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American Journal of Mathematics, Volume 142, Number 2, April 2020, pp. 627-682 (Article)

Published by Johns Hopkins University Press
DOI: *https://doi.org/10.1353/ajm.2020.0017*

HIGHER CHERN CLASSES IN IWASAWA THEORY

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Abstract. We begin a study of mth Chern classes and mth characteristic symbols for Iwasawa modules which are supported in codimension at least m. This extends the classical theory of characteristic ideals and their generators for Iwasawa modules which are torsion, i.e., supported in codimension at least 1. We apply this to an Iwasawa module constructed from an inverse limit of p-parts of ideal class groups of abelian extensions of an imaginary quadratic field. When this module is pseudo-null, which is conjecturally always the case, we determine its second Chern class and show that it has a characteristic symbol given by the Steinberg symbol of two Katz p-adic L-functions.

Contents.

1. Introduction. The main conjecture of Iwasawa theory in its most classical form asserts the equality of two ideals in a formal power series ring. The first is defined through the action of the abelian Galois group of the p-cyclotomic tower over an abelian base field on a limit of p-parts of class groups in the tower. The other is generated by a power series that interpolates values of Dirichlet L-functions. This conjecture was proven by Mazur and Wiles [34] and has since been generalized in a multitude of ways. It has led to the development of a wide range of new methods in number theory, arithmetic geometry and the theory of modular forms: see for

Manuscript received April 20, 2017; revised November 7, 2018.

Research of the first author supported in part by NSF FRG grant DMS-1360621 and NSF grant DMS-1801328; research of the second author supported in part by NSF FRG grant DMS-1360767, NSF FRG grant DMS-1265290, NSF SaTC grants CNS-1513671 and CNS-1701785, Simons Foundation grant 338379 and NSF grant DMS-1107263/1107367/1107452 "RNMS: Geometric Structures and Representation Varieties" (the GEAR Network); research of the third author supported in part by NSF FRG grant DMS-1360902; research of the fourth author supported in part by EPSRC First grant EP/L021986/1; research of the fifth author supported in part by NSF FRG grant DMS-1360733 and NSF grant DMS-1701619; research of the sixth author supported in part by NSF FRG grant DMS-1360583, NSF grants DMS-1401122/1615568 and DMS-1801963, and Simons Foundation grant 304824.

American Journal of Mathematics 142 (2020), 627–682.

example [3, 18, 27] and their references. As we will explain in Section 3, classical main conjectures pertain to the first Chern classes of various complexes of modules over Iwasawa algebras. In this paper, we begin a study of the higher Chern classes of such complexes and their relation to analytic invariants such as p-adic L-functions. This can be seen as studying the behavior in higher codimension of the natural complexes.

Higher Chern classes appear implicitly in some of the earliest work of Iwasawa [22]. Let p be an odd prime, and let F_∞ denote a $\mathbb{Z}_p$-extension of a number field F. Iwasawa showed that for sufficiently large n, the order of the p-part of the ideal class group of the cyclic extension of degree p^n in F_∞ is

$$p^{\mu p^n+\lambda n+\nu} \tag{1.1}$$

for some constants μ, λ and ν. Let L be the maximal abelian unramified pro-p extension of F_∞. Iwasawa's theorem is proved by studying the structure of $X = \mathrm{Gal}(L/F_\infty)$ as a module for the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![t]\!]$ associated to $\Gamma = \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$. Here, Λ is a dimension two unique factorization domain with a unique codimension two prime ideal (p,t), which has residue field $\mathbb{F}_p$. The focus of classical Iwasawa theory is on the invariants μ and λ, which pertain to the support of X in codimension 1 as a torsion finitely generated Λ-module. More precisely, μ and λ are determined by the first Chern class of X as a Λ-module, as will be explained in Subsection 2.5. Suppose now that $\mu = 0 = \lambda$. Then X is either zero or supported in codimension 2 (i.e., X is pseudo-null), and

$$\nu \in \mathbb{Z} = \mathrm{K}_0\big(\mathbb{F}_p\big)$$

may be identified with the (localized) second Chern class of X as a Λ-module. In general, the relevant Chern class is associated to the codimension of the support of an Iwasawa module. This class can be thought of as the leading term in the algebraic description of the module. When one is dealing with complexes of modules, the natural codimension is that of the support of the cohomology of the complex.

1.1. Chern classes and characteristic symbols. There is a general theory of localized Chern classes due to Fulton-MacPherson [12, Chapter 18] based on MacPherson's graph construction (see also [49]). Moreover, Gillet developed a sophisticated theory of Chern classes in K-cohomology with supports in [13]. This pertains to suitable complexes of modules over a Noetherian scheme which are exact off a closed subscheme and requires certain assumptions, including Gersten's conjecture. In this paper, we will restrict to a special situation that can be examined by simpler tools. Suppose that R is a local commutative Noetherian ring and that $\mathcal{C}^\bullet$ is a bounded complex of finitely generated R-modules which is exact in codimension less than m. We now describe an mth Chern class which can be associated to $\mathcal{C}^\bullet$. In our applications, R will be an Iwasawa algebra.

Let $Y = \mathrm{Spec}(R)$, and let $Y^{(m)}$ be the set of codimension m points of Y, i.e., height m prime ideals of R. Denote by $\mathrm{Z}^m(Y)$ the group of cycles of codimension m in Y, i.e., the free abelian group generated by $y \in Y^{(m)}$:

$$\mathrm{Z}^m(Y) = \bigoplus_{y \in Y^{(m)}} \mathbb{Z} \cdot y.$$

For $y \in Y^{(m)}$, let R_y denote the localization of R at y, and set $\mathcal{C}^\bullet_y = \mathcal{C}^\bullet \otimes_R R_y$. Under our condition on $\mathcal{C}^\bullet$, the cohomology groups $\mathrm{H}^i(\mathcal{C}^\bullet_y) = \mathrm{H}^i(\mathcal{C}^\bullet) \otimes_R R_y$ are finite length R_y-modules. We then define a (localized) Chern class $c_m(\mathcal{C}^\bullet)$ in the group $\mathrm{Z}^m(Y)$ by letting the component at y of $c_m(\mathcal{C}^\bullet)$ be the alternating sum of the lengths

$$\sum_i (-1)^i \mathrm{length}_{R_y} \mathrm{H}^i\big(\mathcal{C}^\bullet_y\big).$$

If the codimension of the support of some $\mathrm{H}^i(\mathcal{C}^\bullet_y)$ is exactly m, the Chern class $c_m(\mathcal{C}^\bullet)$ is what we referred to earlier, just before the start of Subsection 1.1, as the leading term of $\mathcal{C}^\bullet$ as a complex of R-modules. This is a very special case of the construction in [49] and [12, Chapter 18]. In particular, if M is a finitely generated R-module which is supported in codimension at least m, we have

$$c_m(M) = \sum_{y \in Y^{(m)}} \mathrm{length}_{R_y}\big(M_y\big) \cdot y.$$

We would now like to relate $c_m(\mathcal{C}^\bullet)$ to analytic invariants. Suppose that R is a regular integral domain, and let Q be the fraction field of R. When $m = 1$ one can use the divisor homomorphism

$$\nu_1 \colon Q^\times \longrightarrow \mathrm{Z}^1(Y) = \bigoplus_{y \in Y^{(1)}} \mathbb{Z} \cdot y.$$

In the language of the classical main conjectures, an element $f \in Q^\times$ such that $\nu_1(f) = c_1(M)$ is a characteristic power series for M when R is a formal power series ring. A main conjecture for M posits that there is such an f which can be constructed analytically, e.g., via p-adic L-functions.

The key to generalizing this is to observe that $Q^\times$ is the first Quillen K-group $\mathrm{K}_1(Q)$ and ν_1 is a tame symbol map. To try to relate $c_m(M)$ to analytic invariants for arbitrary m, one can consider elements of $\mathrm{K}_m(Q)$ which can be described by symbols involving m-tuples of elements of Q associated to L-functions. The homomorphism ν_1 is replaced by a homomorphism ν_m involving compositions of tame symbol maps. We now describe one way to do this.

Suppose that $\underline{\eta} = (\eta_0, \ldots, \eta_m)$ is a sequence of points of Y with $\mathrm{codim}(\eta_i) = i$ and such that η_{i+1} lies in the closure $\overline{\eta}_i$ of η_i for all $i < m$. Denote by $P_m(Y)$ the set of all such sequences. Let $k(\eta_i) = Q(R/\eta_i)$ be the residue field of η_i. Composing

successive tame symbol maps (i.e., connecting maps of localization sequences), we obtain homomorphisms

$$\nu_{\underline{\eta}}\colon \mathrm{K}_m(Q) = \mathrm{K}_m\big(k(\eta_0)\big) \longrightarrow \mathrm{K}_{m-1}\big(k(\eta_1)\big) \longrightarrow \cdots \longrightarrow \mathrm{K}_0\big(k(\eta_m)\big) = \mathbb{Z}.$$

Here, K_i denotes the ith Quillen K-group. We combine these in the following way to give a homomorphism

$$\nu_m\colon \bigoplus_{\underline{\eta}'\in P_{m-1}(Y)} \mathrm{K}_m(Q) \longrightarrow \mathrm{Z}^m(Y) = \bigoplus_{y\in Y^{(m)}} \mathbb{Z}\cdot y.$$

Suppose $a = (a_{\underline{\eta}'})_{\underline{\eta}'\in P_{m-1}(Y)}$. We define the component of $\nu_m(a)$ at y to be the sum of $\nu_{(\eta_0',\eta_1',\ldots,\eta_{m-1}',y)}(a_{\underline{\eta}'})$ over all the sequences

$$\underline{\eta}' = (\eta_0', \eta_1', \ldots, \eta_{m-1}') \in P_{m-1}(Y)$$

such that y is in the closure of η_{m-1}'.

If M is a finitely generated R-module supported in codimension at least m as above, then we refer to any element in $\bigoplus_{\underline{\eta}'\in P_{m-1}(Y)} \mathrm{K}_m(Q)$ that ν_m maps to $c_m(M)$ as a *characteristic symbol* for M. This generalizes the notion of a characteristic power series of a torsion module in classical Iwasawa theory, which can be reinterpreted as the case $m = 1$.

We focus primarily on the case in which $m = 2$ and R is a formal power series ring $A[\![t_1, \ldots, t_r]\!]$ over a mixed characteristic complete discrete valuation ring A. In this case, we show that the symbol map ν_2 gives an isomorphism

$$\frac{\prod'_{\eta_1\in Y^{(1)}} \mathrm{K}_2(Q)}{\mathrm{K}_2(Q)\prod_{\eta_1\in Y^{(1)}} \mathrm{K}_2(R_{\eta_1})} \xrightarrow{\sim} \mathrm{Z}^2(Y). \tag{1.2}$$

This uses the fact that Gersten's conjecture holds for K_2 and R. In the numerator of (1.2), the restricted product $\prod'_{\eta_1\in Y^{(1)}} \mathrm{K}_2(Q)$ is the subgroup of the direct product in which all but a finite number of components belong to $\mathrm{K}_2(R_{\eta_1}) \subset \mathrm{K}_2(Q)$. In the denominator, we have the product of the subgroups $\prod_{\eta_1\in Y^{(1)}} \mathrm{K}_2(R_{\eta_1})$ and $\mathrm{K}_2(Q)$, the second group embedded diagonally in $\prod'_{\eta_1\in Y^{(1)}} \mathrm{K}_2(Q)$. The significance of this formula is that it shows that one can specify elements of $\mathrm{Z}^2(Y)$ through a list of elements of $\mathrm{K}_2(Q)$, one for each codimension one prime η_1 of R, such that the element for η_1 lies in $\mathrm{K}_2(R_{\eta_1})$ for all but finitely many η_1.

1.2. Results. Returning to Iwasawa theory, an optimistic hope one might have is that under certain hypotheses, the second Chern class of an Iwasawa module or complex thereof can be described using (1.2) and Steinberg symbols in $\mathrm{K}_2(Q)$ with arguments that are p-adic L-functions. Our main result, Theorem 5.2.5, is of exactly this kind. In it, we work under the assumption of a conjecture of Greenberg which predicts that certain Iwasawa modules over multi-variable power series rings

are pseudo-null, i.e., that they have trivial support in codimension 1. We recall this conjecture and some evidence for it found by various authors in Subsection 3.4.

More precisely, we consider in Subsection 5.2 an imaginary quadratic field E, and we assume that p is an odd prime that splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ of E. Let $\widetilde{E}$ denote the compositum of all $\mathbb{Z}_p$-extensions of E. Let ψ be a one-dimensional p-adic character of the absolute Galois group of E of finite order prime to p, and denote by K (resp., F) the compositum of the fixed field of ψ with $\widetilde{E}(\mu_p)$ (resp., $E(\mu_p)$). We consider the Iwasawa module $X = \mathrm{Gal}(L/K)$, where L is the maximal abelian unramified pro-p extension of K. Set $\mathcal{G} = \mathrm{Gal}(K/E)$, and let ω be its Teichmüller character. Let $\Delta = \mathrm{Gal}(F/E)$, which we may also view as the largest subgroup of $\mathcal{G}$ of prime-to-p order. For simplicity in this discussion, we suppose that $\psi \neq 1, \omega$.

The Galois group $\mathcal{G}$ has an open maximal pro-p subgroup Γ isomorphic to $\mathbb{Z}_p^2$. Greenberg has conjectured that X is pseudo-null as a module for $\Lambda = \mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![t_1, t_2]\!]$. Our goal is to obtain information about X and its eigenspaces $X^\psi = \mathcal{O}_\psi \otimes_{\mathbb{Z}_p[\Delta]} X$, where $\mathcal{O}_\psi$ is the $\mathbb{Z}_p$-algebra generated by the values of ψ, and $\mathbb{Z}_p[\Delta] \to \mathcal{O}_\psi$ is the surjection induced by ψ. When Greenberg's conjecture is true, the characteristic ideal giving the first Chern class of X^ψ is trivial. It thus makes good sense to consider the second Chern class, which gives information about the height 2 primes in the support of X^ψ.

Consider the Katz p-adic L-functions $\mathcal{L}_{\mathfrak{p},\psi}$ and $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ in the fraction field Q of the ring $R = e_\psi \cdot W[\![\mathcal{G}]\!] \cong W[\![t_1, t_2]\!]$, where $e_\psi \in W[\![\mathcal{G}]\!]$ is the idempotent associated to ψ and W denotes the Witt vectors of an algebraic closure of $\mathbb{F}_p$. We can now define an analytic element c_2^{an} in the group $\mathrm{Z}^2(\mathrm{Spec}(R))$ of (1.2) in the following way. Let c_2^{an} be the image of the element on the left-hand side of (1.2) with component at η_1 the Steinberg symbol

$$\{\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi}\} \in \mathrm{K}_2(Q),$$

if $\mathcal{L}_{\mathfrak{p},\psi}$ is not a unit at η_1, and with other components trivial. This element c_2^{an} does not depend on the ordering of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ (see Remark 2.5.2).

Our main result, Theorem 5.2.5, is that if X is pseudo-null, then

$$c_2^{\mathrm{an}} = c_2\big(X_W^\psi\big) + c_2\big(\big(X_W^{\omega\psi^{-1}}\big)^\iota(1)\big) \tag{1.3}$$

where X_W^ψ and $(X_W^{\omega\psi^{-1}})^\iota(1)$ are the R-modules defined as follows: X_W^ψ is the completed tensor product $W \hat{\otimes}_{\mathcal{O}_\psi} X^\psi$, while $(X_W^{\omega\psi^{-1}})^\iota(1)$ is the Tate twist of the module which results from $X_W^{\omega\psi^{-1}}$ by letting $g \in \mathcal{G}$ act by g^{-1}.

In (1.3), one needs to take completed tensor products of Galois modules with W because the analytic invariant c_2^{an} is only defined over W. Note that the right-hand side of (1.3) concerns two different components of X, namely those associated to ψ and $\omega\psi^{-1}$. It frequently occurs that exactly one of the two is nontrivial: see Example 5.2.8. In fact, one consequence of our main result is a codimension

two elliptic counterpart of the Herbrand-Ribet Theorem (see Corollary 5.2.7): the eigenspaces X^{ψ} and $X^{\omega\psi^{-1}}$ are both trivial if and only if one of $\mathcal{L}_{\mathfrak{p},\psi}$ or $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ is a unit power series.

One can also interpret the right-hand side of (1.3) in the following way. Let $\Omega = \mathbb{Z}_p[\![\mathcal{G}]\!]$ and let $\epsilon\colon \Omega \to \Omega$ be the involution induced by the map $g \to \chi_{\mathrm{cyc}}(g)g^{-1}$ on $\mathcal{G}$ where $\chi_{\mathrm{cyc}}\colon \mathcal{G} \to \mathbb{Z}_p^{\times}$ is the cyclotomic character. Then $(X^{\omega\psi^{-1}})^{\iota}(1)$ is canonically isomorphic to the ψ component $(X_{\epsilon})^{\psi}$ of the twist $X_{\epsilon} = \Omega \otimes_{\epsilon,\Omega} X$ of X by ϵ. Thus X_{ϵ} is isomorphic to X as a $\mathbb{Z}_p$-module but with the action of Ω resulting from precomposing with the involution $\epsilon\colon \Omega \to \Omega$. Then (1.3) can be written

$$c_2^{\mathrm{an}} = c_2\Big(W \hat{\otimes}_{\mathcal{O}_{\psi}} \big(X \oplus X_{\epsilon}\big)^{\psi}\Big). \tag{1.4}$$

We discuss two extensions of (1.3). In Subsection 5.3, we explain how the algebraic part of our result for imaginary quadratic fields extends, under certain additional hypotheses on E and ψ, to number fields E with at most one complex place. In Section 6, we show how when E is imaginary quadratic and K is Galois over $\mathbb{Q}$, we can obtain information about the X above as a module for the non-commutative Iwasawa algebra $\mathbb{Z}_p[\![\mathrm{Gal}(K/\mathbb{Q})]\!]$. This involves a "non-commutative second Chern class" in which, instead of lengths of modules, we consider classes in appropriate Grothendieck groups. Developing counterparts of our results for more general non-commutative Galois groups is a natural goal in view of the non-commutative main conjecture concerning first Chern classes treated in [7].

1.3. Outline of the proof. We now outline the strategy of the proof of (1.3). We first consider the Galois group $\mathfrak{X} = \mathrm{Gal}(N/K)$, with N the maximal abelian pro-p extension of K that is unramified outside of p. One has $X = \mathfrak{X}/(I_{\mathfrak{p}} + I_{\bar{\mathfrak{p}}})$ for $I_{\mathfrak{p}}$ the subgroup of $\mathfrak{X}$ generated by inertia groups of primes of K over $\mathfrak{p}$ and $I_{\bar{\mathfrak{p}}}$ defined similarly for the prime $\bar{\mathfrak{p}}$. A novelty of the proof is that it requires carefully analyzing the discrepancy between the rank one Ω-module $\mathfrak{X}$ and its free reflexive hull.

The reflexive hull of a Λ-module M is $M^{**} = (M^*)^*$ for $M^* = \mathrm{Hom}_{\Lambda}(M, \Lambda)$, and there is a canonical homomorphism $M \to M^{**}$. Iwasawa-theoretic duality results tell us that since X is pseudo-null, the map $\mathfrak{X} \to \mathfrak{X}^{**}$ is injective with an explicit pseudo-null cokernel (in particular, see Proposition 4.1.17). We have a commutative diagram

$$\begin{array}{ccc} I_{\mathfrak{p}} \oplus I_{\bar{\mathfrak{p}}} & \longrightarrow & I_{\mathfrak{p}}^{**} \oplus I_{\bar{\mathfrak{p}}}^{**} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X}^{**}. \end{array} \tag{1.5}$$

Taking cokernels of the vertical homomorphisms in (1.5) yields a homomorphism

$$f\colon X \longrightarrow \mathfrak{X}^{**}/\big(\mathrm{im}\,\big(I_{\mathfrak{p}}^{**}\big) + \mathrm{im}\,\big(I_{\bar{\mathfrak{p}}}^{**}\big)\big),$$

where im denotes the image. A snake lemma argument then tells us that the cokernel of f is the Tate twist of an Iwasawa adjoint $\alpha(X)$ of X which has the same class as X^ι in the quotient of the Grothendieck group of the category of pseudo-null modules by the Grothendieck group of the category of finite modules. Moreover, the map f is injective in its ψ-eigenspace as $\psi \neq \omega$.

The ψ-eigenspaces of $\mathfrak{X}$, $I_{\mathfrak{p}}$ and $I_{\bar{\mathfrak{p}}}$ are of rank one over $\Lambda_\psi = \mathcal{O}_\psi[\![\Gamma]\!]$. They need not be free, but the key point is that their reflexive hulls are. The main conjecture for imaginary quadratic fields proven by Rubin [50, 51] (see also [26]) implies that the p-adic L-function $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ in $\Lambda_W = W[\![\Gamma]\!]$ generates the image of the map

$$W \hat{\otimes}_{\mathcal{O}_\psi} (I_{\mathfrak{p}}^\psi)^{**} \longrightarrow W \hat{\otimes}_{\mathcal{O}_\psi} (\mathfrak{X}^\psi)^{**} \cong \Lambda_W,$$

and similarly switching the roles of the two primes. Putting everything together, we have an exact sequence of Λ_W-modules:

$$0 \longrightarrow X_W^\psi \longrightarrow \frac{\Lambda_W}{\mathcal{L}_{\mathfrak{p},\psi}\Lambda_W + \mathcal{L}_{\bar{\mathfrak{p}},\psi}\Lambda_W} \longrightarrow \alpha(X_W^{\omega\psi^{-1}})(1) \longrightarrow 0. \tag{1.6}$$

The second Chern class of the middle term is c_2^{an}, and the second Chern class of the last term depends only on its class in the Grothendieck group, yielding (1.3).

1.4. Generalizations. We next describe several potential generalizations of (1.3) to other fields and Selmer groups of higher-dimensional Galois representations. We intend them as motivation for further study, leaving details to future work.

Consider first the case of a CM field E of degree $2d$, and suppose that the primes over p in E are split from the maximal totally real subfield. There is then a natural generalization of the analytic class on the left side of (1.4). Fix a p-adic CM type $\mathcal{Q}$ for E, and let $\bar{\mathcal{Q}}$ be the conjugate type. Then for K and F defined as above using a p-adic character ψ of prime-to-p order of the absolute Galois group of E, one has Katz p-adic L-functions $\mathcal{L}_{\mathcal{Q},\psi}$ and $\mathcal{L}_{\bar{\mathcal{Q}},\psi}$ in the algebra $R = \Omega_W^\psi$ for $\Omega = \mathbb{Z}_p[\![\mathrm{Gal}(K/E)]\!]$. Suppose that the quotient

$$\frac{R}{R\mathcal{L}_{\mathcal{Q},\psi} + R\mathcal{L}_{\bar{\mathcal{Q}},\psi}} \tag{1.7}$$

is pseudo-null over R. A generalization of (1.3) would relate the second Chern class $c_{2,\mathcal{Q}}^{\mathrm{an}}$ of (1.7) to a sum of second Chern classes of algebraic objects arising from Galois groups.

The question immediately arises of how to extend our algebraic methods, as the unramified outside p Iwasawa module $\mathfrak{X}$ over K has Ω-rank d. For this, we turn to the use of highest exterior powers, which has a rich tradition in the study of special values of L-functions, notably in conjectures of Stark and Rubin-Stark. Let $I_{\mathcal{Q}}$ denote the subgroup of $\mathfrak{X}$ generated by the inertia groups of the primes in

$\mathcal{Q}$. Recall that the main conjecture states that the quotient $\mathfrak{X}/I_{\mathcal{Q}}$ has first Chern class agreeing with the divisor of $\mathcal{L}_{\mathcal{Q},\psi}$. To obtain a rank one object related to the above p-adic L-functions as before, it is natural to consider the dth exterior power of the reflexive hull of $\mathfrak{X}$, which we may localize at a height 2 prime to ensure its freeness. Under Greenberg's conjecture, the quotient $\bigwedge_{\Omega}^{d}\mathfrak{X}/\bigwedge_{\Omega}^{d} I_{\mathcal{Q}}$ is pseudo-isomorphic to the analogous quotient for exterior powers of reflexive hulls, and the main conjecture becomes the statement that $c_1((\bigwedge_{\Omega}^{d}\mathfrak{X}/\bigwedge_{\Omega}^{d} I_{\mathcal{Q}})^{\psi}_{W})$ is the divisor of $\mathcal{L}_{\mathcal{Q},\psi}$.

This suggests that the proper object for comparison with the analytic class $c^{\mathrm{an}}_{2,\mathcal{Q}}$ is no longer the second Chern class of the ψ-eigenspace of the unramified Iwasawa module X but of the quotient of dth exterior powers

$$Z_{\mathcal{Q}} = \frac{\bigwedge_{\Omega}^{d}\mathfrak{X}}{\bigwedge_{\Omega}^{d} I_{\mathcal{Q}} + \bigwedge_{\Omega}^{d} I_{\bar{\mathcal{Q}}}}. \tag{1.8}$$

Following the approach used when $d = 1$, it is natural to consider the difference between $Z_{\mathcal{Q}}$ and the analogous quotient in which every term is replaced by its reflexive hull. This is the approach taken in [1], where roughly speaking, we show that the analytic class $c^{\mathrm{an}}_{2,\mathcal{Q}}$ is a sum of second Chern classes arising from $Z^{\psi}_{\mathcal{Q}}$ and $X^{\omega\psi^{-1}}$. We also provide a Galois-theoretic interpretation of $Z_{\mathcal{Q}}$ for $d = 2$ as a quotient of the second graded quotient in the lower central series of the Galois group of the maximal pro-p, unramified outside p extension of K.

We can also consider the case in which E is imaginary quadratic but p is inert, so that $\mathfrak{X}$ is again of rank one but the product of corresponding inertia groups over the completions of K at p has rank two. Using Kobayashi's plus/minus Selmer conditions, Pollack and Rubin [45, Section 4] define two rank one submodules I^{+} and I^{-} of $\mathfrak{X}$ that play the role of $I_{\mathfrak{p}}$ and $I_{\bar{\mathfrak{p}}}$. Using the results in the present paper, one can get an analogue of (1.6) with $\mathcal{L}_{\mathfrak{p}}$ and $\mathcal{L}_{\bar{\mathfrak{p}}}$ replaced by characteristic elements of $\mathfrak{X}/I^{+}$ and $\mathfrak{X}/I^{-}$. As a two-variable main conjecture in this setting is lacking, this does not directly yield a relation of the second Chern class with L-functions.

Finally, we turn to the Selmer groups of ordinary p-adic modular forms, which fit in one-variable families known as Hida families, parameterized by the weight of the form. Theorem 5.2.5 is a special case of this framework involving CM newforms. Hida families of residually irreducible newforms give rise to Galois representations with Galois-stable lattices free over an integral extension $\mathcal{I}$ of an Iwasawa algebra in two variables, the so-called weight and cyclotomic variables. These lattices are self-dual up to a twist. We can use the cohomology of such a lattice to define objects analogous to $\mathfrak{X}$ and X. The former is of rank one over $\mathcal{I}$, and the latter as before should be pseudo-null, with the dual Selmer group of the lattice providing an intermediate object between the two. The dual Selmer group is expected to have first Chern class given by a Mazur-Kitagawa p-adic L-function. In this case, the algebraic study goes through without serious additional complication,

and the only obstruction to an exact sequence as in (1.6) is the identification of a second annihilator.

The above examples may be just the tip of an iceberg. In [17], a main conjecture is formulated in a very general context where one considers a Galois representation over a complete Noetherian local ring R with finite residue field of characteristic p. A main conjecture corresponds to a so-called *Panchishkin condition*. It is not uncommon for there to be more than one choice of a Panchishkin condition and hence more than one main conjecture. On the analytic side, the corresponding p-adic L-functions should often have divisors intersecting properly. On the algebraic side, the Pontryagin dual of the intersections of the corresponding Selmer groups should often be supported in codimension 2. It is then tempting to believe that the type of result we consider in this paper would have an analogue in this context and would involve taking highest exterior powers of appropriate R-modules.

There are other situations where one can define more than one Selmer group and more than one p-adic L-function in natural ways, found for example in the work of Pollack [44], Kobayashi [28], Lei-Loeffler-Zerbes [30], Sprung [55], Pottharst [46]. Some of these constructions involve what amounts to a choice of Panchishkin condition after a change of scalars. This begs the question, that we do not address here, of how to define a suitable generalized notion of a Panchishkin condition.

1.5. Organization of the paper. In Section 2, we define Chern classes and characteristic symbols, and we explain how (1.2) follows from certain proven cases of Gersten's conjecture. In Section 3, we recall the formalism of some previous main conjectures in Iwasawa theory. We also recall properties of Katz's p-adic L-functions and Rubin's results on the main conjecture over imaginary quadratic fields. In Subsection 3.4, we recall Greenberg's conjecture and some evidence for it.

In Section 4, we discuss various Iwasawa modules in some generality. The emphasis is on working out Iwasawa-theoretic consequences of Tate, Poitou-Tate and Grothendieck duality. This requires the work in the Appendix, which concerns Ext-groups and Iwasawa adjoints of modules over certain completed group rings.

We begin Section 5 with a discussion of reflection theorems of the kind we will need to discuss Iwasawa theory in codimension two. In Subsection 5.1, we discuss codimension two phenomena in the most classical case of the cyclotomic $\mathbb{Z}_p$-extension of an abelian extension of $\mathbb{Q}$. Our main result over imaginary quadratic fields is proven in Subsection 5.2 using the strategy discussed above. The extension of the algebraic part of the proof to number fields with at most one complex place is given in Subsection 5.3. The non-commutative generalization over imaginary quadratic fields is proved in Section 6.

Acknowledgments. F. Bleher and T. Chinburg would like to thank L'Institut des Hautes Études Scientifiques for support during the Fall of 2015. R. Greenberg

would like to thank La Fondation Sciences Mathématiques de Paris for its support during the Fall of 2015. M. Kakde would like to thank TIFR, Mumbai, for its hospitality during the writing of parts of this paper. T. Chinburg, R. Greenberg, M. Kakde, and R. Sharifi would like to thank the Banff International Research Station for hosting the workshop "Applications of Iwasawa Algebras" in March 2013.

2. Chern classes and characteristic symbols.

2.1. Chern classes. We denote by $\mathrm{K}'_m(R)$ and $\mathrm{K}_m(R)$, the Quillen K-groups [47] of a ring R defined using the categories of finitely generated and finitely generated projective R-modules, respectively. If R is regular and Noetherian, then we can identify $\mathrm{K}_m(R) = \mathrm{K}'_m(R)$.

Suppose that R is a commutative local integral Noetherian ring. Denote by $\mathfrak{m}$ the maximal ideal of R. Set $Y = \mathrm{Spec}(R)$, and denote by $Y^{(i)}$ the set of points of Y of codimension i, i.e., of prime ideals of R of height i. Let Q denote the fraction field $Q(R)$ of R, and denote by η the generic point of Y.

For $m \geq 0$, we set

$$Z^m(Y) = \bigoplus_{y \in Y^{(m)}} \mathbb{Z} \cdot y,$$

the right-hand side being the free abelian group generated by $Y^{(m)}$.

Consider the Grothendieck group $\mathrm{K}_0^{\prime(m)}(R) = \mathrm{K}_0^{\prime(m)}(Y)$ of bounded complexes $\mathcal{E}^\bullet$ of finitely generated R-modules which are exact in codimension less than m, as defined for example in [54, I.3]. This is generated by classes $[\mathcal{E}^\bullet]$ of such complexes with relations given by

(i) $[\mathcal{E}^\bullet] = [\mathcal{F}^\bullet]$ if there is a quasi-isomorphism $\mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet$,

(ii) $[\mathcal{E}^\bullet] = [\mathcal{F}^\bullet] + [\mathcal{G}^\bullet]$ if there is an exact sequences of complexes

$$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{E}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow 0.$$

If M is a finitely generated R-module with support of codimension at least m, we regard it as a complex with only nonzero term M at degree 0.

Suppose that $\mathcal{C}^\bullet$ is a bounded complex of finitely generated R-modules which is exact in codimension less than m. Then for each $y \in Y^{(m)}$, we can consider the complex of R_y-modules given by the localization $\mathcal{C}^\bullet_y = \mathcal{C}^\bullet \otimes_R R_y$. The assumption on $\mathcal{C}^\bullet$ implies that all the homology groups $\mathrm{H}^i(\mathcal{C}^\bullet_y)$ are R_y-modules of finite length and $\mathrm{H}^i(\mathcal{C}^\bullet_y) = 0$ for all but a finite number of $y \in Y^{(m)}$. We set

$$c_m\big(\mathcal{C}^\bullet\big)_y = \sum_i (-1)^i \mathrm{length}_{R_y} \mathrm{H}^i\big(\mathcal{C}^\bullet_y\big)$$

and

$$c_m\big(\mathcal{C}^\bullet\big) = \sum_{y\in Y^{(m)}} c_m\big(\mathcal{C}^\bullet\big)_y \cdot y \in \mathrm{Z}^m(Y).$$

We can easily see that $c_m(\mathcal{C}^\bullet)$ only depends on the class $[\mathcal{C}^\bullet]$ in $\mathrm{K}_0^{\prime(m)}(Y)$ and that it is additive, which is to say that it gives a group homomorphism

$$c_m\colon \mathrm{K}_0^{\prime(m)}(Y) \longrightarrow \mathrm{Z}^m(Y).$$

The element $c_m(\mathcal{C}^\bullet)$ can also be thought of as a localized mth Chern class of $\mathcal{C}^\bullet$. In particular, if M is a finitely generated R-module which is supported in codimension $\geq m$, then we have

$$c_m(M) = \sum_{y\in Y^{(m)}} \mathrm{length}_{R_y}\big(M_y\big)\cdot y.$$

In [49], the element $c_m(M)$ is called the codimension-m cycle associated to M and is denoted by $[M]_{\dim(R)-m}$. The class c_m can also be given as a very special case of the construction in [12, Chapter 18].

In what follows, we will show how to produce elements of $\mathrm{Z}^m(Y)$ starting from elements in $\mathrm{K}_m(Q)$.

2.2. Tame symbols and Parshin chains. Suppose that R is a discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field Q and residue field k. Then, for all $m \geq 1$, the localization sequence of [47, Theorem 5] produces connecting homomorphisms

$$\partial_m\colon \mathrm{K}_m(Q) \longrightarrow \mathrm{K}_{m-1}(k).$$

We will call these homomorphisms ∂_m "tame symbols".

If $m = 1$, then $\partial_1(f) = \mathrm{val}(f) \in \mathrm{K}_0(k) = \mathbb{Z}$. If $m = 2$, then by Matsumoto's theorem, all elements in $\mathrm{K}_2(Q)$ are finite sums of Steinberg symbols $\{f,g\}$ with $f,g \in Q^\times$ (see [37]). We have

$$\partial_2(\{f,g\}) = (-1)^{\mathrm{val}(f)\,\mathrm{val}(g)}\frac{f^{\mathrm{val}(g)}}{g^{\mathrm{val}(f)}} \bmod \mathfrak{m} \in k^\times \tag{2.1}$$

(see for example [15, Cor. 7.13]). In this case, by [9], localization gives a short exact sequence

$$1 \longrightarrow \mathrm{K}_2(R) \longrightarrow \mathrm{K}_2(Q) \xrightarrow{\partial_2} k^\times \longrightarrow 1. \tag{2.2}$$

This exactness is a special case of Gersten's conjecture: see Subsection 2.3.

In what follows, we denote by η_i a point in $Y^{(i)}$, i.e., a prime ideal of codimension i. Suppose that η_i lies in the closure $\overline{\{\eta_{i-1}\}}$, so η_i contains η_{i-1}, and consider

$R/(\eta_{i-1})$. This is a local integral domain with fraction field $k(\eta_{i-1})$, and η_i defines a height 1 prime ideal in $R/(\eta_{i-1})$. The localization $R_{\eta_{i-1},\eta_i} = (R/(\eta_{i-1}))_{\eta_i}$ is a 1-dimensional local ring with fraction field $k(\eta_{i-1})$ and residue field $k(\eta_i)$. The localization sequence in K'-theory applied to R_{η_{i-1},η_i} still gives a connecting homomorphism

$$\partial_m(\eta_{i-1},\eta_i)\colon \mathrm{K}_m\big(k(\eta_{i-1})\big) \longrightarrow \mathrm{K}_{m-1}\big(k(\eta_i)\big).$$

For $m = 1$, by [47, Lemma 5.16] (see also Remark 5.17 therein), or by [15, Corollary 8.3], the homomorphism $\partial_1(\eta_{i-1},\eta_i)\colon k(\eta_{i-1})^\times \to \mathbb{Z}$ is equal to $\mathrm{ord}_{\eta_i}\colon k(\eta_{i-1})^\times \to \mathbb{Z}$ where ord_{η_i} is the unique homomorphism with

$$\mathrm{ord}_{\eta_i}(x) = \mathrm{length}_{R_{\eta_{i-1},\eta_i}}\big(R_{\eta_{i-1},\eta_i}/(x)\big)$$

for all $x \in R_{\eta_{i-1},\eta_i} - \{0\}$.

For any $n \geq 1$, we now consider the set $P_n(Y)$ of ordered sequences of points of Y of the form $\underline{\eta} = (\eta_0,\eta_1,\dots,\eta_n)$, with $\mathrm{codim}(\eta_i) = i$ and $\eta_i \in \overline{\{\eta_{i-1}\}}$, for all i. Such sequences are examples of "Parshin chains" [43]. For $\underline{\eta} = (\eta_0,\eta_1,\dots,\eta_n) \in P_n(Y)$, we define a homomorphism

$$\nu_{\underline{\eta}}\colon \mathrm{K}_n(Q) = \mathrm{K}_n\big(k(\eta_0)\big) \longrightarrow \mathbb{Z} = \mathrm{K}_0\big(k(\eta_n)\big)$$

as the composition of successive symbol maps:

$$\begin{aligned}&\nu_{\underline{\eta}} = \partial_1\big(\eta_{n-1},\eta_n\big)\circ\cdots\circ\partial_{n-1}\big(\eta_1,\eta_2\big)\circ\partial_n\big(\eta_0,\eta_1\big)\colon\\ &\mathrm{K}_n\big(k(\eta_0)\big) \longrightarrow \mathrm{K}_{n-1}\big(k(\eta_1)\big) \longrightarrow \cdots \longrightarrow \mathrm{K}_1\big(k(\eta_{n-1})\big) \longrightarrow \mathrm{K}_0\big(k(\eta_n)\big).\end{aligned}$$

Using this, we can define a homomorphism

$$\nu_m\colon \bigoplus_{\underline{\eta}'\in P_{m-1}(Y)} \mathrm{K}_m(Q) \longrightarrow \mathrm{Z}^m(Y) = \bigoplus_{y\in Y^{(m)}} \mathbb{Z}\cdot y$$

by setting the component of $\nu_m((a_{\underline{\eta}'})_{\underline{\eta}'})$ for $a_{\underline{\eta}'} \in \mathrm{K}_m(Q)$ that corresponds to $y \in Y^{(m)}$ to be the sum

$$\nu_m\Big(\big(a_{\underline{\eta}'}\big)_{\underline{\eta}'}\Big)_y = \sum_{\underline{\eta}'\,|\,y\in\overline{\{\eta'_{m-1}\}}} \nu_{\underline{\eta}'\cup y}\big(a_{\underline{\eta}}\big). \tag{2.3}$$

Here, we set

$$\underline{\eta}'\cup y = \big(\eta'_0,\eta'_1,\dots,\eta'_{m-1}\big)\cup y = \big(\eta'_0,\eta'_1,\dots,\eta'_{m-1},y\big).$$

Only a finite number of terms in the sum are nonzero.

For the remainder of the section, we assume that R is in addition regular.

For $m = 1$, the map ν_m amounts to

$$\nu_1 \colon \mathrm{K}_1(Q) = Q^\times \longrightarrow \mathrm{Z}^1(Y) = \bigoplus_{y \in Y^{(1)}} \mathbb{Z} \cdot y$$

sending $f \in Q^\times$ to its divisor $\mathrm{div}(f)$. Since R is regular, it is a UFD, and ν_1 gives an isomorphism

$$\mathrm{div} \colon Q^\times / R^\times \xrightarrow{\sim} \mathrm{Z}^1(Y). \tag{2.4}$$

For $m = 2$, the map

$$\nu_2 \colon \bigoplus_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q) \longrightarrow \mathrm{Z}^2(Y) = \bigoplus_{y \in Y^{(2)}} \mathbb{Z} \cdot y,$$

satisfies

$$\nu_2(a) = \sum_{\eta_1 \in Y^{(1)}} \mathrm{div}_{\eta_1}\big(\partial_2(a_{\eta_1})\big)$$

for $a = (a_{\eta_1})_{\eta_1}$ with $a_{\eta_1} \in \mathrm{K}_2(Q)$. Here,

$$\mathrm{div}_{\eta_1}(f) = \sum_{y \in \overline{\{\eta_1\}}} \mathrm{ord}_y(f) \cdot y$$

is the divisor of the function $f \in k(\eta_1)^\times$ on $\overline{\{\eta_1\}}$.

2.3. Tame symbols and Gersten's conjecture. In this paragraph we suppose that Gersten's conjecture is true for K_2 and the integral regular local ring R. By this, we mean that we assume that the sequence

$$1 \longrightarrow \mathrm{K}_2(R) \longrightarrow \mathrm{K}_2(Q) \xrightarrow{\vartheta_2} \bigoplus_{\eta_1 \in Y^{(1)}} k\big(\eta_1\big)^\times \xrightarrow{\vartheta_1} \bigoplus_{\eta_2 \in Y^{(2)}} \mathbb{Z} \longrightarrow 0 \tag{2.5}$$

is exact, where the component of ϑ_2 at η_1 is the connecting homomorphism

$$\partial_2\big(\eta_0, \eta_1\big) \colon \mathrm{K}_2(Q) \longrightarrow \mathrm{K}_1\big(k\big(\eta_1\big)\big) = k\big(\eta_1\big)^\times,$$

and ϑ_1 has components $\partial_1(\eta_1, \eta_2) = \mathrm{ord}_{\eta_2} \colon k(\eta_1)^\times \to \mathbb{Z}$.

The sequence (2.5) is exact when the integral regular local ring R is a DVR by Dennis-Stein [9], when R is essentially of finite type over a field by Quillen [47, Theorem 5.11], when R is essentially of finite type and smooth over a mixed characteristic DVR by Gillet-Levine [14] and Bloch [2], and when $R = A[\![t_1, \ldots, t_r]\!]$ is a formal power series ring over a complete DVR A by work of Reid-Sherman [48]. In these last two cases, by examining the proof of [14, Corollary 6] (see also [48, Corollary 3]), one sees that the main theorems of [14] and [48] allow one to reduce the proof to the case of a DVR.

By the result of Dennis and Stein quoted above for the DVR R_{η_1}, we also have

$$1 \longrightarrow \mathrm{K}_2\big(R_{\eta_1}\big) \longrightarrow \mathrm{K}_2(Q) \xrightarrow{\partial_2} k\big(\eta_1\big)^\times \longrightarrow 1. \tag{2.6}$$

Continuing to assume (2.5) is exact, we then obtain that ϑ_1 induces an isomorphism

$$\frac{\bigoplus_{\eta_1 \in Y^{(1)}} k\big(\eta_1\big)^\times}{\vartheta_2\big(\mathrm{K}_2(Q)\big)} \xrightarrow{\sim} \mathrm{Z}^2(Y). \tag{2.7}$$

Combining this with (2.6), we obtain an isomorphism

$$\bar{\nu}_2 \colon \frac{\prod'_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q)}{\mathrm{K}_2(Q) \cdot \prod_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(R_{\eta_1})} \xrightarrow{\sim} \mathrm{Z}^2(Y) = \bigoplus_{\eta_2 \in Y^{(2)}} \mathbb{Z} \cdot \eta_2 \tag{2.8}$$

where the various terms are as in the following paragraph.

In the numerator, the restricted product $\prod'_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q)$ is the subgroup of the direct product in which all but a finite number of components belong to $\mathrm{K}_2(R_{\eta_1})$. In the denominator, we have the product of the subgroups $\prod_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(R_{\eta_1})$ and $\mathrm{K}_2(Q)$, the second group embedded diagonally in $\prod'_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q)$. Note that by the description of elements in $\mathrm{K}_2(Q)$ as symbols, this diagonal embedding of $\mathrm{K}_2(Q)$ lies in the restricted product. The map giving the isomorphism is obtained by

$$\nu_2 \colon \prod_{\eta_1 \in Y^{(1)}}{}' \mathrm{K}_2(Q) \longrightarrow \mathrm{Z}^2(Y), \tag{2.9}$$

which is defined by summing the maps

$$\nu_{(\eta_0,\eta_1,\eta_2)} = \partial_1\big(\eta_1,\eta_2\big) \circ \partial_2\big(\eta_0,\eta_1\big) \colon \mathrm{K}_2(Q) \longrightarrow \mathbb{Z}$$

as in (2.3). The map ν_2 is well defined on the restricted product since $\nu_{(\eta_0,\eta_1,\eta_2)}$ is trivial on $\mathrm{K}_2(R_{\eta_1})$, and it makes sense independently of assuming that (2.5) is exact.

2.4. Characteristic symbols. Suppose that R is a local integral Noetherian ring and that $\mathcal{C}^\bullet$ is a complex of finitely generated R-modules which is exact on codimension $m-1$. We can then consider the mth localized Chern class, as defined in Subsection 2.1

$$c_m\big(\mathcal{C}^\bullet\big) \in \mathrm{Z}^m(Y) = \bigoplus_{\eta_m \in Y^{(m)}} \mathbb{Z} \cdot \eta_m.$$

Definition 2.4.1. An element $(a_{\underline{\eta}'})_{\underline{\eta}'} \in \bigoplus_{\underline{\eta}' \in P_{m-1}(Y)} \mathrm{K}_m(Q)$ such that

$$\nu_m\Big(\big(a_{\underline{\eta}'}\big)_{\underline{\eta}'}\Big) = c_m\big(\mathcal{C}^\bullet\big)$$

in $\mathrm{Z}^m(Y)$ will be called an *mth characteristic symbol* for $\mathcal{C}^\bullet$.

If m is the smallest integer such that $\mathcal{C}^\bullet$ is exact on codimension less than m, we will simply say that $(a_{\underline{\eta}'})_{\underline{\eta}'}$ as above is a characteristic symbol.

2.5. First and second Chern classes and characteristic symbols. We now assume that the integral Noetherian local ring R is, in addition, regular.

Suppose first that $m = 1$, and let $\mathcal{C}^\bullet$ be a complex of finitely generated R-modules which is exact on codimension 0, which is to say that $\mathcal{C}^\bullet \otimes_R Q$ is exact. We can then consider the first Chern class $c_1\big(\mathcal{C}^\bullet\big) \in \mathrm{Z}^1(Y)$. By (2.4), we have $\mathrm{Z}^1(Y) \simeq Q^\times/R^\times$ given by the divisor map. In this case, a first characteristic symbol (or characteristic element) for $\mathcal{C}^\bullet$ is an element $f \in Q^\times$ such that

$$\operatorname{div}(f) = c_1\big(\mathcal{C}^\bullet\big).$$

This extends the classical notion of a characteristic power series of a torsion module in Iwasawa theory, considering the module as a complex of modules supported in degree zero.

In fact, let M be a finitely generated torsion R-module. Let $\mathscr{P}$ be a set of representatives in R for the equivalence classes of irreducibles under multiplication by units so that $\mathscr{P}$ is in bijection with the set of height 1 primes $Y^{(1)}$. For each $\pi \in \mathscr{P}$, let $n_\pi(M)$ be the length of the localization of M at the prime ideal of R generated by π. Then $c_1(M) = \sum_{\pi\in\mathscr{P}} n_\pi \cdot (\pi)$. In the sections that follow, we will also use the symbol $c_1(M)$ to denote the ideal generated by $\prod_{\pi\in\mathscr{P}} \pi^{n_\pi(M)}$; this should not lead to confusion. Note that, with this notation, M is pseudo-null if and only if $c_1(M) = R$. If $R = \mathbb{Z}_p[\![t]\!] = \mathbb{Z}_p[\![\mathbb{Z}_p]\!]$, then $c_1(M)$ is just the usual characteristic ideal of R. This explains the statements in the introduction connecting the growth rate in (1.1) to first Chern classes (e.g., via the proof of Iwasawa's theorem in [56, Theorem 13.13]).

Suppose now that $m = 2$. Let $\mathcal{C}^\bullet$ be a complex of finitely generated R-modules which is exact on codimension ≤ 1. We can then consider the second localized Chern class $c_2(\mathcal{C}^\bullet) \in \mathrm{Z}^2(Y)$. In this case, we can also consider characteristic symbols in a restricted product of K_2-groups. An element $(a_{\eta_1})_{\eta_1} \in \prod'_{\eta_1\in Y^{(1)}} \mathrm{K}_2(Q)$ is a second characteristic symbol for $\mathcal{C}^\bullet$ when we have

$$\nu_2\Big(\big(a_{\eta_1}\big)_{\eta_1}\Big) = c_2\big(\mathcal{C}^\bullet\big).$$

PROPOSITION 2.5.1. *Suppose that f_1, f_2 are two prime elements in R. Assume that f_1/f_2 is not a unit of R. Then a second characteristic symbol of the R-module $R/(f_1, f_2)$ is given by $a = (a_{\eta_1})_{\eta_1}$ with*

$$a_{\eta_1} = \begin{cases} \big\{f_1, f_2\big\}^{-1}, & \text{if } \eta_1 = \big(f_1\big), \\ 1, & \text{if } \eta_1 \neq \big(f_1\big). \end{cases}$$

Proof. Notice that, under our assumptions, $R/(f_1, f_2)$ is supported on codimension 2. We have to calculate the image of the Steinberg symbol $\{f_1, f_2\}$ under

$$\mathrm{K}_2(Q) \xrightarrow{\partial_2} \mathrm{K}_1\big(k(\eta_1)\big) \xrightarrow{\mathrm{div}_{\eta_2}} \mathbb{Z}$$

for $\eta_1 = (f_1)$ and $\eta_2 \in \overline{\{\eta_1\}}$. (The rest of the contributions to $\nu_2((a_{\eta_1})_{\eta_1})$ are obviously trivial.) We have $\mathrm{val}_{\eta_1}(f_1) = 1$, $\mathrm{val}_{\eta_1}(f_2) = 0$, and so

$$\partial_2\big(\{f_1, f_2\}\big) = f_2^{-1} \bmod (f_1) \in k(\eta_1)^{\times}.$$

By definition,

$$\mathrm{div}_{\eta_2}(f_2) = \mathrm{length}_{R'}\big(R'/f_2R'\big),$$

where R' is the localization $R_{(f_1),\eta_2} = (R/(f_1))_{\eta_2}$. We have a surjective homomorphism of local rings $R_{\eta_2} \to R'$. The R_{η_2}-module structure on $(R/(f_1, f_2))_{\eta_2} = R'/f_2\,R'$ factors through $R_{\eta_2} \to R'$, so

$$\mathrm{length}_{R'}\big(R'/f_2\,R'\big) = \mathrm{length}_{R_{\eta_2}}\Big(\big(R/(f_1, f_2)\big)_{\eta_2}\Big).$$

This, taken together with the definition of $c_2(R/(f_1, f_2))$, completes the proof. □

Remark 2.5.2. The same argument shows that a second characteristic symbol of the R-module $R/(f_1, f_2)$ is also given (symmetrically) by $a' = (a'_{\eta_1})_{\eta_1}$ with $a'_{\eta_1} = \{f_2, f_1\}^{-1} = \{f_1, f_2\}$ if $\eta_1 = (f_2)$, and $a'_{\eta_1} = 1$ otherwise. We can actually see directly that the difference $a - a' \in \prod'_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q)$ lies in the denominator of the right-hand side of (2.8). Indeed, $a - a'$ is equal modulo $\prod_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(R_{\eta_1})$ to the image of $\{f_1, f_2\}^{-1} \in \mathrm{K}_2(Q)$ under the diagonal embedding $\mathrm{K}_2(Q) \to \prod'_{\eta_1 \in Y^{(1)}} \mathrm{K}_2(Q)$.

3. Some conjectures in Iwasawa theory.

3.1. Main conjectures. In this subsection, we explain the relationship between the first Chern class (i.e., the case $m = 1$ in Section 2) and main conjectures of Iwasawa theory. First we strip the main conjecture of all its arithmetic content and present an abstract formulation. To make things concrete, we then give two examples.

For the ring R, we take the Iwasawa algebra $\Lambda = \mathcal{O}[\![\Gamma]\!]$ of the group $\Gamma = \mathbb{Z}_p^r$ for a prime p, where $\mathcal{O}$ is the valuation ring of a finite extension of $\mathbb{Q}_p$. That is, $\Lambda = \varprojlim_U \mathcal{O}[\Gamma/U]$, where U ranges over the open subgroups of Γ. In this case, Λ is non-canonically isomorphic to $\mathcal{O}[\![t_1, \ldots, t_r]\!]$, the power series ring in r variables over $\mathcal{O}$. We need two ingredients to formulate a "main conjecture":

(i) a complex of Λ-modules $\mathcal{C}^\bullet$ quasi-isomorphic to a bounded complex of finitely generated free Λ-modules that is exact in codimension zero, and

(ii) a subset $\{a_\rho\colon \rho\in\Xi\}\subset\overline{\mathbb{Q}}_p$ for a dense set Ξ of continuous characters of Γ.

Note that every continuous $\rho\colon \Gamma\to\overline{\mathbb{Q}}_p^\times$ induces a homomorphism $\Lambda\to\overline{\mathbb{Q}}_p$ that can be extended to a map $Q=Q(\Lambda)\to\overline{\mathbb{Q}}_p\cup\{\infty\}$. We denote this by $\zeta\mapsto\zeta(\rho)$ or by $\zeta\mapsto\int_\Gamma\rho d\zeta$. A main conjecture for the data in (i) and (ii) above is the following statement.

Main Conjecture for $\mathcal{C}^\bullet$ and $\{a_\rho\}$. There is an element $\zeta\in Q^\times$ such that
(a) $\zeta(\rho)=a_\rho$ for all $\rho\in\Xi$,
(b) ζ is a characteristic element for $\mathcal{C}^\bullet$, i.e., $c_1(\mathcal{C}^\bullet)=\mathrm{div}(\zeta)$.
Here, the Chern class c_1 and the divisor div are as defined in Section 2.

3.2. The Iwasawa main conjecture over a totally real field. Let E be a totally real number field. Let χ be an even one-dimensional character of the absolute Galois group of E of finite order, and let E_χ denote the fixed field of its kernel. For a prime p which we take here to be odd, we then set $F=E_\chi(\mu_p)$ and $\Delta=\mathrm{Gal}(F/E)$. We assume that the order of Δ is prime to p. We denote the cyclotomic $\mathbb{Z}_p$-extension of F by K. Then $\mathrm{Gal}(K/E)\cong\Delta\times\Gamma$, where $\Gamma\cong\mathbb{Z}_p$. If Leopoldt's conjecture holds for E and p, then K is the only $\mathbb{Z}_p$-extension of F abelian over E. Let L be the maximal abelian unramified pro-p extension of K. Then $\mathrm{Gal}(K/E)$ acts continuously on $X=\mathrm{Gal}(L/K)$, as there is a short exact sequence

$$1\longrightarrow\mathrm{Gal}(L/K)\longrightarrow\mathrm{Gal}(L/E)\longrightarrow\mathrm{Gal}(K/E)\longrightarrow 1.$$

Thus X becomes a module over the Iwasawa algebra $\mathbb{Z}_p[\![\mathrm{Gal}(K/E)]\!]$. For a character ψ of Δ, define $\mathcal{O}_\psi$ to be the $\mathbb{Z}_p$-algebra generated by the values of ψ. The ψ-eigenspace

$$X^\psi=X\otimes_{\mathbb{Z}_p[\Delta]}\mathcal{O}_\psi$$

is a module over $\Lambda_\psi=\mathcal{O}_\psi[\![\Gamma]\!]$. By a result of Iwasawa, X^ψ is known to be a finitely generated torsion Λ_ψ-module.

On the other side, we let $\Xi=\{\chi_{\mathrm{cyc}}^k\mid k\le 0\}$, where χ_{cyc} is the p-adic cyclotomic character on Γ. Define

$$a_{\chi_{\mathrm{cyc}}^k}=L\big(\chi\omega^{k-1},k\big)\prod_{\mathfrak{p}\in S_p}\big(1-\chi\omega^{k-1}(\mathfrak{p})N\mathfrak{p}^{-k}\big),$$

where ω is the Teichmüller character, S_p is the set of primes of E above p, $N\mathfrak{p}$ is the norm of $\mathfrak{p}$, and $L(\chi\omega^{k-1},s)$ is the complex L-function of $\chi\omega^{k-1}$. Then we have the following Iwasawa main conjecture [57].

THEOREM 3.2.1. (Barsky, Cassou-Noguès, Deligne-Ribet, Mazur-Wiles, Wiles) *There is a unique $\mathcal{L}\in Q^\times$ such that*

(a) $\mathcal{L}(\chi_{\mathrm{cyc}}^k) = a_{\chi_{\mathrm{cyc}}^k}$ *for every even positive integer* k,
(b) $c_1(X^{\chi^{-1}\omega}) = \mathrm{div}(\mathcal{L})$.

3.3. The two-variable main conjecture over an imaginary quadratic field. We assume that p is an odd prime that splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in the imaginary quadratic field E. Fix an abelian extension F of E of order prime to p. Let K be the unique abelian extension of E such that $\mathrm{Gal}(K/F) \cong \mathbb{Z}_p^2$. Let $\Delta = \mathrm{Gal}(F/E)$ and $\Gamma = \mathrm{Gal}(K/F)$. Then we have a canonical isomorphism $\mathrm{Gal}(K/E) \cong \Delta \times \Gamma$. Let $\mathfrak{X}_{\mathfrak{p}}$ (resp., $\mathfrak{X}_{\overline{\mathfrak{p}}}$) be the Galois group over K of the maximal abelian pro-p extension of K unramified outside $\mathfrak{p}$ (resp., $\overline{\mathfrak{p}}$). Then, as above, $\mathfrak{X}_{\mathfrak{p}}$ and $\mathfrak{X}_{\overline{\mathfrak{p}}}$ become modules over $\mathbb{Z}_p[\![\Delta \times \Gamma]\!]$. It is proven in [50, Theorem 5.3(ii)] that $\mathfrak{X}_{\mathfrak{p}}$ and $\mathfrak{X}_{\overline{\mathfrak{p}}}$ are finitely generated torsion $\mathbb{Z}_p[\![\Delta \times \Gamma]\!]$-modules. As in Subsection 3.2, for any character ψ of Δ, we let $\mathcal{O}_\psi$ be the extension of $\mathbb{Z}_p$ obtained by adjoining values of ψ and let

$$\mathfrak{X}_{\mathfrak{p}}^{\psi} = \mathfrak{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\psi, \quad \mathfrak{X}_{\overline{\mathfrak{p}}}^{\psi} = \mathfrak{X}_{\overline{\mathfrak{p}}} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\psi.$$

The other side takes the following analytic data: Let $\Xi_{\psi,\mathfrak{p}}$ (resp., $\Xi_{\psi,\bar{\mathfrak{p}}}$) be the set of all Grössencharacters of E factoring through $\Delta \times \Gamma$ of infinity type (k,j) (resp., (j,k)), with $j \le 0 < k$ and with restriction to Δ equal to ψ. Let $\mathfrak{g}$ be the conductor of ψ. Let $-d_E$ be the discriminant of E. For $\chi \in \Xi_{\psi,\mathfrak{p}}$ (resp., $\chi \in \Xi_{\psi,\bar{\mathfrak{p}}}$) of infinity type (k,j), let

$$a_{\chi,\mathfrak{p}} = \frac{\Omega^{j-k}}{\Omega_p^{j-k}} \left(\frac{\sqrt{d_E}}{2\pi}\right)^j G(\chi) \left(1 - \frac{\chi(\mathfrak{p})}{p}\right) L_{\infty,\mathfrak{g}\overline{\mathfrak{p}}}\big(\chi^{-1}, 0\big)$$

$$\left(\text{resp., } a_{\chi,\overline{\mathfrak{p}}} = \frac{\Omega^{j-k}}{\Omega_p^{j-k}} \left(\frac{\sqrt{d_E}}{2\pi}\right)^j G(\chi) \left(1 - \frac{\chi(\overline{\mathfrak{p}})}{p}\right) L_{\infty,\mathfrak{g}\mathfrak{p}}\big(\chi^{-1}, 0\big)\right).$$

Here, Ω and Ω_p are complex and p-adic periods of E, respectively, and $G(\chi)$ is a Gauss sum. Moreover, $L_{\infty,\mathfrak{f}}$ refers to the L-function with the Euler factor at ∞ but without the Euler factors at the primes dividing $\mathfrak{f}$. (For more explanation, see [8, Equation (36), p. 80].) Let W be the ring of Witt vectors of $\overline{\mathbb{F}}_p$. Using work of Yager, deShalit proves in [8, Theorem 4.14] that there are $\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\overline{\mathfrak{p}},\psi} \in W[\![\Gamma]\!]$ such that

$$\mathcal{L}_{\mathfrak{p},\psi}(\chi) = a_{\chi,\mathfrak{p}}, \text{ for every } \chi \in \Xi_{\psi,\mathfrak{p}}, \quad \text{and} \quad \mathcal{L}_{\overline{\mathfrak{p}},\psi}(\chi) = a_{\chi,\overline{\mathfrak{p}}}, \text{ for every } \chi \in \Xi_{\psi,\overline{\mathfrak{p}}}.$$

We have the following result of Rubin [50] on the two-variable main conjecture over E.

THEOREM 3.3.1. (Rubin) *With the notation as above, we have*

$$\operatorname{div}\left(\mathcal{L}_{\mathfrak{p},\psi}\right) = c_1\left(W[\![\Gamma]\!]\hat{\otimes}_{\mathcal{O}_\psi[\![\Gamma]\!]}\mathfrak{X}_{\mathfrak{p}}^{\psi}\right).$$

The above is also true with $\mathfrak{p}$ *replaced by* $\overline{\mathfrak{p}}$.

Let σ denote the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q})$. We obtain an action of σ on $\Delta\times\Gamma$ via conjugation by any lift of σ to $\mathrm{Gal}(K/\mathbb{Q})$. We extend this action $\mathbb{Z}_p$-linearly to a map

$$\sigma\colon \mathbb{Z}_p[\![\Delta\times\Gamma]\!] \longrightarrow \mathbb{Z}_p[\![\Delta\times\Gamma]\!].$$

This homomorphism σ maps $\mathcal{O}_\psi[\![\Gamma]\!]$ isomorphically to $\mathcal{O}_{\psi\circ\sigma}[\![\Gamma]\!]$.

LEMMA 3.3.2. *The two Katz p-adic L-functions are related by*

(a) $\mathcal{L}_{\overline{\mathfrak{p}},\psi} = \sigma(\mathcal{L}_{\mathfrak{p},\psi\circ\sigma})$.

(b) $\mathcal{L}_{\overline{\mathfrak{p}},\psi}(\chi) = (p\text{-adic unit})\cdot\mathcal{L}_{\mathfrak{p},\psi^{-1}\omega}(\chi^{-1}\chi_{\mathrm{cyc}})$, *where* χ_{cyc} *is the p-adic cyclotomic character on* $\Delta\times\Gamma$.

Proof. Assertion (a) is proven simply by interpolating both sides at all elements in $\Xi_{\psi,\overline{\mathfrak{p}}}$. We first note that both $\sigma(\mathcal{L}_{\mathfrak{p},\psi\circ\sigma})$ and $\mathcal{L}_{\overline{\mathfrak{p}},\psi}$ lie in $W\hat{\otimes}_{\mathcal{O}_\psi}\mathcal{O}_\psi[\![\Gamma]\!]$. Then

$$\begin{aligned}
\sigma\left(\mathcal{L}_{\mathfrak{p},\psi\circ\sigma}\right)(\chi) &= \mathcal{L}_{\mathfrak{p},\psi\circ\sigma}(\chi\circ\sigma)\\
&= \frac{\Omega^{j-k}}{\Omega_p^{j-k}}\left(\frac{\sqrt{d_E}}{2\pi}\right)^j G(\chi\circ\sigma)\left(1-\frac{(\chi\circ\sigma)(\mathfrak{p})}{p}\right)L_{\infty,\mathfrak{gp}}\left((\chi\circ\sigma)^{-1},0\right)\\
&= \frac{\Omega^{j-k}}{\Omega_p^{j-k}}\left(\frac{\sqrt{d_E}}{2\pi}\right)^j G(\chi)\left(1-\frac{\chi(\overline{\mathfrak{p}})}{p}\right)L_{\infty,\mathfrak{gp}}\left(\chi^{-1},0\right)\\
&= \mathcal{L}_{\overline{\mathfrak{p}},\psi}(\chi),
\end{aligned}$$

where we use the fact that the infinity type of $\chi\circ\sigma$ is (k,j), and in the third equality we use the obvious equalities $G(\chi)=G(\chi\circ\sigma)$ and $L_{\infty,\mathfrak{g}\overline{\mathfrak{p}}}((\chi\circ\sigma)^{-1},0) = L_{\infty,\mathfrak{gp}}(\chi^{-1},0)$.

To prove (b), we use the functional equation for p-adic L-functions, which says that

$$\mathcal{L}_{\mathfrak{p},\psi}(\chi) = (p\text{-adic unit})\cdot\mathcal{L}_{\mathfrak{p},\overline{\psi}^{-1}\omega}\left(\overline{\chi}^{-1}\chi_{\mathrm{cyc}}\right),$$

where we write $\overline{\psi}$ and $\overline{\chi}$ instead of $\psi\circ\sigma$ and $\chi\circ\sigma$ for convenience (see [8, Equation (9), p. 93]). Using (a) and the functional equation, we obtain

$$\begin{aligned}
\mathcal{L}_{\overline{\mathfrak{p}},\psi}(\chi) &= \sigma\left(\mathcal{L}_{\mathfrak{p},\overline{\psi}}\right)(\chi)\\
&= \mathcal{L}_{\mathfrak{p},\overline{\psi}}(\overline{\chi})\\
&= (p\text{-adic unit})\cdot\mathcal{L}_{\mathfrak{p},\psi^{-1}\omega}\left(\chi^{-1}\chi_{\mathrm{cyc}}\right).
\end{aligned}$$

□

3.4. Greenberg's Conjecture. Let E be an arbitrary number field, and let $\widetilde{E}$ be the compositum of all $\mathbb{Z}_p$-extensions of E. Let $\Gamma = \mathrm{Gal}(\widetilde{E}/E)$ and $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$. Then $\Gamma \cong \mathbb{Z}_p^r$ for some $r \geq r_2(E) + 1$, where $r_2(E)$ is the number of complex places of E. Leopoldt's Conjecture for E and p is the assertion that $r = r_2(E) + 1$. This is known to be true if E is abelian over $\mathbb{Q}$ or over an imaginary quadratic field [4]. The ring Λ is isomorphic (non-canonically) to the formal power series ring over $\mathbb{Z}_p$ in r variables.

Let L be the maximal abelian, unramified pro-p-extension of $\widetilde{E}$, and let $X = \mathrm{Gal}(L/\widetilde{E})$, which is a Λ-module that we will call the unramified Iwasawa module over $\widetilde{E}$. The following conjecture was first stated in print in [18, Conjecture 3.5].

CONJECTURE 3.4.1. (Greenberg) *With the above notation, the Λ-module X is pseudo-null. That is, its localizations at all codimension 1 points of* $\mathrm{Spec}(\Lambda)$ *are trivial.*

Note that if E is totally real, and if Leopoldt's conjecture for E and p is valid, then $\widetilde{E}$ is the cyclotomic $\mathbb{Z}_p$-extension of E, and the conjecture states that X is finite.

In the case that E is totally complex, we have the following reasonable extension of the above conjecture. Let K be any finite extension of $\widetilde{E}$ which is abelian over E. Then $\mathrm{Gal}(K/E) \cong \Delta \times \Gamma$, where Δ is a finite group and Γ (as defined above) is identified with $\mathrm{Gal}(K/F)$ for some finite extension F of E. Let X be the unramified Iwasawa module over K. Then Γ acts on X and so we can again regard X as a Λ-module. The extended conjecture asserts that X is pseudo-null as a Λ-module.

Some evidence for Conjecture 3.4.1 has been given in various special cases. For instance, in [38], Minardi verifies Conjecture 3.4.1 when E is an imaginary quadratic field and p is a prime not dividing the class number of E, and also for many imaginary quadratic fields E when $p = 3$ and does divide the class number. In [21], Hubbard verifies the conjecture when $p = 3$ for a number of biquadratic fields E. In [53], Sharifi gave a criterion for Conjecture 3.4.1 to hold for $\mathbb{Q}(\mu_p)$. By a result of Fukaya-Kato [11, Theorem 7.2.8] on a conjecture of McCallum-Sharifi and related computations in [36], the condition holds for $E = \mathbb{Q}(\mu_p)$ for all $p < 25000$. The results of [53] suggest that X should have an annihilator of very high codimension for $E = \mathbb{Q}(\mu_p)$.

One can construct examples of $\mathbb{Z}_p^r$-extensions K of a suitably chosen number field F such that the unramified Iwasawa module X over K has the ideal (p) in its support as a $\mathbb{Z}_p[\![\mathrm{Gal}(K/F)]\!]$-module. Such examples can be constructed by imitating Iwasawa's construction of $\mathbb{Z}_p$-extensions with positive μ-invariant. This was pointed out to us by T. Kataoka. Such a construction can be done for any positive r. However, if one adds the assumption that K contain the cyclotomic $\mathbb{Z}_p$-extension of F, and $r > 1$, then we actually know of no examples where X is demonstrably not pseudo-null.

Assume that K is a $\mathbb{Z}_p^r$-extension of F with $r \geq 1$ such that K contains all p-power roots of unity and, therefore, the cyclotomic $\mathbb{Z}_p$-extension of F. The class group $\mathrm{Cl}(K)$ is defined as the direct limit under the obvious maps of the ideal class groups of the finite extensions of F contained in K. The assertion that X is pseudo-null as a Λ-module should conjecturally be equivalent to the assertion that the p-primary subgroup $\mathrm{Cl}(K)_p$ of $\mathrm{Cl}(K)$ is actually trivial. We sketch an argument just in one direction. One can show that if X is pseudo-null, then so is $\mathrm{Hom}(\mathrm{Cl}(K)_p, \mathbb{Q}_p/\mathbb{Z}_p)$, the Pontryagin dual of $\mathrm{Cl}(K)_p$. One then employs a standard Kummer theory argument to show that $\mathrm{Hom}(\mathrm{Cl}(K)_p, \mu_{p^\infty}) = \mathrm{Hom}(\mathrm{Cl}(K)_p, \mathbb{Q}_p/\mathbb{Z}_p)(1)$ is isomorphic to a Λ-submodule of $\mathfrak{X} = \mathrm{Gal}(M/K)$, where M denotes the maximal abelian pro-p extension of K unramified outside the primes above p, which we call the unramified outside p Iwasawa module over K. One can then use the result that $\mathfrak{X}$ has no nontrivial pseudo-null Λ-submodules. That result is a consequence of the fact that $\mathfrak{X}$ has rank $r_2(F)$ as a Λ-module known as the *weak Leopoldt conjecture* for K/F, which is satisfied because K contains the cyclotomic $\mathbb{Z}_p$-extension of F. (See [41, Théoreme 3.1] or [16, Proposition 5].) For a partial result in the converse direction, see [42, Théoreme 5.1].

If one assumes in addition that the decomposition subgroups of $\Gamma = \mathrm{Gal}(K/F)$ for primes above p are of $\mathbb{Z}_p$-rank at least 2, then the assertion that X is pseudo-null is equivalent to the assertion that $\mathfrak{X}$ is torsion-free as a Λ-module. This equivalence follows from [29, Proposition 3.6] (see also Remark 4.2.5). The result in [29] is stated in terms of the pseudo-nullity of a certain quotient X' of X, namely $X' = \mathrm{Gal}(L'/K)$, where L' is the maximal unramified abelian pro-p extension of K in which all the primes of K lying over p split completely. However, the kernel of the map $X \to X'$ is pseudo-null under our assumption on the decomposition subgroups by Lemma 4.2.3 below. It follows that X is pseudo-null if and only if X' is pseudo-null.

4. Unramified Iwasawa modules.

4.1. The general setup. Let p be a prime, E be a number field, F a finite Galois extension of E, and $\Delta = \mathrm{Gal}(F/E)$. Let K be a Galois extension of E that is a $\mathbb{Z}_p^r$-extension of F for some $r \geq 1$, and set $\Gamma = \mathrm{Gal}(K/F)$. Set $\mathcal{G} = \mathrm{Gal}(K/E)$, $\Omega = \mathbb{Z}_p[\![\mathcal{G}]\!]$, and $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$. Note that K/F is unramified outside p as a compositum of $\mathbb{Z}_p$-extensions.

Let S be a set of primes of E including those over p and ∞, and let S_f be the set of finite primes in S. For any algebraic extension F' of F, let $G_{F',S}$ denote the Galois group of the maximal extension F'_S of F' that is unramified outside the primes over S. Let $\mathcal{Q} = \mathrm{Gal}(F_S/E)$. For a compact $\mathbb{Z}_p[\![\mathcal{Q}]\!]$-module T, we consider the Iwasawa cohomology group

$$\mathrm{H}^i_{\mathrm{Iw}}(K,T) = \varprojlim_{F' \subset K} \mathrm{H}^i\big(G_{F',S}, T\big)$$

that is the inverse limit of continuous Galois cohomology groups under corestriction maps, with F' running over the finite extensions of F in K. It has the natural structure of an Ω-module.

We will use the following notation. For a locally compact Λ-module M, let us set

$$\mathrm{E}^i_\Lambda(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)$$

for short. This again has a Λ-module structure with $\gamma \in \Gamma$ acting on $f \in \mathrm{E}^0_\Lambda(M)$ by $(\gamma \cdot f)(m) = f(\gamma^{-1}m) = \gamma^{-1}f(m)$. We let $M^\vee$ denote the Pontryagin dual, to which we give a module structure by letting γ act by precomposition by γ^{-1}. If M is a (left) Ω-module, then $M^\vee$ is likewise a (left) Ω-module. Moreover, $\mathrm{E}^i_\Lambda(M) \cong \mathrm{Ext}^i_\Omega(M, \Omega)$ as Λ-modules since Ω is Λ-projective (cf. [40, Proposition 5.4.17]), through which $\mathrm{E}^i_\Lambda(M)$ acquires an Ω-module structure. We set $M^* = \mathrm{E}^0_\Lambda(M) = \mathrm{Hom}_\Lambda(M, \Lambda)$.

The first of the following two spectral sequences is due to Jannsen [25, Theorem 1], and the second to Nekovář [39, Theorem 8.5.6] (though it is assumed there that p is odd or K has no real places). One can find very general versions that imply these in [10, 1.6.12] and [32, Theorem 4.5.1].

THEOREM 4.1.1. (Jannsen, Nekovář) *Let T be a compact $\mathbb{Z}_p[\![\mathcal{Q}]\!]$-module that is finitely generated and free over $\mathbb{Z}_p$. Set $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. There are convergent spectral sequences of Ω-modules*

$$\mathrm{F}_2^{i,j}(T) = \mathrm{E}^i_\Lambda\big(\mathrm{H}^j(G_{K,S}, A)^\vee\big) \Longrightarrow \mathrm{F}^{i+j}(T) = \mathrm{H}^{i+j}_{\mathrm{Iw}}(K, T)$$
$$\mathrm{H}_2^{i,j}(T) = \mathrm{E}^i_\Lambda\big(\mathrm{H}^{2-j}_{\mathrm{Iw}}(K, T)\big) \Longrightarrow \mathrm{H}^{i+j}(T) = \mathrm{H}^{2-i-j}\big(G_{K,S}, A\big)^\vee.$$

We will be interested in the above spectral sequences in the case that $T = \mathbb{Z}_p$. We have a canonical isomorphism

$$\mathrm{H}^1\big(G_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p\big)^\vee \cong \mathfrak{X},$$

where $\mathfrak{X}$ denotes the S-ramified Iwasawa module over K (i.e., the Galois group of the maximal abelian pro-p, unramified outside S extension of K). We study the relationship between $\mathfrak{X}$ and $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)$.

The following nearly immediate consequence of Jannsen's spectral sequence is a mild extension of earlier unpublished results of McCallum [35, Theorems A and B].

THEOREM 4.1.2. (McCallum, Jannsen) *There is a canonical exact sequence of Ω-modules*

$$0 \longrightarrow \mathbb{Z}_p^{\delta_{r,1}} \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big) \longrightarrow \mathfrak{X}^* \longrightarrow \mathbb{Z}_p^{\delta_{r,2}} \longrightarrow \mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big),$$

where $\delta_{j,i} = 1$ if $i = j$ and $\delta_{j,i} = 0$ otherwise. If the weak Leopoldt conjecture holds for K, which is to say that $\mathrm{H}^2(G_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, then this exact sequence extends to

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}_p^{\delta_{r,1}} \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big) \longrightarrow \mathfrak{X}^* \longrightarrow \mathbb{Z}_p^{\delta_{r,2}} \longrightarrow \mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big) \\ \longrightarrow \mathrm{E}^1_\Lambda(\mathfrak{X}) \longrightarrow \mathbb{Z}_p^{\delta_{r,3}} \longrightarrow 0. \end{aligned} \tag{4.1}$$

If p is odd or K has no real places, then there are canonical isomorphisms $\mathrm{E}^i_\Lambda(\mathfrak{X}) \xrightarrow{\sim} \mathbb{Z}_p^{\delta_{r,i+2}}$ for $i \geq 2$.

Proof. The first sequence is just the five-term exact sequence of base terms in Jannsen's spectral sequence for $T = \mathbb{Z}_p$. For this, we remark that

$$\mathrm{F}_2^{i,0}\big(\mathbb{Z}_p\big) = \mathrm{E}^i_\Lambda\big(\mathbb{Z}_p\big) \cong \mathbb{Z}_p^{\delta_{r,i}}$$

by [24, Lemma 5] or Corollary A.13 below. Under weak Leopoldt, $\mathrm{F}_2^{0,2}(\mathbb{Z}_p)$ is zero, so the exact sequence continues as written, the next term being $\mathrm{H}^3_{\mathrm{Iw}}(K, \mathbb{Z}_p) = 0$. If p is odd or K is totally imaginary, then $G_{F',S}$ has p-cohomological dimension 2 for some finite extension F' of F in K, so $\mathrm{H}^j_{\mathrm{Iw}}(K, \mathbb{Z}_p)$ vanishes for $j \geq 3$, in which case the spectral sequence also yields the remaining isomorphisms. □

Remark 4.1.3. The weak Leopoldt conjecture for K is well known to hold in the case that $K(\mu_p)$ contains all p-power roots of unity (see [40, Theorem 10.3.25]).

Remark 4.1.4. For p odd, McCallum proved everything but the exactness at $\mathbb{Z}_p^{\delta_{r,2}}$ in Theorem 4.1.2, supposing both hypotheses listed therein.

COROLLARY 4.1.5. *There is a canonical isomorphism $\mathfrak{X}^{**} \to \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)^*$ of Ω-modules.*

Proof. This follows from Theorem 4.1.2, which provides an isomorphism if $r \geq 3$, or if $r = 2$ and the map $\mathfrak{X}^* \to \mathbb{Z}_p$ is zero. If $r = 1$, then we obtain an exact sequence

$$0 \longrightarrow \mathfrak{X}^{**} \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)^* \longrightarrow \mathrm{E}^0_\Lambda\big(\mathbb{Z}_p\big),$$

and the last term is zero. If $r = 2$ and the map $\mathfrak{X}^* \to \mathbb{Z}_p$ is nonzero, then we obtain an exact sequence

$$0 \longrightarrow \mathrm{E}^0_\Lambda\big(\mathbb{Z}_p\big) \longrightarrow \mathfrak{X}^{**} \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)^* \longrightarrow \mathrm{E}^1_\Lambda\big(\mathbb{Z}_p\big),$$

and $\mathrm{E}^0_\Lambda(\mathbb{Z}_p) = \mathrm{E}^1_\Lambda(\mathbb{Z}_p) = 0$ since $r = 2$. □

Using the second spectral sequence in Theorem 4.1.1, we may use this to obtain the following.

COROLLARY 4.1.6. *Suppose that p is odd or K is totally imaginary. There is an exact sequence*

$$0 \longrightarrow \mathrm{E}^1_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{X}^{**} \longrightarrow \mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \longrightarrow \mathbb{Z}_p \tag{4.2}$$

of Ω-modules. In particular, $\mathrm{E}^1_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p))$ is isomorphic to the Λ-torsion submodule of $\mathfrak{X}$.

Proof. By hypothesis, $G_{F',S}$ has p-cohomological dimension 2 for some finite extension F' of F in K. Therefore, Nekovář's spectral sequence is a first quadrant spectral sequence for any T. For $T = \mathbb{Z}_p$, it provides an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{E}^1_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \longrightarrow \mathrm{H}^1\big(G_{K,S},\mathbb{Q}_p/\mathbb{Z}_p\big)^\vee \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)^* \\ &\longrightarrow \mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \longrightarrow \mathrm{H}^0\big(G_{K,S},\mathbb{Q}_p/\mathbb{Z}_p\big)^\vee \longrightarrow \mathrm{E}^1_\Lambda\big(\mathrm{H}^1_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \\ &\longrightarrow \mathrm{E}^3_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big) \longrightarrow 0 \end{aligned} \tag{4.3}$$

of Ω-modules. In particular, applying Corollary 4.1.5 to get the third term, we have the exact sequence of the statement. □

Remark 4.1.7. In the case that $r = 1$, Corollary 4.1.6 is in a sense implicit in the work of Iwasawa [23] (see Theorem 12 and its proof of Lemma 12). In this case, second Ext-groups are finite, so the map to $\mathbb{Z}_p$ in the corollary is zero.

Remark 4.1.8. It is natural to ask how (4.2) is related to the more abstract exact sequence of [24, (1.8.1)] (see also [40, Prop. 5.4.9]). Under the assumption that K contains all p-power roots of unity, we may answer this as follows. Jannsen defines a functor D on the homotopy category of Λ-modules. Up to homotopy, $\mathrm{D}\mathfrak{X} = \mathrm{coker}(P_0^* \to P_1^*)$ for a choice of projective resolution

$$0 \longrightarrow P_{r+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathfrak{X} \longrightarrow 0 \tag{4.4}$$

of $\mathfrak{X}$. Taking Ext-groups of the four-term exact sequence defining $\mathrm{D}\mathfrak{X}$ leads to the exact sequence

$$0 \longrightarrow \mathrm{E}^1_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{X}^{**} \longrightarrow \mathrm{E}^2_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow 0. \tag{4.5}$$

By definition, one has an injective Λ-module homomorphism $\alpha\colon \mathrm{E}^1_\Lambda(\mathfrak{X}) \to \mathrm{D}\mathfrak{X}$ with cokernel equal to the kernel of $P_2^* \to P_3^*$ (taking $P_3 = 0$ if $r = 1$). By Theorem 4.1.2, the Ext-groups computed by the Λ-dual of (4.4) satisfy $\mathrm{E}^i_\Lambda(\mathfrak{X}) \cong \mathbb{Z}_p^{\delta_{i+2,r}}$ for $i \geq 2$. We also have a map $\beta\colon \mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p) \to \mathrm{E}^1_\Lambda(\mathfrak{X})$ that by (4.1) is an isomorphism if $r = 1$ or $r \geq 4$, an injection with cokernel $\mathbb{Z}_p$ if $r = 3$, and a surjection with kernel a quotient C of $\mathbb{Z}_p$ if $r = 2$. Setting $C = 0$ if $r = 1$ and $C = \mathbb{Z}_p$ if $r \geq 3$, it follows

from these facts that the map $\rho = \alpha \circ \beta\colon \mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p) \to \mathrm{D}\mathfrak{X}$ fits in a complex

$$\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big) \xrightarrow{\rho} \mathrm{D}\mathfrak{X} \longrightarrow P_2^* \longrightarrow \cdots \longrightarrow P_{r+1}^* \tag{4.6}$$

which has cohomology C concentrated in degree $r-2$ counting from 0.

If $r \geq 3$, or if $r = 2$ and $C = \mathbb{Z}_p$, the complex (4.6) just described allows us to compute that we have isomorphisms

$$\mathrm{E}^i_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow \mathrm{E}^i_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big) \tag{4.7}$$

for $i \notin \{2, 3\}$ and an exact sequence

$$0 \longrightarrow \mathrm{E}^2_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow \mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big) \longrightarrow \mathbb{Z}_p \longrightarrow \mathrm{E}^3_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow \mathrm{E}^3_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big).$$

If $r = 2$ and C is finite, the map of (4.7) is an isomorphism for $i = 2$ and an injection with cokernel C for $i = 3$. In all cases, we obtain a map from the exact sequence (4.5) to the exact sequence (4.2) of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathrm{D}\mathfrak{X}) & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X}^{**} & \longrightarrow & \mathrm{E}^2_\Lambda(\mathrm{D}\mathfrak{X}) \longrightarrow 0 \\
 & & \wr\downarrow & & \| & & \| & & \downarrow \\
0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p)) & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X}^{**} & \longrightarrow & \mathrm{E}^2_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p)) \longrightarrow \mathbb{Z}_p,
\end{array}$$

where the leftmost and rightmost vertical maps are induced by $\pm\rho$. (We have only checked commutativity up to the ambiguous signs, as the remark is not used.)

Remark 4.1.9. In Corollary 4.1.6, the map $\mathfrak{X} \to \mathfrak{X}^{**}$ can be taken to be the standard map from $\mathfrak{X}$ to its double dual. That is, both the map $\mathfrak{X} \to \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)^*$ in (4.3) and the map $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p) \to \mathfrak{X}^*$ of Theorem 4.1.2 arise in the standard manner from a Λ-bilinear pairing

$$\mathfrak{X} \times \mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big) \longrightarrow \Lambda$$

defined as follows. Write $\Lambda = \varprojlim_{F'} \Lambda_{F'}$, where $\Lambda_{F'} = \mathbb{Z}_p[\mathrm{Gal}(F'/F)]$ and F' runs over the finite extensions of F in K. Take $\sigma \in \mathfrak{X}$ and $f \in \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)$. Write f as an inverse limit of homomorphisms $f_{F'} \in \mathrm{H}^1(G_{F',S}, \mathbb{Z}_p)$. Then our pairing is given by

$$(\sigma, f) \longmapsto \varprojlim_{F'} \sum_{\tau \in \mathrm{Gal}(F'/F)} f_{F'}\big(\tilde{\tau}^{-1}\sigma\tilde{\tau}\big)[\tau]_{F'},$$

where $\tilde{\tau}$ denotes a lift of τ to $G_{F,S}$, and $[\tau]_{F'}$ denotes the group element of τ in $\Lambda_{F'}$. Thus, the composition of $\mathfrak{X} \to \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)^*$ with the map $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{Z}_p)^* \to \mathfrak{X}^{**}$ of Corollary 4.1.5 is the usual map $\mathfrak{X} \to \mathfrak{X}^{**}$.

Definition 4.1.10. For $\mathfrak{p}$ in the set S_f of finite primes in S, let $\mathcal{G}_\mathfrak{p}$ denote the decomposition group in $\mathcal{G}$ at a place over the prime $\mathfrak{p}$ in K, and set $\mathcal{K}_\mathfrak{p} = \mathbb{Z}_p[\![\mathcal{G}/\mathcal{G}_\mathfrak{p}]\!]$, which has the natural structure of a left Ω-module. We then set

$$\mathcal{K} = \bigoplus_{\mathfrak{p}\in S_f} \mathcal{K}_\mathfrak{p} \quad \text{and} \quad \mathcal{K}_0 = \ker\big(\mathcal{K} \to \mathbb{Z}_p\big),$$

where the map is the sum of augmentation maps.

Remark 4.1.11. If K contains all p-power roots of unity, then the group $\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p)$ is the twist by $\mathbb{Z}_p(-1)$ of $\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p(1))$. As explained in the proof of [52, Lemma 2.1], Poitou-Tate duality provides a canonical exact sequence

$$0 \longrightarrow X' \longrightarrow \mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p(1)\big) \longrightarrow \mathcal{K}_0 \longrightarrow 0, \tag{4.8}$$

where X' is the completely split Iwasawa module over K (i.e., the Galois group of the maximal abelian pro-p extension K that is completely split at all places above S_f).

We next wish to consider local versions of the above results. Let T and $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ be as in Theorem 4.1.1. For $\mathfrak{p} \in S_f$, let

$$\mathrm{H}^i_{\mathrm{Iw},\mathfrak{p}}(K,T) = \varprojlim_{\substack{F'/E \text{ finite} \\ F'\subset K}} \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{H}^i\big(G_{F'_\mathfrak{P}}, T\big),$$

where $G_{F'_\mathfrak{P}}$ denotes the absolute Galois group of the completion $F'_\mathfrak{P}$. If M is a discrete $\mathbb{Z}_p[\![\mathrm{Gal}(F_S/E)]\!]$-module, let $\mathrm{H}^i(G_{K,\mathfrak{p}}, M)$ denote the direct sum of the groups $\mathrm{H}^i(G_{K_\mathfrak{P}}, M)$ over the primes $\mathfrak{P}$ in K over $\mathfrak{p}$. We have the local spectral sequence

$$\mathrm{P}^{i,j}_{2,\mathfrak{p}}(T) = \mathrm{E}^i_\Lambda\Big(\mathrm{H}^j\big(G_{K,\mathfrak{p}}, A\big)^\vee\Big) \Longrightarrow \mathrm{P}^{i+j}_\mathfrak{p}(T) = \mathrm{H}^{i+j}_{\mathrm{Iw},\mathfrak{p}}(K,T)$$

(cf. [32, Theorem 4.2.2]). Note that $\mathrm{H}^j(G_{K,\mathfrak{p}}, A)^\vee \cong \mathrm{H}^{2-j}_{\mathrm{Iw},\mathfrak{p}}(K, T^\dagger)$ by Tate duality, where $T^\dagger = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$.

Remark 4.1.12. Tate and Poitou-Tate duality provide maps between the sum of these local spectral sequences over all $\mathfrak{p} \in S_f$ and the global spectral sequences, supposing for simplicity that p is odd or K is purely imaginary (in general, for real places, one uses Tate cohomology). On E_2-terms, these form a complex

$$\cdots \longrightarrow \mathrm{F}^{i,j}_2(T) \longrightarrow \bigoplus_{\mathfrak{p}\in S_f} \mathrm{P}^{i,j}_{2,\mathfrak{p}}(T) \longrightarrow \mathrm{H}^{i,j}_2\big(T^\dagger\big) \longrightarrow \mathrm{F}^{i,j+1}_2(T) \longrightarrow \cdots.$$

These spectral sequences can be seen in the derived category of complexes of finitely generated Ω-modules, where they form an exact triangle (see [32, Theorem 4.5.1]). To see this, one uses the regularity of $\mathbb{Z}_p$ in order to replace the dualizing complex with $\mathbb{Z}_p$. The cohomology groups in question can then be identified with those we have written by the isomorphisms of [31, Lemmas 5.3.1 and 5.3.2, Theorem 5.4.1].

Let $\Gamma_{\mathfrak{p}} = \mathcal{G}_{\mathfrak{p}} \cap \Gamma$ be the decomposition group in Γ at a prime over $\mathfrak{p}$ in K, and let $r_{\mathfrak{p}} = \operatorname{rank}_{\mathbb{Z}_p} \Gamma_{\mathfrak{p}}$. For an Ω-module M, we let M^{ι} denote the Ω-module which as a compact $\mathbb{Z}_p$-module is M and on which $g \in \mathcal{G}$ now acts by g^{-1}.

LEMMA 4.1.13. *For $j \geq 0$, we have isomorphisms $\mathrm{E}^j_{\Lambda}(\mathcal{K}_{\mathfrak{p}}) \cong (\mathcal{K}^{\iota}_{\mathfrak{p}})^{\delta_{r_{\mathfrak{p}},j}}$ of Ω-modules.*

Proof. This is immediate from Corollary A.13. □

Let $\mathfrak{D}_{\mathfrak{p}}$ denote the Galois group of the maximal abelian, pro-p quotient of the absolute Galois group of the completion $K_{\mathfrak{p}}$ of K at a prime over $\mathfrak{p}$, and consider the completed tensor product

$$D_{\mathfrak{p}} = \Omega \hat{\otimes}_{\mathbb{Z}_p[\![\mathcal{G}_{\mathfrak{p}}]\!]} \mathfrak{D}_{\mathfrak{p}},$$

which has the structure of an Ω-module by left multiplication.

THEOREM 4.1.14. *Suppose that K contains all p-power roots of unity. For each $\mathfrak{p} \in S_f$, we have a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{E}^1_{\Lambda}(\mathcal{K}_{\mathfrak{p}})(1) & \longrightarrow & D_{\mathfrak{p}} & \longrightarrow & D^{**}_{\mathfrak{p}} & \longrightarrow & \mathrm{E}^2_{\Lambda}(\mathcal{K}_{\mathfrak{p}})(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{E}^1_{\Lambda}(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)) & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X}^{**} & \longrightarrow & \mathrm{E}^2_{\Lambda}(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)) & \longrightarrow & \mathbb{Z}_p
\end{array}$$

of Ω-modules in which the vertical maps are the canonical ones.

Proof. We have

$$\mathrm{H}^2_{\mathrm{Iw},\mathfrak{p}}\big(K,\mathbb{Z}_p\big) \cong \mathcal{K}_{\mathfrak{p}}(-1) \quad \text{and} \quad \mathrm{H}^1\big(G_{K,\mathfrak{p}},\mathbb{Q}_p/\mathbb{Z}_p\big)^{\vee} \cong D_{\mathfrak{p}},$$

the first using our assumption on K. We also have $\mathrm{H}^2(G_{K,\mathfrak{p}},\mathbb{Q}_p/\mathbb{Z}_p) = 0$.

The analogue of Theorem 4.1.2 is the exact sequence

$$0 \longrightarrow \mathrm{E}^1_{\Lambda}\big(\mathcal{K}_{\mathfrak{p}}\big) \longrightarrow \mathrm{H}^1_{\mathrm{Iw},\mathfrak{p}}\big(K,\mathbb{Z}_p\big) \longrightarrow D^*_{\mathfrak{p}} \longrightarrow \mathrm{E}^2_{\Lambda}\big(\mathcal{K}_{\mathfrak{p}}\big) \longrightarrow \mathrm{H}^2_{\mathrm{Iw},\mathfrak{p}}\big(K,\mathbb{Z}_p\big). \tag{4.9}$$

We remark that the map

$$\mathrm{E}^2_\Lambda\big(\mathcal{K}_\mathfrak{p}\big) = \big(\mathcal{K}^\iota_\mathfrak{p}\big)^{\delta_{r_\mathfrak{p},2}} \longrightarrow \mathrm{H}^2_{\mathrm{Iw},\mathfrak{p}}\big(K,\mathbb{Z}_p\big) \cong \mathcal{K}_\mathfrak{p}(-1)$$

is zero since $\Gamma_\mathfrak{p}$ acts trivially on $\mathcal{K}^\iota_\mathfrak{p}$ but not on any nonzero element of $\mathcal{K}_\mathfrak{p}(-1)$. Applying Lemma 4.1.13 to (4.9), dualizing, and using the fact that $r_\mathfrak{p} \geq 1$ by assumption on K, we obtain an isomorphism $\mathrm{H}^1_{\mathrm{Iw},\mathfrak{p}}(K,\mathbb{Z}_p)^* \xrightarrow{\sim} D^{**}_\mathfrak{p}$ compatible with Corollary 4.1.5. The analogue of Corollary 4.1.6 is then the exact sequence

$$0 \longrightarrow \mathrm{E}^1_\Lambda\big(\mathcal{K}_\mathfrak{p}\big)(1) \longrightarrow D_\mathfrak{p} \longrightarrow D^{**}_\mathfrak{p} \longrightarrow \mathrm{E}^2_\Lambda\big(\mathcal{K}_\mathfrak{p}\big)(1) \longrightarrow \mathcal{K}_\mathfrak{p}. \tag{4.10}$$

As above, the map $\mathrm{E}^2_\Lambda(\mathcal{K}_\mathfrak{p})(1) \to \mathcal{K}_\mathfrak{p}$ is zero.

The map of exact sequences follows from Remark 4.1.12. □

One might ask whether or not the map $\mathfrak{X}^* \to \mathbb{Z}_p$ in Theorem 4.1.2 is zero in the case $r = 2$.

PROPOSITION 4.1.15. *Suppose that K contains all p-power roots of unity. If Leopoldt's conjecture holds for F, then X' has no Λ-quotient or Λ-submodule isomorphic to $\mathbb{Z}_p(1)$.*

Proof. We claim that if M is a finitely generated Λ-module such that the invariant group M^Γ has positive $\mathbb{Z}_p$-rank, then the coinvariant group M_Γ does as well. To see this, let I be the augmentation ideal in Λ. The annihilator of M^Γ is I, so the annihilator of M is contained in I. By [19, Proposition 2.1] and its proof, there is an ideal J of Λ contained in the annihilator of M such that any prime ideal P of Λ containing J satisfies $\mathrm{rank}_{\Lambda/P} M/PM$ is positive. We then apply this to $P = I$ to obtain the claim.

Applying this to $X'(-1)$, we may suppose that X' has a quotient isomorphic to $\mathbb{Z}_p(1)$. Such a quotient is in particular a locally trivial $\mathbb{Z}_p(1)$-quotient of the Galois group of $\mathfrak{X}$. In other words, we have a subgroup of $\mathrm{H}^1(G_{K,S},\mu_{p^\infty})$ isomorphic to $\mathbb{Z}_p$ and which maps trivially to $\mathrm{H}^1(G_{K,\mathfrak{p}},\mu_{p^\infty})$ for all $\mathfrak{p} \in S_f$.

The maps

$$\mathrm{H}^1\big(G_{F,S},\mu_{p^\infty}\big) \longrightarrow \mathrm{H}^1\big(G_{K,S},\mu_{p^\infty}\big)^\Gamma \quad \text{and} \quad \mathrm{H}^1\big(G_{F,\mathfrak{p}},\mu_{p^\infty}\big) \longrightarrow \mathrm{H}^1\big(G_{K,\mathfrak{p}},\mu_{p^\infty}\big)^\Gamma$$

have p-torsion kernel and cokernel. For instance, the kernel (resp., cokernel) of the first map is (resp., is contained in) $\mathrm{H}^i(\Gamma,\mu_{p^\infty})$ for $i = 1$ (resp., $i = 2$). If $\Phi \cong \mathbb{Z}_p$ is a subgroup of Γ that does not fix $\mathbb{Z}_p(1)$, then the Hochschild-Serre spectral sequence

$$\mathrm{H}^i\big(\Gamma/\Phi,\mathrm{H}^j\big(\Phi,\mu_{p^\infty}\big)\big) \Longrightarrow \mathrm{H}^{i+j}\big(\Gamma,\mu_{p^\infty}\big)$$

gives finiteness of all $\mathrm{H}^k(\Gamma,\mu_{p^\infty})$, as $\mathrm{H}^j(\Phi,\mu_{p^\infty})$ is finite for every j (and zero for every $j \neq 0$).

We may now conclude that $\mathrm{H}^1(G_{F,S},\mu_{p^\infty})$ has a subgroup isomorphic to $\mathbb{Z}_p$ with finite image under the localization map

$$\mathrm{H}^1\big(G_{F,S},\mu_{p^\infty}\big)\longrightarrow\bigoplus_{\mathfrak{p}\in S_f}\mathrm{H}^1\big(G_{F,\mathfrak{p}},\mu_{p^\infty}\big).$$

In other words, Leopoldt's conjecture must fail (see [40, Theorem 10.3.6]). □

Remark 4.1.16. Proposition 4.1.15 also holds for the unramified Iwasawa module X over K in place of X'.

PROPOSITION 4.1.17. *Suppose that $r=2$ and K contains all p-power roots of unity. If Leopoldt's conjecture holds for F, then the sequences*

$$0\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\longrightarrow\mathfrak{X}^*\longrightarrow\mathbb{Z}_p\longrightarrow 0 \tag{4.11}$$

$$0\longrightarrow\mathrm{E}^1_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big)\longrightarrow\mathfrak{X}\longrightarrow\mathfrak{X}^{**}\longrightarrow\mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big)\longrightarrow 0 \tag{4.12}$$

of Theorem 4.1.2 and Corollary 4.1.6 are exact.

Proof. Suppose that Leopoldt's conjecture holds for F. Consider first the map $\phi\colon\mathbb{Z}_p\to\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)$ of Theorem 4.1.2. The image of $\mathbb{Z}_p$ is contained in $\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)^\Gamma$. There is an exact sequence

$$0\longrightarrow X'(-1)^\Gamma\longrightarrow\mathrm{H}^2_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)^\Gamma\longrightarrow\mathcal{K}_0(-1)^\Gamma.$$

Primes in S_f are finitely decomposed in the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}, and the action of the summand $\Gamma_{\mathrm{cyc}}=\mathrm{Gal}(F_{\mathrm{cyc}}/F)$ of Γ on $\mathbb{Z}_p(-1)$ is faithful. It follows that $\mathcal{K}_0(-1)^\Gamma=(\mathcal{K}_0^{\mathrm{Gal}(K/F_{\mathrm{cyc}})}(-1))^{\Gamma_{\mathrm{cyc}}}$ is trivial. Proposition 4.1.15 then implies that ϕ must have finite image, and we have the first exact sequence.

By Corollary A.13, $\mathrm{E}^1_\Lambda(\mathbb{Z}_p)=0$ and $\mathrm{E}^2_\Lambda(\mathbb{Z}_p)\cong\mathbb{Z}_p$. The long exact sequence of Ext-groups for (4.11) reads

$$0\longrightarrow\mathrm{E}^1_\Lambda\big(\mathfrak{X}^*\big)\longrightarrow\mathrm{E}^1_\Lambda\big(\mathrm{H}^1_{\mathrm{Iw}}\big(K,\mathbb{Z}_p\big)\big)\longrightarrow\mathbb{Z}_p\longrightarrow\mathrm{E}^2_\Lambda\big(\mathfrak{X}^*\big).$$

By Corollary A.9(b), this implies that $\mathrm{E}^1_\Lambda(\mathrm{H}^1_{\mathrm{Iw}}(K,\mathbb{Z}_p))\to\mathbb{Z}_p$ is surjective with finite kernel. The map $\mathbb{Z}_p\to\mathrm{E}^1_\Lambda(\mathrm{H}^1_{\mathrm{Iw}}(K,\mathbb{Z}_p))$ of (4.3) is then also forced to be injective, being that it is of finite (i.e., codimension at least 3) cokernel $\mathrm{E}^3_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p))$, for instance by Proposition A.8. Therefore, the map $\mathrm{E}^2_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p))\to\mathbb{Z}_p$ in (4.3) is trivial, and (4.12) is exact. □

4.2. Useful lemmas. It is necessary for our purposes to account for discrepancies between decomposition and inertia groups, and the unramified Iwasawa module X and $H^2_{\mathrm{Iw}}(K,\mathbb{Z}_p(1))$. The following lemmas are designed for this

purpose. For a prime $\mathfrak{p} \in S_f$, we set

$$I_{\mathfrak{p}} = \Omega \hat{\otimes}_{\mathbb{Z}_p[\![\mathcal{G}_{\mathfrak{p}}]\!]} \mathfrak{I}_{\mathfrak{p}},$$

where $\mathfrak{I}_{\mathfrak{p}}$ denotes the inertia subgroup of $\mathfrak{D}_{\mathfrak{p}}$. Then $I_{\mathfrak{p}}$ is an Ω-submodule of $D_{\mathfrak{p}}$.

Remark 4.2.1. The unramified Iwasawa module X over K is the cokernel of the map $\bigoplus_{\mathfrak{p}\in S_f} I_{\mathfrak{p}} \to \mathfrak{X}$, independent of S containing the primes over p. Its completely split-at-S_f quotient is the cokernel of $\bigoplus_{\mathfrak{p}\in S_f} D_{\mathfrak{p}} \to \mathfrak{X}$. The latter Ω-module is the completely split Iwasawa module X' if K contains the cyclotomic $\mathbb{Z}_p$-extension of F.

In the following, we suppose that primes over $\mathfrak{p}$ do not split completely in K/F, which occurs, for instance, if $\mathfrak{p}$ lies over p or K contains the cyclotomic $\mathbb{Z}_p$-extension of F.

LEMMA 4.2.2. *Suppose that $\Gamma_{\mathfrak{p}} \neq 0$. Let $\epsilon_{\mathfrak{p}} = 0$ (resp., 1) if the completion $K_{\mathfrak{p}}$ at a prime over $\mathfrak{p}$ contains (resp., does not contain) the unramified $\mathbb{Z}_p$-extension of $E_{\mathfrak{p}}$. Let $\epsilon'_{\mathfrak{p}} = \epsilon_{\mathfrak{p}}\delta_{r_{\mathfrak{p}},1}$, and if $\epsilon'_{\mathfrak{p}} = 1$, suppose that K contains all p-power roots of unity. We have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathfrak{p}} & \longrightarrow & D_{\mathfrak{p}} & \longrightarrow & \mathcal{K}_{\mathfrak{p}}^{\epsilon_{\mathfrak{p}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{\mathfrak{p}}^{**} & \longrightarrow & D_{\mathfrak{p}}^{**} & \longrightarrow & \mathcal{K}_{\mathfrak{p}}^{\epsilon'_{\mathfrak{p}}} & \longrightarrow & 0 \end{array} \tag{4.13}$$

where the right-hand vertical map is the identity if $\epsilon'_{\mathfrak{p}} = 1$.

Proof. We have exact sequences $0 \to \mathfrak{I}_{\mathfrak{p}} \to \mathfrak{D}_{\mathfrak{p}} \to \mathbb{Z}_p^{\epsilon_{\mathfrak{p}}} \to 0$ by the theory of local fields. These yield the upper exact sequence upon taking the tensor product with Ω over $\mathbb{Z}_p[\![\mathcal{G}_{\mathfrak{p}}]\!]$. Since $\Gamma_{\mathfrak{p}} \neq 0$, we have that $\mathcal{K}_{\mathfrak{p}}$ is a torsion Λ-module. Taking Ext-groups, we obtain an exact sequence

$$0 \longrightarrow D_{\mathfrak{p}}^* \longrightarrow I_{\mathfrak{p}}^* \longrightarrow \mathrm{E}^1_{\Lambda}\big(\mathcal{K}_{\mathfrak{p}}^{\epsilon_{\mathfrak{p}}}\big) \longrightarrow \mathrm{E}^1_{\Lambda}\big(D_{\mathfrak{p}}\big). \tag{4.14}$$

If $r_{\mathfrak{p}} > 1$ or $\epsilon_{\mathfrak{p}} = 0$, then we are done by Lemma 4.1.13 after taking a dual.

Suppose that $r_{\mathfrak{p}} = \epsilon'_{\mathfrak{p}} = 1$. We claim that the last map in (4.14) is trivial. This map is, by Lemma A.12, just the map of Ω-modules

$$\Omega^{\iota} \otimes_{\mathbb{Z}_p[\![\mathcal{G}_{\mathfrak{p}}]\!]} \mathrm{Ext}^1_{\Lambda_{\mathfrak{p}}}\left(\mathbb{Z}_p, \Lambda_{\mathfrak{p}}\right) \longrightarrow \Omega^{\iota} \otimes_{\mathbb{Z}_p[\![\mathcal{G}_{\mathfrak{p}}]\!]} \mathrm{Ext}^1_{\Lambda_{\mathfrak{p}}}\left(\mathfrak{D}_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}\right),$$

where $\Lambda_{\mathfrak{p}} = \mathbb{Z}_p[\![\Gamma_{\mathfrak{p}}]\!]$. For the claim, we may then assume that $r = 1$ and K is the cyclotomic $\mathbb{Z}_p$-extension of F. We then have an exact sequence

$$0 \longrightarrow \mathcal{K}_{\mathfrak{p}}^{\iota}(1) \longrightarrow D_{\mathfrak{p}} \longrightarrow D_{\mathfrak{p}}^{**} \longrightarrow 0$$

from Theorem 4.1.14 and Lemma 4.1.13. Taking Ext-groups yields an exact sequence

$$0 \longrightarrow \mathrm{E}^1_\Lambda\big(D^{**}_\mathfrak{p}\big) \longrightarrow \mathrm{E}^1_\Lambda\big(D_\mathfrak{p}\big) \longrightarrow \mathcal{K}_\mathfrak{p}(-1) \longrightarrow \mathrm{E}^2_\Lambda\big(D^{**}_\mathfrak{p}\big),$$

and the first and last term are trivial by Corollary A.9. As there is no nonzero Λ-module homomorphism $\mathbb{Z}_p \to \mathbb{Z}_p(-1)$, there is no nonzero homomorphism $\mathcal{K}^\iota_\mathfrak{p} \to \mathcal{K}_\mathfrak{p}(-1)$, hence the claim. Finally, taking Ext-groups once again, we have an exact sequence

$$0 \longrightarrow I^{**}_\mathfrak{p} \longrightarrow D^{**}_\mathfrak{p} \longrightarrow \mathcal{K}_\mathfrak{p} \longrightarrow \mathrm{E}^1_\Lambda\big(I^*_\mathfrak{p}\big).$$

By Corollary A.9, $\mathrm{E}^1_\Lambda(I^*_\mathfrak{p}) = 0$, so we have shown the exactness of the second row of (4.13). □

Using Lemma 4.2.2, one can derive exact sequences as in Theorem 4.1.14 with $I_\mathfrak{p}$ in place of $D_\mathfrak{p}$ if we suppose that K contains all p-power roots of unity. When F contains μ_p, this hypothesis is equivalent to K containing the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F.

LEMMA 4.2.3. (a) *If $K_\mathfrak{p}$ contains a $\mathbb{Z}_p^2$-extension of $E_\mathfrak{p}$ for all $\mathfrak{p} \in S_f$ lying over p, then the kernel of the quotient map $X \to X'$ is pseudo-null.*

(b) *If $K_\mathfrak{p}$ contains the unramified $\mathbb{Z}_p$-extension of $E_\mathfrak{p}$ for all $\mathfrak{p} \in S_f$ lying over p, the quotient map $X \to X'$ is an isomorphism.*

Proof. Take S to be the set of primes over p and ∞. We have a canonical surjection

$$\bigoplus_{\mathfrak{p}\in S_f} \big(\Omega \hat{\otimes}_{\mathbb{Z}_p[\![\mathcal{G}_\mathfrak{p}]\!]} \mathfrak{D}_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}\big) \longrightarrow \ker(X \to X')$$

and $\mathfrak{D}_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}$ is zero or $\mathbb{Z}_p$ according as to whether $K_\mathfrak{p}$ does or does not contain the unramified $\mathbb{Z}_p$-extension of $E_\mathfrak{p}$, respectively. This implies part (b) immediately. It also implies part (a), since $\Omega \hat{\otimes}_{\mathbb{Z}_p[\![\mathcal{G}_\mathfrak{p}]\!]} \mathfrak{D}_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}$ is of finite $\mathbb{Z}_p[\![\mathcal{G}/\mathcal{G}_\mathfrak{p}]\!]$-rank and $\mathbb{Z}_p[\![\mathcal{G}/\mathcal{G}_\mathfrak{p}]\!]$ is pseudo-null in case (a). □

The following lemma describes the structure of the Ext-groups of $\mathcal{K}_0$ in terms of those of $\mathcal{K}$.

LEMMA 4.2.4. *Let $\mathfrak{p} \in S_f$.*

(a) *For $0 \le j < r-1$, we have $\mathrm{E}^j_\Lambda(\mathcal{K}_0) \cong \mathrm{E}^j_\Lambda(\mathcal{K})$. For $j \ge r+1$, we have $\mathrm{E}^j_\Lambda(\mathcal{K}) = E^j_\Lambda(\mathcal{K}_0) = 0$.*

(b) *If $r \ne r_\mathfrak{p}$ for all $\mathfrak{p} \in S_f$, then $\mathrm{E}^r_\Lambda(\mathcal{K}) = \mathrm{E}^r_\Lambda(\mathcal{K}_0) = 0$, and we have an exact sequence*

$$0 \longrightarrow \mathrm{E}^{r-1}_\Lambda(\mathcal{K}) \longrightarrow \mathrm{E}^{r-1}_\Lambda\big(\mathcal{K}_0\big) \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

(c) *If $r = r_{\mathfrak{p}}$ for some $\mathfrak{p} \in S_f$, then $\mathrm{E}_\Lambda^{r-1}(\mathcal{K}_0) \cong \mathrm{E}_\Lambda^{r-1}(\mathcal{K})$, and we have an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathrm{E}_\Lambda^r(\mathcal{K}) \longrightarrow \mathrm{E}_\Lambda^r\big(\mathcal{K}_0\big) \longrightarrow 0.$$

Proof. Note that

$$\mathrm{E}_\Lambda^j(\mathcal{K}) \cong \bigoplus_{\mathfrak{p} \in S_f} \big(\mathcal{K}_{\mathfrak{p}}^\iota\big)^{\delta_{j,r_{\mathfrak{p}}}}$$

by Lemma 4.1.13. Moreover, Corollary A.13 tells us that $\mathrm{E}_\Lambda^j(\mathbb{Z}_p) \cong \mathbb{Z}_p^{\delta_{r,j}}$. We are quickly reduced to the case that $r = r_{\mathfrak{p}}$ for some $\mathfrak{p}$. The map $\mathrm{E}_\Lambda^r(\mathbb{Z}_p) \to \mathrm{E}_\Lambda^r(\mathcal{K}_{\mathfrak{p}})$ for such a $\mathfrak{p}$ is the map $\mathbb{Z}_p \to \mathcal{K}_{\mathfrak{p}}^\iota$ that takes 1 to the norm element, hence is injective. □

If K contains all p-power roots of unity, then from (4.8) we have an exact sequence

(4.15)
$$\cdots \longrightarrow \mathrm{E}_\Lambda^j\big(\mathcal{K}_0\big)(1) \longrightarrow \mathrm{E}_\Lambda^j\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big) \longrightarrow \mathrm{E}_\Lambda^j(X')(1) \longrightarrow \mathrm{E}_\Lambda^{j+1}\big(\mathcal{K}_0\big)(1) \longrightarrow \cdots$$

for all j. Lemmas 4.2.3 and 4.2.4 then allow one to study the relationship between the higher Ext-groups of $\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p)$ occurring in Theorem 4.1.14 and the higher Ext-groups of X.

Remark 4.2.5. At the end of Section 3.4, we asserted that if K contains μ_{p^∞} and $r_{\mathfrak{p}} \geq 2$ for all $\mathfrak{p} \in S_f$, then $\mathfrak{X}$ is torsion-free if and only if X is pseudo-null. This may also be seen as follows. By Corollary 4.1.6, the Λ-torsion subgroup of $\mathfrak{X}$ is isomorphic to $\mathrm{E}_\Lambda^1(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p))$. By assumption and Lemmas 4.1.13 and 4.2.4, we have $\mathrm{E}_\Lambda^1(\mathcal{K}_0) = 0$. Thus, by the exact sequence (4.15), the triviality of $\mathrm{E}_\Lambda^1(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p))$ and the triviality of $\mathrm{E}_\Lambda^1(X')$ are equivalent. Since X' is Λ-torsion, $\mathrm{E}_\Lambda^1(X') = 0$ if and only if X' is pseudo-null, which by Lemma 4.2.3 is equivalent to the pseudo-nullity of X.

4.3. Eigenspaces. We end with a discussion of the rank of the Δ-eigenspaces of the global and local Iwasawa modules $\mathfrak{X}$ and $D_{\mathfrak{p}}$. Let us suppose now that $\mathcal{G} = \Gamma \times \Delta$, and for simplicity, that Δ is abelian. Without loss of generality, we shall suppose here that F contains $\mathbb{Q}(\mu_p)$, and we let $\omega \colon \Delta \to \mathbb{Z}_p^\times$ denote the Teichmüller character.

Let ψ be a $\bar{\mathbb{Q}}_p^\times$-valued character of Δ. For a $\mathbb{Z}_p[\Delta]$-module M, we let

$$M^\psi = M \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\psi,$$

where $\mathcal{O}_\psi$ is the $\mathbb{Z}_p$-algebra generated by the values of ψ, and $\mathbb{Z}_p[\Delta] \to \mathcal{O}_\psi$ is the surjection induced by ψ. We set $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ and $\Lambda_\psi = \mathcal{O}_\psi[\![\Gamma]\!]$. Note that $\Omega^\psi \cong \Lambda_\psi$

as compact $\mathcal{O}_\psi$-algebras, but Ω^ψ has the extra structure of an Ω-module on which Δ acts by ψ.

Let $r_2(E)$ denote the number of complex places of E and $r_1^\psi(E)$ the number of real places of E at which ψ is odd. We have the following consequence of Iwasawa-theoretic global and local Euler-Poincaré characteristic formulas, as found in [39, 5.2.11, 5.3.6].

LEMMA 4.3.1. (a) *If weak Leopoldt holds for K, then* $\operatorname{rank}_{\Lambda_\psi} \mathfrak{X}^\psi = r_2(E) + r_1^\psi(E)$.

(b) *If either $\Gamma_\mathfrak{p} \neq 0$ or $\psi|_{\Delta_\mathfrak{p}} \neq 1$, then* $\operatorname{rank}_{\Lambda_\psi} D_\mathfrak{p}^\psi = [E_\mathfrak{p} : \mathbb{Q}_p]$.

Proof. Let Σ be the union of S and the primes that ramify in F/E. Since the primes in $\Sigma \setminus S$ can ramify at most tamely in $\mathfrak{X}_\Sigma$, the Λ_ψ-modules $\mathfrak{X}^\psi$ and $\mathfrak{X}_\Sigma^\psi$ (the Σ-ramified Iwasawa module over K) have the same rank. Endow $\mathcal{O}_{\psi^{-1}}$ (which equals $\mathcal{O}_\psi$ as a $\mathbb{Z}_p$-module) with a $G_{E,\Sigma}$-action through ψ^{-1}. Let $B_\psi = (\Omega^{\psi^{-1}})^\vee \cong \operatorname{Hom}_{\mathbb{Z}_p,\mathrm{cont}}(\Lambda, \mathcal{O}_{\psi^{-1}}^\vee)$, which is a discrete $\Lambda_\psi[\![G_{E,\Sigma}]\!]$-module. Restriction and Shapiro's lemma (see [39, 8.3.3] and [31, 5.2.2, 5.3.1]) provide Λ_ψ-module homomorphisms

$$\mathrm{H}^1\big(G_{E,\Sigma}, B_\psi\big) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(G_{F,\Sigma}, B_\psi)^\Delta \xrightarrow{\sim} \mathrm{H}^1\big(G_{K,\Sigma}, \mathcal{O}_{\psi^{-1}}^\vee\big)^\Delta \xrightarrow{\sim} \big(\mathfrak{X}_\Sigma^\psi\big)^\vee,$$

restriction having cotorsion kernel and cokernel. (The last step passes through the intermediate module $\operatorname{Hom}_{\mathbb{Z}_p[\Delta]}(\mathfrak{X}_\Sigma \otimes_{\mathbb{Z}_p} \mathcal{O}_{\psi^{-1}}, \mathbb{Q}_p/\mathbb{Z}_p)$.) We are therefore reduced to computing the Λ_ψ-corank of $\mathrm{H}^1(G_{E,\Sigma}, B_\psi)$. The global Euler characteristic formula tells us that

$$\sum_{j=0}^{2} (-1)^{j-1} \operatorname{rank}_{\Lambda_\psi} \mathrm{H}^j\big(G_{E,\Sigma}, B_\psi\big)^\vee = \sum_{v \in S - S_f} \operatorname{rank}_{\Lambda_\psi} \big(\Omega^\psi(1)\big)^{G_{E_v}},$$

and $\mathrm{H}^j(G_{E,\Sigma}, B_\psi)$ is Λ_ψ-cotorsion for $j = 0$ and $j = 2$, the latter by weak Leopoldt for K.

Recall that $D_\mathfrak{p} = \mathrm{H}^1(G_{K,\mathfrak{p}}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. Restriction and Shapiro's lemma [31, 5.3.2] again reduce the computation of the Λ_ψ-corank of $H^1(G_{E,\mathfrak{p}}, B_\psi)$, and the local Euler characteristic formula tells us that

$$\sum_{j=0}^{2} (-1)^j \operatorname{rank}_{\Lambda_\psi} \mathrm{H}^j\big(G_{E_\mathfrak{p}}, B_\psi\big)^\vee = \big[E_\mathfrak{p} : \mathbb{Q}_p\big] \cdot \operatorname{rank}_{\Lambda_\psi} \Omega^\psi = \big[E_\mathfrak{p} : \mathbb{Q}_p\big].$$

As $\mathrm{H}^2(G_{E_\mathfrak{p}}, B_\psi)^\vee$ is trivial, and $\mathrm{H}^0(G_{E_\mathfrak{p}}, B_\psi)^\vee \cong (\Omega^\psi)_{G_{E_\mathfrak{p}}}$ is torsion by virtue of the fact that either $\Gamma_\mathfrak{p}$ or $\psi|_{\Delta_\mathfrak{p}}$ is nontrivial, we are done. □

Let us suppose in the following three lemmas that ψ has order prime to p. These following lemmas are variants of the lemmas of the previous section in "good eigenspaces". The proofs are straightforward from what has already been

done and as such are left to the reader. For the second lemma, one can use the following simple fact: for an Ω-module M, we have

$$\left(\mathrm{E}^j_\Lambda(M)(1)\right)^\psi \cong \mathrm{E}^j_\Lambda\left(M^{\omega\psi^{-1}}\right)(1). \tag{4.16}$$

LEMMA 4.3.2. *We have $\mathcal{K}^\psi_{\mathfrak{p}} = 0$ if $\psi|_{\Delta_{\mathfrak{p}}} \neq 1$. We have $\mathcal{K}^\psi \cong \mathcal{K}^\psi_0$ if $\psi \neq 1$.*

LEMMA 4.3.3. *Suppose that K contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F. If $\psi|_{\Delta_{\mathfrak{p}}} \neq 1$ (resp., $\omega\psi^{-1}|_{\Delta_{\mathfrak{p}}} \neq 1$), then $I^\psi_{\mathfrak{p}} \to D^\psi_{\mathfrak{p}}$ (resp., $D^\psi_{\mathfrak{p}} \to (D^\psi_{\mathfrak{p}})^{**}$) is an isomorphism.*

LEMMA 4.3.4. *Suppose that K contains F_{cyc}. Then the maps $X^\psi \twoheadrightarrow (X')^\psi \hookrightarrow \mathrm{H}^2(K, \mathbb{Z}_p(1))^\psi$ are isomorphisms if $\psi|_{\Delta_{\mathfrak{p}}} \neq 1$ for all $\mathfrak{p}$ lying over p.*

5. Reflection-type theorems for Iwasawa modules. In this section, we prove results that relate an Iwasawa module in a given eigenspace with another Iwasawa module in a "reflected" eigenspace. These modules typically appear on opposite sides of a short exact sequence, with the middle term being measured by p-adic L-functions. The method in all cases is the same: we take a sum of the maps of exact sequences at primes over p found in Theorem 4.1.14 and apply the snake lemma to the resulting diagram. Here, we focus especially on cases in which eigenspaces of the unramified outside p Iwasawa modules $\mathfrak{X}$ have rank 1, in order that the corresponding eigenspace of the double dual is free of rank one. Our main result is a symmetric exact sequence for an unramified Iwasawa module and its reflection in the case of an imaginary quadratic field. This sequence gives rise to a computation of second Chern classes (see Subsection 5.2).

We maintain the notation of Section 4. We suppose in this section that p is odd, and we let S be the set of primes of E over p and ∞. We let ψ denote a one-dimensional character of the absolute Galois group of E of finite order prime to p, and let E_ψ denote the fixed field of its kernel. We then set $F = E_\psi(\mu_p)$ and $\Delta = \mathrm{Gal}(F/E)$. Let ω denote the Teichmüller character of Δ.

We now take $\widetilde{E}$ to be the compositum of all $\mathbb{Z}_p$-extensions of E, and we set $r = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(\widetilde{E}/E)$. If Leopoldt's conjecture holds for E, then $r = r_2(E) + 1$. We set $K = F\widetilde{E}$. As before, we take $\mathcal{G} = \mathrm{Gal}(K/E)$ and $\Gamma = \mathrm{Gal}(K/F)$ and set $\Omega = \mathbb{Z}_p[\![\mathcal{G}]\!]$ and $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$.

For a subset Σ of S_f, let us set $\mathcal{K}_\Sigma = \bigoplus_{\mathfrak{p}\in\Sigma} \mathcal{K}_{\mathfrak{p}}$. We set

$$\mathcal{H}_\Sigma = \ker\left(\mathrm{H}^2_{\mathrm{Iw}}\left(K, \mathbb{Z}_p(1)\right) \to \mathcal{K}_{S_f - \Sigma}\right),$$

which for $\Sigma \neq \varnothing$ fits in an exact sequence

$$0 \longrightarrow X' \longrightarrow \mathcal{H}_\Sigma \longrightarrow \mathcal{K}_\Sigma \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{5.1}$$

For $\Sigma = \varnothing$, we have $\mathcal{H}_\Sigma \cong X'$. We shall study the diagram that arises from the sum of exact sequences in Theorem 4.1.14 over primes in $T = S_f - \Sigma$. Setting

$D_T = \bigoplus_{\mathfrak{p}\in T} D_{\mathfrak{p}}$, it reads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathcal{K}_T)(1) & \longrightarrow & D_T & \longrightarrow & D_T^{**} & \longrightarrow & \mathrm{E}^2_\Lambda(\mathcal{K}_T)(1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi_T} & & \downarrow \\ 0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)) & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X}^{**} & \longrightarrow & \mathrm{E}^2_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K,\mathbb{Z}_p)) \longrightarrow \mathbb{Z}_p, \end{array} \tag{5.2}$$

where $\phi_T = \sum_{\mathfrak{p}\in T} \phi_{\mathfrak{p}}$ is the sum of maps $\phi_{\mathfrak{p}}\colon D_{\mathfrak{p}}^{**} \to \mathfrak{X}^{**}$. We take ψ-eigenspaces, on which the map to $\mathbb{Z}_p$ in the diagram will vanish if $\psi \neq 1$, $r = 1$, or $r = 2$ and Leopoldt's conjecture holds for F, the latter by Proposition 4.1.17. The cokernel of $D_T \to \mathfrak{X}$ is the Iwasawa module $\mathfrak{X}_{\Sigma,T}$ which is the Galois group over K of the maximal pro-p abelian extension of K which is unramified outside of Σ and totally split over $T = S_f - \Sigma$. The group $I_T = \bigoplus_{\mathfrak{p}\in T} I_{\mathfrak{p}}$ has the property that the cokernel of $I_T \to \mathfrak{X}$ is the unramified outside of Σ-Iwasawa module $\mathfrak{X}_\Sigma$.

In this section, we focus on examples for which $\operatorname{rank}_{\Lambda_\psi} \mathfrak{X}^\psi = 1$, which forces $r \leq 2$ under Leopoldt's conjecture by Lemma 4.3.1. We have that $(\mathfrak{X}^\psi)^{**}$ is free of rank one over Λ_ψ, by Lemma A.1. If $\mathfrak{p}$ is split in E, then $(D_{\mathfrak{p}}^\psi)^{**}$ is also isomorphic to Λ_ψ, so

$$\phi_{\mathfrak{p}}^\psi\colon \big(D_{\mathfrak{p}}^\psi\big)^{**} \longrightarrow \big(\mathfrak{X}^\psi\big)^{**}$$

is identified with multiplication by an element of Λ_ψ, well defined up to unit. We shall exploit this fact throughout. At times, we will have to distinguish between decomposition and inertia groups, which we will deal with below as the need arises. In our examples, T is always a set of degree one primes, so $r_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in T$. The assumptions on ψ and T make most results cleaner and do well to illustrate the role of second Chern classes, but the methods can be applied for any $\mathbb{Z}_p^r$-extension containing the cyclotomic $\mathbb{Z}_p$-extension and any set of primes over p.

As in Definition A.6, for an Ω-module M that is finitely generated over Λ with annihilator of height at least r, we define the adjoint $\alpha(M)$ of M to be $\mathrm{E}^r_\Lambda(M)$.

5.1. The rational setting. Let us demonstrate the application of the results of Subsection 4.1 in the setting of the classical Iwasawa main conjecture. Suppose that $E = \mathbb{Q}$ and that ψ is odd. For simplicity, we assume $\psi \neq \omega$. We study the unramified Iwasawa module X over K.

THEOREM 5.1.1. *If $(X')^{\omega\psi^{-1}}$ is finite, then there is an exact sequence of Ω-modules*

$$0 \longrightarrow X^\psi \longrightarrow \Omega^\psi/\big(\mathcal{L}_\psi\big) \longrightarrow \big((X')^{\omega\psi^{-1}}\big)^\vee(1) \longrightarrow 0$$

with $\mathcal{L}_\psi$ interpolating the p-adic L-function for $\chi = \omega\psi^{-1}$ as in Theorem 3.2.1.

Proof. Note that $\mathcal{K} = \mathcal{K}_p$. By Lemma 4.1.13, we have $\mathrm{E}^1_\Lambda(\mathcal{K}) \cong \mathcal{K}^\iota$ and $\mathrm{E}^2_\Lambda(\mathcal{K}) = 0$. Since $\psi \neq \omega$, Lemma 4.3.2 tells us that $\mathcal{K}^{\omega\psi^{-1}} \cong \mathcal{K}_0^{\omega\psi^{-1}}$. By (4.15) and Lemma 4.2.4 and our assumption of pseudo-nullity of $X^{\omega\psi^{-1}}$, we have that the natural maps

$$\mathrm{E}^1_\Lambda(\mathcal{K})(1)^\psi \xrightarrow{\sim} \mathrm{E}^1_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big)^\psi \quad \text{and} \quad \mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big)^\psi \xrightarrow{\sim} \mathrm{E}^2_\Lambda(X')(1)^\psi$$

are isomorphisms. By (4.16) and Proposition A.4(a), $\mathrm{E}^2_\Lambda(X')(1)^\psi \cong ((X')^{\omega\psi^{-1}})^\vee(1)$. The diagram of Theorem 4.1.14 with $\mathfrak{p} = p$ reads

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathcal{K}_p)(1)^\psi & \longrightarrow & D_p^\psi & \longrightarrow & (D_p^\psi)^{**} & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{E}^1_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p))^\psi & \longrightarrow & \mathfrak{X}^\psi & \longrightarrow & (\mathfrak{X}^\psi)^{**} & \longrightarrow & ((X')^{\omega\psi^{-1}})^\vee(1) \longrightarrow 0.
\end{array}$$

It follows from Lemma 4.2.2 and the fact that there is no nonzero map $\mathcal{K}_p^\iota(1) \to \mathcal{K}_p$ of Ω-modules that we can replace D_p by I_p in the diagram. By applying the snake lemma to the resulting diagram, we obtain an exact sequence

$$0 \longrightarrow X^\psi \longrightarrow \operatorname{coker}\theta \longrightarrow \big((X')^{\omega\psi^{-1}}\big)^\vee(1) \longrightarrow 0, \tag{5.3}$$

where $\theta\colon (I_p^\psi)^{**} \to (\mathfrak{X}^\psi)^{**}$ is the canonical map (restricting ϕ_p^ψ). The map θ is of free rank one Λ_ψ-modules by Lemmas 4.3.1 and A.1, and it is nonzero, hence injective, as X is torsion. We may identify its image with a nonzero submodule of Ω^ψ. Since $(X')^{\omega\psi^{-1}}$ is pseudo-null, (5.3) tells us that

$$c_1\big(X^\psi\big) = c_1\big(\Omega^\psi / \operatorname{im}\theta\big),$$

and this forces the image of θ to be $c_1(X^\psi)$. By the main conjecture of Theorem 3.2.1, we have $c_1(X^\psi) = (\mathcal{L}_\psi)$. □

Remark 5.1.2. If we do not assume that $(X')^{\omega\psi^{-1}}$ is finite, one may still derive an exact sequence of Λ_ψ-modules

$$0 \longrightarrow \alpha\big(X^{\omega\psi^{-1}}\big)(1) \longrightarrow X^\psi \longrightarrow \Omega^\psi/(\mathcal{M}) \longrightarrow \big(\big(X'_{\mathrm{fin}}\big)^{\omega\psi^{-1}}\big)^\vee(1) \longrightarrow 0$$

for some $\mathcal{M} \in \Omega^\psi$ such that $(\mathcal{M})c_1((X^{\omega\psi^{-1}})^\iota(1)) = (\mathcal{L}_\psi)$.

5.2. The imaginary quadratic setting. In this subsection, we take our base field E to be imaginary quadratic. Let p be an odd prime that splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in E, so $S_f = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. Since Leopoldt's conjecture holds for E, we have $\Gamma = \mathrm{Gal}(K/F) \cong \mathbb{Z}_p^2$. We let $\mathfrak{X}_\mathfrak{p}$ denote the $\mathfrak{p}$-ramified (i.e., unramified outside of the primes over $\mathfrak{p}$) Iwasawa module over K, and similarly for $\bar{\mathfrak{p}}$.

We will prove the following result and derive some consequences of it.

THEOREM 5.2.1. *Suppose that E is imaginary quadratic and p splits in E. If $X^{\omega\psi^{-1}}$ is pseudo-null as a Λ_ψ-module, then there is a canonical exact sequence of Ω-modules*

$$0 \longrightarrow \big(X/X_{\text{fin}}\big)^{\psi} \longrightarrow \frac{\Omega^{\psi}}{c_1\big(\mathfrak{X}_{\mathfrak{p}}^{\psi}\big) + c_1\big(\mathfrak{X}_{\bar{\mathfrak{p}}}^{\psi}\big)} \longrightarrow \alpha\big(X^{\omega\psi^{-1}}\big)(1) \longrightarrow 0. \tag{5.4}$$

Moreover, we have $X_{\text{fin}}^{\psi} = 0$ unless $\psi = \omega$, and X_{fin}^{ω} is cyclic.

We require some lemmas.

LEMMA 5.2.2. *The completely split Iwasawa module X' over K is equal to the unramified Iwasawa module X over K, and the map $I_{\mathfrak{p}} \to D_{\mathfrak{p}}$ is an isomorphism. Moreover, we have $\mathrm{E}_\Lambda^1(\mathcal{K}_{\mathfrak{p}}) = 0$ and $\mathrm{E}_\Lambda^2(\mathcal{K}_{\mathfrak{p}}) \cong \mathcal{K}_{\mathfrak{p}}^{\iota}$.*

Proof. The prime $\mathfrak{p}$ is infinitely ramified and has infinite residue field extension in $\widetilde{E}$, so $r_{\mathfrak{p}} = 2$. The statements follow from Lemma 4.2.3(b), Lemma 4.2.2, and Lemma 4.1.13 respectively. □

Note that $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \oplus \mathcal{K}_{\bar{\mathfrak{p}}}$ and $\mathcal{K}_0 = \ker(\mathcal{K} \to \mathbb{Z}_p)$ by the definition of Remark 4.1.11.

LEMMA 5.2.3. *If $X^{\omega\psi^{-1}}$ is pseudo-null as a Λ_ψ-module, then so is $\mathrm{H}^2_{\text{Iw}}(K, \mathbb{Z}_p)^{\omega\psi^{-1}}$, and we have an exact sequence of Ω-modules*

$$0 \longrightarrow \mathbb{Z}_p(1)^{\psi} \longrightarrow \big(\mathcal{K}^{\omega\psi^{-1}}\big)^{\iota}(1) \longrightarrow \mathrm{E}_\Lambda^2\big(\mathrm{H}^2_{\text{Iw}}\big(K, \mathbb{Z}_p\big)\big)^{\psi} \longrightarrow \mathrm{E}_\Lambda^2\big(X^{\omega\psi^{-1}}\big)(1) \longrightarrow 0.$$

Proof. Lemmas 5.2.2 and 4.2.4 tell us that $E_\Lambda^i(\mathcal{K}_0)(1) = 0$ for $i \neq 2$ and provide an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathcal{K}^{\iota}(1) \longrightarrow \mathrm{E}_\Lambda^2\big(\mathcal{K}_0\big)(1) \longrightarrow 0. \tag{5.5}$$

The exact sequence (4.15) has the form

$$\begin{aligned} 0 \longrightarrow \mathrm{E}_\Lambda^1\big(\mathrm{H}^2_{\text{Iw}}\big(K, \mathbb{Z}_p\big)\big) \longrightarrow \mathrm{E}_\Lambda^1(X)(1) \longrightarrow \mathrm{E}_\Lambda^2\big(\mathcal{K}_0\big)(1) \longrightarrow \mathrm{E}_\Lambda^2\big(\mathrm{H}^2_{\text{Iw}}\big(K, \mathbb{Z}_p\big)\big) \\ \longrightarrow \mathrm{E}_\Lambda^2(X)(1) \longrightarrow 0, \end{aligned}$$

noting that $X = X'$ by Lemma 5.2.2. The ψ-eigenspaces of the first two terms are zero by (4.16) and the pseudo-nullity of $X^{\omega\psi^{-1}}$, yielding the first assertion and leaving us with a short exact sequence. Splicing this together with the ψ-eigenspace of the sequence (5.5) and applying (4.16) to the last term, we obtain the exact sequence of the statement. □

The main conjecture for imaginary quadratic fields is concerned with the unramified outside $\mathfrak{p}$ Iwasawa module $\mathfrak{X}_{\mathfrak{p}}$ over K. For it, we have the following result on first Chern classes.

PROPOSITION 5.2.4. *If $X^{\omega\psi^{-1}}$ is pseudo-null as a Λ_ψ-module, then there is an injective pseudo-isomorphism $\mathfrak{X}_{\mathfrak{p}}^{\psi} \to \Omega^{\psi}/c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})$ of Ω-modules.*

Proof. We apply the snake lemma to the ψ-eigenspaces of the diagram of Theorem 4.1.14. By Lemma 5.2.2 and the pseudo-nullity in Lemma 5.2.3, one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\bar{\mathfrak{p}}}^{\psi} & \longrightarrow & (I_{\bar{\mathfrak{p}}}^{\psi})^{**} & \longrightarrow & \mathrm{E}_{\Lambda}^{2}(\mathcal{K}_{\bar{\mathfrak{p}}}^{\omega\psi^{-1}})(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_{\bar{\mathfrak{p}}}^{\psi}} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{X}^{\psi} & \longrightarrow & (\mathfrak{X}^{\psi})^{**} & \longrightarrow & \mathrm{E}_{\Lambda}^{2}(\mathrm{H}_{\mathrm{Iw}}^{2}(K,\mathbb{Z}_p))^{\psi} & \longrightarrow & 0, \end{array} \tag{5.6}$$

the right exactness of the bottom row following from Proposition 4.1.17. We immediately obtain an exact sequence

$$0 \longrightarrow \mathfrak{X}_{\mathfrak{p}}^{\psi} \longrightarrow \operatorname{coker}\left(\phi_{\bar{\mathfrak{p}}}^{\psi}\right) \longrightarrow C \longrightarrow 0$$

for $\phi_{\bar{\mathfrak{p}}}^{\psi}$ defined to be as in the diagram (5.6), with C a pseudo-null Ω-module that by Lemmas 4.1.13 and 5.2.3 fits in an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1)^{\psi} \longrightarrow \left(\mathcal{K}_{\mathfrak{p}}^{\omega\psi^{-1}}\right)^{\iota}(1) \longrightarrow C \longrightarrow \alpha\left(X^{\omega\psi^{-1}}\right)(1) \longrightarrow 0.$$

The map $\phi_{\bar{\mathfrak{p}}}^{\psi}\colon (I_{\bar{\mathfrak{p}}}^{\psi})^{**} \to (\mathfrak{X}^{\psi})^{**}$ is an injective homomorphism of free rank one Λ_ψ-modules. Since C is pseudo-null, the image of $\phi_{\bar{\mathfrak{p}}}^{\psi}$ is $c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})$, as required. □

We now prove our main result.

Proof of Theorem 5.2.1. Consider (5.2) for $T = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. From (5.6) one has

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathfrak{p}}^{\psi} \oplus I_{\bar{\mathfrak{p}}}^{\psi} & \longrightarrow & (I_{\mathfrak{p}}^{\psi})^{**} \oplus (I_{\bar{\mathfrak{p}}}^{\psi})^{**} & \longrightarrow & \mathrm{E}_{\Lambda}^{2}(\mathcal{K}^{\omega\psi^{-1}})(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_{\mathfrak{p}}^{\psi}+\phi_{\bar{\mathfrak{p}}}^{\psi}} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{X}^{\psi} & \longrightarrow & (\mathfrak{X}^{\psi})^{**} & \longrightarrow & \mathrm{E}_{\Lambda}^{2}(\mathrm{H}_{\mathrm{Iw}}^{2}(K,\mathbb{Z}_p))^{\psi} & \longrightarrow & 0. \end{array} \tag{5.7}$$

The snake lemma applied to (5.7) produces an exact sequence

$$\mathbb{Z}_p(1)^{\psi} \longrightarrow X^{\psi} \longrightarrow \frac{\left(\mathfrak{X}^{\psi}\right)^{**}}{\left(I_{\mathfrak{p}}^{\psi}\right)^{**} + \left(I_{\bar{\mathfrak{p}}}^{\psi}\right)^{**}} \longrightarrow \alpha\left(X^{\omega\psi^{-1}}\right)(1) \longrightarrow 0, \tag{5.8}$$

where the first and last terms follow from the exact sequence of Lemma 5.2.3. In the proof of Proposition 5.2.4, we showed that $\phi_{\bar{\mathfrak{p}}}^{\psi}$ is injective with image $c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})$ in the free rank one Ω^{ψ}-module $(\mathfrak{X}^{\psi})^{**}$, and similarly upon switching $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Thus,

we have an isomorphism

$$\frac{\left(\mathfrak{X}^{\psi}\right)^{**}}{\left(I_{\mathfrak{p}}^{\psi}\right)^{**}+\left(I_{\bar{\mathfrak{p}}}^{\psi}\right)^{**}} \cong \frac{\Omega^{\psi}}{c_1\left(\mathfrak{X}_{\mathfrak{p}}^{\psi}\right)+c_1\left(\mathfrak{X}_{\bar{\mathfrak{p}}}^{\psi}\right)}. \tag{5.9}$$

If $\psi \neq \omega$, then $\mathbb{Z}_p(1)^{\psi} = 0$. For $\psi = \omega$, we claim that the image of the map $\mathbb{Z}_p(1) \to X^{\omega}$ of (5.8) is finite cyclic. Since $\Omega^{\psi}/(c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})+c_1(\mathfrak{X}_{\bar{\mathfrak{p}}}^{\psi}))$ has no nontrivial finite submodule by Lemma A.3, the result then follows from (5.8) and (5.9).

To prove the claim, we identify $I_{\mathfrak{p}}^{\omega}$ and $I_{\bar{\mathfrak{p}}}^{\omega}$ with their isomorphic images in $\mathfrak{X}^{\omega}$, so the kernel of $I_{\mathfrak{p}}^{\omega} \oplus I_{\bar{\mathfrak{p}}}^{\omega} \to \mathfrak{X}^{\omega}$ is identified with $(I_{\mathfrak{p}} \cap I_{\bar{\mathfrak{p}}})^{\omega}$, and similarly with the double duals. By the exact sequence

$$0 \longrightarrow I_{\mathfrak{p}}^{\omega} \longrightarrow \left(I_{\mathfrak{p}}^{\omega}\right)^{**} \longrightarrow \mathbb{Z}_p[\![\Gamma/\Gamma_{\mathfrak{p}}]\!]^{\iota}(1) \longrightarrow 0$$

that follows from (4.10), we see that $I_{\mathfrak{p}}^{\omega}$ is contained in the ideal I of $\Lambda \cong (I_{\mathfrak{p}}^{**})^{\omega}$ with $\Lambda/I \cong \mathbb{Z}_p(1)$. This means that the intersection $(I_{\mathfrak{p}} \cap I_{\bar{\mathfrak{p}}})^{\omega}$ is contained in I times the free rank one Λ-submodule $(I_{\mathfrak{p}}^{**} \cap I_{\bar{\mathfrak{p}}}^{**})^{\omega}$ of $(\mathfrak{X}^{\omega})^{**}$. As the kernel of $\mathbb{Z}_p(1) \to X^{\omega}$ is isomorphic to $(I_{\mathfrak{p}}^{**} \cap I_{\bar{\mathfrak{p}}}^{**})^{\omega}/(I_{\mathfrak{p}} \cap I_{\bar{\mathfrak{p}}})^{\omega}$, which has $\mathbb{Z}_p(1)$ as a quotient, the claim follows. □

Let $\Omega_W = W[\![\mathcal{G}]\!]$ and $\Lambda_W = W[\![\Gamma]\!]$, where W denotes the Witt vectors of $\bar{\mathbb{F}}_p$. Let $\mathcal{L}_{\mathfrak{p},\psi}$ denote the element of $\Lambda_W \cong \Omega_W^{\psi}$ that determines the two-variable p-adic L-function for $\mathfrak{p}$ and $\omega\psi^{-1}$. Let X_W^{ψ} denote the completed tensor product of X^{ψ} with W over $\mathcal{O}_{\psi}$. Together with the Iwasawa main conjecture for K, Theorem 5.2.1 implies the following result.

THEOREM 5.2.5. *Suppose that E is imaginary quadratic and p splits in E. If both X^{ψ} and $X^{\omega\psi^{-1}}$ are pseudo-null Λ_{ψ}-modules, then there is an equality of second Chern classes*

$$c_2\left(\frac{\Lambda_W}{\left(\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi}\right)}\right) = c_2\left(X_W^{\psi}\right) + c_2\left(\left(X_W^{\omega\psi^{-1}}\right)^{\iota}(1)\right). \tag{5.10}$$

These Chern classes have a characteristic symbol with component at a codimension one prime P of Λ_W equal to the Steinberg symbol $\{\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi}\}$ if $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ is not a unit at P, and with other components trivial.

Proof. By [8, Corollary III.1.11], the main conjecture as proven in [51, Theorem 2(i)] implies that $\mathcal{L}_{\mathfrak{p},\psi}$ generates $c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})\Lambda_W$. We have

$$c_2\left(\alpha\left(X_W^{\omega\psi^{-1}}\right)(1)\right) = c_2\left(\left(X_W^{\omega\psi^{-1}}\right)^{\iota}(1)\right)$$

by Proposition A.11. We then apply Proposition 2.5.1. □

Remark 5.2.6. Supposing that both X^{ψ} and $X^{\omega\psi^{-1}}$ are pseudo-null, the Tate twist of the result of applying ι to the sequence (5.4) reads exactly as the analogous

sequence for the character $\omega\psi^{-1}$ in place of ψ. The functional equation of Lemma 3.3.2(b) yields an isomorphism

$$\big(\Omega_W^{\psi}/\big(\mathcal{L}_{\mathfrak{p},\psi},\mathcal{L}_{\bar{\mathfrak{p}},\psi}\big)\big)^{\iota}(1) \cong \Omega_W^{\omega\psi^{-1}}/\big(\mathcal{L}_{\bar{\mathfrak{p}},\omega\psi^{-1}},\mathcal{L}_{\mathfrak{p},\omega\psi^{-1}}\big),$$

of the middle terms of these sequences.

This implies the following codimension two Iwasawa-theoretic analogue of the Herbrand-Ribet theorem in the imaginary quadratic setting, as mentioned in the introduction. Note that in this analogue we must treat the eigenspaces X^{ψ} and $X^{\omega\psi^{-1}}$ together.

COROLLARY 5.2.7. *Suppose that $\psi \neq 1,\omega$. The Iwasawa modules X^{ψ} and $X^{\omega\psi^{-1}}$ are both trivial if and only if at least one of $\mathcal{L}_{\mathfrak{p},\psi}$ or $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ is a unit in Λ_W.*

Proof. If X^{ψ} is not pseudo-null, then so are both $\mathfrak{X}_{\mathfrak{p}}^{\psi}$ and $\mathfrak{X}_{\bar{\mathfrak{p}}}^{\psi}$. So, by the main conjecture proven by Rubin (see Theorem 3.3.1), neither $\mathcal{L}_{\mathfrak{p},\psi}$ nor $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ are units. If $X^{\omega\psi^{-1}}$ is not pseudo-null, then $\mathcal{L}_{\mathfrak{p},\omega\psi^{-1}}$ and $\mathcal{L}_{\bar{\mathfrak{p}},\omega\psi^{-1}}$ are similarly not units. By the functional equation of Lemma 3.3.2(b), this implies that $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ nor $\mathcal{L}_{\mathfrak{p},\psi}$ are non-units as well.

If X^{ψ} and $X^{\omega\psi^{-1}}$ are both pseudo-null, then the exact sequence (5.4) of Theorem 5.2.5 shows that X^{ψ} and $X^{\omega\psi^{-1}}$ are both finite if and only if the quotient $\Omega^{\psi}/(c_1(\mathfrak{X}_{\mathfrak{p}}^{\psi})+c_1(\mathfrak{X}_{\bar{\mathfrak{p}}}^{\psi}))$ is finite, which cannot happen unless it is trivial by Lemma A.3. Since $\psi \neq \omega$, again noting (5.4), this happens if and only if both X^{ψ} and $X^{\omega\psi^{-1}}$ are trivial as well. By the main conjecture, the quotient is trivial if and only at least one of $\mathcal{L}_{\mathfrak{p},\psi}$ and $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$ is a unit in Λ_W. □

Example 5.2.8. Suppose that ψ is cyclotomic, so extends to an abelian character of $\widetilde{\Delta} = \mathrm{Gal}(F/\mathbb{Q})$, that $E \not\subseteq \mathbb{Q}(\mu_p)$, and that $\psi \neq 1,\omega$. Then X^{ψ} is nontrivial if and only if the ψ-eigenspace under Δ of the unramified Iwasawa module X_{cyc} over the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F is nontrivial. That is, since all primes over p are unramified in K/F_{cyc}, the map from the $\mathrm{Gal}(K/F_{\mathrm{cyc}})$-coinvariants of X to X_{cyc} is injective with cokernel isomorphic to $\mathrm{Gal}(K/F_{\mathrm{cyc}})$, which has trivial Δ-action. We extend ψ and ω in a unique way to odd characters $\tilde{\psi}$ and $\tilde{\omega}$ of $\widetilde{\Delta}$. Identify the quadratic character κ of $\mathrm{Gal}(E/\mathbb{Q})$ with a character of $\widetilde{\Delta}$ that is trivial on Δ.

The ψ-eigenspace of X_{cyc} under Δ is the direct sum of the two eigenspaces $X_{\mathrm{cyc}}^{\tilde{\psi}}$ and $X_{\mathrm{cyc}}^{\tilde{\psi}\kappa}$ under $\widetilde{\Delta}$. By the cyclotomic main conjecture (Theorem 3.2.1), the Iwasawa module $X_{\mathrm{cyc}}^{\tilde{\psi}}$ is nontrivial if and only if the appropriate Kubota-Leopoldt p-adic L-function is not a unit. This in turn occurs if and only if p divides the Kubota-Leopoldt p-adic L-value

$$L_p\big(\tilde{\omega}\tilde{\psi}^{-1},0\big) = \big(1-\tilde{\psi}^{-1}(p)\big)L\big(\tilde{\psi}^{-1},0\big).$$

The value $L(\tilde{\psi}^{-1},0)$ is the negative of the generalized Bernoulli number $B_{1,\tilde{\psi}^{-1}}$. We have $\tilde{\psi}^{-1}(p)=1$ if and only if $\tilde{\psi}$ is locally trivial at p, in which case the p-adic L-function is said to have an exceptional zero. By the usual reflection principle (see also Remark 5.1.2), if $X_{\mathrm{cyc}}^{\tilde{\psi}\kappa}$ is nonzero, then so is $X_{\mathrm{cyc}}^{\tilde{\omega}\tilde{\psi}^{-1}\kappa}$.

Similarly, the unique extension of $\omega\psi^{-1}$ to an odd character of $\widetilde{\Delta}$ is $\tilde{\omega}\tilde{\psi}^{-1}\kappa$, and $X_{\mathrm{cyc}}^{\tilde{\omega}\tilde{\psi}^{-1}\kappa}$ is nontrivial if and only if $\mathcal{L}_{\tilde{\omega}\tilde{\psi}^{-1}\kappa}$ is not a unit, which is to say that p divides $L_p(\tilde{\psi}\kappa,0)$, or equivalently that either $p \mid B_{1,\tilde{\omega}^{-1}\tilde{\psi}\kappa}$ or $\tilde{\omega}\tilde{\psi}^{-1}\kappa$ is locally trivial at p. If $X_{\mathrm{cyc}}^{\tilde{\omega}\tilde{\psi}^{-1}} \neq 0$, then $X_{\mathrm{cyc}}^{\tilde{\psi}} \neq 0$.

Typically, when $X_{\mathrm{cyc}}^{\tilde{\psi}}$ is nonzero, $X_{\mathrm{cyc}}^{\tilde{\omega}\tilde{\psi}^{-1}}$ and $X_{\mathrm{cyc}}^{\tilde{\omega}\tilde{\psi}^{-1}\kappa}$ are trivial. For example, if $p=37$, then $37 \mid B_{1,\tilde{\omega}^{31}}$ (and $X_{\mathrm{cyc}}^{\tilde{\omega}^{31}}=0$), but $37 \nmid B_{1,\tilde{\omega}^{5}\kappa}$ for κ the quadratic character of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$. However, it can occur, though relatively infrequently, that both $B_{1,\tilde{\psi}^{-1}}$ and $B_{1,\tilde{\omega}^{-1}\tilde{\psi}\kappa}$ are divisible by p. A cursory computer search using this κ revealed many examples in the case one of the p-adic L-functions has an exceptional zero, e.g., for $p=5$ and $\tilde{\psi}$ a character of conductor 28 and order 6, and other examples in the cases that neither does, e.g., with $p=5$ and $\tilde{\psi}$ a character of conductor 555 and order 4.

5.3. Two further rank one cases. We will briefly indicate generalizations of Theorem 5.2.1 which can be proved in the remaining two cases when $\mathrm{rank}_{\Lambda_\psi} \mathfrak{X}^{\psi}=1$. Our field E will have at most one complex place, but it will not be $\mathbb{Q}$ or imaginary quadratic. In view of Lemma 4.3.1, the two cases to consider are when (i) E has exactly one complex place and the character ψ is even at all real places, and (ii) E is totally real and ψ is odd at exactly one real place.

For any set of primes T of E, we let $\mathfrak{X}_T$ denote the T-ramified Iwasawa module over K. If $T=\{\mathfrak{p}\}$, we set $\mathfrak{X}_{\mathfrak{p}}=\mathfrak{X}_T$. Suppose that we are given n degree one primes $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ of E over p, and set $T=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$, $\Sigma=S_f-T$ and $\Sigma_i=S_f-\{\mathfrak{p}_i\}$ for $i\in\{1,\dots,n\}$.

THEOREM 5.3.1. *Let E be a number field with exactly one complex place and at least one real place, and suppose that ψ is even at all real places of E. Assume that Leopoldt's conjecture holds for E, so $r=2$. Furthermore, suppose that $\mathfrak{X}_{\Sigma_i}^{\psi}$ is Λ_ψ-torsion for all $i\in\{1,\dots,n\}$. Assume that $r_{\mathfrak{p}}=2$ for all $\mathfrak{p}\in S_f$. If $X^{\omega\psi^{-1}}$ is pseudo-null, then there is an exact sequence of Ω-modules*

$$0 \longrightarrow \mathfrak{X}_{\Sigma}^{\psi} \longrightarrow \frac{\Omega^{\psi}}{\sum_{i=1}^{n} c_1\big(\mathfrak{X}_{\Sigma_i}^{\psi}\big)} \longrightarrow \alpha\big(\mathcal{H}_{\Sigma}^{\omega\psi^{-1}}\big)(1) \longrightarrow 0$$

where $\mathcal{H}_\Sigma$ is as in (5.1).

Proof. As in the imaginary quadratic case, the strategy is to control the terms and vertical homomorphisms of the diagram (5.2). The three steps needed to do this are (i) show that decomposition groups can be replaced by inertia groups, (ii)

show that the appropriate eigenspaces of the E^1_Λ groups in (5.2) are trivial, and (iii) use Iwasawa cohomology groups to relate the E^2_Λ groups in (5.2) to $\mathcal{H}_\Sigma$, which is an extension of an unramified Iwasawa module.

Note that $\mathbb{Z}_p(1)^\psi = 0$ since ψ is even at a real place. For $\mathfrak{p} \in T$, the field $K_\mathfrak{p}$ contains the unramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$ because $\mathfrak{p}$ has degree 1 and $r_\mathfrak{p} = 2 = r$. By assumption and Lemmas 4.2.3(a), 4.2.2, and 4.1.13, the map $X \to X'$ has pseudo-null kernel, $I_T = D_T$ and $\mathrm{E}^i_\Lambda(\mathcal{K}_T) = 0$ for $i \neq 2$. By our pseudo-nullity assumption, we have $\mathrm{E}^1_\Lambda(X^{\omega\psi^{-1}}) = 0$. It follows from Lemma 4.2.4 and (4.15) that $\mathrm{E}^1_\Lambda(\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p))^\psi = 0$. Since $\mathcal{H}_\Sigma^{\omega\psi^{-1}}$ is a submodule of the pseudo-null module $\mathrm{H}^2_{\mathrm{Iw}}(K, \mathbb{Z}_p(1))^{\omega\psi^{-1}}$, we have $\mathrm{E}^1_\Lambda(\mathcal{H}_\Sigma^{\omega\psi^{-1}}) = 0$ as well.

As $\mathbb{Z}_p(1)^\psi = 0$, we have $(\mathcal{K}_T)_0^{\omega\psi^{-1}} = \mathcal{K}_T^{\omega\psi^{-1}}$. Hence, we have an exact sequence

$$0 \longrightarrow \big(\mathrm{E}^2_\Lambda\big(\mathcal{K}_T\big)(1)\big)^\psi \longrightarrow \big(\mathrm{E}^2_\Lambda\big(\mathrm{H}^2_{\mathrm{Iw}}\big(K, \mathbb{Z}_p\big)\big)\big)^\psi \longrightarrow \big(\mathrm{E}^2_\Lambda\big(\mathcal{H}_\Sigma\big)(1)\big)^\psi \longrightarrow 0.$$

Taking ψ-eigenspaces of the terms of diagram (5.2) and applying the snake lemma, we obtain an exact sequence

$$0 \longrightarrow \mathfrak{X}_T^\psi \longrightarrow \operatorname{coker} \phi_T^\psi \longrightarrow \mathrm{E}^2_\Lambda\big(\mathcal{H}_\Sigma^{\omega\psi^{-1}}\big)(1) \longrightarrow 0.$$

As $\mathfrak{X}^\psi$ and $D_\mathfrak{p}^\psi$ for all $\mathfrak{p} \in T$ have Λ_ψ-rank one by Lemma 4.3.1, the argument is now just as before, the assumption on $\mathfrak{X}^\psi_{\Sigma_i}$ ensuring the injectivity of $\phi^\psi_{\mathfrak{p}_i}$. □

This yields the following statement on second Chern classes.

COROLLARY 5.3.2. *Let the notation and hypotheses be as in Theorem 5.3.1. Suppose in addition that $n = 2$ and $\mathfrak{X}^\psi_\Sigma$ is pseudo-null. Let f_i be a generator for the ideal $c_1((\mathfrak{X}_{\Sigma_i})^\psi)$ of Λ_ψ. Then the sum of second Chern classes*

$$c_2\big(\mathfrak{X}^\psi_\Sigma\big) + c_2\big(\big(\mathcal{K}_\Sigma^{\omega\psi^{-1}}\big)^\iota(1)\big) + c_2\big(\big((X')^{\omega\psi^{-1}}\big)^\iota(1)\big)$$

has a characteristic symbol with component at a codimension one prime P of Λ_ψ the Steinberg symbol $\{f_1, f_2\}$ if f_2 is not a unit at P, and trivial otherwise.

Proof. By the exact sequence (5.1) for $\mathcal{H}_\Sigma$, Lemma A.7, and the fact that $\mathbb{Z}_p^{\omega\psi^{-1}} = 0$, we have

$$c_2\big(\alpha\big(\mathcal{H}_\Sigma^{\omega\psi^{-1}}\big)(1)\big) = c_2\big(\alpha\big(\mathcal{K}_\Sigma^{\omega\psi^{-1}}\big)(1)\big) + c_2\big(\alpha\big((X')^{\omega\psi^{-1}}\big)(1)\big).$$

The result then follows from Theorem 5.3.1, as in the proof of Theorem 5.2.5. □

In the following, it is not necessary to suppose Leopoldt's conjecture, if one simply allows K to be the cyclotomic $\mathbb{Z}_p$-extension of F.

THEOREM 5.3.3. *Let E be a totally real field other than $\mathbb{Q}$, and let ψ be odd at exactly one real place of E. Assume that Leopoldt's conjecture holds, so $r = 1$. Furthermore, suppose that $\mathfrak{X}_{\Sigma_i}^{\psi}$ is Λ_ψ-torsion for all $i \in \{1, \dots, n\}$. If $(X')^{\omega\psi^{-1}}$ is finite, then there is an exact sequence of Ω-modules*

$$\left(\mathcal{K}_{\Sigma}^{\omega\psi^{-1}}\right)^{\iota}(1) \longrightarrow \mathfrak{X}_{\Sigma}^{\psi} \longrightarrow \frac{\Omega^{\psi}}{\sum_{i=1}^{n} c_1\left(\mathfrak{X}_{\Sigma_i}^{\psi}\right)} \longrightarrow \left((X')^{\omega\psi^{-1}}\right)^{\vee}(1) \longrightarrow 0.$$

Proof. The argument is much as before: we take the ψ-eigenspace of the terms of diagram (5.2) with $T = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. We have $\mathrm{E}_{\Lambda}^{2}(\mathcal{K}) = 0$ and $\mathrm{E}_{\Lambda}^{1}(\mathcal{K}) \cong \mathcal{K}^{\iota}$. The map $D_T \to D_T^{**}$ in the diagram can be replaced by $I_T \to I_T^{**}$, as in the proof of Theorem 5.1.1. Applying the snake lemma gives the stated sequence. □

Although this is somewhat less strong than our other results in general (when $\omega\psi^{-1}|_{\Delta_{\mathfrak{p}}} = 1$ for some $\mathfrak{p} \in \Sigma$), we have the following interesting corollary.

COROLLARY 5.3.4. *Suppose that E is real quadratic, p is split in E into two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and the character ψ is odd at exactly one place of E. If X^{ψ} and $(X')^{\omega\psi^{-1}}$ are finite, then there is an exact sequence of finite Ω-modules*

$$0 \longrightarrow X^{\psi} \longrightarrow \frac{\Omega^{\psi}}{c_1\left(\mathfrak{X}_{\mathfrak{p}_1}^{\psi}\right) + c_1\left(\mathfrak{X}_{\mathfrak{p}_2}^{\psi}\right)} \longrightarrow \left((X')^{\omega\psi^{-1}}\right)^{\vee}(1) \longrightarrow 0.$$

We can make this even more symmetric, replacing X by X' on the left, if we also replace $\mathfrak{X}_{\mathfrak{p}_i}^{\psi}$ by its maximal split-at-$\mathfrak{p}_{3-i}$ quotient for $i \in \{1, 2\}$, and supposing only that $(X')^{\psi}$ is finite. We of course have the corresponding statement on second Chern classes.

6. A non-commutative generalization. The study of non-commutative generalizations of the first Chern class main conjectures discussed in Section 3 has been very fruitful. See [7], for example, and its references. We now indicate briefly a non-commutative generalization of Theorems 5.2.1 and 5.2.5 concerning second Chern classes.

We make the same assumptions as in Subsection 5.2. Namely, E is imaginary quadratic, and p is an odd prime that splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in E. Let ψ be a one-dimensional p-adic character of the absolute Galois group of E of finite order prime to p with fixed field of its kernel E_ψ. Let $F = E_\psi(\mu_p)$. Let ω denote the Teichmüller character of $\Delta = \mathrm{Gal}(F/E)$. Let $\widetilde{E}$ denote the compositum of all $\mathbb{Z}_p$-extensions of E, and let K be the compositum of $\widetilde{E}$ with F. Let S be the set of primes of E above p and ∞, so $S_f = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$.

We suppose in addition that F is Galois over $\mathbb{Q}$. Let σ be a complex conjugation in $\mathrm{Gal}(F/\mathbb{Q})$, and let $H = \{e, \sigma\}$. Then $\widetilde{\Delta} = \mathrm{Gal}(F/\mathbb{Q})$ is a semi-direct product of the abelian group Δ with H. The group H acts on Δ and $\Gamma = \mathrm{Gal}(K/F) \cong \mathbb{Z}_p^2$ by conjugation. Let τ be the character of an n-dimensional irreducible p-adic

representation of $\widetilde{\Delta}$. Then $n \in \{1,2\}$. If $n=1$, then τ restricts to a one-dimensional character ψ of Δ. If $n=2$, then the representation corresponding to τ restricts to a direct sum of two one-dimensional representations ψ and $\psi\circ\sigma$ of Δ. So, the orbit of ψ under the action of σ has order n.

Let A_τ denote the direct factor of $\mathcal{O}_\psi \otimes_{\mathbb{Z}_p} \Omega = \mathcal{O}_\psi[\![\mathcal{G}]\!]$ obtained by applying the idempotent in $\mathcal{O}_\psi[\widetilde{\Delta}]$-attached to τ, where $\mathcal{O}_\psi$ is as before. Then $A_\tau = \Omega^\psi$ if $n=1$ and $A_\tau = \Omega^\psi \times \Omega^{\psi\circ\sigma}$ if $n=2$. The H-action on A_τ is compatible with the Ω-module structure and the action of H on Ω. Thus, A_τ is a module over the twisted group ring $B_\tau = A_\tau\langle H\rangle$, which itself is a direct factor of $\mathcal{O}_\psi[\![\mathrm{Gal}(K/\mathbb{Q})]\!]$.

The following non-commutative generalization of Theorem 5.2.1 follows from the compatibility with the H-action of the arguments used in the proof of said theorem.

PROPOSITION 6.1. *Suppose that $X^{\omega\psi^{-1}}$ is pseudo-null as a Λ_ψ-module. If $n=1$, then the sequence* (5.4) *for ψ is an exact sequence of modules for the non-commutative ring B_τ. If $n=2$, then the direct sum of the sequences* (5.4) *for ψ and $\psi\circ\sigma$ is an exact sequence of B_τ-modules.*

To generalize Theorem 5.2.5, we first extend the approach to Chern classes used in Subsection 2.1 to the context of non-commutative algebras which are finite over their centers. (For related work on non-commutative Chern classes, see [6].)

The twisted group algebra B_τ is a free rank four module over its center $Z_\tau = A_\tau^H$. Suppose that M is a finitely generated module for B_τ with support as a Z_τ-module of codimension at least 2. Let $Y = \mathrm{Spec}(Z_\tau)$, and let $Y^{(2)}$ be the set of codimension two primes in Y. The localization $M_y = (Z_\tau)_y \otimes_{Z_\tau} M$ of M at $y \in Y^{(2)}$ has finite length over the localization $(B_\tau)_y$. Let $k(y)$ be the residue field of y, and let $B_\tau(y) = k(y)\otimes_{Z_\tau} B_\tau$. Then $B_\tau(y)$ has dimension 4 as a $k(y)$-algebra. From a composition series for M_y as a $B_\tau(y)$-module, we can define a class $[M_y]$ in the Grothendieck group $\mathrm{K}_0'(B_\tau(y))$ of all finitely generated $B_\tau(y)$-modules. This leads to a second Chern class

$$c_{2,B_\tau}(M) = \sum_{y\in Y^{(2)}} \big[M_y\big]\cdot y \tag{6.1}$$

in the group

$$\mathrm{Z}^2\big(B_\tau\big) = \bigoplus_{y\in Y^{(2)}} \mathrm{K}_0'\big(B_\tau(y)\big).$$

For $y \in Y^{(2)}$, note that $A_\tau(y) = k(y)\otimes_{Z_\tau} A_\tau$ is a $k(y)$-algebra of dimension 2 with an action of H over $k(y)$, and $B_\tau(y)$ is the twisted group algebra $A_\tau(y)\langle H\rangle$. Moreover, we have

$$k(y)\otimes_{Z_\tau} Z_\tau[H] \cong k(y)[H],$$

and in this way, both $A_\tau(y)$ and $k(y)[H]$ are commutative $k(y)$-subalgebras of $B_\tau(y)$.

LEMMA 6.2. *For $y \in Y^{(2)}$, the forgetful functors on finitely generated module categories produced by restricting operators from $B_\tau(y)$ to either $A_\tau(y)$ or $k(y)[H]$ induce injections*

$$\mathrm{K}_0'\big(B_\tau(y)\big) \longrightarrow \mathrm{K}_0'\big(A_\tau(y)\big) = \mathbb{Z} \oplus \mathbb{Z} \qquad \text{if } n = 2, \text{ and} \tag{6.2}$$

$$\mathrm{K}_0'\big(B_\tau(y)\big) \longrightarrow \mathrm{K}_0'\big(k(y)[H]\big) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{if } n = 1. \tag{6.3}$$

Proof. Since H has order 2 and A_τ is a $\mathbb{Z}_p$-algebra for an odd prime p, the surjection $A_\tau \to A_\tau(y)$ gives a surjection $Z_\tau = A_\tau^H \to (A_\tau(y))^H$. Thus $k(y) = A_\tau(y)^H$. Therefore either $A_\tau(y)$ is a Galois étale H-algebra over $k(y)$ or $A_\tau(y)$ is isomorphic to the dual numbers $k(y)[\epsilon]/(\epsilon^2)$ in such a way that the generator σ for H sends ϵ to $-\epsilon$.

There is a homomorphism $\pi^*\colon K_0'(k(y)) \to K_0'(B_\tau(y))$ which sends a finitely generated $k(y)$-module M to the $B_\tau(y)$-module with underlying $A_\tau(y)$-module $A_\tau(y) \otimes_{k(y)} M$ and action of H induced by the action of H on $A_\tau(y)$.

Suppose to begin with that $A_\tau(y)$ is an étale algebra over $k(y)$, so $A_\tau(y)$ is a Galois étale $k(y)$-algebra with Galois group H. It is shown by descent in the paragraph just before [5, Lemma 8.4] that π^* is an isomorphism. The composition of π^* with the forgetful homomorphism $f_1\colon \mathrm{K}_0'(B_\tau(y)) \to \mathrm{K}_0'(A_\tau(y))$ is the homomorphism $h\colon K_0'(k(y)) \to K_0(A_\tau(y))$ induced by tensoring with $A_\tau(y)$ over $k(y)$. Since h is injective, f_1 is injective.

If $n = 2$, then H permutes the two algebra components of A_τ, and $A_\tau(y)$ is isomorphic to the étale $k(y)$-algebra $k(y) \times k(y)$, so we have shown (6.2).

Suppose now that $n = 1$ and that $A_\tau(y)$ is étale over $k(y)$. We have shown $A_\tau(y)$ is then a Galois étale $k(y)$-algebra with Galois group H. Thus $A_\tau(y)$ has two one-dimensional $k(y)$-eigenspaces under the action of H. We have also shown that the map $\pi^*\colon K_0'(k(y)) \to K_0'(B_\tau(y))$ induced by tensoring with $A_\tau(y)$ over $k(y)$ is an isomorphism. So from the definition of π^*, we see that restricting operators from $B_\tau(y)$ to $k(y)[H]$ gives an injection

$$K_0'\big(B_\tau(y)\big) \longrightarrow K_0\big(k(y)[H]\big) = \mathbb{Z} \oplus \mathbb{Z}$$

whose image is the diagonal embedding of $\mathbb{Z}$.

Finally, suppose $n = 1$ and that $A_\tau(y)$ is not étale over $k(y)$. We have shown that $A_\tau(y)$ is isomorphic in this case to the dual numbers $k(y)[\epsilon]/(\epsilon^2)$ in such a way that a generator for H sends ϵ to $-\epsilon$. All simple $B_\tau(y)$-modules are annihilated by ϵ. Since $B_\tau(y)/\epsilon B_\tau(y)$ is isomorphic to $k(y)[H]$, we find that the map (6.3) is injective. □

As in Theorem 5.2.5, we must take completed tensor products over $\mathcal{O}_\psi$ with the Witt vectors W over $\overline{\mathbb{F}}_p$. In what follows, we abuse notation and omit this W

from the notation of the completed tensor products. That is, from now on we let Z_τ denote $W \hat{\otimes}_{\mathcal{O}_\psi} Z_\tau$, and similarly with A_τ and B_τ. We then let $Y = \mathrm{Spec}(Z_\tau)$, and we use $k(y)$ to denote the residue field of Z_τ at $y \in Y$, and we define $A_\tau(y)$ and $B_\tau(y)$ as before. Note that the analogue of Lemma 6.2 holds for $y \in Y^{(2)}$, with $Y^{(2)}$ the subset of codimension 2 primes in Y.

We suppose for the remainder of this section that X^ψ and $X^{\omega\psi^{-1}}$ are pseudo-null as Λ_ψ-modules. In view of Proposition 6.1, we have by Theorem 5.2.5 the following identity among non-commutative second Chern classes

$$c_{2,B_\tau}\left(\bigoplus_{\chi\in T}\frac{\Omega_W^\chi}{\left(\mathcal{L}_{\mathfrak{p},\chi},\mathcal{L}_{\bar{\mathfrak{p}},\chi}\right)}\right)=c_{2,B_\tau}\left(\bigoplus_{\chi\in T}X_W^\chi\right)+c_{2,B_\tau}\left(\bigoplus_{\chi\in T}\left(X_W^{\omega\chi^{-1}}\right)^\iota(1)\right), \tag{6.4}$$

where T denotes the orbit of ψ (of order 1 or 2). In view of Lemma 6.2, to compute (6.4) in terms of p-adic L-functions, it suffices to compute the analogous abelian second Chern classes via L-functions when B_τ is replaced by A_τ and by $Z_\tau[H]$ and we view the latter two as quadratic algebras over Z_τ.

In the case that $n = 2$, a prime $y \in Y^{(2)}$ gives rise to one prime in each of the two factors of $A_\tau = \Omega_W^\psi \times \Omega_W^{\psi\circ\sigma}$ by projection. Note that we can identify Ω_W^ψ and $\Omega_W^{\psi\circ\sigma}$ with Λ_W so that Z_τ is identified with the diagonal in Λ_W^2, and these two primes of Λ_W are then equal. We have

$$\mathrm{K}_0'\big(A_\tau(y)\big) \cong \mathbb{Z}\oplus\mathbb{Z}, \tag{6.5}$$

the terms being K_0' of the residue fields of Ω_W^ψ and $\Omega_W^{\psi\circ\sigma}$ for y, respectively.

PROPOSITION 6.3. *If $n = 2$, then under the injective map*

$$\bigoplus_{y\in Y^{(2)}}\mathrm{K}_0'\big(B_\tau(y)\big)\longrightarrow\bigoplus_{y\in Y^{(2)}}(\mathbb{Z}\oplus\mathbb{Z})$$

induced by (6.2) *and* (6.5)*, the class in* (6.4) *is sent to an element with both components having a characteristic symbol which at $P \in Y^{(1)}$ is equal to the Steinberg symbol*

$$\left\{\mathcal{L}_{\mathfrak{p},\psi},\mathcal{L}_{\bar{\mathfrak{p}},\psi}\right\}\in\mathrm{K}_2\big(\mathrm{Frac}\left(\Lambda_W\right)\big)\quad \textit{if } \mathcal{L}_{\bar{\mathfrak{p}},\psi} \textit{ is not a unit at } P,$$

and is zero otherwise.

Proof. This is immediate from Theorem 5.2.5 in the first coordinate. The second coordinate is the same by Lemma 3.3.2(a) and the above identification of $\Omega_W^{\psi\circ\sigma}$ with Λ_W, recalling Remark 2.5.2. □

In the case that $n = 1$, so $\psi = \tau|_\Delta$, we have an algebra decomposition

$$Z_\tau[H] = Z_\tau^+ \times Z_\tau^- \tag{6.6}$$

with the summands corresponding to the trivial and nontrivial one-dimensional characters of H. These summands are isomorphic to Z_τ as Z_τ-algebras. We then have a decomposition

$$\mathrm{K}_0'\big(k(y)[H]\big) \cong \mathbb{Z} \oplus \mathbb{Z} \tag{6.7}$$

with the terms being K_0' of the residue fields of Z_τ^+ and Z_τ^-, respectively, for the images of y.

There are pro-generators $\gamma_1, \gamma_2 \in \mathrm{Gal}(K/F)$ such that $\sigma(\gamma_1) = \gamma_1$ and $\sigma(\gamma_2) = \gamma_2^{-1}$. The ring $A_\tau = \Omega^\psi$ is $Z_\tau[\lambda]$, where $\lambda = \gamma_2 - \gamma_2^{-1}$. Note that $\sigma(\lambda) = -\lambda$ and $\lambda^2 \in Z_\tau$. It follows that we have an isomorphism $Z_\tau[H] \xrightarrow{\sim} \Omega^\psi$ of $Z_\tau[H]$-modules taking 1 to $(1+\lambda)/2$ and σ to $(1-\lambda)/2$.

The element σ permutes the p-adic L-functions $\mathcal{L}_{\mathfrak{p},\psi}$ and $\mathcal{L}_{\bar{\mathfrak{p}},\psi}$. Define

$$\mathcal{L}_\tau^+ = \mathcal{L}_{\mathfrak{p},\psi} + \mathcal{L}_{\bar{\mathfrak{p}},\psi} \quad \text{and} \quad \mathcal{L}_\tau^- = \mathcal{L}_{\mathfrak{p},\psi} - \mathcal{L}_{\bar{\mathfrak{p}},\psi}.$$

PROPOSITION 6.4. *If $n = 1$, then under the injective map*

$$\bigoplus_{y \in Y^{(2)}} \mathrm{K}_0'\big(B_\tau(y)\big) \longrightarrow \bigoplus_{y \in Y^{(2)}} (\mathbb{Z} \oplus \mathbb{Z})$$

induced by (6.3) *and* (6.7)*, the class in* (6.4) *is sent to an element that in the first and second components, respectively, has a characteristic symbol which at $P \in Y^{(1)}$ is equal to*

$$\{\lambda\mathcal{L}_\tau^-, \mathcal{L}_\tau^+\} \in \mathrm{K}_2(\mathrm{Frac}(Z_\tau^+)) \quad \textit{if } \mathcal{L}_\tau^+ \textit{ is not a unit at } P,$$
$$\{\mathcal{L}_\tau^-, \lambda\mathcal{L}_\tau^+\} \in \mathrm{K}_2(\mathrm{Frac}(Z_\tau^-)) \quad \textit{if } \lambda\mathcal{L}_\tau^+ \textit{ is not a unit at } P,$$

and is zero otherwise.

Proof. The decomposition (6.6) and the isomorphism $\Omega^\psi \cong Z_\tau[H]$ induce an isomorphism

$$\frac{\Omega^\psi}{\big(\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi}\big)} \cong \frac{Z_\tau^+}{\big(\mathcal{L}_\tau^+, \lambda\mathcal{L}_\tau^-\big)} \oplus \frac{Z_\tau^-}{\big(\lambda\mathcal{L}_\tau^+, \mathcal{L}_\tau^-\big)}.$$

From the two summands on the right, together with Proposition 2.5.1, we arrive at the two components of the non-commutative second Chern class of (6.4), as in the statement of the proposition. □

Appendix A. Results on Ext-groups. In this appendix, we derive some facts about modules over power series rings. For our purposes, let $\mathcal{O}$ be the valuation ring of a finite extension of $\mathbb{Q}_p$. Let $\Gamma = \mathbb{Z}_p^r$ for some $r \geq 1$, and denote its standard topological generators by γ_i for $1 \leq i \leq r$. Set $\Lambda = \mathcal{O}[\![\Gamma]\!] = \mathcal{O}[\![t_1, \ldots, t_r]\!]$, where $t_i = \gamma_i - 1$.

As in Subsection 4.1, we use the following notation for a finitely generated Λ-module M. We set $\mathrm{E}^i_\Lambda(M) = \mathrm{Ext}^i_\Lambda(M,\Lambda)$, and we set $M^* = \mathrm{E}^0_\Lambda(M) = \mathrm{Hom}_\Lambda(M,\Lambda)$. Moreover, $M^\vee$ denotes the Pontryagin dual, and M_{tor} denotes the Λ-torsion submodule of M.

We will be particularly concerned with Λ-modules of large codimension, but we first recall a known result on much larger modules.

LEMMA A.1. *Let M be a Λ-module of rank one. Then M^{**} is free.*

Proof. The canonical map $(M/M_{\mathrm{tor}})^* \to M^*$ is an isomorphism, so we may assume that $M_{\mathrm{tor}} = 0$. We may then identify M with a nonzero ideal of Λ. The dual of a finitely generated module is reflexive, so we are reduced to showing that a reflexive ideal I of Λ is principal. For each height one ideal P of Λ, let π_P be a uniformizer of Λ_P, and let $n_P \geq 0$ be such that $\pi_P^{n_P}$ generates I_P. Let s be the finite product of the $\pi_P^{n_P}$. Then the principal ideal $J = s\Lambda$ is obviously reflexive and has the same localizations at height one primes as I. As I and J are reflexive, they are the intersections of their localizations at height one primes, so $I = J$. □

For a finitely generated Λ-module M, we have $\mathrm{E}^i_\Lambda(M) = 0$ for all $i > r+1$. Since Λ is Cohen-Macaulay (in fact, regular), the minimal $j = j(M)$ such that $\mathrm{E}^j_\Lambda(M) \neq 0$ is also the height of the annihilator of M. (We take $j = \infty$ for $M = 0$.) In particular, M is torsion (resp., pseudo-null) if $j \geq 1$ (resp., $j \geq 2$), and M is finite if $j = r+1$.

LEMMA A.2. *For $j \geq 0$, let $G_0(j)$ be the Grothendieck group of the category of finitely generated Λ-modules M with $j(M) \geq j$. The quotient of $G_0(j)$ by the image of the natural homomorphism $G_0(j+1) \to G_0(j)$ is generated by the classes of modules of the form Λ/P with P a prime ideal of height j.*

Proof. Suppose M is a finitely generated Λ-module with $j(M) \geq j$. The codimension of the support of M is then at least j, and the localization of M at every prime P of codimension j is of finite length over Λ_P. If P is in the support of M, then P is an associated prime of M by [33, (7.D) Thm. 9]. Hence there is an $m \in M$ such that $\Lambda \cdot m$ is isomorphic to Λ/P. Thus there is an exact sequence

$$0 \longrightarrow \Lambda/P \longrightarrow M \longrightarrow M' \longrightarrow 0$$

in which $j(M') \geq j$ and the sum $s(M')$ of the lengths of M' at codimension j primes of Λ is one less than $s(M)$. The lemma now follows by induction on $s(M)$. □

We also have the following:

LEMMA A.3. *Let $1 \leq d \leq r$, and let f_i for $1 \leq i \leq d$ be elements of Λ such that $(f_1, \ldots, f_d)$ has height d. Then $M = \Lambda/(f_1, \ldots, f_d)$ has no nonzero Λ-submodule N with $j(N) \geq d+1$.*

Proof. Since Λ is a Cohen-Macaulay local ring, we know from [33, Theorem 17.4(iii)] that the ideal $(f_1, \dots, f_d)$ has height d if and only if $f_1, \dots, f_d$ form a regular sequence in Λ. Then M is a Cohen-Macaulay module by [33, Theorem 17.3(ii)], and it has no embedded prime ideals by [33, Theorem 17.3(i)]. If M has a nonzero Λ-submodule N with $j(N) \geq d+1$, then a prime ideal of Λ of height strictly greater than d will be the annihilator of a nonzero element of M. This contradicts the fact that M has no embedded primes. □

Let $\mathcal{G}$ be a profinite group containing Γ as an open normal subgroup, and set $\Omega = \mathcal{O}[\![\mathcal{G}]\!]$. For a left (resp., right) Ω-module M, the groups $\mathrm{E}^i_\Lambda(M)$ have the structure of right (resp., left) Ω-modules (see [32, Proposition 2.1.2], for instance).

We will say that a finitely generated Ω-module M is small if $j(M) \geq r$ as a (finitely generated) Λ-module. We use the notation (finite) to denote an unspecified finite module occurring in an exact sequence, and the notation M_{fin} to denote the maximal finite Λ-submodule of M. Let $M^\dagger = (M \otimes_{\mathbb{Z}_p} \bigwedge^r \Gamma)^\vee$, which is isomorphic to $M^\vee$ if $\mathcal{G}$ is abelian.

We derive the following from the general study of Jannsen [24]. In [25, Lemma 5], a form of this is proven for modules finitely generated over $\mathbb{Z}_p$. Its part (b) gives an explicit description of the Iwasawa adjoint of a small Ω-module M in the case that Ω has sufficiently large center. We do not use this in the rest of the paper, but for comparison with the classical theory, the explicit description appears to be of interest.

PROPOSITION A.4. *Let M be a small (left) Ω-module.*

(a) *There exist canonical right Ω-module isomorphisms $\mathrm{E}^{r+1}_\Lambda(M) \cong M^\dagger_{\text{fin}}$, and these are natural in M.*

(b) *Given a non-unit $f \in \Lambda$ that is central in Ω and not contained in any height r prime ideal in the support of M, there exists a canonical right Ω-module homomorphism*

$$\mathrm{E}^r_\Lambda(M) \cong \varprojlim_n \left(M/f^n M\right)^\dagger,$$

the inverse limit taken with respect to maps $(M/f^{n+1}M)^\vee \to (M/f^n M)^\vee$ induced by multiplication by f. The maximal finite submodule of $\mathrm{E}^r_\Lambda(M)$ is zero.

Proof. For $i \geq 0$ and a locally compact Ω-module A, set

$$\mathrm{D}_i(A) = \varinjlim_U \mathrm{H}^i_{\text{cont}}(U, A)^\vee,$$

where the direct limit is with respect to duals of restriction maps over all open subgroups U of finite index in Γ. The group Γ is a duality group (see [40, Theorem

3.4.4]) of strict cohomological dimension r, and its dualizing module is the Ω-bimodule

$$\mathrm{D}_r\big(\mathbb{Z}_p\big) \cong \varinjlim_U \mathrm{Hom}_{\mathbb{Z}_p}\left(\Lambda^r U, \mathbb{Z}_p\right)^\vee \cong \bigwedge^r \Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p.$$

We have $\mathrm{D}_i(M^\vee) = 0$ for $i > r$ and, by duality, we have the first isomorphism in

$$\mathrm{D}_r\big(M^\vee\big) \cong \mathrm{Hom}_\Lambda\left(M^\vee, \mathrm{D}_r\big(\mathbb{Z}_p\big)\right) \cong \Big(\varinjlim_U M^U\Big) \otimes_{\mathbb{Z}_p} \bigwedge^r \Gamma.$$

By [24, Theorem 2.1], we then have canonical and natural isomorphisms

$$\mathrm{E}_\Lambda^{r+1}(M) \cong \big(\mathrm{D}_r\big(M^\vee\big)\big[p^\infty\big]\big)^\vee \cong \Big(\Big(\varinjlim_U M^U\Big)\big[p^\infty\big]\Big)^\dagger. \tag{A.1}$$

Moreover, by [24, Corollary 2.6b], we have that $\mathrm{E}_\Lambda^i(M) = 0$ for $i \neq r+1$ if M happens to be finite.

We claim that

$$\Big(\varinjlim_U M^U\Big)\big[p^\infty\big] = M_{\mathrm{fin}} \tag{A.2}$$

which will finish the proof of part (a). As Γ acts continuously on M, the left-hand side contains M_{fin}, so it suffices to show that $(\varinjlim_U M^U)[p^\infty]$ is finite. As M is compact, there exist an open subgroup V and $k \geq 1$ such that $M^V[p^k] = \varinjlim_U M^U[p^\infty]$. As $V \cong \Gamma$, it suffices to show that $M^\Gamma[p]$ is finite.

By Lemma A.2 and the right exactness of E_Λ^{r+1}, we are recursively reduced to considering M of the form Λ/P with P a prime ideal of height r. If $p \notin P$, then M has no p-power torsion and (A.2) is clear. If $p \in P$, then Λ/P is isomorphic to $\mathbb{F}_q[\![\overline{t}_1, \ldots, \overline{t}_r]\!]/P'$ for a prime ideal P' of height $r-1$ and some finite field $\mathbb{F}_q$ of characteristic p, where the $\overline{t}_i$ are the images of the $t_i = \gamma_i - 1$ for topological generators γ_i of Γ. The Γ-invariants of $(\Lambda/P)^\Gamma$ are annihilated by all $\overline{t}_i$. If this invariant group had a nonzero element, it would be annihilated by all $\overline{t}_i$. The primality of P' would then force P' to contain all $\overline{t}_i$. Since P' is not maximal, this proves the claim, and hence part (a).

Suppose we are given an element $f \in \Lambda$ which is not a unit in Λ and is central in Ω but is not in any prime ideal of codimension r in the support of M. As M/f^nM and $M[f^n]$ are supported in codimension $r+1$, these Λ-modules are finite. It follows that we have isomorphisms $\mathrm{E}_\Lambda^r(M/M[f^n]) \cong \mathrm{E}_\Lambda^r(M)$ and then exact sequences

$$0 \longrightarrow \mathrm{E}_\Lambda^r(M) \xrightarrow{f^n} \mathrm{E}_\Lambda^r(M) \longrightarrow \mathrm{E}_\Lambda^{r+1}\big(M/f^nM\big) \longrightarrow 0.$$

We write

$$\mathrm{E}^r_\Lambda(M) \cong \varprojlim_n \mathrm{E}^r_\Lambda(M)/f^n\mathrm{E}^r_\Lambda(M) \cong \varprojlim_n \mathrm{E}^{r+1}_\Lambda\big(M/f^nM\big) \cong \varprojlim_n (M/f^nM)^\dagger,$$

where multiplication by f induces the map $(M/f^{n+1}M)^\dagger \to (M/f^nM)^\dagger$, which is the twist by the inverse of $\bigwedge^r \Gamma$ of $M^\vee[f^{n+1}] \to M^\vee[f^n]$. It is clear from the latter description that $\mathrm{E}^r_\Lambda(M)$ can have no nonzero finite submodule (and for this, it suffices to prove the statement as a Λ-module, in which case the existence of f is guaranteed), so we have part (b). □

Remark A.5. A non-unit $f \in \Lambda$ as in Proposition A.4(b) always exists. That is, consider the finite set of height r prime ideals conjugate under $\mathcal{G}$ to a prime ideal in the support of M. The union of these primes is not the maximal ideal of Λ, so we may always find a non-unit $b \in \Lambda$ not contained in any prime in the set. The product of the distinct $\mathcal{G}$-conjugates of b is the desired f. Given a morphism $M \to N$ of small Ω-modules, we obtain a canonical morphism between the isomorphisms of Proposition A.4(b) for M and N by choosing f to be the same element for both modules.

For a small Ω-module M and an f as in Proposition A.4(b), the quotient M/fM is finite, M itself is finitely generated and torsion over $\mathbb{Z}_p[\![f]\!]$. The description of $\mathrm{E}^r_\Lambda(M)$ in Proposition A.4(b) then coincides (up to choice of a $\mathbb{Z}_p$-generator of $\bigwedge^r \Gamma$) with the usual definition of the Iwasawa adjoint as a $\mathbb{Z}_p[\![f]\!]$-module. In view of this, we make the following definition.

Definition A.6. The Iwasawa adjoint $\alpha(M)$ of a small Ω-module M is $\mathrm{E}^r_\Lambda(M)$.

We then have the following simple lemma (cf. [56, Proposition 15.29]).

LEMMA A.7. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of small* Ω*-modules. The long exact sequence of* Ext*-groups yields an exact sequence*

$$0 \longrightarrow \alpha\big(M_3\big) \longrightarrow \alpha\big(M_2\big) \longrightarrow \alpha\big(M_1\big) \longrightarrow (\text{finite})$$

of right Ω*-modules, where* (finite) *is the zero module if* $(M_3)_{\text{fin}} = 0$.

We recall the following consequence of Grothendieck duality [20, Chapter V], noting that Λ is its own dualizing module in that Λ is regular (and that Ω is a finitely generated, free Λ-module).

PROPOSITION A.8. *For a finitely generated* Ω*-module* M*, there is a convergent spectral sequence*

$$\mathrm{E}^p_\Lambda\big(\mathrm{E}^{r+1-q}_\Lambda(M)\big) \Longrightarrow M^{\delta_{p+q,r+1}}, \tag{A.3}$$

natural in M*, of right* Ω*-modules, where* $\delta_{i,j} = 1$ *if* $i = j$ *and* $\delta_{i,j} = 0$ *if* $i \neq j$. *Moreover,* $\mathrm{E}^i_\Lambda(\mathrm{E}^j_\Lambda(M)) = 0$ *for* $i < j$ *and for* $i > r+1$.

This implies the following:

COROLLARY A.9. *Let M be a finitely generated Ω-module.*

(a) *For $r = 1$, one has $\mathrm{E}^i_\Lambda(M^*) = 0$ for all $i \geq 1$. Hence, M^* is Λ-free for any M.*

(b) *For $r = 2$, one has $\mathrm{E}^2_\Lambda(M^*) = 0$ and $\mathrm{E}^1_\Lambda(M^*) \cong \mathrm{E}^3_\Lambda(\mathrm{E}^1_\Lambda(M))$, so $\mathrm{E}^1_\Lambda(M^*)$ is finite.*

(c) *If M is small, then there is an exact sequence of Ω-modules*

$$0 \longrightarrow \mathrm{E}^{r+1}_\Lambda\big(\mathrm{E}^{r+1}_\Lambda(M)\big) \longrightarrow M \longrightarrow \mathrm{E}^r_\Lambda\big(\mathrm{E}^r_\Lambda(M)\big) \longrightarrow 0.$$

That is, $\alpha(\alpha(M)) \cong M/M_{\mathrm{fin}}$ as Ω-modules.

For a left (resp., right) Ω-module M, we let M^ι denote the right (resp., left) Ω-module that is M as an $\mathcal{O}$-module and on which $g \in \mathcal{G}$ acts as g^{-1} does on M. The following is a consequence of the theory of Iwasawa adjoints for $r = 1$ (see [24, Lemma 3.1]), in which case Λ-small means Λ-torsion.

LEMMA A.10. *Let $d \geq 1$, and let f_i for $1 \leq i \leq d$ be elements of Λ such that $(f_1, \ldots, f_d)$ has height d. Set $M = \Lambda/(f_1, \ldots, f_d)$. Then $\mathrm{E}^i_\Lambda(M) \cong (M^\iota)^{\delta_{i,d}}$ for all $i \geq 0$.*

Proof. This is clearly true for $d = 0$. Let $d \geq 1$, and set $N = \Lambda/(f_1, \ldots, f_{d-1})$ so that $M = N/(f_d)$. The exact sequence

$$0 \longrightarrow N \xrightarrow{f_d} N \longrightarrow M \longrightarrow 0$$

that is a consequence of Lemma A.3 gives rise to a long exact sequence of Ext-groups. By induction on d, the only nonzero terms of that sequence form a short exact sequence

$$0 \longrightarrow N^\iota \xrightarrow{(f_d)^\iota} N^\iota \longrightarrow \mathrm{E}^d_\Lambda(M) \longrightarrow 0,$$

and the result follows. □

For more general Ω-modules, we can for instance prove the following.

PROPOSITION A.11. *Suppose that $\mathcal{G} \cong \Gamma \times \Delta$, where Δ is abelian of order prime to p. Let M be a small Ω-module. Then $(M/M_{\mathrm{fin}})^\iota$ and $\alpha(M)$ have the same class in the quotient of the Grothendieck group of the category of small right Ω-modules by the image of the Grothendieck group of the category of finite right Ω-modules. In particular, as Λ-modules, their rth localized Chern classes agree.*

Proof. By taking Δ-eigenspaces of M (passing to a coefficient ring containing $|\Delta|$th roots of unity), we can reduce to the case that Δ is trivial. It then suffices by Lemmas A.2 and A.7 to show the first statement for $M = \Lambda/P$, where P is a height

r prime. Let $\iota\colon \Lambda \to \Lambda$ be the involution determined by inversion of group elements. We can compute $\alpha(M) = \mathrm{E}^r_\Lambda(M) = \mathrm{Ext}^r_\Lambda(M,\Lambda)$ by an injective resolution of Λ by Λ-modules. Every group in the resulting complex of homomorphism groups will be killed by $\iota(P)$, so $\alpha(M)$ will be annihilated by $\iota(P)$. Clearly, $\iota(P)$ is the only codimension r prime possibly in the support of $\mathrm{E}^r_\Lambda(M)$, and it is in the support since $\alpha(\alpha(M)) \cong M/M_{\mathrm{fin}}$ by Corollary A.9(c). □

The following particular computation is of interest to us. Let $\mathcal{G}'$ be a closed subgroup of $\mathcal{G}$, and let M be a finitely generated left $\Omega' = \mathbb{Z}_p[\![\mathcal{G}']\!]$-module. Set $\Gamma' = \mathcal{G}' \cap \Gamma$, and let $\Lambda' = \mathbb{Z}_p[\![\Gamma']\!]$. For $i = 0$ we have a right action of $g \in \Gamma'$ on $f \in \mathrm{E}^0_{\Lambda'}(M) = \mathrm{Hom}_{\Lambda'}(M,\Lambda')$ given by setting $(fg)(m) = f(m)g$ and a left action of g on f given by $(gf)(m) = f(m)g^{-1}$. This extends functorially to right and left actions of Γ' on $\mathrm{E}^i_{\Lambda'}(M)$ for all i.

LEMMA A.12. *With the above notation, we have for all $i \geq 0$ an isomorphism of right Ω-modules*

$$\mathrm{E}^i_\Lambda\big(\Omega \otimes_{\Omega'} M\big) \cong \mathrm{E}^i_{\Lambda'}(M) \otimes_{\Omega'} \Omega \tag{A.4}$$

and an isomorphism of left Ω-modules

$$\mathrm{E}^i_\Lambda\big(\Omega \otimes_{\Omega'} M\big) \cong \Omega^\iota \otimes_{\Omega'} \mathrm{E}^i_{\Lambda'}(M). \tag{A.5}$$

Proof. Let us first show that Ω is flat over Ω'. This follows, for instance, from [32, Lemma 2.4.3(a)], since Ω is a free profinite Ω'-module, i.e., a topological direct product of copies of Ω', on a set right coset representatives for $\mathcal{G}'$ in $\mathcal{G}$. By [32, Lemma 2.1.6] and [32, Lemma 2.1.7] there are isomorphisms of right Ω-modules

$$\mathrm{Ext}^i_\Omega\big(\Omega \otimes_{\Omega'} M, \Omega\big) \cong \mathrm{Ext}^i_{\Omega'}(M,\Omega) \cong \mathrm{Ext}^i_{\Omega'}(M,\Omega') \otimes_{\Omega'} \Omega.$$

The isomorphism (A.4) follows, as the left term is $\mathrm{E}^i_\Lambda(\Omega \otimes_{\Omega'} M)$ and the right term is $\mathrm{E}^i_{\Lambda'}(M) \otimes_{\Omega'} \Omega$. The isomorphism (A.5) follows from (A.4). □

COROLLARY A.13. *Let $\mathcal{G}'$ be a closed normal subgroup of $\mathcal{G}$, and let N denote the left Ω-module $\mathbb{Z}_p[\![\mathcal{G}/\mathcal{G}']\!]$. Let $r' = \mathrm{rank}_{\mathbb{Z}_p}(\mathcal{G}' \cap \Gamma)$. Then $\mathrm{E}^i_\Lambda(N) \cong (N^\iota)^{\delta_{i,r'}}$ as right Ω-modules.*

Proof. Let $\Omega' = \mathbb{Z}_p[\![\mathcal{G}']\!]$. Note that $N \cong \Omega \otimes_{\Omega'} \mathbb{Z}_p$, so $N^\iota \cong \mathbb{Z}_p \otimes_{\Omega'} \Omega$ as right Ω-modules. By Lemmas A.12 and A.4, we have

$$\mathrm{E}^i_\Lambda(N) \cong \mathrm{E}^i_{\Lambda'}\big(\mathbb{Z}_p\big) \otimes_{\Omega'} \Omega \cong \big(\mathbb{Z}_p \otimes_{\Omega'} \Omega\big)^{\delta_{i,r'}}.$$ □

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