

TESTING FOR DENSE SUBSETS IN A GRAPH VIA THE PARTITION FUNCTION

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ABSTRACT. For a set S of vertices of a graph G , we define its density $0 \leq \sigma(S) \leq 1$ as the ratio of the number of edges of G spanned by the vertices of S to $\binom{|S|}{2}$. We show that, given a graph G with n vertices and an integer $m \ll n$, the partition function $\sum_S \exp\{\gamma m \sigma(S)\}$, where the sum is taken over all m -subsets S of vertices and $0 < \gamma < 1$ is fixed in advance, can be approximated within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ time. We discuss numerical experiments and observe that for the random graph $G(n, 1/2)$ one can afford a much larger γ , provided the ratio n/m is sufficiently large.

1. INTRODUCTION AND MAIN RESULTS

Let $G = (V, E)$ be an undirected graph, without loops or multiple edges. For a non-empty subset $S \subset V$ of vertices, we define the *density* $\sigma(S)$ as the fraction of the pairs of vertices of S that span an edge of G :

$$\sigma(S) = \frac{\left| \binom{S}{2} \cap E \right|}{\binom{|S|}{2}},$$

where $\binom{S}{2}$ is the set of all unordered pairs of vertices from S . Hence $0 \leq \sigma(S) \leq 1$ for all subsets, $\sigma(S) = 0$ if S is an *independent set* and $\sigma(S) = 1$ if S is a *clique*.

We are interested in the following general problem: given a graph $G = (V, E)$ with $|V| = n$ vertices and an integer $m \leq n$, estimate the highest density of an m -subset $S \subset V$. This is, of course, a hard problem: for example, testing whether a given graph contains a clique of a given size, or even estimating the size of the largest clique within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, fixed in advance, is already an NP-hard problem [Hå99], [Zu07]. Moreover, modulo some plausible complexity assumptions,

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it is hard to approximate the highest density of an m -subset for a given m , within a constant factor, fixed in advance [Bh12]. The best known efficient approximation achieves the factor of $n^{1/4}$ in quasi-polynomial $n^{O(\ln n)}$ time [B+10]. There are indications that the factor $n^{1/4}$ might be hard to beat [B+12]. We note that the most interesting case is when m grows and $n \gg m$, since the highest density of an m -subset can be computed in polynomial time up to an additive error of $\epsilon n^2/m^2$ for any $\epsilon > 0$, fixed in advance [FK99] (and if m is fixed in advance, the densest m -subset can be found by the exhaustive search in polynomial time).

(1.1) Partition function. In this paper, we approach the problem of finding the densest, or just a reasonably dense subset, via computing the *partition function*

$$(1.1.1) \quad \text{den}_m(G; \gamma) = \binom{n}{m}^{-1} \sum_{\substack{S \subset V: \\ |S|=m}} \exp \{ \gamma m \sigma(S) \},$$

where $\gamma > 0$ is a parameter. We are interested in computing (approximating) $\text{den}_m(G; \gamma)$ efficiently. The *exponential tilting*, $\sigma(S) \mapsto \exp \{ \gamma m \sigma(S) \}$, see for example, Section 13.7 of [Te99], puts greater emphasis on the sets of higher density. Let us consider the set $\binom{V}{m}$ of all m -subsets of V as a probability space with the uniform measure. By the Markov inequality, for any $0 < \sigma_0 < 1$, we have

$$(1.1.2) \quad \sigma_0 + \frac{\ln \mathbf{P}(\sigma(S) \geq \sigma_0)}{\gamma m} \leq \frac{\ln \text{den}_m(G; \gamma)}{\gamma m} \leq \max_{\substack{S \subset V: \\ |S|=m}} \sigma(S),$$

so the larger γ we can afford, the better approximation for the densest m -subset we get. In particular, if we could choose $\gamma \gg \ln n$ then from (1.1.2) we could approximate the highest density of an m -subset within an arbitrarily small additive error.

The partition function (1.1.1) was introduced in [Ba15], where an algorithm of quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ complexity was constructed to compute (1.1.1) within relative error $0 < \epsilon < 1$, when $\gamma = 0.07$ and when $\gamma = 0.27$, under additional assumptions that $n \geq 8m$ and $m \geq 10$. It follows from (1.1.2) that if the probability to hit an m -subset S of density at least σ_0 at random is $e^{-o(m)}$ then we can certify the existence of an m -subset of density at least $\sigma_0 - o(1)$ in quasi-polynomial time, just by computing (1.1.1). It is also shown in [Ba15] that by successive conditioning, one can find in quasi-polynomial time an m -subset S with density at least as high as certified by the value of (1.1.1).

In this paper, we present an algorithm, which, for any $0 < \gamma < 1$, fixed in advance, and a given $0 < \epsilon < 1$, computes the value of (1.1.1) within relative error ϵ in quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ time, provided $n > \omega(\gamma)m$ for some constant $\omega(\gamma) > 1$. This improvement from $\gamma = 0.27$ to $\gamma = 1$ makes the algorithm competitive in some situations where it was not competitive before. Suppose, for example, we want to separate efficiently the graphs that have sufficiently many m -cliques

from the graphs that are sufficiently far from having a single m -clique. Below we show that for $\gamma < 0.5$ our algorithm is inferior to a simple test based on the Kruskal - Katona Theorem, while for $\gamma > 0.5$ the former can cover a greater range than the latter.

(1.2) Example: testing graphs for m -cliques. Let us fix two numbers $0 < \delta < 1$ and $\alpha > 0$ and consider the following two mutually exclusive conditions:

(1.2.1) For every $S \subset V$ such that $|S| = m$ we have $\sigma(S) \leq 1 - \delta$

and

(1.2.2) If $S \subset V$ is a random subset, sampled uniformly from the set $\binom{V}{m}$ of all m -sets of vertices, then the probability that S is a clique is at least $e^{-\alpha m}$.

Suppose further, we are presented with a graph $G = (V, E)$ and told that either condition (1.2.1) or condition (1.2.2) holds. Our goal is to decide which one. This is somewhat in the spirit of “property testing” [Go17].

We observe that if (1.2.1) holds then $\text{den}_m(G; \gamma) \leq e^{\gamma m(1-\delta)}$ and if (1.2.2) holds then $\text{den}_m(G; \gamma) \geq e^{(\gamma-\alpha)m}$. Consequently, if

$$(1.2.3) \quad \alpha < \gamma\delta$$

and we can approximate $\text{den}_m(G; \gamma)$ efficiently, we can efficiently tell (1.2.1) and (1.2.2) apart.

An anonymous referee to [Ba15] noticed that another, much simpler, algorithm can be inferred from the Kruskal - Katona Theorem. Let $|V| = n$. If (1.2.1) holds then $|E| \leq (1-\delta)\binom{n}{2}$. The Kruskal - Katona Theorem (see, for example, Section 5 of [Bo86]) implies that if (1.2.2) holds, then for every k such that $\binom{k}{m} \leq e^{-\alpha m}\binom{n}{m}$, we must have $|E| \geq \binom{k}{2}$, the model case being a graph G consisting of a k -clique and $n - k$ isolated vertices. A computation shows that as $n \rightarrow \infty$, we can tell (1.2.1) and (1.2.2) apart just by counting the edges of G , provided

$$(1.2.4) \quad \alpha < -\frac{1}{2} \ln(1 - \delta).$$

Comparing (1.2.3) and (1.2.4), we observe that the algorithm based on computing the partition function $\text{den}_m(G; \gamma)$ is not competitive as long as $\gamma < 0.5$, which is the case in [Ba15], but becomes competitive at least for small values of δ as soon as $\gamma > 0.5$. Numerical estimates show that as long as we can choose $\gamma > 0$ arbitrarily close 1, the condition (1.2.3) serves a wider range of α than the condition (1.2.4) provided $\delta < 0.7968$.

We still don’t know, however, if (1.1.1) can be efficiently computed for *any* $\gamma > 0$, fixed in advance, and as we remarked above, it is unlikely that (1.1.1) can be efficiently computed for $\gamma \gg \ln n$. Our numerical experiments seem to indicate that we can afford a substantially larger γ . This can be partially explained by the

fact that for the Erdős - Rényi random graph $G(n, 0.5)$ indeed a much larger γ can be used with high probability, see Theorem 1.5 below.

The improvement from $\gamma = 0.27$ to an arbitrary $\gamma < 1$ required the addition of some new ideas to the technique of [Ba15]. The approach of [Ba15] and of this paper are based on the “interpolation method” [Ba16]. As applied to our case, the idea of the method is to consider $\text{den}_m(G; z)$ for a *complex* parameter z . We can efficiently approximate $\text{den}_m(G; z)$ at $z = \gamma$ if there is a connected open set $U \subset \mathbb{C}$, not dependent on m or G , such that $0 \in U$, $\gamma \in U$ and $\text{den}_m(G; z) \neq 0$ for all $z \in U$. In [Ba15], the set U is a disk centered at $z = 0$, whereas in the current paper it is a thin neighborhood of the interval $[0, \gamma]$, which allows us to reach larger γ , but also requires a more refined analysis to establish zero-freeness. We give some more details now.

(1.3) Multivariate partition function. Given $n \times n$ symmetric complex matrix $Z = (z_{ij})$ and $2 \leq m \leq n$, we define

$$(1.3.1) \quad P_m(Z) = \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \exp \left\{ \sum_{\substack{\{i,j\} \subset S \\ i \neq j}} z_{ij} \right\}.$$

Note that the diagonal entries of Z are irrelevant, so we assume that $z_{ii} = 0$ for all i .

Given a graph $G = (V, E)$ with set $V = \{1, \dots, n\}$ of vertices and $\gamma > 0$, we define $Z_0 = (z_{ij})$ by

$$z_{ij} = \begin{cases} \frac{\gamma}{m-1} & \text{if } \{i, j\} \in E \\ -\frac{\gamma}{m-1} & \text{if } \{i, j\} \notin E, \end{cases}$$

and observe that

$$(1.3.2) \quad \begin{aligned} P_m(Z_0) &= \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \exp \left\{ m\gamma\sigma(S) - \frac{\gamma m}{2} \right\} \\ &= \exp \left\{ -\frac{\gamma m}{2} \right\} \binom{n}{m} \text{den}_m(G; \gamma). \end{aligned}$$

Hence to compute (1.1.1) it suffices to compute $P_m(Z_0)$. We compute $P_m(Z_0)$ by interpolation, see [Ba15], [Ba16]. For that, it suffices to show that $P_m(Z) \neq 0$ in some neighborhood of a path connecting the zero matrix to Z_0 in the space of complex matrices.

We prove the following result.

(1.4) Theorem. *For any $0 < \delta < 1$ there exist $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ such that if $n \geq \omega m$ then $P_m(Z) \neq 0$ for any $n \times n$ symmetric complex matrix $Z = (z_{ij})$ such that*

$$|\Re z_{ij}| \leq \frac{\delta}{m-1} \quad \text{and} \quad |\Im z_{ij}| \leq \frac{\eta}{m-1} \quad \text{for all } 1 \leq i \neq j \leq n.$$

We prove Theorem 1.4 in Sections 2 and 3. Using Theorem 1.4, in Section 4 we present an algorithm of quasi-polynomial $n^{O(\ln m)}$ complexity to compute $P_m(Z_0)$ and hence $\text{den}_m(G; \gamma)$ for any $0 < \gamma < 1$, fixed in advance.

In [Ba15] it was established that $P_m(Z) \neq 0$ in a polydisc

$$\mathcal{D}_{m,n} = \left\{ Z = (z_{ij}) : |z_{ij}| \leq \frac{0.27}{m-1} \text{ for all } 1 \leq i \neq j \leq n \right\}$$

provided $n \gg m$ and m is large enough. In Theorem 1.4, we establish that $P_m(Z) \neq 0$ in a more “economical” domain, “stretched” along the real part of the complex space of matrices. This allows us to improve the constant γ for which $\text{den}_m(G; \gamma)$ is still efficiently computable.

In Section 5, we discuss some results of our numerical experiments, which seem to indicate that we can afford an essentially bigger δ in Theorem 1.4. This can be partially explained by the fact that for the Erdős - Rényi random graph $G(n, 0.5)$ this is indeed the case. Namely, we prove the following result in Section 6.

(1.5) Theorem. *Let us choose positive integers n and $2 \leq m \leq n$. For $n \times n$ symmetric matrix $W = (w_{ij})$ of independent random variables, where*

$$\mathbf{P}(w_{ij} = 1) = \mathbf{P}(w_{ij} = -1) = \frac{1}{2},$$

we define the polynomial

$$h_W(z) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \prod_{\{i,j\} \subset S} (1 + zw_{ij}).$$

Let $r > 0$ and $\tau > 1$ be real numbers. If $n \geq 2m^2(1 + r^2)^m + 2m$ then the probability that $h_W(z)$ has a root in the disc $|z| < r/\sqrt{2\tau}$ does not exceed $1/\tau$.

In particular, if $n \gg m^2$ then with high probability $h_W(z)$ has no roots in the disc $|z| < c/\sqrt{m}$, for an arbitrary large $c > 0$, fixed in advance. Similarly, if $\ln n \gg m$ then with high probability $h_W(z)$ has no roots in the disc $|z| < c$ for an arbitrary large $c > 0$, fixed in advance.

The polynomial $h_W(z)$ is easily translated into the partition function $\text{den}_m(G; \gamma)$, where G is the graph with set $V = \{1, \dots, n\}$ of vertices and two vertices $\{i, j\}$ span an edge if and only if $w_{ij} = 1$: for $0 < \alpha < 1$, we have

$$(1.5.1) \quad h_W(\alpha) = (1 - \alpha)^{\binom{m}{2}} \text{den}_m(G; \gamma) \quad \text{where} \quad \gamma = \frac{m-1}{2} \ln \frac{1+\alpha}{1-\alpha}.$$

Consequently, with high probability we can approximate $\text{den}_m(G; \gamma)$ in quasi-polynomial time for γ as large as $\gamma = \sqrt{m}$ provided $n \gg m^2$ and as large as $\gamma = m$

provided $\ln n \gg m$. Since the graphs we experimented on were to a large degree random (but not necessarily Erdős - Rényi $G(n, 0.5)$), we may have obtained overly optimistic numerical evidence.

As is easily seen, $\mathbf{E} h_W(\alpha) = 1$ and from our proof in Section 6 it follows that $h_W(\alpha)$ is strongly concentrated. For example, in the regime of $n = \Omega(m^2)$ and $\alpha = 1/\sqrt{m}$, we have $\mathbf{var} h_W(\alpha) = O(1)$. This concentration, however, does not allow us to predict with high probability the value of $h_W(\alpha)$ with the precision that the interpolation technique based on Theorem 1.5 allows for.

In Section 6, we also discuss what may happen if G is a random graph $G(n, 0.5)$ with a planted m -clique.

2. PRELIMINARIES

We consider the partition function P_m of Section 1.3 within a family of partition functions, which will allow us to prove Theorem 1.4 by induction.

(2.1) Functionals $P_\Omega(Z)$. Let us fix integers n and $2 \leq m \leq n$. For a subset $\Omega \subset \{1, \dots, n\}$ and $n \times n$ complex symmetric matrix $Z = (z_{ij})$, we define

$$P_\Omega(Z) = \sum_{\substack{S \subset \{1, \dots, n\}: \\ |S|=m, \Omega \subset S}} \exp \left\{ \sum_{\substack{\{i,j\} \subset S \\ i \neq j}} z_{ij} \right\}$$

where we agree that $P_\Omega(Z) = 0$ if $|\Omega| > m$. In other words, we restrict the sum (1.3.1) defining $P_m(Z)$ onto subsets S containing a given set Ω . In particular,

$$P_\Omega(Z) = P_m(Z) \quad \text{if} \quad \Omega = \emptyset.$$

The induction will be built on the following straightforward formulas:

$$(2.1.1) \quad P_\Omega(Z) = \frac{1}{m - |\Omega|} \sum_{j \in \{1, \dots, n\} \setminus \Omega} P_{\Omega \cup \{j\}}(Z) \quad \text{provided} \quad |\Omega| < m$$

and for $i \neq j$, we have

$$(2.1.2) \quad \frac{\partial}{\partial z_{ij}} P_\Omega(Z) = \begin{cases} P_\Omega(Z) & \text{if } i, j \in \Omega, \\ P_{\Omega \cup \{j\}}(Z) & \text{if } i \in \Omega, j \notin \Omega, \\ P_{\Omega \cup \{i\}}(Z) & \text{if } i \notin \Omega, j \in \Omega, \\ P_{\Omega \cup \{i, j\}}(Z) & \text{if } i, j \notin \Omega. \end{cases}$$

We will often consider complex numbers as vectors in the plane, by identifying $\mathbb{C} = \mathbb{R}^2$ and measuring, in particular, angles between non-zero complex numbers. We will use the following geometric lemma.

(2.2) Lemma. *Let $u_1, \dots, u_n \in \mathbb{C}$ be non-zero complex numbers such that the angle between any two does not exceed θ for some $0 < \theta < \pi/2$. Suppose that*

$$\Im \left(\sum_{j=1}^n u_j \right) = 0 \quad \text{and} \quad \sum_{j=1}^n |u_j| = c.$$

Then

$$\sum_{j=1}^n |\Im u_j| \leq c \sin \frac{\theta}{2}.$$

Proof. Scaling u_j , if necessary, without loss of generality we assume that $c = 1$.

Without loss of generality, we assume that $\arg u_j \neq 0$ for $j = 1, \dots, n$. Indeed, if $\arg u_j = 0$ for some j , we can remove the vector from the collection, which would make the sum

$$(2.2.1) \quad \sum_{j=1}^n |u_j|$$

only smaller. Rescaling $u_j \mapsto \tau u_j$ for some real $\tau > 1$, we make (2.2.1) equal to 1 and increase

$$(2.2.2) \quad \sum_{j=1}^n |\Im u_j|.$$

Reflecting the vectors u_j in the coordinate axes if necessary, without loss of generality we may assume that $\Re u_1 \geq 0$ and $\Im u_1 > 0$. Hence there is a vector, say u_2 , such that $\Im u_2 < 0$. We necessarily have $\Re u_2 \geq 0$, since otherwise the angle between u_1 and u_2 exceeds $\pi/2$. Then for any vector u_j , we must have $\Re u_j \geq 0$, since otherwise one of the angles formed by u_j with u_1 or u_2 will exceed $\pi/2$.

Hence without loss of generality, we assume that $\Re u_j > 0$ for $j = 1, \dots, n$. Let

$$\alpha = \max_{j=1, \dots, n} \arg u_j,$$

so that

$$0 < \alpha < \theta$$

and let

$$-\beta = \min_{j=1, \dots, n} \arg u_j < 0.$$

Then $\alpha + \beta \leq \theta$.

Let

$$J_+ = \{j : \arg u_j > 0\} \quad \text{and} \quad J_- = \{j : \arg u_j < 0\}.$$

Next, without loss of generality, we assume that $\arg u_j = \alpha$ for all $j \in J_+$ and that $\arg u_j = -\beta$ for all $j \in J_-$. Indeed, suppose that $\arg u_1 = \alpha_1$ where $0 < \alpha_1 < \alpha$. We can modify

$$u_1 \mapsto \frac{\sin \alpha_1}{\sin \alpha} e^{i(\alpha - \alpha_1)} u_1$$

(we rotate and shrink u_1 so as to make its argument equal to α and leave $\Im u_1$ intact). The sum (2.2.1) gets smaller while all other conditions and the sum (2.2.2) remain intact. Rescaling $u_j \mapsto \tau u_j$ for some real $\tau > 1$, we make (2.2.1) equal to 1 and increase (2.2.2), while keeping other constraints of the lemma intact. The case of $\arg u_j > -\beta$ for some $j \in J_-$ is handled similarly.

Next, without loss of generality, we assume that $\alpha + \beta = \theta$. Indeed, if $\alpha + \beta < \theta$, we can rotate and scale vectors u_j as above, so that the sum (2.2.2) increases while all other conditions are satisfied.

Now, let

$$u_+ = \sum_{j \in J_+} u_j \quad \text{and} \quad u_- = \sum_{j \in J_-} u_j.$$

Then $\arg u_+ = \alpha$, $\arg u_- = -\beta$, $\Im(u_+ + u_-) = 0$, $|u_+| + |u_-| = 1$ and (2.2.2) is equal to $|\Im u_+| + |\Im u_-|$.

Denoting $a = |u_+|$ and $b = |u_-|$, we have $a + b = 1$ and $a \sin \alpha - b \sin \beta = 0$, from which

$$a = \frac{\sin \beta}{\sin \alpha + \sin \beta} \quad \text{and} \quad b = \frac{\sin \alpha}{\sin \alpha + \sin \beta}$$

and so

$$|\Im u_+| + |\Im u_-| = \frac{2 \sin \alpha \sin \beta}{\sin \alpha + \sin \beta}.$$

Now, the function

$$\alpha \mapsto \frac{1}{\sin \alpha} \quad \text{for} \quad 0 \leq \alpha \leq \frac{\pi}{2}$$

is convex and hence the minimum of

$$\frac{\sin \alpha + \sin \beta}{\sin \alpha \sin \beta} = \frac{1}{\sin \alpha} + \frac{1}{\sin \beta}$$

on the interval $\alpha + \beta = \theta$, $\alpha, \beta \geq 0$, is attained at $\alpha = \beta = \theta/2$. The proof now follows. $\square$

We need another geometric lemma.

(2.3) Lemma. *Let $u_1, \dots, u_n \in \mathbb{C}$ be non-zero complex numbers such that the angle between any two does not exceed θ for some $0 \leq \theta < 2\pi/3$. Let $u = u_1 + \dots + u_n$. Then*

$$|u| \geq \left(\cos \frac{\theta}{2} \right) \sum_{k=1}^n |u_k|.$$

Proof. This is Lemma 3.1 of [Ba15] and Lemma 3.6.3 of [Ba16]. $\square$

3. PROOF OF THEOREM 1.4

We identify the space of $n \times n$ zero-diagonal complex symmetric matrices $Z = (z_{ij})$ with $\mathbb{C}^{\binom{n}{2}}$. Given $\delta \geq \eta > 0$, we define a domain $\mathcal{U}(\delta, \eta) = \mathcal{U}_{n,m}(\delta, \eta) \subset \mathbb{C}^{\binom{n}{2}}$ by

$$\mathcal{U}(\delta, \eta) = \left\{ Z = (z_{ij}) : |\Re z_{ij}| \leq \frac{\delta}{m-1} \quad \text{and} \quad |\Im z_{ij}| \leq \frac{\eta}{m-1} \right\}.$$

If $Z' = (z'_{ij})$ and $Z'' = (z''_{ij})$ are two matrices from $\mathcal{U}(\delta, \tau)$ then

$$|z'_{ij} - z''_{ij}| \leq \frac{\sqrt{(2\delta)^2 + (2\eta)^2}}{m-1} \leq \frac{2\sqrt{2}\delta}{m-1} \quad \text{for all } i, j.$$

We will prove by descending induction on $|\Omega|$ that $P_\Omega(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$ and that, moreover, a number of stronger conditions are met. The induction is based on the following two lemmas that describe how $P_\Omega(Z)$ changes when only the entries in the i -th row and column of Z change. The first lemma deals with the case of $i \in \Omega$.

(3.1) Lemma. *Let us fix $\Omega \subset \{1, \dots, n\}$ such that $|\Omega| < m$. Suppose that for any $Z \in \mathcal{U}(\delta, \eta)$ and any $j, k \notin \Omega$, we have $P_{\Omega \cup \{j\}}(Z) \neq 0$, $P_{\Omega \cup \{k\}}(Z) \neq 0$ and the angle between the two non-zero complex numbers does not exceed θ for some $0 < \theta \leq \pi/2$. Then*

(1) *We have*

$$P_\Omega(Z) \neq 0 \quad \text{for all } Z \in \mathcal{U}(\delta, \eta).$$

(2) *Suppose additionally, that $\Omega \neq \emptyset$ and let us fix an $i \in \Omega$. Let $Z', Z'' \in \mathcal{U}(\delta, \eta)$ be two matrices that differ only in the coordinates $z_{ij} = z_{ji}$ for $j \neq i$. Then*

$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \leq e^{6\delta}$$

and the angle between $P_\Omega(Z') \neq 0$ and $P_\Omega(Z'') \neq 0$ does not exceed

$$2\delta \tan \frac{\theta}{2} + 5\eta.$$

Proof. It follows from (2.1.1) and Lemma 2.3 that

$$(3.1.1) \quad |P_\Omega(Z)| \geq \frac{\cos(\theta/2)}{m-|\Omega|} \sum_{j \notin \Omega} |P_{\Omega \cup \{j\}}(Z)| \geq \frac{1}{(m-1)\sqrt{2}} \sum_{j \notin \Omega} |P_{\Omega \cup \{j\}}(Z)|.$$

In particular, Part (1) follows.

To prove Part (2), let us choose a branch of $\ln P_\Omega(Z)$ for $Z \in \mathcal{U}(\delta, \eta)$. For $0 \leq t \leq 1$, let $Z(t) = tZ'' + (1-t)Z'$. Then

$$\begin{aligned} \ln P_\Omega(Z'') - \ln P_\Omega(Z') &= \int_0^1 \frac{d}{dt} \ln P_\Omega(Z(t)) dt \\ &= \int_0^1 \sum_{j: j \neq i} (z''_{ij} - z'_{ij}) \frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) \Big|_{Z=Z(t)} dt. \end{aligned}$$

Using (2.1.2), we conclude that

$$\frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) = \begin{cases} 1 & \text{if } j \in \Omega, \\ P_{\Omega \cup \{j\}}(Z)/P_\Omega(Z) & \text{if } j \notin \Omega, \end{cases}$$

and hence

$$\begin{aligned} \ln P_\Omega(Z'') - \ln P_\Omega(Z') &= \sum_{j \in \Omega, j \neq i} (z''_{ij} - z'_{ij}) \\ (3.1.2) \quad &+ \int_0^1 \sum_{j \notin \Omega} (z''_{ij} - z'_{ij}) \frac{P_{\Omega \cup \{j\}}(Z(t))}{P_\Omega(Z(t))} dt. \end{aligned}$$

Using (3.1.1), we get from (3.1.2) that

$$\begin{aligned} |\Re \ln P_\Omega(Z'') - \Re \ln P_\Omega(Z')| &\leq 2\delta + (m-1)\sqrt{2} \max_{j \notin \Omega} |z''_{ij} - z'_{ij}| \\ &\leq 2\delta + 4\delta = 6\delta \end{aligned}$$

and hence

$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \leq e^{6\delta},$$

as claimed.

From (2.1.1), for all $Z \in \mathcal{U}(\delta, \eta)$ we have that

$$\sum_{j \notin \Omega} \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} = m - |\Omega|$$

is real, while from (3.1.1), we conclude that

$$\sum_{j \notin \Omega} \left| \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} \right| \leq \frac{m - |\Omega|}{\cos(\theta/2)} \leq \frac{m-1}{\cos(\theta/2)}.$$

Applying Lemma 2.2 with $u_j = P_{\Omega \cup \{j\}}(Z)/P_\Omega(Z)$, we conclude that

$$\sum_{j \notin \Omega} \left| \Im \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} \right| \leq (m-1) \tan \frac{\theta}{2}.$$

Therefore, from (3.1.2),

$$\begin{aligned}
|\Im \ln P_\Omega(Z'') - \Im \ln P_\Omega(Z')| &\leq 2\eta + (m-1) \tan \frac{\theta}{2} \max_{j \notin \Omega} |\Re z''_{ij} - \Re z'_{ij}| \\
&\quad + (m-1) \sqrt{2} \max_{j \notin \Omega} |\Im z''_{ij} - \Im z'_{ij}| \\
&\leq 2\delta \tan \frac{\theta}{2} + 5\eta.
\end{aligned}$$

Hence the angle between $P_\Omega(Z'')$ and $P_\Omega(Z')$ does not exceed $2\delta \tan \frac{\theta}{2} + 5\eta$, as claimed. $\square$

The second lemma shows that $P_\Omega(Z)$ does not change much if only the entries of Z in the i -th row and column are changed for some $i \notin \Omega$, assuming that $n \gg m$.

(3.2) Lemma. *Let us fix an $\Omega \subset \{1, \dots, n\}$, $|\Omega| \leq m-1$. Suppose for any $i, j \notin \Omega$ and all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega \cup \{i\}}(Z) \neq 0$, $P_{\Omega \cup \{j\}}(Z) \neq 0$ and the angle between the two complex numbers does not exceed $\pi/2$ and that*

$$\left| \frac{P_{\Omega \cup \{i\}}(Z)}{P_{\Omega \cup \{j\}}(Z)} \right| \leq \lambda$$

for some $\lambda \geq 1$.

In addition, suppose that if $|\Omega| \leq m-2$ then for any distinct $i, j, k \notin \Omega$ and all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega \cup \{i, j\}}(Z) \neq 0$, $P_{\Omega \cup \{i, k\}}(Z) \neq 0$ and the angle between the two complex numbers does not exceed $\pi/2$.

Let us fix an $i \notin \Omega$ and let $Z', Z'' \in \mathcal{U}(\delta, \eta)$ be two matrices that differ only in the coordinates $z_{ij} = z_{ji}$ for $j \neq i$. Then

$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \leq \exp \left\{ \frac{10\delta\lambda m}{n-1} \right\}$$

and the angle between $P_\Omega(Z') \neq 0$ and $P_\Omega(Z'') \neq 0$ does not exceed

$$\frac{10\delta\lambda m}{n-1}.$$

Proof. It follows from Lemma 3.1 that $P_\Omega(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$.

Arguing as in the proof of Lemma 3.1, we introduce $Z(t) = tZ'' + (1-t)Z'$ and write

$$\ln P_\Omega(Z'') - \ln P_\Omega(Z') = \int_0^1 \sum_{j: j \neq i} (z''_{ij} - z'_{ij}) \frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) \Big|_{Z=Z(t)} dt.$$

From (2.1.2), we write

$$(3.2.1) \quad \begin{aligned} \ln P_\Omega(Z'') - \ln P_\Omega(Z') &= \int_0^1 \sum_{j \in \Omega} (z''_{ij} - z'_{ij}) \frac{P_{\Omega \cup \{i\}}(Z(t))}{P_\Omega(Z(t))} \\ &\quad + \sum_{j \notin \Omega, j \neq i} (z''_{ij} - z'_{ij}) \frac{P_{\Omega \cup \{i, j\}}(Z(t))}{P_\Omega(Z(t))} dt. \end{aligned}$$

Suppose first that $|\Omega| \leq m - 2$. From (2.1.1), we have

$$P_{\Omega \cup \{i\}}(Z) = \frac{1}{m - |\Omega| - 1} \sum_{j \notin \Omega, j \neq i} P_{\Omega \cup \{i, j\}}(Z).$$

Applying Lemma 2.3, we get that

$$(3.2.2) \quad \sum_{j \notin \Omega, j \neq i} |P_{\Omega \cup \{i, j\}}(Z)| \leq (m - 1)\sqrt{2} |P_{\Omega \cup \{i\}}(Z)|$$

for all $Z \in \mathcal{U}(\delta, \eta)$.

Since by (2.1.1) we also have

$$P_\Omega(Z) = \frac{1}{m - |\Omega|} \sum_{j \notin \Omega} P_{\Omega \cup \{j\}}(Z),$$

applying Lemma 2.3, we conclude that

$$\sum_{j \notin \Omega} |P_{\Omega \cup \{j\}}(Z)| \leq (m - |\Omega|)\sqrt{2} |P_\Omega(Z)|.$$

Hence for all $i \notin \Omega$, we have

$$(3.2.3) \quad |P_{\Omega \cup \{i\}}(Z)| \leq \frac{\lambda(m - |\Omega|)\sqrt{2}}{n - |\Omega|} |P_\Omega(Z)| \leq \frac{\lambda m \sqrt{2}}{n} |P_\Omega(Z)|.$$

Combining (3.2.3) and (3.2.2), we get

$$(3.2.4) \quad \sum_{j \notin \Omega, j \neq i} |P_{\Omega \cup \{i, j\}}(Z)| \leq \frac{2\lambda m(m - 1)}{n} |P_\Omega(Z)|.$$

Combining (3.2.1), (3.2.2), (3.2.3) and (3.2.4), we get

$$\begin{aligned} |\ln P_\Omega(Z'') - \ln P_\Omega(Z')| &\leq \frac{2\sqrt{2}\delta}{m - 1} \cdot \frac{\lambda|\Omega|(m - |\Omega|)\sqrt{2}}{n - |\Omega|} + \frac{2\sqrt{2}\delta}{m - 1} \cdot \frac{2\lambda m(m - 1)}{n} \\ &\leq \frac{4\delta\lambda m}{n - 1} + \frac{4\sqrt{2}\delta\lambda m}{n} \leq \frac{10\delta\lambda m}{n - 1}. \end{aligned}$$

If $|\Omega| = m - 1$ then from (3.2.1) and (3.2.3), we get

$$|\ln P_\Omega(Z'') - \ln P_\Omega(Z')| \leq \frac{2\sqrt{2}\delta}{m - 1} \cdot \frac{\lambda m \sqrt{2}}{n} \leq \frac{4\delta\lambda m}{n - 1},$$

which concludes the proof. $\square$

Now we are ready to prove Theorem 1.4.

(3.3) Proof of Theorem 1.4. Given $0 < \delta < 1$, we choose $0 < \theta < \pi/2$ so that

$$2\delta \tan \frac{\theta}{2} < \theta.$$

We then choose $\eta > 0$ such that

$$2\delta \tan \frac{\theta}{2} + 5\eta < \theta.$$

We choose

$$\lambda > e^{6\delta}$$

and choose $\omega > 1$ so that

$$2\delta \tan \frac{\theta}{2} + 5\eta + \frac{10\delta\lambda m}{n-1} \leq \theta \quad \text{and} \quad \exp \left\{ 6\delta + \frac{10\delta\lambda m}{n-1} \right\} \leq \lambda$$

whenever $n \geq \omega m$.

Suppose that $n \geq \omega m$. We prove by descending induction on $r = m, m-1, \dots, 1$ that if $\Omega_1, \Omega_2 \in \{1, \dots, n\}$ are two sets such that $|\Omega_1| = |\Omega_2| = r$ and $|\Omega_1 \Delta \Omega_2| = 2$ then for all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega_1}(Z) \neq 0$, $P_{\Omega_2}(Z) \neq 0$, the angle between $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ does not exceed θ while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed λ .

Assume that $r = m$. Without loss of generality, we assume that $\Omega_1 = \Omega \cup \{1\}$ and $\Omega_2 = \Omega \cup \{2\}$ for some $\Omega \subset \{3, \dots, n\}$ such that $|\Omega| = m-1$. We have

$$\begin{aligned} P_{\Omega_1}(Z) &= \exp \left\{ \sum_{\{i,j\} \subset \Omega} z_{ij} \right\} \exp \left\{ \sum_{i \in \Omega} z_{1i} \right\} \quad \text{and} \\ P_{\Omega_2}(Z) &= \exp \left\{ \sum_{\{i,j\} \subset \Omega} z_{ij} \right\} \exp \left\{ \sum_{i \in \Omega} z_{2i} \right\}. \end{aligned}$$

Clearly, $P_{\Omega_1}(Z) \neq 0$, $P_{\Omega_2}(Z) \neq 0$, the angle between $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ does not exceed $2\eta \leq \theta$ while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed $e^{2\delta} \leq \lambda$.

Suppose now that the statements hold for all subsets $\Omega \subset \{1, \dots, n\}$ of cardinality at least $r+1$ for some $r \leq m-1$ and let $\Omega_1, \Omega_2 \subset \{1, \dots, n\}$ be two subsets of cardinality $r \geq 1$ such that $|\Omega_1 \Delta \Omega_2| = 2$. Again, without loss of generality, we assume that $\Omega_1 = \Omega \cup \{1\}$ and $\Omega_2 = \Omega \cup \{2\}$ for some $\Omega \subset \{3, \dots, n\}$ such that $|\Omega| = r-1$. Then we observe that $P_{\Omega_2}(Z) = P_{\Omega_1}(Z')$, where

$$z'_{1i} = z'_{i1} = z_{2i} = z_{i2} \quad \text{and} \quad z'_{2i} = z'_{i2} = z_{1i} = z_{i1} \quad \text{for } i \neq 1, 2,$$

while all other entries of Z and Z' coincide. Applying Lemma 3.1 and Lemma 3.2 and the induction hypothesis to sets $\Omega_1 \cup \{j\}$ for $j \notin \Omega_1$ and $\Omega_1 \cup \{j, k\}$ for

$j, k \notin \Omega_1$, we conclude that the angle between $P_{\Omega_1}(Z) \neq 0$ and $P_{\Omega_2}(Z) \neq 0$ does not exceed

$$2\delta \tan \frac{\theta}{2} + 5\eta + \frac{10\delta\lambda m}{n-1} \leq \theta,$$

while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed

$$\exp \left\{ 6\delta + \frac{10\delta\lambda m}{n-1} \right\} \leq \lambda.$$

This proves that $P_{\{i\}}(Z) \neq 0$ for all $i \in \{1, \dots, n\}$ and all $Z \in \mathcal{U}(\delta, \eta)$ and that the angle between $P_{\{i\}}(Z) \neq 0$ and $P_{\{j\}}(Z) \neq 0$ does not exceed θ for all $i, j \in \{1, \dots, n\}$. From (2.1.1) we conclude that $P_m(Z) = P_\emptyset(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$. $\square$

4. COMPUTING THE PARTITION FUNCTION

Here we show how to compute the density partition function $\text{den}_m(G; \gamma)$. First, we make a change of coordinates to convert the partition function $P_m(Z)$ of Section 1.3 into a multivariate polynomial.

(4.1) A polynomial version of $P_m(Z)$. For an $n \times n$ complex symmetric matrix $W = (w_{ij})$ with zero diagonal, we define

$$p_m(W) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} (1 + w_{ij}).$$

Hence $p_m(W)$ is a polynomial of degree $\binom{m}{2}$ in the entries w_{ij} and, assuming that $|w_{ij}| < 1$ for all i, j , we can write

$$p_m(W) = \binom{n}{m}^{-1} P_m(Z) \quad \text{where} \quad Z = (z_{ij}) \quad \text{and} \quad z_{ij} = \ln(1 + w_{ij})$$

(we choose the standard branch of the logarithm in the right half-plane of $\mathbb{C}$). Theorem 1.4 implies that for every $0 < \delta < 1$ there is $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ such that

$$(4.1.1) \quad \begin{aligned} p_m(W) \neq 0 \quad \text{whenever} \quad & |\Re \ln(1 + w_{ij})| \leq \frac{\delta}{m-1}, \\ & |\Im \ln(1 + w_{ij})| \leq \frac{\eta}{m-1} \quad \text{and} \\ & n \geq \omega m. \end{aligned}$$

To compute $\text{den}_m(G; \gamma)$ for a given $0 < \gamma < 1$ and a given graph $G = (V, E)$, we define

$$(4.1.2) \quad w_{ij} = \begin{cases} \exp \left\{ \frac{\gamma}{m-1} \right\} - 1 & \text{if } \{i, j\} \in E, \\ \exp \left\{ -\frac{\gamma}{m-1} \right\} - 1 & \text{if } \{i, j\} \notin E. \end{cases}$$

Then, by (1.3.2), we have

$$(4.1.3) \quad \text{den}_m(G; \gamma) = \exp \left\{ \frac{\gamma m}{2} \right\} p_m(W).$$

The interpolation method is based on the following simple lemma.

(4.2) Lemma. *Let $g : \mathbb{C} \rightarrow \mathbb{C}$ be a univariate polynomial and suppose that $g(z) \neq 0$ provided $|z| < \beta$ where $\beta > 1$ is some real number. Let us choose a branch of $f(z) = \ln g(z)$ in the disc $|z| < \beta$ and let*

$$T_r(z) = f(0) + \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} z^k$$

be the Taylor polynomial of f of degree r computed at $z = 0$. Then

$$|f(1) - T_r(1)| \leq \frac{\deg g}{\beta^r (\beta - 1)(r + 1)}.$$

Proof. This is Lemma 2.2.1 of [Ba16], see also Lemma 1.1 of [Ba15]. $\square$

The gist of Lemma 4.2 is that to approximate $f(1)$ within an additive error ϵ , it suffices to compute the Taylor polynomial of $f(z)$ at 0 of degree $r = O_\beta(\ln \deg g - \ln \epsilon)$, where the implicit constant in the “ O ” notation depends on β alone. We would like to apply Lemma 4.2 to the univariate polynomial

$$(4.2.1) \quad h(z) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} (1 + zw_{ij}),$$

where w_{ij} are defined by (4.1.2). Indeed, the value we are ultimately interested is $h(1) = p_m(W)$. However, Lemma 4.2 requires that $h(z) \neq 0$ in a disc of some radius $\beta > 1$, whereas (4.1.1) only guarantees that $h(z) \neq 0$ for z in a neighborhood of the interval $[0, 1] \subset \mathbb{C}$. To remedy this, we compose h with a polynomial $\phi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and ϕ maps the disc $|z| < \beta$ for some $\beta > 1$ inside the prescribed neighborhood of $[0, 1] \subset \mathbb{C}$. We then apply Lemma 4.2 to the composition $g(z) = h(\phi(z))$. The following lemma provides an explicit construction of ϕ .

(4.3) Lemma. *For $0 < \rho < 1$, we define*

$$\begin{aligned} \alpha = \alpha(\rho) &= 1 - e^{-\frac{1}{\rho}}, \quad \beta = \beta(\rho) = \frac{1 - e^{-1 - \frac{1}{\rho}}}{1 - e^{-\frac{1}{\rho}}} > 1, \\ N = N(\rho) &= \left\lceil \left(1 + \frac{1}{\rho}\right) e^{1 + \frac{1}{\rho}} \right\rceil, \quad \sigma = \sigma(\rho) = \sum_{k=1}^N \frac{\alpha^k}{k} \quad \text{and} \\ \phi(z) = \phi_\rho(z) &= \frac{1}{\sigma} \sum_{k=1}^N \frac{(\alpha z)^k}{k}. \end{aligned}$$

Then $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial of degree N such that $\phi(0) = 0$, $\phi(1) = 1$,

$$-\rho \leq \Re \phi(z) \leq 1 + 2\rho \quad \text{and} \quad |\Im \phi(z)| \leq 2\rho$$

provided $|z| \leq \beta$.

Proof. This is Lemma 2.2.3 of [Ba16]. $\square$

Lemma 4.2 also requires the derivatives $f^{(k)}(0)$ of $f(z) = \ln g(z)$ at $z = 0$. Those, however, can be easily computed from the derivatives $g^{(k)}(0)$, as described in Section 2.2.2 of [Ba16], see also Section 2.1 of [Ba15]. We briefly sketch how.

(4.4) Computing derivatives. Suppose that $f(z) = \ln g(z)$ as in Lemma 4.2. Then

$$f'(z) = \frac{g'(z)}{g(z)} \quad \text{and} \quad g'(z) = f'(z)g(z).$$

Differentiating the product $k - 1$ times, we obtain

$$(4.4.1) \quad g^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0) g^{(j)}(0) \quad \text{for } k = 1, \dots, r.$$

We interpret (4.4.1) as a system of linear equations in variables $f^{(k)}(0)$ for $k = 1, \dots, r$ with coefficients $g^{(k)}(0)$ for $k = 0, \dots, r$. This is a triangular system of linear equations with non-zero entries $g^{(0)}(0) = g(0)$ on the diagonal, that can be solved in $O(r^2)$ time, provided the values of $g^{(k)}(0)$ are known.

To supply the last ingredient of the algorithm, we show how to compute $h^{(k)}(0)$ for $k = 0, \dots, r$, where h is the polynomial defined by (4.2.1). This is also done in [Ba15], but we reproduce it here for completeness.

We have

$$h^{(k)}(0) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} \sum_{\{i_1, j_1\}, \dots, \{i_k, j_k\} \subset S} w_{i_1 j_1} \cdots w_{i_k j_k},$$

where the inner sum is taken over all ordered collections of distinct unordered pairs $\{i_1, j_1\}, \dots, \{i_k, j_k\} \subset S$. For such a collection, say I , let $\nu(I)$ be the number of distinct vertices among $i_1, j_1, \dots, i_k, j_k$. Then there are exactly $\binom{n-\nu(I)}{m-\nu(I)}$ different m -subsets S containing the edges from I and we can rewrite the above sum as

$$(4.4.2) \quad h^{(k)}(0) = \binom{n}{m}^{-1} \sum_{I=(\{i_1, j_1\}, \dots, \{i_k, j_k\})} \binom{n-\nu(I)}{m-\nu(I)} w_{i_1 j_1} \cdots w_{i_k j_k},$$

where the sum is taken over all ordered collections of k unordered pairs $\{i_s, j_s\}$. It is clear now that $h^{(k)}(0)$ can be computed in $n^{O(k)}$ time by the exhaustive enumeration of all possible collections of k pairs.

In Section 5 we present faster formulas for computing $h^{(2)}(0)$ and $h^{(3)}(0)$ that we used for our numerical experiments.

(4.5) The algorithm. Let us fix $0 < \gamma < 1$. Below we summarize the algorithm for computing $\text{den}_m(G; \gamma)$ within relative error $0 < \epsilon < 1$, by which we understand computing $\ln \text{den}_m(G; \gamma)$ within additive error ϵ . We assume that $m \geq 4$ and that $n \geq \omega m$ for some $\omega = \omega(\gamma) > 1$, to be specified below.

Given a graph $G = (V, E)$ with set $V = \{1, \dots, n\}$ of vertices, and an integer $m \leq n$, we compute the $n \times n$ symmetric matrix $W = (w_{ij})$ by (4.1.2). Since $m \geq 4$, we have $|w_{ij}| \leq 0.4$ for all i, j .

Our goal is to compute $p_m(W) = h(1)$, where h is the univariate polynomial defined by (4.2.1). We note that $\deg h = \binom{m}{2}$.

Let us choose $1 > \delta > \gamma$ and let $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ be the numbers of Theorem 1.4 and in (4.1.1). We find $\rho = \rho(\delta) > 0$ such that

$$|\Re \ln(1 + zw_{ij})| \leq \frac{\delta}{m-1} \quad \text{and} \quad |\Im \ln(1 + zw_{ij})| \leq \frac{\eta}{m-1}$$

as long as

$$(4.5.1) \quad -\rho \leq \Re z \leq 1 + \rho \quad \text{and} \quad |\Im z| \leq \rho.$$

Indeed, if $z \in [0, 1]$ then

$$-\frac{\gamma}{m-1} \leq \ln(1 + zw_{ij}) \leq \frac{\gamma}{m-1}$$

and for $|z| \leq 2$, we have

$$\left| \frac{d}{dz} \ln(1 + zw_{ij}) \right| = \left| \frac{w_{ij}}{1 + zw_{ij}} \right| \leq \frac{10}{m-1}$$

so the desired ρ can indeed be found.

It follows by (4.1.1) that $h(z) \neq 0$ as long as $n \geq \omega m$ and (4.5.1) holds.

Using Lemma 4.3, we construct a polynomial $\phi : \mathbb{C} \rightarrow \mathbb{C}$ of some degree $N = N(\rho) = N(\delta)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and

$$-\rho \leq \Re \phi(z) \leq 1 + \rho \quad \text{and} \quad |\Im \phi(z)| \leq \rho$$

as long as $|z| \leq \beta$ for some $\beta = \beta(\rho) = \beta(\delta) > 1$. We define

$$g(z) = h(\phi(z))$$

and our goal is to compute $g(1) = h(\phi(1))$. We note that

$$\deg g \leq N \deg h = N \binom{m}{2}.$$

We choose a branch of $f(z) = \ln g(z)$ for z satisfying (4.5.1).

Using Lemma 4.2, we find an integer $r = O_\rho(\ln m - \ln \epsilon) = O_\delta(\ln m - \ln \epsilon)$ such that

$$|T_r(1) - f(1)| \leq \epsilon,$$

where $T_r(z)$ is the Taylor polynomial of $f(z)$ of degree r , computed at $z = 0$. The implicit constant in the “ O ” notation depends only on ρ , which in turn depends only on δ . Hence our goal is to compute $T_r(1)$, for which we need to compute $f^{(k)}(0)$ for $k = 1, \dots, r$. As in Section 4.4, we reduce it in $O(r^2)$ time to computing $g^{(k)}(0)$ for $k = 1, \dots, r$. Note that

$$g(0) = h(\phi(0)) = h(0) = 1.$$

Let $\phi_r(z)$ be the truncation of the polynomial $\phi(z)$ obtained by discarding all monomials of degree higher than r . Similarly, let $h_r(z)$ be the truncation of the polynomial $h(z)$, obtained by discarding all monomial of degree higher than r . We compute $h_r(z)$ as in Section 4.4 in $n^{O(r)}$ time. Finally, we compute the truncation of the composition $h_r(\phi_r(z))$. A fast (polynomial in r) way to do it, is to use Horner’s method: assuming that

$$h_r(z) = \sum_{k=0}^r b_k z^k,$$

we successively compute

$$\begin{aligned} & b_r \phi_r(z) + b_{r-1}, \quad (b_r \phi_r(z) + b_{r-1}) \phi_r(z) + b_{r-2}, \\ & ((b_r \phi_r(z) + b_{r-1}) \phi_r(z) + b_{r-2}) \phi_r(z) + b_{r-3}, \dots \end{aligned}$$

discarding on the way all monomials of degree higher than r . In the end, we have computed $g^{(k)}(0)$ for $k = 0, \dots, r$ and hence $f^{(k)}(0)$ for $k = 0, \dots, r$ and hence $T_m(1)$ approximating $f(1) = \ln h(1)$ within additive error ϵ . From (4.1.3), we compute

$$\text{den}_m(G; \gamma) = \exp \left\{ \frac{\gamma m}{2} \right\} h(1)$$

within relative error $\epsilon > 0$.

5. REMARKS ON THE PRACTICAL IMPLEMENTATION

We implemented a *much* simplified version of the algorithm. Given a graph $G = (V, E)$ with set $V = \{1, \dots, n\}$ of vertices and an integer $2 \leq m \leq n$, we define the $n \times n$ matrix w_{ij} by

$$w_{ij} = \begin{cases} \alpha & \text{if } \{i, j\} \in E \\ -\alpha & \text{if } \{i, j\} \notin E, \end{cases}$$

where $0 < \alpha < 1$ is a parameter.

We consider the polynomial $h(z)$ defined by (4.2.1) and let $f(z) = \ln h(z)$. Our goal is to approximate $f(1) = \ln h(1)$ and hence

$$\begin{aligned} h(1) &= \sum_{\substack{S \subset \{1, \dots, n\} \\ |S|=m}} (1 + \alpha)^{\binom{m}{2} \sigma(S)} (1 - \alpha)^{\binom{m}{2} (1 - \sigma(S))} \\ &= (1 - \alpha)^{\binom{m}{2}} \text{den}_m(G; \gamma), \quad \text{where} \quad \gamma = \frac{m-1}{2} \ln \frac{1+\alpha}{1-\alpha}. \end{aligned}$$

We approximate $f(1)$ by the degree r Taylor polynomial of $f(z)$ computed at $z = 0$. The results of [Ba15] suggest that for $\alpha = O(1/m)$, we should get a reasonable approximation if we use $r \sim \ln m$. The results of our numerical experiments suggest that we get reasonable approximations if we use $\alpha = \Omega(1)$ and $r = 2$ or $r = 3$. In short, on the examples we tested, the quality of approximation was more consistent with the quality of the Taylor polynomial approximation of $\ln(1 \pm \alpha)$.

More precisely, we ran the algorithm typically with parameters $n = 50, 100$ and $m = 10$, although occasionally we chose n as large as $n = 300$. For the parameters $n = 50$ and $m = 10$ we were able to compare our approximation with the exact value. Typically, choosing $\alpha = 0.5$ or lower produced an approximation of $f(1)$ within 1% accuracy. For $\alpha = 0.7$, the accuracy went down to 10% – 20% and for $\alpha > 0.7$ the approximation was not accurate. For higher values of n , where the exact value of $f(1)$ was unavailable, we compared the approximations obtained for $r = 2$ and $r = 3$. If the approximations were close to each other, we considered it as an indication that they are also close to the true value of $f(1)$. Again, we observed that up to $\alpha = 0.5$, the approximations agreed, but were beginning to essentially differ at $\alpha = 0.7$ and higher. For the graphs, we used the Erdős -Rényi models $G(n, 0.5)$, $G(n, 0.4)$, those graphs with planted cliques of size m , and occasionally manually constructed “random-looking” graphs.

We provide below the explicit formulas for the approximations up to degree 3, in case the reader will be interested to do some numerical experiments. We interpret w_{ij} as weights on the edges of a complete graph with n vertices. Borrowing an idea from [PR17], we express the derivatives $f^{(k)}(0)$ in terms of various sums associated with *connected* subgraphs, since it improves the computational complexity of the algorithm. We remark, however, that it looks unlikely that the methods of [PR17] can be pushed to improve the complexity of our algorithm in the general situation from quasi-polynomial to genuinely polynomial, since we work with graphs of unbounded degrees.

It is convenient to introduce the following sums:

$$A_1 = \sum_{\{i,j\}} w_{ij},$$

where the sum is taken over all unordered pairs $\{i, j\}$ of distinct indices;

$$B_1 = \sum_{\{i,j\}} w_{ij}^2, \quad B_2 = \sum_{j, \{i,k\}} w_{ij} w_{jk},$$

where in the formula for B_1 the sum is taken over all unordered pairs $\{i, j\}$ of distinct indices and in B_2 the sum is taken over all pairs consisting of an index j and an unordered pair $\{i, k\}$, so that all three indices are distinct; and

$$\begin{aligned} C_1 &= \sum_{\{i,j\}} w_{ij}^3, & C_2 &= \sum_{(i,j,k)} w_{ij}^2 w_{jk}, & C_3 &= \sum_{\{i,j,k\}} w_{ij} w_{jk} w_{ki}, \\ C_4 &= \sum_{(i,j,k,l)} w_{ij} w_{jk} w_{kl}, & C_5 &= \sum_{\{j,k,l\},i} w_{il} w_{ij} w_{ik}, \end{aligned}$$

where in C_1 the sum is taken over all unordered pairs $\{i, j\}$ of distinct indices, in C_2 the sum is taken over all ordered triples (i, j, k) of distinct indices, in C_3 the sum is taken over all unordered triples of distinct integers, in C_4 , the sum is taken over all ordered 4-tuples (i, j, k, l) of distinct indices, and in C_5 the sum is taken over all pairs consisting of an index i and an unordered triple $\{j, k, l\}$ so that all four indices $\{i, j, k, l\}$ are distinct.

(5.1) First-order approximation. Clearly, $h(0) = 1$. From (4.4.2), we have

$$h'(0) = \binom{n}{m}^{-1} \binom{n-2}{m-2} \sum_{\{i,j\} \subset \{1, \dots, n\}} w_{ij} = \frac{m(m-1)}{n(n-1)} A_1.$$

Since $f(0) = \ln h(0) = 0$ and $f'(0) = h'(0)/h(0) = h'(0)$, we obtain the first order approximation

$$f(1) \approx h'(0),$$

where $h'(0)$ is defined as above. The complexity of computing the first order approximation is $O(n^2)$.

(5.2) Second-order approximation. From (4.4.2), we have

$$h''(0) = \binom{n}{m}^{-1} \sum_{I=(\{i_1, j_1\}, \{i_2, j_2\})} \binom{n-\nu(I)}{m-\nu(I)} w_{i_1 j_1} w_{i_2 j_2}.$$

Here $\nu(I) = 4$ if the pairs $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are pairwise disjoint and $\nu(I) = 3$ if they share exactly one index. Hence we can write

$$\begin{aligned} h''(0) &= \binom{n}{m}^{-1} \left(2 \binom{n-3}{m-3} B_2 + \binom{n-4}{m-4} (A_1^2 - 2B_2 - B_1) \right) \\ &= 2 \frac{m(m-1)(m-2)}{n(n-1)(n-2)} B_2 + \frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)} (A_1^2 - 2B_2 - B_1). \end{aligned}$$

Since

$$f''(0) = h''(0) - (h'(0))^2,$$

we obtain the second order approximation:

$$f(1) \approx f'(0) + \frac{1}{2} f''(0) = h'(0) - \frac{1}{2} (h'(0))^2 + \frac{1}{2} h''(0),$$

where $h'(0)$ and $h''(0)$ are defined as above. The complexity of computing the second order approximation is $O(n^3)$.

(5.3) Third-order approximation. From (4.4.2), one can deduce that

$$\begin{aligned} h'''(0) = & 6 \frac{m(m-1)(m-2)}{n(n-1)(n-2)} C_3 + \frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)} (6C_5 + 3C_4) \\ & + 6 \frac{m(m-1)(m-2)(m-3)(m-4)}{n(n-1)(n-2)(n-3)(n-4)} (A_1 B_2 - 3C_5 - 3C_3 - C_4 - C_2) \\ & + \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \left(A_1^3 + 12C_3 - 6A_1 B_2 \right. \\ & \quad \left. + 12C_5 + 3C_4 + 6C_2 - 3A_1 B_1 + 2C_1 \right). \end{aligned}$$

Since we have

$$f'''(0) = h'''(0) - 2f''(0)h'(0) - f'(0)h''(0) = 2(h'(0))^3 - 3h'(0)h''(0) + h'''(0),$$

we obtain the third order approximation

$$\begin{aligned} f(1) & \approx f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(0) \\ & = h'(0) - \frac{1}{2}(h'(0))^2 + \frac{1}{2}h''(0) + \frac{1}{3}(h'(0))^3 - \frac{1}{2}h'(0)h''(0) + \frac{1}{6}h'''(0). \end{aligned}$$

The complexity of computing the third order approximation is $O(n^4)$.

6. PROOF OF THEOREM 1.5 AND CONCLUDING REMARKS

We got the idea of the proof from [EM18], where a similar question about complex zeros of the permanents of matrices with independent random entries was treated.

Applying Jensen's formula, see for example, Section 5.3 of [Ah78], we obtain

$$(6.1) \quad \ln |h_W(0)| = \sum_{s=1}^N \ln \frac{|a_{s,W}|}{r} + \frac{1}{2\pi} \int_0^{2\pi} \ln |h_W(re^{i\theta})| d\theta,$$

where $a_{s,W}$, $s = 1, \dots, N$ are the roots of the polynomial $h_W(z)$ in the disc $|z| < r$ and we assume that $h_W(z)$ has no zeros on the circle $|z| = r$ (since there are only finitely many values of r with roots on the circle $|z| = r$, this assumption is not restrictive). We have

$$\ln |h_W(0)| = 0$$

and furthermore, applying Jensen's inequality, we bound:

$$\begin{aligned} (6.2) \quad \frac{1}{2\pi} \int_0^{2\pi} \ln |h_W(re^{i\theta})| d\theta & = \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \ln |h_W(re^{i\theta})|^2 d\theta \\ & \leq \frac{1}{2} \ln \left(\frac{1}{2\pi} \int_0^{2\pi} |h_W(re^{i\theta})|^2 d\theta \right). \end{aligned}$$

For a fixed $\theta \in [0, 2\pi]$, we compute the expectation

$$\begin{aligned} \mathbf{E} |h_W(re^{i\theta})|^2 &= \binom{n}{m}^{-2} \sum_{\substack{S_1, S_2 \subset \{1, \dots, n\} \\ |S_1|=|S_2|=m}} \mathbf{E} \left(\prod_{\{j,k\} \subset S_1} (1 + re^{i\theta} w_{jk}) \right. \\ &\quad \left. \times \prod_{\{j,k\} \subset S_2} (1 + re^{-i\theta} w_{jk}) \right) \\ &= \binom{n}{m}^{-2} \sum_{\substack{S_1, S_2 \subset \{1, \dots, n\} \\ |S_1|=|S_2|=m}} (1 + r^2)^{\binom{|S_1 \cap S_2|}{2}}. \end{aligned}$$

A subset $S \subset \{1, \dots, n\}$ of cardinality $l = |S| \leq m$ can be represented as the intersection $S = S_1 \cap S_2$ of m -subsets S_1, S_2 in $\binom{n-l}{m-l} \binom{n-m}{m-l}$ ways. Hence

$$(6.3) \quad \mathbf{E} |h_W(re^{i\theta})|^2 = \binom{n}{m}^{-2} \sum_{l=0}^m \binom{n}{l} \binom{n-l}{m-l} \binom{n-m}{m-l} (1 + r^2)^{\binom{l}{2}}.$$

To bound (6.3), we consider the ratio of the $(l+1)$ -st term to the l -th term:

$$\begin{aligned} \frac{n-l}{l+1} \cdot \frac{m-l}{n-l} \cdot \frac{m-l}{n-2m+l+1} \cdot (1 + r^2)^l &= \frac{(m-l)^2 (1 + r^2)^l}{(l+1)(n-2m+l+1)} \\ &\leq \frac{m^2(1 + r^2)^m}{n-2m+1}. \end{aligned}$$

In particular, if

$$(6.4) \quad n \geq 2m^2(1 + r^2)^m + 2m,$$

the ratio does not exceed $1/2$ and hence we can bound the sum (6.3) by

$$\mathbf{E} |h_W(re^{i\theta})|^2 \leq 2 \binom{n}{m}^{-2} \binom{n}{m} \binom{n-m}{m} \leq 2.$$

Integrating over θ , we conclude that if (6.4) holds then

$$\mathbf{E} \left(\frac{1}{2\pi} \int_0^{2\pi} |h_W(re^{i\theta})| d\theta \right) \leq 2.$$

By the Markov inequality, for any $\tau \geq 1$, we get

$$\mathbf{P} \left(\frac{1}{2\pi} \int_0^{2\pi} |h_W(re^{i\theta})| d\theta \geq 2\tau \right) \leq \frac{1}{\tau}.$$

Consequently, from (6.1) and (6.2), we have

$$\mathbf{P} \left(\sum_{s=1}^N \ln \frac{|a_{s,W}|}{r} \leq -\frac{1}{2} \ln 2\tau \right) \leq \frac{1}{\tau}.$$

and the proof follows. $\square$

An anonymous referee asked what happens if G is a random graph $G(n, 0.5)$ with a planted m -clique. The most interesting asymptotic regime is when $m^2 \ll n \leq m^{O(1)}$ and m grows, see [A+98] for results and references. Here we are interested in a polynomial time algorithm which, with high probability, tells G from $G(n, 0.5)$. A quasi-polynomial time algorithm is readily available (by an exhaustive search for a clique of size at least $3 \log_2 n$, say). Our proof of Theorem 1.5 does not seem to extend to random graphs with a planted clique. We note, however, that if the radius of zero-free region is roughly the same $r = \Omega(1/\sqrt{m})$ as in Theorem 1.5 or even weaker, $r = \Omega(m^{-1+\epsilon})$ for some $\epsilon > 0$, we do obtain a desired polynomial time algorithm. Indeed, in the latter case, we can choose $\gamma = m^{\epsilon'}$ with some $0 < \epsilon' < \epsilon$. If G is a graph with a planted m -clique, we have

$$\text{den}_m(G; \gamma) \geq \exp \left\{ m^{1+\epsilon'} - O(m \ln m) \right\},$$

cf. (1.1.2). If G is a random graph $G(n, 0.5)$, our proof Theorem 1.5 implies that

$$\text{den}_m(G; \gamma) \leq \exp \left\{ \frac{m^{1+\epsilon'}}{2} + O(1) \right\}$$

with high probability, cf. (1.5.1). Note that by choosing $\epsilon' < \epsilon$, we choose γ sufficiently “deep” inside the purported zero-free region, and hence we can get a genuinely polynomial, as opposed to a quasi-polynomial, algorithm by computing a constant, as opposed to logarithmic, number of terms in the Taylor polynomial approximation, cf. Lemma 4.2.

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