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K-theoretic Tate–Poitou duality and the fiber of the cyclotomic trace

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Abstract Let $p \in \mathbb{Z}$ be an odd prime. We prove a spectral version of Tate–Poitou duality for the algebraic K-theory spectra of number rings with p inverted. This identifies the homotopy type of the fiber of the cyclotomic trace $K(\mathcal{O}_F)^{\wedge}_p \to TC(\mathcal{O}_F)^{\wedge}_p$ after taking a suitably connective cover. As an application, we identify the homotopy type at odd primes of the homotopy fiber of the cyclotomic trace for the sphere spectrum in terms of the algebraic K-theory of $\mathbb{Z}$.

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Introduction

Tate–Poitou duality describes the relationship between the étale cohomology of S-integers in number fields and their completions in terms of a long exact sequence where the third term is a Pontryagin dual related to the first term. In

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the most basic case, for $p > 2$ a prime in $\mathbb{Z}$ and a number field F, we get a long exact sequence

$$\begin{array}{l} 0 \to H^0_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}]; \mathbb{Z}^\wedge_p(k)) \to \prod_{\nu|p} H^0_{\text{ét}}(F^\wedge_\nu; \mathbb{Z}^\wedge_p(k)) \to (H^2_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}/p^\infty(1-k)))^* \\ \to H^1_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}]; \mathbb{Z}^\wedge_p(k)) \to \prod_{\nu|p} H^1_{\text{ét}}(F^\wedge_\nu; \mathbb{Z}^\wedge_p(k)) \to (H^1_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}/p^\infty(1-k)))^* \\ \to H^2_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}]; \mathbb{Z}^\wedge_p(k)) \to \prod_{\nu|p} H^2_{\text{ét}}(F^\wedge_\nu; \mathbb{Z}^\wedge_p(k)) \to (H^0_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}/p^\infty(1-k)))^* \to 0 \end{array}$$

where $\mathcal{O}_F$ denotes the ring of integers, $F^\wedge_\nu$ denotes completion at the valuation ν, and $(-)^*$ denotes Pontryagin dual. The purpose of this paper is to describe a spectrum-level "K-theoretic" version of Tate–Poitou duality encoding the behavior of the completion map in the algebraic K-theory of rings of integers in number fields and use it to study the algebraic K-theory of the sphere spectrum.

Thomason's work [21] on the Quillen–Lichtenbaum conjecture identifies the étale cohomology groups in the above sequence as the homotopy groups of the $K(1)$-*localization* of algebraic K-theory spectra. Specifically, Thomason [21, Theorem 4.1, App. A] shows

$$\pi_n(L_{K(1)}K(R)) \cong \begin{cases} H^0_{\text{ét}}(R; \mathbb{Z}_p(\tfrac{n}{2})) \oplus H^2_{\text{ét}}(R; \mathbb{Z}_p(\tfrac{n}{2}+1)) & n \text{ even} \\ H^1_{\text{ét}}(R; \mathbb{Z}_p(\tfrac{n+1}{2})) & n \text{ odd} \end{cases}$$

for $R = \mathcal{O}_F[1/p]$ or $F^\wedge_\nu$. Letting $M_{\mathbb{Z}/p^\infty}$ denote the Moore spectrum for $\mathbb{Z}/p^\infty$, we also have

$$\pi_n(L_{K(1)}K(R) \wedge M_{\mathbb{Z}/p^\infty}) \cong \begin{cases} H^0_{\text{ét}}(R; \mathbb{Z}/p^\infty(\tfrac{n}{2})) \oplus H^2_{\text{ét}}(R; \mathbb{Z}/p^\infty(\tfrac{n}{2}+1)) & n \text{ even} \\ H^1_{\text{ét}}(R; \mathbb{Z}/p^\infty(\tfrac{n+1}{2})) & n \text{ odd} \end{cases}$$

for $R = \mathcal{O}_F[1/p]$. Algebraically, we can then use these isomorphisms to rewrite the Tate–Poitou sequence as the long exact sequence

$$\begin{array}{l} \cdots \to (\pi_{-1-n}(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge M_{\mathbb{Z}/p^\infty}))^* \\ \to \pi_n L_{K(1)}K(\mathcal{O}_F[1/p]) \to \prod \pi_n L_{K(1)}K(F^\wedge_\nu) \to \cdots \end{array}$$

with the first term (on the left) the homotopy groups of the $K(1)$-localization of $K(\mathcal{O}_F[1/p])$ and the second term the homotopy groups of the product of the $K(1)$-localizations of the K-theory of the completed fields. We can interpret the third term as the homotopy groups of a spectrum as well, using Brown–Comenetz duality or Anderson duality:

$$\begin{aligned}\pi_{-1-n}(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge M_{\mathbb{Z}/p^\infty})^* &\cong \pi_n(\Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(\mathbb{Z}) \wedge M_{\mathbb{Z}/p^\infty})) \\ &\cong \pi_n(\Sigma^{-1} I_{\mathbb{Z}_p^\wedge} L_{K(1)}K(\mathbb{Z})),\end{aligned}$$

where $I_{\mathbb{Q}/\mathbb{Z}}$ denotes the Brown–Comenetz dual and $I_{\mathbb{Z}_p^\wedge}$ denotes the Anderson dual of p-complete spectra. Our main result lifts this exact sequence to a cofiber sequence on the spectrum level.

Theorem (K-Theoretic Tate–Poitou Duality) *Let $p > 2$ be a prime number. Let F be a number field, $\mathcal{O}_F$ its ring of integers, and S the set of primes of $\mathcal{O}_F$ above p. For $\nu \in S$, write $F_\nu^\wedge$ for the ν-completion of F. The homotopy fiber Fib(κ) of the completion map in $K(1)$-local algebraic K-theory*

$$\kappa\colon L_{K(1)}K(\mathcal{O}_F[\tfrac{1}{p}]) \longrightarrow \prod_{\nu\in S} L_{K(1)}K(F_\nu^\wedge)$$

is weakly equivalent to

$$\Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(\mathcal{O}_F[\tfrac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} L_{K(1)}K(\mathcal{O}_F[\tfrac{1}{p}]).$$

The weak equivalence Fib$(\kappa) \to \Sigma^{-1} I_{\mathbb{Z}_p} L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}])$ is adjoint to the map

$$L_{K(1)}K(\mathcal{O}_F[\tfrac{1}{p}]) \wedge Fib(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S}$$

induced by the $L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}])$-module structure map $L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \wedge$ Fib$(\kappa) \to$ Fib(κ) and a map

$$u_{\mathcal{O}_F}\colon Fib(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S}$$

constructed as (1.7).

The map $u_{\mathcal{O}_F}$ above is "canonical" in that given the standard conventions for the Hasse invariant (see [20, XIII§3,p. 193]), the construction involves no further choices.

The previous theorem establishes a global arithmetic duality for algebraic K-theory. Clausen's MIT thesis [3] contains work in this direction; in particular, Clausen produces a duality map of a similar type to the one in the main theorem (with different details), presumably related through Gross-Hopkins duality. Work in progress of Schlank and Stojanoska [19] establishes arithmetic duality results for a much wider range of theories.

For us, the main interest is in the case $F = \mathbb{Q}$, $S = \{p\}$, where the main theorem (and some fiddling in low dimensions) identifies the homotopy fiber of the cyclotomic trace on the sphere spectrum.

Corollary *Let p be an odd prime. The connective cover of the homotopy fiber of the cyclotomic trace $K(\mathbb{S})^{\wedge}_p \to TC(\mathbb{S})^{\wedge}_p$ is weakly equivalent to the connective cover of $\Sigma^{-1} I_{\mathbb{Z}^{\wedge}_p}(L_{K(1)}K(\mathbb{Z}))$.*

Rognes [17,18] had previously identified the homotopy type of the homotopy fiber of the cyclotomic trace at regular primes, although not in these terms.

To deduce the corollary from the main theorem, we apply the work of Dundas [7] and Hesselholt–Madsen [9] together with the (affirmed) Quillen–Lichtenbaum conjecture. The linearization map $\mathbb{S} \to \mathbb{Z}$ and the cyclotomic trace $K \to TC$ fit together in a commutative square

$$\begin{tikzcd}
K(\mathbb{S})^{\wedge}_p \arrow[r] \arrow[d] & K(\mathbb{Z})^{\wedge}_p \arrow[d] \\
TC(\mathbb{S})^{\wedge}_p \arrow[r] & TC(\mathbb{Z})^{\wedge}_p
\end{tikzcd}$$

that Dundas [7] shows is homotopy cartesian, and it follows that the homotopy fiber of the cyclotomic trace for $\mathbb{S}$ is weakly equivalent to the homotopy fiber of the cyclotomic trace for $\mathbb{Z}$. Hesselholt-Madsen [9] shows that the completion map $TC(\mathbb{Z})^{\wedge}_p \to TC(\mathbb{Z}^{\wedge}_p)^{\wedge}_p$ is a weak equivalence and the cyclotomic trace $K(\mathbb{Z}^{\wedge}_p)^{\wedge}_p \to TC(\mathbb{Z}^{\wedge}_p)^{\wedge}_p$ is a connective cover. It follows that the connective cover of the homotopy fiber of the cyclotomic trace is weakly equivalent to the homotopy fiber of the completion map $K(\mathbb{Z})^{\wedge}_p \to K(\mathbb{Z}^{\wedge}_p)^{\wedge}_p$. Quillen's localization sequence for $\mathbb{Z} \to \mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}^{\wedge}_p \to \mathbb{Q}^{\wedge}_p$,

$$\begin{tikzcd}
K(\mathbb{Z}/p) \arrow[r] \arrow[d, "\mathrm{id}"'] & K(\mathbb{Z}) \arrow[r] \arrow[d] & K(\mathbb{Z}[1/p]) \arrow[r] \arrow[d] & \Sigma \cdots \\
K(\mathbb{Z}/p) \arrow[r] & K(\mathbb{Z}^{\wedge}_p) \arrow[r] & K(\mathbb{Q}^{\wedge}_p) \arrow[r] & \Sigma \cdots
\end{tikzcd}$$

then shows that the homotopy fiber of the completion map $K(\mathbb{Z}) \to K(\mathbb{Z}^{\wedge}_p)$ is weakly equivalent to the homotopy fiber of the completion map $K(\mathbb{Z}[\frac{1}{p}])^{\wedge}_p \to K(\mathbb{Q}^{\wedge}_p)^{\wedge}_p$. The (affirmed) Quillen–Lichtenbaum conjecture [22, VI.8.2] implies that the maps

$$K(\mathbb{Z}[\tfrac{1}{p}])^{\wedge}_p \longrightarrow L_{K(1)}K(\mathbb{Z}[\tfrac{1}{p}]), \qquad K(\mathbb{Q}^{\wedge}_p)^{\wedge}_p \longrightarrow L_{K(1)}K(\mathbb{Q}^{\wedge}_p),$$

are weak equivalences after taking 1-connected covers. Looking in low dimensions, $K(\mathbb{Z}[\frac{1}{p}])^\wedge_p \to L_{K(1)}K(\mathbb{Z}[\frac{1}{p}])$ is actually a connective cover, while $K(\mathbb{Q}^\wedge_p)^\wedge_p \to L_{K(1)}K(\mathbb{Q}^\wedge_p)$ induces an isomorphism on homotopy groups in degrees > 0 and an injection in degree 0. It then follows that the (connective) homotopy fiber of $K(\mathbb{Z}[\frac{1}{p}])^\wedge_p \to K(\mathbb{Q}^\wedge_p)^\wedge_p$ is weakly equivalent to the connective cover of the homotopy fiber of $L_{K(1)}K(\mathbb{Z}[\frac{1}{p}]) \to L_{K(1)}K(\mathbb{Q}^\wedge_p)$. Applying the main theorem, the corollary now follows.

Returning to the case of a number field F, the part of the discussion above that is general still obtains. Writing $R^\wedge_\nu$ for the completion of $\mathcal{O}_F$ at the prime ν, we have Quillen's localization sequence for $\mathcal{O}_F \to \mathcal{O}_F[\frac{1}{p}]$ and $\prod R^\wedge_\nu \to \prod F^\wedge_\nu$,

$$\begin{tikzcd}
\prod\limits_{\nu\in S} K(\mathcal{O}_F/\nu) \arrow[r] \arrow[d, "\cong"] & K(\mathcal{O}_F) \arrow[r] \arrow[d] & K(\mathcal{O}_F) \arrow[r] \arrow[d] & \Sigma\cdots \\
\prod\limits_{\nu\in S} K(R^\wedge_\nu/\nu) \arrow[r] & K(\prod\limits_{\nu\in S} R^\wedge_\nu) \arrow[r] & K(\prod\limits_{\nu\in S} F^\wedge_\nu) \arrow[r] & \Sigma\cdots ,
\end{tikzcd}$$

which identifies the homotopy fiber of the completion map $K(\mathcal{O}_F) \to K(\prod R^\wedge_\nu)$ as weakly equivalent to the homotopy fiber of the completion map $K(\mathcal{O}_F[\frac{1}{p}]) \to K(\prod F^\wedge_\nu)$. When S is the set of divisors of p, each $\mathcal{O}_F/\nu$ is a finite field of characteristic p, and so $K(\mathcal{O}_F/\nu)^\wedge_p$ is an Eilenberg–Mac Lane spectrum. The Quillen localization sequence then implies that the map $L_{K(1)}K(\mathcal{O}_F) \to L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}])$ is a weak equivalence. Since

$$\prod R^\wedge_\nu \cong (\mathcal{O}_F)^\wedge_p \cong \mathcal{O}_F \otimes \mathbb{Z}^\wedge_p,$$

Hesselholt-Madsen [9, Add. 6.2] shows that the map $TC(\mathcal{O}_F)^\wedge_p \to TC(\prod R^\wedge_\nu)^\wedge_p$ is a weak equivalence. Hesselholt-Madsen [9, Theorem D] shows that the map $K(\prod R^\wedge_\nu)^\wedge_p \to TC(\prod R^\wedge_\nu)$ is a connective cover. Applying the main theorem above and the (affirmed) Quillen–Lichtenbaum conjecture, we obtain the following corollary.

Corollary *Let F be a number and let $p \in \mathbb{Z}$ be an odd prime. Then there is a canonical map in the stable category from the homotopy fiber of the cyclotomic trace $K(\mathcal{O}_F)^\wedge_p \to TC(\mathcal{O}_F)^\wedge_p$ to $\Sigma^{-1}I_{\mathbb{Z}^\wedge_p}(L_{K(1)}K(\mathcal{O}_F))$ that induces an isomorphism on homotopy groups in dimensions ≥ 2.*

Finally, we note that the main theorem implies in the affirmative a conjecture of Calegari [2, 1.5] regarding the homotopy groups of the homotopy fiber on algebraic K-theory of the completion map. Calegari was interested in the completed cohomology of SL, and our result yields the construction of a spectrum with homotopy groups $\tilde{K}_*(\mathcal{O}_F)$ and continuous spectrum homology given by $\tilde{H}_*(\mathrm{SL}, \mathbb{Z}_p)$ [2, 1.19]. Moreover, the affirmed conjecture sharpens [2, 0.2] (an explicit calculation of completed homology) by making the conclusion unconditional.

1 *K*-theoretic local duality and the construction of the map $\mathrm{Fib}(\kappa) \to \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S}$

The Tate–Poitou duality theorem in global arithmetic derives from a much easier local duality theorem. We have a corresponding K-theoretic local duality theorem. We state and prove the K-theoretic local duality theorem in this section, deducing it from the local duality theorem in arithmetic. The argument is parallel to the argument used in Sect. 3 to prove the K-theoretic Tate–Poitou duality theorem and explains the construction of the canonical map $\mathrm{Fib}(\kappa) \to \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S}$, which is characterized by its relationship to a corresponding map $L_{K(1)}K(\mathbb{Q}_p) \to I_{\mathbb{Z}_p}\mathbb{S}$ in the K-theoretic local duality of $\mathbb{Q}_p$.

In arithmetic, local duality is an isomorphism

$$H^i_{\text{ét}}(k; M) \xrightarrow{\cong} (H^{2-i}_{\text{ét}}(k; M^*(1)))^*$$

where k is the field of fractions of a complete discrete valuation ring whose residue field is finite (e.g., a finite extension of $\mathbb{Q}_p$), M is a finite Galois module, and $(-)^*$ denotes the Pontryagin dual. The map is induced by the cup product pairing

$$H^i_{\text{ét}}(k; M) \otimes H^{2-i}_{\text{ét}}(k; M^*(1)) \longrightarrow H^2_{\text{ét}}(k; M \otimes M^*(1)) \longrightarrow H^2_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(1))$$

and the canonical isomorphism

$$H^2_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{\cong} H^2_{\text{ét}}(k, \mathbb{G}_m) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}. \tag{1.1}$$

Letting $M = \mathbb{Z}/p^n(j)$ and taking the limit $n \to \infty$, we get an isomorphism

$$H^i_{\text{ét}}(k; \mathbb{Z}_p(j)) \cong (H^{2-i}_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(1-j)))^*$$

(where the group on the left is Jannsen's continuous étale cohomology, which in the case of a field as above is equivalent to Galois cohomology).

For K-theory, the E_∞ multiplication induces a map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}$$

and we have a map

$$L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S} \tag{1.2}$$

essentially induced by the Hasse invariant as follows. Such a map is uniquely determined by specifying a homomorphism

$$\pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Thomason's descent spectral sequence puts $\pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$ into a short exact sequence

$$\begin{aligned} 0 \longrightarrow H^2_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \\ \longrightarrow H^0_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0 \end{aligned}$$

which is split by the map $\mathbb{Q}_p/\mathbb{Z}_p = H^0_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p) \to \pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$ induced by the unit $\mathbb{S} \to K(k)$. This gives us a retraction

$$\pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H^2_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(1))$$

that we compose with (1.1) to obtain a map

$$\pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{1.3}$$

The map (1.2) represents the map (1.3).

Theorem 1.4 (K-Theoretic Local Duality) *Let k be the field of fractions of a complete discrete valuation ring whose residue field is finite. The map*

$$L_{K(1)}K(k) \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq I_{\mathbb{Z}_p}(L_{K(1)}K(k))$$

adjoint to the composite map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

described above is a weak equivalence.

Proof Because $L_{K(1)}K(k)$ and $I_{\mathbb{Z}_p}(L_{K(1)}K(k))$ are both p-complete, it suffices to check that the map is a weak equivalence after taking the derived smash product with the mod p Moore spectrum M_p, or equivalently taking the homotopy cofiber of multiplication by p. Using the canonical weak equivalence

$$I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})/p \simeq I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k)/p),$$

the induced map

$$L_{K(1)}K(k)/p \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k)/p)$$

is adjoint to the composite map

$$\begin{aligned} L_{K(1)}K(k)/p \wedge L_{K(1)}K(k)/p &\longrightarrow L_{K(1)}K(k)/p \\ &\longrightarrow L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S} \end{aligned}$$

induced by the E_∞ pairing on $L_{K(1)}K(k)$ and the usual pairing $M_p \wedge M_p \to M_p$ for the first map and by the usual inclusion of M_p in $M_{\mathbb{Q}_p/\mathbb{Z}_p}$ for the second map. Thus, it suffices to check that the pairing above induces a perfect pairing

$$\pi_q(L_{K(1)}K(k)/p) \otimes \pi_{-q}(L_{K(1)}K(k)/p) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Thomason's descent spectral sequence [21, 4.1] is multiplicative with the multiplication on the E_2-term induced by the cup product in étale cohomology. We therefore have a perfect pairing on the $E_2 = E_\infty$-term by local duality in arithmetic, and it follows that we have a perfect pairing on homotopy groups. □

The proof of the global K-theoretic Tate–Poitou duality theorem in Sect. 3 follows the same general outline as the proof of the local duality theorem above. However, instead of using the multiplication on a single $K(1)$-localized K-theory spectrum, we study the pairing of $L_{K(1)}K(\mathcal{O}_F[1/p])$ with $\mathrm{Fib}(\kappa)$, where κ denotes the map $L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \to \prod L_{K(1)}K(F_\nu^\wedge)$ as in the main theorem on page 2. In local duality, the map $\pi_0(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \to \mathbb{Q}/\mathbb{Z}$ comes from the Hasse invariant isomorphism $H^2_{\text{ét}}(k; \mathbb{Q}/\mathbb{Z}(1)) \cong \mathbb{Q}/\mathbb{Z}$, or in terms of p-torsion, the isomorphism $H^2_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathbb{Q}_p/\mathbb{Z}_p$. For global duality, the map is related to the Albert–Brauer–Hasse–Noether sequence for $\mathcal{O}_F[1/p]$: the p-torsion version of this sequence takes the form of an exact sequence

$$\begin{aligned} 0 \longrightarrow H^2_{\text{ét}}(\mathcal{O}_F[\tfrac{1}{p}]; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \prod_{\nu \in S} H^2_{\text{ét}}(F_\nu^\wedge; \mathbb{Q}_p/\mathbb{Z}_p(1)) \\ \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0 \end{aligned}$$

where S is the set of primes lying above p and $p > 2$. Looking at the map of short exact sequences from Thomason's descent spectral sequence

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^2_{\text{ét}}(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Q}_p/\mathbb{Z}_p(1)) & \longrightarrow & \prod H^2_{\text{ét}}(F^\wedge_\nu; \mathbb{Q}_p/\mathbb{Z}_p(1)) \\
\downarrow & & \downarrow \\
\pi_0(L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) & \longrightarrow & \prod \pi_0(L_{K(1)}K(F^\wedge_\nu) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \\
\downarrow & & \downarrow \\
H^0_{\text{ét}}(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & \prod H^0_{\text{ét}}(F^\wedge_\nu; \mathbb{Q}_p/\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

we get an induced map from $\mathbb{Q}_p/\mathbb{Z}_p$ to the cokernel

$$C = \operatorname{coker}(\pi_0(L_{K(1)}K(\mathcal{O}_F[\tfrac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \to \pi_0(\prod L_{K(1)}K(F^\wedge_\nu) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})).$$

This and the long exact sequence on homotopy groups gives us a canonical map from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\pi_{-1}(\text{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$.

Theorem 1.5 *The canonical map $\mathbb{Q}_p/\mathbb{Z}_p \to \pi_{-1}(Fib(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$ is a split injection and has a unique retraction that commutes with the K-theory transfer associated to inclusions of number fields.*

Proof First we note that the K-theory transfer associated to the inclusion of number fields extends to a well-defined map in the stable category on $\text{Fib}(\kappa)$: For $F \subset E$ an inclusion of number fields, $\mathcal{O}_E[1/p]$ is a finitely generated projective $\mathcal{O}_F[1/p]$-module and we have an associated K-theory transfer map $K(\mathcal{O}_E[1/p]) \to K(\mathcal{O}_F[1/p])$ induced by regarding a finitely generated projective $\mathcal{O}_E[1/p]$-module as a finitely generated projective $\mathcal{O}_F[1/p]$-module. For the p-completions

$$\mathcal{O}_E \otimes \mathbb{Q}_p \cong \prod_{\nu \in S_E} E^\wedge_\nu, \qquad \mathcal{O}_F \otimes \mathbb{Q}_p \cong \prod_{\nu \in S_F} F^\wedge_\nu,$$

we have an associated K-theory transfer map and in the standard models for K-theory, the diagram

$$
\begin{array}{ccc}
K(\mathcal{O}_E[\frac{1}{p}]) & \longrightarrow & K(\mathcal{O}_E \otimes \mathbb{Q}_p) \\
\downarrow & & \downarrow \\
K(\mathcal{O}_F[\frac{1}{p}]) & \longrightarrow & K(\mathcal{O}_F \otimes \mathbb{Q}_p)
\end{array}
$$

commutes up to canonical homotopy since for a finitely generated projective $\mathcal{O}_E[1/p]$-module P, the underlying $(\mathcal{O}_F \otimes \mathbb{Q}_p)$-module of $P \otimes \mathbb{Q}_p$ is canonically isomorphic to the underlying $\mathcal{O}_F[1/p]$-module of P tensored with $\mathbb{Q}_p$.

This is enough structure to specify a canonical map in the stable category on the homotopy fibers.

Uniqueness is clear because in the case $F = \mathbb{Q}$, the inclusion of $\mathbb{Q}_p/\mathbb{Z}_p$ in $\pi_{-1}(\mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$ is an isomorphism. To see this note that the map from $\mathbb{Q}_p/\mathbb{Z}_p$ to the cokernel C is an isomorphism (because the map

$$H^0(\mathbb{Z}[1/p]; \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^0(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p)$$

is an isomorphism) and the inclusion of the cokernel C in $\pi_{-1}(\mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$ from the long exact sequence of homotopy groups is surjective because the map

$$\pi_{-1}(L_{K(1)}K(\mathbb{Z}[1/p]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow \pi_{-1}(L_{K(1)}K(\mathbb{Q}_p) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})$$

is injective (because the map $H^1(\mathbb{Z}[1/p]; \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p)$ is injective by abelianized Galois group considerations).

For existence of the splitting, since $\mathbb{Q}$ is initial among number fields, we just need to know that for an inclusion of number fields $F \subset E$, the diagram

$$\begin{tikzcd} \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "\mathrm{id}"'] & C_E \arrow[d] \\ \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & C_F \end{tikzcd}$$

commutes where C_E and C_F are the cokernels C associated to E and F above and the map is induced by the K-theory transfer. Because $H^2_{\text{ét}}(F^\wedge_\nu; \mathbb{Q}_p/\mathbb{Z}_p(1))$ is the p-torsion in $H^2_{\text{ét}}(F^\wedge_\nu; \mathbb{G}_m)$, the basic properties of a class formation (q.v. Proposition 1(ii) in [20, XI§2]) imply that it is enough to see that the diagram

$$\begin{tikzcd} H^2_{\text{ét}}(\mathcal{O}_E \otimes \mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p(1)) \arrow[r] \arrow[d] & \pi_0(L_{K(1)}(\mathcal{O}_E \otimes \mathbb{Q}_p) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \arrow[d] \\ H^2_{\text{ét}}(\mathcal{O}_F \otimes \mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p(1)) \arrow[r] & \pi_0(L_{K(1)}(\mathcal{O}_F \otimes \mathbb{Q}_p) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \end{tikzcd}$$

commutes where the left vertical map is the transfer in étale cohomology. This follows from the well-known result that the K-theory transfer for Galois extensions induces the étale cohomology transfer on the E_2-page of Thomason's descent spectral sequence. □

The composite of the map $\pi_{-1}(\mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \to \mathbb{Q}_p/\mathbb{Z}_p$ in the previous theorem with the inclusion of p-torsion $\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ now specifies a map

$$\mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}. \tag{1.6}$$

Adjoint to this map is the map

$$u_{\mathcal{O}_F} \colon \mathrm{Fib}(\kappa) \longrightarrow F(M_{\mathbb{Q}_p/\mathbb{Z}_p}, \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}) \simeq \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S} \tag{1.7}$$

in the statement of the K-theoretic Tate–Poitou duality theorem on page 2. In particular, we have constructed $u_{\mathcal{O}_F}$ to be compatible with the corresponding map

$$v_{F_\nu^\wedge} \colon L_{K(1)} K(F_\nu^\wedge) \longrightarrow I_{\mathbb{Z}_p}\mathbb{S} \tag{1.8}$$

for local duality adjoint to the map (1.2) (for $k = F_\nu^\wedge$). They are compatible in the sense that $v_{F_\nu^\wedge}$ is the composite

$$L_{K(1)} K(F_\nu^\wedge) \longrightarrow \Sigma\mathrm{Fib}(\kappa) \xrightarrow{\Sigma u_{\mathcal{O}_F}} \Sigma\Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S} \cong I_{\mathbb{Z}_p}\mathbb{S},$$

where the first map is a component of the map

$$\prod_{\nu \in S} L_{K(1)} K(F_\nu^\wedge) \longrightarrow \Sigma\mathrm{Fib}(\kappa)$$

in the cofiber sequence (associated to the fiber sequence) defining $\mathrm{Fib}(\kappa)$.

2 Fib(κ) as hypercohomology and $j_!$

The proof of the K-theoretic Tate–Poitou duality theorem relies on an étale hypercohomological interpretation of $\mathrm{Fib}(\kappa)$. Arithmetic Tate–Poitou duality arises from a duality pairing plus a long exact sequence arising from *recollement*. The purpose of this section is to give a spectral lifting of this setup. We begin with a terse review.

For a fixed number field F and S the set of primes lying over p, let $Y = \mathrm{spec}\mathcal{O}_F$, let U be the open subscheme $Y \backslash S = \mathrm{spec}(\mathcal{O}_F[1/p])$, and let Z be the reduced closed subscheme $Y \backslash U = \coprod \mathrm{spec}(\mathcal{O}_F/\nu)$. Writing i for the inclusion of Z in Y and j for the inclusion of U in Y, we have various adjoint functors on sheaves of abelian groups on the étale sites:

$$\mathrm{Sh}(Z_{\text{ét}}, \mathcal{A}b) \;\begin{matrix} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow{\;i^!\;} \end{matrix}\; \mathrm{Sh}(Y_{\text{ét}}, \mathcal{A}b) \;\begin{matrix} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow{\;j_*\;} \end{matrix}\; \mathrm{Sh}(U_{\text{ét}}, \mathcal{A}b), \tag{2.1}$$

where each functor is the left adjoint of the functor below it. One consequence of recollement is that for any sheaf or complex of sheaves $\mathcal{F}$ on $Y_{\text{ét}}$, the unit of the i_*, i^* adjunction and the counit of the $j_!, j^*$ adjunction fit into a short exact sequence

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0. \tag{2.2}$$

Now we take $\mathcal{F}$ to be a complex modeling $Rj_*(\mathbb{Z}/p^n(t))$, the total right derived functor of j_* applied to the locally constant sheaf $\mathbb{Z}/p^n(t)$ on $U_{\text{ét}}$. We can identify the terms in the resulting long exact sequence on hypercohomology as

$$\begin{aligned} \cdots \longrightarrow H^s(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t)) &\longrightarrow H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t)) \\ &\longrightarrow \prod_{\nu\in S} H^s_{\text{ét}}(F^h_\nu; \mathbb{Z}/p^n(t)) \longrightarrow \cdots \end{aligned} \tag{2.3}$$

(cf. [15, II.2.3(a)]), where F^h_ν denotes the field of fractions of the henselization R^h_ν of the discrete valuation ring $(\mathcal{O}_F)_{(\nu)}$. F^h_ν consists of the elements in the completion $F^\wedge_\nu$ that are algebraic over F. Because the inclusion of $F^h_\nu \to F^\wedge_\nu$ induces an isomorphism of absolute Galois groups, it induces an isomorphism $H^s_{\text{ét}}(F^h_\nu; \mathbb{Z}/p^n(t)) \to H^s_{\text{ét}}(F^\wedge_\nu; \mathbb{Z}/p^n(t))$.

Tate–Poitou duality is a consequence of the long exact sequence (2.3) and the perfect pairing [15, II.3.2–3]

$$\begin{aligned} H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t)) \otimes H^{3-s}(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(1-t)) &\longrightarrow H^3(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(1)) \\ &\longrightarrow H^3(Y_{\text{ét}}, j_!\mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}. \end{aligned} \tag{2.4}$$

Here the isomorphism $H^3(Y_{\text{ét}}, j_!\mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ is induced by the map from

$$C' = \operatorname{coker}(H^2(U_{\text{ét}}; \mathbb{G}_m) \to \prod H^2_{\text{ét}}(F^h_\nu; \mathbb{G}_m))$$

to $H^3(Y_{\text{ét}}, j_!\mathbb{G}_m)$ (in the corresponding long exact sequence for $\mathcal{F} = Rj_*(\mathbb{G}_m)$), which is an isomorphism, together with the canonical isomorphism from C' to

$$C = \operatorname{coker}(H^2(U_{\text{ét}}; \mathbb{G}_m) \to \prod H^2_{\text{ét}}(F^\wedge_\nu; \mathbb{G}_m))$$

and the Albert–Brauer–Hasse–Noether isomorphism from C to $\mathbb{Q}/\mathbb{Z}$.

In light of the above, the first step for K-theoretic Tate–Poitou duality is to identify $\mathrm{Fib}(\kappa)$ in terms of a spectral version of $j_!$ applied to the $K(1)$-local algebraic K-theory hypersheaf on $U_{\text{ét}}$. For a Grothendieck site $\mathcal{T}$, we write $\mathrm{Hyp}(\mathcal{T}, \mathrm{Sp})$ for the ∞-category of *hypersheaves of spectra* on $\mathcal{T}$; we understand this as the full subcategory of the ∞-category $\mathrm{Pre}(\mathcal{T}, \mathrm{Sp})$ of presheaves of spectra that satisfy hypercover descent. This also admits a description in terms of localization: For a presheaf of spectra $\mathcal{F}$, let $\tilde{\pi}_n\mathcal{F}$ denote the sheafification of the presheaf of abelian groups $\pi_n\mathcal{F}$ (homotopy groups applied objectwise). Work of Jardine [10,11] and Dugger–Hollander–Isaksen [6, 1.1] identifies $\mathrm{Hyp}(\mathcal{T}, \mathrm{Sp})$ as the localization of $\mathrm{Pre}(\mathcal{T}, \mathrm{Sp})$ obtained by formally inverting the maps that are isomorphisms on $\tilde{\pi}_n$ for all n; cf. [14, 6.5.3.13]. The localization functor $\mathrm{Pre}(\mathcal{T}, \mathrm{Sp}) \to \mathrm{Hyp}(\mathcal{T}, \mathrm{Sp})$ is called *hypersheafification*.

As an example, Thomason [21, 2.45 or 2.50] shows that under the hypotheses that hold there and in particular in our current setting, $K(1)$-local K-theory is a hypersheaf on the small étale site. We write $\mathcal{K}$ for the $K(1)$-localized K-theory functor and $\mathcal{K}_{U_{\text{ét}}}$ for the hypersheaf on $U_{\text{ét}}$. It will also be convenient to write $\mathcal{K}_{U_{\text{ét}}}^{/p^n}$ for $\mathcal{K}_{U_{\text{ét}}}/p^n \simeq \mathcal{K}_{U_{\text{ét}}} \wedge M_{p^n}$ and $\mathcal{K}^{/p^n}$ for $\mathcal{K}/p^n$.

The recollement above extends to the context of hypersheaves of spectra; see [13, A.8.20,A.8.19]). We have a diagram of adjoint pairs of functors of hypersheaves of spectra

$$\mathrm{Hyp}(Z_{\text{ét}}, \mathrm{Sp}) \underset{\xleftarrow{i^!}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}} \mathrm{Hyp}(Y_{\text{ét}}, \mathrm{Sp}) \underset{\xleftarrow{j_*}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}} \mathrm{Hyp}(U_{\text{ét}}, \mathrm{Sp})$$

mostly analogous to the diagram of adjoint pairs of functors of sheaves of abelian groups pictured in (2.1), or more precisely analogous to the derived category extension. (The functors $i^!$ and j_* are the analogs of the right derived functors $Ri^!$ and Rj_* on the derived categories of sheaves of abelian groups.) The analogue of (2.2) also holds:

$$j_! j^* \mathcal{F} \simeq \mathrm{Fib}(\mathcal{F} \to i_* i^* \mathcal{F})$$

(see the proof of (b) in [13, A.8.20], where the equivalent adjoint formula is proved). In particular, taking $\mathcal{F}$ to be $j_* \mathcal{K}_{U_{\text{ét}}}$, we have

$$j_! \mathcal{K}_{U_{\text{ét}}} \simeq \mathrm{Fib}(j_* \mathcal{K}_{U_{\text{ét}}} \to i_* i^* j_* \mathcal{K}_{U_{\text{ét}}}).$$

The following theorem relates $j_! \mathcal{K}_{U_{\text{ét}}}$ to $\mathrm{Fib}(\kappa)$, the homotopy fiber of the completion map.

Theorem 2.5 *The spectrum $j_! \mathcal{K}_{U_{\acute{e}t}}(Y_{\acute{e}t})$ of global sections of $j_! \mathcal{K}_{U_{\acute{e}t}}$ is p-equivalent to $Fib(\kappa)$.*

Proof We construct a commutative diagram

$$\begin{tikzcd} j_* \mathcal{K}_{U_{\text{ét}}}(Y_{\text{ét}}) \arrow[r] \arrow[d, "\simeq"'] & i_* i^* j_* \mathcal{K}_{U_{\text{ét}}}(Y_{\text{ét}}) \arrow[d] \\ \mathcal{K}(U_{\text{ét}}) \arrow[r] & \prod_{v \in S} \mathcal{K}(F_v^h) \end{tikzcd}$$

with the left vertical map the tautological equivalence and the right vertical map a p-equivalence constructed below. This will then complete the argument since the completion map κ factors as the bottom horizontal map followed by the map

$$\prod_{\nu\in S}\mathcal{K}(F_\nu^h)\longrightarrow\prod_{\nu\in S}\mathcal{K}(F_\nu^\wedge)$$

induced by the inclusions $F_\nu^h\to F_\nu^\wedge$, which is a p-equivalence by Thomason's theorem [21, 4.1] proving the $K(1)$-local Quillen–Lichtenbaum conjecture.

Construction of the righthand map essentially amounts to understanding the hypersheaf $i_*i^*j_*\mathcal{K}_{U_{\text{ét}}}$. Write i^*_{pre} for the inverse image functor $\text{Pre}(Y_{\text{ét}},\text{Sp})\to\text{Pre}(Z_{\text{ét}},\text{Sp})$. Then for $\nu\in S$, let R_ν^h denote the henselization of $(\mathcal{O}_F)_{(\nu)}$, and for any finite separable extension k of $\mathcal{O}_F/\nu$, let $R_\nu^h(k)$ denote the corresponding étale R_ν^h-algebra (under the usual equivalence of categories [16, I.4.4]). Then we have

$$i^*_{\text{pre}}j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}(\text{spec}k)\simeq\mathcal{K}^{/p^n}((\text{spec}R_\nu^h(k))\times_Y U)\simeq\mathcal{K}^{/p^n}(F_\nu^h(k))$$

(cf. [21, 1.44]) where $F_\nu^h(k)=R_\nu^h(k)[1/p]$ is the quotient field. It follows that $i^*_{\text{pre}}j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}$ satisfies hypercover descent and so computes $i^*j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}$. In particular $i_*i^*j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}})\simeq\prod\mathcal{K}^{/p^n}_{U_{\text{ét}}}(F_\nu^h)$. Similarly, $(i^*j_*\mathcal{K}_{U_{\text{ét}}})^\wedge_p\simeq(i^*_{\text{pre}}j_*\mathcal{K}_{U_{\text{ét}}})^\wedge_p$, and this induces the p-equivalence in the diagram. Since the map

$$\mathcal{K}(\mathcal{O}_F[1/p])\simeq j_*\mathcal{K}_{U_{\text{ét}}}(Y_{\text{ét}})\longrightarrow i_*i^*_{\text{pre}}j_*\mathcal{K}_{U_{\text{ét}}}(Y_{\text{ét}})\longrightarrow\mathcal{K}(F_\nu^h)$$

is induced by the map $\mathcal{O}_F[1/p]\to F_\nu^h$, the diagram commutes. □

In order to apply the previous theorem, we also need to know how the pairing $\mathcal{K}(\mathcal{O}_F[1/p])\wedge\text{Fib}(\kappa)\to\text{Fib}(\kappa)$ relates to the interpretation of $\text{Fib}(\kappa)$ as $j_!\mathcal{K}_{U_{\text{ét}}}(Y_{\text{ét}})^\wedge_p$. Recent work of Clausen–Mathew [4, 2.17] proves that the hypersheafication functor on any Grothendieck site is lax symmetric monoidal; it follows that the ∞-categories of hypersheaves of spectra on Grothendieck sites are symmetric monoidal ∞-categories. The direct image functor is then a symmetric monoidal functor; in particular, the natural multiplication on K-theory induces pairings of hypersheaves (natural in n)

$$\begin{gathered}j_*\mathcal{K}_{U_{\text{ét}}}\wedge j_*\mathcal{K}_{U_{\text{ét}}}\longrightarrow j_*\mathcal{K}_{U_{\text{ét}}}\\ j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}\wedge j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}\longrightarrow j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}}\end{gathered}$$

compatible with the usual pairings

$$\begin{gathered}\mathcal{K}(\mathcal{O}_F[1/p])\wedge\mathcal{K}(\mathcal{O}_F[1/p])\longrightarrow\mathcal{K}(\mathcal{O}_F[1/p])\\ \mathcal{K}^{/p^n}(\mathcal{O}_F[1/p])\wedge\mathcal{K}^{/p^n}(\mathcal{O}_F[1/p])\longrightarrow\mathcal{K}^{/p^n}(\mathcal{O}_F[1/p])\end{gathered}$$

when passing to global sections. The usual (equivalence on stalks) argument shows that for any hypersheaves of spectra $\mathcal{F}$ and $\mathcal{G}$,

$$j_*\mathcal{F} \wedge j_!\mathcal{G} \simeq j_!(\mathcal{F} \wedge \mathcal{G}),$$

and in our context, this and the pairings above give pairings (natural in n)

$$\begin{aligned} j_*\mathcal{K}_{U_{\text{ét}}} \wedge j_!\mathcal{K}_{U_{\text{ét}}} &\longrightarrow j_!(\mathcal{K}_{U_{\text{ét}}} \wedge \mathcal{K}_{U_{\text{ét}}}) \longrightarrow j_!\mathcal{K}_{U_{\text{ét}}} \\ j_*\mathcal{K}^{/p^n}_{U_{\text{ét}}} \wedge j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}} &\longrightarrow j_!(\mathcal{K}^{/p^n}_{U_{\text{ét}}} \wedge \mathcal{K}^{/p^n}_{U_{\text{ét}}}) \longrightarrow j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}. \end{aligned} \tag{2.6}$$

Theorem 2.7 *Under the equivalence of Theorem 2.5, the pairing $\mathcal{K}(\mathcal{O}_F[1/p]) \wedge Fib(\kappa) \to Fib(\kappa)$ is the induced pairing on global sections from the pairing of hypersheaves of (2.6).*

Proof The equivalence $j_*\mathcal{F} \wedge j_!\mathcal{G} \simeq j_!(\mathcal{F} \wedge \mathcal{G})$ is the inverse of the map adjoint to the equivalence

$$j^*(j_*\mathcal{F} \wedge j_!\mathcal{G}) \simeq j^*j_*\mathcal{F} \wedge j^*j_!\mathcal{G} \simeq \mathcal{F} \wedge \mathcal{G}; \tag{2.8}$$

we need to see that it is the map

$$\begin{aligned} &j_*\mathcal{F} \wedge \mathrm{Fib}(j_*\mathcal{G} \to i_*i^*j_*\mathcal{G}) \simeq \mathrm{Fib}\big((j_*\mathcal{F} \wedge j_*\mathcal{G}) \to (j_*\mathcal{F} \wedge i_*i^*j_*\mathcal{G})\big) \longrightarrow \\ &\mathrm{Fib}\big((j_*\mathcal{F} \wedge j_*\mathcal{G}) \to (i_*i^*j_*\mathcal{F} \wedge i_*i^*j_*\mathcal{G})\big) \simeq \mathrm{Fib}\big(j_*(\mathcal{F} \wedge \mathcal{G}) \to i_*i^*j_*(\mathcal{F} \wedge \mathcal{G})\big) \end{aligned} \tag{2.9}$$

under the usual identification of $j_!$ with the fiber. Now it is easy to see that (2.9) is inverse to the adjoint of (2.8) by applying j^*. □

3 Proof of the K-theoretic Tate–Poitou duality theorem

In this section we prove the K-theoretic Tate–Poitou duality theorem. We deduce the result from the classical Tate–Poitou duality theorem; more precisely, we use the formulation in terms of Artin–Verdier duality (2.4). We argue in terms of a pairing of étale descent spectral sequences.

Theorem 3.1 *The descent spectral sequences*

$$\begin{aligned} E_2^{s,t}(U_{\text{ét}}; \mathcal{K}^{/p^n}_{U_{\text{ét}}}) = H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) &\implies \pi_{-s+t}\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}}) \quad \textit{and} \\ E_2^{s,t}(Y_{\text{ét}}; j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}) = H^s(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2)) &\implies \pi_{-s+t}j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) \end{aligned}$$

admit a pairing of the form

$$E_r^{s,t}\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}}) \otimes E_r^{s',t'}j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) \longrightarrow E_r^{s+s',t+t'}j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}})$$

which converges to the pairing

$$\pi_{-s+t}\mathcal{K}_{U_{\text{ét}}}^{/p^n}(U_{\text{ét}}) \otimes \pi_{-s'+t'} j_!\mathcal{K}_{U_{\text{ét}}}^{/p^n}(Y_{\text{ét}}) \longrightarrow \pi_{-(s+s')+(t+t')} j_!\mathcal{K}_{U_{\text{ét}}}^{/p^n}(Y_{\text{ét}})$$

induced from the weak equivalences

$$\mathcal{K}_{U_{\text{ét}}}^{/p^n}(U_{\text{ét}}) \simeq \mathcal{K}^{/p^n}(spec\mathcal{O}_F[\tfrac{1}{p}]), \qquad and \qquad j_!\mathcal{K}_{U_{\text{ét}}}^{/p^n}(Y_{\text{ét}}) \simeq Fib(\kappa)^{/p^n}$$

and the pairing $\mathcal{K}^{/p^n}(spec\mathcal{O}_F[\frac{1}{p}]) \wedge Fib(\kappa)^{/p^n} \to Fib(\kappa)^{/p^n}$.

We prove this theorem in the next section. In order to apply it, we need to related the pairing on the E^2-term with the pairing in Artin–Verdier duality. We also prove the following theorem in the next section.

Theorem 3.2 *Under the canonical isomorphism*

$$\begin{aligned} H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) &\cong Ext^*_{U_{\text{ét}}}(\mathbb{Z}/p^n(t'/2), \mathbb{Z}/p^n(t/2+t'/2)) \\ &\cong Ext^*_{Y_{\text{ét}}}(j_!\mathbb{Z}/p^n(t'/2), j_!\mathbb{Z}/p^n(t/2+t'/2)), \end{aligned}$$

the multiplication on the E_2-term in Theorem 3.1

$$H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \longrightarrow H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2+t'/2))$$

coincides with the Yoneda pairing

$$\begin{aligned} Ext^*_{Y_{\text{ét}}}(j_!\mathbb{Z}/p^n(t'/2), j_!\mathbb{Z}/p^n(t/2+t'/2)) \otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \\ \longrightarrow H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2+t'/2)) \end{aligned}$$

The previous two theorems give all the ingredients we need to prove the K-theoretic Tate–Poitou duality theorem.

Proof of the K-theoretic Tate–Poitou duality theorem The pairing

$$\mathcal{K}(\text{spec}\mathcal{O}_F[\tfrac{1}{p}]) \wedge \text{Fib}(\kappa) \longrightarrow \text{Fib}(\kappa)$$

and the map

$$u_{\mathcal{O}_F} : \text{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S}$$

of (1.7) give a pairing

$$\mathcal{K}(\text{spec}\mathcal{O}_F[\tfrac{1}{p}]) \wedge \text{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p}\mathbb{S},$$

which induces a map

$$\mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p}(\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}])) \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}). \tag{3.3}$$

We need to see that it is a weak equivalence. Since both sides are p-complete, it suffices to check that (3.3) becomes a weak equivalence after smashing with the mod p Moore spectrum M_p on both sides. Then we are looking at the map

$$\begin{aligned}\mathrm{Fib}(\kappa)/p \longrightarrow\ & \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})/p \\ & \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}])/p),\end{aligned} \tag{3.4}$$

which is adjoint to a map

$$\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}])/p \wedge \mathrm{Fib}(\kappa)/p \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}. \tag{3.5}$$

Naturality and the fact that for an odd prime the map

$$M_p \wedge M_p \simeq F(M_p, M_{\mathbb{Q}_p/\mathbb{Z}_p}) \wedge M_p \longrightarrow M_{\mathbb{Q}_p/\mathbb{Z}_p}$$

induced by evaluation is the same as the composite of the multiplication on M_p and the inclusion of M_p in $M_{\mathbb{Q}_p/\mathbb{Z}_p}$ imply that the map (3.5) is the composite of the multiplication

$$\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}])/p \wedge \mathrm{Fib}(\kappa)/p \longrightarrow \mathrm{Fib}(\kappa)/p$$

and the map

$$\mathrm{Fib}(\kappa)/p \longrightarrow \mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

(where $\mathrm{Fib}(\kappa) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \to \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$ is the map (1.6) adjoint to $u_{\mathcal{O}_F}$). Because (2.4) is a perfect pairing, Theorem 3.1 and Theorem 3.2 imply that (3.5) induces a perfect pairing on homotopy groups

$$\pi_q(\mathcal{K}(\mathrm{spec}\mathcal{O}_F[\tfrac{1}{p}])/p) \otimes \pi_{-1-q}(\mathrm{Fib}(\kappa)/p) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

This implies that (3.4) is a weak equivalence, and we conclude that (3.3) is a weak equivalence. □

4 Construction and analysis of the spectral sequence

This section proves Theorems 3.1 and 3.2. For the proof of 3.1, we use a modern take on the descent spectral sequence of Jardine [12, §6.1] based on Postnikov towers rather than the original approach of Thomason based on the Godement construction; these are well-known to be isomorphic from E^2 onwards. As explained by Dugger [5, §4], using Whitehead towers in place of Postnikov towers leads to the same spectral sequence but with better multiplicative properties, and we take this approach.

Work of Hedenlund et al. [8] expands on Dugger's observations on pairing of spectral sequences and puts them in modern language. Although the category of bigraded spectral sequences of abelian groups does not form a symmetric monoidal category, it does form an ∞-operad under the usual notion of multilinear pairing. The main theorem of [8] is that the usual construction of a spectral sequence from a tower of spectra assembles to a map of ∞-operads. Here the ∞-category of towers is the ∞-category of spectral presheaves on the ordered set $\mathbb{Z}$ (for the increasing order) and the ∞-operad structure is the Day convolution symmetric monoidal structure for addition [13, 4.8.1.13]. A pairing of towers $A^\bullet \wedge B^\bullet \to C^\bullet$ in this structure amounts to (homotopy coherent) maps

$$A^t \wedge B^{t'} \longrightarrow C^{t+t'}.$$

A pairing of towers gives a pairing of spectral sequences in the classical sense.

The Whitehead tower is the name for the functor obtained by assembling into a tower the truncations $\tau^{\geq n}$ in a t-structure. We will write this functor as $W^\bullet$, i.e., $W^n := \tau^{\geq n}$. In the present context, $W^\bullet$ is a functor from the ∞-category of spectra (with the standard t-structure) to the ∞-category of towers of spectra. Because the smash product of spectra adds connectivities, this can be enhanced to a lax symmetric monoidal functor of symmetric monoidal ∞-categories. Because hypersheafication from presheaves of spectra on $U_{\text{ét}}$ to étale hypersheaves of spectra on $U_{\text{ét}}$ is a symmetric monoidal functor [4, 2.17], the Whitehead tower functor followed by global sections gives a symmetric monoidal functor from the ∞-category of étale hypersheaves on $Y_{\text{ét}}$ to the ∞-category of towers of spectra.

The descent spectral sequences of Theorem 3.1 apply the above composite map of ∞-operads from étale hypersheaves of spectra to spectral sequences. Concretely, the first spectral sequence in 3.1 is the homotopy group spectral sequence of the tower of spectra

$$\cdots \longrightarrow (W^{t+1}\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}}) \longrightarrow (W^{t}\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}}) \longrightarrow \cdots .$$

As an abbreviation in the work below, we write

$$C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}} = (W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}, W^{t+1}\mathcal{K}^{/p^n}_{U_{\text{ét}}})$$

for the pair $W^{t+1}\mathcal{K}^{/p^n}_{U_{\text{ét}}} \to W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}$. In particular,

$$\pi_*(C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}})) \cong \pi_*\mathrm{Cof}\big((W^{t+1}\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}}) \to (W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}})\big).$$

Since $\tilde{\pi}_t\mathcal{K}^{/p^n}_{U_{\text{ét}}} \cong \mathbb{Z}/p^n(t/2)$, we get a canonical isomorphism

$$\pi_{-s+t}(C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}})) \cong H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)).$$

The spectral sequence then has E_2-term (with the standard Whitehead/Postnikov Atiyah–Hirzebruch renumbering)

$$E_2^{s,t} := \pi_{-s+t}(C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}})) \cong H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2))$$

and abuts to the colimit

$$\mathrm{colim}\ \pi_{-s+t}\big((W^\bullet\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}})\big) \cong \pi_{-s+t}\big(\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}})\big).$$

Because $\mathrm{holim}(W^\bullet\mathcal{K}^{/p^n}_{U_{\text{ét}}})(U_{\text{ét}}) \simeq *$, the spectral sequence converges conditionally [1, 5.10]. Because $H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2))$ is only non-zero in a finite range, the spectral sequence converges strongly [1, 6.1] to the abutment $\pi_{-s+t}\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}})$.

For the spectral sequence on $\mathrm{Fib}(\kappa)$, we use the tower

$$\cdots \longrightarrow (j_!W^{t+1}\mathcal{K}^{/p^n}_{U_{\text{ét}}})(Y_{\text{ét}}) \longrightarrow (j_!W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}})(Y_{\text{ét}}) \longrightarrow \cdots .$$

We abbreviate

$$\mathcal{X}^t := j_!W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}} \simeq \mathrm{Fib}(j_*W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}} \to i_*i^*j_*W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}})$$

and write

$$C^t\mathcal{X} = (\mathcal{X}^t, \mathcal{X}^{t+1})$$

for the pair. We then have

$$\pi_{-s+t}C^t\mathcal{X}(Y_{\text{ét}}) \cong H^s(Y_{\text{ét}}; j_!\tilde{\pi}_{-t}\mathcal{K}^{/p^n}_{U_{\text{ét}}})$$

and the tower of spectra $\mathcal{X}^\bullet(Y_{\text{ét}})$ gives a spectral sequence with

$$E_2^{s,t} := \pi_{-s+t}(\mathcal{X}^t(Y_{\text{ét}}), \mathcal{X}^{t+1}(Y_{\text{ét}})) \cong H^s(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2))$$

that abuts to the colimit

$$\operatorname{colim} \pi_{-s+t}\mathcal{X}^\bullet(Y_{\text{ét}}) \cong \pi_{-s+t}\big(j_!\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}})\big).$$

Again because the homotopy limit of $\mathcal{X}^\bullet$ is trivial and $H^s(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2))$ is only non-zero in a finite range, the spectral sequence converges strongly, and $\pi_{-s+t}\operatorname{colim}\mathcal{X}^\bullet(Y_{\text{ét}})$ is isomorphic to $\pi_{-s+t}\mathrm{Fib}(\kappa)$ by the comparison map. The pairing property of $W^\bullet$ induces a pairing

$$\begin{aligned} &W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(U_{\text{ét}}) \wedge \mathcal{X}^{t'}(Y_{\text{ét}}) = j_*W^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) \wedge j_!W^{t'}\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) \\ &\longrightarrow j_!W^{t+t'}(\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}})\wedge\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}})) \longrightarrow j_!W^{t+t'}\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) = \mathcal{X}^{t+t'}(Y_{\text{ét}}), \end{aligned}$$

inducing a pairing of spectral sequences. Theorem 2.7 identifies this with our standard model

$$\mathcal{K}^{/p^n}(\operatorname{spec}\mathcal{O}_F[\tfrac{1}{p}]) \wedge \mathrm{Fib}(\kappa)^{/p^n} \longrightarrow \mathrm{Fib}(\kappa)^{/p^n}$$

for the pairing of the $K(1)$-local mod p^n algebraic K-theory and the fiber.

This completes the proof of Theorem 3.1. Next we need to identify the multiplication on the E_2-term, which takes the form

$$H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^{s'}(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \longrightarrow H^{s+s'}(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2+t'/2)).$$

In the notation above, the multiplication is induced by the map of pairs

$$j_*C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}(Y_{\text{ét}}) \wedge C^{t'}\mathcal{X}(Y_{\text{ét}}) \longrightarrow C^{t+t'}\mathcal{X}(Y_{\text{ét}}).$$

By construction the homotopy cofiber of the pair $j_*C^t\mathcal{K}^{/p^n}_{U_{\text{ét}}}$ is a model for the étale hypersheaf $\Sigma^t j_*H\mathbb{Z}/p^n(t/2)$ on $Y_{\text{ét}}$ and the homotopy cofiber of the pair $C^{t'}\mathcal{X}$ is a model for the étale hypersheaf $\Sigma^{t'} j_!H\mathbb{Z}/p^n(t'/2)$ on $Y_{\text{ét}}$. Thus, we can identify the induced map on homotopy groups of global sections as the composite of the cup product

$$\begin{aligned} &H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t'/2)) \otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2)) \\ &\quad\cong H^*(Y_{\text{ét}}; j_*H\mathbb{Z}/p^n(t'/2)) \otimes H^*(Y_{\text{ét}}; j_!H\mathbb{Z}/p^n(t/2)) \\ &\qquad\longrightarrow H^*(Y_{\text{ét}}; j_*H\mathbb{Z}/p^n(t'/2) \wedge j_!H\mathbb{Z}/p^n(t/2)) \end{aligned}$$

and the map of étale hypersheaves

$$\begin{aligned} &j_*H\mathbb{Z}/p^n(t'/2) \wedge j_!H\mathbb{Z}/p^n(t/2) \longrightarrow j_!H\mathbb{Z}/p^n(t/2+t'/2) \\ &\quad\simeq Hj_!\mathbb{Z}/p^n(t/2+t'/2) \end{aligned}$$

induced by the pairing. Using the equivalence of étale hypersheaves of spectra

$$j_* H\mathbb{Z}/p^n(t/2) \wedge j_! H\mathbb{Z}/p^n(t'/2) \simeq j_!(H\mathbb{Z}/p^n(t/2) \wedge H\mathbb{Z}/p^n(t'/2)),$$

since the target

$$j_! H\mathbb{Z}/p^n(t/2 + t'/2) \simeq H j_! \mathbb{Z}/p^n(t/2 + t'/2)$$

is an Eilenberg–Mac Lane presheaf, the map is determined by the factorization through the coconnective cover

$$j_!(H\mathbb{Z}/p^n(t/2) \wedge H\mathbb{Z}/p^n(t'/2))(-\infty, 0] \simeq j_!(H\mathbb{Z}/p^n(t/2 + t'/2)).$$

By looking at stalks, we see that the self-map of $j_!(H\mathbb{Z}/p^n(t/2+t'/2))$ is the identity. As a consequence, it follows that the map

$$H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \longrightarrow H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t/2 + t'/2))$$

on the E_2-term in Theorem 3.1 factors through the corresponding cup product map in the derived category of sheaves of abelian groups on $Y_{\text{ét}}$,

$$\begin{aligned}
H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) &\otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \\
&= \mathbb{H}^*(Y_{\text{ét}}; Rj_*\mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2)) \\
&\longrightarrow H^*(\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; Rj_*\mathbb{Z}/p^n(t/2)) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; j_!\mathbb{Z}/p^n(t'/2))) \\
&\longrightarrow \mathbb{H}^*(Y_{\text{ét}}; Rj_*\mathbb{Z}/p^n(t/2) \otimes^L j_!\mathbb{Z}/p^n(t'/2)) \\
&\cong \mathbb{H}^*(Y_{\text{ét}}; j_!(\mathbb{Z}/p^n(t/2) \otimes^L \mathbb{Z}/p^n(t'/2))) \\
&\longrightarrow H^*(Y_{\text{ét}}, j_!(\mathbb{Z}/p^n(t/2) \otimes \mathbb{Z}/p^n(t'/2))) \\
&\cong H^*(Y_{\text{ét}}, j_!\mathbb{Z}/p^n(t/2 + t'/2)).
\end{aligned}$$

Here for ease of comparison to algebraic conventions, we have switched to derived category notation and written $\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; -)$ for the hypercohomology object of a sheaf of abelian groups (i.e., the sections of the hypersheaf, viewed as an object of the derived category of abelian groups) and $\mathbb{H}^*(Y_{\text{ét}}; -)$ for its hypercohomology groups $H^*(\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; -))$.

This identifies the multiplication on the E_2 term in terms of the cup product, and Theorem 3.2 now follows from the basic relationship between the cup product and the Yoneda product in the derived category of sheaves of abelian groups on $Y_{\text{ét}}$, cf. [16, §5.1]: For sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$, the following diagram in the derived category commutes

$$\begin{tikzcd}
\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; R\,\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{F}) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; R\,\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G}) \otimes^L \mathcal{F}) \arrow[d] \\
R\mathrm{Hom}(\mathcal{F},\mathcal{G}) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{F}) \arrow[r] & \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{G})
\end{tikzcd}$$

where the top arrow is the cup product and the bottom arrow and righthand arrows are the appropriate evaluation maps. This completes the proof of Theorem 3.2.

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