Randomized Asynchronous Recursions with a Sinusoidal Input

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Abstract—**This study considers a randomized asynchronous form of the discrete time-invariant state-space models, in which only a random subset of the state variables is updated in each iteration. When the system has a single input in the form of a complex exponential, it is shown that the output signal still behaves like an exponential in a statistical sense. The study presents the necessary and sufficient condition that ensures the stability of a randomized asynchronous system, which does not necessarily require the stability of the state transition matrix.**

Index Terms—**Asynchronous models, randomized iterations, state-space models, autonomous networks, graph filters.**

I. Introduction

Linear time-invariant discrete systems are well studied, and they find useful applications in a large number of different fields ranging from mathematical finance to implementation of digital filters [1]–[4]. They are also used in the recent area of graph signal processing [5], [6], in which the state variables are assumed to represent the nodes of a graph, and the state transition matrix is considered as a local operator of the underlying graph, e.g., the adjacency matrix, the graph Laplacian, etc. With this formalism, the studies in [7]–[10] extended infinite impulse response (IIR) digital filters to the case of graphs.

In the context of graph signal processing state recursions are interpreted as repeated communication between the neighboring nodes, in which all the nodes are assumed to exchange data simultaneously, or wait for each other, before starting the next round of communication. This type of implementation clearly requires a synchronization over the whole network (graph). In order to eliminate the need for such a synchronization, the recent studies [11]–[14] considered a randomized asynchronous variant of the state recursions and provided sufficiency conditions under which the iterations are guaranteed to converge. In fact, non-random variants of asynchronous state recursions (possibly with non-linear updates) are well-studied problems with early results dating back to late 60's [15]–[17].

Whether random or non-random, the studies [13]–[17] considered the asynchronous updates from a fixed point iteration viewpoint, in which the iterant (the state vector) is shown to converge to a fixed point of the update model. Such an approach corresponds to a state-space model with a *constant* input. Differently in this study, we will consider randomized asynchronous state recursions with a single input in the form of a complex exponential that varies over iterations. Thus, the state vector, hence the output signal, does not converge to a point. Nevertheless, we show that the output signal, which is a random quantity due to the randomness of the update scheme, still behaves like a complex exponential in a *statistical sense*.

This work was supported in parts by the ONR grant N00014-18-1-2390, the NSF grant CCF-1712633, and the Electrical Engineering Carver Mead Research Seed Fund of the California Institute of Technology.

We further show that the output signal "oscillates" at the same frequency as the input signal on average, which allows us to consider the "frequency response" of the randomized asynchronous state recursions. In this regard, we first present the necessary and sufficient condition under which the expectation of the output behaves like an exponential signal. Then, we present the necessary and sufficient condition under which the output has a bounded covariance, which ensures the stability of the randomized asynchronous system.

Interestingly, we observe that the stability of the randomized asynchronous recursions does *not* necessarily require the stability of the transition matrix of the state-space model. That is to say, an unstable synchronous system may get stable with a randomized asynchronicity, which is a remarkable property observed also in [13], [14] for the case of zero-input. More generally we conclude that the stability of the randomized asynchronous updates and the stability of the synchronous updates do not imply each other.

Section II presents the randomized asynchronous state recursions and shows its convergence behavior (Lemmas 2 and 3). In Section III we provide a numerical example that visualizes the behavior of the randomized asynchronous recursions.

A. Preliminaries and Notation

We will use $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ to denote the probability and expectation, respectively. For a matrix $\mathbf{X}$ we will use $\mathbf{X}^*$ and $\mathbf{X}^{\rm H}$ to denote its conjugate and conjugate transpose, respectively and $\rho(\mathbf{X})$ to denote its spectral radius (the largest eigenvalue in absolute sense). For a matrix $\mathbf{X}\in\mathbb{C}^{N\times M}$ we will use $\mathrm{vec}(\mathbf{X})\in\mathbb{C}^{NM}$ to denote a vector obtained by cascading the columns of $\mathbf{X}$. For a vector $\mathbf{x}$ we will use $\mathrm{diag}(\mathbf{x})$ to denote a diagonal matrix with entries of $\mathbf{x}$ on the diagonal. We will use $\otimes$ to denote the Kronecker product.

We will use $\mathcal{T}$ to denote a subset of $\{1,\cdots,N\}$, and use $\mathbf{D}_{\mathcal{T}}\in\mathbb{R}^{N\times N}$ to denote a diagonal matrix that has value 1 only at the indices specified by the set $\mathcal{T}$. That is,

$$\mathbf{D}_{\mathcal{T}} = \sum_{i\in\mathcal{T}} \mathbf{e}_i\,\mathbf{e}_i^{\rm H}, \tag{1}$$

where $\mathbf{e}_i\in\mathbb{R}^N$ is the i^{th} standard vector that has 1 at the i^{th} index and 0 elsewhere.

II. Asynchronous State Recursions with a Single Exponential Input

In this study, we consider a discrete time-invariant system with a single input and possibly with multiple outputs whose state-space description is given as follows:

$$\mathbf{x}_{k+1} = \mathbf{A}\,\mathbf{x}_k + \mathbf{B}\,u_k, \tag{2}$$

$$\mathbf{y}_k = \mathbf{C}\,\mathbf{x}_k + \mathbf{D}\,u_k, \tag{3}$$

where $\mathbf{x}_0$ denotes the initial state vector (initial condition), and the size of the matrices are as follows:

$$\mathbf{A} \in \mathbb{C}^{N\times N}, \quad \mathbf{B} \in \mathbb{C}^{N\times 1}, \quad \mathbf{C} \in \mathbb{C}^{q\times N}, \quad \mathbf{D} \in \mathbb{C}^{q\times 1}, \tag{4}$$

where $\mathbf{A}$ is referred to as the state transition matrix. We further assume that the input signal u_k has the following form:

$$u_k = e^{j\omega k}, \tag{5}$$

where $0 \leqslant \omega < 2\pi$ represents the frequency of the input.

It is well-known from linear system theory that the output vector $\mathbf{y}_k \in \mathbb{C}^q$ in (3) can be written as follows:

$$\mathbf{y}_k = \mathbf{y}_k^{\text{ss}} + \mathbf{y}_k^{\text{tr}}, \tag{6}$$

where $\mathbf{y}_k^{\text{ss}}$ denotes the steady-state component, and $\mathbf{y}_k^{\text{tr}}$ denotes the transient component that are given as follows:

$$\mathbf{y}_k^{\text{ss}} = \mathbf{H}(e^{j\omega})\, e^{j\omega k}, \qquad \mathbf{y}_k^{\text{tr}} = \mathbf{C}\,\mathbf{A}^k \left(\mathbf{x}_0 - \mathbf{x}_0^{\text{ss}}\right), \tag{7}$$

where $\mathbf{H}(e^{j\omega})$ is referred to as the frequency response of the system, and it is given as follows:

$$\mathbf{H}(e^{j\omega}) = \mathbf{D} + \mathbf{C}\big(e^{j\omega}\mathbf{I} - \mathbf{A}\big)^{-1}\mathbf{B}, \quad \mathbf{x}_0^{\text{ss}} = (e^{j\omega}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}. \tag{8}$$

It is clear from (7) that when the state transition matrix $\mathbf{A}$ is a stable matrix, i.e., the following holds true:

$$\rho(\mathbf{A}) < 1, \tag{9}$$

then the transient component $\mathbf{y}_k^{\text{tr}}$ converges to zero as the iterations progress leaving only the steady-state component $\mathbf{y}_k^{\text{ss}}$ in the output signal. In fact, the stability of $\mathbf{A}$ is also necessary for the transient part to converge to zero.

Discrete state-space models find many applications in various fields, one of which is the recent area of graph signal processing [5], [6]. In this context the state transition matrix $\mathbf{A}$ is assumed to be a local operator (shift matrix) on the graph of interest. Assuming that each state variable corresponds a node on the graph, the $(i,j)^{th}$ entry of $\mathbf{A}$ is non-zero if and only if there is a path from the node j to the node i on the graph. With this formalism, the state recursions in the form of (2) are used for implementing IIR graph filters [7]–[11], in which the graph signal is assumed to be constant, that is, $\omega = 0$, hence $u_k = 1$.

In the context of graph signal processing, an iteration in the form of (2) can be implemented on the graph as a data exchange between the neighboring nodes. That is, (2) can be written as follows:

$$(\mathbf{x}_{k+1})_i = \sum_{j \in \mathcal{N}_{\text{in}}(i)} A_{i,j}\ (\mathbf{x}_k)_j + B_i, \qquad \forall\, i, \tag{10}$$

where $\mathcal{N}_{\text{in}}(i)$ denotes the incoming neighbors of the node i. In this setting, $\mathbf{B}$ is considered as a signal defined on the graph, where the nodes will be the "domain" analogous to time. The index k will denote the round of communication, so the graph signal $\mathbf{B}$ does not have any dependency on the iteration index.

Although the updates in (10) can be performed locally by the nodes, its implementation requires a synchronization mechanism among the nodes. That is, all the nodes should send and receive data at the same time instance, or nodes should wait until all the communications are terminated before proceeding to the next iteration. In order to eliminate the need for synchronization, the studies in [11]–[13] considered a randomized asynchronous variant of (10) and presented conditions that are sufficient to ensure the convergence.

Rather than focusing on the specific area of graph signal processing, in this study we will consider the general state-space models from a randomized asynchronous viewpoint, in which the input signal is assumed to be a complex exponential. More precisely, we consider the following randomized model:

$$(\mathbf{x}_{k+1})_i = \begin{cases} (\mathbf{A}\,\mathbf{x}_k)_i + B_i\ u_k, & i \in \mathcal{T}_{k+1}, \\ (\mathbf{x}_k)_i, & i \notin \mathcal{T}_{k+1}, \end{cases} \tag{11}$$

$$\mathbf{y}_k = \mathbf{C}\,\mathbf{x}_k + \mathbf{D}\,u_k \tag{12}$$

where $\mathcal{T}_k$ denotes the set of indices updated at the k^{th} iteration.

Furthermore, the update set $\mathcal{T}_k$ is assumed to be selected *randomly and independently* among all possible 2^N different subsets of $\{1, \cdots, N\}$ in every iteration of (11). The specific stochastic model considered in this study regarding the selection of the update sets will be elaborated next.

A. Random Selection of the Update Sets

In the asynchronous model considered in (11), we assume that the i^{th} index is updated independently with probability p_i in every iteration. Thus, p_i denotes the probability of i being an element of the update set $\mathcal{T}_k$. More precisely,

$$\mathbb{P}[i \in \mathcal{T}_k] = p_i. \tag{13}$$

As a result, both the content and the size of $\mathcal{T}_k$ are random variables. We will use $\mathbf{P}$ to denote the average index selection matrix for the index selection model in (13). More precisely,

$$\mathbf{P} \triangleq \mathbb{E}\big[\mathbf{D}_{\mathcal{T}_k}\big] = \mathbb{E}\Big[\sum_{i\in\mathcal{T}_k} \mathbf{e}_i\,\mathbf{e}_i^{\text{H}}\Big] = \sum_{i=1}^{N} p_i\,\mathbf{e}_i\,\mathbf{e}_i^{\text{H}} \tag{14}$$

$$= \operatorname{diag}\big([p_1 \quad p_2 \quad \cdots \quad p_N]\big), \tag{15}$$

which shows that $\mathbf{P}$ is a diagonal matrix consisting of the probability p_i's. We note that $\mathbf{P}$ satisfies $\mathbf{0} \prec \mathbf{P} \preccurlyeq \mathbf{I}$, where the positive definiteness follows from the fact that no index is left out on average during the updates of (11). Moreover, the case of $\mathbf{P} = \mathbf{I}$ implies that all the indices are updated in every iteration, which corresponds to the synchronous case in (2).

B. Frequency Response in the Mean

Due to the random selection of the update sets it is clear from (11) that the state vector $\mathbf{x}_k$ is a random vector. Thus, the output signal $\mathbf{y}_k$ in (12) is also a random vector. As a result, even when the input signal is exponential as in (5), the output signal will not be an exponential signal unlike the synchronous case. Nevertheless, we can still consider the behavior of the output vector from a statistical viewpoint. In particular, we claim that $\mathbf{y}_k$ still behaves like an exponential signal *on average*, that is, the expectation of $\mathbf{y}_k$ (with respect to the random selection of the update sets) can be decomposed into steady-state and transient parts similar to (6). The following lemma presents this observation formally:

Lemma 1. *The expectation of the output of the randomized asynchronous state recursions described in* (12) *is as follows:*

$$\mathbb{E}[\mathbf{y}_k] = \mathbf{y}_k^{\text{ss}} + \mathbf{y}_k^{\text{tr}}, \tag{16}$$

where

$$\mathbf{y}_k^{\text{ss}} = \bar{\mathbf{H}}(e^{j\omega})\, e^{j\omega k}, \qquad \mathbf{y}_k^{\text{tr}} = \mathbf{C}\,\bar{\mathbf{A}}^k \left(\mathbf{x}_0 - \mathbf{x}_0^{\text{ss}}\right), \tag{17}$$

where

$$\bar{\mathbf{H}}(e^{j\omega}) = \mathbf{D} + \mathbf{C}\big(e^{j\omega}\mathbf{I} - \bar{\mathbf{A}}\big)^{-1}\bar{\mathbf{B}}, \quad \mathbf{x}_0^{\text{ss}} = (e^{j\omega}\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}, \tag{18}$$

and

$$\bar{\mathbf{A}} = \mathbf{I} + \mathbf{P}\,(\mathbf{A} - \mathbf{I}), \qquad \bar{\mathbf{B}} = \mathbf{P}\,\mathbf{B}. \tag{19}$$

Despite the random nature of the output vector $\mathbf{y}_k$, Lemma 1 shows that the steady-state component of $\mathbb{E}[\mathbf{y}_k]$ remains an exponential signal with the same frequency as the input, which allows us to talk about a "frequency response" of the randomized asynchronous recursions. In this regard, we consider the following quantity:

$$\mathbf{r}_k = \mathbf{y}_k - \mathbf{y}_k^{\mathrm{ss}}, \tag{20}$$

which will be referred to as the error term since it is the difference between the output signal $\mathbf{y}_k$ itself and the steady-state component of its mean.

Proof of Lemma 1: Due to asynchronous updates described in (11), state vector $\mathbf{x}_k$ can be written as follows:

$$\begin{aligned}\mathbf{x}_k &= \sum_{i\notin\mathcal{T}_k} \mathbf{e}_i\,\mathbf{e}_i^{\mathrm{H}}\,\mathbf{x}_{k\text{-}1} + \sum_{i\in\mathcal{T}_k} \mathbf{e}_i\,\mathbf{e}_i^{\mathrm{H}}\,\big(\mathbf{A}\,\mathbf{x}_{k\text{-}1} + \mathbf{B}\,e^{j\omega(k\text{-}1)}\big)\\ &= \big(\mathbf{I} + \mathbf{D}_{\mathcal{T}_k}\,(\mathbf{A}-\mathbf{I})\big)\mathbf{x}_{k\text{-}1} + \mathbf{D}_{\mathcal{T}_k}\,\mathbf{B}\,e^{j\omega(k\text{-}1)}. \end{aligned}\tag{21}$$

Taking expectation of (21) and using the fact that update sets are selected independently, we have the following:

$$\begin{aligned}\mathbb{E}[\mathbf{x}_k] &= \bar{\mathbf{A}}\,\mathbb{E}[\mathbf{x}_{k\text{-}1}] + \bar{\mathbf{B}}\,e^{j\omega(k\text{-}1)}\\ &= \bar{\mathbf{A}}^k\,\mathbf{x}_0 + \sum_{i=0}^{k-1}\bar{\mathbf{A}}^i\,\bar{\mathbf{B}}\,e^{j\omega(k\text{-}1\text{-}i)} \qquad (22)\\ &= (e^{j\omega}\mathbf{I}-\bar{\mathbf{A}})^{\text{-}1}\,\bar{\mathbf{B}}\,e^{j\omega k} + \bar{\mathbf{A}}^k\,(\mathbf{x}_0 - (e^{j\omega}\mathbf{I}-\bar{\mathbf{A}})^{\text{-}1}\,\bar{\mathbf{B}}),\end{aligned}$$

where $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are as in (19).

Then, due to (12) the expectation of $\mathbf{y}_k$ is given as follows:

$$\mathbb{E}[\mathbf{y}_k] = \mathbf{C}\,\mathbb{E}[\mathbf{x}_k] + \mathbf{D}\,z^k = \mathbf{y}_k^{\mathrm{ss}} + \mathbf{y}_k^{\mathrm{tr}}, \tag{23}$$

where $\mathbf{y}_k^{\mathrm{ss}}$ and $\mathbf{y}_k^{\mathrm{tr}}$ are as in (17). ■

In the case of synchronous updates, the error term defined as in (20) is a deterministic quantity that corresponds to the transient component of the output vector in (6). As a result, the condition $\rho(\mathbf{A}) < 1$ is both necessary and sufficient to ensure that the transient part (hence, the error term $\mathbf{r}_k$) converges to zero, in which case $\mathbf{y}_k$ approaches the steady-state component resulting in an exponential output signal.

In the asynchronous case the error term is a random quantity due to the random nature of the updates in (11). In general, $\mathbf{r}_k$ itself may not converge to zero as the iterations progress. Nevertheless, the following lemma provides the necessary and sufficient condition that ensures that $\mathbb{E}[\mathbf{r}_k]$ converges to zero:

Lemma 2. *The following holds true irrespective of* $\mathbf{C}$, *the input frequency* ω *and the initial condition* $\mathbf{x}_0$*:*

$$\lim_{k\to\infty}\ \mathbb{E}[\mathbf{r}_k] = \mathbf{0} \tag{24}$$

if and only if

$$\rho(\bar{\mathbf{A}}) < 1. \tag{25}$$

Proof: We note that

$$\mathbb{E}[\mathbf{r}_k] = \mathbb{E}[\mathbf{y}_k] - \mathbf{y}_k^{\mathrm{ss}} = \mathbf{y}_k^{\mathrm{tr}} = \mathbf{C}\,\bar{\mathbf{A}}^k\,\big(\mathbf{x}_0 - \mathbf{x}_0^{\mathrm{ss}}\big), \tag{26}$$

which follows simply from (16) and (17). Thus, the condition (25) is necessary and sufficient to ensure that $\mathbb{E}[\mathbf{r}_k]$ converges to zero irrespective of the value of $\mathbf{C}$, $\mathbf{x}_0$ and ω. ■

We first note that Lemma 2 is consistent with the synchronous case: when all the indices are updated simultaneously we have $\mathbf{P} = \mathbf{I}$, in which case $\bar{\mathbf{A}} = \mathbf{A}$, thus the condition (25) reduces to the stability of the state transition matrix $\mathbf{A}$.

Importance of Lemma 2 follows from the fact that the convergence of $\mathbb{E}[\mathbf{r}_k]$ to zero implies that the output signal $\mathbf{y}_k$ behaves the same as the steady-state component $\mathbf{y}_k^{\mathrm{ss}}$ on average after a sufficient number of iterations. More precisely,

$$\mathbb{E}[\mathbf{y}_k] \xrightarrow{k\to\infty} \mathbf{y}_k^{\mathrm{ss}} = \bar{\mathbf{H}}(e^{j\omega})\,e^{j\omega k}, \tag{27}$$

which shows that when (25) is satisfied an exponential input results in an exponential output *on average* even with the randomized asynchronous state recursions. (See Figure 2 to be explained later in Section III.) Furthermore, the term $\bar{\mathbf{H}}(e^{j\omega})$ can be considered as the "frequency response" of the randomized asynchronous system.

From (18) it is clear that the frequency response of the randomized model depends on the matrix $\mathbf{P}$, namely update probability of the state variables. In fact, from the frequency response viewpoint random asynchronous updates running on a system denoted with $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ can be represented equivalently as $(\bar{\mathbf{A}},\bar{\mathbf{B}},\mathbf{C},\mathbf{D})$. We note that $\bar{\mathbf{A}}$ corresponds to a row-wise convex combination of $\mathbf{A}$ and the identity matrix with combination coefficients being the update probabilities of the state variables, and the matrix $\bar{\mathbf{B}}$ is a row-wise scaled version of $\mathbf{B}$ with scaling coefficients being the update probabilities. As a result, even when a single state variable updates its value with a different probability, the response of the overall system changes. Nevertheless, the output stays as a complex exponential (on average) with frequency ω irrespective of the underlying update probabilities.

Although the condition (25) allows us to interpret the average behavior of the output signal from a frequency response viewpoint, it should be noted that $\mathbf{y}_k$ itself is not an exponential signal due to it being random. As a result, the system (11)-(12) does not produce an exponential output even with an exponential input. The condition (25) ensures merely that $\mathbb{E}[\mathbf{r}_k]$ converges to zero, hence $\mathbb{E}[\mathbf{y}_k]$ behaves like an exponential signal, which does not necessarily imply that the error term has a bounded covariance as the iterations progress. In addition to (25), if $\mathbf{r}_k$ is guaranteed not to diverge, only then the interpretation of $\mathbb{E}[\mathbf{y}_k]$ from a frequency response viewpoint is meaningful.

In the next section we will analyze the correlation matrix of the error term defined in (20) and provide the necessary and sufficient condition for $\mathbf{r}_k$ to have a bounded covariance.

C. Convergence of the Error Correlation Matrix

In order to analyze the second order characteristics of the error term defined in (20), we consider the following error correlation matrix:

$$\mathbf{R}_k = \mathbb{E}\big[\mathbf{r}_k\,\mathbf{r}_k^{\mathrm{H}}\big] \in \mathbb{C}^{q\times q}. \tag{28}$$

Since the error correlation matrix having bounded entries implies that the error term $\mathbf{r}_k$ has a bounded covariance, the matrix $\mathbf{R}_k$ should be guaranteed to stay bounded as the iterations progress. Namely, we need to ensure the following:

$$\lim_{k\to\infty}\ \mathbf{R}_k < \infty, \tag{29}$$

so that $\mathbb{E}[\mathbf{y}_k]$ is meaningful to interpret from a frequency response viewpoint.

It should be noted that the error correlation matrix depends on the iteration index k, and in general $\mathbf{R}_k$ may not converge to a point, i.e., the limit considered in (29) may not exist. In fact, when the input signal is a linear combination of different frequencies the error correlation matrix shows an oscillatory behavior, yet it can still stay bounded. Due to its intricate details, the case of multiple frequencies will be elaborated in a later study. Nevertheless, when the input signal consists of a single frequency (as considered in this study), the error correlation matrix either converges to a point or increases unboundedly. In this regard, we define the following:

$$\lim_{k\to\infty}\ \mathbf{R}_k \triangleq \mathbf{R}. \tag{30}$$

The following lemma, whose proof is omitted due to space limitations, provides the necessary and sufficient condition for $\mathbf{R}$ to be finite and provides its exact closed form expression:

Lemma 3. *If the following holds true:*

$$\rho(\mathbf{S}) < 1, \tag{31}$$

then, the limit of the error correlation matrix is as follows:

$$\mathrm{vec}(\mathbf{R}) = 4\sin^2(\omega/2) \cdot \tag{32}$$
$$(\mathbf{C}^* \otimes \mathbf{C})\ (\mathbf{I} - \mathbf{S})^{-1}\ \mathbf{J}\ \mathrm{vec}\left(\mathbf{x}_0^{\mathrm{ss}}\ (\mathbf{x}_0^{\mathrm{ss}})^{\mathrm{H}}\,(\mathbf{P}^{-1} - \mathbf{I})\right),$$

where ω denotes the frequency of the input signal, and the matrices $\mathbf{S} \in \mathbb{C}^{N^2 \times N^2}$ and $\mathbf{J} \in \mathbb{R}^{N^2 \times N^2}$ are as follows:

$$\mathbf{S} = \bar{\mathbf{A}}^* \otimes \bar{\mathbf{A}} + \Big((\mathbf{I}-\mathbf{P}) \otimes \mathbf{P}\Big)\ \mathbf{J}\ \Big((\mathbf{A}^* - \mathbf{I}) \otimes (\mathbf{A} - \mathbf{I})\Big), \tag{33}$$

$$\mathbf{J} = \sum_{i=1}^{N} (\mathbf{e}_i\ \mathbf{e}_i^{\mathrm{H}}) \otimes (\mathbf{e}_i\ \mathbf{e}_i^{\mathrm{H}}) = \mathrm{diag}(\mathrm{vec}(\mathbf{I})) \in \mathbb{R}^{N^2 \times N^2}. \tag{34}$$

If the condition (31) *is violated, then $\mathbf{R}_k$ increases unboundedly as k goes to infinity, that is, $\mathbf{R}$ is not bounded.*

It can be shown that the condition (31) is more restrictive than the one in (25). As a result, (31) is sufficient to make $\mathbb{E}[\mathbf{y}_k]$ to behave like an exponential as in (27), whereas (25) alone does not guarantee that the error term has a bounded covariance. (See Figure 1.)

More importantly, the stability of the matrix $\mathbf{S}$, i.e., the condition (31), does not necessarily imply the stability of the state transition matrix $\mathbf{A}$, i.e., the condition (9). That is, even when $\rho(\mathbf{A}) \geqslant 1$ it may be possible to find a set of probabilities $\mathbf{P}$ such that $\rho(\mathbf{S}) < 1$. Thus, even when the state recursions defined in (2) are unstable it may be possible to *stabilize* them using randomized asynchronicity. This is a remarkable property of the randomized asynchronous updates, which is observed also in [12], [13] for the case of zero input $\mathbf{B} = \mathbf{0}$. We also note that the study [13] provides only a sufficiency condition that is valid only when $\mathbf{A}$ is a normal matrix, whereas Lemma 3 presents the necessary and sufficient condition that is valid for any $\mathbf{A}$.

It should be clear that not every unstable synchronous system gets stable with randomized asynchronicity. Conversely, a stable synchronous system may get unstable with asynchronicity as well. In short, we conclude that *the stability of the randomized asynchronous updates and the stability of the synchronous updates do not imply each other in general*. In order to support this claim we consider the following:

$$\mathbf{A}_1 = \begin{bmatrix} -0.9 & 0.8 \\ 0.8 & -0.3 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 1.2 & -1.1 \\ 0.3 & 0.4 \end{bmatrix}, \tag{35}$$

which can be verified to satisfy $\rho(\mathbf{A}_2) < 1 < \rho(\mathbf{A}_1)$. Then, we construct the matrices $\bar{\mathbf{A}}$ and $\mathbf{S}$ as in (19) and (33), respectively for both $\mathbf{A}_1$ and $\mathbf{A}_2$ for all possible values of $\mathbf{P} = \mathrm{diag}([p_1\ \ p_2])$. Figure 1 presents the regions of $\mathbf{P}$ for which $\bar{\mathbf{A}}$ satisfies (25), or $\mathbf{S}$ satisfies (31).

We first note that synchronous updates correspond to $\mathbf{P} = \mathbf{I}$, in which case the matrix $\mathbf{S}$ reduces to $\mathbf{S} = \mathbf{A}^* \otimes \mathbf{A}$ with $\rho(\mathbf{S}) = \rho^2(\mathbf{A})$. As a result, the condition (31) reduces to the stability of the matrix $\mathbf{A}$. This can be seen clearly from Figures 1(a) and 1(b) since the top-right corner represents $\mathbf{P} = \mathbf{I}$ and $\mathbf{A}_2$ is a stable matrix, whereas $\mathbf{A}_1$ is not.

Although $\mathbf{A}_1$ itself is unstable Figure 1(a) shows that there exist some set of probabilities for which the randomized asynchronous updates on $\mathbf{A}_1$ remain stable, that is, the error term has a bounded covariance. On the other hand, the matrix $\mathbf{A}_2$ itself is stable and the randomized asynchronous updates are stable for some set of probabilities as well. However, it should also be noted that asynchronous updates may get unstable for both matrices for some set of probabilities. Thus,

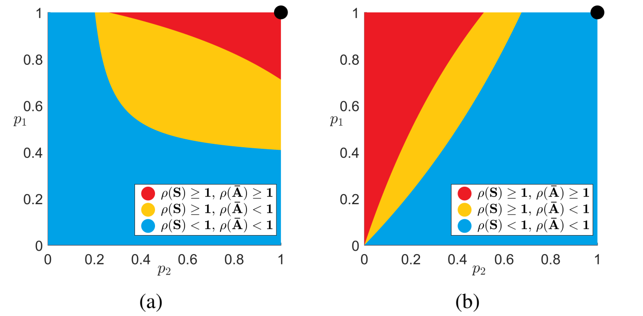


Fig. 1. The set of probabilities that ensures the stability of the randomized asynchronous updates for the matrices (a) $\mathbf{A}_1$, (b) $\mathbf{A}_2$ defined in (35). The top-right corner indicates $\mathbf{P} = \mathbf{I}$, which corresponds to the synchronous case.

we conclude that the stability of the updates depends heavily on the update probabilities as well as the transition matrix.

At this point it is important to point out that non-random asynchronous state recursions in the form of (11) (with $\omega = 0$, $\mathbf{C} = \mathbf{I}$, $\mathbf{D} = \mathbf{0}$) is a well-studied problem in the literature [15]–[17]. In fact, the first analysis of the problem can be traced back to the study in [15] (and references therein), which assumes that only one index is updated per iteration and allows the use of the past values of the iterant. The study showed that the following condition is both necessary and sufficient for the convergence of the updates:

$$\rho(|\mathbf{A}|) < 1, \tag{36}$$

where $|\mathbf{A}|$ is the matrix obtained by replacing the elements of $\mathbf{A}$ by their absolute values. It can be shown that the condition (36) is more restrictive than the stability of the matrix $\mathbf{A}$. Thus, if $\mathbf{A}$ is unstable, then (36) is not satisfied, and [15] proved that the updates do not converge in the sense that there exits an index sequence for which iterations diverge. On the other hand, Lemma 3 shows that the convergence can be achieved for some set of probabilities. (See Figure 1(a).) Although these results appear to be contradictory, the key difference is the notion of convergence. The study [15] ensures the convergence for any index sequence, whereas Lemma 3 considers the convergence in a statistical mean squared sense (for the case of $\omega = 0$).

As a final note, although the condition (31) does not depend on the input frequency ω, the limit of the error correlation matrix given in (32) *does* depend on the input frequency. That is, the covariance of the error term $\mathbf{r}_k$ depends on the input frequency. Numerical examples show that $\mathbf{R}$ does not necessarily decrease monotonically with the input frequency due to the implicit dependence on ω through the term $\mathbf{x}_0^{\mathrm{ss}}$ defined in (18). Nevertheless, as the frequency gets lower $\mathbf{R}$ tends to be smaller. That is, when the input oscillates less, the covariance of the error term $\mathbf{r}_k$ tends to be less. In the particular case of $\omega = 0$, which corresponds to a constant input, it is clear from (32) that $\mathbf{R} = \mathbf{0}$ irrespective of the value of $\mathbf{P}$. That is, the random vector $\mathbf{y}_k$ converges to $\mathbf{D} + \mathbf{C}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}$ in the mean squared sense as long as the condition (31) is satisfied.

III. Numerical Examples

In this section, we will visualize the behavior of the output signal $\mathbf{y}_k$ with respect to the iteration index k and the update probabilities for the state variables. We consider the following state-space model of size $N = 4$:

$$\mathbf{A} = \frac{1}{10}\begin{bmatrix} -4 & -1 & 2 & -6 \\ 4 & -6 & -5 & 3 \\ 2 & -2 & 7 & 2 \\ 5 & 9 & -3 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^{\mathrm{T}}, \ \mathbf{D} = 0, \tag{37}$$

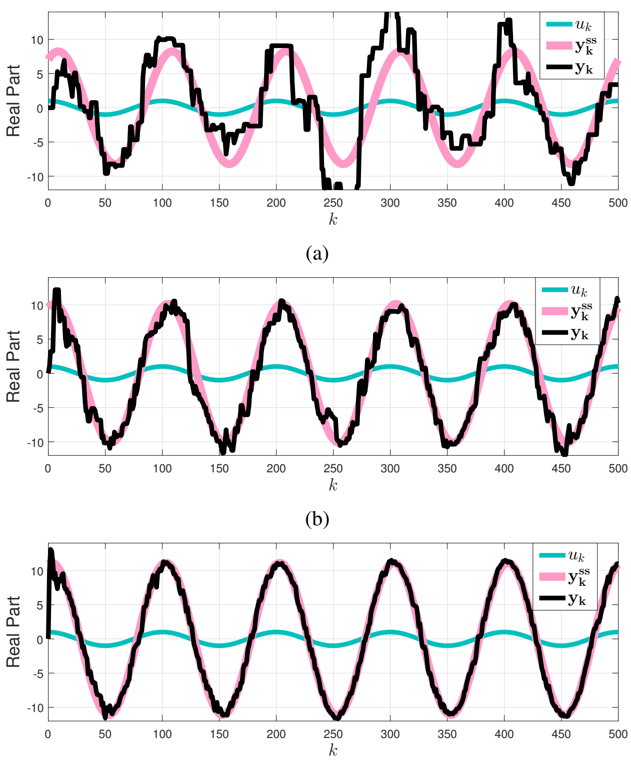
where the matrix $\mathbf{A}$ itself is *not* stable since $\rho(\mathbf{A}) \approx 1.0441$.

For the randomized model in (11) we assume that all the variables are updated with equal probabilities and assume that the input signal has the following frequency:

$$\omega = 2\pi/100, \qquad \text{and} \qquad \mathbf{P} = p\,\mathbf{I}, \tag{38}$$

where p denotes the probability of an index being updated during the iterations. We have numerically verified that as long as $0 < p \leqslant 0.9542$ is satisfied the condition (31) is met, thus the error term keeps having a bounded covariance.

In Figure 2 we visualize *a realization* of the output signal $\mathbf{y}_k$ together with the steady-state component $\mathbf{y}_k^{\text{ss}}$ as well as the input signal u_k for three different update probabilities, namely, $p \in \{0.1,\, 0.3,\, 0.6\}$, all of which are guaranteed to result in a bounded error correlation matrix. We also note that the figure shows only the real part of the signals for convenience.



(a)

(b)

(c)

Fig. 2. A realization of the output signal with the state-space model in (37), the frequency in (38), and the probabilities (a) $p=0.1$, (b) $p=0.3$, (c) $p=0.6$.

From Figure 2 it is clear that $\mathbf{y}_k$ is not exponential in a strict sense, yet it still behaves like an exponential (in expectation) as in (27). We also note that $\mathbf{y}_k$ has the same "frequency" as the input signal irrespective of the update probabilities. However, it should be noted that the probabilities do affect the covariance of the error term, which is apparent both from (32) and Figure 2. Namely, as the indices get updated with higher probabilities, the matrix $\mathbf{R}$ tends to be smaller. For the particular example considered here the value of $\mathbf{R}$ can be found approximately as 28, 3.1, 0.46 for the probabilities 0.1, 0.3, 0.6, respectively, which explain why Figure 2(a) has the highest error variance, and Figure 2(c) has the smallest.

It should also be noted from Figure 2 that the steady-state component $\mathbf{y}_k^{\text{ss}}$ is different in each case. This follows from the fact that the response of the asynchronous system does depend on the update probabilities as given in (18). Thus, different update probabilities result in different "frequency responses" while leaving the output frequency unchanged.

IV. Concluding Remarks & Future Work

In this paper, we considered a randomized asynchronous form of the discrete time-invariant state space models, in which only a random subset of the state variables is updated in each iteration and the remaining ones are kept unchanged. Although the input signal is assumed to be a complex exponential, the output of the system is not exponential in a strict sense due to its being random. We presented the necessary and sufficient condition under which the output of the randomized asynchronous system behaves like an exponential in expectation. In addition, we presented the necessary and sufficient condition under which the error correlation matrix stays bounded and converges to a point whose closed form expression is also provided. We provided numerical examples in which an unstable synchronous system became stable for some specific set of update probabilities. We also visualized a realization of the random output signal and presented its exponential-like behavior with an exponential input.

In future work we plan to investigate the relation between the presented stability condition and the state transition matrix more thoroughly. In particular, we will investigate the cases for which an unstable system gets stable with a randomized asynchronicity. We will study the relation between the input frequency and the update probabilities, and their combined effect on the error correlation matrix. We will also extend the results to the case of multiple input frequencies.

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