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Srinivasa Ramanujan and signal-processing problems

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The Ramanujan sum $c_q(n)$ has been used by mathematicians to derive many important infinite series expansions for arithmetic-functions in number theory. Interestingly, this sum has many properties which are attractive from the point of view of digital signal processing. One of these is that $c_q(n)$ is periodic with period q, and another is that it is always integer-valued in spite of the presence of complex roots of unity in the definition. Engineers and physicists have in the past used the Ramanujan-sum to extract periodicity information from signals. In recent years, this idea has been developed further by introducing the concept of Ramanujan-subspaces. Based on this, Ramanujan dictionaries and filter banks have been developed, which are very useful to identify integer-valued periods in possibly complex-valued signals. This paper gives an overview of these developments from the view point of signal processing.

This article is part of a discussion meeting issue 'Srinivasa Ramanujan: in celebration of the centenary of his election as FRS'.

1. Introduction

In 1918, Srinivasa Ramanujan introduced a summation, known today as the Ramanujan sum [1]. The qth Ramanujan sum is defined by

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn}, \tag{1.1}$$

where $W_q = \mathrm{e}^{-\mathrm{j}2\pi/q}$ and (k, q) denotes the gcd of k and q. So the sum runs over those k that are *coprime* to q. For example, if $q = 10$, then $k \in \{1, 3, 7, 9\}$ so that

$$c_{10}(n) = \mathrm{e}^{\mathrm{j}2\pi n/10} + \mathrm{e}^{\mathrm{j}6\pi n/10} + \mathrm{e}^{\mathrm{j}14\pi n/10} + \mathrm{e}^{\mathrm{j}18\pi n/10}.$$

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Ramanujan used this sum to derive many important infinite series expansions for arithmetic functions in number theory [2–4].

Many decades later, some scientists and engineers began to recognize the value of Ramanujan-sums in signal processing [5–9]. It was found useful in identifying periodicities in signals, and extracting periodic components. The relevance of such ideas was also noticed in the physics community [10,11]. However, there were only some limited activities in this direction, and deeper potentialities of the Ramanujan sum for signal representation largely remained unexplored. More recently *Ramanujan subspaces* were introduced in [12,13], which gave rise to a new representation for periodic signals. From this, it was possible to develop new dictionaries called *Ramanujan periodicity dictionaries* to represent and extract periodic components [14].

Ramanujan-sum based methods for identifying periodicities work differently from traditional Fourier based ones, and offer advantages when the data length is short, and multiple hidden periodicites have to be identified, especially when the periods are integer-valued (although the signals themselves might have non-integer, even complex, values). Furthermore, when periodicities change with time, one can track the changes by generating a time-period plane plot using a so-called Ramanujan filter bank.

(a) Paper outline

In this paper, we present an overview of some of these recent results. The main goal is to provide a lucid introduction so that the reader may be inspired to study the references in depth and go deeper into the proofs and details. These details can be found in the papers cited in the various sections, and in the website 'Ramanujan Periodicity Project' maintained by the authors, which can be accessed at (http://systems.caltech.edu/dsp/students/srikanth/Ramanujan/, The Ramanujan Periodicity Project). Section 2 describes the basic limitations of the discrete Fourier transform (DFT) in the representation of signals with integer periodicities. Section 3 introduces Ramanujan subspaces and their role in representing periodic signals. Ramanujan dictionaries, which offer a practical method to find such representations, are introduced in §4. Section 5 discusses Ramanujan filter banks which are very useful to analyse periodic signals whose period is localized and varies with time. Some further developments are described in §6, and §7 concludes the paper.

(b) Notations

The following notations are used throughout.

— Discrete-time signals are denoted by notations such as $x(n)$ where n is an integer representing integer time. One can merely regard these as sequences indexed by n.

— The quantity W_q is defined as

$$W_q = \mathrm{e}^{-\mathrm{j}2\pi/q}, \tag{1.2}$$

where $\mathrm{j} = \sqrt{-1}$. (In keeping with electrical engineering tradition, we use j instead of i.) W_q is a root of unity, since $W_q^q = 1$. The subscript q in W_q is sometimes deleted if it is clear from the context.

— $\mathbb{C}^q$ is the q-dimensional space of complex vectors. For a matrix $\mathbf{A}$, the transpose, conjugate and transpose-conjugate are denoted as $\mathbf{A}^T, \mathbf{A}^*$ and $\mathbf{A}^\dagger$, respectively.

— The abbreviations *lcm* and *gcd* stand for least common multiple and greatest common divisor, respectively.

— The notation (k, q) represents the gcd of the integers k and q. So $(k, q) = 1$ means that k and q are coprime.

— The quantity $\phi(q)$ is the *Euler's totient function* [3]. It is equal to the number of integers k in $1 \leq k \leq q$ satisfying $(k, q) = 1$. For example if $q = 10$, then the integers $1, 3, 7$ and 9 are coprime to q, so $\phi(10) = 4$.

— The notation $q_i|q$ means that q_i is a divisor of q (i.e. a factor of q). For example, if $q = 10$, the divisors are $q_1 = 1, q_2 = 2, q_3 = 5$ and $q_4 = 10$. A *proper* divisor of q is a divisor of q less than q.
— $\mathcal{S}_q$ denotes the Ramanujan subspace associated with the integer q (defined in §3a).

(c) Preliminaries

A discrete-time signal $x(n)$ is periodic if there exists an integer R such that

$$x(n) = x(n + R) \tag{1.3}$$

for all n, and the integer R is called a repetition interval. The period P (an integer) is the *smallest* positive repetition interval [15]. It can be shown that any repetition interval is an integer multiple of P (i.e. P is a divisor of any repetition interval). If $x_1(n)$ and $x_2(n)$ have periods P_1 and P_2, their sum $x_1(n) + x_2(n)$ has a repetition interval $R = \text{lcm}\,(P_1, P_2)$, so its period P is either this lcm or a *proper divisor* of it. Next, we mention some important properties of Ramanujan sums. More are summarized in [12].

(i) Since $(k, q) = 1$, each term in (1.1) has period exactly q (it cannot be smaller, as there are no uncancelled factors between k and q). Most importantly, it can be shown, though not obvious [12], that the Ramanujan sum (1.1) has *period exactly equal to* q (and not a proper divisor of q).

(ii) Secondly, $c_q(n)$ is real and *integer valued*, inspite of the complex exponentials in the definition [1,12]. For example

$$c_{10}(n) = \{4, 1, -1, 1, -1, -4, -1, 1, -1, 1\} \tag{1.4}$$

in the first period $0 \leq n \leq 9$. The integer property is very useful in implementing $c_q(n)$ as a digital filter [15,16]. Note that $|c_q(n)| \leq \phi(q)$ from (1.1).

(iii) Thirdly, the periodic functions $c_{q_1}(n)$ and $c_{q_2}(n)$ are *orthogonal* for $q_1 \neq q_2$, that is, $\sum_{n=0}^{l-1} c_{q_1}(n) c_{q_2}(n) = 0$, where l is any common multiple (e.g. lcm) of q_1 and q_2.

2. Fourier methods and integer periodicities

Assume we are given N samples of a discrete-time signal $x(n)$, namely $x(n), 0 \leq n \leq N - 1$. Let

$$X[k] = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leq k \leq N - 1 \tag{2.1}$$

be the N-point DFT [15] of these N samples of $x(n)$. Then the DFT representation of $x(n), 0 \leq n \leq N - 1$ is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi nk/N}. \tag{2.2}$$

If $x(n)$ has period $P < N$, can we always find P by examining a plot of $|X[k]|$? The answer depends on the relation between P and N.

If P is a proper divisor of N, that is $N = PK$ for some integer $K > 1$, then it can be shown that only terms of the form $e^{j2\pi ln/P}$, $l =$ integer, can be non-zero in (2.2). Since

$$e^{j2\pi ln/P} = e^{j2\pi Kln/N}, \tag{2.3}$$

this means that only the DFT coefficients $X[Kl]$, i.e. $X[(N/P)l]$, are possibly non-zero. For example, if $N = 32$ and $P = 8$, then only $X[4l]$ are non-zero as demonstrated in figure 1*a*. Thus, by examining this systematic sparse pattern we can identify the period P. On the other hand, if P *is not a divisor of* N, this is not so easy, as the DFT becomes non-sparse. Figure 1*b* shows this for $N = 32$, $P = 9$. In this case, it is not possible to estimate the period from examining the plot of $|X[k]|$. If the DFT has

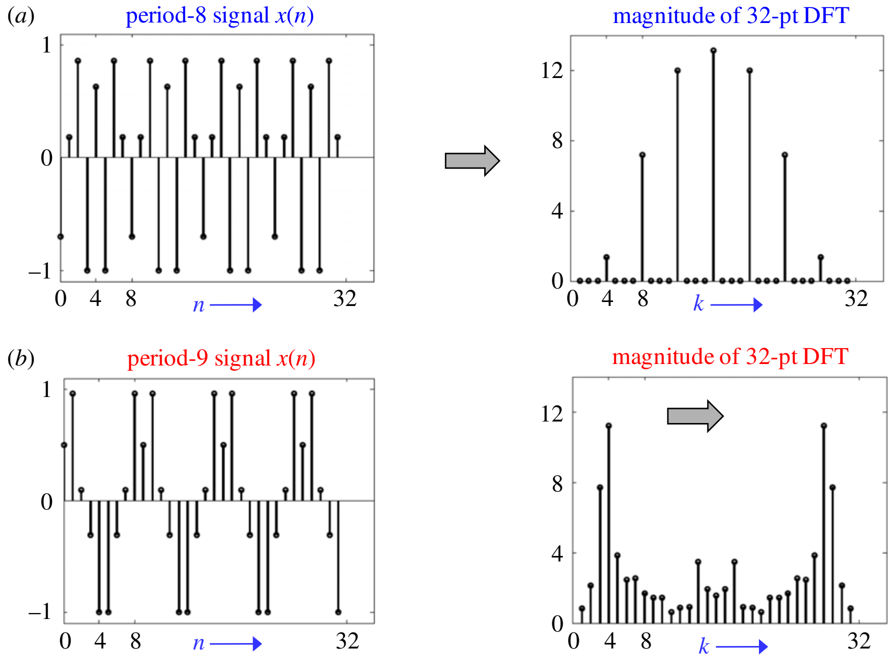

Figure 1. (*a*) A period-8 signal and (*b*) a period-9 signal. The 32-point DFT magnitudes are also shown. In (*a*), the period can be inferred from the sparse structure of the DFT plot because 8 is a divisor of 32, but this is not so in part (*b*) [13].

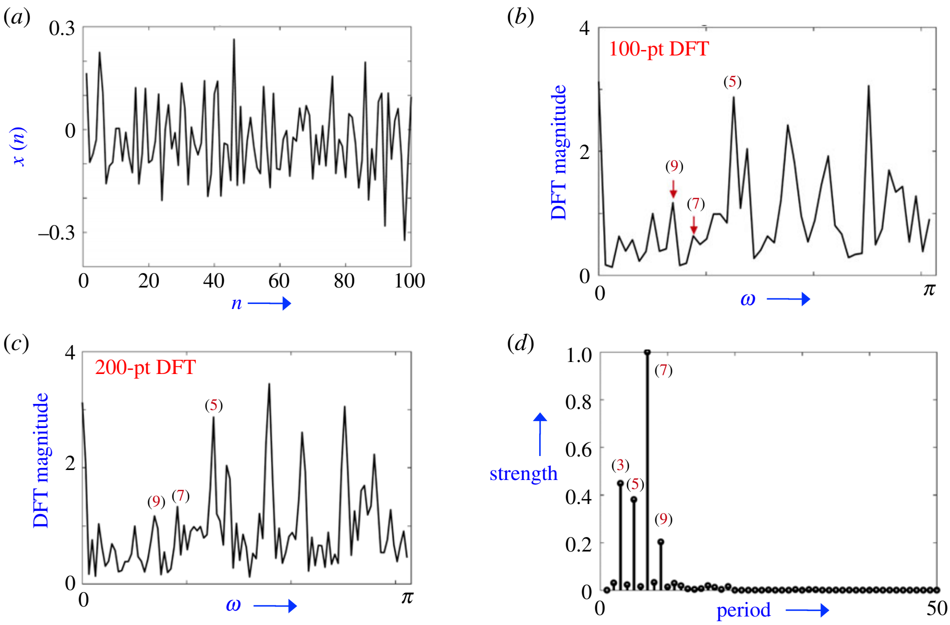

Figure 2. (*a*) A short segment ($N = 100$ samples) of a sum of three periodic signals, with hidden periods 5, 7 and 9, (*b*) magnitude of the 100-point DFT, (*c*) magnitude of the 200-point DFT obtained using zero-padding and (*d*) the strength versus period plot using the Ramanujan dictionary. (Online version in colour.)

a peak at $k = k_0$ and multiples thereof, we can estimate the period to be 'approximately' N/k_0 but this estimate is in general not an integer. For small N especially, peaks in the DFT are not reliable indicators of integer periodicities in $x(n)$.

The problem becomes even more complicated if $x(n)$ is a *sum of periodic components.* Multiple hidden periodicities produce multiple sets of peaks in the DFT with overlaps, and can completely fill the DFT domain, so that no obvious patterns are discernible. Say

$$x(n) = x_1(n) + x_2(n) + x_3(n), \tag{2.4}$$

where the components $x_i(n)$ have periods $P_i < N$. Then $x(n)$ has period P equal to either $\text{lcm}(P_1, P_2, P_3)$, or a proper divisor of this lcm. If $P > N$, then $x(n)$ does not even 'look' periodic in $0 \le n \le N-1$. This is demonstrated in figure 2*a* for $P_1 = 5$, $P_2 = 7$ and $P_3 = 9$, with $N = 100$. It is hard to judge the hidden periods P_i by examining the peaks in the plot of the 100-point DFT (figure 2*b*). Computing the value of $2\pi/\omega$ at the peaks we find that some of them correspond to periods $5, 7$ and 9 as indicated, but the peak corresponding to period 7 is not sharp. We can use zero-padding to compute a 200-point DFT, which reveals period 7 more clearly (part (*c*)), but there are many new peaks, making it hard to find true periods. We will see, however, that Ramanujan subspace methods can identify the periods more easily, and perfectly. Thus, for the same signal $x(n)$, figure 2*d* shows the so-called strength-versus-period plot, obtained using Ramanujan subspace dictionaries [14]. The hidden periods are clearly seen. The extra peak at 3 merely indicates a harmonic in the period-9 component. The non-zero background is due to small noise added to the periodic signal in the experiement (5 dB SNR).

3. Ramanujan subspace representations

We now review the fundamental principles behind Ramanujan-subspace representation of signals [12–14]. First observe that the N basis functions in the DFT representation are

$$x_k^{dft}(n) = W_N^{kn}, \quad 0 \le k \le N-1, \tag{3.1}$$

where the data length is N. These basis functions are periodic signals with periods equal to divisors of N. For example if $N = 10$, then only the four periods $1, 2, 5$ and 10 are present in the 10 basis functions. The periods $3, 4, 6, 7, 8, 9$ are not represented in the basis. Instead of the DFT, suppose we develop a redundant representation (a frame rather than basis) with possibly many more 'atoms' than the data length:[1]

$$x(n) = \sum_{q=1}^{Q} \sum_{k=0}^{q-1} a_{qk} W_q^{kn}. \tag{3.2}$$

Here, the atoms W_q^{kn} include functions that have *all the periodicities* in the range $1 \le q \le Q < N$. Thus, there are basis vectors available to represent period q whether q is a divisor of data length N or not. Note that for every $[k, q]$ pair, we can write

$$W_q^k = \mathrm{e}^{-\mathrm{j}2\pi k/q} = \mathrm{e}^{-\mathrm{j}2\pi \widehat{k}/\widehat{q}}, \tag{3.3}$$

where $(\widehat{k}, \widehat{q}) = 1$. So there are many repetitions in the set of all W_q^k in (3.2). If all repetitions are removed, a minimal set of atoms needed to represent all periods in $1 \le q \le Q$ will emerge. This includes only those W_q^{kn} for which k and q are coprime. So (3.2) can be replaced with [17]

$$x(n) = \sum_{q=1}^{Q} \underbrace{\sum_{\substack{1 \le k \le q \\ (k,q)=1}} b_{qk} W_q^{kn}}_{\text{call this } x_q(n)} = \sum_{q=1}^{Q} \sum_{\substack{1 \le k \le q \\ (k,q)=1}} b_{qk}\, \mathrm{e}^{-\mathrm{j}2\pi kn/q}. \tag{3.4}$$

Now we are ready to explain how the Ramanujan subspace comes about.

[1]It is more convenient to use W_q^{kn} instead of its conjugate W_q^{-kn} as in (2.2). This does not affect any conclusions.

(a) Ramanujan subspace, Farey series and lcm property

The Ramanujan subspace [12], denoted as $\mathcal{S}_q$, is defined as the span of

$$\{W_q^{-kn} = \mathrm{e}^{\mathrm{j}2\pi(k/q)n}\}, \quad \text{where } 1 \le k \le q,\ (k,q) = 1. \tag{3.5}$$

Equivalently, we can take it to be the span of $\{W_q^{kn} = \mathrm{e}^{-\mathrm{j}2\pi(k/q)n}\}$. Thus, the signal $x_q(n)$ defined in equation (3.4) belongs to $\mathcal{S}_q$. The space $\mathcal{S}_q$ has dimension $\phi(q)$ (Euler totient). Now let us write (3.4) in the simpler form

$$x(n) = \sum_{q=1}^{Q} x_q(n), \quad x_q(n) \in \mathcal{S}_q. \tag{3.6}$$

Here, Q is the maximum value of the period to be represented. The following properties of $\mathcal{S}_q$ are proved in [12].

(i) Any non-zero signal in $\mathcal{S}_q$ has period exactly q (i.e. it cannot be a proper divisor of q).
(ii) The spaces $\mathcal{S}_{q_i}$ and $\mathcal{S}_{q_j}$ are orthogonal for $q_i \neq q_j$, over any interval that is a common multiple of q_i and q_j.

So (3.6) *is a decomposition of* $x(n)$ *in terms of an orthogonal set of signals where the component* $x_q(n)$ *has period* q *and belongs to the* $\phi(q)$*-dimensional Ramanujan subspace* $\mathcal{S}_q$. But since $\mathcal{S}_q$ only has dimension $\phi(q)$, it does not include all period-q signals. In fact, an arbitrary period-q signal can be decomposed into a sum of components in $\mathcal{S}_{q_i}$ where q_i are divisors of q [13].[2] The beauty of the decomposition (3.6) is the so-called lcm property [13]. Namely, once we have obtained a unique decomposition (3.6), suppose the non-zero terms are

$$x_{q_1}(n), x_{q_2}(n), \ldots, x_{q_K}(n). \tag{3.7}$$

Then the period P of $x(n)$ is given precisely by

$$P = \mathrm{lcm}(q_1, q_2, \ldots, q_K). \tag{3.8}$$

It *cannot* be smaller, that is it cannot be a *proper* divisor of this lcm, as proved in [13].

Note that the set of rational numbers k/q in the frequencies $2\pi k/q$ represented in (3.4) includes *all rationals* of the form

$$\frac{k}{q}, \quad 1 \le k \le q,\ 1 \le q \le Q. \tag{3.9}$$

The restriction $(k,q) = 1$ merely eliminates duplicate entries. The sequence of numbers in (3.9) is called a *Farey series* and has interesting applications in number theory [3]. The total number of terms in (3.4) will be denoted as $\Phi(Q)$. Thus

$$\Phi(Q) = \sum_{q=1}^{Q} \phi(q) = \frac{3Q^2}{\pi^2} + O(Q\log Q) \approx 3Q^2/\pi^2. \tag{3.10}$$

The closed form expression on the right is well known [3]. Thus there are $O(Q^2)$ terms in (3.4) when compared with the DFT representation which has N terms.

(b) Relation to Ramanujan sums

Recall that the Ramanujan sum is defined as in (1.1). Clearly, *the Ramanujan sum* $c_q(n)$ *belongs to the Ramanujan space* $\mathcal{S}_q$. In fact, $c_q(n)$ can be used to generate a real, integer, basis for the Ramanujan

[2]The elements of $\mathcal{S}_q$ can either be regarded as length-q vectors or as infinite length vectors with elements repeating with period q, depending on what is more convenient in a particular context.

space $\mathcal{S}_q$. Thus, it can be shown [12] that the $\phi(q)$ successively shifted signals

$$c_q(n), c_q(n-1), \ldots, c_q(n-\phi(q)+1) \tag{3.11}$$

are linearly independent and form a basis for $\mathcal{S}_q$. Since $c_q(n)$ has period q, the sequence $c_q(n-k)$ is a circularly right-shifted version of $c_q(n)$, the amount of shift being k samples. For example, if $q=10$ then $\phi(q)=4$ and the four sequences which form the basis for $\mathcal{S}_{10}$ are

$$\left.\begin{aligned} c_{10}(n) &\equiv \{4,1,-1,1,-1,-4,-1,1,-1,1\},\\ c_{10}(n-1) &\equiv \{1,4,1,-1,1,-1,-4,-1,1,-1\},\\ c_{10}(n-2) &\equiv \{-1,1,4,1,-1,1,-1,-4,-1,1\}\\ \text{and}\quad c_{10}(n-3) &\equiv \{1,-1,1,4,1,-1,1,-1,-4,-1\}, \end{aligned}\right\} \tag{3.12}$$

where the numerical values are shown for the region $0 \le n \le 9$. These form a basis for $\mathcal{S}_{10}$. Since (3.11) is also a basis for $\mathcal{S}_q$, the representation (3.4) has equivalent form

$$x(n) = \sum_{q=1}^{Q} \sum_{m=0}^{\phi(q)-1} d_{qm} c_q(n-m). \tag{3.13}$$

The total number of terms in either of the representations (3.4) or (3.13) is $\Phi(Q)$ (equation (3.10)). The complex basis (3.5) is orthonormal, but the real integer basis (3.11) is not, unless q is a power of two [12]. Earlier methods which developed periodicity transforms include the pioneering work by Sethares & Staley [18] and the work by Muresan & Parks [19]. The former is unrelated to Ramanujan subspaces, while the latter can be shown to be closely related. Further details can be found in [20]. Other interesting early work (again, not related to Ramanujan sums) includes [21].

(c) Identifying multiple hidden periods

In figure 2*a*, we showed a segment of a signal which is a sum of three periodic signals, with periods 5, 7 and 9. We mentioned that 5, 7 and 9 are hidden periods, but how does one exactly *define* hidden periods? Some reflection will show that it is not trivial. For example, we can always add and subtract an arbitrary periodic signal $p_1(n)$ to $x(n)$ and claim that $p_1(n)$ and $x(n)-p_1(n)$ are the periodic components.

To avoid such degeneracies, we compute the non-zero orthogonal periodic components $x_{q_i}(n)$ belonging to the Ramanujan subspaces $\mathcal{S}_{q_i}$, in the representation (3.6). It is tempting to regard the set $\{q_i\}$ as the hidden periods of $x(n)$. However, note that if q_k is a proper divisor of q_i, then $x_{q_i}(n)+x_{q_k}(n)$ has period q_i, so the smaller number q_k can be dropped. (Think of q_k as a harmonic of the fundamental period q_i.) So, we simply extract a subset of integers $\{p_i\}$ from $\{q_i\}$ such that p_i is not a proper divisor of p_j if $j \neq i$. *This smaller set* $\{p_i\}$ *is considered to be the set of hidden periods of* $x(n)$. Summarizing, we can write $x(n)$ in two ways:

$$x(n) = \sum_{i=1}^{K} x_{q_i}(n) = \sum_{i=1}^{L} y_{p_i}(n), \tag{3.14}$$

where the set $\{p_i\}$ is a subset of the set $\{q_i\}$ (so $L \le K$), such that p_i is not a proper divisor of p_j for distinct i, j. Here, $x_{q_i}(n) \in \mathcal{S}_{q_i}$ and has period q_i. While $y_{p_i}(n)$ also has period p_i, it may not be in $\mathcal{S}_{p_i}$. The lcm of q_i-s reveals the period of $x(n)$ whereas the set $\{p_i\}$ is the set of *all hidden periods* in $x(n)$. For example if $\{q_i\}=\{3,4,12,16\}$, then $\{p_i\}=\{12,16\}$, so the period of $x(n)$ is 48 and the hidden periods are 12 and 16. So we can rewrite

$$x(n) = x_3(n) + x_4(n) + x_{12}(n) + x_{16}(n) = y_{12}(n) + y_{16}(n). \tag{3.15}$$

Note that even though the $x_{q_i}(n)$-s above are unique (namely, projections of $x(n)$ onto $\mathcal{S}_{q_i}$), the signals $y_{p_i}(n)$ are not: the harmonics that are common to both 12 and 16 can be divided between these two terms arbitrarily. Thus, since 4 is a divisor of both 12 and 16, we can have $\alpha x_4(n)$

included in $y_{12}(n)$, and $(1-\alpha)x_4(n)$ included in $y_{16}(n)$, where α is arbitrary. In short, *hidden periods* p_i, defined according to the above procedure, always yield unique answers, but the *hidden periodic components* y_{p_i} *are not unique;* this ambiguity is called *harmonic ambiguity*.

4. Dictionaries for period identification

With the insights gained from Ramanujan-space representations, we now explain how the hidden periods can be estimated. In practice, we only have a finite number N of samples of $x(n)$. In this case, it is convenient to rewrite the representation (3.4) as follows:

$$\mathbf{x} = \underbrace{\mathbf{A}}_{N\times\Phi(Q)}\mathbf{s}. \tag{4.1}$$

Here $\mathbf{x} = [x(0)\ x(1)\ \cdots\ x(N-1)]^{\mathrm{T}}$ and the vector $\mathbf{s} \in \mathbb{C}^{\Phi(Q)}$ contains the coefficients b_{qk} of (3.4) in some order. $\Phi(Q)$ is the number of coefficients b_{qk} in (3.4), as given by equation (3.10). The elements of the matrix $\mathbf{A}$ are W_q^{kn}. For example, if $N = Q = 6$, the matrix is as shown below

$$\begin{array}{cc} q\rightarrow & \begin{array}{cccccccccccc} 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 \end{array} \\ \mathbf{A} = & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_2 & W_3 & W_3^2 & W_4 & W_4^3 & W_5 & W_5^2 & W_5^3 & W_5^4 & W_6 & W_6^5 \\ 1 & W_2^2 & W_3^2 & W_3^4 & W_4^2 & W_4^6 & W_5^2 & W_5^4 & W_5^6 & W_5^8 & W_6^2 & W_6^{10} \\ 1 & W_2^3 & W_3^3 & W_3^6 & W_4^3 & W_4^9 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} & W_6^3 & W_6^{15} \\ 1 & W_2^4 & W_3^4 & W_3^8 & W_4^4 & W_4^{12} & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} & W_6^4 & W_6^{20} \\ 1 & W_2^5 & W_3^5 & W_3^{10} & W_4^5 & W_4^{15} & W_5^5 & W_5^{10} & W_5^{15} & W_5^{20} & W_6^5 & W_6^{25} \end{pmatrix}. \end{array} \tag{4.2}$$

This matrix is called a *dictionary* and its columns are usually referred to as *atoms* of the dictionary.[3] Since there are other possible choices of dictionaries for period estimation, the matrix in (4.2) is specifically referred to as the *Farey dictionary* [17] (inspired by 'Farey series' in (3.9)). Note that the set of columns of this matrix corresponding to any specific q span a space $\widehat{\mathcal{S}}_q$ whose elements are truncated versions of $\mathcal{S}_q$ (truncated to $0 \le n \le N-1$). Because of this truncation, non-zero vectors in $\widehat{\mathcal{S}}_{q_1}$ and $\widehat{\mathcal{S}}_{q_2}$ are not necessarily orthogonal.

(a) The basic principle

The basic principle behind the use of dictionaries for identifying periodicity is as follows: given the infinite sequence $x(n)$, $-\infty < n < \infty$, the representation in equation (3.4) is unique because the representation of the periodic sequence in terms of Fourier exponentials $\mathrm{e}^{\mathrm{j}\omega_i n}$ is unique (Theorem 2 in [23]). That is, if $\mathbf{x}$ is a doubly infinite vector in (4.1) then the solution $\mathbf{s}$ is unique. Call this the ideal solution $\mathbf{s}^{(I)}$. In practice when we are given a *finite sequence* $\mathbf{x}$ and the dictionary $\mathbf{A}$, the solution $\mathbf{s} = \mathbf{s}^{(I)}$ is still valid because equation (4.1) represents a subset of the infinity of equations in (3.4). But there may be other solutions because $\mathbf{A}$ is usually a fat matrix (i.e. $\Phi(Q) > N$, since $\Phi(Q)$ is $O(N^2)$). Assume for a moment that we somehow identify the ideal solution $\mathbf{s} = \mathbf{s}^{(I)}$. Then we can identify the period of $x(n)$ from the support of $\mathbf{s}$, that is from the indices ℓ of the components s_ℓ which are non-zero. To see this observe that these indices indicate the columns of $\mathbf{A}$ which participate in the representation of $\mathbf{x}$. This helps to identify those values of q_i (i.e. those Ramanujan subspaces $\mathcal{S}_{q_i}$) that have non-zero components in the representation. By taking the lcm of these q_i-s, we can find the period of $x(n)$. By using the non-zero components s_i of $\mathbf{s}$, the components $x_{q_i}(n)$ in the representation $x(n) = \sum_i x_{q_i}(n)$ can therefore be identified where $x_{q_i}(n) \in \mathcal{S}_{q_i}$.

[3]The applicability of dictionary methods for period estimation has been known since the work of Nakashizuka [22] although these early constructions of dictionaries were not based on Ramanujan subspaces.

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We can also find hidden periods by eliminating those q_i which are divisors of larger numbers in the set $\{q_i\}$. This reduced set $\{p_i\}$ represents the hidden periods as explained earlier in §3c. The above representation can then be rewritten in two ways as in (3.14) where $y_{p_i}(n)$ are hidden periodic components (which do not necessarily belong to a specific Ramanujan subspace). Given a signal $x(n)$, we can arrive at the above representation, and plot the energies of the components $y_{q_i}(n)$ versus q_i. This is how the strength-versus-period plot in figure 2(d) was obtained. As explained before, although $\{p_i\}$ is unique, the components $y_{p_i}(n)$ are in general not unique because of harmonic ambiguity.

(b) Unique identifiability of periods using the Farey dictionary

To show that the period of $x(n)$ (or the hidden periods) can be identified uniquely, it only remains to explain how to obtain the ideal solution $\mathbf{s} = \mathbf{s}^{(I)}$ from the set of all solutions. Before we do this, some comments are in order. The Farey dictionary $\mathbf{A}$ is *Vandemonde*, and the elements W_q^k in the second row are distinct. Now there are two possibilities. (i) When $\Phi(Q) > N$, any set of N columns is linearly independent (because of the Vandermonde property), that is the matrix has Kruskal rank N. So we can choose to represent $\mathbf{x}$ with any set of N columns of $\mathbf{A}$ which means that there are many solutions $\mathbf{s}$, each with at most N non-zero entries. (ii) When $N \geq \Phi(Q)$ the matrix has full column rank and solution $\mathbf{s}$ is unique (and should therefore be equal to $\mathbf{s}^{(I)}$).

Since $\Phi(Q) = O(N^2)$, the case $N \geq \Phi(Q)$ is not of much practical interest. The case $\Phi(Q) > N$ is more practical. In this case, equation (4.1) has multiple solutions $\mathbf{s}$. However, it turns out that the *sparsest solutions*, that is any solution with the smallest number ρ of non-zero elements s_i, can be used to estimate periods and hidden periods, as explained next. Thus consider any solution to the following problem:

$$\mathbf{s} = \arg\min_{\mathbf{s}} \|\mathbf{s}\|_0 \quad \text{s.t.} \quad \mathbf{x} = \mathbf{A}\mathbf{s}, \tag{4.3}$$

where $\|\mathbf{s}\|_0 =$ number of non-zero elements in $\mathbf{s}$. It is shown in [23] that the support of this solution is equal to the support of the ideal solution $\mathbf{s}^{(I)}$ as long as $N \geq L_{\min}$ where the threshold $L_{\min}$ is derived in [23]. (This is true regardless of whether the sparsest solution is unique or not.) Thus, as long as the number of samples N is large enough, we can identify the period of $x(n)$ from the samples in $\mathbf{x}$. Similarly, the hidden periods can be identified as well (although the required $L_{\min}$ for this is different).

To be more specific, consider the example where we have the *a priori* information that the period of $x(n)$ belongs in the set $\{1, 2, 3, \ldots Q\}$. Then $L_{\min} = 2Q$ (Theorem 8 of [23]). On the other hand, if $x(n)$ has M hidden periods with each hidden period belonging to the known set $\{1, 2, 3, \ldots Q\}$ then the period of $x(n)$ itself may not belong in this set. In this case, $L_{\min}$ has a more complicated expression, and it is given in Theorem 9 of [23].

To give a numerical example, assume we have *a priori* knowledge that $x(n)$ contains only two periodic components ($M = 2$) with periods p_i in the range $1 \leq p_i \leq 15$. So $Q = 15$ and $\mathbf{A}$ has $\Phi(15) = 72$ columns. It can be shown (from Theorem 9 in [23]) that $L_{\min} = 56$ in this case. With $N = L_{\min}$, $\mathbf{A}$ is a fat matrix of size 56×72.

A final remark on details is in order. In practice, it turns out that the number of non-zero components s_i in $\mathbf{s}$ is less than $L_{\min}/2$, and, furthermore, the matrix $\mathbf{A}$ is fat. In this case, since the Kruskal rank of $\mathbf{A}$ is $L_{\min}$ it follows that the sparse solution $\mathbf{s}$ with $< L_{\min}/2$ non-zero components is itself unique.

(c) Ramanujan dictionary, and more general dictionaries

For any q in $1 \leq q \leq Q$, there are $\phi(q)$ columns with period q in the matrix (4.2); each column is a Vandermonde vector. Since the complex basis functions (3.5) for $\mathcal{S}_q$ can be replaced with the real integer basis (3.11) of Ramanujan sums, the Farey dictionary can be replaced with an equivalent real, *integer dictionary* containing the integers $c_q(n-l)$. This is called the *Ramanujan dictionary* [14]. When $A = 6$ the Ramanujan dictionary is

$$\begin{array}{c} q \rightarrow \\ \\ \mathbf{A} = \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 & -1 & 2 & 0 & 4 & 4 & -1 & -1 & 2 & 1 \\ 1 & -1 & -1 & 2 & 0 & 2 & -1 & 4 & 4 & -1 & 1 & 2 \\ 1 & 1 & -1 & -1 & -2 & 0 & -1 & -1 & 4 & 4 & -1 & 1 \\ 1 & -1 & 2 & -1 & 0 & -2 & -1 & -1 & -1 & 4 & -2 & -1 \\ 1 & 1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1 & -1 & -2 \\ 1 & -1 & -1 & -1 & 0 & 2 & 4 & -1 & -1 & -1 & 1 & -1 \end{pmatrix} \end{array}. \tag{4.4}$$

There are many other equivalent periodicity dictionaries as shown in [14]. All these are matrices of the form

$$\mathbf{A} = \begin{array}{c} \begin{matrix} \phi(1) & \phi(2) & \cdots & \phi(Q) \end{matrix} \\ \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_Q \end{pmatrix} \end{array}, \tag{4.5}$$

where $\mathbf{A}_i$ is $N \times \phi(i)$ with rank $\phi(i)$. A unified theory of such periodic representations can be found in [20], based on the concept of *nested periodic matrices*. This unifies a number of apparently unrelated results such as the *exactly periodic subspaces* (EPS) representation of Muresan & Parks [19], and the *intrinsic integer-periodic functions* of Pei & Lu [24]. These other dictionaries can also be used for period identification, and in practice they work very well. It should be mentioned however that the unique identifiability result for the case of the Farey dictionary (§4b) is in general not applicable to these other dictionaries because they do not have the Vandermonde structure.

5. Ramanujan filter banks

The theory and applications of filter banks are well known in digital signal processing [16,25]. Ramanujan-sum filter banks were introduced in [26,27]. They produce a time-period plane plot, which is well suited for tracking localized periodic behaviour, often yielding more useful results than short-time Fourier transforms (STFT) [15].

(a) The basic principle

First consider a period-P signal $x(n.)$ We know its Fourier transform is a line spectrum with lines at $2\pi k/P$ where k takes integer values in $0 \leq k \leq P-1$ (figure 3*a*). Assume $1 \leq P \leq N$. One might think that P can be 'identified' by using a bank of ideal comb filters

$$H_q(e^{j\omega}), \quad 1 \leq q \leq N, \tag{5.1}$$

with the frequency response of the qth filter chosen as in figure 3*b*. However, the period-P input $x(n)$ can produce a non-zero output for many of these filters besides $H_P(e^{j\omega})$.

Even assuming the filters suppress zero frequency or 'DC' component, we can prove the following [27]: *assume the P-point DFT of $x(n)$ is non-zero for all non-zero DFT frequencies. Then $H_q(e^{j\omega})$ has non-zero output for every filter with index q not coprime to P.*

Thus, a comb filter bank is not well suited to identify periodicities (unless we use sophisticated *adaptive* comb filters [28]). Now assume we use a filter bank with frequency responses

$$C_q(e^{j\omega}), \quad 1 \leq q \leq N, \tag{5.2}$$

where $C_q(e^{j\omega})$ is the Fourier transform of the qth Ramanujan sum $c_q(n)$ (figure 4*a*). This is called a Ramanujan analysis filter bank. The qth filter has the response shown in figure 3*c*. It is a line spectrum because $c_q(n)$ is periodic. By definition of $c_q(n)$, the line spectrum is non-zero only at $2\pi k_i/q$ where k_i is coprime to q. It can be shown [27] that this property enables us to identify the period of the input.

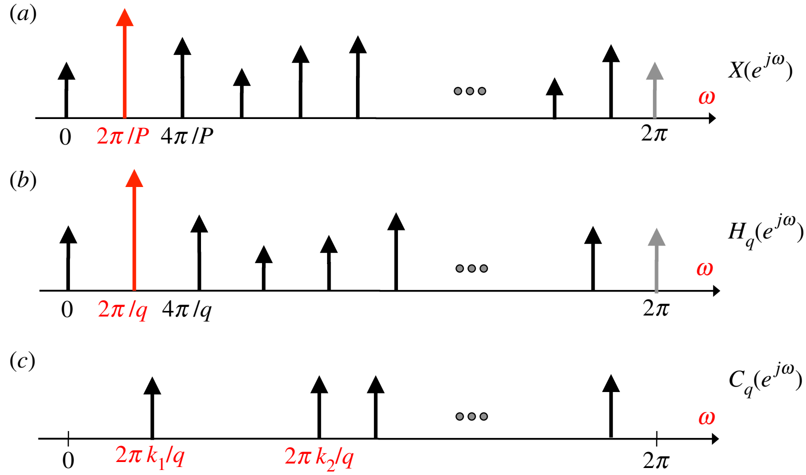

Figure 3. Fourier transforms of (*a*) a period-*P* signal, (*b*) a period-*q* comb filter, and (*c*) a period-*q* Ramanujan filter. This structure of the Ramanujan filter is more suited to identifying periods [27]. See text. (Online version in colour.)

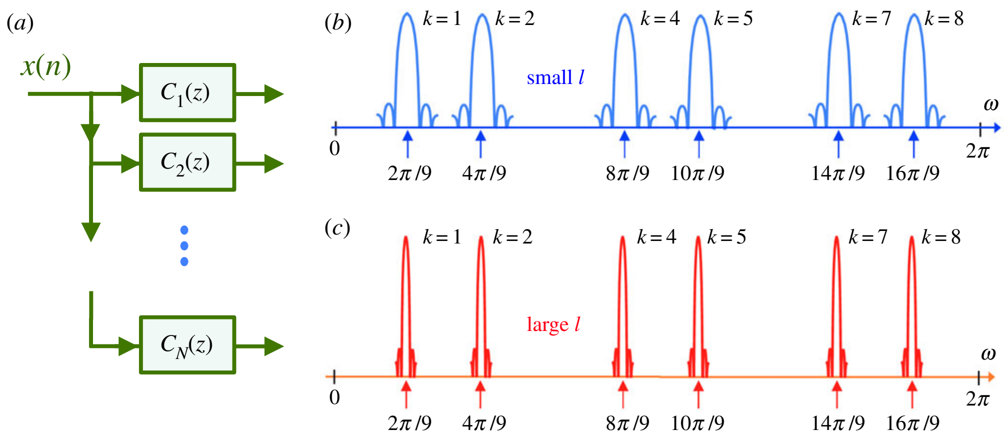

Figure 4. (*a*) The Ramanujan filter bank, and (*b*,*c*) frequency response magnitudes for $C_9^{(l)}(z)$ for different values of *l* [27,31]. Here, *z* in $C_q(z)$ stands for $e^{j\omega}$. (Online version in colour.)

Theorem 5.1 (Ramanujan filter banks). *Consider a Ramanujan analysis filter bank* $\{c_q(n)\}$ *with* $1 \leq q \leq N$ (figure 4*a*), *and let* $x(n)$ *be a period-P input signal with* $1 \leq P \leq N$. *Then non-zero outputs can only be produced by those filters* $c_q(n)$ *such that the filter index q is a divisor of P. Furthermore, assuming non-zero outputs are produced by the subset of filters* $c_{q_i}(n)$ *with periods* $q_1, q_2, \ldots, q_K$, *the input period P is exactly equal to* $\mathrm{lcm}(q_1, q_2, \ldots, q_K)$.

From the filter indices $q_1, q_2, \ldots, q_K$, we can also identify multiple hidden periods $\{p_i\}$ in $x(n)$ by using the method described in §3c. In practice, each filter is usually truncated to l periods:

$$C_q^{(l)}(z) = \sum_{n=0}^{ql-1} c_q(n) z^{-n}, \tag{5.3}$$

so that the lines are spread out (figure 4*b*,*c*). Since the filters have finite duration impulse responses $c_q^{(l)}(n)$, this is called an *FIR Ramanujan filter bank*. Since $c_q^{(l)}(n)$ is localized to a duration of ql, this helps to extract localization information: if $x(n)$ has various periodic components localized at

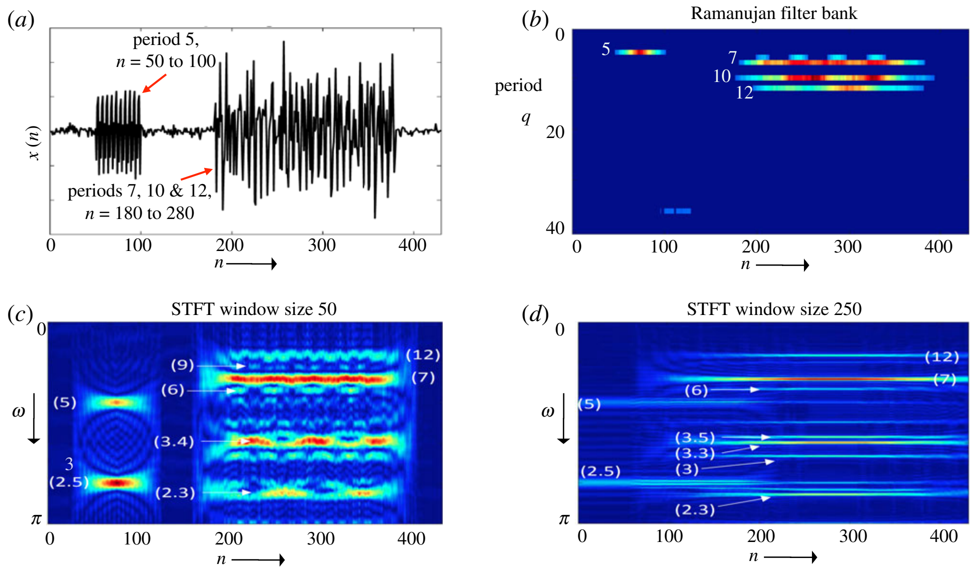

Figure 5. (*a*) Signal with periodic components buried in noise. (*b–d*) Time-period planes generated by the Ramanujan filter bank, and the short-time Fourier transform (STFT) [27,31]. (Online version in colour.)

different times, then by examining the filter-bank output as a function of time, we can obtain a time-period plane plot which contains the localization information. For fixed l, the time-spread of the qth filter is ql. Thus short periods (small q) are localized more accurately, which is desirable.

Figure 5*a* shows an example of $x(n)$ with a period-5 component localized at $50 \leq n \leq 100$ and period-7, period-10 and period-12 components localized at $180 \leq n \leq 280$. These components are buried in noise (SNR 5 dB). The time-period plane plot obtained by analysing the output of the Ramanujan filter bank with $N = 40$, $l = 10$ is shown in figure 5*b*. The periodic components can be seen very clearly as the horizontal lines which also reveal time-localization information. Details about how these plots are obtained can be found in [26].

For comparison, the STFT plot (or time-dependent Fourier transform [15]) is also shown. This is obtained by computing the magnitude of the Fourier transform over short windows of data, and sliding the window in time, in order to obtain a time-frequency plot. Figure 5*c*,*d* shows these for two window sizes. We have also indicated, in round brackets, the value of $2\pi/\omega$ at the location of the bright bands, which indicates the approximate periods revealed by the plot. For window size 50, periods 5, 7 and 12 are visible, and so are some other spurious bands which typically correspond to harmonics of these. But the period 10 cannot be spotted, and furthermore, the lines are spread too thick, that is the frequency resolution is poor. The frequency resolution can be improved by increasing the STFT window size to 250 (part (*d*)), but the time domain resolution is highly compromised as seen from the figure: while the bright lines are thinner, they are also longer. Since the STFT can only have a fixed window size unlike the Ramanujan filter bank whose filter lengths are adjusted according to the periods, the STFT does not perform well. In this sense, the Ramanujan filter bank is more analogous to a 'wavelet' type of filter bank, which offers basis functions with shorter durations for higher frequencies [16,25,29,30]. But unlike wavelets, the coefficients of the Ramanujan filter bank are specifically chosen to identify integer periods effectively.

(b) Efficient implementations

As mentioned earlier, it is well known that $c_q(n)$ is *integer valued* [1,12] in spite of the complex trigonometric functions in its definition (1.1). For example, see equation (1.4) which shows one period of $c_{10}(n)$. To some extent this helps to reduce the multiplier complexity in hardware implementations. More significantly, a completely multiplierless implementation of the Ramanujan filter bank is possible, as explained next. It was shown in [12] that Ramanujan sums satisfy the beautiful recursive relation

$$c_q(n) = q\delta_q(n) - \sum_{\substack{q_k|q \\ q_k<q}} c_{q_k}(n). \tag{5.4}$$

Here $\delta_q(n)$ is the periodic impulse, where the '1' occurs at locations that are multiples of q, e.g. $\delta_3(n) = \ldots 0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1 \ldots$ Starting from this, it can be shown (Theorem 2 in [31]) that the Ramanujan-sum can be expressed in the form

$$c_q(n) = \sum_{q_k|q} \alpha_{q_k} \times q_k\delta_{q_k}(n), \tag{5.5}$$

where $\alpha_{q_k} \in \{0, 1, -1\}$. This leads to a efficient digital implementation of the Ramanujan filter bank, which is essentially free of multipliers. The detailed derivation of such a structure can be found in [31]. The final form of the multiplierless structure is shown in fig. 4 of [31] using standard signal processing notations.

6. Minimum data length for integer period estimation

It is shown in [32] that the minimum number of dictionary atoms needed to identify the period of a signal is given by $\Phi(Q)$ in equation (3.10). So the number of columns in the dictionary should be at least $\Phi(Q)$. As explained in §4b, the minimum number of rows required $L_{\min}$ has also been derived in [23,33]. This gives the number of samples necessary to identify periods using dictionary methods. More fundamentally, what is the minimum number of consecutive samples necessary regardless of methods used for period identification? To be specific, assume we have the *a priori* knowledge that $x(n)$ is periodic with integer period

$$P \in \{P_1, P_2, \ldots, P_Q\}. \tag{6.1}$$

Then what is the *data length* i.e. number $L_{\min}$ of *consecutive samples*

$$x(0), x(1), \ldots, x(L_{\min} - 1) \tag{6.2}$$

required, in order to identify P through any computational means at all (not necessarily using the above dictionaries)? It is shown in [23] that

$$L_{\min} = \max_{P_i, P_j \in \{P_1, P_2, \ldots, P_Q\}} (P_i + P_j - (P_i, P_j)). \tag{6.3}$$

For example suppose $x(n)$ is known to have period either 6 or 14. To tell what its period is, we need at least $L_{\min} = 6 + 14 - (6, 14) = 18$ samples. Figure 6 shows a case of two signals with periods 6 and 14, where the first 17 samples are identical, demonstrating that these samples are not enough to tell which of the two signals they came from. As another example, if we have the *a priori* information that the period of $x(n)$ belongs in the set $\{1, 2, 3, \ldots Q\}$ then [23] $L_{\min} = 2Q - 2$ which is slightly smaller than the minimum number of samples ($L_{\min} = 2Q$) required in the dictionary approach (§4b).

These results have also been extended to the case where multiple hidden periods have to be identified. It has also been shown that if the samples of the signal are allowed to be non-contiguous then the minimum required number of samples can be considerably reduced [34]. Finally, some limited extensions to the case of multidimensional periodic lattices have been reported in [35].

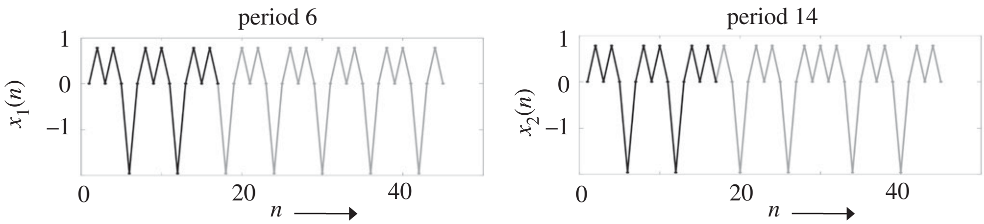

Figure 6. (*a*,*b*) Signals with period 6 and 15. The first 17 samples are identical for the two signals, so it is not possible to know the true signal based only on the first 17 samples.

7. Conclusion

In this paper, we presented an overview of the impact of Ramanujan sums in signal processing, especially integer-period estimation in real or complex signals. These methods have recently been used in identification of integer periodicities in DNA molecules [36] and in protein molecules [37]. As shown in these references, these new methods are quite competitive and often work better than existing state of the art methods. Ramanujan filter banks have also been shown to be applicable in the identification of epileptic seizures in patients, which are characterized by sudden appearance of periodic waveforms in the measured EEG records [38]. Other interesting applications have recently been reported by a number of authors such as, for example in source-separation [39], RF communications [40], ECG signal processing [41] and brain-computer interfacing [42,43]. For details, the reader is requested to consult the above papers and references therein. More recently, a well-known algorithm called the MUSIC algorithm [44,45], which is popularly used for identifying sinusoids in noise, has been extended to the case of integer period identification using Ramanujan-subspace ideas [46]. This method, known as iMUSIC, has also been compared with other well-known methods for multipitch estimation [47], and the details can be found in [46].

It is satisfying indeed when one finds that a well-known mathematical concept has practical impact in engineering, even though the original mathematical ideas may not have been inspired by any such application. Engineers have seen this happening over and over again in disciplines such as information theory, coding, digital communications, system theory and machine learning. Indeed, the classical view that pure mathematics of the highest quality is bound to be 'useless' for real-life applications [48] is evidently not valid as the last several decades of science and engineering have amply demonstrated. What is often regarded as pure mathematics sometimes impacts engineering in wonderful ways. A classic example is the theory of finite fields and rings, which has impacted the practice of error-correction coding in digital communications, data compression, and digital storage. Other examples include graph theory which has had many engineering, network and signal processing applications. Yet another is number theory which has impacted nearly all aspects of science and engineering such as acoustical hall designs, computer music, and so forth [49,50].

Data accessibility. This article has no additional data.
Authors' contributions. P.P.V. introduced the idea of Ramanujan subspaces, studied their properties for periodic signals, and introduced the Farey dictionary for signal representation. S.T. further developed the idea of periodicity dictionaries and proved a number of results for them. S.T. also developed Ramanujan filter banks, established the minimum data length results, developed the applications mentioned and produced all computer simulation plots. Both authors read and approved the manuscript.
Competing interests. We declare we have no competing interests.
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