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C^∞ Smoothing for Weak Solutions of the Inhomogeneous Landau Equation

Christopher Henderson & Stanley Snelson

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Abstract

We consider the spatially inhomogeneous Landau equation with initial data that is bounded by a Gaussian in the velocity variable. In the case of moderately soft potentials, we show that weak solutions immediately become smooth, and remain smooth as long as the mass, energy, and entropy densities remain under control. For very soft potentials, we obtain the same conclusion with the additional assumption that a sufficiently high moment of the solution in the velocity variable remains bounded. Our proof relies on the iteration of local Schauder-type estimates.

1. Introduction

The Landau equation from plasma physics models the evolution of a particle density $f(t, x, v) \geqq 0$ in phase space, see e.g. [4,16]. In spatial dimension d, the equation is given by

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= Q_L(f, f) \\ &:= \nabla_v \cdot \left(\int_{\mathbb{R}^d} a(v - w)[f(w)\nabla f(v) - f(v)\nabla f(w)] \, \mathrm{d}w \right), \end{aligned} \tag{1.1}$$

$$a(z) = a_{d,\gamma} |z|^{\gamma+2} \left(I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right). \tag{1.2}$$

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Here, $t \in [0, T_0]$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, $\gamma \geqq -d$, and $a_{d,\gamma} > 0$ is a physical constant. The Landau equation arises as the limit of the Boltzmann equation as grazing collisions predominate [1]. We are interested in both the case of *moderately soft potentials*, $\gamma \in (-2, 0)$ and *very soft potentials*, $\gamma \in [-d, -2]$. The case $d = 3, \gamma = -3$ corresponds to Coulomb interaction between particles at small scales.

As opposed to the Boltzmann collision operator, which is a purely integro-differential operator of fractional order, Q_L is an operator of diffusion type whose coefficients depend nonlocally on f. In particular, the Landau equation (1.1) can be written in divergence form

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot [\overline{a}(t, x, v) \nabla_v f] + \overline{b}(t, x, v) \cdot \nabla_v f + \overline{c}(t, x, v) f, \tag{1.3}$$

or in nondivergence form

$$\partial_t f + v \cdot \nabla_x f = \operatorname{tr}\left[\overline{a}(t, x, v) D_v^2 f\right] + \overline{c}(t, x, v) f, \tag{1.4}$$

with the coefficients $\overline{a}(t, x, v) \in \mathbb{R}^{d \times d}$, $\overline{b}(t, x, v) \in \mathbb{R}^d$, and $\overline{c}(t, x, v) \in \mathbb{R}$ defined by

$$\overline{a}(t, x, v) := a_{d,\gamma} \int_{\mathbb{R}^d} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(t, x, v - w) \, \mathrm{d}w, \tag{1.5}$$

$$\overline{b}(t, x, v) := b_{d,\gamma} \int_{\mathbb{R}^d} |w|^{\gamma} w f(t, x, v - w) \, \mathrm{d}w, \tag{1.6}$$

$$\overline{c}(t, x, v) := c_{d,\gamma} \int_{\mathbb{R}^d} |w|^{\gamma} f(t, x, v - w) \, \mathrm{d}w, \tag{1.7}$$

for some constants $a_{d,\gamma}$, $b_{d,\gamma}$, and $c_{d,\gamma}$. When $\gamma = -d$, the expression for $\overline{c}$ must be replaced by $c_{d,\gamma} f$. We use both formulations (1.3) and (1.4), which are equivalent as long as, say, $f \in C^2_{v,\mathrm{loc}}$ and f has enough decay so that $\overline{a}$, $\overline{b}$, and $\overline{c}$ are well-defined.

We make the following assumptions on the mass density, energy density, and entropy density:

$$0 < m_0 \leqq \int_{\mathbb{R}^d} f(t, x, v) \, \mathrm{d}v \leqq M_0, \tag{1.8}$$

$$\int_{\mathbb{R}^d} |v|^2 f(t, x, v) \, \mathrm{d}v \leqq E_0, \qquad \text{and} \tag{1.9}$$

$$\int_{\mathbb{R}^d} f(t, x, v) \log f(t, x, v) \, \mathrm{d}v \leqq H_0, \tag{1.10}$$

uniformly in $t \geqq 0$ and $x \in \mathbb{R}^d$. In the spatially homogeneous case, i.e. when f is assumed to be independent of x, the mass and energy are conserved, and the entropy is monotonically decreasing; hence, in this case, it would suffice to assume that the initial data have finite mass, energy, and entropy. It is not currently known whether these hydrodynamic quantities stay under control for $t > 0$ in the inhomogeneous case, so we include (1.8), (1.9), and (1.10) as *a priori* assumptions.

We are interested in the regularity of weak solutions to (1.1). We use the following notion of weak solution, which is implicitly used in [9] and [3]:

Definition 1.1. We say $f : [0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a weak solution of (1.3) if $f, \nabla_v f, \partial_t f + v \cdot \nabla_x f \in L^2_{loc}(\mathbb{R}^{2d+1})$, the coefficients $\overline{a}$, $\overline{b}$, and $\overline{c}$ are well-defined, and

$$\int_{\mathbb{R}^{2d+1}} (\partial_t f + v \cdot \nabla_x f)\phi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_{\mathbb{R}^{2d+1}} \left(-\langle \overline{a} \nabla_v f, \nabla_v \phi \rangle + (\overline{b} \cdot \nabla_v f + \overline{c} f)\phi \right) \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t$$

for all $\phi \in H^1_0(\mathbb{R}^{2d+1})$.

Our main result states that weak solutions immediately become smooth, for any initial data that is bounded by a Gaussian and regular enough for a weak solution to exist:

Theorem 1.2. *Let $\gamma \in (-2, 0)$, and let $f : [0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be a bounded weak solution of the Landau equation (1.1) satisfying the bounds (1.8), (1.9), and (1.10). There exists $\mu_0 > 0$ depending on d, γ, m_0, M_0, E_0, and H_0 such that if the initial data f_{in} satisfies*

$$f_{\text{in}}(x, v) \leqq C_0 e^{-\mu |v|^2}$$

for some $C_0 > 0$ and $\mu > 0$, then $f \in C^\infty((0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d)$, and for any $\mu' < \min\{\mu_0, \mu\}$, any integer $j \geqq 0$, and any multi-indices β and η with non-negative integer coordinates, the partial derivatives of f satisfy the pointwise estimates

$$|\partial_t^j \partial_x^\beta \partial_v^\eta f(t, x, v)| \leqq C \left(1 + t^{-q}\right) e^{-\mu' |v|^2}. \tag{1.11}$$

The constants $C, q \geqq 0$ depend on d, γ, m_0, M_0, E_0, H_0, μ', j, $|\beta|$, $|\eta|$, and C_0.

For $\gamma \in [-d, -2]$, if we make the additional assumption that for all $t \in [0, T_0]$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |v|^p f(t, x, v) \, \mathrm{d}v \leqq P_0, \tag{1.12}$$

where p is the smallest integer such that $p > \dfrac{d|\gamma|}{2 + \gamma + d}$, then the same conclusion holds, with all constants depending additionally on P_0 and $\|f\|_{L^\infty([0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d)}$. If $\gamma \leqq -d/2 - 1$, the constants also depend on T_0.

The question of global-in-time existence of smooth solutions to (1.1) for non-perturbative initial data remains a challenging open problem. In the case of moderately soft potentials, Theorem 1.2 implies a physically meaningful continuation criterion: any loss of smoothness of f can be detected at the macroscopic level by a breakdown of the bounds on the mass, energy, or entropy density.

Our proof of Theorem 1.2 relies on three elements:

1. The local Hölder continuity of solutions to (1.1), which was established in [23] and [9].

2. Decay of the solution f for large velocities, and corresponding decay in the local estimates, which is needed to pass regularity of f to regularity of the coefficients $\overline{a}$ and $\overline{c}$ in (1.4).
3. Local Schauder-type estimates for kinetic Fokker-Planck equations with Hölder continuous coefficients, which we prove in Section 2 and apply iteratively in Section 3.

The second point is where our assumption that f_{in} is bounded by a Gaussian comes in. In [3], it was shown that this upper bound is propagated for all $t \in (0, T_0]$ when $\gamma \in (-2, 0)$. We extend this to $\gamma \in [-d, -2]$ in Theorem 3.4, under more restrictive assumptions; however, if we could guarantee by any other method that sufficiently high moments of the solution are finite (as in the hypotheses of [5] and [17], see below), our proof would still go through. It was shown in [3] that solutions of (1.1) satisfying the hydrodynamic bounds (1.8), (1.9), and (1.10) satisfy *a priori* pointwise decay proportional to $(1 + |v|)^{-1}$ for arbitrary initial data, but this is not strong enough for our purposes because of the slowly decaying kernels in (1.5) and (1.7). It was also shown in [3] that *a priori* Gaussian decay cannot hold without any decay assumption on $f_{\text{in}}(x, v)$.

1.1. Related Work

In [5], the authors show that classical solutions of (1.1) defined on a three-dimensional torus are C^∞ in all three variables, provided that infinitely many moments of the solution and its first eight derivatives in x and v remain bounded uniformly in time and provided that the solution remains bounded away from vacuum. A corresponding result for solutions defined on $\mathbb{R}^3$ was shown in [17], in the case $\gamma \in [-3, -2)$. Our Theorem 1.2 extends these results in the case where f_{in} is bounded by a Gaussian. The assumptions (1.8), (1.9), and (1.10) are much weaker than the *a priori* regularity hypotheses of [5] and [17], and are defined in terms of physically relevant hydrodynamic quanitites. At least in the case $\gamma \in (-2, 0)$, our estimates do not depend quantitatively on the L^∞ norm of f.

Local Hölder estimates for kinetic equations with rough coefficients were proven by WANG–ZHANG [23] and GOLSE–IMBERT–MOUHOT–VASSEUR [9], and this is the starting point for the application of our Schauder estimates. The first global regularity estimates for (1.1) in this setting (weak solutions with bounded mass, energy, and entropy) were established in [3]. The ellipticity constants of the diffusion operator Q_L degenerate as $|v| \to \infty$ in a non-isotropic way (see "Appendix A"). To deal with this, we use a change of variables derived in [3] to obtain an equation with universal ellipticity constants in a small cylinder (see Lemma 3.1).

Regarding the existence theory for (1.1), global-in-time classical solutions have only been constructed in the close-to-equilibrium setting: see the work of GUO [10] in the x-periodic case, and MOUHOT–NEUMANN [19] in the whole space. For general initial data, VILLANI [21] constructed so-called renormalized solutions with defect measure for the Landau equation. More recently, HE–YANG [12] established the short-time existence of spatially periodic classical solutions to (1.1) in the Coulomb case ($\gamma = -d$) with initial data in a weighted $H^7_{x,v}$ space, by taking the

grazing collisions limit in their estimates on the Boltzmann collision operator. They assume that the mass density of the initial data is uniformly bounded away from zero. Since this lower bound along with the bounds (1.8), (1.9), (1.10), and (1.12) can be shown to propagate for a short time, our Theorem 1.2 combined with [12] provides a C^∞ solution to the Cauchy problem for suitable initial data. However, on physical grounds, the equation should be expected to be well-posed even with vacuum regions in the initial data. We explore this issue, as well as short-time existence for a broader range of γ, in a forthcoming paper.

For the spatially homogeneous Landau equation, C^∞ smoothing was established in [6] in the case $\gamma > 0$ and [22] in the $\gamma = 0$ case. For $\gamma \in (-2, 0)$, the upper bounds of [20] also imply smoothing via parabolic regularity theory. For $\gamma \in [-d, -2]$, the result of Theorem 1.2 is new even in the space homogeneous case, to the best of our knowledge.

1.2. Schauder Estimates

Our main technical tools are local Schauder-type estimates for linear kinetic Fokker-Planck equations of the form

$$\partial_t u + v \cdot \nabla_x u = \operatorname{tr}(A D_v^2 u) + g, \tag{1.13}$$

with A and g Hölder continuous (see Theorem 2.12 below). Schauder estimates have been established in the more general setting of ultraparabolic equations by Manfredini [18], DiFrancesco–Polidoro [7], and Bramanti–Brandolini [2], among others. However, there are two complications involved in bootstrapping regularity estimates in this context: based on the natural scaling of the equation, Schauder estimates should be expected to bound two derivatives in v, one derivative in t, and two-thirds of a derivative in x (i.e. the $\frac{2}{3}$-Hölder norm in x) of u, which is not enough to directly conclude u is a classical solution. Even worse, Schauder estimates do not provide C^α estimates on $\partial_t u$, but rather on $\partial_t u + v \cdot \nabla_x u$. This is related to the non-symmetric Lie group structure of the equation, which shows up in the representation formula (2.3) of the solution. To get around this, we prove a second estimate that bounds $\partial_t u$ and $\nabla_x u$ in terms of the $C^{1+\alpha}$-norm of g. We give elementary proofs of the estimates we need, using the explicit fundamental solution for constant-coefficient equations.

1.3. Organization of the Paper

In Section 2, we prove regularity estimates for kinetic equations with Hölder continuous coefficients. In Section 3, we apply these estimates iteratively to weak solutions of the Landau equation. In "Appendix A", we review the bounds on the coefficients $\overline{a}$, $\overline{b}$, and $\overline{c}$ in (1.4).

1.4. Notation

We let $z = (t, x, v)$ denote a point in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. For any $z_0 = (t_0, x_0, v_0)$, define the Galilean transformation

$$\mathcal{S}_{z_0}(t, x, v) := (t_0 + t, x_0 + x + t v_0, v_0 + v).$$

We also have

$$\mathcal{S}_{z_0}^{-1}(t,x,v) := (t-t_0, x-x_0-(t-t_0)v_0, v-v_0).$$

For $r>0$, define the scaling δ_r by

$$\delta_r(t,x,v) = (r^2 t, r^3 x, rv).$$

The class of equations of the form (1.13) is invariant under S_{z_0} and δ_r. We also define the quasimetric

$$\rho(z,z') := \|\mathcal{S}_z^{-1} z'\| = |t'-t|^{1/2} + |x'-x-(t'-t)v|^{1/3} + |v'-v|,$$

where

$$\|z-z'\| := |t-t'|^{1/2} + |x-x'|^{1/3} + |v-v'|.$$

For any $r>0$ and $z_0=(t_0,x_0,v_0)$, let

$$Q_r(z_0) := (t_0-r^2, t_0] \times \{x : |x-x_0-(t-t_0)v_0| < r^3\} \times B_r(v_0),$$

and $Q_r = Q_r(0,0,0)$.

We say a constant is universal if it depends only on γ, d, m_0, M_0, E_0, and H_0 when $\gamma \in (-2,0)$. When $\gamma \in [-d,-2]$, we also allow universal constants to depend on P_0 and $\|f\|_{L^\infty([0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d)}$. The notation $A \lesssim B$ means that $A \leqq CB$ for a constant C that depends on the quantities listed in the statement of the given lemma or theorem, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Schauder Estimates for Linear Kinetic Equations

In this section, we obtain regularity estimates for equations of the form (1.13). We begin by defining Hölder norms and semi-norms that correspond to ρ.

Definition 2.1. Let $Q \subseteq \mathbb{R}^{2d+1}$. For $u : Q \to \mathbb{R}$, define

$$[u]_{\alpha,Q} := \sup_{\substack{z,z'\in Q,\\ z\neq z'}} \frac{|u(z)-u(z')|}{\rho(z,z')^\alpha}$$

$$[u]_{\alpha,x,Q} := \sup_{\substack{(t,x,v),(t,x',v)\in Q,\\ x\neq x'}} \frac{|u(t,x,v)-u(t,x',v)|}{|x-x'|^\alpha},$$

$$[u]_{\alpha,t,Q} := \sup_{\substack{(t,x,v),(t',x,v)\in Q,\\ t\neq t'}} \frac{|u(t,x,v)-u(t',x,v)|}{|t-t'|^\alpha + |(t'-t)v|^{2\alpha/3}}$$

$$|u|_{0,Q} := \sup_{z\in Q} |u(z)|$$

$$|u|_{\alpha,Q} := |u|_{0,Q} + [u]_{\alpha,Q}$$

$$[u]_{1+\alpha,Q} := [\nabla_v u]_{\alpha,Q_1} + [u]_{(1+\alpha)/2,t,Q} + [u]_{(1+\alpha)/3,x,Q}$$

$$|u|_{1+\alpha,Q} := |u|_{0,Q} + |\nabla_v u|_{0,Q} + [u]_{1+\alpha,Q}$$

$$[u]_{2+\alpha,Q} := [D_v^2 u]_{\alpha,Q} + [\partial_t u]_{\alpha,Q} + [u]_{(2+\alpha)/3,x,Q}$$

$$|u|_{2+\alpha,Q} := |u|_{0,Q} + |\partial_t u|_{0,Q} + |\nabla_v u|_{0,Q} + |D_v^2 u|_{0,Q} + [u]_{2+\alpha,Q}.$$

For $\beta \in (0, 3)$, if $|u|_{\beta,Q} < \infty$, we say $u \in C^\beta(Q)$.

If u is in $C^\alpha(Q)$ by this definition, then in particular, u is $\frac{\alpha}{3}$-Hölder continuous in the Euclidean metric on $\mathbb{R}^{2d+1}$. We use the following lemma repeatedly:

Lemma 2.2. (Interpolation Inequalities) *Let $Q = Q_r(z_0)$ for some $z_0 \in \mathbb{R}^{2d+1}$ and $r > 0$, and let $u \in C^{2+\alpha}(Q)$. There exists a constant C, depending only on the dimension, such that for any $\varepsilon > 0$,*

$$\begin{aligned}
[u]_{\alpha,Q} &\leqq \varepsilon^2 [u]_{2+\alpha,Q} + C\varepsilon^{-\alpha}|u|_{0,Q},\\
|\partial_t u|_{0,Q} &\leqq \varepsilon^\alpha [\partial_t u]_{\alpha,Q} + C\varepsilon^{-2}|u|_{0,Q},\\
|\nabla_v u|_{0,Q} &\leqq \varepsilon^{1+\alpha}[u]_{2+\alpha,Q} + C\varepsilon^{-1}|u|_{0,Q},\\
[\nabla_v u]_{\alpha,Q} &\leqq \varepsilon [u]_{2+\alpha,Q} + C\varepsilon^{-(1+\alpha)}|u|_{0,Q},\\
|D_v^2 u|_{0,Q} &\leqq \varepsilon^\alpha [u]_{2+\alpha,Q} + C\varepsilon^{-2}|u|_{0,Q}.
\end{aligned}$$

If $D_v^3 u, \nabla_x u \in C^\alpha(Q)$, we also have

$$\begin{aligned}
|D_v^3 u|_{0,Q} &\leqq \varepsilon^\alpha [D_v^3 u]_{\alpha,Q} + C\varepsilon^{-3}|u|_{0,Q}\\
|\nabla_x u|_{0,Q} &\leqq \varepsilon^\alpha [\nabla_x u]_{\alpha,Q} + C\varepsilon^{-3}|u|_{0,Q}.
\end{aligned}$$

The method of proving inequalities of this type is standard. (See, for example, [18] or [14, Theorem 8.8.1]). Briefly, it suffices to prove the case $\varepsilon = 1$ by scaling. To prove the first inequality, one estimates $|u(z) - u(z')|$ by writing $z - z'$ as a sum of segments parallel to the coordinate axes, and applying the mean value inequality. The details are omitted.

Finally, we define the non-scale-invariant Hölder seminorms that correspond to our regularity estimates as follows:

Definition 2.3. For $Q \subseteq \mathbb{R}^{2d+1}$, $u : Q \to \mathbb{R}$, and $\alpha, \beta \in (0, 1)$, define

$$\begin{aligned}
[u]'_{2+\alpha,\beta,Q} &:= [D_v^2 u]_{\alpha,Q} + [u]_{(2+\alpha)/3,x,Q} + [u]_{\beta,t,Q},\\
[u]''_{3+\alpha,Q} &:= [\partial_t u]_{\alpha,Q} + [\nabla_x u]_{\alpha,Q} + [D_v^3 u]_{\alpha,Q}.
\end{aligned}$$

2.1. Constant Coefficients

Consider the equation

$$u_t + v \cdot \nabla_x u - \Delta_v u = g \tag{2.1}$$

in $(-1, 0] \times \mathbb{R}^d \times \mathbb{R}^d$ with zero initial data at $t = -1$. The explicit fundamental solution for this equation is given by

$$\Gamma(z) := \begin{cases} \dfrac{C_d}{t^{2d}} \exp\left(-\dfrac{|v|^2}{t} - \dfrac{3v \cdot x}{t^2} - \dfrac{3|x|^2}{t^3}\right), & t > 0,\\ 0, & t \leqq 0, \end{cases} \tag{2.2}$$

where $C_d = (\sqrt{3}/(2\pi))^d$. More precisely, if g is, say, continuous, bounded, and has support contained in $\{t > -1\}$ then (2.1) is uniquely solved by

$$u(z) = \int_{\mathbb{R}^{2d+1}} \Gamma\left(\mathcal{S}_\zeta^{-1} z\right) g(\zeta)\, \mathrm{d}\zeta, \tag{2.3}$$

where $\zeta = (s, y, w)$ and $\mathcal{S}_\zeta^{-1} z = (t-s, x-y-(t-s)w, v-w)$. The fundamental solution Γ is a special case of the solution constructed by HÖRMANDER [13] for more general hypoelliptic equations. (See also [15,18].) The following lemma provides a useful characterization of the homogeneity of the fundamental solution:

Lemma 2.4. *For any partial derivative $\partial_t^j \partial_x^\beta \partial_v^\eta \Gamma$ of Γ, with β a multi-index of order $k \geqq 0$ and η a multi-index of order $\ell \geqq 0$, there exists a constant $C = C(d, j, k, \ell)$ such that for all $t > 0$ and $p, q \geqq 0$,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_t^j \partial_x^\beta \partial_v^\eta \Gamma(t+\xi_1, y+\xi_2, w+\xi_3)||y|^p |w|^q \,\mathrm{d}w\,\mathrm{d}y \leqq C t^{-(\ell/2+j+3k/2)+3p/2+q/2}.$$

Further, if $\xi \in [0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ and $\|\xi\| \leqq t^{1/2}/2$, then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_t^j \partial_x^\beta \partial_v^\eta \Gamma(z+\xi)||y|^p |w|^q \,\mathrm{d}w\,\mathrm{d}y \leqq C t^{-(\ell/2+j+3k/2)+3p/2+q/2},$$

where $z = (t, x, v)$.

Proof. It is straightforward to show by induction that every partial derivative of Γ can be written

$$\partial_t^j \partial_x^\beta \partial_v^\eta \Gamma(t, y, w) = P_{j,\beta,\eta}\left(\frac{1}{t^{1/2}}, \frac{y_1}{t^2}, \ldots, \frac{y_d}{t^2}, \frac{w_1}{t}, \ldots, \frac{w_d}{t}\right) \Gamma(t, y, w),$$

with $P_{j,\beta,\eta}$ a homogeneous polynomial where each term is of degree exactly $\ell + 2j + 3k$. Since $\exp(-|w|^2/t - 3w\cdot y/t^2 - 3|y|^2/t^3) \leqq \exp(-|w|^2/(16t) - 3|y|^2/(5t^3))$, formula (2.2) for Γ implies

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_t^j \partial_x^\beta \partial_v^\eta \Gamma(t, y, w)||y|^p |w|^q \,\mathrm{d}w\,\mathrm{d}y \\
&= \frac{C_d}{t^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| P_{j,\beta,\eta}\left(\frac{1}{t^{1/2}}, \frac{y_1}{t^2}, \ldots, \frac{y_d}{t^2}, \frac{w_1}{t}, \ldots, \frac{w_d}{t}\right) \right. \\
&\quad \left. \Gamma(t, y, w)\right| |y|^p |w|^q \,\mathrm{d}w\,\mathrm{d}y \\
&\leqq C_d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| P_{j,\beta,\eta}\left(\frac{1}{t^{1/2}}, \frac{y_1}{t^{1/2}}, \ldots, \frac{y_d}{t^{1/2}}, \frac{w_1}{t^{1/2}}, \ldots, \frac{w_d}{t^{1/2}}\right) \right. \\
&\quad \left. \times \exp\left(-\frac{|\overline{w}|^2}{16} - \frac{3|\overline{y}|^2}{5}\right)\right| t^{3p/2+q/2} |\overline{y}|^p |\overline{w}|^q \,\mathrm{d}\overline{w}\,\mathrm{d}\overline{y} \\
&\lesssim \left(\frac{1}{t^{1/2}}\right)^{\ell+2j+3k} t^{3p/2+q/2},
\end{aligned}$$

where $\overline{w} = w/t^{1/2}$ and $\overline{y} = y/t^{3/2}$. The proof of the second claim is almost identical, using the fact that $t \lesssim t + \xi_t \lesssim t$, where $\xi := (\xi_t, \xi_x, \xi_v)$. □

We now prove our main regularity estimates in the constant-coefficient case.

Lemma 2.5. *Suppose that $g \in C^\alpha(Q_1)$ has compact support in Q_1, for some $\alpha \in (0,1)$. Then the solution u of (2.1) in Q_1 satisfies*

$$[D_v^2 u]_{\alpha,Q_1} + [u]_{(2+\alpha)/3,x,Q_1} \lesssim [g]_{\alpha,Q_1},$$

where the implied constant depends only on α and the dimension d. We also have $[u]_{\beta,t,Q_1} \lesssim [g]_{\alpha,Q_1}$ for any $\beta \in (0,1)$, so that

$$[u]'_{2+\alpha,\beta,Q_1} \lesssim [g]_{\alpha,Q_1},$$

with $[\cdot]_{2+\alpha,\beta,Q_1}$ as in Definition 2.3. In particular, $[u]_{1+\beta,Q_1} \lesssim [g]_{\alpha,Q_1}$ for any $\beta \in (0,1)$.

Proof. First, we estimate $[D_v^2 u]_{\alpha,Q_1}$. Since g has compact support in Q_1, (2.3) implies that, for any $(t,x,v) \in Q_1$,

$$\begin{aligned}
&\partial_{v_i v_j} u(z) \\
&\quad = \int_{-1}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(t-s, x-y-(t-s)w, v-w) g(s,y,w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \int_{0}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(s,y,w) g(t-s, x-y-s(v-w), v-w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned}$$

for $1 \leqq i, j \leqq d$. Let $z = (t,x,v)$ and $z' = (t',x',v')$ be fixed points in Q_1 with $t \leqq t'$. Further, let $h = \rho(z,z')$ and fix any $i, j \in \{1,\dots,d\}$. We write

$$\begin{aligned}
&\partial_{v_i v_j} u(z) - \partial_{v_i v_j} u(z') \\
&\quad = \left(\int_0^{2h^2} + \int_{2h^2}^{1+t} \right) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(s,y,w) \delta g(s,y,w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad - \int_{1+t}^{1+t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(s,y,w) g(t'-s, x'-y-s(v'-w), v'-w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \\
&\quad =: I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
\delta g(s,y,w) := {} & g(t-s, x-y-s(v-w), v-w) \\
& - g(t'-s, x'-y-s(v'-w), v'-w).
\end{aligned}$$

We make the convention that if $2h^2 \geqq 1+t$, then $I_2 = 0$.

Since $\mathrm{spt}(g) \subset Q_1$, we have $|\delta g(s,y,w) - \delta g(s,y,0)| \leqq 2[g]_{\alpha,Q_1}((s|w|)^{\alpha/3} + |w|^\alpha)$. Observe that for any $s > 0$, $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(s,y,w)\,\mathrm{d}w = 0.$$

This allows us to estimate I_1 as follows:

$$\begin{aligned}
|I_1| &= \left| \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(s, y, w)[\delta g(s, y, w) - \delta g(s, y, 0)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \right| \\
&\leqq 2[g]_{\alpha, Q_1} \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{v_i v_j} \Gamma(s, y, w)|((s|w|)^{\alpha/3} + |w|^\alpha) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim [g]_{\alpha, Q_1} \int_0^{2h^2} s^{\alpha/2 - 1} \, \mathrm{d}s \lesssim [g]_{\alpha, Q_1} h^\alpha,
\end{aligned}$$

where the second-to-last inequality follows from Lemma 2.4.

Changing variables in I_2 and adding and subtracting a term, we have

$$\begin{aligned}
I_2 &= \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{v_i v_j} \Gamma(t - s, x - y, v - w) g(s, y - (t - s)w, w) \\
&\quad - \partial_{v_i v_j} \Gamma(t' - s, x' - y, v' - w) g(s, y - (t' - s)w, w)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&= \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(t - s, x - y, v - w) \\
&\quad \times [g(s, y - (t - s)w, w) - g(s, y - (t' - s)w, w)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&\quad + \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{v_i v_j} \Gamma(t - s, x - y, v - w) \\
&\quad - \partial_{v_i v_j} \Gamma(t' - s, x' - y, v' - w)] \\
&\quad \times g(s, y - (t' - s)w, w) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&=: I_2' + I_2''.
\end{aligned}$$

Re-defining $\delta g(s, y, w) := g(s, y - (t - s)w, w) - g(s, y - (t' - s)w, w)$, we have

$$\begin{aligned}
&|\delta g(s, y, w) - \delta g(s, y, v)| \\
&\quad \leqq [g]_{\alpha, Q_1} \left((|t - s|^{1/3} + |t' - s|^{1/3}) |v - w|^{1/3} + 2|v - w| \right),
\end{aligned}$$

which implies

$$\begin{aligned}
|I_2'| &= \left| \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma(t - s, x - y, v - w)[\delta g(s, y, w) \right. \\
&\quad \left. - \delta g(s, y, v)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \right| \\
&\lesssim [g]_{\alpha, Q_1} \int_{2h^2}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{v_i v_j} \Gamma(s, y, w)| \\
&\quad \times \left(\left(s^{\alpha/3} + |t' - t + s|^{\alpha/3} \right) |w|^{\alpha/3} + |w|^\alpha \right) \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim [g]_{\alpha, Q_1} \int_{2h^2}^{1+t} s^{-1} \left(s^{\alpha/2} + h^{2\alpha/3} s^{\alpha/6} \right) \mathrm{d}s \lesssim [g]_{\alpha, Q_1} h^\alpha,
\end{aligned}$$

by Lemma 2.4. For I_2'', first note that

$$I_2'' = \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{v_i v_j} \Gamma(t-s, x-y, v-w) - \Gamma(t'-s, x'-y, v'-w)] \times [g(s, y-(t'-s)w, w) - g(s, y-(t'-s)v, v)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s.$$

We next note that, with $\zeta = (s, y, w)$,

$$\begin{aligned} &|\partial_{v_i v_j} \Gamma(t-s, x-y, v-w) - \partial_{v_i v_j} \Gamma(t'-s, x'-y, v'-w)| \\ &\leqq \max_{\|\xi\| \leqq h, \xi_1 \geqq 0} \Big(h^2 |\partial_t \partial_{v_i v_j} \Gamma(z-\zeta+\xi)| \\ &\quad + h^3 |\nabla_x \partial_{v_i v_j} \Gamma(z-\zeta+\xi)| + h |\nabla_v \partial_{v_i v_j} \Gamma(z-\zeta+\xi)| \Big), \end{aligned}$$

where we denote $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

Using these two facts along with the second half of Lemma 2.4, we have

$$\begin{aligned} |I_2''| &\lesssim [g]_{\alpha, Q_1} \int_{2h^2}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{\|\xi\| \leqq h, \xi_1 \geqq 0} \big[h^2 |\partial_t \partial_{v_i v_j} \Gamma(\zeta+\xi)| \\ &\quad + h^3 |\nabla_x \partial_{v_i v_j} \Gamma(\zeta+\xi)| + h |\nabla_v \partial_{v_i v_j} \Gamma(\zeta+\xi)| \big] \\ &\quad \Big(|t'-t+s+\xi_1|^{\alpha/3} |w-\xi_3|^{\alpha/3} + |w-\xi_3|^\alpha \Big) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\ &\lesssim [g]_{\alpha, Q_1} \int_{2h^2}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{\|\xi\| \leqq h, \xi_1 \geqq 0} \big[h^2 |\partial_t \partial_{v_i v_j} \Gamma(\zeta+\xi)| \\ &\quad + h^3 |\nabla_x \partial_{v_i v_j} \Gamma(\zeta+\xi)| \\ &\quad + h |\nabla_v \partial_{v_i v_j} \Gamma(\zeta+\xi)| \big] \Big((h^2+s)^{\alpha/3} (|w|+h)^{\alpha/3} + |w|^\alpha + h^\alpha \Big) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\ &\lesssim [g]_{\alpha, Q_1} h^\alpha. \end{aligned}$$

Proceeding as in our estimate of I_1, with $g(t'-s, x'-y-s(v'-w), v'-w)$ playing the role of $\delta g(s, y, w)$, we obtain

$$|I_3| \lesssim [g]_{\alpha, Q_1} \Big((1+t')^{\alpha/2} - (1+t)^{\alpha/2} \Big) \lesssim [g]_{\alpha, Q_1} |t'-t|^{\alpha/2} \lesssim [g]_{\alpha, Q_1} h^\alpha,$$

completing the estimate of $[D_v^2 u]_{\alpha, Q_1}$.

To estimate the $C^{(2+\alpha)/3}$ norm of u in the x variable, we define $h = |x'-x|$ and write

$$\begin{aligned} &u(t, x', v) - u(t, x, v) \\ &= \left(\int_0^{h^{2/3}} + \int_{h^{2/3}}^{1+t} \right) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma(s, y, w) \\ &\quad \times [g(t-s, x'-y-s(v-w), v-w) \\ &\quad - g(t-s, x-y-s(v-w), v-w)] \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\ &=: J_1 + J_2. \end{aligned}$$

Since $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(s,y,w)\,dw\,dy = 1$ for any $s>0$, we have

$$\begin{aligned}|J_1| &\leqq [g]_{\alpha,Q_1}h^{\alpha/3}\int_0^{h^{2/3}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(s,y,w)\,dw\,dy\,ds\\ &\leqq [g]_{\alpha,Q_1}h^{(2+\alpha)/3}.\end{aligned}$$

For J_2, we use a change of variables and then the fact that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(s,x'-y,w)\,dy\,dw = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(s,x-y,w)\,dy\,dw$$

to rewrite the convolution as follows:

$$\begin{aligned}|J_2| &= \left|\int_{h^{2/3}}^{1+t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\Gamma(s,x'-y,w)-\Gamma(s,x-y,w)]\right.\\ &\quad\left.\times g(t-s,y-s(v-w),v-w)\,dy\,dw\,ds\right|\\ &= \left|\int_{h^{2/3}}^{1+t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\Gamma(s,x'-y,w)-\Gamma(s,x-y,w)]\right.\\ &\quad\times[g(t-s,y-s(v-w),v-w)\\ &\quad\left.-g(t-s,x-s(v-w),v-w)]\,dy\,dw\,ds\right|\\ &\leqq [g]_{\alpha,Q_1}h\int_{h^{2/3}}^{1+t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\max_{|\xi|\leqq h}|\nabla_x\Gamma(s,x-y+\xi,w)|\right)\\ &\quad\times|x-y|^{\alpha/3}\,dy\,dw\,ds\\ &\lesssim [g]_{\alpha,Q_1}h\max_{|\xi|\leqq h}\int_{h^{2/3}}^{1+t}\left(s^{-3/2+\alpha/2}+s^{-3/2}|\xi|^{\alpha/3}\right)ds \lesssim [g]_{\alpha,Q_1}h^{(2+\alpha)/3},\end{aligned}$$

using Lemma 2.4, that $|\xi|\leqq h$, and that $h\leqq s^{3/2}$ on the domain of integration.

The proof that $[u]_{\beta,t,Q_1}\leqq[g]_{\alpha,Q_1}$ follows a similar outline, and is omitted. □

Lemma 2.6. *With g and u as in Lemma 2.5, assume in addition that $g\in C^{1+\alpha}(Q_1)$ for some $\alpha\in(0,1)$. Then u satisfies*

$$[u]''_{3+\alpha,Q_1} = [\partial_t u]_{\alpha,Q_1}+[\nabla_x u]_{\alpha,Q_1}+[D_v^3 u]_{\alpha,Q_1}\leqq C[g]_{1+\alpha,Q_1},$$

where the constant depends on α and d.

Proof. First, we show the estimate $[\nabla_x u]_{\alpha,Q_1}\leqq C[g]_{1+\alpha,Q_1}$. We proceed as in the previous lemma, taking advantage of the regularity of g in x. We have

$$\partial_{x_i}u(z) = \int_0^{1+t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{x_i}\Gamma(s,y,w)g(t-s,x-y-s(v-w),v-w)\,dw\,dy\,ds$$

for $1 \leqq i \leqq d$. Let $z, z' \in Q_1$ with $t \leqq t'$, and let $h = \rho(z, z')$. We write

$$\begin{aligned}
&\partial_{x_i} u(z) - \partial_{x_i} u(z') \\
&= \left(\int_0^{2h^2} + \int_{2h^2}^{1+t} \right) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{x_i} \Gamma(s, y, w) \delta g(s, y, w) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&\quad - \int_{1+t}^{1+t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{x_i} \Gamma(s, y, w) g(t' - s, x' - y - s(v' - w), v' - w) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
\delta g(s, y, w) &:= g(t - s, x - y - s(v - w), v - w) \\
&\quad - g(t' - s, x' - y - s(v' - w), v' - w).
\end{aligned}$$

We make the convention that if $2h^2 \geqq 1 + t$, then $I_2 = 0$.

Since $\mathrm{spt}(g) \subset Q_1$, we have $|\delta g(s, y, w) - \delta g(s, 0, w)| \leqq 2[g]_{\alpha, Q_1} |y|^{(1+\alpha)/3}$. Observe that for any $s > 0$, $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \partial_{x_i} \Gamma(s, y, w) \, \mathrm{d}y = 0.$$

This allows us to estimate I_1 as follows:

$$\begin{aligned}
|I_1| &= \left| \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{x_i} \Gamma(s, y, w) [\delta g(s, y, w) - \delta g(s, 0, w)] \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s \right| \\
&\leqq 2[g]_{\alpha, Q_1} \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{x_i} \Gamma(s, y, w)| |y|^{(1+\alpha)/3} \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s \\
&\lesssim [g]_{\alpha, Q_1} \int_0^{2h^2} s^{-3/2 + (1+\alpha)/2} \, \mathrm{d}s \lesssim [g]_{\alpha, Q_1} h^\alpha,
\end{aligned}$$

by Lemma 2.4.

Changing variables in I_2, we have

$$\begin{aligned}
I_2 &= \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{x_i} \Gamma(t - s, x - y, v - w) g(s, y - (t - s)w, w) \\
&\quad - \partial_{x_i} \Gamma(t' - s, x' - y, v' - w) g(s, y - (t' - s)w, w)] \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s \\
&= \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{x_i} \Gamma(t - s, x - y, v - w) \\
&\quad \times [g(s, y - (t - s)w, w) - g(s, y - (t' - s)w, w)] \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s \\
&\quad + \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{x_i} \Gamma(t - s, x - y, v - w) \\
&\quad - \partial_{x_i} \Gamma(t' - s, x' - y, v' - w)] \\
&\quad \times g(s, y - (t' - s)w, w) \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s \\
&=: I_2' + I_2''.
\end{aligned}$$

Re-defining $\delta g(s,y,w) := g(s,y-(t-s)w,w) - g(s,y-(t'-s)w,w)$, we have

$$|\delta g(s,y,w) - \delta g(s,x,w)| \leqq 2[g]_{1+\alpha,Q_1}|x-y|^{(1+\alpha)/3},$$

which implies

$$\begin{aligned}
|I_2'| &= \left| \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{x_i}\Gamma(t-s,x-y,v-w)[\delta g(s,y,w) \right. \\
&\quad \left. - \delta g(s,x,w)]\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \right| \\
&\lesssim [g]_{\alpha,Q_1} \int_{2h^2}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{x_i}\Gamma(s,y,w)||y|^{(1+\alpha)/3}\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\lesssim [g]_{\alpha,Q_1} \int_{2h^2}^{1+t} s^{-3/2+(1+\alpha)/2}\,\mathrm{d}s \lesssim [g]_{\alpha,Q_1} h^\alpha,
\end{aligned}$$

by Lemma 2.4. For I_2'', first note that with $\zeta = (s,y,w)$,

$$\begin{aligned}
&|\partial_{x_i}\Gamma(t-s,x-y,v-w) - \partial_{x_i}\Gamma(t'-s,x'-y,v'-w)| \\
&\quad \leqq \max_{\|\xi\|\leqq h} \Big(h^2|\partial_t\partial_{x_i}\Gamma(z-\zeta+\xi)| \\
&\qquad + h^3|\nabla_x\partial_{x_i}\Gamma(z-\zeta+\xi)| + h|\nabla_v\partial_{x_i}\Gamma(z-\zeta+\xi)| \Big).
\end{aligned}$$

By applying Lemma 2.4 again and arguing as in the proof of Lemma 2.5, we have

$$\begin{aligned}
|I_2''| &= \left| \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\partial_{x_i}\Gamma(t-s,x-y,v-w) - \partial_{x_i}\Gamma(t'-s,x'-y,v'-w)] \right. \\
&\quad \left. \times\, [g(s,y-(t'-s)w,w) - g(s,x-(t'-s)w,w)]\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \right| \\
&\lesssim [g]_{1+\alpha,Q_1} \int_{2h^2}^{1+t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{\|\xi\|\leqq h} [h^2|\partial_t\partial_{x_i}\Gamma(s,y,w)| + h^3|\nabla_x\partial_{x_i}\Gamma(s,y,w)| \\
&\quad + h|\nabla_v\partial_{x_i}\Gamma(s,y,w)|]|y-\xi_2|^{(1+\alpha)/3}\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\lesssim [g]_{1+\alpha,Q_1} h^\alpha.
\end{aligned}$$

Proceeding as in our estimate of I_1, with $g(t'-s,x'-y-s(v'-w),v'-w)$ playing the role of $\delta g(s,y,w)$, we obtain

$$\begin{aligned}
|I_3| &\leqq C[g]_{1+\alpha,Q_1} \left((1+t')^{\alpha/2} - (1+t)^{\alpha/2} \right) \\
&\leqq C[g]_{1+\alpha,Q_1}|t'-t|^{\alpha/2} \leqq C[g]_{1+\alpha,Q_1} h^\alpha,
\end{aligned}$$

and the proof of the estimate on $[\nabla_x u]_{\alpha,Q_1}$ is complete.

Equation (2.1) and Lemma 2.5 imply the estimate on $[\partial_t u]_{\alpha,Q_1}$. We complete the proof by differentiating (2.1) in v and applying Lemma 2.5 to estimate $[D_v^3 u]_{\alpha,Q_1}$, using our already-established estimate on $\nabla_x u$. □

Next, let A_0 be a (constant) symmetric, strictly positive definite, $d \times d$ matrix. Assume that $\sigma(A_0) \subset [\lambda, \Lambda]$ where $0 < \lambda < \Lambda$.

Lemma 2.7. *If $g \in C^\alpha(Q_1)$ for some $\alpha \in (0, 1)$, and g has compact support in Q_1, then the solution u of*

$$\partial_t u + v \cdot \nabla_x u - tr(A_0 D_v^2 u) = g$$

satisfies

$$[u]'_{2+\alpha,\beta,Q_1} \leqq C[g]_{\alpha,Q_1},$$

for any $\beta \in (0, 1)$. If, in addition, $g \in C^{1+\alpha}(Q_1)$, then

$$[u]''_{3+\alpha,Q_1} \leqq C[g]_{1+\alpha,Q_1}.$$

The constants C depend on d, α, β, λ, and Λ.

Proof. Let P be such that $P^2 = A_0$, and define $u_P(t, x, v) := u(t, Px, Pv)$. Notice that $\sigma(P) \subset [\sqrt{\lambda}, \sqrt{\Lambda}]$. Then

$$\begin{aligned}\partial_t u_P + v \cdot \nabla_x u_P - \Delta_v u_P &= (\partial_t u + v \cdot \nabla_x u - \Delta_v u)(t, Px, Pv) \\ &= g(t, Px, Pv) =: g_P(t, x, v),\end{aligned}$$

and we can apply Lemma 2.5 to $u_P = \int \Gamma(\mathcal{S}_\zeta^{-1} z) g_P(\zeta)\, d\zeta$ to obtain

$$[u]_{2+\alpha,P(Q_1)} \leqq C(P)[g]_{\alpha,P(Q_1)},$$

where $P(Q_1) := (-1, 0] \times P(B_1) \times P(B_1)$. To get an estimate on Q_1, we replace u with $u(R^2 t, R^3 x, Rv)$, where $R > 0$ depends only on λ and Λ. Similarly, if $D_v^3 u, \nabla_x u, \partial_t u \in C^\alpha(Q_1)$, we apply Lemma 2.6 to u_P. □

2.2. Variable Coefficients

Let L be an operator of the form

$$Lu = \operatorname{tr}(A(z) D_v^2 u),$$

where $A \in C^\alpha(Q_1)$, and $0 < \lambda I \leqq A(z) \leqq \Lambda I$ for all $z \in Q_1$. We now study equations of the form

$$\partial_t u + v \cdot \nabla_x u - Lu = g. \tag{2.4}$$

As is standard, we extend Lemma 2.7 to solutions of (2.4) by freezing the coefficients at a point z and taking advantage of the closeness of L to $L(z)$ in a small cylinder around z, where $L(z)$ refers to the operator $\operatorname{tr}(A(z) D_v^2 u)$ with z "frozen". We also remove the assumption that u has compact support, which requires tracking how interior estimates on Q_r scale for $r \in (0, 1]$. For this, we need the following technical lemma:

Lemma 2.8. *Let $\omega(r) \geqq 0$ be bounded in $[r_0, r_1]$ with $r_0 \geqq 0$. Suppose for $r_0 \leqq r < R \leqq r_1$, we have*

$$\omega(r) \leqq \mu\omega(R) + \frac{A}{(R-r)^p} + B$$

for some $\mu \in [0, 1)$ and $A, B, p \geqq 0$. Then for any $r_0 \leqq r < R \leqq r_1$, there holds

$$\omega(r) \lesssim \left(\frac{A}{(R-r)^p} + B \right),$$

where the implied constant depends only on μ and p.

Proof. See [11, Lemma 4.3]. □

Theorem 2.9. *Fix $\alpha \in (0, 1)$. Suppose that, $[u]'_{2+\alpha,\beta,Q_1} < \infty$ for all $\beta \in (0, 1)$, and $A \in C^\alpha(Q_1)$. Then*

$$[u]'_{2+\alpha,\beta,Q_{1/2}} \lesssim \left([g]_{\alpha,Q_1} + |A|^{3+\alpha+2/\alpha}_{\alpha,Q_1} |u|_{0,Q_1} \right),$$

where $g := \partial_t u + v \cdot \nabla_x u - Lu$. The implied constant depends only on $d, \alpha, \beta, \lambda$, and Λ.

Proof. For $r \in (0, 1]$, recall that

$$[u]'_{2+\alpha,\beta,Q_r} = [D_v^2 u]_{\alpha,Q_r} + [u]_{(2+\alpha)/3,x,Q_r} + [u]_{\beta,t,Q_r}.$$

Let $r \in [\frac{1}{4}, \frac{3}{4}]$ be arbitrary. For $1 \leqq i, j \leqq d$, pick $z, z' \in Q_r$ such that

$$\frac{|\partial_{v_i v_j} u(z) - \partial_{v_i v_j} u(z')|}{\rho(z, z')^\alpha} \geqq \frac{1}{2} [\partial_{v_i v_j} u]_{\alpha,Q_r}.$$

Let $\theta \in (0, 1/8)$ be a constant, to be chosen later. If $\rho(z, z') \geqq \theta$, then by the interpolation inequalities in Lemma 2.2,

$$[\partial_{v_i v_j} u]_{\alpha,Q_r} \leqq 2\theta^{-\alpha} |D_v^2 u|_{0,Q_r} \leqq \frac{1}{12d^2} [u]'_{2+\alpha,\beta,Q_r} + C\theta^{-2} |u|_{0,Q_r}. \quad (2.5)$$

On the other hand, if $\rho(z, z') < \theta$, let χ be a smooth cutoff such that $\chi(\tilde{z}) = 1$ if $\rho(\tilde{z}, z') < \theta$ and $\chi(\tilde{z}) = 0$ if $\rho(\tilde{z}, z') \geqq 2\theta$. We can choose χ such that

$$\begin{aligned} &|\nabla_v \chi|_{0,Q_1} \lesssim \theta^{-1}, \quad [\nabla_v \chi]_{0,Q_1} \lesssim \theta^{-1-\alpha}, \quad |\partial_t \chi + v \cdot \nabla_x \chi|_{0,Q_1} \\ &\quad + |D_v^2 \chi|_{0,Q_1} \lesssim \theta^{-2}, \\ &\text{and} \quad [\partial_t \chi + v \cdot \nabla_x \chi]_{\alpha,Q_1} + [D_v^2 \chi]_{\alpha,Q_1} \lesssim \theta^{-2-\alpha}. \end{aligned}$$

Using Lemma 2.7, we now have

$$\begin{aligned} &[\partial_{v_i v_j} u]_{\alpha,Q_r} \\ &\quad \leqq 2[\chi u]'_{2+\alpha,\beta,Q_{r+2\theta}} \\ &\quad \lesssim [\partial_t(\chi u) + v \cdot \nabla_x(\chi u) - L(z')(\chi u)]_{\alpha,Q_{r+2\theta}} \\ &\quad \lesssim [\partial_t(\chi u) + v \cdot \nabla_x(\chi u) - L(\chi u)]_{\alpha,Q_{r+2\theta}} + [(L - L(z'))(\chi u)]_{\alpha,Q_{r+2\theta}}. \end{aligned}$$

Let $R = r + 2\theta$. To estimate the first term on the last line, note that

$$\partial_t(\chi u) + v \cdot \nabla_x(\chi u) - L(\chi u) = \chi g + u(\partial_t + v \cdot \nabla_x - L)\chi - 2(A(z)\nabla_v u) \cdot \nabla_v \chi.$$

By the interpolation inequalities in Lemma 2.2,

$$\begin{aligned} &[\partial_t(\chi u) + v \cdot \nabla_x(\chi u) - L(\chi u)]_{\alpha, Q_R} \\ &\quad \lesssim ([g]_{\alpha, Q_R} + (1 + |A|_{0, Q_1})(\theta^{-2}[u]_{\alpha, Q_R} + \theta^{-1}[\nabla_v u]_{\alpha, Q_R})) \\ &\quad \lesssim [g]_{\alpha, Q_R} + (1 + |A|_{0, Q_1}) \left(\theta^\alpha [u]'_{2+\alpha, \beta, Q_R} + C\theta^{-2-\alpha(2+\alpha)} |u|_{0, Q_R}\right). \end{aligned} \tag{2.6}$$

For the second term, note that $(L - L(z'))(\chi u) = \operatorname{tr}((A(\tilde{z}) - A(z'))D_v^2(\chi u))$ for all $\tilde{z} \in Q_1$. Since $\operatorname{spt}(\chi) \subset \{\tilde{z} : \rho(\tilde{z}, z') \leqq 2\theta\}$, we have

$$\begin{aligned} [(L - L(z'))(\chi u)]_{\alpha, Q_R} &\lesssim [A]_{\alpha, Q_1} \theta^\alpha \left([D_v^2 u]_{\alpha, Q_R} + |D_v^2 u|_{0, Q_R}\right) \\ &\lesssim [A]_{\alpha, Q_1} \theta^\alpha \left([u]'_{2+\alpha, \beta, Q_R} + \theta^{-2} |u|_{0, Q_R}\right), \end{aligned} \tag{2.7}$$

using the interpolation inequalities again. Combining (2.6) and (2.7), we obtain, when $\rho(z, z') < \theta$,

$$[\partial_{v_i v_j} u]_{\alpha, Q_r} \lesssim |A|_{\alpha, Q_1} \theta^\alpha \left([u]'_{2+\alpha, \beta, Q_R} + [g]_{\alpha, Q_R} + \theta^{-p} |A|_{\alpha, Q_1} |u|_{0, Q_1}\right), \tag{2.8}$$

with $p = 2 + \alpha(2 + \alpha)$.

The combination of (2.5) and (2.8) implies that, for any fixed $\theta \in (0, 1/8)$,

$$\begin{aligned} [\partial_{v_i v_j} u]_{\alpha, Q_r} \leqq &\left(C|A|_{\alpha, Q_1} \theta^\alpha + \frac{1}{12d^2}\right) [u]'_{2+\alpha, \beta, Q_R} \\ &+ C[g]_{\alpha, Q_R} + C\theta^{-p} |A|_{\alpha, Q_1} |u|_{0, Q_1}. \end{aligned}$$

Summing over i and j, and applying a similar argument to $[u]_{(2+\alpha)/3, x, Q_r}$ and $[u]_{\beta, t, Q_r}$, we obtain

$$\begin{aligned} [u]'_{2+\alpha, \beta, Q_r} \leqq &\left(C|A|_{\alpha, Q_1} \theta^\alpha + \frac{1}{4}\right) [u]'_{2+\alpha, \beta, Q_R} + C[g]_{\alpha, Q_R} \\ &+ C\theta^{-p} |A|_{\alpha, Q_1} |u|_{0, Q_1}. \end{aligned}$$

Fix $\theta_0 > 0$ such that $C|A|_{\alpha, Q_1} \theta^\alpha < 1/4$ for all $\theta \in (0, \theta_0)$. Then, for each $R \in (r, r + 2\theta_0)$, we have

$$[u]'_{2+\alpha, \beta, Q_r} \leqq \frac{1}{2} [u]'_{2+\alpha, \beta, Q_R} + C[g]_{Q_R} + C(R - r)^{-p} |A|_{\alpha, Q_1} |u|_{0, Q_1}.$$

Recall that $r \in [\frac{1}{4}, \frac{3}{4}]$ was arbitrary. Lemma 2.8 with $\omega(s) = [u]'_{2+\alpha, \beta, Q_s}$, $r_0 = 1/2$, and $r_1 = 1/2 + 2\theta_0$ implies

$$[u]'_{2+\alpha, \beta, Q_r} \leqq C([g]_{\alpha, Q_1} + (R - r)^{-p} |A|_{\alpha, Q_1} |u|_{0, Q_1}),$$

for each $\frac{1}{2} \leqq r < R \leqq \frac{1}{2} + 2\theta_0$. Choose $r = \frac{1}{2}$ and $R = \frac{1}{2} + \theta_0$, and the proof is complete. □

Next, we extend the estimate of Lemma 2.6 to the variable-coefficient case. Here, we need to assume $A(z)$ in the operator L is in $C^{1+\alpha}(Q_1)$.

Theorem 2.10. *Assume that $D_v^3 u, \nabla_x u, \partial_t u \in C^\alpha(Q_1)$. Then*

$$[u]''_{3+\alpha, Q_{1/2}} \leqq C \left(|g|_{1+\alpha, Q_1} + |A|^{5+\alpha+6/\alpha}_{1+\alpha, Q_1} |u|_{0, Q_1} \right),$$

where $g := \partial_t u + v \cdot \nabla_x u - tr(A D_v^2 u)$. The constant C depends on d, α, λ, and Λ.

Proof. For $r \in (0, 1]$, recall

$$[u]''_{3+\alpha, Q_r} = [\partial_t u]_{\alpha, Q_r} + [\nabla_x u]_{\alpha, Q_r} + [D_v^3 u]_{\alpha, Q_r}.$$

With r, θ, and R as in the proof of Theorem 2.9, we can follow the argument of that proof to show

$$\begin{aligned}[u]''_{3+\alpha, Q_r} \leqq \left(C|A|_{1+\alpha, Q_1} \theta^\alpha + \frac{1}{4} \right) [u]''_{3+\alpha, Q_R} + C|g|_{1+\alpha, Q_R} \\ + C\theta^{\alpha(4+\alpha)+6} |A|_{1+\alpha, Q_1} |u|_{0, Q_1}.\end{aligned}$$

The conclusion of the proof is the same as Theorem 2.9. □

In the previous theorems, we assumed that solutions exist. This is verified by the following theorem:

Proposition 2.11. *Given $g \in C^\alpha((-1, 0] \times \mathbb{R}^d \times \mathbb{R}^d)$ with compact support in Q_1, then there exists a unique weak solution u in $C^{2+\alpha}|((-1, 0] \times \mathbb{R}^d \times \mathbb{R}^d)$ of (2.4). Furthermore, $[u]'_{2+\alpha, \beta, Q_1} < \infty$. If $g \in C^{1+\alpha}(Q_1)$, the same conclusion holds with $[u]''_{3+\alpha, Q_1} < \infty$, where $[\cdot]'_{2+\alpha, \beta, Q_1}$ and $[\cdot]''_{3+\alpha, Q_1}$ are as in Definition 2.3.*

Proof. Fix any $\beta \in (0, 1)$ and assume that the matrix A is uniformly bounded and coercive on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. Define the norm

$$\begin{aligned}\|u\|_{\mathcal{B}} := \max \Big\{ |u|_{\alpha, Q_1(z_0)} + [u]_{2+\alpha, \beta, Q_1(z_0)} + [\partial_t u + v \cdot \nabla_x u]_{\alpha, Q_1(z_0)} : z_0 \\ = (0, x_0, v_0), x_0, v_0 \in \mathbb{R}^d \Big\},\end{aligned}$$

and the Banach space

$$\mathcal{B} := \{u \in C^{2+\alpha}([-1, 0] \times \mathbb{R}^d \times \mathbb{R}^d) : \| \cdot \|_{\mathcal{B}} < \infty\},$$

endowed with $\| \cdot \|_{\mathcal{B}}$, and

$$\mathcal{V} := \{u \in C^\alpha([-1, 0] \times \mathbb{R}^d \times \mathbb{R}^d) : u(-1, \cdot, \cdot) \equiv 0\},$$

endowed with the obvious norm.

For any $\theta \in [0, 1]$, define the operator $E_\theta : \mathcal{B} \to \mathcal{V}$ by

$$E_\theta u := u_t + v \cdot \nabla_x u - (1 - \theta) \Delta_v u - \theta L u.$$

From Theorem 2.9, we see that

$$\|u\|_{\mathcal{B}} \lesssim \|E_\theta u\|_{\mathcal{V}}$$

for all $u \in \mathcal{B}$. Linearity and the above inequality imply that E_θ is injective. Also, from (2.2), we see that E_0 is onto. Applying the method of continuity as in [8, Theorem 5.2], we obtain that E_1 is onto as well. Hence, E_1 is invertible.

This finishes the first claim. The same argument applies in the second case when g has one more derivative, using Theorem 2.10. □

We collect all estimates above and use the equation that u solves in order to derive an estimate on $(\partial_t + v \cdot \nabla_x)u$ to obtain the following theorem:

Theorem 2.12. *Let u be such that*

$$\partial_t u + v \cdot \nabla_x u - Lu = g$$

in Q_1, with $L = tr(AD_v^2 u)$ and $\lambda I \leqq A \leq \Lambda I$.

(a) *If $g, A \in C^\alpha(Q_1)$ for some $\alpha \in (0,1)$, we have the estimate*

$$[D_v^2 u]_{\alpha,Q_{1/2}} + [u]_{(2+\alpha)/3,x,Q_{1/2}} + [u]_{\beta,t,Q_{1/2}} + [(\partial_t + v \cdot \nabla_x)u]_{\alpha,Q_{1/2}} \lesssim ([g]_{\alpha,Q_1} + |A|^p_{\alpha,Q_1}|u|_{0,Q_1}),$$

for any $\beta \in (0,1)$.

(b) *If $g, A \in C^{1+\alpha}(Q_1)$ for some $\alpha \in (0,1)$, then*

$$[\partial_t u]_{\alpha,Q_{1/2}} + [\nabla_x u]_{\alpha,Q_{1/2}} + [D_v^3 u]_{\alpha,Q_{1/2}} \lesssim (|g|_{1+\alpha,Q_1} + |A|^q_{1+\alpha,Q_1}|u|_{0,Q_1}).$$

The implied constants depend on d, α, β, λ, and Λ. The exponents $p, q > 0$ depend only on α.

3. Smoothing for Weak Solutions of the Landau Equation

In this section, we apply the estimates of Section 2 to the Landau equation. The diffusion operator $\mathrm{tr}(\overline{a}(z)D_v^2 f)$ (or in divergence form, $\nabla_v \cdot (\overline{a}(z)\nabla_v f)$) is uniformly elliptic in any bounded set, but the ellipticity constants degenerate as $|v| \to \infty$. (See "Appendix A".) To deal with this, we apply a change of variables in a small cylinder around a given point z_0, which yields an equation with ellipticity constants that are independent of z_0. In the sequel, we undo this transformation to explicitly see the dependence of the estimates on $|v|$.

The following lemma was first proven in [3] in the case of moderately soft potentials:

Lemma 3.1. *Let $z_0 = (t_0, x_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ be such that $|v_0| \geqq 2$, and let T be the linear transformation such that*

$$Te = \begin{cases} |v_0|^{1+\gamma/2} e, & e \cdot v_0 = 0 \\ |v_0|^{\gamma/2} e, & e \cdot v_0 = |v_0|. \end{cases}$$

Let $\tilde{T}(t,x,v) = (t, Tx, Tv)$, *and define*

$$\begin{aligned}\mathcal{T}_{z_0}(t,x,v) &:= \mathcal{S}_{z_0} \circ \tilde{T}(t,x,v)\\ &= (t_0 + t, x_0 + Tx + tv_0, v_0 + Tv).\end{aligned}$$

Then:

(a) *There exists a constant* $C > 0$ *independent of* $v_0 \in \mathbb{R}^d \setminus B_2$ *such that for all* $v \in B_1$,

$$C^{-1}|v_0| \leqq |v_0 + Tv| \leqq C|v_0|.$$

(b) *Let* f *be a weak solution of the Landau equation* (1.3) *satisfying* (1.8), (1.9), *and* (1.10), *and if* $\gamma < -2$, *assume that* f *satisfies* (1.12). *Then there exists a radius*

$$r_1 = c_1 |v_0|^{-(1+\gamma/2)_+} \min\left(1, \sqrt{t_0/2}\right),$$

with c_1 *universal, such that for any* $r \in (0, r_1]$, *the function* $f_{z_0}(t,x,v) := f(\mathcal{T}_{z_0}(r^2 t, r^3 x, rv))$ *satisfies*

$$\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \nabla_v \cdot \left(\overline{A}(z) \nabla_v f_{z_0}\right) + \overline{B}(z) \cdot \nabla_v f_{z_0} + \overline{C}(z) f_{z_0}, \quad (3.1)$$

or equivalently,

$$\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \operatorname{tr}\left(\overline{A}(z) D_v^2 f_{z_0}\right) + \overline{C}(z) f_{z_0}, \quad (3.2)$$

in Q_1, *and the coefficients*

$$\overline{A}(z) = T^{-1} \overline{a}(\mathcal{T}_{z_0}(\delta_r(z))) T^{-1}, \quad \overline{B}(z) = r T^{-1} \overline{b}(\mathcal{T}_{z_0}(\delta_r(z))), \quad \text{and}$$
$$\overline{C}(z) = r^2 \overline{c}(\mathcal{T}_{z_0}(\delta_r(z)))$$

satisfy

$$\lambda I \leqq \overline{A}(z) \leqq \Lambda I,$$

$$|\overline{B}(z)| \lesssim \begin{cases} 1, & -1 \leqq \gamma < 0,\\ |v_0|^{\min\{1+\gamma/2, 0\}} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\theta(v))}\right)^{-(\gamma+1)/d}, & -2 \leqq \gamma < -1,\\ |v_0|^{-\gamma/2 - 2 - 2(\gamma+1)/d} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\theta(v))}\right)^{-(\gamma+1)/d}, & -d \leqq \gamma < -2, \end{cases}$$

$$|\overline{C}(v)| \lesssim \begin{cases} |v_0|^{-2} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\theta(v))}\right)^{-\gamma/d}, & \frac{-2d}{d+2} \leqq \gamma < 0,\\ |v_0|^{-(2+\gamma)_+ - 2 - 2\gamma/d} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\theta(v))}\right)^{-\gamma/d}, & -d < \gamma < \frac{-2d}{d+2}, \end{cases}$$

with λ *and* Λ *universal, and* $\theta \lesssim 1 + |v_0|^{-2/d}$.

Proof. For $\gamma \in (-2, 0)$, this lemma is proven in [3, Lemma 4.1]. In fact, that proof does not use $\gamma > -2$ in an essential way. The necessary ingredients are the upper and lower bounds of Proposition A.1 and Lemma A.3 from the "Appendix", which hold under our assumptions on f. The bounds on $\overline{B}$ and $\overline{C}$ come from Proposition A.2 and Lemma A.3. □

The coefficients $\overline{A}$, $\overline{B}$, and $\overline{C}$ are dependent on z_0, which we refer to as the "base point," and r.

For any $z_0 = (t_0, x_0, v_0)$ with $|v_0| \leqq 2$, we define $f_{z_0}(z) = f(\mathcal{S}_{z_0}\delta_{r_1} z)$, with r_1 as in Lemma 3.1(b). Note that in the notation of [3], our $f_{z_0}(t, x, v)$ is equal to $f_T(r_1^2 t, r_1^3 x, r_1 v)$. The following proposition shows how the regularity of f depends on the regularity of f_{z_0}.

Proposition 3.2. *Let $f : [0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ for some $T_0 > 0$. If f_{z_0} is defined with base point $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$, and some partial derivative $\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}$ of order $M = 2j + 3|\beta| + |\eta|$ exists in $C^\alpha(Q_1)$ for some $\alpha \in (0, 1)$, then*

$$\begin{aligned}|\partial_t^j \partial_x^\beta \partial_v^\eta f|_{\alpha, Q_{r_1}(z_0)} &\lesssim r_1^{-M-\alpha}(1 + |v_0|)^{-\gamma\alpha/2} |\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}|_{\alpha, Q_1} \\ &\lesssim \left(1 + t_0^{-(M+\alpha)/2}\right)(1 + |v_0|)^{M(1+\gamma/2)+\alpha} |\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}|_{\alpha, Q_1},\end{aligned}$$

with r_1 as in Lemma 3.1.

Proof. Let $\partial = \partial_t^j \partial_x^\beta \partial_v^\eta$. For $z, z' \in Q_{r_1}(z_0)$ with $|v_0| \geqq 2$, we have

$$\begin{aligned}|\partial f(z) - \partial f(z')| &= r_1^{-M} |\partial f_{z_0}(\delta_{r_1}^{-1} \mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z) - \partial f_{z_0}(\delta_{r_1}^{-1} \mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z')| \\ &\leqq [\partial f_{z_0}]_{\alpha, Q_1} r_1^{-M} \rho(\delta_{r_1}^{-1} \mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z, \delta_{r_1}^{-1} \mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z')^\alpha \\ &= [\partial f_{z_0}]_{\alpha, Q_1} r_1^{-M-\alpha} \rho(\mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z, \mathcal{S}_{z_0}^{-1} \tilde{T}^{-1} z')^\alpha \\ &\leqq [\partial f_{z_0}]_{\alpha, Q_1} r_1^{-M-\alpha} \rho(\tilde{T}^{-1} z, \tilde{T}^{-1} z')^\alpha \\ &\leqq [\partial f_{z_0}]_{\alpha, Q_1} r_1^{-M-\alpha} |v_0|^{-\gamma\alpha/2} \rho(z, z')^\alpha.\end{aligned}$$

In the case $|v_0| \leqq 2$, we have $f(z) = f_{z_0}(\delta_{r_1}^{-1} \mathcal{S}_{z_0}^{-1} z)$, and a similar calculation applies. □

Next, we show that if the regularity estimates of f_{z_0} decay sufficiently quickly as $|v| \to \infty$, they imply regularity of the coefficients of (3.2). Although it is enough to show that partial derivatives of $\overline{A}$ and $\overline{C}$ grow at most polynomially, we derive explicit rates for the sake of concreteness.

Lemma 3.3. *Let f_{z_0} be as in Lemma 3.1. Assume that some partial derivative $\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}$ of order $M = j + |\beta| + |\eta|$ exists in $C^\alpha(Q_1)$ for every $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$, and satisfies*

$$[\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}]_{\alpha, Q_1} \leqq C_0 \left(1 + t_0^{-p}\right)(1 + |v_0|)^{-q}$$

for some $p \geqq 0$ and $q > d + 2 + \gamma(1 - \alpha/2) + \alpha/3$. Then $\overline{A}(t, x, v)$ and $\overline{C}(t, x, v)$ enjoy the same regularity as f_{z_0}, and for any $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$, one has

$$\begin{aligned}\left[\partial_t^j \partial_x^\beta \partial_v^\eta \overline{A}\right]_{2\alpha/3, Q_1} &\lesssim \left(1 + t_0^{-M/2-p}\right)(1 + |v_0|)^{(M+\alpha/3)(1+\gamma/2)_+ + 2 + \alpha + \alpha\gamma/3} \\ \left[\partial_t^j \partial_x^\beta \partial_v^\eta \overline{C}\right]_{2\alpha/3, Q_1} &\lesssim \left(1 + t_0^{-M/2-p+1}\right)(1 + |v_0|)^{(M+\alpha/3-2)(1+\gamma/2)_+ + \alpha + \alpha\gamma/3},\end{aligned}$$

where $\overline{A}$ and $\overline{C}$ are defined with base point z_0, and r_1 is as in Lemma 3.1. The implied constant depends on d, γ, q, and C_0.

Proof. Let $\partial = \partial_t^j \partial_x^\beta \partial_v^\eta$. For some base point z_0 with $|v_0| \geqq 2$, fix $z, z' \in Q_1$ and let $\tilde{z} = (\tilde{t}, \tilde{x}, \tilde{v}) = \mathcal{T}_{z_0}(\delta_{r_1} z)$ and $\tilde{z}' = \mathcal{T}_{z_0}(\delta_{r_1} z')$, with r_1 as in Lemma 3.1. For $w \in \mathbb{R}^d$, Proposition 3.2 implies

$$\begin{aligned}
&|\partial f(\tilde{t}, \tilde{x}, \tilde{v} - w) - \partial f(\tilde{t}', \tilde{x}', \tilde{v}' - w)| \\
&\quad \leqq [\partial f]_{\alpha, Q_{r_1}(t_0, x_0, v_0 - w)} \rho((\tilde{t}, \tilde{x}, \tilde{v} - w), (\tilde{t}', \tilde{x}', \tilde{v}' - w))^\alpha \\
&\quad \lesssim (1 + t_0^{-p}) r_1^{-M-\alpha} (1 + |v_0 - w|)^{-q - \gamma\alpha/2} (\rho(\tilde{z}, \tilde{z}')^\alpha + |w|^{\alpha/3} \rho(\tilde{z}, \tilde{z}')^{2\alpha/3}).
\end{aligned}$$

Recall $\overline{A}(z) = T^{-1} \overline{a}(\mathcal{T}_{z_0}(\delta_{r_1} z)) T^{-1}$. The formula (1.5) for $\overline{a}$ implies

$$\begin{aligned}
|\partial \overline{A}(z) - \partial \overline{A}(z')| &\leqq |v_0|^{-\gamma} \int_{\mathbb{R}^d} |w|^{\gamma+2} |\partial f(\tilde{t}, \tilde{x}, \tilde{v} - w) - \partial f(\tilde{t}', \tilde{x}', \tilde{v}' - w)| \, \mathrm{d}w \\
&\lesssim (1 + t_0^{-p}) |v_0|^{-\gamma} r_1^{-M-\alpha} \rho(\tilde{z}, \tilde{z}')^{2\alpha/3} \\
&\quad \times \int_{\mathbb{R}^d} |w|^{\gamma+2+\alpha/3} (1 + |v_0 - w|)^{-q - \gamma\alpha/2} \, \mathrm{d}w \\
&\lesssim (1 + t_0^{-p}) r_1^{-M-\alpha} \rho(\tilde{z}, \tilde{z}')^{2\alpha/3} |v_0|^{2+\alpha/3} \\
&\lesssim (1 + t_0^{-p}) r_1^{-M-\alpha/3} \rho(z, z')^{2\alpha/3} |v_0|^{2+\alpha+\alpha\gamma/3},
\end{aligned}$$

where we have used $\rho(\tilde{z}, \tilde{z}') \lesssim |v_0|^{1+\gamma/2} r_1 \rho(z, z')$. A similar calculation applies to $\overline{C}(z) = r_1^2 \overline{c}(\mathcal{T}_{z_0} \delta_{r_1} z)$. In the borderline case $\gamma = -d$, we have $\overline{C}(z) = c_{d,\gamma} r_1^2 f_{z_0}(z)$, and the conclusion of the lemma follows from the even stronger decay of ∂f_{z_0}. □

Remark. The decay in the estimates of Lemma 3.3 can be improved when $|\eta| > 0$ by integrating by parts in w. However, this would still not grant us enough decay to conclude $f \in C^\infty$ without any decay assumption on the initial data.

Next, we show that Gaussian bounds in the initial data are propagated. This result was established in the case $\gamma \in (-2, 0)$ in [3, Theorem 1.2] , under the assumption that the hydrodynamic bounds (1.8), (1.9), and (1.10) hold. To prove such a result when $\gamma \in [-d, -2]$, we also need *a priori* bounds on $\|f\|_{L^\infty}$ and on sufficiently high moments of f.

Theorem 3.4. *Let $\gamma \in [-d, -2]$, and let f be a bounded weak solution of the Landau equation (1.4) satisfying the hydrodynamic bounds (1.8), (1.9), and (1.10). Assume, in addition, that*

$$\int_{\mathbb{R}^d} |v|^p f(v) \, \mathrm{d}v \leqq P_0,$$

where p is the smallest integer such that $p > \dfrac{d|\gamma|}{2 + \gamma + d}$. Then there exists $\mu_0 > 0$ such that if

$$f_{\text{in}}(x, v) \leqq C_0 e^{-\mu |v|^2}$$

for all $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$ and $\mu > 0$, then

$$f(t,x,v) \lesssim e^{-\min\{\mu_0,\mu\}|v|^2}, \tag{3.3}$$

where μ_0 and the implied constant in (3.3) depend on C_0, M_0, E_0, and $\|f\|_{L^\infty([0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d)}$. If $\gamma \leqq -d/2-1$, then the implied constant in (3.3) also depends on the time of existence T_0.

Proof. First, assume that $\gamma \in (-d/2-1, -2]$. Fix $\mu_0 > 0$ to be determined and let $\overline{\mu} = \min\{\mu, \mu_0\}$. Proceeding as in the proof of [3, Theorem 1.2], we claim that $\phi(t,x,v) = e^{-\overline{\mu}|v|^2}$ is a supersolution to the linear Landau equation

$$\partial_t \phi + v \cdot \nabla_x \phi = \operatorname{tr}(\overline{a} D_v^2 \phi) + \overline{c}\phi, \tag{3.4}$$

for $|v|$ large, where $\overline{a}$ and $\overline{c}$ are defined in terms of f. Since ϕ is radial in v, we have

$$\begin{aligned}\partial_{v_i}\partial_{v_j}\phi &= \frac{\partial_{rr}\phi}{|v|^2}v_i v_j + \frac{\partial_r\phi}{|v|}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)\\ &= \left[\frac{4\overline{\mu}^2|v|^2 - 2\overline{\mu}}{|v|^2}v_i v_j - 2\overline{\mu}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)\right]e^{-\overline{\mu}|v|^2}.\end{aligned}$$

Proposition A.1 and Lemma A.3 from the appendix imply

$$\begin{aligned}\overline{a}_{ij}\partial_{v_i}\partial_{v_j}\phi &\leqq \left[(4\overline{\mu}^2(1+|v|)^2 - 2\overline{\mu})C_1(1+|v|)^\gamma - 2\overline{\mu}C_2(1+|v|)^{\gamma+2}\right]e^{-\overline{\mu}|v|^2}\\ &= \left((4\overline{\mu}^2C_1 - 2\overline{\mu}C_2)(1+|v|)^{\gamma+2} - 2\overline{\mu}C_1(1+|v|)^\gamma\right)e^{-\overline{\mu}|v|^2}\\ &\leqq -C(1+|v|)^{\gamma+2}\phi(v),\end{aligned}$$

for $|v|$ sufficiently large, provided that we choose $\mu_0 < C_2/(2C_1)$, where we use the convention that repeated indices are summed over. With the bound on $\overline{c}$ from Lemma A.3, this implies

$$\overline{a}_{ij}\partial_{v_i}\partial_{v_j}\phi + \overline{c}\phi \leqq \left[-C(1+|v|)^{\gamma+2} + C(1+|v|)^{\gamma+2-\varepsilon}\right]\phi(v).$$

The first term on the right-hand side dominates for large $|v|$, and we have

$$\overline{a}_{ij}\partial_{v_i}\partial_{v_j}\phi + \overline{c}\phi \leqq -C|v|^{\gamma+2}\phi \tag{3.5}$$

for $|v| \geqq R_0$ for some large R_0. Choose C_f such that $C_f\phi(t,x,v) > \|f\|_{L^\infty}$ for all $|v| \leqq R_0$ and such that $C_f\phi(0,x,v) > f(0,x,v)$ for all $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. In the second inequality we used that $\overline{\mu} \leqq \mu$. Define the function

$$g(t,x,v) := [f(t,x,v) - C_f\phi(t,x,v)]_+.$$

If $|v| \leqq R_0$, then $g(t,x,v) = 0$ by our choice of C_1. If $|v| > R_0$, then by (3.5), ϕ is a supersolution to (3.4). We conclude $g(t,x,v)$ is a subsolution of $\partial_t g + v \cdot \nabla_x g \leqq \overline{a}_{ij}\partial_{v_i v_j} g + \overline{c}g$ in its entire domain; hence, by the maximum

principle [3, Lemma A.2], we have $g \leqq 0$ for all $t > 0$, so $f(t,x,v) \leqq C_1\phi(t,x,v)$ for all $t > 0$ for which f is defined.

If $\gamma \leqq -d/2 - 1$, the above argument does not apply because we do not have enough *a priori* decay in $\overline{c}$ to conclude (3.5). For this case, we define $h(t,x,v) = f(t,x,v)e^{\mu|v|^2}$. From the equation (1.4) for f, we have

$$\begin{aligned}\partial_t h + v\cdot\nabla_x h &= e^{\mu|v|^2}\left(\operatorname{tr}\left[\overline{a}D_v^2(e^{-\mu|v|^2}h)\right] + \overline{c}e^{-\mu|v|^2}h\right)\\ &= \operatorname{tr}\left[\overline{a}D_v^2 h\right] - 4\mu v\cdot(\overline{a}\nabla_v h) + \left(\overline{c} - 2\mu\operatorname{tr}(\overline{a}) + 4\mu^2\overline{a}_{ij}v_iv_j\right)h.\end{aligned}$$

Lemma A.3 implies that $\|\overline{c} - 2\mu\operatorname{tr}(\overline{a}) + 4\mu^2\overline{a}_{ij}v_iv_j\|_{L^\infty([0,T_0]\times\mathbb{R}^{2d})} \leqq C_0$ for some C_0, so that $\tilde{h}(t,x,v) = e^{-C_0t}h(t,x,v)$ is a supersolution of $\partial_t\tilde{h} + v\cdot\nabla_x\tilde{h} = \operatorname{tr}(\overline{a}D_v^2\tilde{h}) + \tilde{b}\cdot\nabla_v\tilde{h}$ with bounded drift $\tilde{b}_j = -4\mu v_i\overline{a}_{ij}$. The maximum principle for this class of equations (see for example [3, Proposition A.1]) implies $h(t,x,v) \leqq e^{C_0t}f_{\text{in}}(x,v)e^{\mu|v|^2}$, which is uniformly bounded on any finite time interval. Note that, since $\|f\|_{L^\infty([0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d)}$ is finite, this argument also applies in the case $\gamma = -d$. □

We are now in a position to prove our main result.

Proof of Theorem 1.2. Let f be a weak solution of the Landau equation (1.3) such that $f_{\text{in}}(x,v) = f(0,x,v) \lesssim e^{-\mu|v|^2}$ for some $\mu > 0$. Without loss of generality, we may assume $\mu \leqq \mu_0$, with μ_0 as in the statement of the theorem. By applying [3, Theorem 1.2] if $\gamma \in (-2, 0)$ or Theorem 3.4 if $\gamma \in [-d, -2]$, we see that, for all $(t,x,v) \in [0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$f(t,x,v) \lesssim e^{-\mu|v|^2}, \tag{3.6}$$

where the implied constant is independent of T_0 if $\gamma > -d/2 - 1$. The dependence of the implied constant in (3.6) on T_0 in the case $\gamma \leqq -d/2 - 1$ propagate to the rest of our estimates. Throughout this proof, as we absorb algebraic-in-v factors into factors with Gaussian decay in v, μ' denotes a changing, positive constant, with $\mu' < \mu \leqq \mu_0$. The constant μ' changes only finitely many times, by an arbitrarily small amount, so the final conclusion is valid for any $\mu' < \mu$.

Let f_{z_0} be as in Lemma 3.1 with base point $z_0 \in [0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$. Since Lemma 3.1 locally controls the coefficients in the equation for f_{z_0} (3.1), we may apply [9, Theorem 2] to obtain:

$$|f_{z_0}|_{\alpha,Q_{1/2}} \lesssim \|f_{z_0}\|_{L^2(Q_1)} + |\overline{C}f_{z_0}|_{0,Q_1},$$

for some $\alpha \in (0,1)$. Using the Gaussian decay of f (3.6), this implies $|f_{z_0}|_{\alpha,Q_{1/2}} \lesssim e^{-\mu'|v_0|^2}$. By rescaling, we have $|f_{z_0}|_{\alpha,Q_1} \lesssim e^{-\mu'|v_0|^2}$. Next, Lemma 3.3 with $M = p = 0$, along with the local upper bounds on $\overline{A}$ and $\overline{C}$ in Lemma 3.1, implies that the coefficients $\overline{A}$ and $\overline{C}$ in (3.2) satisfy

$$\left|\overline{A}\right|_{2\alpha/3,Q_1} + \left|\overline{C}\right|_{2\alpha/3,Q_1} \lesssim (1+|v_0|)^{k_0},$$

for some $k_0 \in \mathbb{R}$, with α as above. We apply the Schauder estimate, Theorem 2.12(a), to f_{z_0} in Q_1 with $\alpha' = 2\alpha/3$ to obtain

$$[f_{z_0}]_{1+\alpha'/3, Q_{1/2}} \leqq C([\overline{C} f_{z_0}]_{\alpha', Q_1} + |\overline{A}|^p_{\alpha', Q_1} |f_{z_0}|_{0, Q_1}) \lesssim e^{-\mu' |v_0|^2}$$

for any $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$, where $p > 0$ depends on α[1]. By Lemma 3.3 again, this implies $\overline{A}, \overline{C} \in C^{1+\alpha''}(Q_{1/2})$, with $\alpha'' = 2\alpha'/3$ and

$$\begin{aligned} |\overline{A}|_{1+\alpha'', Q_{1/2}} &\lesssim r_1^{-2}(1 + |v_0|)^{k_1} \lesssim (1 + t_0^{-1})(1 + |v_0|)^{k_1}, \\ |\overline{C}|_{1+\alpha'', Q_{1/2}} &\lesssim (1 + |v_0|)^{\ell_1}, \end{aligned}$$

for $k_1, \ell_1 \in \mathbb{R}$. We can now apply Theorem 2.12(b) to obtain

$$\begin{aligned} &[\partial_t f_{z_0}]_{\alpha'', Q_{1/4}} + [\nabla_x f_{z_0}]_{\alpha'', Q_{1/4}} + [D_v^3 f_{z_0}]_{\alpha'', Q_{1/4}} \\ &\quad \lesssim (|\overline{C} f_{z_0}|_{1+\alpha'', Q_{1/2}} + |\overline{A}|^q_{1+\alpha'', Q_{1/2}} |f_{z_0}|_{0, Q_{1/2}}) \\ &\quad \lesssim (1 + t_0^{-q}) e^{-\mu' |v_0|^2}. \end{aligned}$$

where $q > 0$ depends on α. Again, by taking a larger constant we have

$$[D_v^3 f_{z_0}]_{\alpha'', Q_1} + [\partial_t f_{z_0}]_{\alpha'', Q_1} + [\nabla_x f_{z_0}]_{\alpha'', Q_1} \lesssim (1 + t_0^{-q}) e^{-\mu' |v_0|^2}.$$

From here, we can inductively apply Theorem 2.12(a) and (b) to conclude $f_{z_0} \in C^\infty(Q_1)$. In more detail, assume that all partial derivatives $\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}$ with

$$2j + 3|\beta| + |\eta| \leqq M \tag{3.7}$$

exist in $C^\alpha(Q_1)$ for some $\alpha > 0$, and that for every such partial derivative ∂f_{z_0} and $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$, we have

$$[\partial f_{z_0}]_{\alpha, Q_1} \leqq C \left(1 + t_0^{-q}\right) e^{-\mu' |v_0|^2}, \tag{3.8}$$

for some $q > 0$. Then Lemma 3.3 implies that $\overline{A}$ and $\overline{C}$ in (3.2) satisfy

$$\begin{aligned} \left[\partial \overline{A}\right]_{1+\alpha', Q_{1/2}} &\lesssim (1 + t_0^{-q'})(1 + |v_0|)^k \\ \left[\partial \overline{C}\right]_{1+\alpha', Q_{1/2}} &\lesssim (1 + t_0^{-q'+1})(1 + |v_0|)^\ell, \end{aligned} \tag{3.9}$$

for some $q' > 0$ and $k, \ell \in \mathbb{R}$. Letting $\partial = \partial_t^j \partial_x^\beta \partial_v^\eta$ be a partial derivative satisfying (3.7), we can therefore differentiate equation (3.2) to obtain an equation for ∂f_{z_0} of the form

$$\partial_t(\partial f_{z_0}) + v \cdot \nabla_x(\partial f_{z_0}) = \operatorname{tr}(\overline{A}(z) \partial f_{z_0}) + \overline{C}(z) \partial f_{z_0} + \mathcal{F}(f_{z_0}(z), \overline{A}(z), \overline{C}(z)),$$

[1] Technically, Theorem 2.12 does not apply to f since it is not sufficiently regular; however, a standard mollification argument allows us to sidestep this potential issue. We omit the details.

for some differential operator $\mathcal{F}$ of order at most M (counted with the scaling of (3.7)). Applying Theorem 2.12(a) and our inductive hypothesis (3.8), we have

$$[\partial f_{z_0}]_{1+\alpha',Q_{1/2}} \lesssim \left([\overline{C}(z)\partial f_{z_0} + \mathcal{F}(f_{z_0}(z), \overline{A}(z), \overline{C}(z))]_{\alpha',Q_1} + |A|^p_{\alpha',Q_1} |f_{z_0}|_{0,Q_1}\right)$$
$$\lesssim \left(1 + t_0^{-q''}\right) e^{-\mu'|v_0|^2},$$

with $q'' > 0$. By (3.9), we have enough regularity of $\overline{C}(z)$ and $\mathcal{F}(f_{z_0}(z), \overline{A}(z), \overline{C}(z))$ to apply Theorem 2.12(b):

$$[D_v^3 \partial f_{z_0}]_{\alpha'',Q_{1/4}} + [\partial_t \partial f_{z_0}]_{\alpha'',Q_{1/4}} + [\nabla_x \partial f_{z_0}]_{\alpha'',Q_{1/4}} \lesssim \left(1 + t_0^{-q'''}\right) e^{-\mu'|v_0|^2}.$$

As above, we may replace $Q_{1/4}$ with Q_1 by taking a larger implied constant. Such an estimate holds for each partial derivative ∂f_{z_0} satisfying (3.7), so we have shown (3.8) holds with some $q > 0$ for $\partial_t^j \partial_x^\beta \partial_v^\eta f_{z_0}$ whenever

$$2j + 3|\beta| + |\eta| \leqq M + 3.$$

We conclude $f_{z_0} \in C^\infty(Q_1)$ for any $z_0 \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d$. By Proposition 3.2, we have that $f \in C^\infty((0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d))$ with the pointwise estimates (1.11). □

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A. Bounds on the Coefficients of the Landau Equation

In this appendix, we collect the available bounds on the coefficients $\overline{a}$, $\overline{b}$, and $\overline{c}$ in the Landau equation (1.3) with soft potentials ($\gamma \in [-d, 0)$). The estimates in Propositions A.1 and A.2 were derived in [20] and [3]. Earlier, corresponding bounds in the case $\gamma \geqq 0$ were shown in [6].

Proposition A.1. *Let $f : [0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ satisfy the bounds (1.8), (1.9), and (1.10), and let $\overline{a}$ be defined by (1.5). If $\gamma \in [-d, 0)$, then for unit vectors $e \in \mathbb{R}^d$,*

$$\overline{a}_{ij}(t,x,v)e_ie_j \geqq c\begin{cases}(1+|v|)^\gamma, & e\in\mathbb{S}^{d-1},\\ (1+|v|)^{\gamma+2}, & e\cdot v=0.\end{cases} \tag{A.1}$$

If $\gamma\in[-2,0)$, then $\overline{a}$ satisfies the upper bound

$$\overline{a}_{ij}(t,x,v)e_ie_j \leqq C\begin{cases}(1+|v|)^{\gamma+2}, & e\in\mathbb{S}^{d-1},\\ (1+|v|)^{\gamma}, & e\cdot v=|v|,\end{cases} \tag{A.2}$$

and if $\gamma\in[-d,-2)$,

$$\overline{a}_{ij}(t,x,v)e_ie_j \leqq C\|f(t,x,\cdot)\|_{L^\infty(\mathbb{R}^d)}^{-(\gamma+2)/d}, \quad e\in\mathbb{S}^{d-1}. \tag{A.3}$$

The constants c and C depend on d, γ, m_0, M_0, E_0, and H_0.

Proposition A.2. *Let f be as in Proposition A.1. The coefficients $\overline{b}$ and $\overline{c}$ defined by (1.6) and (1.7) respectively, satisfy the upper bounds*

$$\begin{aligned}&\left|\overline{b}(t,x,v)\right|\\ &\quad\leqq C\begin{cases}(1+|v|)^{\gamma+1}, & -1\leqq\gamma<0,\\ (1+|v|)^{\gamma+1}(1+\|f\|_{L^\infty(B_1(v))})^{-(\gamma+1)/d}, & \dfrac{-3d-2}{d+2}\leqq\gamma<-1,\\ (1+|v|)^{-2-2(\gamma+1)/d}\left(1+\|f\|_{L^\infty(B_1(v))}\right)^{-\gamma/d}, & -d\leqq\gamma<\dfrac{-3d-2}{d+2},\end{cases}\end{aligned} \tag{A.4}$$

and

$$\overline{c}(t,x,v)\leqq C\begin{cases}(1+|v|)^{\gamma}(1+\|f\|_{L^\infty(B_1(v))})^{-\gamma/d}, & \dfrac{-2d}{d+2}\leqq\gamma<0,\\ (1+|v|)^{-2-2\gamma/d}\left(1+\|f\|_{L^\infty(B_1(v))}\right)^{-\gamma/d}, & -d<\gamma<\dfrac{-2d}{d+2},\end{cases} \tag{A.5}$$

where the constants depend on d, γ, M_0, and E_0.

Finally, we show that when $\gamma\in[-d,-2]$, the coefficients $\overline{a}$ and $\overline{c}$ still have the appropriate decay to prove Theorem 3.4, if sufficiently many moments of f are finite.

Lemma A.3. *Let $\gamma\in[-d,-2]$, and let $f:[0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be a bounded function satisfying (1.8) and (1.9). Assume in addition that*

$$\int_{\mathbb{R}^d}|v|^p f(v)\,\mathrm{d}v\leqq P_0,$$

where p is the smallest integer such that $p > \dfrac{d|\gamma|}{2+\gamma+d}$. Then the upper bounds (A.2) *hold, with constants depending on d, γ, M_0, E_0, P_0, and $\|f\|_{L^\infty([0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d)}$. If, in addition, $\gamma > -d/2 - 1$, there is an $\varepsilon > 0$ depending on d and γ such that*

$$\overline{c}(t,x,v) \leqq C(1+|v|)^{\gamma+2-\varepsilon},$$

with C depending on the same quantities.

Proof. For any $e \in \mathbb{S}^{d-1}$, the formula (1.5) implies

$$\begin{aligned}\overline{a}_{ij}(t,x,v)e_ie_j &= a_{d,\gamma}\int_{\mathbb{R}^d}\left(1-\left(\frac{w\cdot e}{|w|}\right)^2\right)|w|^{\gamma+2}f(v-w)\,\mathrm{d}w\\ &\lesssim \int_{\mathbb{R}^d}|w|^{\gamma+2}f(v-w)\,\mathrm{d}w.\end{aligned}$$

Let $r := \frac{1}{2}|v|^{(\gamma+2)/(\gamma+2+d)}$, $R = |v|/2$, and define

$$I_1 = \int_{B_r}|w|^{\gamma+2}f(v-w)\,\mathrm{d}w, \quad I_2 = \int_{B_R\setminus B_r}|w|^{\gamma+2}f(v-w)\,\mathrm{d}w,$$
$$I_3 = \int_{\mathbb{R}^d\setminus B_R}|w|^{\gamma+2}f(v-w)\,\mathrm{d}w.$$

We have

$$I_1 \lesssim \|f\|_{L^\infty}r^{d+\gamma+2} \lesssim |v|^{\gamma+2},$$
$$I_2 \lesssim r^{\gamma+2}|v|^{-p}\int_{B_R}|v-w|^p f(v-w)\,\mathrm{d}w \lesssim P_0|v|^{-p+(\gamma+2)^2/(d+\gamma+2)}.$$

Our choice of p implies $-p < d(\gamma+2)/(d+\gamma+2)$, so that $I_2 \lesssim |v|^{\gamma+2}$. Finally, for $|w| \geqq |v|/2$, we have $|w|^{\gamma+2} \lesssim |v|^{\gamma+2}$, and

$$I_3 \lesssim |v|^{\gamma+2}\int_{\mathbb{R}^d\setminus B_R}f(v-w)\,\mathrm{d}w \leqq M_0|v|^{\gamma+2}.$$

If e is parallel to v, then proceeding as in [3, Lemma 2.1], we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(1-\left(\frac{w\cdot e}{|w|}\right)^2\right)|w|^{\gamma+2}f(v-w)\,\mathrm{d}w\\ &\quad= \int_{\mathbb{R}^d}\left(1-\left(\frac{(v-z)\cdot e}{|v-z|}\right)^2\right)|v-z|^{\gamma+2}f(z)\,\mathrm{d}z\\ &\quad= \int_{\mathbb{R}^d}\left(|v-z|^2-(|v|-z\cdot e)^2\right)|v-z|^{\gamma}f(z)\,\mathrm{d}z\\ &\quad= \int_{\mathbb{R}^d}\left(|z|^2-(z\cdot e)^2\right)|v-z|^{\gamma}f(z)\,\mathrm{d}z\\ &\quad= \int_{\mathbb{R}^d}|z|^2\sin^2\theta|v-z|^{\gamma}f(z)\,\mathrm{d}z,\end{aligned}$$

where θ is the angle between v and z. We may assume $|v| > 2$. Let $R = |v|/2$ and $q = \dfrac{p(p-2)}{p+d}$. By our choice of p, we have $(\gamma+2)p/q > -d$. If $z \in B_R(v)$, then $|\sin\theta| \leqq |v-z|/|v|$, $|z| \lesssim |v|$, and

$$\begin{aligned}
&\int_{B_R(v)} |z|^2 \sin^2\theta |v-z|^\gamma f(z)\,\mathrm{d}z \\
&\quad\leqq |v|^{-2} \int_{B_R(v)} |z|^2 |v-z|^{\gamma+2} f(z)\,\mathrm{d}z \\
&\quad\leqq |v|^{-p+q} \|f\|_{L^\infty}^{q/p} \int_{B_R(v)} |z|^{p-q} f(z)^{(p-q)/p} |v-z|^{\gamma+2}\,\mathrm{d}z \\
&\quad\lesssim |v|^{-p+q} \left(\int_{B_R(v)} |z|^p f(z)\,\mathrm{d}z\right)^{(p-q)/p} \left(\int_{B_R(v)} |v-z|^{(\gamma+2)p/q}\,\mathrm{d}z\right)^{q/p} \\
&\quad\lesssim |v|^{-p+q} E_0^{(p-q)/p} \left(|v|^{(\gamma+2)p/q+d}\right)^{q/p} \lesssim |v|^\gamma.
\end{aligned}$$

If $|v-z| \geqq R = |v|/2$, then $|v-z|^\gamma \lesssim |v|^\gamma$, and we have

$$\int_{\mathbb{R}^d \setminus B_R(v)} |z|^2 \sin^2\theta |v-z|^\gamma f(z)\,\mathrm{d}z \lesssim |v|^\gamma \int_{\mathbb{R}^d \setminus B_R(v)} |z|^2 f(z)\,\mathrm{d}z \lesssim E_0 |v|^\gamma.$$

For $\overline{c}$, our choice of p and the restriction that $\gamma > -d/2 - 1$ implies there is an $\varepsilon > 0$ such that $-p + \dfrac{\gamma(\gamma+2-\varepsilon)}{d+\gamma} < \gamma + 2$. Define $r = \frac{1}{2}|v|^{(\gamma+2-\varepsilon)/(d+\gamma)}$, $R = |v|/2$, and I_1, I_2, I_3 as above. The same method implies that $I_1 + I_2 + I_3 \lesssim |v|^{\gamma+2-\varepsilon}$. □

References

1. Alexandre, R., Villani, C.: On the Landau approximation in plasma physics. *Annales de l'Institut Henri Poincare (C) Non Linear Anal*. **21**(1), 61–95, 2004
2. Bramanti , M., Brandolini , L.: Schauder estimates for parabolic nondivergence operators of Hörmander type. *J. Differ. Equ*. **234**(1), 177–245, 2007
3. Cameron , S., Silvestre , L., Snelson , S.: Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials. *Annales de l'Institut Henri Poincaré (C) Analyse Non Linéare* **35**(3), 625–642, 2018
4. Chapman, S., Cowling, T.G.: *The Mathematical Theory of Non-uniform gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, 3rd edn. Cambridge University Press, Cambridge 1970
5. Chen , Y., Desvillettes , L., He , L.: Smoothing effects for classical solutions of the full Landau equation. *Arch. Ration. Mech. Anal*. **193**(1), 21–55, 2009
6. Desvillettes , L., Villani , C.: On the spatially homogeneous Landau equation for hard potentials part I: existence, uniqueness and smoothness. *Commun. Partial Differ. Equ*. **25**(1–2), 179–259, 2000
7. Di Francesco , M., Polidoro , S.: Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv. Differ. Equ*. **11**(11), 1261–1320, 2006

8. GILBARG, D., TRUDINGER, N.S.: *Elliptic Partial Differential Equations of Second Order*, 2nd edn. Springer, Berlin 2001
9. GOLSE , F., IMBERT , C., MOUHOT , C., VASSEUR , A.: Harnack inequality for kinetic Fokker–Planck equations with rough coefficients and application to the Landau equation. *Annali della Scuola Normale Superiore di Pisa* **XIX**(1), 253–295, 2019
10. GUO , Y.: The Landau equation in a periodic box. *Commun. Math. Phys*. **231**(3), 391–434, 2002
11. HAN , Q., LIN , F.-H.: *Elliptic Partial Differential Equations. Courant Lecture Notes*, 2nd edn. Courant Institute of Mathematical Sciences, New York University, New York 2011
12. HE , L., YANG , X.: Well-posedness and asymptotics of grazing collisions limit of Boltzmann equation with Coulomb interaction. *SIAM J. Math. Anal*. **46**(6), 4104–4165, 2014
13. HÖRMANDER , L.: Hypoelliptic second order differential equations. *Acta Math*. **119**, 147–171, 1967
14. KRYLOV, N.V.: *Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Graduate Studies in Mathematics*, vol. 12. American Mathematical Society, Providence 1996
15. LANCONELLI , E., POLIDORO , S.: On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino* **52**(1), 29–63, 1994. (Partial differential equations, II (Turin, 1993))
16. LIFSHITZ, E.M., PITAEVSKII, L.P.: *Course of Theoretical Physics: Physical Kinetics*, vol. 10, 1st edn. Butterworth-Heinemann, Oxford 1981
17. LIU , S., MA , X.: Regularizing effects for the classical solutions to the Landau equation in the whole space. *J. Math. Anal. Appl*. **417**(1), 123–143, 2014
18. MANFREDINI , M.: The Dirichlet problem for a class of ultraparabolic equations. *Adv. Differ. Equ*. **2**(5), 831–866, 1997
19. MOUHOT , C., NEUMANN , L.: Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity* **19**(4), 969, 2006
20. SILVESTRE , L.: Upper bounds for parabolic equations and the Landau equation. *J. Differ. Equ*. **262**(3), 3034–3055, 2017
21. VILLANI , C.: On the Cauchy problem for Landau equation: sequential stability, global existence. *Adv. Differ. Equ*. **1**(5), 793–816, 1996
22. VILLANI , C.: On the spatially homogeneous Landau equation for Maxwellian molecules. *Math. Models Methods Appl. Sci*. **08**(06), 957–983, 1998
23. WANG , W., ZHANG , L.: The C^α regularity of weak solutions of ultraparabolic equations. *Discrete Contin. Dyn. Syst*. **29**(3), 1261–1275, 2011

C. Henderson, S. Snelson
Department of Mathematics,
University of Chicago,
5734 S. University Ave.,
Chicago
IL
60637 USA.
e-mail: ckhenderson@math.arizona.edu
e-mail: ssnelson@fit.edu

and

C. Henderson
Present Address:
Department of Mathematics,
University of Arizona,
Tucson
AZ
85721 USA.

and

S. Snelson
Present Address:
Department of Mathematical Sciences,
Florida Insitute of Technology,
Melbourne
FL
32901 USA.

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