The dual Minkowski problem for symmetric convex bodies

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1. Introduction

Geometric measures of convex bodies in Euclidean space and their associated Minkowski problems are of central interest in the subject of convex geometric analysis. In the classical Minkowski problem, it is the surface area measure of a convex body that is prescribed (in the smooth case, it is the Gauss curvature). The solution to the classical Minkowski problem has had many applications in various fields of analysis and geometry. See Section 8.2 in Schneider [47] for an overview. The Christoffel-Minkowski problem (prescribing j-th surface area measures) and the Aleksandrov problem of prescribing j-th curvature measures are two other important Minkowski problems in convex geometric analysis that are still unsolved. See, for example, Sections 8.4 and 8.5 in [47]. These Minkowski problems belong to the classical Brunn-Minkowski theory.

More recently, Lutwak [40] introduced the L_p Brunn-Minkowski theory, where $p = 1$ is the classical theory cited above, and posed the L_p Minkowski problem (prescribing L_p surface area measure) as a fundamental question. The most important (and therefore most challenging) cases include, when $p = 0$, the logarithmic Minkowski problem (see Böröczky-Lutwak-Yang-Zhang [10]) and, when $p = -n$, the centro-affine Minkowski problem (see Chou-Wang [16] and Zhu [58]). The L_p Minkowski problem when $p > 1$ was solved by Lutwak [40] for symmetric convex bodies and by Chou-Wang [16] in the general case. Alternate proofs were given by Hug-Lutwak-Yang-Zhang [32]. The case where $p < 1$ is still largely open (see Böröczky-Lutwak-Yang-Zhang [10], Huang-Liu-Xu [29], Jian-Lu-Wang [33], and Zhu [57, 59]). For other recent progress on the L_p-Minkowski problem, see Böröczky-Trinh [12] and Chen-Li-Zhu [13, 14]. The L_p Minkowski problem also plays a key role in establishing affine Sobolev inequalities (see, for example, Lutwak-Yang-Zhang [41, 42], Cianchi-Lutwak-Yang-Zhang [15], and Haberl-Schuster [27]).

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Very recently, Huang-Lutwak-Yang-Zhang [30] introduced *dual curvature measures* $\widetilde{C}_q$, where $q \in \mathbb{R}$, as the natural duals to Federer's curvature measures. These are fundamental in the dual Brunn-Minkowski theory and the analogs of the surface area measures in Brunn-Minkowski theory. This leads naturally to the *dual Minkowski problem* of prescribing dual curvature measures. Remarkably, the family of dual Minkowski problems connects the well-known Aleksandrov problem ($q = 0$) to the logarithmic Minkowski problem ($q = n$) mentioned above. Here we present a complete solution to the dual Minkowski problem within the class of origin-symmetric convex bodies for the critical strip $0 < q < n$.

The dual Brunn-Minkowski theory was first introduced by Lutwak, based on a conceptual but mysterious duality[3] in convex geometry (see Schneider [47], p. 507 for a lucid explanation). The power of the theory was demonstrated when intersection bodies, which are central to the dual Brunn-Minkowski theory, played a crucial role in the solution to the well-known Busemann-Petty problem. The solution relied on connections between the dual theory and harmonic analysis. See, for example, Bourgain [5], Gardner [18], Gardner-Koldobsky-Schlumprecht [20], Lutwak [39], and Zhang [53], and see Gardner [17] and Koldobsky [34] for additional references.

Dual curvature measures, parameterized by $q \in \mathbb{R}$, are the analogues in the dual Brunn-Minkowski theory of Federer's curvature measures in the classical Brunn-Minkowski theory. The 0-th dual curvature measure is (a constant multiple of) Aleksandrov's integral curvature of the polar body. The n-th dual curvature measure is the cone volume measure studied in Barthe, Guédon, Mendelson-Naor [4], Böröczky-Henk [7], Henk-Linke [28], Ludwig-Reitzner [37], Stancu [50, 51], and Zou-Xiong [60]. Dual curvature measures encode the geometry of a convex body's interior, while their counterparts in the Brunn-Minkowski theory reflect the geometry of the boundary. Dual curvature measures are a new class of valuations (i.e., finitely additive geometric invariants of convex bodies) that are dual to their counterparts in the Brunn-Minkowski theory. The latter have been studied extensively in recent years. See, for example, Böröczky-Ludwig [9], Haberl [23], Haberl-Ludwig [24], Haberl-Parapatits [25, 26], Ludwig [35, 36], Ludwig-Reitzner [37], Schuster [48, 49], Zhao [54] and the references therein.

The dual Minkowski problem for dual curvature measures proposed in Huang-Lutwak-Yang-Zhang [30] states:

The Dual Minkowski Problem. *Given a finite Borel measure μ on the unit sphere S^{n-1} and a real number q, find necessary and sufficient conditions on μ so that there exists a convex body $K \subset \mathbb{R}^n$ that solves,*

$$\widetilde{C}_q(K, \cdot) = \mu, \tag{1.1}$$

where $\widetilde{C}_q(K, \cdot)$ is the q-th dual curvature measure of K.

In the special case when the given measure (which is called the "*data*") has a density f, then (1.1) reduces to the Monge-Ampère type equation on S^{n-1} given by

$$\det(\overline{\nabla}^2 h + hI) = n(|\overline{\nabla} h|^2 + h^2)^{(n-q)/2} h^{-1} f, \tag{1.2}$$

[3]Although Lutwak's duality is motivated by the duality between intersections and projections in projective geometry, it is a duality of concepts (such as mixed volumes) instead of the usual duality between points and hyperplanes in a vector space.

where f is a given non-negative integrable function, h is the unknown function, I is the standard Riemannian metric on S^{n-1}, while $\overline{\nabla}h$ and $\overline{\nabla}^2 h$ are the gradient and Hessian of h, with respect to I, respectively.

Dual Minkowski problems, including the logarithmic Minkowski problem, are more challenging than previously solved Minkowski problems. This arises from the phenomenon of measure concentration, which implies that there are singular prescribed measures for which no solutions are possible. Thus, there is no straightforward way to solve these general problems by first solving the smooth data case of (1.2) and then using an approximation argument to solve (1.1).

When $q = 0$, the dual Minkowski problem is the classical Aleksandrov problem, which was posed and solved by Aleksandrov [1], using a topological argument. See Guan-Li [22], Oliker [43], [44], [45], and Wang [52] for other work on this problem and its variants. The L_p version of the Aleksandrov problem was introduced and studied by Huang-Lutwak-Yang-Zhang [31].

When $q = n$, the dual Minkowski problem is the logarithmic Minkowski problem, which was solved for even data (a measure that assumes the same value on antipodal Borel subsets of S^{n-1}) by Böröczky-Lutwak-Yang-Zhang [10]. The logarithmic Minkowski problem remains open for data that is not even (see, for example, Böröczky-Hegedűs-Zhu [6], Stancu [50, 51], Zhu [57]). Surprisingly, the logarithmic Minkowski problem is closely connected to isotropic measures (see Böröczky-Lutwak-Yang-Zhang [11]) and also to curvature flows (see Andrews [2, 3]). It was discovered that a measure concentration condition (described in the next paragraph) is the precise obstruction to the existence of solutions to this singular Monge-Ampère equation.

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if

$$\frac{\mu(\xi \cap S^{n-1})}{\mu(S^{n-1})} \leq \frac{\dim \xi}{n}, \tag{1.3}$$

for each proper subspace $\xi \subset \mathbb{R}^n$ and, if equality holds for a subspace ξ, there exists a subspace $\xi' \subset \mathbb{R}^n$ complementary to ξ such that μ is concentrated on $S^{n-1} \cap (\xi \cup \xi')$. Böröczky-Lutwak-Yang-Zhang [10] proved that if μ is an even finite Borel measure, then there exists an origin-symmetric convex body whose cone volume measure is equal to μ if and only if μ satisfies the subspace concentration condition.

A similar phenomenon arose in the attempt of Huang-Lutwak-Yang-Zhang [30] to solve the dual Minkowski problem for symmetric convex bodies. However, the conditions presented in the attempt of Huang-Lutwak-Yang-Zhang [30] turned out to be sufficient but not necessary. A more refined subspace mass inequality, which first appeared in [8, 55], is the following:

Subspace Mass Inequality. For $0 < q < n$, a finite Borel measure μ on S^{n-1} is said to satisfy *the q-th subspace mass inequality*, if

$$\frac{\mu(\xi(i) \cap S^{n-1})}{\mu(S^{n-1})} < \begin{cases} i/q, & \text{when } i < q, \\ 1, & \text{when } i \geq q, \end{cases} \tag{1.4}$$

for each proper i-dimensional subspace $\xi(i) \subset \mathbb{R}^n$. Böröczky-Henk-Pollehn [8] showed that, when $1 < q < n$, the q-th subspace mass inequality is a necessary condition for the existence of solutions to the dual Minkowski problem within the class of origin-symmetric convex bodies. That is, the q-th dual curvature measure of every origin-symmetric convex body satisfies the q-th subspace mass inequality. Zhao [55] showed that, when $q \in \{2, \ldots, n-1\}$, the q-th subspace mass inequality is also a sufficient condition for the existence of solutions to the dual Minkowski problem. That is, every even finite Borel measure satisfying the q-th subspace mass inequality is the q-th dual curvature measure of an origin-symmetric convex body. This provides a complete solution to the dual Minkowski problem for even data and integer $q \in \{2, \ldots, n-1\}$ within the class of origin-symmetric convex bodies.

The aim of this paper is to give a complete solution to the dual Minkowski problem for even data and all real $q \in (0, n)$.

Theorem 1.1. *Let $0 < q < n$ and μ be a non-zero even finite Borel measure on S^{n-1}. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $\widetilde{C}_q(K, \cdot) = \mu$ if and only if μ satisfies the q-th subspace mass inequality* (1.4).

When $0 < q \leq 1$, the q-th subspace mass inequality says that the measure μ cannot be concentrated on any great hypersphere. Theorem 1.1 for this case was established in [30]. When $1 < q < n$, the necessity of the q-th subspace mass inequality was proved in [8], and its sufficiency, when q is an integer such that $1 < q < n$, was established in [55].

The solution to the dual Minkowski problem for $q < 0$ does not require any non-trivial measure concentration conditions as shown by Zhao [56]. The dual Minkowski problem for even data and $q = 0$ is equivalent to the Aleksandrov problem for even data, which was solved by Aleksandrov himself. Alternate approaches appear in [31] and [45] (see also [43], [44]). When $q = n$, the dual Minkowski problem for even data is the even logarithmic Minkowski problem, which was solved in [10].

Unlike the classical Minkowski problem, it is difficult to see how it might be possible to reduce the case of the dual Minkowski problem where $q > 0$ to the case where the measure has a density. Moreover, estimates for the dual quermassintegrals of degree $q > 0$, but $q \neq n$, are much more difficult to obtain than when $q = n$, where the dual quermassintegral is just volume and only an entropy estimate is needed. More delicate estimates for both entropy and the dual quermassintegrals are required when $q > 0$ but $q \neq n$.

The proof presented here uses a variational approach. The maximization problem associated with the dual Minkowski problem is described in Section 3. Its solution requires two crucial estimates. In Section 4, we prove an estimate for an entropy integral using the technique of spherical partitions introduced in [10].

The role of barrier bodies in obtaining integral estimates is the same as that of barrier functions in obtaining PDE estimates. Choosing a proper barrier body and establishing a sharp estimate are critical in showing that the q-th subspace mass inequality is both necessary and sufficient for solving the dual Minkowski problem. However, for a dual quermassintegral of real degree $q > 0$, choosing an appropriate barrier is considerably more difficult than it was in [30] and [55]. In Section 5 we use a Gaussian integral trick to establish the needed estimate.

In [30], a cross-polytope was used as the barrier to establish a sufficient condition for the cases considered there. In [55], using the Cartesian product of an ellipsoid and a ball as the barrier showed that (1.4) is both necessary and sufficient. Unfortunately, this works only for integer $q \in \{2, \ldots, n-1\}$. In this work, that (1.4) is both necessary and sufficient will be established for all real $q \in (0, n)$ by taking as a barrier a Cartesian product of an ellipsoid, a line segment, and a ball. The estimates of its dual quermassintegrals appear in Section 5.

The work presented here extends significantly the results and techniques in [10], [30] and [55].

2. Preliminaries

The needed basics from the theory of convex bodies will be reviewed in this section. Details can be found in the books [17] and [47].

We will work in $\mathbb{R}^n$ equipped with the standard Euclidean norm. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the inner product of x and y, and let $|x| = \sqrt{x \cdot x}$.

We shall write $C(S^{n-1})$ for the vector space of continuous functions on the unit sphere S^{n-1} equipped with the max norm; i.e., $\|f\| = \max\{|f(u)| : u \in S^{n-1}\}$, for each $f \in C(S^{n-1})$. Let $C^+(S^{n-1}) \subset C(S^{n-1})$ denote the cone of strictly positive functions, $C_e(S^{n-1}) \subset C(S^{n-1})$ the subspace of even functions, and $C_e^+(S^{n-1}) = C^+(S^{n-1}) \cap C_e(S^{n-1})$.

We always assume that a measure is nonzero and finite. A measure is said to be even, if its value on a measurable set is equal to that of its antipodal set. The total measure of a measure μ will be written as $|\mu|$. Throughout the paper, an expression $c_{\ldots}$, with a subscript containing a list of parameters, represents a "constant", whose exact value depends on the parameters listed but may change from line to line. For example, $c_{n,k,q}$ depends only on n, k, q, and nothing else. At times, this may also be written as $c(n, k, q)$. Denote by $\lfloor q \rfloor$ the floor function whose value is the largest integer less than or equal to q.

We say that $K \subset \mathbb{R}^n$ is a *convex body* if it is a compact convex set with non-empty interior. The boundary of K is written as ∂K. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. The set of all convex bodies containing the origin in the interior is denoted by $\mathcal{K}_o^n$, and the set of all origin-symmetric convex bodies by $\mathcal{K}_e^n$. Obviously, $\mathcal{K}_e^n \subset \mathcal{K}_o^n \subset \mathcal{K}^n$.

Associated with a compact convex $K \subset \mathbb{R}^n$ is its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.1}$$

For real $c > 0$, define the compact convex set cK by $h_{cK} = ch_K$. A sequence of convex bodies K_l is said to converge to a compact convex set $K \subset \mathbb{R}^n$ with respect to the *Hausdorff metric* provided that

$$\|h_{K_l} - h_K\| \to 0.$$

If $K \subset \mathbb{R}^n$ is compact and star-shaped with respect to the origin, its *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined, for $x \in \mathbb{R}^n \setminus \{0\}$, by

$$\rho_K(x) = \max\{t \geq 0 : tx \in K\}. \tag{2.2}$$

If ρ_K is positive and continuous, K is called a *star body*, and the set of star bodies is denoted by $\mathcal{S}_o^n$. Obviously, $\mathcal{K}_o^n \subset \mathcal{S}_o^n$. We shall need the trivial observation that if $K \in \mathcal{S}_o^k$

and $L \in \mathcal{S}_o^{n-k}$, then, for $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $x, y \neq o$,

$$\rho_{K\times L}(x, y) = \min\{\rho_K(x), \rho_L(y)\}. \tag{2.3}$$

Note that, for $K \in \mathcal{K}_o^n$, both h_K and ρ_K are positive. The volume (i.e., Lebesgue measure) of K will be denoted $V_n(K)$. When it is clear that the ambient dimension n is, the subscript is often suppressed, and we will write simply $V(K)$. It is easily shown that

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^n du, \tag{2.4}$$

where du represents the spherical Lebesgue measure.

For each $h \in C^+(S^{n-1})$, the *Wulff shape* generated by h, denoted $[h]$, is the convex body defined by

$$[h] = \{x \in \mathbb{R}^n : x \cdot v \leq h(v), \text{ for all } v \in S^{n-1}\}.$$

The Wulff shape, also known as the *Aleksandrov body*, is a key ingredient in Aleksandrov's variational formula for volume, which is an essential ingredient in solving the classical Minkowski problem. It is easy to see that

$$h_{[h]} \leq h, \tag{2.5}$$

and that, for $K \in \mathcal{K}_o^n$, we have

$$[h_K] = K. \tag{2.6}$$

Obviously,

$$[ch] = c\,[h], \tag{2.7}$$

for real $c > 0$. We shall make use of the trivial observation that

$$f \in C_e^+(S^{n-1}) \quad \Longrightarrow \quad [f] \in \mathcal{K}_e^n. \tag{2.8}$$

Suppose $h_0 \in C^+(S^{n-1})$ and $f \in C(S^{n-1})$, while $\delta > 0$ and, for each $t \in (-\delta, \delta)$, the function $o(t, \cdot) \in C(S^{n-1})$ satisfies

$$\lim_{t\to 0} \frac{\|o(t, \cdot)\|}{t} = 0.$$

Then, for each $t \in (-\delta, \delta)$, define $h_t : S^{n-1} \to (0, \infty)$ by

$$\log h_t(v) = \log h_0(v) + tf(v) + o(t, v), \tag{2.9}$$

for $v \in S^{n-1}$. The family of Wulff shapes generated by h_t is called *a family of logarithmic Wulff shapes generated by* h_0 *and* f. We sometimes denote the family $[h_t]$ by $[h_0, f, t]$, or, when h_0 is the support function of a convex body K, denote it simply by $[K, f, t]$.

The *supporting hyperplane* of $K \in \mathcal{K}_o^n$ in the direction $v \in S^{n-1}$ is given by

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}.$$

A vector $v \in S^{n-1}$ is called an *outer unit normal* of K at the point $x \in \partial K$ provided $x \in H_K(v)$.

If $K \in \mathcal{K}_0^n$ and $\omega \subset S^{n-1}$, then the *radial Gauss image* $\boldsymbol{\alpha}_K(\omega)$ is the set of all outer unit normals of K at the boundary points $\rho_K(u)u$ where $u \in \omega$, i.e.,

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{u\in\omega} \{v \in S^{n-1} : \rho_K(u)u \cdot v = h_K(v)\}. \tag{2.10}$$

If $\eta \subset S^{n-1}$, then the *reverse radial Gauss image* $\boldsymbol{\alpha}_K^*(\eta)$ is the set of all radial directions $u \in S^{n-1}$, such that the boundary point $\rho_K(u)u$ has at least one element in η as its outer unit normal, i.e.,

$$\boldsymbol{\alpha}_K^*(\eta) = \bigcup_{v\in\eta} \{u \in S^{n-1} : \rho_K(u)u \cdot v = h_K(v)\}. \tag{2.11}$$

Lemma 2.2.14 of Schneider [47] (see also Lemma 2.1 in [30]) tells us that, when η is a Borel set, the set $\boldsymbol{\alpha}_K^*(\eta)$ is measurable with respect to spherical Lebesgue measure.

Dual quermassintegrals, which include volume as a special case, are fundamental geometric invariants in the dual Brunn-Minkowski theory. For $i = 1, \dots, n$, the $(n-i)$-th *dual quermassintegral* $\widetilde{W}_{n-i}^{(n)}(K)$ of $K \in \mathcal{S}_o^n$ is proportional to the mean of i-dimensional volumes of the intersections of K with i-dimensional subspaces. That is,

$$\widetilde{W}_{n-i}^{(n)}(K) = \frac{\omega_n}{\omega_i} \int_{G(n,i)} V_i(K \cap \xi) d\xi, \tag{2.12}$$

where V_i denotes i-dimensional volume, $G(n,i)$ is the Grassmannian manifold of i-dimensional linear subspaces $\xi \subset \mathbb{R}^n$, and the integration is with respect to the Haar measure on $G(n,i)$. The dual quermassintegrals have the following integral representation (see [38]),

$$\widetilde{W}_{n-i}^{(n)}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^i du. \tag{2.13}$$

Using this, $\widetilde{W}_{n-q}^{(n)}$ is defined in the obvious manner for all $q \in \mathbb{R}$:

$$\widetilde{W}_{n-q}^{(n)}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^q du. \tag{2.14}$$

When the dimension n of the ambient space is clear, we will write $\widetilde{W}_{n-q}^{(n)}(K)$ simply as $\widetilde{W}_{n-q}(K)$, omitting the superscript.

It is easy to see that the $(n-q)$-th dual quermassintegral is homogeneous of degree q; i.e., for $c > 0$ and $K \in \mathcal{S}_o^n$,

$$\widetilde{W}_{n-q}(cK) = c^q \, \widetilde{W}_{n-q}(K),$$

since $\rho_{cK} = c\rho_K$. The origin-centered unit ball in $\mathbb{R}^n$ shall be denoted by B^n and its volume by $\omega_n = V_n(B^n)$. If there is no ambiguity about its dimension, we shall write simply B rather than B^n. Note that, for all q,

$$\widetilde{W}_{n-q}(B^n) = \omega_n. \tag{2.15}$$

If μ is a Borel measure on S^{n-1}, then the *entropy functional* of μ, $E_\mu : C^+(S^{n-1}) \to \mathbb{R}$, is defined by

$$E_\mu(f) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log f(v)\, d\mu(v), \tag{2.16}$$

for $f \in C^+(S^{n-1})$. We shall make use of the trivial observation that

$$E_\mu(cf) = E_\mu(f) - \log c, \tag{2.17}$$

for real $c > 0$. When f is the support function h_K of a convex body K, define

$$E_\mu(K) = E_\mu(h_K). \tag{2.18}$$

Since $h_B = 1$,

$$E_\mu(B) = 0. \tag{2.19}$$

3. The even dual Minkowski problem via maximization

The dual curvature measures of the dual Brunn-Minkowski theory are the counterparts of Federer's curvature measures in the classical Brunn-Minkowski theory. Huang-Lutwak-Yang-Zhang [30] reformulated the dual Minkowski problem as the maximization problems described below.

For $K \in \mathcal{K}_o^n$ and real $q \neq 0$, the q-th dual curvature measure $\widetilde{C}_q(K, \cdot)$ of K, can be defined via the integral representation

$$\frac{d}{dt}\widetilde{W}_{n-q}([K, f, t])\Big|_{t=0} = q\int_{S^{n-1}} f(v)\, \widetilde{C}_q(K, v),$$

for each $f \in C(S^{n-1})$. There is a similar integral representation for the case where $q = 0$. The q-th dual curvature measure has the following explicit definition,

$$\widetilde{C}_q(K, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K(u)^q du, \tag{3.1}$$

for each Borel set $\eta \subset S^{n-1}$. There are also Steiner-type formulas associated with dual curvature measures, similar to the Steiner formulas for area and curvature measures. See [30] for details.

Huang-Lutwak-Yang-Zhang [30] posed the *dual Minkowski problem*: Given $q \in \mathbb{R}$, what are necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that the measure is precisely the q-th dual curvature measure of some convex body in $\mathbb{R}^n$. Since the unit balls of finite dimensional Banach spaces are origin-symmetric convex bodies and the dual curvature measures of origin-symmetric convex bodies are even, it is of great interest to study the *even dual Minkowski problem.*

The Even Dual Minkowski Problem. *Given $q \in \mathbb{R}$ and an even Borel measure μ on S^{n-1}, find necessary and sufficient conditions on μ so that there exists a $K \in \mathcal{K}_e^n$ such that*

$$\widetilde{C}_q(K, \cdot) = \mu.$$

When $q = 0$, the even dual Minkowski problem is the even Aleksandrov problem, whose solution was given by Aleksandrov. When $q = n$, the even dual Minkowski problem is the even logarithmic Minkowski problem, whose solution was given by Böröczky-Lutwak-Yang-Zhang [10].

The q-th subspace mass condition (1.4) was discovered and defined independently in [8] and [55]. In [8], it was shown that, when $1 < q < n$, (1.4) is a necessary condition. In [55], it was shown that, for $q = 2, \dots, n-1$, (1.4) is also a sufficient condition.

It is the aim of this work to give a complete solution to the even dual Minkowski problem for $q \in (1, n)$. Specifically, we shall prove that, when $1 < q < n$, the q-th subspace mass condition is both necessary and sufficient for the existence of a solution to the even dual Minkowski problem.

We use the variational method to solve the even dual Minkowski problem. Here, for completeness, we recall results from [30], but give a slightly different treatment.

The maximization problem whose Euler-Lagrange equation is (1.1) was formulated in [30]. To derive the Euler-Lagrange equation for the maximization problem, the following variational formula established in [30] is critical. If $q \neq 0$, then

$$\frac{d}{dt}\widetilde{W}_{n-q}([h_0, f, t])\bigg|_{t=0} = q\int_{S^{n-1}} f(v)\,d\widetilde{C}_q([h_0], v), \tag{3.2}$$

for each $h_0 \in C^+(S^{n-1})$ and $f \in C(S^{n-1})$. Here $[h_0, f, t]$ is the logarithmic family of Wulff shapes generated by h_0 and f, as defined in Section 2. The corresponding formula when $q = 0$ is also given in [30].

Let μ be an even Borel measure on S^{n-1} and $q \neq 0$. Define the functional

$$\Phi_\mu : C_e^+(S^{n-1}) \longrightarrow \mathbb{R}$$

by

$$\Phi_\mu(f) = E_\mu(f) + \frac{1}{q}\log \widetilde{W}_{n-q}([f]), \tag{3.3}$$

for $f \in C_e^+(S^{n-1})$. Observe that, since $\widetilde{W}_{n-q}$ is homogeneous of degree q, it follows immediately from (2.7) and (2.17) that

$$\Phi_\mu(cf) = \Phi_\mu(f), \tag{3.4}$$

for all real $c > 0$.

Maximization Problem I. Given an even Borel measure μ on S^{n-1}, does there exist an $f_0 \in C_e^+(S^{n-1})$ such that

$$\sup\{\Phi_\mu(f) : f \in C_e^+(S^{n-1})\} = \Phi_\mu(f_0)? \tag{3.5}$$

Note that the set of support functions of convex bodies in $\mathcal{K}_e^n$ is a convex sub-cone of $C_e^+(S^{n-1})$. If the functional Φ_μ is restricted to this sub-cone and the support function of a convex body is identified with the convex body, the functional Φ_μ can be treated as a functional on $\mathcal{K}_e^n$,

$$\Phi_\mu : \mathcal{K}_e^n \longrightarrow \mathbb{R},$$

given by

$$\Phi_\mu(K) = E_\mu(K) + \frac{1}{q} \log \widetilde{W}_{n-q}(K), \tag{3.6}$$

for $K \in \mathcal{K}_e^n$. Thus,

$$\Phi_\mu(K) = \Phi_\mu(h_K). \tag{3.7}$$

Note that, from (2.15) and (2.19), we see that, for fixed $q \neq 0$,

$$\Phi_\mu(B) = \frac{1}{q} \log \omega_n. \tag{3.8}$$

This leads to the following variational problem.

Maximization Problem II. For real $q \neq 0$, and a given even Borel measure μ on S^{n-1}, does there exist a convex body $K_0 \in \mathcal{K}_e^n$ such that

$$\sup\{\Phi_\mu(K) : K \in \mathcal{K}_e^n\} = \Phi_\mu(K_0)? \tag{3.9}$$

The following lemma shows that if we identify a convex body K with its support function h_K, then a solution to Maximization Problem II is a solution to Maximization Problem I.

Lemma 3.1. *Suppose q is a nonzero real number and μ is an even Borel measure on S^{n-1}. If there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\Phi_\mu(K_0) = \sup\{\Phi_\mu(K) : K \in \mathcal{K}_e^n\}, \tag{3.10}$$

then

$$\Phi_\mu(h_{K_0}) = \sup\{\Phi_\mu(f) : f \in C_e^+(S^{n-1})\}. \tag{3.11}$$

Proof. Let $f \in C_e^+(S^{n-1})$. From (2.8) we know that $[f] \in \mathcal{K}_e^n$. From (3.7) and (3.10), we have

$$\Phi_\mu(h_{K_0}) = \Phi_\mu(K_0) \geq \Phi_\mu([f]).$$

From (2.5) and definition (2.16), it immediately follows that

$$E_\mu([f]) \geq E_\mu(f). \tag{3.12}$$

But (3.12) and definition (3.3) yield

$$\begin{aligned}
\Phi_\mu([f]) &= E_\mu([f]) + \frac{1}{q} \log \widetilde{W}_{n-q}([f]) \\
&\geq E_\mu(f) + \frac{1}{q} \log \widetilde{W}_{n-q}([f]) \\
&= \Phi_\mu(f).
\end{aligned}$$

Hence, $\Phi_\mu(h_{K_0}) \geq \Phi_\mu(f)$, for all $f \in C_e^+(S^{n-1})$, as was desired. □

The next lemma shows that a solution to Maximization Problem I is a solution to the even dual Minkowski problem.

Lemma 3.2. *Suppose q is a nonzero real number and μ is an even Borel measure on S^{n-1}. If there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\Phi_\mu(h_{K_0}) = \sup\{\Phi_\mu(f) : f \in C_e^+(S^{n-1})\},$$

then there exists $c > 0$ such that

$$\mu = \widetilde{C}_q(cK_0, \cdot).$$

Proof. Since the $(n-q)$-th dual quermassintegral is homogeneous of degree $q \neq 0$, we can choose $c > 0$ so that

$$\widetilde{W}_{n-q}(cK_0) = c^q\, \widetilde{W}_{n-q}(K_0) = |\mu|. \tag{3.13}$$

Since $h_{cK_0} = ch_{K_0}$, from (3.4), we have

$$\Phi_\mu(h_{cK_0}) = \Phi_\mu(h_{K_0}) = \sup\{\Phi_\mu(f) : f \in C_e^+(S^{n-1})\}. \tag{3.14}$$

Suppose $g \in C_e(S^{n-1})$. Define $h_t \in C_e^+(S^{n-1})$ by

$$\log h_t = \log h_{cK_0} + tg. \tag{3.15}$$

Now (3.15) and definition (2.16) yield

$$E_\mu(h_t) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{cK_0}\, d\mu - t\frac{1}{|\mu|} \int_{S^{n-1}} g\, d\mu. \tag{3.16}$$

From (3.14) and (3.15), we know that,

$$\Phi_\mu(h_0) = \Phi_\mu(h_{cK_0}) \geq \Phi_\mu(h_t).$$

From this fact, together with (3.3), the definition of Φ_μ, together with (3.15), (3.2), and (3.13), it follows that

$$\begin{aligned}
0 &= \frac{d}{dt}\Phi_\mu(h_t)\Big|_{t=0} \\
&= \frac{d}{dt}\left(E_\mu(h_t) + \frac{1}{q}\log \widetilde{W}_{n-q}([h_t])\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(E_\mu(h_t) + \frac{1}{q}\log \widetilde{W}_{n-q}([cK_0, g, t])\right)\Big|_{t=0} \\
&= -\frac{1}{|\mu|}\int_{S^{n-1}} g(v)\, d\mu(v) + \frac{1}{\widetilde{W}_{n-q}(cK_0)}\int_{S^{n-1}} g(v)\, d\widetilde{C}_q(cK_0, v) \\
&= \frac{1}{|\mu|}\left(-\int_{S^{n-1}} g(v)\, d\mu(v) + \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(cK_0, v)\right).
\end{aligned}$$

Since this holds for arbitrary $g \in C_e(S^{n-1})$, it follows that

$$\mu = \widetilde{C}_q(cK_0, \cdot).$$

□

From Lemmas 3.2 and 3.1, we see that a solution to Maximization Problem II is a solution to the even dual Minkowski problem. This is now stated formally in the following lemma.

Lemma 3.3. *Suppose q is a nonzero real number and μ is an even Boreal measure on S^{n-1}. If there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\Phi_\mu(K_0) = \sup \{\Phi_\mu(K) : K \in \mathcal{K}_e^n\},$$

then there exists $c > 0$ such that

$$\mu = \widetilde{C}_q(cK_0, \cdot).$$

Therefore, to solve the even dual Minkowski problem, it suffices to solve Maximization Problem II. Solving Maximization Problem II requires delicate estimates for the functional E_μ and the dual quermassintegral $\widetilde{W}_{n-q}$, which will be dealt with in the next two sections.

4. Estimates for the entropy functional E_μ

In this section, we will estimate the functional E_μ under the assumption that μ satisfies the subspace mass inequality (1.4).

Let $q > 0$ be a real number. Recall that an even Borel measure μ on S^{n-1} is said to satisfy the *q-th subspace mass inequality* provided

$$\frac{\mu(\xi(i) \cap S^{n-1})}{|\mu|} < \begin{cases} \frac{i}{q}, & \text{when } i < q, \\ 1, & \text{when } i \geq q, \end{cases} \tag{4.1}$$

for each proper i-dimensional subspace $\xi(i) \subset \mathbb{R}^n$. We assume, for the rest of this section, that $1 < q < n$ and μ is an even Borel measure on S^{n-1} that satisfies the q-th subspace mass inequality (4.1).

The key technique for estimating E_μ is to use an appropriate spherical partition. This general approach was introduced in [10].

Let $e_1, \dots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. For each $\delta \in (0, \frac{1}{\sqrt{n}})$, define the partition $\{\Omega_{i,\delta}\}_{i=1}^n$ of S^{n-1}, with respect to $e_1, \dots, e_n$, by

$$\Omega_{i,\delta} = \{v \in S^{n-1} : |v \cdot e_i| \geq \delta \text{ and } |v \cdot e_j| < \delta, \text{ for all } j > i\}. \tag{4.2}$$

For notational convenience, let

$$\xi_i = \operatorname{span}\{e_1, \dots, e_i\}, \quad i = 1, \dots, n,$$

and $\xi_0 = \{0\}$. It was shown in [10] that, for any Borel measure μ on S^{n-1},

$$\lim_{\delta \to 0^+} \mu(\Omega_{i,\delta}) = \mu((\xi_i \setminus \xi_{i-1}) \cap S^{n-1}) \tag{4.3}$$

and, therefore,

$$\lim_{\delta \to 0^+} \big(\mu(\Omega_{1,\delta}) + \cdots + \mu(\Omega_{i,\delta})\big) = \mu(\xi_i \cap S^{n-1}). \tag{4.4}$$

We also will need the following elementary lemma.

Lemma 4.1. *Suppose* $\lambda_1, \ldots, \lambda_m \in [0,1]$ *are such that*

$$\lambda_1 + \cdots + \lambda_m = 1.$$

Suppose further that $a_1, \ldots, a_m \in \mathbb{R}$ *are such that*

$$a_1 \leq a_2 \leq \cdots \leq a_m.$$

Assume there exists $\sigma_0, \sigma_1, \ldots, \sigma_m \in [0, \infty)$, *with* $\sigma_0 = 0$, $\sigma_m = 1$, *such that*

$$\lambda_1 + \cdots + \lambda_i \leq \sigma_i, \qquad \text{for} \quad i = 1, \ldots, m. \tag{4.5}$$

Then

$$\sum_{i=1}^{m} \lambda_i a_i \geq \sum_{i=1}^{m} (\sigma_i - \sigma_{i-1}) a_i.$$

Proof. Let $s_0 = 0$, and, for $i = 1, \ldots, m$,

$$s_i = \lambda_1 + \cdots + \lambda_i. \tag{4.6}$$

Observe that

$$\lambda_i = s_i - s_{i-1}, \qquad \text{for} \quad i = 1, \ldots, m. \tag{4.7}$$

From this and the facts that $s_0 = 0$ and $s_m = 1$, we see that,

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i a_i &= \sum_{i=1}^{m} (s_i - s_{i-1}) a_i \\ &= \sum_{i=1}^{m} s_i a_i - \sum_{i=1}^{m-1} s_i a_{i+1} \\ &= a_m + \sum_{i=1}^{m-1} s_i (a_i - a_{i+1}). \end{aligned} \tag{4.8}$$

Note that $\sigma_0 = 0$, $s_m = \sigma_m = 1$, and $s_i \leq \sigma_i$, when $1 \leq i \leq m-1$. Since $a_1 \leq a_2 \leq \cdots \leq a_m$, it follows from (4.6), (4.8), and the facts stated immediately above that

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i a_i &\geq a_m + \sum_{i=1}^{m-1} \sigma_i (a_i - a_{i+1}) \\ &= a_m + \sum_{i=1}^{m-1} \sigma_i a_i - \sum_{i=2}^{m} \sigma_{i-1} a_i \\ &= \sum_{i=1}^{m} \sigma_i a_i - \sum_{i=1}^{m} \sigma_{i-1} a_i \\ &= \sum_{i=1}^{m} (\sigma_i - \sigma_{i-1}) a_i, \end{aligned}$$

which is the desired conclusion. ∎

The following lemma provides the key estimate for E_μ.

Lemma 4.2. *Suppose $q \in (1, n)$ and $\varepsilon_0 > 0$. Suppose further that $(e_{1l}, \dots, e_{nl})$, where $l = 1, 2, \dots$, is a sequence of ordered orthonormal bases of $\mathbb{R}^n$ converging to the ordered orthonormal basis $(e_1, \dots, e_n)$, while $(a_{1l}, \dots, a_{nl})$ is a sequence of n-tuples satisfying, for all l,*

$$0 < a_{1l} \le a_{2l} \le \cdots \le a_{nl} \text{ and } a_{nl} > \varepsilon_0.$$

For each $l = 1, 2, \dots$, let

$$Q_l = \{x \in \mathbb{R}^n : |x \cdot e_{1l}|^2/a_{1l}^2 + \cdots + |x \cdot e_{nl}|^2/a_{nl}^2 \le 1\}$$

denote the ellipsoid generated by the $(e_{1l}, \dots, e_{nl})$ and $(a_{1l}, \dots, a_{nl})$. Let μ be an even Borel measure on S^{n-1} that satisfies the q-th subspace mass inequality

$$\frac{\mu(\xi(i) \cap S^{n-1})}{|\mu|} < \begin{cases} \frac{i}{q}, & \text{when } i < q, \\ 1, & \text{when } i \ge q, \end{cases}$$

for each proper i-dimensional subspace $\xi(i)$. Then there exist $t_0, \delta_0, l_0 > 0$ and $c_{q,\varepsilon_0,t_0,\delta_0}$ and $c_{\varepsilon_0,t_0,\delta_0}$, both independent of l, such that, for each $l > l_0$,

$$E_\mu(Q_l) \le -\frac{\log(a_{1l} \cdots a_{\lfloor q \rfloor l})}{q} - \frac{\log a_{\lfloor q \rfloor + 1, l}}{q/(q - \lfloor q \rfloor)} + t_0 \log a_{1l} + c_{\varepsilon_0,t_0,\delta_0}, \tag{4.9}$$

when $q \in (1, n-1)$, while

$$E_\mu(Q_l) \le -\frac{\log(a_{1,l} \cdots a_{n-1,l})}{q} + t_0 \log a_{1l} + c_{q,\varepsilon_0,t_0,\delta_0}, \tag{4.10}$$

when $q \in [n-1, n)$.

Proof. Fix $q \in (1, n)$.

For each $\delta \in (0, 1/\sqrt{n})$, define the partition $\{\Omega_{i,\delta}\}_{i=1}^n$ of S^{n-1}, with respect to the orthonormal basis $e_1, \dots, e_n$, as in (4.2):

$$\Omega_{i,\delta} = \{v \in S^{n-1} : |v \cdot e_i| \ge \delta \text{ and } |v \cdot e_j| < \delta, \text{ for all } j > i\}.$$

Let

$$\lambda_{i,\delta} = \frac{\mu(\Omega_{i,\delta})}{|\mu|}. \tag{4.11}$$

Note that

$$\lambda_{1,\delta} + \cdots + \lambda_{n,\delta} = 1. \tag{4.12}$$

Letting $\xi_i = \operatorname{span}\{e_1, \dots, e_i\}$, it follows from (4.4) and (4.1) that

$$\lim_{\delta \to 0^+} (\lambda_{1,\delta} + \cdots + \lambda_{i,\delta}) = \frac{\mu(\xi_i \cap S^{n-1})}{|\mu|} < \min\{i/q, 1\}, \tag{4.13}$$

for $i = 1, 2, \dots, n-1$. Since the inequality is strict, we may choose $t_0, \delta_0 > 0$ such that

$$\lambda_{1,\delta_0} + \cdots + \lambda_{i,\delta_0} < \min\{i/q, 1\} - t_0 := \sigma_i, \tag{4.14}$$

for $i = 1, 2, \ldots, n-1$. Since $\lim_{l\to\infty} e_{il} = e_i$ for each $i = 1, \ldots, n$, we may choose $l_0 > 0$ such that

$$|e_i - e_{il}| < \frac{\delta_0}{2}, \quad \text{for each } i = 1, \ldots, n, \text{ and each } l > l_0. \tag{4.15}$$

We shall assume, for the rest of the proof, that it is always the case that $l > l_0$, so that (4.15) always holds.

Suppose $v \in \Omega_{i,\delta_0}$. Since $(e_{1l}, \ldots, e_{nl})$ is orthonormal, it follows immediately from the definition of Q_l that $\pm a_{il} e_{il} \in Q_l$ and, hence, $h_{Q_l}(v) \geq a_{il}|e_{il} \cdot v|$. This, together with the fact that v is a unit vector, the definition of Ω_{i,δ_0}, and (4.15), give

$$h_{Q_l}(v) \geq a_{il}|e_{il} \cdot v| \geq a_{il} \left(|e_i \cdot v| - |e_i - e_{il}|\right) \geq a_{il} \frac{\delta_0}{2}. \tag{4.16}$$

From the definition of E_μ, the fact that $\{\Omega_{i,\delta_0}\}_{i=1}^n$ is a partition of S^{n-1}, together with (4.16), and finally (4.11), we have

$$\begin{aligned} E_\mu(Q_l) &\leq -\sum_{i=1}^n \frac{1}{|\mu|} \int_{\Omega_{i,\delta_0}} (\log a_{i,l} + \log \frac{\delta_0}{2})\, d\mu(v) \\ &= -\log \frac{\delta_0}{2} - \sum_{i=1}^n \lambda_{i,\delta_0} \log a_{i,l}. \end{aligned} \tag{4.17}$$

Let $\sigma_0 = 0$ and let $\sigma_n = 1$, and recall that

$$\sigma_i = \min\{i/q, 1\} - t_0,$$

for $i = 1, \ldots, n-1$. Observe that, when $1 < q < n-1$,

$$\sigma_i - \sigma_{i-1} = \begin{cases} 1/q - t_0, & \text{when } i = 1, \\ 1/q, & \text{when } 1 < i \leq \lfloor q \rfloor, \\ 1 - \lfloor q \rfloor / q, & \text{when } i = \lfloor q \rfloor + 1, \\ 0, & \text{when } \lfloor q \rfloor + 1 < i < n, \\ t_0, & \text{when } i = n, \end{cases} \tag{4.18}$$

and, when $n - 1 \leq q < n$,

$$\sigma_i - \sigma_{i-1} = \begin{cases} 1/q - t_0, & \text{when } i = 1, \\ 1/q, & \text{when } 1 < i \leq n-1, \\ 1 - (n-1)/q + t_0, & \text{when } i = n. \end{cases} \tag{4.19}$$

The fact that $0 < a_{1l} \leq \cdots \leq a_{nl}$, together with (4.12) and (4.14), shows that the hypothesis of Lemma 4.1 is satisfied and thus, we have, for $1 < q < n$,

$$\sum_{i=1}^n \lambda_{i,\delta_0} \log a_{il} \geq \sum_{i=1}^n (\sigma_i - \sigma_{i-1}) \log a_{il}. \tag{4.20}$$

When $1 < q < n-1$, from (4.20) together with (4.18), we have

$$\begin{aligned}\sum_{i=1}^{n}&\lambda_{i,\delta_0}\log a_{il}\\ &\geq (\frac{1}{q}-t_0)\log a_{1l}+\sum_{i=2}^{\lfloor q\rfloor}\frac{1}{q}\log a_{il}+(1-\frac{\lfloor q\rfloor}{q})\log a_{\lfloor q\rfloor+1,l}+t_0\log a_{nl}.\end{aligned} \tag{4.21}$$

When $n-1 \leq q < n$, from (4.20) together with (4.19), we have

$$\sum_{i=1}^{n}\lambda_{i,\delta_0}\log a_{il} \geq (\frac{1}{q}-t_0)\log a_{1l}+\sum_{i=2}^{n-1}\frac{1}{q}\log a_{il}+(1-\frac{n-1}{q}+t_0)\log a_{nl}. \tag{4.22}$$

When $1 < q < n-1$, combine (4.17) and (4.21) to get

$$\begin{aligned}E_\mu(Q_l) \leq &-\log\frac{\delta_0}{2}-(\frac{1}{q}-t_0)\log a_{1l}-\sum_{i=2}^{\lfloor q\rfloor}\frac{1}{q}\log a_{il}\\ &-(1-\frac{\lfloor q\rfloor}{q})\log a_{\lfloor q\rfloor+1,l}-t_0\log a_{nl}\\ =&-\log\frac{\delta_0}{2}+t_0\log a_{1l}-\frac{\log(a_{1l}\cdots a_{\lfloor q\rfloor l})}{q}\\ &-\frac{\log a_{\lfloor q\rfloor+1,l}}{q/(q-\lfloor q\rfloor)}-t_0\log a_{nl}.\end{aligned} \tag{4.23}$$

When $1 < q < n-1$, equation (4.23) and the fact that $a_{nl} > \varepsilon_0$ give (4.9).

When $n-1 \leq q < n$, combine (4.17) and (4.22) to get

$$\begin{aligned}&E_\mu(Q_l)\\ &\leq -\log\frac{\delta_0}{2}-(\frac{1}{q}-t_0)\log a_{1l}-\sum_{i=2}^{n-1}\frac{1}{q}\log a_{il}-(1-\frac{n-1}{q}+t_0)\log a_{nl}\\ &= -\log\frac{\delta_0}{2}+t_0\log a_{1l}-\frac{\log(a_{1,l}\cdots a_{n-1,l})}{q}-\frac{\log a_{n,l}}{q/(q-n+1)}-t_0\log a_{nl}.\end{aligned} \tag{4.24}$$

Again, the fact that $a_{nl} > \varepsilon_0$ together with (4.24) give (4.10). □

5. Estimates for dual quermassintegrals

Solving the even dual Minkowski problem when $1 < q < n$ requires estimates for dual quermassintegrals, which are in general difficult to establish. One indication of this is that, when q is an integer, the dual quermassintegrals involve lower dimensional cross sections of a convex body and are defined using integration over Grassmannians, as shown by (2.12). This is a new obstacle that is not present in the logarithmic Minkowski problem. In [30] this was overcome by choosing a barrier convex body and bounding the dual quermassintegral by using general spherical coordinates to decompose the dual quermassintegral into a sum of integrals and estimating each integral separately.

When $n-1 \leq q < n$, we use a Cartesian product of an ellipsoid and a ball as the barrier. The following lemma was proved in [55]. It also follows from Lemma 5.3 below.

Lemma 5.1. *Suppose $k \in [1, n-1]$ is an integer and $k < q \leq n$. Let $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$, while $a_1, \ldots, a_k > 0$, and define*

$$T = \left\{ x \in \mathbb{R}^n : \frac{|x \cdot e_1|^2}{a_1^2} + \cdots + \frac{|x \cdot e_k|^2}{a_k^2} \leq 1,\ |x \cdot e_{k+1}|^2 + \cdots + |x \cdot e_n|^2 \leq 1 \right\}.$$

Then there exists a $c_{n,k,q} > 0$ such that

$$\widetilde{W}_{n-q}(T) \leq c_{n,k,q}\, a_1 \cdots a_k.$$

Although Lemma 5.1 is sufficient for solving the even dual Minkowski problem when $q \in \{1, 2, \ldots, n-1\}$ (see [55]) and when $n-1 \leq q \leq n$, (see Lemma 6.1 in Section 6), stronger estimates are needed for non-integer $q \in (1, n-1)$. This requires a more careful choice of the barrier convex body and sufficiently sharp estimates for the dual quermassintegrals of this body. The rest of this section will focus on deriving these estimates. These estimates can be proved using the same approach used in proofs of earlier results, such as Lemma 5.1, but the calculations are quite complicated. Instead, we show below how the estimate can be obtained more easily using what we call the Gaussian integral trick.

For the rest of this section we always assume that the dimension n is at least 3.

The following lemma shows that the dual quermassintegral, defined in (2.14), can also be written as a Gaussian integral.

Lemma 5.2. *Given $q < n$,*

$$\int_{\mathbb{R}^n} \rho_S(z)^q e^{-|z|^2}\, dz = c_0(n,q) \widetilde{W}_{n-q}(S),$$

where

$$c_0(n,q) = n \int_0^\infty e^{-r^2} r^{-q+n-1}\, dr,$$

for each star body $S \in \mathcal{S}_o^n$.

Proof.

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_S(z)^q e^{-|z|^2}\, dz &= \int_{S^{n-1}} \int_0^\infty \rho_S(u)^q e^{-r^2} r^{-q+n-1}\, dr\, du \\ &= \int_{S^{n-1}} \rho_S(u)^q\, du \int_0^\infty e^{-r^2} r^{-q+n-1}\, dr \\ &= c_0(n,q)\, \frac{1}{n} \int_{S^{n-1}} \rho_S(u)^q\, du \\ &= c_0(n,q)\, \widetilde{W}_{n-q}(S). \end{aligned} \tag{5.1}$$

□

We begin with an upper bound for the dual quermassintegral of the Cartesian product of two convex bodies.

Lemma 5.3. *If $1 \le k < q < n$, then, for each $K \in \mathcal{K}_o^k$ and $L \in \mathcal{K}_o^{n-k}$,*

$$c_0(n,q)\,\widetilde{W}_{n-q}(K{\times}L) = \int_{z\in\mathbb{R}^n} \rho_{K\times L}(z)^q e^{-|z|^2}\,dz \le \frac{q}{q-k} c_0(n{-}k,q{-}k)\,V_k(K)\,\widetilde{W}^{(n-k)}_{n-q}(L).$$

Proof. From (2.3) we know that

$$\int_{z\in\mathbb{R}^n} \rho_{K\times L}(z)^q e^{-|z|^2}\,dz = \int_{(x,y)\in\mathbb{R}^k\times\mathbb{R}^{n-k}} [\min(\rho_K(x),\rho_L(y))]^q\, e^{-|x|^2-|y|^2}\,dx\,dy. \quad (5.2)$$

We shall decompose the integral in (5.2) as the sum of two integrals, I_1 and I_2, over the sub-regions of $(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, one being where the characteristic function $\mathbb{1}_{\{\rho_K(x)\le\rho_L(y)\}}$ is positive and the other where $\mathbb{1}_{\{\rho_K(x)>\rho_L(y)\}}$ is positive.

Using the fact that the radial function is homogeneous of degree -1 and, in the end, Lemma 5.2,

$$\begin{aligned}
I_1 &= \int_{y\in\mathbb{R}^{n-k}} \int_{x\in\mathbb{R}^k} \mathbb{1}_{\{\rho_K(x)\le\rho_L(y)\})}\, \rho_K(x)^q e^{-|x|^2-|y|^2}\,dx\,dy\\
&= \int_{y\in\mathbb{R}^{n-k}} e^{-|y|^2} \left(\int_{x\in\mathbb{R}^k} \mathbb{1}_{\{\rho_K(x)\le\rho_L(y)\}}\, \rho_K(x)^q e^{-|x|^2}\,dx\right) dy\\
&\le \int_{y\in\mathbb{R}^{n-k}} e^{-|y|^2} \left(\int_{S^{k-1}} \int_0^\infty \mathbb{1}_{\{\rho_K(r\theta)\le\rho_L(y)\}}\, \rho_K(r\theta)^q r^{k-1}\,dr\,d\theta\right) dy\\
&= \int_{y\in\mathbb{R}^{n-k}} e^{-|y|^2} \left(\int_{S^{k-1}} \rho_K(\theta)^q \int_0^\infty \mathbb{1}_{\{\rho_K(\theta)/\rho_L(y)\le r\}}\, r^{k-q-1}\,dr\,d\theta\right) dy\\
&= \int_{y\in\mathbb{R}^{n-k}} e^{-|y|^2} \left(\int_{S^{k-1}} \rho_K(\theta)^q \int_{\frac{\rho_K(\theta)}{\rho_L(y)}}^\infty r^{k-1-q}\,dr\,d\theta\right) dy\\
&= \frac{1}{q-k}\int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^{q-k} e^{-|y|^2}\,dy \int_{S^{k-1}} \rho_K(\theta)^k\,d\theta\\
&= \frac{k}{q-k} c_0(n-k,q-k)\,V_k(K)\,\widetilde{W}^{(n-k)}_{n-q}(L).
\end{aligned}$$

On the other hand, using the fact that the radial function is homogeneous of degree -1

and, in the end, Lemma 5.2,

$$\begin{aligned}
I_2 &= \int_{y\in\mathbb{R}^{n-k}} \int_{x\in\mathbb{R}^k} \mathbb{1}_{\{\rho_K(x)>\rho_L(y)\})}\, \rho_L(y)^q e^{-|x|^2-|y|^2}\, dx\, dy \\
&\le \int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^q e^{-|y|^2} \int_{x\in\mathbb{R}^k} \mathbb{1}_{\{\rho_K(x)>\rho_L(y)\}}\, dx\, dy \\
&= \int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^q e^{-|y|^2} \int_{S^{k-1}} \int_0^\infty \mathbb{1}_{\{\rho_K(r\theta)>\rho_L(y)\}} r^{k-1}\, dr\, d\theta\, dy \\
&= \int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^q e^{-|y|^2} \int_{S^{k-1}} \int_0^{\frac{\rho_K(\theta)}{\rho_L(y)}} r^{k-1}\, dr\, d\theta\, dy \\
&= \frac{1}{k} \int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^q e^{-|y|^2} \int_{S^{k-1}} \left(\frac{\rho_K(\theta)}{\rho_L(y)}\right)^k d\theta\, dy \\
&= \frac{1}{k} \int_{y\in\mathbb{R}^{n-k}} \rho_L(y)^{q-k} e^{-|y|^2}\, dy \int_{S^{k-1}} \rho_K(\theta)^k d\theta \\
&= c_0(n-k, q-k)\, V_k(K)\, \widetilde{W}_{n-q}^{(n-k)}(L).
\end{aligned}$$

□

We now state and prove the estimate needed for the proof to the main theorem. It is, however, convenient to introduce some notation first. Given $a = (a_1, \ldots, a_d) \in (0, \infty)^d$, let $E_d[a]$ denote the origin-centered d-dimensional ellipsoid

$$E_d[a] = \{(x_1/a_1)^2 + \cdots + (x_d/a_d)^2 \le 1\}$$

and, given a real $w > 0$, let $I[w]$ denote the origin-centered line segment $[-w, w]$.

Lemma 5.4. *If k is an integer such that $1 \le k < q < k+1 < n$, then there exists $C(n,k,q) > 0$ such that, for all $a = (a_1, \ldots, a_k) \in (0,\infty)^k$ and $0 < b \le 1$,*

$$\widetilde{W}_{n-q}(E_k[a] \times I[b] \times B^{n-k-1}) \le C(n,k,q) a_1 a_2 \cdots a_k b^{q-k}.$$

Proof. For simplicity we denote $E = E_k[a]$, $I = I[b]$, and $B = B^{n-k-1}$. Recall that

$$V_k(E) = \omega_k a_1 \cdots a_k.$$

From Lemma 5.3, we know there exists a $c'(n,k,q) > 0$ such that

$$\begin{aligned}
\widetilde{W}_{n-q}(E \times I \times B) &\le c'(n,k,q) V_k(E) \int_{\mathbb{R}^{n-k}} \rho_{I\times B}^{q-k}(y) e^{-|y|^2}\, dy \\
&= c'(n,k,q) \omega_k a_1 \cdots a_k \int_{\mathbb{R}^{n-k}} \rho_{I\times B}^{q-k}(y) e^{-|y|^2}\, dy
\end{aligned}$$

It therefore suffices to prove that there exists a constant $c(n,k,q) > 0$ such that

$$\int_{\mathbb{R}^{n-k}} \rho_{I\times B}(y)^{q-k} e^{-|y|^2}\, dy < c(n,k,q) b^{q-k}. \tag{5.3}$$

Recall that

$$\rho_I(t) = \frac{b}{|t|} \text{ and } \rho_B(x) = \frac{1}{|x|}. \tag{5.4}$$

We shall decompose the integral in (5.3) as the sum of two integrals, I_1 and I_2, over the sub-regions of $(t, x) \in \mathbb{R} \times \mathbb{R}^{n-k-1}$, one being where the characteristic function $\mathbb{1}_{\{\rho_I(t)<\rho_B(x)\}}$ is positive and the other where $\mathbb{1}_{\{\rho_I(t)\geq\rho_B(x)\}}$ is positive.

From (5.4), we see that, for real $t \neq 0$, real $r > 0$, and $\theta \in S^{n-k-2}$, we have

$$\mathbb{1}_{\{\rho_I(t)<\rho_B(r\theta)\}} = \mathbb{1}_{\{b/|t|<\rho_B(r\theta)\}} = \mathbb{1}_{\{|t|>rb\}}.$$

Thus, from (2.3), while keeping in mind that $-1 < k - q < 0$ and $-1 < n - k - 2$,

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-k-1}} \mathbb{1}_{\{\rho_I(t)<\rho_B(x)\}} \rho_I(t)^{q-k} e^{-t^2-|x|^2}\, dx\, dt \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-k-2}} \mathbb{1}_{\{\rho_I(t)<\rho_B(r\theta)\}} \rho_I(t)^{q-k} r^{n-k-2} e^{-t^2-|r\theta|^2}\, d\theta\, dr\, dt \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-k-2}} \mathbb{1}_{\{|t|>rb\}} b^{q-k} |t|^{k-q} r^{n-k-2} e^{-t^2-r^2}\, d\theta\, dr\, dt \\
&= 2b^{q-k} \int_{S^{n-k-2}} d\theta \int_0^{\infty} \int_{rb}^{\infty} t^{k-q} e^{-t^2} r^{n-k-2} e^{-r^2}\, dt\, dr \\
&\leq 2b^{q-k}(n-k-1)\omega_{n-k-1} \int_0^{\infty} t^{k-q} e^{-t^2}\, dt \int_0^{\infty} r^{n-k-2} e^{-r^2}\, dr
\end{aligned}$$

Similarly, from (5.4), we see that, for real $t \neq 0$, real $r > 0$, and $\theta \in S^{n-k-2}$, we have

$$\mathbb{1}_{\{\rho_I(t)\geq\rho_B(r\theta)\}} = \mathbb{1}_{\{b/|t|\geq 1/r\}} = \mathbb{1}_{\{|t|\leq rb\}}.$$

Thus, from (2.3), while keeping in mind that $0 < q - k < 1$ and $0 < b \leq 1$,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-k-1}} \mathbb{1}_{\{\rho_I(t)\geq\rho_B(x)\}} \rho_B(x)^{q-k} e^{-t^2-|x|^2}\, dx\, dt \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-k-2}} \mathbb{1}_{\{\rho_I(t)\geq\rho_B(r\theta)\}} \rho_B(r\theta)^{q-k} r^{n-k-2} e^{-t^2-|r\theta|^2}\, d\theta\, dr\, dt \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S^{n-k-2}} \mathbb{1}_{\{|t|\leq rb\}} r^{n-q-2} e^{-t^2-r^2}\, d\theta\, dr\, dt \\
&= \int_{S^{n-k-2}} d\theta \int_0^{\infty} r^{n-q-2} e^{-r^2} \left(\int_{-rb}^{rb} e^{-t^2}\, dt \right) dr \\
&\leq \int_{S^{n-k-2}} d\theta \int_0^{\infty} r^{n-q-2} e^{-r^2}\, 2br\, dr \\
&\leq 2b^{q-k}(n-k-1)\omega_{n-k-1} \int_0^{\infty} r^{n-q-1} e^{-r^2}\, dr.
\end{aligned}$$

□

6. Solutions to the dual Minkowski problem

In this section we present the solution to Maximization Problem II and thus, via Lemma 3.3, solve the even dual Minkowski problem for $1 < q < n$. The solution for $n-1 \le q < n$ relies on Lemmas 4.2 and 5.1, while the solution for $1 < q < n-1$ uses Lemmas 4.2 and 5.4.

The following lemma gives a positive answer to Maximum Problem II.

Lemma 6.1. *Let μ be an even Borel measure on S^{n-1} and $1 < q < n$. If μ satisfies the q-th subspace mass inequality, then there exists $K' \in \mathcal{K}_e^n$ such that*

$$\Phi_\mu(K') = \sup\{\Phi_\mu(K) : K \in \mathcal{K}_e^n\}. \tag{6.1}$$

Proof. Let $\{K_l\}$ be a maximizing sequence; i.e., $K_l \in \mathcal{K}_e^n$ and

$$\lim_{l\to\infty} \Phi_\mu(K_l) = \sup\{\Phi_\mu(K) : K \in \mathcal{K}_e^n\} \ge \Phi_\mu(B) = \frac{1}{q}\log \omega_n.$$

Since Φ_μ is homogeneous of degree 0, we may assume that each K_l has diameter 1. By Blaschke's selection theorem, there is a subsequence that converges to an origin-symmetric compact convex set K_0. The continuity of Φ_μ with respect to the Hausdorff metric shows that if K_0 has nonempty interior, then $K' = K_0$ satisfies (6.1), establishing the lemma. To prove that K_0 has nonempty interior, we argue by contradiction and assume that K_0 is contained in some proper subspace of $\mathbb{R}^n$.

For each K_l, we let Q_l to be the John ellipsoid associated with K_l; i.e., Q_l is the ellipsoid of largest volume that's contained in K_l. We choose an orthonormal basis $e_{1l}, \dots, e_{nl}$ and real numbers $0 < a_{1l} \le a_{2l} \le \cdots \le a_{nl} < 1$ such that

$$Q_l = \{x \in \mathbb{R}^n : |x\cdot e_{1l}|^2/a_{1l}^2 + \cdots + |x\cdot e_{nl}|^2/a_{nl}^2 \le 1\}.$$

John's theorem (see, e.g., Schneider [47]) tells us that, since K_l is origin symmetric,

$$Q_l \subset K_l \subset \sqrt{n}\, Q_l. \tag{6.2}$$

Since the diameter of K_l is 1, the diameter of $\sqrt{n}Q_l$ is greater than 1. But the diameter of Q_l is $2a_{nl}$, and, therefore, $a_{nl} \ge \frac{1}{2\sqrt{n}}$. By taking subsequences, we may assume that the sequence of orthonormal bases $\{e_{1l}, \dots, e_{nl}\}$ and the sequences $\{a_{1l}\}, \dots, \{a_{nl}\}$ converge. Since K_0 is contained in some proper subspace of $\mathbb{R}^n$, there must exist an integer k, where $1 \le k \le n-1$, such that, as $l \to \infty$, $a_{il} \to 0$ for $1 \le i \le k$, and $a_{il} \to a_i > 0$ for $k+1 \le i \le n$.

We first consider the case of $n-1 \le q < n$. From (6.2) and Lemma 4.2, we conclude that there exist $t_0, \delta_0, l_0 > 0$ such that, for all $l > l_0$,

$$E_\mu(K_l) \le E_\mu(Q_l) \le -\frac{1}{q}\log(a_{1l}\cdots a_{n-1,l}) + t_0 \log a_{1l} + c_{n,q,t_0,\delta_0}. \tag{6.3}$$

Define the ellipsoidal cylinder,

$$T_l = \left\{ x \in \mathbb{R}^n : \frac{|x\cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x\cdot e_{n-1,l}|^2}{a_{n-1,l}^2} \le 1 \text{ and } |x\cdot e_{nl}| \le 1 \right\}.$$

Since $a_{nl} \le 1$, we have

$$K_l \subset \sqrt{n} Q_l \subset \sqrt{n} T_l. \tag{6.4}$$

Since $t_0 > 0$, one can choose q_0 so that $q < q_0 < n$ and

$$(n-1)\left(\frac{1}{q_0} - \frac{1}{q}\right) + t_0 > 0. \tag{6.5}$$

By (2.14), the monotonicity of L_p norms with the fact that $q_0 > q$, (6.4), the homogeneity of a dual quermassintegral, and Lemma 5.1, we have

$$\begin{aligned}
\frac{1}{q}\log \widetilde{W}_{n-q}(K_l) &= \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{K_l}(u)^q\, du\right)^{\frac{1}{q}} + \frac{1}{q}\log\omega_n \\
&\le \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{K_l}(u)^{q_0}\, du\right)^{\frac{1}{q_0}} + \frac{1}{q}\log\omega_n \\
&\le \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{\sqrt{n}T_l}(u)^{q_0}\, du\right)^{\frac{1}{q_0}} + \frac{1}{q}\log\omega_n \\
&= \frac{1}{q_0}\log \widetilde{W}_{n-q_0}(T_l) + c_{n,q,q_0} \\
&\le \frac{1}{q_0}\log(a_{1,l}\cdots a_{n-1,l}) + c_{n,q,q_0}.
\end{aligned} \tag{6.6}$$

From (3.6), (6.3), (6.6), the fact that $q_0 > q$, together with $a_{1l} \le \cdots \le a_{n-1,l}$ and (6.5), now imply that

$$\begin{aligned}
\Phi_\mu(K_l) &= E_\mu(K_l) + \frac{1}{q}\log \widetilde{W}_{n-q}(K_l) \\
&\le \left(\frac{1}{q_0} - \frac{1}{q}\right)\log(a_{1l}\cdots a_{n-1,l}) + t_0 \log a_{1l} + c_{n,q,q_0,t_0,\delta_0} \\
&\le \left((n-1)\left(\frac{1}{q_0} - \frac{1}{q}\right) + t_0\right)\log a_{1l} + c_{n,q,q_0,t_0,\delta_0} \\
&\to -\infty,
\end{aligned}$$

as $l \to \infty$. The last step follows since $a_{1l} \to 0$. But

$$-\infty = \lim_{l\to\infty} \Phi_\mu(K_l) = \Phi_\mu(K_0) \ge \Phi_\mu(B) = \frac{1}{q}\log\omega_n$$

is the contradiction that shows that our assumption that K_0 is contained in some proper subspace of $\mathbb{R}^n$ is impossible.

Next, we consider the case when $1 < q < n-1$. If $n = 2$, there is nothing to show. We therefore consider only the case where $n \ge 3$.

From (6.2) and Lemma 4.2, we see that there exists $t_0, \delta_0, l_0 > 0$ such that, for each $l > l_0$, we have

$$E_\mu(K_l) \le E_\mu(Q_l) \le -\frac{\log(a_{1l}\cdots a_{\lfloor q\rfloor l})}{q} - \frac{\log a_{\lfloor q\rfloor+1,l}}{q/(q-\lfloor q\rfloor)} + t_0\log a_{1l} + c_{n,t_0,\delta_0}. \tag{6.7}$$

Since $t_0 > 0$, there exists $q_0 \in (q, n-1)$ sufficiently close to q so that q_0 is a non-integer satisfying $\lfloor q_0 \rfloor = \lfloor q \rfloor$ and

$$(n-2)\left(\frac{1}{q_0} - \frac{1}{q}\right) + t_0 > 0. \tag{6.8}$$

Let k_0 be the integer so that $q_0 - 1 < k_0 < q_0$, that is,

$$k_0 = \lfloor q_0 \rfloor = \lfloor q \rfloor. \tag{6.9}$$

Let E_l be the ellipsoid defined by

$$E_l = \left\{x \in \mathbb{R}^{k_0} : |x \cdot e_{1l}|^2/a_{1l}^2 + \cdots + |x \cdot e_{k_0 l}|^2/a_{k_0 l}^2 \le 1\right\},$$

where $\mathbb{R}^{k_0}$ is the span of $e_{1l}, \ldots, e_{k_0 l}$. Let I_l be the segment defined by

$$I_l = [-a_{k_0+1,l}e_{k_0+1,l}, a_{k_0+1,l}e_{k_0+1,l}].$$

Let B_l be the ball defined by

$$B_l = \left\{x \in \mathbb{R}^{n-k_0-1} : |x \cdot e_{k_0+2,l}|^2 + \cdots + |x \cdot e_{nl}|^2 \le 1\right\},$$

where $\mathbb{R}^{n-k_0-1}$ is the span of $e_{k_0+2,l}, \ldots, e_{nl}$. Let

$$G_l = E_l \times I_l \times B_l.$$

Note that since $a_{1l} \le \cdots \le a_{nl} < 1$, we have $Q_l \subset G_l$. By (6.2),

$$K_l \subset \sqrt{n} Q_l \subset \sqrt{n} G_l. \tag{6.10}$$

Note that $1 \le k_0 \le n-2$. By (2.14), the monotonicity of L_p norms with the fact that $q_0 > q$, (6.10), the homogeneity of dual quermassintegral, Lemma 5.4, and (6.9),

$$\begin{aligned}
\frac{1}{q}\log \widetilde{W}_{n-q}(K_l) &= \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{K_l}(u)^q\,du\right)^{\frac{1}{q}} + \frac{1}{q}\log\omega_n \\
&\le \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{K_l}(u)^{q_0}\,du\right)^{\frac{1}{q_0}} + \frac{1}{q}\log\omega_n \\
&\le \log\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho_{\sqrt{n}G_l}(u)^{q_0}\,du\right)^{\frac{1}{q_0}} + \frac{1}{q}\log\omega_n \\
&= \frac{1}{q_0}\log \widetilde{W}_{n-q_0}(G_l) + c_{n,q,q_0} \\
&\le \frac{1}{q_0}\log(a_{1l}\cdots a_{k_0 l}) + \frac{q_0-k_0}{q_0}\log a_{k_0+1,l} + c_{n,k_0,q,q_0} \\
&= \frac{1}{q_0}\log(a_{1l}\cdots a_{\lfloor q\rfloor,l}) + \frac{q_0-\lfloor q\rfloor}{q_0}\log a_{\lfloor q\rfloor+1,l} + c_{n,q,q_0}.
\end{aligned} \tag{6.11}$$

From (6.7), (6.11), the fact that $q < q_0 < n-1$, the fact that $0 < a_{1l} \le \cdots \le a_{nl} < 1$, and (6.8), we conclude that, when $l > l_0$,

$$\begin{aligned}
\Phi_\mu(K_l) &= E_\mu(K_l) + \frac{1}{q}\log \widetilde{W}_{n-q}(K_l) \\
&\le \left(\frac{1}{q_0} - \frac{1}{q}\right)\log(a_{1l}\cdots a_{\lfloor q\rfloor,l}) + \lfloor q\rfloor \left(\frac{1}{q} - \frac{1}{q_0}\right)\log a_{\lfloor q\rfloor+1,l} + t_0 \log a_{1l} + c_{n,\delta_0,t_0,q,q_0} \\
&\le \left(\frac{1}{q_0} - \frac{1}{q}\right)\log(a_{1l}\cdots a_{\lfloor q\rfloor,l}) + t_0 \log a_{1l} + c_{n,\delta_0,t_0,q,q_0} \\
&\le \lfloor q\rfloor \left(\frac{1}{q_0} - \frac{1}{q}\right)\log a_{1l} + t_0 \log a_{1l} + c_{n,\delta_0,t_0,q,q_0} \\
&\le \left((n-2)\Big(\frac{1}{q_0} - \frac{1}{q}\Big) + t_0\right)\log a_{1l} + c_{n,\delta_0,t_0,q,q_0} \\
&\to -\infty,
\end{aligned}$$

as $l \to \infty$, where the last step uses the fact that $\lim_{l\to\infty} a_{1l} = 0$. As before, this contradicts the assumption that K_0 is contained in some proper subspace of $\mathbb{R}^n$, thereby establishing the lemma. □

Lemma 6.1 together with Lemma 3.3 gives a complete solution to the even dual Minkowski problem for $1 < q < n$. When this is combined with the solution of the even dual Minkowski problem for $q \in (0, 1]$, given in [30], the result is:

Theorem 6.2. *If $0 < q < n$ and μ is an even Borel measure on S^{n-1}, then there exists $K \in \mathcal{K}_e^n$ such that $\mu = \widetilde{C}_q(K, \cdot)$ if and only if μ satisfies the q-th subspace mass inequality* (4.1).

The necessary condition of Theorem 6.2, when $1 < q < n$, was proved in [8], and the sufficient condition of Theorem 6.2, when $q \in \{2, \ldots, n-1\}$, was established in [55].

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