On ESPRIT with Multiple Coprime-Invariances

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***Abstract*—In conventional ESPRIT, a single translational invariance in a sensor array is used to obtain high-resolution direction-of-arrival (DOA) estimation. However, when the invariance is greater than the classical sensor spacing $\lambda/2$, spatial frequency ambiguity may occur. In this paper, we propose to use multiple setwise coprime invariances to resolve this ambiguity. While special cases of this were known in the literature, our algorithm is more general in that we consider any number of invariances, and that it can perfectly recover any number of DOAs (limited only in terms of number of sensors) if infinite snapshots are available. We also demonstrate through simulation that our algorithm works well in a practical setting where only finite snapshots are available.**[1]

***Index Terms*—Linear sensor arrays, ESPRIT, multiple invariance, coprime invariance, pairing problem.**

I. Introduction

High-resolution direction-of-arrival (DOA) estimation is an important topic in many sensor systems such as radar and sonar [1]. There are various DOA estimators in the literature, including MUSIC [2] and ESPRIT [3]. By requiring that the sensor array possesses a translational invariance between subarrays, ESPRIT obtains high-resolution DOA estimation at a low cost in computation because ESPRIT does not require searching over the parameter space [3]. ESPRIT also has the advantage that array calibration is not required [3], and continues to be of current research interest [4], [5].

When the translational invariance is greater than the classical sensor spacing $\lambda/2$, where λ is the wavelength of the incoming monochromatic source signals, spatial frequency ambiguity may occur [6]. This poses a limitation to the application of ESPRIT. However, there is an incentive to use a large invariance because this can yield a higher spatial resolution and a lower estimation error variance [6]–[8]. Several previous works [6]–[8] tackle this problem by using multiple invariances. In [7], it is shown that a sensor array with two invariances $M_1\lambda/2$ and $M_2\lambda/2$ can resolve the ambiguity for a single source if M_1 and M_2 are coprime integers. In [8], two invariances $\lambda/2$ and $M_2\lambda/2$, where $M_2 > 1$, are used. The half-wavelength invariance yields unambiguous but high-variance DOA estimates, which are used to disambiguate the low-variance but ambiguous estimates from the larger invariance. Note that the half-wavelength invariance itself already offers unambiguous estimates, so the purpose of using the second invariance in [8] is to lower the error variance only.

A nontrivial pairing problem arises for such multiple invariance scheme when there are *multiple sources* [6], [8]. From each invariance that is greater than $\lambda/2$, ESPRIT yields a set of ambiguous estimates, or equivalently a set of residues in modular arithmetic language, for the sources. However, the correspondence between the elements in the set and the sources is unknown at this point (see Sec. III below for clearer explanation). Hence, we have to pair the elements in the set obtained from one invariance with those obtained from another invariance so that the estimates that correspond to the same source are correctly grouped together. It is claimed in [7] that this pairing problem can be solved for two sources if three pairwise coprime invariances are used, but only up to two sources are considered in [7]. In [8], another method for solving the pairing problem for any number of sources is proposed when the two particular invariances $\lambda/2$ and $M_2\lambda/2$ with $M_2 > 1$ are used.

In this paper, we generalize the idea of using multiple coprime invariances in [7] to resolve the spatial frequency ambiguity. We consider a linear array with sensors located at $n\lambda/2$, where n belongs to an integer set $\mathcal{S}$. Thus, the array is fully defined by this set of integers $\mathcal{S}$. The array $\mathcal{S}$ is said to have an invariance $M\lambda/2$ for some positive integer M if and only if there exist two subarrays $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S}_2 = \mathcal{S}_1 + M$, and is said to have invariances $M_1\lambda/2, \ldots, M_N\lambda/2$ if and only if there exist $\mathcal{S}_{k1}, \mathcal{S}_{k2} \subset \mathcal{S}$ such that $\mathcal{S}_{k2} = \mathcal{S}_{k1} + M_k$ for $k = 1, \ldots, N$.

To avoid the limitations on the number of sources and requirement of pairwise coprimality in [7], we solve the pairing problem by extending the method in [8] to the most general case. In particular, we show that if infinite snapshots are available, the DOA of each source can be perfectly recovered for up to D sources given that (a) a set of N invariances $M_1\lambda/2, \ldots, M_N\lambda/2$ that are setwise coprime, i.e., $(M_1, \ldots, M_N) = 1$, are available to use, and (b) all subarrays contain at least D sensors. In the above, $(M_1, \ldots, M_N)$ denotes the GCD of $M_1, \ldots, M_N$. Except for these conditions, D, N, and M_k are arbitrary positive integers. Setwise coprimality is a milder requirement than pairwise coprimality, so our work broadens the applicability of ESPRIT. A practical algorithm is also proposed for the finite-snapshot regime.

This paper is organized as follows. The signal model and ESPRIT are reviewed in Sec. II. Then, our algorithm for ESPRIT with multiple coprime invariances is introduced in Sec. III. The algorithm for resolving ambiguity for a single source is also presented. Multiple sources are considered in Sec. IV, and the algorithm for solving the pairing problem is described therein. In Sec. V, we show through simulation that our algorithm works well in a practical setting with noise.

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II. Review of ESPRIT with One Invariance

Consider a linear array $\mathcal{S}$. Assume D sources with distinct DOAs $\theta_1, \ldots, \theta_D \in (-\pi/2, \pi/2)$ impinge on the array, with θ measured from the normal to the line of array. Then the received signal vector $\mathbf{x}$ is modeled as

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{w}, \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\omega_1)\mathbf{a}(\omega_2)\cdots\mathbf{a}(\omega_D)]$ with the steering vector $\mathbf{a}(\omega)$ being a column vector containing entries $e^{j\omega n}$ for $n \in \mathcal{S}$ with $\omega = \pi \sin\theta$, $\mathbf{s}$ contains source amplitudes s_i, and $\mathbf{w}$ contains additive noise terms. Assume the noise $\mathbf{w}$ is white with variance σ^2 and uncorrelated with the source signals $\mathbf{s}$. Then, the covariance of $\mathbf{x}$ is

$$\mathbf{R}_{xx} = \mathrm{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I}, \tag{2}$$

where $\mathbf{R}_{ss} = \mathrm{E}[\mathbf{s}\mathbf{s}^H]$ is of rank D. In practice, we obtain an estimate of the data covariance matrix $\mathbf{R}_{xx}$ using a finite number K of snapshots,

$$\widehat{\mathbf{R}}_{xx} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}(k)\mathbf{x}^H(k). \tag{3}$$

The problem is to estimate the DOAs θ_i given the data covariance estimate $\widehat{\mathbf{R}}_{xx}$.

The essence of ESPRIT lies in using a translational invariance buried in the geometry of an array. Suppose there exist two subarrays $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S}_2 = \mathcal{S}_1 + M$, and that each subarray contains at least D sensors [3]. Let $\mathbf{J}_i$ denote the selection matrix such that $\mathbf{J}_i\mathbf{a}(\omega)$ contains entries $e^{j\omega n}$ for $n \in \mathcal{S}_i$, $i = 1, 2$. For example, if $\mathcal{S} = \{0, 1, 2, 3\}$, $\mathcal{S}_1 = \{0, 1, 2\}$, and $\mathcal{S}_2 = \{1, 2, 3\}$, then $\mathbf{J}_1 = [\mathbf{I}_3\ \mathbf{0}]$ and $\mathbf{J}_2 = [\mathbf{0}\ \mathbf{I}_3]$. Since $\mathcal{S}_2 = \mathcal{S}_1 + M$, it can be shown that

$$\mathbf{J}_2\mathbf{A} = \mathbf{J}_1\mathbf{A}\boldsymbol{\Phi}, \tag{4}$$

where $\boldsymbol{\Phi} = \mathrm{diag}\{e^{jM\omega_1}, \ldots, e^{jM\omega_D}\}$. It is assumed in the literature that $\mathbf{J}_i\mathbf{A}$ has full column rank, $i = 1, 2$.

Let $\mathbf{R}_{xx} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^H$ be the eigenvalue decomposition of $\mathbf{R}_{xx}$, where the eigenvalues in $\boldsymbol{\Lambda}$ are in descending order. Partition the columns of $\mathbf{E}$ so that $\mathbf{E} = [\mathbf{E}_s|\mathbf{E}_n]$, where $\mathbf{E}_s$ contains the first D columns of $\mathbf{E}$, and $\mathbf{E}_n$ contains the remaining columns. We call $\mathcal{R}\{\mathbf{E}_s\}$ the signal subspace and $\mathcal{R}\{\mathbf{E}_n\}$ the noise subspace, where $\mathcal{R}\{\mathbf{B}\}$ denotes the column space (range) of the matrix $\mathbf{B}$. From (2) we can derive that $\mathcal{R}\{\mathbf{E}_s\} = \mathcal{R}\{\mathbf{A}\}$. So there exists a unique invertible $D \times D$ matrix $\mathbf{T}$ such that

$$\mathbf{E}_s = \mathbf{A}\mathbf{T}. \tag{5}$$

Using this and (4), we have

$$\mathbf{J}_2\mathbf{E}_s = \mathbf{J}_2\mathbf{A}\mathbf{T} = \mathbf{J}_1\mathbf{A}\boldsymbol{\Phi}\mathbf{T} = \mathbf{J}_1\mathbf{E}_s\mathbf{T}^{-1}\boldsymbol{\Phi}\mathbf{T}. \tag{6}$$

In other words, $\mathbf{J}_1\mathbf{E}_s$ and $\mathbf{J}_2\mathbf{E}_s$ share the same column space. Hence, there exists a unique invertible matrix $\boldsymbol{\Psi}$ such that

$$\mathbf{J}_2\mathbf{E}_s = \mathbf{J}_1\mathbf{E}_s\boldsymbol{\Psi}. \tag{7}$$

Since $\mathbf{T}^{-1}\boldsymbol{\Phi}\mathbf{T}$ is the eigenvalue decomposition of $\boldsymbol{\Psi}$, we can obtain $M\omega_i \mod 2\pi$ from the phase of the i-th eigenvalue of $\boldsymbol{\Psi}$ and thus obtain

$$r_i = \omega_i \mod \frac{2\pi}{M}. \tag{8}$$

Since $\omega_i = \pi \sin\theta_i \in (-\pi, \pi)$, we can obtain the DOAs θ_i without ambiguity only if $M = 1$. Otherwise, ω_i is determined only modulo $2\pi/M$.

In practice, we can only compute the estimated signal subspace $\widehat{\mathbf{E}}_s$ from the data covariance estimate $\widehat{\mathbf{R}}_{xx}$. Then, $\mathcal{R}\{\mathbf{J}_1\widehat{\mathbf{E}}_s\} \neq \mathcal{R}\{\mathbf{J}_2\widehat{\mathbf{E}}_s\}$ with probability one. Instead of solving a set of exact equations $\mathbf{J}_2\widehat{\mathbf{E}}_s = \mathbf{J}_1\widehat{\mathbf{E}}_s\boldsymbol{\Psi}$, we identify $\boldsymbol{\Psi}$ such that these equations are satisfied approximately based on the total least-squares (TLS) criterion [3]. It can be shown [3], [9] that by computing the eigenvalue decomposition

$$[\mathbf{J}_1\widehat{\mathbf{E}}_s|\mathbf{J}_2\widehat{\mathbf{E}}_s]^H[\mathbf{J}_1\widehat{\mathbf{E}}_s|\mathbf{J}_2\widehat{\mathbf{E}}_s] = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^H \tag{9}$$

and partitioning $\mathbf{Q}$ into $D \times D$ submatrices

$$\mathbf{Q} = \begin{bmatrix}\mathbf{Q}_{11} & \mathbf{Q}_{12}\\ \mathbf{Q}_{21} & \mathbf{Q}_{22}\end{bmatrix}, \tag{10}$$

we can obtain the TLS solution $\boldsymbol{\Psi} = -\mathbf{Q}_{12}\mathbf{Q}_{22}^{-1}$.

III. ESPRIT with Multiple Coprime Invariances

As shown in (8), if the array has a single invariance $M\lambda/2$, there is spatial frequency ambiguity unless $M = 1$. In this paper, we will prove:

Theorem 1: If (a) the array has a set of N setwise coprime invariances $M_1\lambda/2, \ldots, M_N\lambda/2$, i.e., $(M_1, \ldots, M_N) = 1$ for some N, and (b) each subarray contains at least D sensors, then the ambiguity can be resolved for D sources.

In the above, D, N, and M_k are arbitrary positive integers except for the conditions mentioned in the theorem.

A nontrivial pairing problem arises if there are multiple sources. The ESPRIT algorithm based on invariance M_k produces a set of residues $\{r_1^{(M_k)}, \ldots, r_D^{(M_k)}\}$, where

$$r_i^{(M_k)} = \omega_i \mod \frac{2\pi}{M_k} \tag{11}$$

for $i = 1, \ldots, D$, and $k = 1, \ldots, N$. The residues are derived from the eigenvalues of $\boldsymbol{\Psi}$ in (7), so the correspondence between the elements in this set and the sources is unknown at this point. In particular, if we order the residues $r_i^{(M_k)}$ in increasing order and order ω_i in increasing order, it is not in general true that the nth element in the first list came from the nth element in the second list. Hence, we have to pair the elements in the set obtained from one invariance with those obtained from another invariance so that the elements corresponding to the same source are correctly grouped together.

We focus on a single DOA in this section and then deal with multiple sources and the pairing problem in Sec. IV. In essence, our constructive proof of Theorem 1 (hence an algorithm) for multiple sources and the pairing problem is composed of three steps:

1) For each invariance, obtain a set of residues by ESPRIT as presented in Sec. II.
2) Solve the pairing problem as described in Sec. IV.
3) Determine each DOA by the method in Sec. III-B.

A. Proof of Uniqueness for a Single DOA

We first consider a single source with DOA ω. Applying the ESPRIT algorithm for each invariance M_k, we obtain

$$r^{(M_k)} = \omega \mod \frac{2\pi}{M_k} \tag{12}$$

for $k = 1, \ldots, N$. We show that ω is uniquely determined by the N residues $r^{(M_1)}, \ldots, r^{(M_N)}$.

Theorem 2: A single DOA $\omega \in (-\pi, \pi)$ is uniquely determined by the residues

$$r^{(M_k)} = \omega \mod \frac{2\pi}{M_k}, \tag{13}$$

$k = 1, \ldots, N$ if the positive integers $M_1, \ldots, M_N$ are setwise coprime, i.e., $(M_1, \ldots, M_N) = 1$.

Proof: Suppose for the sake of a contradiction that there exist integers $l_k, l'_k \in [0, M_k - 1]$ with $l_k \neq l'_k$, $k = 1, \ldots, N$ such that

$$r^{(M_1)} + \frac{2\pi l_1}{M_1} = r^{(M_2)} + \frac{2\pi l_2}{M_2} = \cdots = r^{(M_N)} + \frac{2\pi l_N}{M_N} \tag{14}$$

and

$$r^{(M_1)} + \frac{2\pi l'_1}{M_1} = r^{(M_2)} + \frac{2\pi l'_2}{M_2} = \cdots = r^{(M_N)} + \frac{2\pi l'_N}{M_N}. \tag{15}$$

Subtracting (15) from (14) and dividing the result by 2π, we obtain

$$\frac{l_1 - l'_1}{M_1} = \frac{l_2 - l'_2}{M_2} = \cdots = \frac{l_N - l'_N}{M_N}. \tag{16}$$

Since $|l_k - l'_k| < M_k$ for each k, we can obtain the irreducible fraction

$$\frac{h}{g} = \frac{l_1 - l'_1}{M_1} = \frac{l_2 - l'_2}{M_2} = \cdots = \frac{l_N - l'_N}{M_N} \tag{17}$$

such that $(g, h) = 1$ and $g > 1$. Thus, $M_k = g(l_k - l'_k)/h$ for $k = 1, \ldots, N$, which implies that $g > 1$ is a common divisor of $M_1, \ldots, M_N$. This contradicts to $(M_1, \ldots, M_N) = 1$, completing the proof. ■

B. Method for Determining a Single DOA

In Sec. III-A, we proved that a single DOA ω is uniquely determined by the residue set $\{r^{(M_1)}, \ldots, r^{(M_N)}\}$. In this subsection, we offer a method to recover ω from the residue set. We first consider the case of two coprime invariances M_1 and M_2 and then extend the method to more than two invariances.

Suppose we have

$$\omega = r^{(M_1)} + \frac{2\pi l_1}{M_1} = r^{(M_2)} + \frac{2\pi l_2}{M_2}, \tag{18}$$

where $r^{(M_1)}$ and $r^{(M_2)}$ are known, while the integers $l_k \in [0, M_k - 1]$, $k = 1, 2$, and thus ω are to be determined. A naive way is to do an exhaustive search of all possible combinations of l_1 and l_2, but the complexity is quite large, $O(M_1 M_2)$. Moreover, this naive way does not work if we only have estimates of $r^{(M_1)}$ and $r^{(M_2)}$. Instead of doing this, we write from (18)

$$d \triangleq \frac{(r^{(M_1)} - r^{(M_2)}) M_1 M_2}{2\pi} = l_2 M_1 - l_1 M_2. \tag{19}$$

Since $(M_1, M_2) = 1$, we can use the extended Euclidean algorithm to find integers s and t such that $sM_1 + tM_2 = 1$, or $dsM_1 + dtM_2 = d$. Then, we can show that the solution is

$$l_1 = -dt \mod M_1, l_2 = ds \mod M_2, \tag{20}$$

and thus

$$\omega = r^{(M_1)} + (r^{(M_2)} - r^{(M_1)}) t M_2 \mod 2\pi \tag{21}$$

$$= r^{(M_2)} + (r^{(M_1)} - r^{(M_2)}) s M_1 \mod 2\pi. \tag{22}$$

Both (21) and (22) yield the same solution of ω, so we can use either of them. Besides, in practice, when we only have the estimates of $r^{(M_1)}$ and $r^{(M_2)}$, these can still be applied to get an estimate of ω.

Now suppose for some $N > 2$ we have

$$\omega = r^{(M_1)} + \frac{2\pi l_1}{M_1} = \cdots = r^{(M_N)} + \frac{2\pi l_N}{M_N}, \tag{23}$$

where $r^{(M_1)}, \ldots, r^{(M_N)}$ are known, while the integers $l_k \in [0, M_k - 1]$, $k = 1, \ldots, N$, and thus ω are to be determined. Here we only have setwise coprimality $(M_1, \ldots, M_N) = 1$. Suppose $(M_1, M_2) = P_1$ and $M_k = P_1 Q_k$ for $k = 1, 2$. Using this and (23), we obtain

$$P_1 \omega = P_1 r^{(M_1)} + \frac{2\pi l_1}{Q_1} = P_1 r^{(M_2)} + \frac{2\pi l_2}{Q_2}. \tag{24}$$

Since $(Q_1, Q_2) = 1$, we can use an expression similar to (21) or (22) to obtain $P_1 \omega \mod 2\pi$. Thus, we can obtain

$$\omega = r^{(M_1, M_2)} + \frac{2\pi p_1}{P_1}, \tag{25}$$

where $r^{(M_1, M_2)}$ is known, while the integer $p_1 \in [0, P_1 - 1]$ is to be determined. Combining (25) and

$$\omega = r^{(M_3)} + \frac{2\pi l_3}{M_3} \tag{26}$$

from (23), we can then similarly obtain $P_2 \omega \mod 2\pi$, where $P_2 = (P_1, M_3) = (M_1, M_2, M_3)$. Continuing this process, we can finally obtain $P_{N-1} \omega \mod 2\pi$, where $P_{N-1} = (P_{N-2}, M_N) = (M_1, \ldots, M_N) = 1$. That is, the unambiguous ω is obtained.

In our proposed method above, we need to compute either (21) or (22) for $N - 1$ times, so its complexity is $O(N)$, much lower than $O(\Pi_{k=1}^{N} M_k)$ of the exhaustive search. Moreover, the integers s and t from the extended Euclidean algorithm can be precomputed because they do not depend on $r^{(M_1)}, \ldots, r^{(M_N)}$.

IV. Pairing Problem for Multiple Sources

In this section, we propose an algorithm to solve the pairing problem described in the beginning of Sec. III. The ESPRIT algorithm based on each invariance M_k produces a set of residues $\{r_1^{(M_k)}, \ldots, r_D^{(M_k)}\}$ as defined in (11). We have to pair the elements in the set obtained from one invariance with those obtained from another invariance so that the elements corresponding to the same source are correctly grouped together. Note that there are $(D!)^{N-1}$ possible ways of pairing for D sources and N invariances.

Although not all ways of pairing can result in feasible solutions, there are indeed scenarios where multiple ways of pairing are all feasible, i.e., ambiguity occurs [7]. In [8], it is proposed that the pairing problem for two invariances $M_1\lambda/2$ and $M_2\lambda/2$ with $M_1 = 1, M_2 > 1$ can be solved for any number of sources by using the matrix $\mathbf{T}$ in (5) in the ESPRIT algorithm. We extend the method in [8] to solve the pairing problem for the most general case, i.e., any number D of sources and any number N of setwise coprime invariances $M_1\lambda/2, \ldots, M_N\lambda/2$ with $(M_1, \ldots, M_N) = 1$.

A. Solution to the Pairing Problem

As shown in Sec. II, for each invariance M_k, the set of residues $\{r_1^{(M_k)}, \ldots, r_D^{(M_k)}\}$ are obtained through the eigenvalues of the matrix $\mathbf{\Psi}^{(M_k)}$ that satisfies $\mathbf{J}_2^{(M_k)}\mathbf{E}_s = \mathbf{J}_1^{(M_k)}\mathbf{E}_s\mathbf{\Psi}^{(M_k)}$. An important observation is that according to (6), $\mathbf{\Psi}^{(M_k)}$ can always be decomposed into $\mathbf{\Psi}^{(M_k)} = \mathbf{T}^{-1}\mathbf{\Phi}^{(M_k)}\mathbf{T}$ for $k = 1, \ldots, N$, where $\mathbf{T}$ is the unique matrix satisfying (5). If each $\mathbf{\Psi}^{(M_k)}$ has distinct eigenvalues, then its eigenvalue decomposition $\mathbf{\Psi}^{(M_k)} = [\mathbf{T}^{(M_k)}]^{-1}\mathbf{\Phi}^{(M_k)}\mathbf{T}^{(M_k)}$ is unique up to row permutations of $\mathbf{T}^{(M_k)}$. Thus, by matching the rows of $\mathbf{T}^{(M_k)}$ for all k, we can pair the eigenvalues contained in $\mathbf{\Phi}^{(M_k)}$ for different k, and thus pair the elements in the set of residues $\{r_1^{(M_k)}, \ldots, r_D^{(M_k)}\}$ from one invariance with another.

In general, $\mathbf{\Phi}^{(M_k)}$ may have some eigenvalues with multiplicity greater than one. Suppose there are R_k distinct eigenvalues in $\mathbf{\Phi}^{(M_k)}$, grouped as

$$\mathbf{\Phi}^{(M_k)} = \text{diag}\Big\{\underbrace{\lambda_1^{(M_k)}, \ldots, \lambda_1^{(M_k)}}_{n_1^{(M_k)}}, \underbrace{\lambda_2^{(M_k)}, \ldots, \lambda_2^{(M_k)}}_{n_2^{(M_k)}}, \ldots, \underbrace{\lambda_{R_k}^{(M_k)}, \ldots, \lambda_{R_k}^{(M_k)}}_{n_{R_k}^{(M_k)}}\Big\}, \tag{27}$$

and

$$[\mathbf{T}^{(M_k)}]^T = \left[[\mathbf{T}_1^{(M_k)}]^T \cdots [\mathbf{T}_{R_k}^{(M_k)}]^T\right], \tag{28}$$

where rows of $\mathbf{T}_m^{(M_k)}$ contain the (left) eigenvectors corresponding to the eigenvalue $\lambda_m^{(M_k)}$. When each $\mathbf{\Phi}^{(M_k)}$ has all distinct eigenvalues, each $\mathbf{T}^{(M_k)}$ is equal to $\mathbf{T}^{(M_1)}$ after some row permutations. When some of the eigenvalues have multiplicities, the corresponding eigenvectors can be any set of vectors that form a basis of the eigenspace. Therefore, for each m, k, there must exist invertible matrices $\mathbf{U}_m^{(M_k)}, \mathbf{V}_1^{(M_1)}, \ldots, \mathbf{V}_{R_1}^{(M_1)}$ and row selection matrices $\mathbf{J}_{m,1}^{(M_k)}, \ldots, \mathbf{J}_{m,R_1}^{(M_k)}$ such that

$$\mathbf{U}_m^{(M_k)}\mathbf{T}_m^{(M_k)} = \begin{bmatrix} \mathbf{J}_{m,1}^{(M_k)}\mathbf{V}_1^{(M_1)}\mathbf{T}_1^{(M_1)} \\ \vdots \\ \mathbf{J}_{m,R_1}^{(M_k)}\mathbf{V}_{R_1}^{(M_1)}\mathbf{T}_{R_1}^{(M_1)} \end{bmatrix}, \tag{29}$$

where $m = 1, \ldots, R_k$. Let $\mathcal{S}_m^{(M_k)}$ denote the eigenspace corresponding to $\lambda_m^{(M_k)}$. Eq. (29) implies that

$$\#\text{rows}\left(\mathbf{J}_{m,n}^{(M_k)}\right) = \dim\left(\mathcal{S}_m^{(M_k)} \cap \mathcal{S}_n^{(M_1)}\right), \tag{30}$$

which equals the number of times for which we should pair $\lambda_m^{(M_k)}$ and $\lambda_n^{(M_1)}$. We show how to obtain this number in the following theorem.

Theorem 3: For each k, m, n, define

$$d_{mn}^{(M_k)} = \text{rank}\left(\left(\mathbf{T}^{(M_k)}[\mathbf{T}^{(M_1)}]^{-1}\right)_{\mathcal{I}_m^{(M_k)}, \mathcal{I}_n^{(M_1)}}\right), \tag{31}$$

where $\mathcal{I}_m^{(M_k)} = \left\{\sum_{l=1}^{m-1} n_l^{(M_k)} + 1, \sum_{l=1}^{m} n_l^{(M_k)}\right\}$ is the index set corresponding to the eigenvalue $\lambda_m^{(M_k)}$, and $\mathbf{B}_{\mathcal{I},\mathcal{J}}$ denotes the submatrix of the matrix $\mathbf{B}$ corresponding to the rows and columns specified by the index sets $\mathcal{I}$ and $\mathcal{J}$, respectively. Then, we have

$$d_{mn}^{(M_k)} = \dim\left(\mathcal{S}_m^{(M_k)} \cap \mathcal{S}_n^{(M_1)}\right). \tag{32}$$

Proof: Multiplying (29) by $[\mathbf{T}^{(M_1)}]^{-1}$, we obtain

$$\begin{aligned} &\mathbf{U}_m^{(M_k)}\mathbf{T}_m^{(M_k)}[\mathbf{T}^{(M_1)}]^{-1} \\ &= \begin{bmatrix} \mathbf{J}_{m,1}^{(M_k)}\mathbf{V}_1^{(M_1)}\mathbf{T}_1^{(M_1)} \\ \vdots \\ \mathbf{J}_{m,R_1}^{(M_k)}\mathbf{V}_{R_1}^{(M_1)}\mathbf{T}_{R_1}^{(M_1)} \end{bmatrix} [\mathbf{T}^{(M_1)}]^{-1} \qquad (33) \\ &= \begin{bmatrix} \mathbf{J}_{m,1}^{(M_k)}\mathbf{V}_1^{(M_1)} & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & \mathbf{J}_{m,R_1}^{(M_k)}\mathbf{V}_{R_1}^{(M_1)} \end{bmatrix}. \qquad (34) \end{aligned}$$

Thus, using (30), (31), (34), and the fact that $\mathbf{U}_m^{(M_k)}$ and $\mathbf{V}_n^{(M_1)}$ are invertible, we can derive

$$\begin{aligned} d_{mn}^{(M_k)} &= \text{rank}\left(\left(\mathbf{T}_m^{(M_k)}[\mathbf{T}^{(M_1)}]^{-1}\right)_{:,\mathcal{I}_n^{(M_1)}}\right) && (35) \\ &= \text{rank}\left(\left(\mathbf{U}_m^{(M_k)}\mathbf{T}_m^{(M_k)}[\mathbf{T}^{(M_1)}]^{-1}\right)_{:,\mathcal{I}_n^{(M_1)}}\right) && (36) \\ &= \text{rank}\left(\begin{bmatrix} \mathbf{O} \\ \mathbf{J}_{m,n}^{(M_k)}\mathbf{V}_n^{(M_1)} \\ \mathbf{O} \end{bmatrix}\right) && (37) \\ &= \#\text{rows}\left(\mathbf{J}_{m,n}^{(M_k)}\right) && (38) \\ &= \dim\left(\mathcal{S}_m^{(M_k)} \cap \mathcal{S}_n^{(M_1)}\right), && (39) \end{aligned}$$

where $\mathbf{B}_{:,\mathcal{J}}$ denotes the submatrix of the matrix $\mathbf{B}$ corresponding to the columns specified by the index set $\mathcal{J}$. ■

According to Theorem 3, if the exact signal subspace $\mathbf{E}_s$ (obtained from infinite snapshots) is available, then by computing and grouping the exact eigenvalues and eigenvectors as in (27) and (28) and computing $d_{mn}^{(M_k)}$ as defined in (31), we solve the pairing problem by pairing $\lambda_m^{(M_k)}$ and $\lambda_n^{(M_1)}$ for $d_{mn}^{(M_k)}$ times, for $m = 1, \ldots, R_k, n = 1, \ldots, R_1$ and $k = 2, \ldots, N$.

B. Practical Algorithm in the Finite-Snapshot Regime

In practice, we only have the estimated $\widehat{\mathbf{E}}_s$ (obtained from finite snapshots) and thus estimated eigenvalues in (27). Hence, the eigenvalues obtained are all distinct with probability one. However, when two eigenvalues are close to each other, the corresponding eigenvectors are likely to influence each other

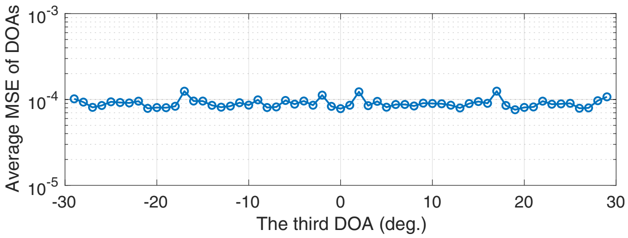

Fig. 1. Average MSE of DOAs.

due to effects of finite snapshots and noise. Thus, it is still beneficial to regard these eigenvalues as equal and group them together. Notice that the eigenvalues in (27) ideally have unit magnitudes in the infinite-snapshot case. So in our algorithm, we propose to regard two eigenvalues as equal if their phases are within a threshold η. Then we replace the eigenvalues in the same group by the average of them. How to set the threshold η is a design problem and should depend on the error variances of the eigenvalues. One may refer to [6] for the analysis of the error variances of the eigenvalues, but the expressions are very complex, and the marginal effect of incorporating them into our algorithm is not very significant.

After grouping the eigenvalues based on the threshold η, instead of computing rank as in (31), we compute the singular values of the R_1 matrices

$$\mathbf{S}_{mn}^{(M_k)} = \left(\mathbf{T}^{(M_k)}[\mathbf{T}^{(M_1)}]^{-1}\right)_{\mathcal{I}_m^{(M_k)},\mathcal{I}_n^{(M_1)}}, \tag{40}$$

$n = 1, \ldots, R_1$. Then, among all these singular values, we take the $n_m^{(M_k)}$ largest ones. We pair $\lambda_m^{(M_k)}$ and $\lambda_n^{(M_1)}$ for $\hat{d}_{mn}^{(M_k)}$ times if there are $\hat{d}_{mn}^{(M_k)}$ singular values of $\mathbf{S}_{mn}^{(M_k)}$ in the set of the $n_m^{(M_k)}$ singular values mentioned above. Repeat the method for $m = 1, \ldots, R_k$ and $k = 2, \ldots, N$.

Note that according to (27) and (32), we can derive

$$n_m^{(M_k)} = \sum_{n=1}^{R_1} d_{mn}^{(M_k)}, n_n^{(M_1)} = \sum_{m=1}^{R_m} d_{mn}^{(M_k)}. \tag{41}$$

By design, our algorithm always yield $\hat{d}_{mn}^{(M_k)}$ satisfying

$$n_m^{(M_k)} = \sum_{n=1}^{R_1} \hat{d}_{mn}^{(M_k)}, \tag{42}$$

as in (41). However, $\hat{d}_{mn}^{(M_k)}$ may not satisfy

$$n_n^{(M_1)} = \sum_{m=1}^{R_m} \hat{d}_{mn}^{(M_k)}, \tag{43}$$

as in (41). Hence, for each Monte Carlo trial, if (43) is not satisfied for some n, we declare failure. According to our simulations, with a suitable threshold η, the failure rate can be essentially zero, which shows that our algorithm is practical.

V. Simulations

A numerical example is given in Fig. 1 to show the effectiveness of our algorithm in a practical setting. Only $N = 2$ coprime invariances, $M_1 = 2$ and $M_2 = 3$, are used to estimate $D = 5$ sources. The sensor array is given by $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where $\mathcal{S}_1 = \{0, 10, \ldots, 70, 79, 88, \ldots, 133\}$, $\mathcal{S}_2 = \mathcal{S}_1 + 2$, and $\mathcal{S}_3 = \mathcal{S}_2 + 3$. (Sensor spacing is measured in $\lambda/2$.) Note that there are no pair of sensors in $\mathcal{S}$ that are separated by $\lambda/2$. Standard methods for DOA estimation will therefore only yield ambiguous results, and are not compared with our method in Fig. 1. The DOAs are $-45°, -30°, \theta, 30°, 45°$, where the third DOA θ is varied from $-29°$ to $29°$. The sources are uncorrelated with equal power $\sigma_i^2 = 1$, and the noise power is $\sigma^2 = 0.1$. The number of snapshots is $K = 500$. The average mean square error (MSE) of the DOAs, defined as $(\sum_{i=1}^{D}\sum_{l=1}^{L}(\hat{\omega}_i(l) - \omega_i)^2)/DL$, is computed using $L = 500$ Monte Carlo trials, where $\hat{\omega}_i(l)$ is the estimate of $\omega_i = \pi \sin\theta_i$ obtained from the l-th trial. The threshold for grouping the eigenvalues is $\eta = 0.04$. With this threshold, the failure rates for (43) are *all zero* in this example. Moreover, as shown in Fig. 1, over the whole range of DOAs experimented, the average RMSEs are reasonably small. As presented in Sec. III, in our algorithm any number of setwise coprime invariances can be used to resolve D sources as long as each subarray contains at least D sensors. In particular, 2 coprime invariances are sufficient. This example shows that our algorithm does quite well for resolving 5 sources using only 2 coprime invariances in a practical setting.

VI. Conclusion

A scheme for ESPRIT using multiple setwise coprime invariances is proposed. While special cases of this were known in the literature, our algorithm is more general in that we consider any number of invariances, and that it can perfectly recover any number of DOAs (limited only in terms of number of sensors) if infinite snapshots are available. A numerical example also shows that our algorithm works well in a practical finite-snapshot regime. Future directions include designing a systematic way to setting the threshold η for grouping the eigenvalues.

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