$ by market clearing. While the agents in $J$ hold positions $\alpha_+ \mathcal{L}^i v(t, x)$ of different magnitude depending on how optimistic they are, the entire group $J$ determines the price. Agents in $J^c$, however, hold zero units and their precise characteristics do not enter price formation. For instance, if we replace a pessimistic type $i \in J^c$ by an even more pessimistic type, the equilibrium price will not change.

## III. Speculation

In this section, we highlight the impact of nonlinear costs-of-carry and short selling on the pricing mechanism by comparing the above "dynamic" equilibrium price at time $t = 0$ with a "static" equilibrium price, that is, an equilibrium without speculation. We shall see that, as in previous models, the dynamic price dominates the static price when cost-of-carry and short selling are removed from our model. This can be attributed to the resale option. However, we show that the cost-of-carry (i.e., risk aversion) gives rise to a delay option that may act in opposition to the resale option and in extreme cases may reverse the order of the prices—even if short selling is prohibited. Moreover, we illustrate that the possibility of short selling tends to depress the dynamic price as it gives rise to a repurchase option for pessimists.

### A. Equilibrium without Speculation

Consider a situation in which trading occurs only at the initial time $t = 0$, that is, agents are forced to use buy-and-hold strategies and speculation is ruled out. The agents use the same models $Q_i$ for the dynamics (1) of the state process $X$ and maximize the same expected net payoff (3). However, the admissible portfolios $\Phi$ are restricted to be constant; we use the letter $q$ to denote a generic portfolio. This market can clear only if the exogenous supply $S \equiv s$ is constant, so we restrict our attention to that case. A *static equilibrium price* is defined like the dynamic equilibrium price above, except that we only look for a constant $p_{\text{sta}} \in \mathbb{R}$ at time $t = 0$ at which the trading happens.

PROPOSITION 5: *(i) There exists a unique static equilibrium price and it is given by*

$$p_{\text{sta}} = \max_{I \subseteq \{1,\dots,n\}} \left( \frac{\alpha_-}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i \in I} e_i + \frac{\alpha_+}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i \in I^c} e_i - \frac{sT}{|I|\alpha_- + |I^c|\alpha_+} \right), \quad (13)$$

*where $e_i = E_i[f(X(T))]$. The corresponding optimal static portfolios are unique and given by*

$$q_i = \alpha_{\text{sign}(e_i - p_{\text{sta}})} T^{-1} (e_i - p_{\text{sta}}). \quad (14)$$

The formula for the static price is the direct analog of the PDE (6) for the dynamic price. Indeed, the PDE considers the difference between a weighted average of instantaneous holding premia and the instantaneous marginal cost of carrying positions. Equation (13) can be read in the same way, after bringing $p_{\text{sta}}$ to the right-hand side: it considers the difference between the weighted average of the holding premia $e_i - p_{\text{sta}} = E_i[f(X(T))] - p_{\text{sta}}$ over the whole interval and the marginal cost of carrying positions over that same interval. Informally, we may think of the PDE as describing a repeated version of the static problem over infinitesimal intervals.

Following our analysis above, we consider limiting cases for the cost coefficients in the static case. We denote by $p_{\text{sta}}^{\alpha_-,\alpha_+}$ the static equilibrium price for cost parameters $\alpha_-$, $\alpha_+$, and initial value $X(0) = x$ as given by (13).

PROPOSITION 6: *(i) In the limit $\alpha_+ \to \infty$ with zero cost for long positions, the price $p_{\text{sta}}^{\alpha_-,\alpha_+}$ converges to*

$$p_{\text{sta}}^{\infty} = \max_{i\in\{1,\dots,n\}} E_i[f(X(T))]. \tag{15}$$

*(ii) In the limit $\alpha_- \to 0$ with infinite cost for short positions, the price $p_{\text{sta}}^{\alpha_-,\alpha_+}$ converges to*

$$p_{\text{sta}}^{0,\alpha_+} = \max_{\emptyset\neq J\subseteq\{1,\dots,n\}} \left(\frac{1}{|J|}\sum_{i\in J} E_i[f(X(T))] - \frac{sT}{|J|\alpha_+}\right). \tag{16}$$

The intuition is the same as in Section B. Without a cost for holding long positions, optimists are indifferent with respect to nonnegative portfolios and the price is determined solely by the most optimistic agents. When shorting is ruled out, the price is determined as an average over a group of relatively more optimistic agents, while the complementary group of more pessimistic agents does not influence the price directly. We omit the proof of Proposition 15 since it is similar to those in Section B.

### *B. Resale and Delay Options*

Next, we compare the dynamic equilibrium price $p_{\text{dyn}} := P(0)$ at time $t = 0$ with the static equilibrium price $p_{\text{sta}}$. For the latter to be well defined, we assume that the supply $s$ is constant. We discuss two options that are present under dynamic trading and that are valued by agents—the resale and delay options—and the effect on prices of eliminating these options by forcing agents to trade only at time zero. In particular, we shall see that the ordering of $p_{\text{dyn}}$ and $p_{\text{sta}}$ may be different than in earlier models.

Previous papers, starting with Harrison and Kreps (1978), consider models with risk-neutral agents who face a constant marginal cost-of-carry for long positions (the interest rate) and cannot sell short. In such models, the dynamic equilibrium price exceeds the static one, with the difference attributed to the "resale option." The possibility of reselling the asset increases the price—agents may want to buy today in order to resell to agents who are more optimistic tomorrow. In these "classical" models, agents may also plan to buy additional units of the asset in some future states of the world. This possibility, however, does not alter the ranking between the dynamic and static equilibrium prices. Indeed, since agents are risk neutral and the marginal cost of carrying a long position is independent of the size of the position, we may assume generically that only one type $i$ would acquire the asset in the static equilibrium and pay its marginal valuation at time zero. When retrading is allowed, $i$'s marginal valuation for holding the full supply of the asset at time zero is at least as large, since an agent can always choose a buy-and-hold strategy. As the market price must exceed the marginal valuation of any type, the dynamic equilibrium price must exceed the static equilibrium price.

The next two results confirm this intuition by showing how this mechanism carries over to limiting cases of our model. First, we show that when the marginal cost of long positions is constant, the dynamic price exceeds the static one. This holds even when shorting is allowed because in this extreme case, only the most optimistic agents contribute to the price formation, just as in the classical models (see also Propositions 3 and 6).

PROPOSITION 7: *In the limit $\alpha_+ \to \infty$, the dynamic equilibrium price dominates the static price: $p^{\infty}_{\text{dyn}} \geq p^{\infty}_{\text{sta}}$.*

Next, we show that if short-sales are prohibited *and* if in the static equilibrium, only one type holds the asset,[28] the dynamic equilibrium price again exceeds the static price, even when longs face an increasing marginal cost-of-carry.

PROPOSITION 8: *In the limit $\alpha_- \to 0$ with no short selling, suppose that type $i$ holds the entire market in the static equilibrium, that is, $q_j = 0$ for all $j \neq i$. Then the dynamic equilibrium price dominates the static price: $p^{0,\alpha_+}_{\text{dyn}} \geq p^{0,\alpha_+}_{\text{sta}}$.*

We now turn to the case in which both marginal costs are increasing and finite. Here, the same options to resell and to delay are present, but the effects are more subtle. The option to delay now affects equilibrium prices because the marginal valuation of buyers varies with the size of their position. More importantly, trading may occur in the dynamic equilibrium even though one type remains the most optimistic. Indeed, in the classical models (and the limiting model of Proposition 3), the most optimistic type always holds the full supply and trading requires that relative optimism changes sign. When the marginal cost-of-carry for long positions is increasing, the magnitude of relative optimism determines equilibrium holdings—it is no longer true that a less optimistic type would always hold a nonpositive amount. Example 1 below illustrates that the delay option may have an important effect on prices and even reverse the ordering of dynamic and static prices.

If shorting is allowed, buying today in order to resell tomorrow needs to be compared with entering a short position tomorrow. The choice will depend on,

[28] Only one type will hold the asset if that type is sufficiently more optimistic than the others and the supply is small enough.

among other factors, the costs-of-carry for long and short positions. The option to resell a short position, that is, the option to cover shorts, and then decreases the minimum amount that pessimists would be willing to receive to short the asset, and this puts downward pressure on prices. Shorts may also exercise the option to delay by building up a short position over time. Example 2 below illustrates how the ordering of dynamic and static prices can be reversed if the cost of shorting is sufficiently low.

REMARK 4: *In the remainder of this section, we use a quadratic payoff function $f$ to obtain explicit formulas. This violates our assumption that $f$ is bounded but our results still apply with the appropriate modifications. In particular, the equilibrium price function $v$ and the admissible portfolios $\phi_i$ exhibit polynomial growth instead of being bounded. The formulas in our examples can also be verified by direct calculation.*

### *C. Illustrating the Effect of the Delay Option*

In this section, we show that the static price may dominate the dynamic price even when short selling is prohibited. This cannot be explained with a resale option; instead, it highlights the delay option. Consider first the dynamic equilibrium and suppose that type $i$ expects their portfolio $\Phi_i(t)$ to increase over time with high probability. If only buy-and-hold strategies are allowed, an agent of type $i$ would consider anticipating the increase of the portfolio at time $t = 0$, and if the additional expected gains outweigh the additional costs-of-carry, the agent would have higher buy-and-hold demand at the previous equilibrium price $p_{\mathrm{dyn}}$. Other types may reduce their positions at the price $p_{\mathrm{dyn}}$ because they are anticipating a decrease in position or because they are indifferent to the amount they are holding (see also Example E2 in Appendix E).

To show that the static price may exceed the dynamic price even when short selling is prohibited ($\alpha_- = 0$), we impose a positive cost for long positions ($\alpha_+ = 1$) and construct an example in which some agents expect to increase their positions over time but no agent expects to decrease their position.[29] To obtain explicit formulas despite the nonlinear context, we consider the limiting case of zero volatility but show later (Proposition 9) that this is indeed the continuous limit for equilibria with small volatility coefficients $\sigma_i$. In particular, the qualitative conclusions of the example extend to examples with diffusion risk. The zero-volatility case violates our assumption of uniformly parabolic coefficients (indeed, $v$ is not smooth in this example) but the formulas can be verified by direct calculation.

EXAMPLE 1: Consider $n = 2$ types with volatility coefficients $\sigma_i = 0$ and constant, opposing drifts

$$b_1 = 1, \qquad b_2 = -1.$$

[29] Since types disagree, it may indeed be the case that all agents expect to increase their positions over time in the dynamic case, without contradicting the market-clearing condition.

The payoff function is $f(y) = y^2$ and supply $s > 0$ is constant. Moreover, $\alpha_- = 0$ and $\alpha_+ = 1$. Then, as we show in Appendix E, the static equilibrium price exceeds the dynamic price. More precisely,

$$p_{\text{sta}} - p_{\text{dyn}} = \begin{cases} T^2, & |x| \leq s/4 - T/2, \\ (s/2 - 2|x|)T, & s/4 - T/2 < |x| < s/4, \\ 0, & |x| \geq s/4. \end{cases}$$

In the regimes $s/4 - T/2 < |x| < s/4$ and $|x| \leq s/4 - T/2$, at least one of the types has a dynamic portfolio that is increasing over time. These agents exercise the delay option when retrading is allowed and have an anticipatory motive when they can trade only at $t = 0$. A price increase is necessary to clear the static market, leading to $p_{\text{sta}} > p_{\text{dyn}}$. In Appendix E, we detail the asset allocation in all regimes and show how the delay option explains the difference $p_{\text{sta}} - p_{\text{dyn}}$.

It remains to prove that the conclusions of the example also hold when volatilities are small but positive, rather than vanishing.

PROPOSITION 9: *Consider the setting of Example 1 with constant volatilities* $\sigma := \sigma_1 = \sigma_2 \geq 0$ *and denote the corresponding static and dynamic equilibrium prices by* $p^\sigma_{\text{sta}}$ *and* $p^\sigma_{\text{dyn}}$*, respectively. Then* $p^\sigma_{\text{sta}} \downarrow p^0_{\text{sta}}$ *and* $p^\sigma_{\text{dyn}} \downarrow p^0_{\text{dyn}}$ *as* $\sigma \downarrow 0$. *As a consequence, we have*

$$p^\sigma_{\text{sta}} - p^\sigma_{\text{dyn}} \to \begin{cases} T^2, & |x| \leq s/4 - T/2, \\ (s/2 - 2|x|)T, & s/4 - T/2 < |x| < s/4, \\ 0, & |x| \geq s/4. \end{cases}$$

The above example of the delay option effect should be contrasted with Proposition 7, where we have seen that when there is no cost-of-carry for long positions ($\alpha_+ = \infty$), the dynamic equilibrium price always exceeds the static one, even if short selling is possible. Example E2 in Appendix E illustrates the mechanics of the delay option in the latter situation. In Example E2, pessimists plan to close their short position over time in the dynamic equilibrium. When forced to buy-and-hold, they decrease their initial short position. However, in contrast to Example 1, this has no effect on the static price because, as we have argued, optimists are indifferent to the size of their own position in the absence of increasing marginal costs.

### D. Illustrating the Effect of Shorting

The following example illustrates that when shorting is allowed, the static price may exceed the dynamic price—this is quite natural once we observe the symmetry between optimists and pessimists in the extreme case $\alpha_- = \alpha_+$. The difference between the dynamic price and the static price has been identified in the previous literature as the size of the “speculative bubble.” If we maintain this identification, the example can be used to illustrate how decreasing the cost of shorting can lead not only to a bubble implosion but also to a negative bubble.

EXAMPLE 2: To facilitate computations, we assume symmetric costs-of-carry $\alpha_- = \alpha_+ = 1$. Consider $n = 2$ types with constant coefficients $b_i \in \mathbb{R}$ and $\sigma_i > 0$, and an asset in zero aggregate supply with payoff $f(y) = y^2$. Writing $\Sigma^2 := (\sigma_1^2 + \sigma_2^2)/2$ and $\mu := (b_1 + b_2)/2$, the dynamic and static equilibrium prices at $t = 0$ for the initial value $X(0) = x$ are

$$p_{\text{dyn}} = x^2 + 2x\mu T + \Sigma^2 T + \left(\frac{b_1 + b_2}{2}\right)^2 T^2,$$

$$p_{\text{sta}} = x^2 + 2x\mu T + \Sigma^2 T + \frac{b_1^2 + b_2^2}{2} T^2;$$

see Appendix F for the calculations. In particular,

$$p_{\text{dyn}} - p_{\text{sta}} = \left[\left(\frac{b_1 + b_2}{2}\right)^2 - \frac{b_1^2 + b_2^2}{2}\right] T^2 \leq 0.$$

The optimal dynamic and static portfolios are given by

$$\phi_i(t, x) = x(b_i - b_j) + \frac{1}{2}(T - t)(b_i^2 - b_j^2) + \frac{1}{2}(\sigma_i^2 - \sigma_j^2),$$

$$q_i = x(b_i - b_j) + \frac{1}{2} T(b_i^2 - b_j^2) + \frac{1}{2}(\sigma_i^2 - \sigma_j^2),$$

where $j = 2$ if $i = 1$ and vice versa; in particular, the demands at $t = 0$ coincide. In the special case in which all agents agree on the drift, $b_1 = b_2$, we have $p_{\text{dyn}} = p_{\text{sta}}$ and the demands coincide at all times. Whenever $b_1 \neq b_2$, a continuity result similar to the results established in Section B guarantees that $p_{\text{dyn}} < p_{\text{sta}}$ for cost parameters close to $\alpha_- = \alpha_+ = 1$.

To obtain some intuition for this example, consider the case in which $\sigma_1 = \sigma_2$, $b_1 > 0$, and $b_2 = 0$. If $x > 0$, type 1 is long and type 2 is short when retrading is allowed. Notice that an agent who is short expects on average to cover some of her shorts in the future. When retrading is ruled out, she prefers to cut her short position at time 0. This would place upward pressure on the static price. The long party also expects to reduce his position if $X(t)$ were to stay constant, but because $b_1 > 0$, he expects the state $X(t)$ to grow, dampening his need to anticipate the reduction when retrading is ruled out. In other words, the long party values the resale option less than the short party values the repurchase option. As a result, the static market presents excess demand at price $p_{\text{dyn}}$, and thus, the static price must rise to clear the market.

## IV. A Planner with Limited Instruments

In this section, we show that our equilibrium can be explained through a planner's problem that sheds light on the PDE for the equilibrium in Theorem 1.

We first explain why this requires a planner with limited instruments. Consider a planner that can allocate the supply arbitrarily across types at any time and state. In addition, she can make arbitrary lump-sum numeraire transfers $\theta_i(T,\omega)$ to agents of type $i = 1, \dots, n$ as well as a transfer $\theta_0(T,\omega)$ to the agents who are originally endowed with the supply, provided that these transfers add up to zero. Criterion (3) implies that the traders' utility functions are separable and linear in numeraire transfers. Hence, the convexity of the cost function for holding assets guarantees that the supply is equally distributed across types in any Pareto optimum, that is, $y_i(t,x) = \frac{s(t,x)}{n}$. This property of the asset allocation holds in the equilibrium of Theorem 1 when traders have homogeneous beliefs ($Q_1 = \cdots = Q_n$). In this case, it is clear that the equilibrium allocation is actually a Pareto optimum; the functional form of the utility function compensates for the lack of complete markets. However, this optimality does not hold when traders are heterogeneous and hold different asset positions in equilibrium—gambling using the asset has real costs and a social planner would like to rule them out. This general nonoptimality of our equilibrium also holds if we use the "belief neutral" Pareto inefficiency criteria in Brunnermeier, Simsek, and Xiong (2014).

Although our equilibrium is not Pareto optimal, we can characterize the equilibrium price as the optimal value for a planner with limited instruments and the equilibrium allocations as the associated allocations induced by this planner. Consider a planner that can use two instruments. The first is to assign "total cost coefficients" $\alpha_i(t,x) \in [\alpha_-, \alpha_+]$ for each type $i$ at each date and state $(t,x)$. If agent $i$ decides to go short $y$ units, she will be subsidized so that her effective cost is $c_i(t,x,y) = \frac{1}{2\alpha_i(t,x)}y^2$, whereas if she goes long, she will be taxed to have the same effective cost. The second instrument is to give lump-sum numeraire subsidies or charge lump-sum taxes $\theta_i(T,\omega)$, $i = 1, \dots, n$ to each type. The planner must break even so that any aggregate taxes collected must equal the net subsidies provided.

Given the assigned cost coefficients and lump-sum transfers, agents choose asset positions taking prices as given and the market settles on prices that equilibrate supply and demand. Since the objective function is separable in the numeraire, the optimal positions are independent of the lump-sum transfers: agent $i$ maximizes the expected net payoff $E_i[\int_0^T \Phi(t)\,dP(t) - \int_0^T c_i(t, X(t), \Phi(t))\,dt]$ from trading, which is analogous to (3) except that the cost is now given by $c_i$.

Recall that $I_*$ denotes the group of agents who go short in the equilibrium of Theorem 1; see (10).

THEOREM 2: *(i) For any sufficiently regular assignment $\alpha = (\alpha_1, \dots, \alpha_n)$ of the planner,* [30] *there exists a unique equilibrium with a price function $v_\alpha \in C_b^{1,2}$. This function can be characterized by a linear PDE and by a Feynman-Kac representation; see (F12) and (F13) in Appendix F.*

[30] See Appendix F for further details. In particular, the assignment defined in (ii) is sufficiently regular in this sense.

*(ii) The planner can maximize the initial price by choosing $\alpha_i(t,x) = \alpha_-$ when $i \in I_*(t,x)$ and $\alpha_i(t,x) = \alpha_+$ when $i \in I_*^c(t,x)$. Under this assignment, the price and the asset allocation coincide with the equilibrium of Theorem 1. Agents assigned $\alpha_+$ choose to go long and agents assigned $\alpha_-$ choose to go short, so that no taxes, subsidies, or transfers are collected.*

This theorem states that a planner who faces the constraint $\alpha_i \in [\alpha_-, \alpha_+]$ on the total cost coefficients and who wishes to maximize the initial price of the asset (or, when $s > 0$, wishes to maximize the welfare of the initial asset holders) would assign $\alpha_i = \alpha_-$ to the agents who in our original equilibrium choose to go short and $\alpha_+$ to the remaining agents. Given the assignment, equilibrium prices will be identical to those obtained in our original equilibrium. This confirms the intuition from Proposition 2, which states that the price rises if costs for optimists (longs) are reduced and costs for pessimists (shorts) are increased: within the constraint, this allocation is the most favorable for the optimists and the least favorable for the pessimists. If the planner were not constrained to the interval $[\alpha_-, \alpha_+]$, she could typically attain an even higher initial price—she would put a tax on the pessimists and subsidize the optimists. In fact, Example 3 below shows that unless the planner is constrained to $[\alpha_-, \alpha_+]$, she can make all agents, including the initial asset holders, better off.

In the proof, we show that the assertion of the theorem holds not only for the initial price but also for the price $v(t,x)$ at any time and state: the planner is *time-consistent*, that is, there is no need for a commitment device.

REMARK 5: *The planner's problem helps explain the PDE for the equilibrium in Theorem 1. Indeed, (6) can be seen as a supremum of linear PDEs parameterized by the groups $I$. The linear PDE for a fixed group $I$ is exactly the equation for the equilibrium price $v_\alpha$ resulting from the assignment given by $\alpha_i = \alpha_-$ when $i \in I$ and $\alpha_i^* = \alpha_-$ when $i \in I^c$. Thus, (6) can be understood as an optimization over assignments of $\alpha_-$ and $\alpha_+$ to the different types.*

The following example shows that without the constraint $\alpha_i \in [\alpha_-, \alpha_+]$, the planner may be able to improve the utility of all agents relative to the equilibrium utility.

EXAMPLE 3: Consider $n = 2$ types in a market with constant supply $s = 0$. The equilibrium portfolios satisfy $\phi_1 = -\phi_2$ by market clearing, and we may assume that they are not identically equal to zero. The precise views and costs are not important for this example; for instance, we can take $b_1 = -b_2 > 0$, $\sigma_1 = \sigma_2 > 0$, and $\alpha := \alpha_+ = \alpha_- > 0$.

Consider a planner who can charge a tax on all types so that the agents face an effective cost coefficient $\tilde{\alpha} = \alpha/2$. It follows from Remark 1 that the equilibrium price is unaffected by this symmetric scaling: $\tilde{v} = v$. In particular, the portfolios are related by $\tilde{\phi}_i = \tilde{\alpha}\mathcal{L}^i\tilde{v} = (\alpha/2)\mathcal{L}^i v = \phi_i/2$. As a consequence, the trading gains of type $i$, $X_i = \int_0^T \Phi_i(t)\,dP(t)$, become $\tilde{X}_i = X_i/2$ in the new equilibrium. Moreover, as $\tilde{c}(\tilde{\phi}_i) = \frac{1}{2\tilde{\alpha}}\tilde{\phi}_i^2 = \frac{1}{2}\frac{1}{2\alpha}\phi_i^2 = c(\phi_i)/2$, the holding costs are also cut by half. As $\phi_1 = -\phi_2$, we have that $X_1 = -X_2$ and $\tilde{X}_1 = -\tilde{X}_2$. In particular, the difference $\Delta_i = X_i - \tilde{X}_i$ satisfies $\Delta_1 = -\Delta_2$. As a result, the planner can transfer $\Delta_i$ to agent $i$ at a zero net cost. After taking this transfer into account, the utility of both types is increased since the total gains are the same as before but the holding costs are cut by half. Moreover, the planner is left with the revenue from the taxes. She may, for example, distribute the revenue to the agents as an additional lump sum.

Using a continuity argument, a similar example can be constructed with supply $s > 0$. In that case, the planner may distribute some of the revenue to the agents initially endowed with the supply to ensure that their utility is also increased.

## V. Conclusion

In this paper, we consider a continuous-time model of trading among risk-neutral agents with heterogeneous beliefs. Agents face quadratic costs-of-carry, and as a consequence, their marginal valuation of the asset decreases when the magnitude of their position increases, as would be the case for risk-averse agents. In previous models of heterogeneous beliefs, it was assumed that agents face a constant marginal cost-of-carry for a positive position and an infinite cost for a negative position. As a result, buyers benefit from a resale option and are willing to pay for an asset in excess of their own valuation of the dividends of that asset. Moreover, the supply does not affect the equilibrium price. We show that when buyers face an increasing marginal cost-of-carry, in equilibrium, they may also value an option to delay. We illustrate with an example that even when shorting is impossible, this delay option may cause the market to equilibrate below the price that would prevail if agents were restricted to buy-and-hold strategies. We introduce the possibility of short selling and show that this gives pessimists the analogous options. In our model, the price depends on the supply.

We characterize the unique equilibrium of our model as the solution to a Hamilton-Jacobi-Bellman equation of a novel form and use this to derive several comparative statics: the price decreases with an increase in the supply of the asset, with an increase in the cost of carrying long positions, and with a decrease in the cost of carrying short positions. The conclusions of earlier models are shown to hold in the limiting case in which the quadratic cost-of-carry for long positions converges to zero. An example shows that a decrease in the cost of shorting and the consequent increase in the supply of shorts can deflate the bubble.

In our model, the demand for the asset is satisfied by the sum of the exogenous supply and the short positions of market participants. While the shorts are determined endogenously, supply is independent of the current price and agents' beliefs. The data in Cordell, Huang, and Williams (2011) suggest that shorting played the dominant role in pricking the CDO bubble, but in other episodes, such as the Internet bubble, investments in projects underlying the asset class and sales by insiders played an important role in satisfying the demand by optimists. For such episodes, one would need to supplement the theory in this paper with an equilibrium model of supply.

Critics have indicted synthetic CDOs for the inordinate damage caused by the subprime implosion, but it is not obvious what would have happened if synthetics had not been created. The spreads in the "safe" tranches of cash CDOs would have been even more compressed. More ominously, the numbers reported in Cordell, Huang, and Williams (2011) suggest that generating the amount of BBB Home Equity (HE) bonds referenced in the synthetic CDOs would have required making an additional 2.5 *trillion* dollars of subprime mortgage loans. This would have probably resulted in substantially more new house construction and mortgage defaults. The model in this paper suggests that if a mechanism for shorting BBB HE bonds and CDO tranches had been created earlier, the subprime bubble would have been smaller.

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## Appendix A: Adding Linear Costs

In this section, we generalize the cost-of-carry by adding linear terms and we discuss the corresponding changes in our main results. Broadly speaking, the generalized model does not affect the economic conclusions.

Let

$$c(y) = \begin{cases} \frac{1}{2\alpha_+} y^2 + \beta_+ y, & y \geq 0, \\ \frac{1}{2\alpha_-} y^2 + \beta_- |y|, & y < 0, \end{cases} \tag{A1}$$

where $\beta_-, \beta_+ \geq 0$ are constants; as discussed in the introduction, the main case of interest is $\beta_- > 0$ and $\beta_+ = 0$. While this cost function is still strictly convex, it fails to be differentiable at $y = 0$ unless $\beta_- = \beta_+ = 0$.

Following the proof of Lemma F2, the optimal portfolio (F1) becomes

$$\phi_i(t, x) = \begin{cases} \alpha_+(\mathcal{L}^i v(t, x) - \beta_+), & \mathcal{L}^i v(t, x) \geq \beta_+, \\ \alpha_-(\mathcal{L}^i v(t, x) + \beta_-), & \mathcal{L}^i v(t, x) \leq -\beta_-, \\ 0, & \text{otherwise.} \end{cases} \tag{A2}$$

That is, there is an interval $[-\beta_-, \beta_+]$ of values of $\mathcal{L}^i v(t, x)$ over which it is optimal to have a zero position, due to the kink in the function $c$.

The main PDE (6) needs to be adapted correspondingly. Indeed, instead of considering only the group $I$ of agents who hold a short position, we now need to distinguish a group $J$ of agents who hold a (strict) long position—the group $J$ may be smaller than the complement $I^c$. More precisely, the generalized PDE (6) reads as follows (the proof is analogous to that of Theorem 1).

THEOREM A1: *The unique equilibrium price function $v \in C_b^{1,2}$ can be characterized as the unique solution of the PDE*

$$\partial_t v(t,x) + \sup_{I\cap J=\emptyset}\left(\mu_{I,J}(t,x)\partial_x v(t,x) + \frac{1}{2}\operatorname{Tr}\Sigma_{I,J}^2(t,x)\partial_{xx}v(t,x) - \kappa_{I,J}(t,x)\right) = 0 \tag{A3}$$

*on $[0,T)\times\mathbb{R}^d$ with terminal condition $v(T,x) = f(x)$, where the supremum is taken over all disjoint subsets $I, J \subseteq \{1,\dots,n\}$ and the coefficients are defined as*

$$\mu_{I,J}(t,x) = \frac{\alpha_-}{|I|\alpha_- + |J|\alpha_+}\sum_{i\in I} b_i(t,x) + \frac{\alpha_+}{|I|\alpha_- + |J|\alpha_+}\sum_{i\in J} b_i(t,x),$$

$$\Sigma_{I,J}^2(t,x) = \frac{\alpha_-}{|I|\alpha_- + |J|\alpha_+}\sum_{i\in I} \sigma_i^2(t,x) + \frac{\alpha_+}{|I|\alpha_- + |J|\alpha_+}\sum_{i\in J} \sigma_i^2(t,x),$$

$$\kappa_{I,J}(t,x) = \frac{s(t,x) - |I|\alpha_-\beta_- + |J|\alpha_+\beta_+}{|I|\alpha_- + |J|\alpha_+}.$$

In particular, the additional constants $\beta_-$, $\beta_+$ enter only through the running cost $\kappa_{I,J}$. It follows that the results on the comparative statics in Propositions 1 and 2 remain valid, and in addition, the equilibrium price function $v$ is increasing with respect to $\beta_-$ and decreasing with respect to $\beta_+$.

In the limiting case of zero cost for long positions, we now need to send $\alpha_+ \to \infty$ and $\beta_+ \to 0$. The result of Proposition 3 is unchanged, that is, the limiting equilibrium price function is the solution of

$$\partial_t v + \sup_{i\in\{1,\dots,n\}}\left(b_i\partial_x v + \frac{1}{2}\operatorname{Tr}\sigma_i^2\partial_{xx}v\right) = 0.$$

On the other hand, for the limit $\alpha_- \to 0$ of infinite cost for shorting, the result of Proposition 4 changes slightly because the long positions are subject to $\beta_+$, which becomes an additional running cost in the limiting equation

$$\partial_t v + \sup_{\emptyset\neq J\subseteq\{1,\dots,n\}}\left(\frac{1}{|J|}\sum_{i\in J} b_i\partial_x v + \frac{1}{2}\operatorname{Tr}\frac{1}{|J|}\sum_{i\in J}\sigma_i^2\partial_{xx}v - \frac{s}{|J|\alpha_+} - \beta_+\right) = 0.$$

The results for the static equilibrium problem can be generalized with analogous changes.

## Appendix B: Heterogeneous Costs

In this section, we show how the equilibrium of Theorem 1 changes if the cost coefficients $\alpha_-, \alpha_+$ depend on the type rather than being the same for all agents. We write $\alpha_-^i, \alpha_+^i$ for the coefficients of type $i$. The following result shows that while the structure of the equilibrium remains similar, agents with lower costs have a larger influence on the coefficients of the PDE that determines the equilibrium price.

THEOREM B1: *The unique equilibrium price function $v \in C_b^{1,2}$ can be characterized as the unique solution of the PDE (6) with coefficients*

$$\mu_I(t,x) = \frac{1}{\sum_{i\in I}\alpha_-^i + \sum_{i\in I^c}\alpha_+^i}\left(\sum_{i\in I}\alpha_-^i b_i(t,x) + \sum_{i\in I^c}\alpha_+^i b_i(t,x)\right),$$

$$\Sigma_I^2(t,x) = \frac{1}{\sum_{i\in I}\alpha_-^i + \sum_{i\in I^c}\alpha_+^i}\left(\sum_{i\in I}\alpha_-^i \sigma_i^2(t,x) + \sum_{i\in I^c}\alpha_+^i \sigma_i^2(t,x)\right),$$

$$\kappa_I(t,x) = \frac{s(t,x)}{\sum_{i\in I}\alpha_-^i + \sum_{i\in I^c}\alpha_+^i}.$$

The proof is analogous to Theorem 1. As in Lemma F2, the optimal portfolios are given by $\alpha_\pm^i \mathcal{L}^i v(t,x)$. Thus, as expected, types with lower costs hold larger positions.

## Appendix C: Quadratic Costs on Values of Positions

In this section, we briefly explain what changes if costs are quadratic in the monetary value of the portfolio rather than the size, that is, the instantaneous cost-of-carry is

$$c(P(t)\Phi(t)) \qquad \text{instead of} \qquad c(\Phi(t)),$$

where $c$ is quadratic as in (2). If the price $P(t)$ becomes zero, these costs become zero, which leads to infinite demand by the agents and thus to nonexistence of equilibria. This discussion therefore pertains to assets with a strictly positive price.

As in Lemma F2, we can derive the first-order condition of optimality for the portfolio function $\phi_i$ of agent $i$ as

$$\phi_i(t,x) = \frac{\alpha_{\mathrm{sign}(\mathcal{L}^i v(t,x))}}{v(t,x)^2}\mathcal{L}^i v(t,x),$$

which is similar to the expression in Lemma F2 except for the additional division by $v^2$. Using market clearing as in the proof of Theorem 1 then produces a PDE where the term $\kappa_I$ in (9) receives an additional factor $v^2$. This new term can no longer be interpreted as a running cost and, in general, the PDE cannot be written as an Hamilton-Jacobi-Bellman equation similar to (6) because in such an equation, the maximization is necessarily carried out over terms that are linear in the $v$-variable. Thus, we do not expect to have an interpretation of equilibria through a stochastic control problem or a social planner. A remarkable exception is $\kappa_I \equiv 0$, which occurs in the case of zero net supply. In that case, the PDE is exactly the same as (6), and hence, the equilibrium price is also the same. The actual portfolios of the agents are not identical, but they differ only by the factor $v^2$. (To ensure a priori that equilibrium prices are positive, it suffices to assume that the payoff $f$ is positive and bounded away from zero. The comparison principle then shows that prices remain bounded away from zero at all times.)

## Appendix D: Optimal Control Representation

The PDE (6) is the Hamilton-Jacobi-Bellman equation of a stochastic optimal control problem where the controller can choose a subset $I \subseteq \{1, \ldots, n\}$ at any time and state, and that choice determines the instantaneous drift and volatility coefficients $\mu_I$ and $\Sigma_I$ as well as the running cost $\kappa_I$.

To formulate this problem precisely, consider a filtered probability space carrying a $d'$-dimensional Brownian motion $W$ and let $\Theta$ be the collection of all (progressively measurable) processes $\mathcal{I}$ with values in the family of all subsets of $\{1, \ldots, n\}$.[31] For each $\mathcal{I} \in \Theta$, let $X^{t,x}_{\mathcal{I}}(r)$, $r \in [t, T]$ be the solution of the SDE

$$dX(r) = \mu_{\mathcal{I}(r)}(r, X(r))\,dr + \Sigma_{\mathcal{I}(r)}(r, X(r))\,dW(r), \quad X(t) = x \tag{D1}$$

on the time interval $[t, T]$. It follows from the assumptions on the coefficients $b_i, \sigma_i$ that this SDE with random coefficients has a unique strong solution.[32] Therefore, we may consider the control problem

$$V(t, x) = \sup_{\mathcal{I} \in \Theta} E\left[ f(X^{t,x}_{\mathcal{I}}(T)) - \int_t^T \kappa_{\mathcal{I}(r)}(r, X^{t,x}_{\mathcal{I}}(r))\,dr \right] \tag{D2}$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, which gives rise to a second characterization for the equilibrium price function $v$.

PROPOSITION D1: *The equilibrium price function $v$ from Theorem 1 coincides with the value function $V$ of (D2). Moreover, an optimal control for (D2) is given by $\mathcal{I}_*(t) = I_*(t, X(t))$, where, as in (10),*

$$I_*(t, x) = \{i \in \{1, \ldots, n\} : \mathcal{L}^i v(t, x) < 0\}. \tag{D3}$$

PROOF: By Theorem 1, the function $v \in C^{1,2}_b$ is a solution of the PDE (6), which is the Hamilton-Jacobi-Bellman equation of the control problem (D2). Moreover, $I_*(t, x)$ maximizes the Hamiltonian as noted after (F4). Thus, the verification theorem of stochastic control (see Fleming and Soner (2006, Theorem IV.3.1, p. 157)) shows that $v$ is the value function and $\mathcal{I}_*$ is an optimal control. ■

[31] While this collection of control processes appears somewhat nonstandard, there is no difficulty involved in defining it—this family of subsets is simply a discrete set with $2^n$ elements; it can be identified with $\{0, 1\}^n$.

[32] The coefficients $\mu_{\mathcal{I}}$ and $\Sigma_{\mathcal{I}}$ may be quite irregular as stochastic processes but the dependence with respect to the $x$-variable is Lipschitz continuous. See Krylov (1980, Theorem 2.5.7, p. 82) for a general result on existence and uniqueness under Lipschitz conditions.

## Appendix E: Examples

In this section, we discuss two examples in more detail. The calculations are provided in Appendix F, together with the rest of the proofs.

The first example, already outlined in Example 1, shows that the static price may exceed the dynamic price, even when short selling is prohibited.

EXAMPLE E1: Consider $n = 2$ types with volatility coefficients $\sigma_i = 0$ and constant opposing drifts

$$b_1 = 1, \qquad b_2 = -1.$$

The payoff function is $f(y) = y^2$ and supply $s > 0$ is constant. Moreover, $\alpha_- = 0$ and $\alpha_+ = 1$. Then the dynamic equilibrium price is

$$p_{\text{dyn}} = \begin{cases} x^2 - sT/2, & |x| + T/2 \le s/4, \\ (|x| + T)^2 - sT, & |x| + T/2 > s/4, \end{cases}$$

and corresponding optimal portfolios in feedback form are given by

$$\phi_1(t,x) = \begin{cases} 0, & |x| + (T-t)/2 > s/4,\ x < 0, \\ s/2 + 2x, & |x| + (T-t)/2 \le s/4, \\ s, & |x| + (T-t)/2 > s/4,\ x > 0, \end{cases}$$

$$\phi_2(t,x) = \begin{cases} s, & |x| + (T-t)/2 > s/4,\ x < 0, \\ s/2 - 2x, & |x| + (T-t)/2 \le s/4, \\ 0, & |x| + (T-t)/2 > s/4,\ x > 0. \end{cases}$$

The static equilibrium price is

$$p_{\text{sta}} = \begin{cases} x^2 + T^2 - sT/2, & |x| \le s/4, \\ x^2 + T^2 + 2|x|T - sT, & |x| > s/4, \end{cases}$$

and corresponding optimal portfolios are given by

$$q_1 = \begin{cases} 0, & x < -s/4, \\ s/2 + 2x, & |x| \le s/4, \\ s, & x > s/4, \end{cases} \qquad q_2 = \begin{cases} s, & x < -s/4, \\ s/2 - 2x, & |x| \le s/4, \\ 0, & x > s/4. \end{cases}$$

The static equilibrium price exceeds the dynamic price, more precisely,

$$p_{\text{sta}} - p_{\text{dyn}} = \begin{cases} T^2, & |x| \le s/4 - T/2, \\ (s/2 - 2|x|)T, & s/4 - T/2 < |x| < s/4, \\ 0, & |x| \ge s/4. \end{cases}$$

Next, we discuss in more detail how the delay option effect explains the difference $p_{\text{sta}} - p_{\text{dyn}}$ in this example. To that end, it will be useful to record the portfolios as expected by the agents: since $X(t) = x + b_i t$ $Q_i$-a.s. and $\Phi_i(t) = \phi_i(t, X(t))$, we have

$$Q_1\text{-a.s.,}\quad \Phi_1(t) = \begin{cases} 0, & |x+t| + (T-t)/2 > s/4,\ x+t<0, \\ s/2 + 2t + 2x, & |x+t| + (T-t)/2 \le s/4, \\ s, & |x+t| + (T-t)/2 > s/4,\ x+t>0, \end{cases}$$

$$Q_2\text{-a.s.,}\quad \Phi_2(t) = \begin{cases} s, & |x-t| + (T-t)/2 > s/4,\ x-t<0, \\ s/2 + 2t - 2x, & |x-t| + (T-t)/2 \le s/4, \\ 0, & |x-t| + (T-t)/2 > s/4,\ x-t>0. \end{cases}$$

Below, we abuse this notation and simply write $\Phi_1(t)$ and $\Phi_2(t)$ for the expressions on the right-hand side. We consider various intervals for the initial value $x$; by symmetry, we may focus on $x \ge 0$ without loss of generality. We also assume that $s > T$, mainly to avoid distinguishing even more cases.

*Case 1*: $x \ge s/4 + T$. In this regime, the expected dynamic portfolios $\Phi_1$ and $\Phi_2$ are constant, and thus, the delay option is never exercised. The static portfolios coincide with their initial values, $q_1 = s = \Phi_1(0)$ and $q_2 = 0 = \Phi_2(0)$, and the static and dynamic prices are equal: $p_{\text{sta}} = p_{\text{dyn}}$.

*Case 2*: $s/4 \le x < s/4 + T$. As before, $q_1 = s = \Phi_1(0)$ and $\Phi_1$ is constant. However, $\Phi_2(t)$ equals zero initially but may become positive for $t$ close to $T$ (for suitable parameter values). Nevertheless, type 2 does not choose to anticipate her trading in the static case because the cost-of-carry outweighs the expected gains—we still have $q_2 = 0 = \Phi_2(0)$ and $p_{\text{sta}} = p_{\text{dyn}}$.

*Case 3*: $(s/4 - T/2)^+ < x < s/4$. Once again, $\Phi_1 \equiv s$ is constant, $\Phi_2(0) = 0$, and $\Phi_2$ increases for some $t > 0$. Furthermore, the increase in type 2's position is larger for smaller $x$. Type 2 now does anticipate some of that increase in the static case and for this reason $p_{\text{dyn}}$ is now too low to be an equilibrium price. The increase in price changes the optimal portfolio for agents of type 1. We are in the mixed case where portfolios and prices adjust. Type 1 decreases his initial position to $q_1 = s/2 + 2x < s = \Phi_1(0)$ and type 2 increases her position to $q_2 = s/2 - 2x > 0 = \Phi_2(0)$. At the same time, the static equilibrium price is raised, $p_{\text{sta}} - p_{\text{dyn}} = (2x - s/2)T > 0$. As $x$ decreases from $s/4$ to $s/4 - T/2$, this difference increases linearly from zero to $T^2$, and the portfolios $(q_1, q_2)$ change linearly from (s,0) to $(s - T, T)$. In summary, the elimination of the delay option in the static case results in portfolio adjustments and a price increase.

*Case 4*: $0 \le x \le s/4 - T/2$. In this last regime, both $\Phi_1$ and $\Phi_2$ are increasing in time, so both types exercise the delay option when retrading is allowed and have an anticipatory motive when they can trade only at $t = 0$. The initial dynamic portfolios are $\Phi_1(0) = s/2 + 2x > 0$ and $\Phi_2(0) = s/2 - 2x > 0$. Since both types want to anticipate in the static case, the static price must be higher. More precisely, the aggregate excess demand at price $p_{\mathrm{dyn}}$ equals $2T^2$ and thus is independent of $x$. Since we are in the region in which both types have positive demand, the marginal effect of an increase in price is $-1$, for each type. Thus, the price adjustment that is necessary to clear the static market is exactly $T^2$ for each value of $x$ in this region.

The next example illustrates the mechanics of the delay option when there is no cost-of-carry for long positions: the most optimistic agent holds the entire market and the dynamic equilibrium price always exceeds the static one.

EXAMPLE E2: Let $\alpha_+ = \infty$ and $\alpha_- = 1$. We consider $n = 2$ types with drift coefficients

$$b_1 = 1, \qquad b_2 = 0$$

and volatility coefficients $\sigma_1 = \sigma_2 = 0$. The payoff is $f(y) = y^2$ and the initial value is $x = 0$, so that the first type is more optimistic at any time.

As in Proposition 6, the static equilibrium price is given by the optimist's expectation $e_1 = E_1[f(X(T))] = T^2$. Following Proposition 3, the same holds for the dynamic price, so that $p_{\mathrm{sta}} = p_{\mathrm{dyn}}$. The static and dynamic portfolios of the pessimist are given by

$$q_2 = T^{-1}(e_2 - p_{\mathrm{sta}}) = -T, \qquad \phi_2(t,x) = \partial_t v(t,x) = -2(x + T - t).$$

Under $Q_2$, the state process $X \equiv 0$ is constant, so that $\Phi_2(t) = \phi_2(t, X(t)) = -2(T - t)$ a.s. Thus, the static position $q_2 = -T$ anticipates some of the increase from $\Phi_2(0) = -2T$ to $\Phi_2(T) = 0$. However, this does not affect the static equilibrium price because an optimistic agent is indifferent to the size of her (nonnegative) position—the absence of a cost-of-carry for long positions allows the portfolios to adjust without affecting the prices.

## Appendix F: Proofs

This appendix combines the proofs for Sections I through IV and Appendix E.

### A. *Proofs for Section I*

Before proving the main result of Theorem 1, we record two lemmas for later reference. The first one guarantees the passage from almost-sure to pointwise identities.

LEMMA F1: *For all $i \in \{1, \dots, n\}$ and all $t \in (0, T]$, the support of $X(t)$ under $Q_i$ is the full space $\mathbb{R}^d$.*

PROOF: Recall that $X$ is the coordinate-mapping process on $\Omega = C([0,T], \mathbb{R}^d)$. Since $b_i$ is bounded and $\sigma_i^2$ is uniformly parabolic, the support of $Q_i$ is in the set of all paths $\omega$ with $\omega(0) = x$ (see Stroock and Varadhan (1972, Theorem 3.1)). The claim is a direct consequence. ■

The second lemma provides an expression for the optimal portfolios.

LEMMA F2: *Let $v \in C_b^{1,2}$ and consider the (price) process $P(t) = v(t, X(t))$. The portfolio defined by $\Phi_i(t) = \phi_i(t, X(t))$, where*

$$\phi_i(t, x) = \alpha_{\mathrm{sign}(\mathcal{L}^i v(t,x))} \mathcal{L}^i v(t, x) \tag{F1}$$

*is the unique optimal portfolio for type $i$.*[33]

PROOF: We note that $\Phi_i$ is admissible since $v \in C_b^{1,2}$. Let $\Phi$ be any admissible portfolio. By Itô's formula,

$$\int_0^T \Phi(t)\, dP(t) - \int_0^T c(\Phi(t))\, dt = \int_0^T \{\Phi(t)\mathcal{L}^i v(t, X(t)) - c(\Phi(t))\}\, dt + M_i(T),$$

where $M_i(T)$ is the terminal value of a (true) martingale with vanishing expectation—recall that $\sigma_i$ and $\partial_x v$ are bounded. Thus, the expected final payoff (3) is given by

$$E_i\left[\int_0^T \{\Phi(t)\mathcal{L}^i v(t, X(t)) - c(\Phi(t))\}\, dt\right].$$

As a result, $\Phi$ is optimal if and only if it maximizes the above integrand (up to $(Q_i \times dt)$-nullsets). The unique maximizer is given by $\Phi_i$, and the claim follows. ■

We can now prove the main result on the pricing PDE.

PROOF OF THEOREM 1:

(a) We first show that a given equilibrium price function $v \in C_b^{1,2}$ solves the PDE. Since $v(T, X(T)) = f(X(T))$ $Q_i$-a.s. for all $i$, the terminal condition $v(T, \cdot) = f$ follows from Lemma F1. At any state $(t, x)$, we introduce the set

$$I_*(t, x) = \{i \in \{1, \ldots, n\} : \mathcal{L}^i v(t, x) < 0\}. \tag{F2}$$

Next, we recall the unique optimal portfolios $\Phi_i$ from Lemma F2. Using again Lemma F1, the market-clearing condition $\sum_i \Phi_i = S$ can be stated as

$$\alpha_- \sum_{i \in I_*} \mathcal{L}^i v + \alpha_+ \sum_{i \in I_*^c} \mathcal{L}^i v = s. \tag{F3}$$

If $i \in I_*$, then $\mathcal{L}^i v \leq 0$ and $\alpha_- \leq \alpha_+$ implies $\alpha_- \mathcal{L}^i v \geq \alpha_+ \mathcal{L}^i v$. Conversely, if $i \in I_*^c$, then $\mathcal{L}^i v \geq 0$ and $\alpha_+ \mathcal{L}^i v \geq \alpha_- \mathcal{L}^i v$. It follows that the set $I_*$ of (F2)

[33] We recall that uniqueness is understood up to $(Q_i \times dt)$-nullsets.

maximizes the left-hand side of (F3) among all subsets $I \subseteq \{1, \dots, n\}$. That is,

$$\max_{I \subseteq \{1,\dots,n\}} \left( \alpha_- \sum_{i \in I} \mathcal{L}^i v + \alpha_+ \sum_{i \in I^c} \mathcal{L}^i v - s \right) = 0, \tag{F4}$$

and the set $I_*$ is a maximizer, or equivalently,

$$\max_{I \subseteq \{1,\dots,n\}} \frac{1}{|I|\alpha_- + |I^c|\alpha_+} \left( \alpha_- \sum_{i \in I} \mathcal{L}^i v + \alpha_+ \sum_{i \in I^c} \mathcal{L}^i v - s \right) = 0. \tag{F5}$$

After plugging in the definition of $\mathcal{L}^i v$ and using the definitions of $\mu_I$, $\Sigma_I$, and $\kappa_I$ in (7) to (9), which is the desired PDE (6).

(b) Conversely, let $v \in C_b^{1,2}$ be a solution of the PDE (6) with terminal condition $f$ and define $\Phi_i, \phi_i$ as in part (i) of Theorem 1. Then the terminal condition $v(T, X(T)) = f(X(T))$ is satisfied and $\Phi_i$ are optimal by Lemma F2. Since $v$ is a solution of the equivalent PDE (F4) and $I_*$ of (F2) is a maximizer, we have that

$$\sum_{1 \le i \le n} \phi_i = \alpha_- \sum_{i \in I_*} \mathcal{L}^i v + \alpha_+ \sum_{i \in I_*^c} \mathcal{L}^i v = s,$$

that is, the market clears. This shows that $v$ is an equilibrium price function.

(c) Since (a) and (b) established a one-to-one correspondence between equilibria and solutions of the PDE (6) with terminal condition $f$, it remains to observe that the latter has a unique solution in $C_b^{1,2}$. Indeed, existence holds by Krylov (1987, Theorem 6.4.3, p. 301);[34] the conditions in the cited theorem can be verified as in Krylov (1987, Example 6.1.4, p. 279).

Uniqueness holds by the comparison principle for parabolic PDEs. See Fleming and Soner (2006, Theorem V.9.1, p. 223). ■

## *B. Proofs for Section II*

We start with the comparative statics for the dependence of the price on supply.

PROOF OF PROPOSITION 1: Since the function $s$ enters linearly in the running cost (9) of the control problem (D2) and nowhere else, it follows immediately that the value function $V$ is monotone decreasing in $s$. The claim then follows from Proposition D1. ■

Next, we consider the dependence on the cost coefficients.

[34] The gist of this rather technical result is that a second-order parabolic PDE of Hamilton-Jacobi-Bellman type has a solution in $C_b^{1,2}$ as soon as the second-order term is uniformly parabolic and all coefficients and boundary conditions are sufficiently smooth and bounded.

PROOF OF PROPOSITION 2 (and Remark 2): Let $\alpha_- \le \alpha_+$ and $\alpha'_- \le \alpha'_+$ be two pairs of cost coefficients and let $v$ and $v'$ be the corresponding equilibrium price functions. Let $I_*$ be the optimal feedback control for $\alpha_\pm$ as defined in (F2). Then by (F4), we have

$$\alpha_- \sum_{i \in I_*} \mathcal{L}^i v + \alpha_+ \sum_{i \in I_*^c} \mathcal{L}^i v - s = 0.$$

If $\alpha'_- \le \alpha_-$ and $\alpha'_+ \ge \alpha_+$, then $\sum_{i \in I_*} \mathcal{L}^i v \le 0$ and $\sum_{i \in I_*^c} \mathcal{L}^i v \ge 0$ yield

$$\alpha'_- \sum_{i \in I_*} \mathcal{L}^i v + \alpha'_+ \sum_{i \in I_*^c} \mathcal{L}^i v - s \ge 0.$$

For the special case in which $s \equiv 0$, this conclusion also holds under the weaker condition that $\alpha_+/\alpha_- \le \alpha'_+/\alpha'_-$, which covers case (iii), and the same holds under the conditions of Remark 2. It then follows that

$$\max_{I \subseteq \{1,\dots,n\}} \left( \alpha'_- \sum_{i \in I} \mathcal{L}^i v + \alpha'_+ \sum_{i \in I^c} \mathcal{L}^i v - s \right) \ge 0,$$

which is a version of (F4) with inequality instead of equality, for the coefficients $\alpha'_\pm$. Following the same steps as after (F4), we deduce that

$$\partial_t v + \sup_{I \subseteq \{1,\dots,n\}} \left( \mu'_I \partial_x v + \frac{1}{2} \operatorname{Tr} \Sigma_I'^2 \partial_{xx} v - \kappa'_I \right) \ge 0,$$

where $\mu'_I$, $\Sigma'_I$, $\kappa'_I$ are defined as in (7) to (9) but with $\alpha'_\pm$ instead of $\alpha_\pm$. In other words, $v$ is a subsolution of the PDE satisfied by $v'$.[35] As $v$ and $v'$ satisfy the same terminal condition $f$, the comparison principle (see Fleming and Soner (2006, Theorem V.9.1, p. 223)) implies that $v \le v'$. ■

We continue with our result on the limit $\alpha_+ \to \infty$.

PROOF OF PROPOSITION 3: We first notice that since $s \ge 0$, the optimal set $I_*$ of (F2) for the Hamiltonian of the PDE (6) must satisfy $|I_*| < n$ due to the market-clearing condition—at least one agent has to hold a nonnegative position. As a result, the PDE (6) remains the same if the supremum is taken only over sets $I$ with $|I^c| > 0$.

Taking that into account, the limiting PDE for (6) as $\alpha_+ \to \infty$ is

$$\partial_t v + \sup_{\emptyset \ne J \subseteq \{1,\dots,n\}} \frac{1}{|J|} \sum_{i \in J} \left( b_i \partial_x v + \frac{1}{2} \operatorname{Tr} \sigma_i^2 \partial_{xx} v \right) = 0. \tag{F6}$$

Notice that given a set of real numbers, the largest average over a subset is, in fact, equal to the largest number in the set. As a result, (F6) coincides with (11). Using again Krylov (1987, Theorem 6.4.3, p. 301) and Fleming and Soner (2006,

[35] Note that the sign convention chosen here is opposite to that of Fleming and Soner (2006), so that a subsolution corresponds to the inequality $\ge 0$ in the PDE.

Theorem V.9.1, p. 223), this equation has a unique solution $v^\infty \in C_b^{1,2}$ for the terminal condition $f$, and the solution is independent of $\alpha_-$ and $s$ since these quantities do not appear in (11).

To see that $v^{\alpha_-,\alpha_+}(t,x) \to v^\infty(t,x)$, one can apply a PDE technique called the Barles-Perthame procedure to the equations under consideration; see Fleming and Soner (2006, Section VII.3). Alternatively, and to give a more concise proof, we may use the representation (D2) of $v^{\alpha_-,\alpha_+}$ as a value function as well as the corresponding representation for $v^\infty$. A result on the stability of value functions (see Krylov (1980, Corollary 3.1.13, p. 138)) then shows that $v^{\alpha_-,\alpha_+} \to v^\infty$ locally uniformly, that is,

$$\sup_{(t,x)\in[0,T]\times K} |v^{\alpha_-,\alpha_+}(t,x) - v^\infty(t,x)| \to 0$$

for any compact set $K \subseteq \mathbb{R}^d$. The monotonicity property of the limit follows from Proposition 2. ■

Finally, we turn to the limit $\alpha_- \to 0$.

PROOF OF PROPOSITION 4: The arguments are similar to those for Proposition 3. In this case, the limiting PDE for (6) as $\alpha_- \to 0$ is (12). As in the proof of Proposition 3, we have that the limiting PDE has a unique solution $v^{0,\alpha_+} \in C_b^{1,2}$ and $v^{\alpha_-,\alpha_+}(t,x) \to v^{0,\alpha_+}(t,x)$ locally uniformly, with monotonicity in $\alpha_-$. In the special case in which $s = 0$, the PDE (12) coincides with (F6), and thus, with (11) as shown in the proof of Proposition 3. ■

### *C. Proofs for Section III and Appendix E*

We first prove our formula for the static equilibrium price.

PROOF OF PROPOSITION 5: We set $e_i = E_i[f(X(T))]$. Given any price $p$, the expected net payoff for agent $i$ using portfolio $q$ is

$$q(e_i - p) - \frac{T}{2\alpha_{\mathrm{sign}(q)}} q^2,$$

and the unique maximizer is $q_i = \alpha_{\mathrm{sign}(e_i-p)} T^{-1}(e_i - p)$ as stated in (14).

Let $p$ be a static equilibrium price. Setting $I_* = \{i \in \{1,\dots,n\} : e_i < p\}$, the market-clearing condition $\sum_i q_i = s$ for these optimal portfolios yields

$$\alpha_- \sum_{i\in I_*}(e_i - p) + \alpha_+ \sum_{i\in I_*^c}(e_i - p) = sT, \tag{F7}$$

and we observe that $I_*$ maximizes the left-hand side; that is,

$$\max_{I\subseteq\{1,\dots,n\}} \left( \alpha_- \sum_{i\in I}(e_i - p) + \alpha_+ \sum_{i\in I^c}(e_i - p) - sT \right) = 0.$$

This is equivalent to the claimed representation (13) for $p$.

Conversely, define $p$ by (13) and $q_i$ by (14). Then $q_i$ is optimal for agent $i$ as mentioned in the beginning of the proof. Moreover, reversing the above, $p$ satisfies (F7), and thus,

$$\sum_{i=1}^{n} q_i = \alpha_- \sum_{i \in I_*} T^{-1}(e_i - p) + \alpha_+ \sum_{i \in I_*^c} T^{-1}(e_i - p) = s,$$

establishing market clearing. ■

We can now deduce the formulas for the limiting cases of the static equilibrium.

PROOF OF PROPOSITION 6: Equation (15) follows by taking the limit $\alpha_+ \to \infty$ in (13). Similarly, (16) is obtained by taking the limit $\alpha_- \to 0$ in (13). ■

Next, we show that in the limit $\alpha_+ \to \infty$ with no cost on long positions, the dynamic price exceeds the static price.

PROOF OF PROPOSITION 7: By equation (15) for $p^{\infty}_{\text{sta}}$, it suffices to verify that $E_i[f(X(T))] \leq p^{\infty}_{\text{dyn}}$ for fixed $i \in \{1, \ldots, n\}$. Let $u = u_i \in C_b^{1,2}$ be the unique solution of

$$\partial_t u + b_i \partial_x u + \frac{1}{2} \operatorname{Tr} \sigma_i^2 \partial_{xx} u = 0, \quad u(T, \cdot) = f.$$

Then by the Feynman-Kac formula (Karatzas and Shreve (1991, Theorem 5.7.6, p. 366)), we have $u(0, x) = E_i[f(X(T))]$. Moreover, $u$ is clearly a subsolution of the PDE (11) for $v^{\infty}$, and now the comparison principle (Fleming and Soner (2006, Theorem V.9.1, p. 223)) yields that $E_i[f(X(T))] = u(0, x) \leq v^{\infty}(0, x) = p^{\infty}_{\text{dyn}}$ as claimed. ■

In what follows, we show that in the limit $\alpha_- \to 0$ where short selling is prohibited, the same inequality holds, provided that one agents holds the static market.

PROOF OF PROPOSITION 8: In view of (16), we have $p^{0,\alpha_+}_{\text{sta}} = E_i[f(X(T))] - \frac{sT}{\alpha_+}$ since the maximizing set is $J = \{i\}$. Using again the Feynman-Kac formula (Karatzas and Shreve (1991, Theorem V.9.1, p. 223)), we deduce that $p^{0,\alpha_+}_{\text{sta}} = u(0, x)$, where $u \in C_b^{1,2}$ is the solution of

$$\partial_t u + b_i \partial_x u + \frac{1}{2} \operatorname{Tr} \sigma_i^2 \partial_{xx} u - \frac{sT}{\alpha_+} = 0, \quad u(T, \cdot) = f.$$

In particular, $u$ is a subsolution of the PDE (12) for $v^{0,\alpha_+}$, and now the comparison principle (Fleming and Soner (2006, Theorem V.9.1, p. 223)) yields that $p^{0,\alpha_+}_{\text{sta}} = u(0, x) \leq v^{0,\alpha_+}(0, x) = p^{0,\alpha_+}_{\text{dyn}}$ as desired. ■

We next turn to our example in which the static price exceeds the dynamic price due to the delay option effect.

PROOFS FOR EXAMPLE 1 (Example E1): We begin with the static case. For later use, we consider the more general situation in which $\sigma := \sigma_1 = \sigma_2$ may be positive (but constant). We have $e_i = E_i[f(X(T))] = x^2 + 2xb_iT + T^2 + \sigma^2 T$, and thus, as in (16), the static price $p_{\mathrm{sta}}$ is

$$p_{\mathrm{sta}} = \max_{\emptyset \neq J \subseteq \{1,2\}} \left( \frac{1}{|J|} \sum_{i \in J} e_i - \frac{sT}{|J|} \right)$$
$$= x^2 + \sigma^2 T + \max\left\{T^2 - sT/2,\, T^2 + 2|x|T - sT\right\}$$

or

$$p_{\mathrm{sta}} = \begin{cases} x^2 + \sigma^2 T + T^2 - sT/2, & |x| \leq s/4, \\ x^2 + \sigma^2 T + T^2 + 2|x|T - sT, & |x| > s/4, \end{cases} \tag{F8}$$

and the portfolios $q_i$ are as stated in Example E1.

Turning to the dynamic case, we restrict attention to $\sigma = 0$. The limiting equation for (12) is

$$\partial_t v + \max\left(|\partial_x v| - s, -s/2\right) = 0, \quad v(T, \cdot) = f. \tag{F9}$$

In analogy to Proposition D1, this can be seen as the Hamilton-Jacobi equation of a deterministic control problem where the drift $\mu$ of the controlled state $dX(t) = \mu(t, X(t))\,dt$ can be chosen to be $\pm 1$ or 0 and the running cost is $s$ or $s/2$, respectively. We can check directly that an optimal control for this problem is

$$\mu(t,x) = \begin{cases} \mathrm{sign}(x), & |x| + (T-t)/2 > s/4, \\ 0, & |x| + (T-t)/2 \leq s/4, \end{cases}$$

in which case the value function is found to be

$$v(t,x) = \begin{cases} (|x| + T - t)^2 - s(T-t), & |x| + (T-t)/2 > s/4, \\ x^2 - s(T-t)/2, & |x| + (T-t)/2 \leq s/4. \end{cases}$$

Indeed, $v$ is continuous and the unique viscosity solution of (F9).[36] The indicated formulas for $p_{\mathrm{dyn}} - p_{\mathrm{sta}} = v(0,x) - p_{\mathrm{sta}}$ and for the optimal controls $\phi_i$ follow. ■

Next, we prove the continuity of the prices in the small volatility limit.

PROOF OF PROPOSITION 9: For the static case, the formula for $p^{\sigma}_{\mathrm{sta}}$ stated in (F8) shows that $p^{\sigma}_{\mathrm{sta}} - p^{0}_{\mathrm{sta}} = \sigma^2 T \downarrow 0$. Turning to the dynamic case, we first show that $p^{\sigma}_{\mathrm{dyn}} = v^{\sigma}(0,x)$ is monotone with respect to $\sigma$. Since $f$ is convex, $x \mapsto v^{\sigma}(t,x)$ is convex and thus $\partial_{xx} v^{\sigma} \geq 0$. Given $\sigma_1 \geq \sigma_2 > 0$, it follows that $v^{\sigma_2}$ is a subsolution to equation (12) for $v^{\sigma_1}$, and thus, the comparison principle for parabolic PDEs implies that $v^{\sigma_1} \geq v^{\sigma_2}$. To see that $v^{\sigma}(t,x) \to v^{0}(t,x)$, and in

[36] As is often the case for deterministic control problems, the value function is not $C^{1,1}$ and (F9) has no classical solution.

particular $p^{\sigma}_{\text{dyn}} \to p^{0}_{\text{dyn}}$, we may again use a general result on the stability of value functions (see Krylov (1980, Corollary 3.1.13, p. 138)). ■

It remains to provide the calculations for our symmetric example with $\alpha_- = \alpha_+ = 1$.

PROOFS FOR EXAMPLE 2: Following Remark 3, the equilibrium price function in the dynamic case is

$$v(t,x) = E[f(x + \mu\tau + \Sigma B_\tau)], \quad \text{where} \quad \tau := T - t,$$

and $B_\tau$ is a centered Gaussian with variance $\tau$. As $f(y) = y^2$,

$$v(t,x) = x^2 + 2x\mu\tau + \mu^2\tau^2 + \Sigma^2\tau,$$

and the optimal portfolios in feedback form are given by

$$\phi_i(t,x) = \mathcal{L}^i v(t,x) = x(b_i - b_j) + \frac{1}{2}\tau(b_i^2 - b_j^2) + \frac{1}{2}(\sigma_i^2 - \sigma_j^2).$$

For the static case, we have

$$e_i = x^2 + 2xb_iT + b_i^2T^2 + \sigma_i^2T,$$

and thus,

$$p_{\text{sta}} = \frac{e_1 + e_2}{2} = x^2 + 2x\mu T + \frac{b_1^2 + b_2^2}{2}T^2 + \Sigma^2 T$$

as well as

$$q_i = T^{-1}(e_i - p_{\text{sta}}) = T^{-1}\frac{e_i - e_j}{2} = x(b_i - b_j) + \frac{1}{2}T(b_i^2 - b_j^2) + \frac{1}{2}(\sigma_i^2 - \sigma_j^2).$$

■

### *D. Proofs for Section IV*

In this section, we discuss the planner's problem.

PROOF OF THEOREM 2: Let $\mathcal{A}$ be the set of all assignments, that is, all measurable functions $\alpha(t,x) = (\alpha_1(t,x),\ldots,\alpha_n(t,x))$ with $\alpha_- \le \alpha_i \le \alpha_+$.

(i) Fix $\alpha \in \mathcal{A}$ and denote $\|\alpha\| = \sum_{i=1}^n \alpha_i$. Suppose first that $w \in C_b^{1,2}$ is a given equilibrium price function for $\alpha$. As in Lemma F2, the unique optimal portfolio for agent $i$ is $\Phi_i(t) = \phi_i(t, X(t))$, where

$$\phi_i(t,x) = \alpha_i(t,x)\mathcal{L}^i w(t,x). \tag{F10}$$

The market-clearing condition then implies $\sum_{i=1}^{n} \alpha_i \mathcal{L}^i w = s$, which is equivalent to

$$\frac{1}{\|\alpha\|}\left(\sum_{i=1}^{n} \alpha_i \mathcal{L}^i w - s\right) = 0 \tag{F11}$$

or

$$\partial_t w + \frac{1}{\|\alpha\|}\sum_{i=1}^{n} \alpha_i b_i \partial_x w + \frac{1}{2}\operatorname{Tr}\frac{1}{\|\alpha\|}\sum_{i=1}^{n} \alpha_i \sigma_i^2 \partial_{xx} w - \frac{s}{\|\alpha\|} = 0. \tag{F12}$$

Together with the terminal condition $w(T,\cdot) = f$, this implies by Itô's formula that $w$ has the Feynman-Kac representation

$$w(t,x) = E\left[ f(X_\alpha^{t,x}(T)) - \int_t^T \kappa_\alpha(r, X_\alpha^{t,x}(r))\, dr\right], \tag{F13}$$

where $\kappa_\alpha = s/\|\alpha\|$ and $X_\alpha^{t,x}$ is a diffusion with initial condition $X^{t,x}(t) = x$, drift $\mu_\alpha = \frac{1}{\|\alpha\|}\sum_{i=1}^{n} \alpha_i b_i$, and volatility $\sigma_\alpha = \frac{1}{\|\alpha\|}\sum_{i=1}^{n} \alpha_i \sigma_i$.
Conversely, suppose that $\alpha$ is sufficiently regular so that (F12) has a solution $w \in C_b^{1,2}$. Then reversing the above arguments shows that $w$ is an equilibrium price function given the assignment $\alpha$. For examples of sufficient regularity conditions on $\alpha$, see, for example, Friedman (1975, p. 147).

(ii) Consider the nonlinear PDE

$$\sup_{\alpha \in \mathcal{A}} \sum_{i=1}^{n} \alpha_i \mathcal{L}^i w = s.$$

Noting that the supremum is attained for $\alpha_i = \alpha_{\operatorname{sign}(\mathcal{L}^i w)}$, we observe that this is the same PDE as (F3). To wit, after stating it in the equivalent form

$$\sup_{\alpha \in \mathcal{A}} \frac{1}{\|\alpha\|}\left(\sum_{i=1}^{n} \alpha_i \mathcal{L}^i w - s\right) = 0, \tag{F14}$$

we see that this is just a rewriting of (6). In particular, for the terminal condition $f$, the unique solution of (F14) in $C_b^{1,2}$ is given by the equilibrium price function $v$ of Theorem 1, and the stated assignment associated with $I_*$ attains that price. (As $v \in C_b^{1,2}$, this assignment is indeed "sufficiently regular" in the sense used in (i) above.)
To see that any other assignment leads to a lower price, consider a fixed (and sufficiently regular) assignment $\alpha = (\alpha_1, \ldots, \alpha_n)$ and its equilibrium price $v_\alpha$. As $v_\alpha$ solves (F11),

$$\sup_{\alpha' \in \mathcal{A}} \frac{1}{\|\alpha'\|}\left(\sum_{i=1}^{n} \alpha'_i \mathcal{L}^i v_\alpha - s\right) \geq \frac{1}{\|\alpha\|}\left(\sum_{i=1}^{n} \alpha_i \mathcal{L}^i v_\alpha - s\right) = 0.$$

This shows that $v_\alpha$ is a subsolution of (6), and hence, $v_\alpha \leq v$ by the comparison principle of Fleming and Soner (2006, Theorem V.9.1, p. 223). ■

REMARK F1: *The difference for a general (measurable) $\alpha$ is that the smoothness of the solution to the PDE (F12) is not clear. However, one may substitute the classical solution of the PDE by a suitable weaker concept to derive the conclusions of Theorem 2. We sketch this for the case when the types disagree on the drift but agree on the volatility $\sigma := \sigma_i$. Let $Q_0$ be a probability under which*

$$dX(t) = \sigma(t, X(t))\, dW^0(t),$$

*where $W^0$ is a $Q_0$-Brownian motion. For $1 \leq i \leq n$, let $Q_i$ be an equivalent probability such that $dW^i(t) := dW^0(t) - \sigma^{-1}(t, X(t)) b_i(t, X(t))\, dt$ is a Brownian motion under $Q_i$ and thus $dX(t) = b_i(t, X(t))\, dt + \sigma(t, X(t))\, dW^i(t)$ under $Q_i$ as desired for type i. Consider under $Q_0$ the linear backward stochastic differential equation (BSDE)*

$$dY(t) = g(t, X(t), Z(t))\, dt + Z(t)\, dW^0(t), \quad Y(T) = f(X(T)),$$

*where*

$$g(t, x, z) = \kappa_\alpha(t, x) - \mu_\alpha(t, x)\sigma(t, x)^{-1} z.$$

*This equation has a unique square-integrable solution $(Y, Z)$ (see El Karoui, Peng, and Quenez (1997, Proposition 2.2)), and, in fact, $Y$ is bounded in the present case. We remark that this solution corresponds to (F12) in the sense that if (F12) has a smooth solution $w$, then $Y(t) = w(t, X(t))$ and $Z(t) = \sigma(t, X(t))\partial_x w(t, X(t))$. Under the measure $Q_i$ of agent i, we have*

$$dY(t) = [g(t, X(t), Z(t)) + Z(t)\sigma^{-1}(t, X(t)) b_i(t, X(t))]\, dt + Z(t)\, dW^i(t). \tag{F15}$$

*Similarly as in (F10), this implies that if Y is the price process, the optimal portfolio for agent i is*

$$\Phi_i(t) = \alpha_i(t, X(t))[g(t, X(t), Z(t)) + Z(t)\sigma^{-1}(t, X(t)) b_i(t, X(t))].$$

*Moreover, the definition of g yields that these portfolios satisfy the market-clearing condition $\sum_i \Phi_i(t) = S(t)$, that is, $P := Y$ is an equilibrium price process. Conversely, any bounded equilibrium price process $P'$ induces a square-integrable solution of (F15) and hence $P' = Y$ by uniqueness.*

*Since the BSDE is Markovian, one can show that the process Y is necessarily of the form $Y(t) = v_\alpha(t, X(t))$ for a deterministic function $v_\alpha$. Even if $v_\alpha$ is not necessarily smooth, it is still a viscosity solution of the related PDE, which is sufficient to apply the comparison principle as in part (ii) of the above proof to see that the equilibrium price of Theorem 1 dominates $Y(0)$. Alternatively, one can apply the comparison principle of BSDEs (see El Karoui, Peng, and Quenez (1997)).*

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