Journal of Combinatorial Theory, Series A 173 (2020) 105235

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Characterization of queer supercrystals

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ARTICLE INFO

Article history:
Received 21 September 2018
Received in revised form 4 September 2019
Accepted 4 February 2020
Available online xxxx

Keywords:
Crystal graphs
Queer Lie superalgebras
Stembridge axioms

ABSTRACT

We provide a characterization of the crystal bases for the quantum queer superalgebra recently introduced by Grantcharov et al. This characterization is a combination of local queer axioms generalizing Stembridge's local axioms for crystal bases for simply-laced root systems, which were recently introduced by Assaf and Oguz, with further axioms and a new graph G characterizing the relations of the type A components of the queer supercrystal. We provide a counterexample to Assaf's and Oguz' conjecture that the local queer axioms uniquely characterize the queer supercrystal. We obtain a combinatorial description of the graph G on the type A components by providing explicit combinatorial rules for the odd queer operators on certain highest weight elements. This also yields a new combinatorial description of the Schur expansion of the Schur P-polynomials.

1. Introduction

The representation theory of Lie algebras is of fundamental importance, and hence combinatorial models for representations, especially those amenable to computation, are

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https://doi.org/10.1016/j.jcta.2020.105235

of great use. In the 1990's, Kashiwara [16] showed that integrable highest weight representations of the Drinfeld–Jimbo quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra, in the $q \to 0$ limit result in a combinatorial skeleton of the integrable representation. He coined the term crystal bases, reflecting the fact that q corresponds to the temperature of the underlying physical system. Since then, crystal bases have appeared in many areas of mathematics, including algebraic geometry, combinatorics, mathematical physics, representation theory, and number theory. One of the major advances in the theory of crystals for simply-laced Lie algebras was the discovery by Stembridge [23] of local axioms that uniquely characterize the crystal graphs corresponding to Lie algebra representations. These local axioms provide a completely combinatorial approach to the theory of crystals; this viewpoint was taken in [4].

Lie superalgebras [15] arose in physics in theories that unify bosons and fermions. They are essential in modern string theories [7] and appear in other areas of mathematics, such as the projective representations of the symmetric group. The crystal basis theory has been developed for various quantum superalgebras [3,11,8–10,12,17,18]. In this paper, we are in particular interested in the queer superalgebra $\mathfrak{q}(n)$ (see for example [6]). A theory of highest weight crystals for the queer superalgebra $\mathfrak{q}(n)$ was recently developed by Grantcharov et al. [8–10]. They provide an explicit combinatorial realization of the highest weight crystal bases in terms of semistandard decomposition tableaux and show how these crystals can be derived from a tensor product rule and the vector representation. They also use the tensor product rule to derive a Littlewood–Richardson rule. Choi and Kwon [5] provide a new characterization of Littlewood–Richardson–Stembridge tableaux for Schur P-functions by using the theory of $\mathfrak{q}(n)$-crystals. Independently, Hiroshima [13] and Assaf and Oguz [1,2] defined a queer supercrystal structure on semistandard shifted tableaux, extending the type A crystal structure of [14] on these tableaux.

In this paper, we provide a characterization of the queer supercrystals. Assaf and Oguz [1,2] conjecture a local characterization of queer supercrystals in the spirit of Stembridge's [23] characterization of crystals associated to classical simply-laced root systems, which involves local relations between the odd crystal operator f_{-1} with the type A_{n-1} crystal operators f_i for $1 \leqslant i < n$. However, we provide a counterexample to [2, Conjecture 4.16], which conjectures that these local axioms uniquely characterize the queer supercrystals. Instead, we define a new graph $G(\mathcal{C})$ on the relations between the type A components of the queer supercrystal $\mathcal{C}$, which together with Assaf's and Oguz' local queer axioms and further new axioms uniquely fixes the queer supercrystal structure (see Theorem 5.1). We provide a combinatorial description of $G(\mathcal{C})$ by providing the combinatorial rules for all odd queer supercrystal operators f_{-i} and e_{-i} on certain highest weight elements for $1 \leqslant i < n$.

This paper is structured as follows. In Section 2, we review the combinatorial definition of the queer supercrystals by [8–10] and prove several results that are needed later for the combinatorial description of the graph $G(\mathcal{C})$. In particular, Theorems 2.12 and 2.16 provide explicit combinatorial descriptions of the odd queer crystal operators f_{-i} and e_{-i} on highest weight elements. In Section 3, we state the local queer axioms by Assaf and Oguz [1,2] and provide a counterexample to [2, Conjecture 4.16]. The graph $G(\mathcal{C})$ is introduced in Section 4. Theorem 4.9 allows us to transform $G(\mathcal{C})$ into combinatorial graphs $\overline{G}(\mathcal{C})$ and $\widetilde{G}(\mathcal{C})$, which together with the local queer axioms of Definition 3.1 and new connectivity axioms of Definition 4.4 uniquely characterize the queer supercrystals as stated in Theorem 5.1. The graph $G(\mathcal{C})$ also yields a new combinatorial description of the Schur expansion of the Schur P-polynomials (see Remark 4.10).

Acknowledgments

We are grateful to Sami Assaf, Dan Bump, Zach Hamaker, Ezgi Oguz, and Travis Scrimshaw for helpful discussions. We would also like to thank Dimitar Grantcharov, Ji-Hye Jung, and Masaki Kashiwara for answering our questions about their work. The last two authors have implemented the queer supercrystals in SageMath [20,19]. This work benefited from experimentations in SageMath.

This work was partially supported by NSF grants DMS–1500050, DMS–1760329, DMS–1764153 and NSF MSPRF grant PDRF 1604262.

2. Queer supercrystals

In Section 2.1, we review the queer supercrystals constructed in [8–10]. In Section 2.2, we review some properties of queer supercrystals discovered in [1,2]. In Section 2.3, we provide new explicit combinatorial descriptions of f_{-i} and e_{-i} on certain highest weight elements, which will be used in Section 4 to construct the graph $G(\mathcal{C})$. In Section 2.4, we provide relations between e_{-i} when acting on certain highest weight elements, which will be used in Section 4 to deal with "by-pass arrows" in the component graph $G(\mathcal{C})$.

2.1. Definition of queer supercrystals

An *(abstract) crystal* of type A_n is a nonempty set B together with the maps

$$\begin{aligned} e_i, f_i &\colon B \to B \sqcup \{0\} \qquad \text{for } i \in I, \\ \mathrm{wt} &\colon B \to \Lambda, \end{aligned} \tag{2.1}$$

where $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$ is the weight lattice of the root of type A_n and $I = \{1, 2, \ldots, n\}$ is the index set, subject to several conditions. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I$ the simple roots of type A_n, where ϵ_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Then we require:

A1. For $b, b' \in B$, we have $f_i b = b'$ if and only if $b = e_i b'$. In this case $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.

For $b \in B$, we also define

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\} \qquad \text{and} \qquad \varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}.$$

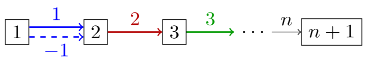


Fig. 1. $\mathfrak{q}(n+1)$-crystal of letters $\mathcal{B}$.

For further details, see for example [4, Definition 2.13].

There is an action of the symmetric group S_n on a type A_n crystal B given by the operators

$$s_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0, \end{cases} \tag{2.2}$$

for $b \in B$, where $k = \varphi_i(b) - \varepsilon_i(b)$.

An element $b \in B$ is called *highest weight* if $e_i(b) = 0$ for all $i \in I$. Similarly, b is called *lowest weight* if $f_i(b) = 0$ for all $i \in I$. For a subset $J \subseteq I$, we say that b is J-highest weight if $e_i(b) = 0$ for all $i \in J$ and similarly b is J-lowest weight if $f_i(b) = 0$ for all $i \in J$.

We are now ready to define an abstract queer supercrystal.

Definition 2.1. [9, Definition 1.9] An *abstract* $\mathfrak{q}(n+1)$*-crystal* is a type A_n crystal B together with the maps $e_{-1}, f_{-1} \colon B \to B \sqcup \{0\}$ satisfying the following conditions:

Q1. $\mathrm{wt}(B) \subset \Lambda$;
Q2. $\mathrm{wt}(e_{-1}b) = \mathrm{wt}(b) + \alpha_1$ and $\mathrm{wt}(f_{-1}b) = \mathrm{wt}(b) - \alpha_1$;
Q3. for all $b, b' \in B$, $f_{-1}b = b'$ if and only if $b = e_{-1}b'$;
Q4. if $3 \leqslant i \leqslant n$, we have
  (a) the crystal operators e_{-1} and f_{-1} commute with e_i and f_i;
  (b) if $e_{-1}b \in B$, then $\varepsilon_i(e_{-1}b) = \varepsilon_i(b)$ and $\varphi_i(e_{-1}b) = \varphi_i(b)$.

Given two $\mathfrak{q}(n+1)$-crystals B_1 and B_2, Grantcharov et al. [9, Theorem 1.8] provide a crystal on the tensor product $B_1 \otimes B_2$, which we state here in reverse convention. It consists of the type A_n tensor product rule (see for example [4, Section 2.3]) and the *tensor product rule* for $b_1 \otimes b_2 \in B_1 \otimes B_2$

$$\begin{aligned} e_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes e_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ e_{-1}b_1 \otimes b_2 & \text{otherwise,} \end{cases} \\ f_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes f_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ f_{-1}b_1 \otimes b_2 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.3}$$

The crystals of interest are the *crystals of words* $\mathcal{B}^{\otimes \ell}$, where $\mathcal{B}$ is the $\mathfrak{q}(n+1)$-crystal of letters depicted in Fig. 1.

In addition to the queer supercrystal operators $f_{-1}, f_1, \dots, f_n$ and $e_{-1}, e_1, \dots, e_n$, we define the crystal operators for $1 < i \leqslant n$

$$f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i} \qquad \text{and} \qquad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}, \tag{2.4}$$

where $s_{w_i} = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and s_i is the reflection along the i-string in the crystal defined in (2.2). Furthermore for $i \in I_0 := \{1, 2, \dots, n\}$

$$f_{-i'} := s_{w_0} e_{-(n+1-i)} s_{w_0} \qquad \text{and} \qquad e_{-i'} := s_{w_0} f_{-(n+1-i)} s_{w_0}, \tag{2.5}$$

where w_0 is the longest word in the symmetric group S_{n+1}. By [9, Theorem 1.14], with all operators e_i, f_i for $i \in \{-1, -2, \dots, -n, 1, 2, \dots, n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique highest weight vector and with all operators e_i, f_i for $i \in \{-1', -2', \dots, -n', 1, 2, \dots, n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique lowest weight vector.

2.2. Properties of queer supercrystals

We now review and prove several properties about the queer supercrystal operators.

Lemma 2.2. *For $1 \leqslant i < n$, we have*

$$\begin{aligned} f_{-(i+1)} &= (s_i s_{i+1})\, f_{-i}\, (s_{i+1} s_i), \\ e_{-(i+1)} &= (s_i s_{i+1})\, e_{-i}\, (s_{i+1} s_i). \end{aligned} \tag{2.6}$$

Proof. We use the definition (2.4). Note that the following recursion holds

$$s_{w_{i+1}} = (s_2 \cdots s_{i+1})(s_1 \cdots s_i) = (s_2 \cdots s_i)(s_1 \cdots s_{i-1}) s_{i+1} s_i = s_{w_i} s_{i+1} s_i, \tag{2.7}$$

which implies the statement. $\square$

Remark 2.3. The operators f_i for $i \in I_0$ have an easy combinatorial description on $b \in \mathcal{B}^{\otimes \ell}$ given by the *signature rule*, which can be directly derived from the tensor product rule (see for example [4, Section 2.4]). One can consider b as a word in the alphabet $\{1, 2, \dots, n+1\}$. Consider the subword of b consisting only of the letters i and $i+1$. Pair (or bracket) any consecutive letters $i+1, i$ in this order, remove this pair, and repeat. Then f_i changes the rightmost unpaired i to $i+1$; if there is no such letter $f_i(b) = 0$. Similarly, e_i changes the leftmost unpaired $i+1$ to i; if there is no such letter $e_i(b) = 0$.

Remark 2.4. From (2.3), one may also derive a simple combinatorial rule for f_{-1} and e_{-1}. Consider the subword v of $b \in \mathcal{B}^{\otimes \ell}$ consisting of the letters 1 and 2. The crystal operator f_{-1} on b is defined if the leftmost letter of v is a 1, in which case it turns it into a 2. Otherwise $f_{-1}(b) = 0$. Similarly, e_{-1} on b is defined if the leftmost letter of v is a 2, in which case it turns it into a 1. Otherwise $e_{-1}(b) = 0$.

Lemmas 2.5 and 2.6 have appeared in [1,2]. We provide proofs for completeness.

Lemma 2.5. *Let $b \in \mathcal{B}^{\otimes \ell}$. The following holds:*

(1) *If $\varphi_1(b) \geqslant 2$ and $\varphi_{-1}(b) = 1$, we have $\varphi_1(b) = \varphi_1(f_{-1}(b)) + 2$ and $\varepsilon_1(b) = \varepsilon_1(f_{-1}(b))$. If furthermore $\varphi_1(b) > 2$, then*

$$f_1 f_{-1}(b) = f_{-1} f_1(b).$$

(2) *If $\varphi_1(b) = \varphi_{-1}(b) = 1$, we have*

$$f_1(b) = f_{-1}(b).$$

(3) *If $\varepsilon_1(b), \varepsilon_{-1}(b) > 0$ and $e_1(b) \neq e_{-1}(b)$, we have $\varepsilon_1(b) = \varepsilon_1(e_{-1}(b))$, $\varphi_1(b) = \varphi_1(e_{-1}(b)) - 2$, and*

$$e_1 e_{-1}(b) = e_{-1} e_1(b).$$

Proof. Let $p = \varphi_1(b)$ and $q = \varepsilon_1(b)$. Consider the subword v consisting of all letters 1 and 2 in b. After performing 1,2-bracketing onto v according to the signature rule, we have a subword of unbracketed letters in b as

$$v_{i_1} v_{i_2} \dots v_{i_p} v_{j_1} \dots v_{j_q}, \tag{2.8}$$

where $v_{i_k} = 1$ for all $1 \leqslant k \leqslant p$ and $v_{j_k} = 2$ for all $1 \leqslant k \leqslant q$.

(1) We assume that $\varphi_{-1}(b) > 0$, so that $f_{-1}(b)$ is defined. This implies $v_1 = 1$. Since v_1 is necessarily unbracketed, $i_1 = 1$ as well. The word $b' = f_{-1}(b)$ is formed by changing the leftmost 1 in b, namely v_{i_1}, into 2. This introduces a new bracketed 1,2-pair formed by $v_1 = 2$ and $v_{i_2} = 1$. The subword of unbracketed letters in b' now becomes

$$v_{i_3} \dots v_{i_p} v_{j_1} \dots v_{j_q}$$

so that $\varphi_1(f_{-1}(b)) = p - 2 = \varphi_1(b) - 2$ and $\varepsilon_1(f_{-1}(b)) = q = \varepsilon_1(b)$. This establishes the first assertion.

Now, assume in addition that $p = \varphi_1(b) > 2$. Using the sequence of unbracketed letters in b as in the preceding paragraph, f_1 changes the rightmost unbracketed 1 in b, namely v_{i_p}, into 2. We still have v_1 to be 1 after the change, so that $f_{-1}(f_1(b))$ is defined and the leftmost 1 in $f_1(b)$, namely v_1, is changed into 2 under f_{-1}. On the other hand, $f_1(f_{-1}(b))$ is defined precisely because $p > 2$, and the rightmost unbracketed 1 in $f_{-1}(b)$, namely v_{i_p}, is changed into 2 under f_1. As the changes introduced in b to form $f_{-1}(f_1(b))$ are the same as in those of $f_1(f_{-1}(b))$, we conclude that $f_1(f_{-1}(b)) = f_{-1}(f_1(b))$, proving the second assertion.

(2) We assume $\varphi_1(b) = 1$, so that (2.8) is of the form $v_{i_1}v_{j_1}\dots v_{j_q}$. Furthermore, as $\varphi_{-1}(b) = 1$, $f_{-1}(b)$ is defined and $v_1 = 1$. As v_1 is necessarily unbracketed, $i_1 = 1$ as well. Therefore, we see that $f_1(b) = f_{-1}(b)$, since the rightmost unbracketed 1 in b and the leftmost 1 in b are the same, namely $v_{i_1} = v_1$.

(3) We assume that $\varepsilon_{-1}(b) > 0$, so that $e_{-1}(b)$ is defined. This implies $v_1 = 2$. However, since $e_{-1}(b) \neq e_1(b)$, e_{-1} and e_1 must change a 2 in b at different locations, so we have $j_1 > 1$. Consequently v_1 is a bracketed 2 and hence must be paired with some $v_h = 1$ where $h < i_1 < j_1$ (in case $p = 0$, $h < j_1$ still holds). The word $b' = e_{-1}(b)$ is obtained by changing the leftmost 2 in b, namely v_1, to 1. This introduces two new unbracketed 1's, namely, v_1 and v_h. The subword of unbracketed letters in b' is now

$$v_1 v_h v_{i_1} \dots v_{i_p} v_{j_1} \dots v_{j_q}$$

so that $\varepsilon_1(b) = q = \varepsilon_1(e_{-1}(b))$ and $\varphi_1(e_{-1}(b)) = p + 2 = \varphi_1(b) + 2$. This establishes the first two equalities.

Now, $e_1(e_{-1}(b))$ is the word formed by changing the leftmost unbracketed 2 in $b' = e_{-1}(b)$, namely v_{j_1}, to 1. On the other hand, using the subword of v in b containing unbracketed letters as described in the preceding paragraph, $e_1(b)$ changes the leftmost unbracketed 2 in b, namely v_{j_1}, into a 1. We still have $v_1 = 2$ and $v_h = 1$ after the change, so that $e_{-1}(e_1(b))$ is defined, with the leftmost 2 in $e_1(b)$, namely v_1, being changed into 1 under e_{-1}. As the changes introduced in b to form $e_{-1}(e_1(b))$ are the same as in those of $e_1(e_{-1}(b))$, we conclude that $e_1(e_{-1}(b)) = e_{-1}(e_1(b))$, thereby proving the final relation. □

Lemma 2.6. *Let $b \in \mathcal{B}^{\otimes \ell}$. The following holds:*

(1) *If $\varphi_2(b), \varphi_{-1}(b) > 0$, we have $\varphi_2(b) = \varphi_2(f_{-1}(b)) - 1$, $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$ and*

$$f_2 f_{-1}(b) = f_{-1} f_2(b).$$

(2) *If $\varphi_2(b) = 0$ and $\varphi_{-1}(b) > 0$, we have either*
   (a) *$\varphi_2(f_{-1}(b)) = 1$ and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$, or*
   (b) *$\varphi_2(f_{-1}(b)) = 0$ and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b)) + 1$.*

(3) *If $\varepsilon_2(b), \varepsilon_{-1}(b) > 0$, we have either*
   (a) *$\varepsilon_2(e_{-1}(b)) = \varepsilon_2(b) + 1$, $\varphi_2(b) = \varphi_2(e_{-1}(b)) = 0$, or*
   (b) *$\varepsilon_2(e_{-1}(b)) = \varepsilon_2(b)$, $\varphi_2(b) = \varphi_2(e_{-1}(b)) + 1$, and*

$$e_{-1} e_2(b) = e_2 e_{-1}(b).$$

Proof. We prove each part separately.

(1) Assume that $\varphi_2(b), \varphi_{-1}(b) > 0$, so that $f_2(b)$ and $f_{-1}(b)$ are both nonzero. Let $b' = f_{-1}(b)$ and $b'' = f_2(b)$.
By the signature rule, $\varphi_2(b)$ is the number of unbracketed 2 entries in the $2,3$-bracketing of b. Since $\varphi_2(b) > 0$, there exists a rightmost unbracketed 2, say b_j. As in Remark 2.4 $b' = f_{-1}(b)$ is formed by changing the leftmost 1, say b_i, to $b'_i = 2$, where b_i is the leftmost of all 1 and 2 entries (so in particular $i < j$).
Since $\varphi_{-1}(b) > 0$, every 2 must be to the right of b_i. Assume that there is a 3 left of b_i bracketed with a 2 to the right of b_i, and let $b_{s_1} \cdots b_{s_r} b_{t_1} \cdots b_{t_r} = 3^r 2^r$ be the subsequence of all 3 and 2 entries bracketed with each other for which $s_k < i$ and $i < t_k$ for all k. Then in b', we have that b'_{s_r} brackets with b'_i rather than b'_{t_1}, and $b'_{s_{r-1}}$ brackets with b'_{t_1}, and so on, leaving b'_{t_r} a new unbracketed 2. Thus we always have $\varphi_2(b') = \varphi_2(b) + 1$. Furthermore, since the number of unbracketed 3 entries remains unchanged, we have $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$.
For the commutativity relation, note that since $j > i$, so $b'_j = 2$ is still the rightmost unbracketed 2 in b' and $b''_i = 1$ is the leftmost 1 in b'' without a 2 to the left of b''_i. Thus both $f_2(f_{-1}(b))$ and $f_{-1}(f_2(b))$ are formed by changing b_i to 2 and b_j to 3. Hence

$$f_2(f_{-1}(b)) = f_{-1}(f_2(b))$$

as desired.

(2) Assume $\varphi_2(b) = 0$ and $\varphi_{-1}(b) > 0$, so that $b' = f_{-1}(b)$ is defined but $f_2(b)$ is not. Then there is an entry $b_i = 1$ with no 1 or 2 left of it that changes to 2 to form b'. There are also no unbracketed 2 entries in the $2,3$ bracketing.
We consider two cases. First, suppose that every 3 to the left of b_i in b is bracketed with some 2 to its right. Then in b' with $b'_i = 2$, the bracketed pairs for the entries $b'_{s_i} = 3$ to the left of b'_i shift left as in part (1) above, leaving a new unbracketed 2 and exactly the same number of unbracketed 3 entries. Thus $\varphi_2(b') = 1$ and $\varepsilon_2(b') = \varepsilon_2(b)$ in this case.
If instead there is an unbracketed 3 to the left of b_i, then this 3 becomes bracketed with a 2 (after the same shift in bracketed pairs) and we have $\varphi_2(b') = 0$ and $\varepsilon_2(b') = \varepsilon_2(b) - 1$, as desired.

(3) Suppose $\varepsilon_2(b), \varepsilon_{-1}(b) > 0$. Then the leftmost 1 or 2 in b is $b_i = 2$ for some i, and $b' := e_{-1}(b)$ is formed by changing b_i to 1. Since $e_2(b)$ is defined, there also exists a leftmost unbracketed 3, say $b_j = 3$.
We consider two cases. First suppose $\varphi_2(b) = 0$, meaning that every 2 is bracketed in the $2,3$-bracketing of b. Then in particular b_i is bracketed; let $b_{s_1} \cdots b_{s_r} b_i b_{t_1} \cdots b_{t_{r-1}} = 3^r 2^r$ be the subsequence consisting of all bracketed 3's (b_{s_i}) to the left of b_i along with the entries they are bracketed with ($b_{t_{r-i}}$ where $t_0 = i$). Then after lowering b_i to 1 to form b', we have that b'_{s_i} brackets with $b'_{t_{r-i+1}}$

for $i \geqslant 2$, and b'_{s_1} is an unbracketed 3. All other bracketed pairs are the same as in b, so there is only one more 3 among the unbracketed letters. It follows that $\varepsilon_2(b') = \varepsilon_2(b) + 1$ and $\varphi_2(b') = \varphi_2(b) = 0$.

For the second case, suppose $\varphi_2(b) > 0$. Then there is some unbracketed 2 in b; let b_k be the leftmost unbracketed 2. Note that $k \geqslant i$ because b_i is the leftmost 2, and note also that $k < j$ because b_j is the leftmost unbracketed 3. Thus $i < j$.

Now, lowering b_i to 1 to form b' results in shifting the bracketing as in the cases above, which makes b'_k be bracketed (and all other bracketings the same). Thus there is one less unbracketed 2 in b' as b, and the same number of unbracketed 3's. It follows that $\varepsilon_2(b') = \varepsilon_2(b)$ and $\varphi_2(b') = \varphi_2(b) - 1$. Furthermore, b'_j is still the leftmost unbracketed 3 in b', and so both $e_{-1}e_2(b)$ and $e_2e_{-1}(b)$ are formed by changing b_i to 1 and b_j to 2. The result follows. □

2.3. *Explicit description of f_{-i} and e_{-i}*

In this section, we give explicit descriptions of $\varphi_{-i}(b)$, $\varepsilon_{-i}(b)$, $f_{-i}b$, and $e_{-i}b$ for J-highest-weight elements $b \in \mathcal{B}^{\otimes \ell}$ for certain $J \subseteq I_0$ (see Proposition 2.9 and Theorems 2.12 and 2.16). We will need these results in Section 4 when we characterize certain graphs on the type A components of the queer supercrystal.

Lemma 2.7. *Let $i \in I_0$ and $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i-1\}$-highest weight. If the first letter in the $(i, i+1)$-subword of b is $i+1$, then $\varepsilon_{-i}(b) = 1$.*

Proof. The statement is true for $i = 1$ by Remark 2.4. Now suppose that by induction on i the statement of the lemma is true for $1, 2, \ldots, i-1$. By Lemma 2.2, we have $e_{-i} = s_{i-1}s_i e_{-(i-1)} s_i s_{i-1}$. Let $u = i+1$ be the leftmost $i+1$ in b and $v = i$ be the leftmost i in b. By assumption, u appears to the left of v and hence v is bracketed in the $(i, i+1)$-bracketing. Since by assumption b is $\{1, 2, \ldots, i-1\}$-highest weight, in the $(i-1, i)$-bracketing there are no unbracketed i and s_{i-1} raises all unbracketed $i-1$ to i. In particular, all $i-1$ to the left of v are raised to i since v is the leftmost i. In turn, s_i acts on unbracketed i and $i+1$ in the $(i, i+1)$-bracketing. Since v is bracketed and there are no $i-1$ to the left of v, the first letter in the $(i-1, i)$-subword of $s_i s_{i-1}(b)$ is i. Also, $s_i s_{i-1}(b)$ is $\{1, 2, \ldots, i-2\}$-highest weight. Hence by induction $\varepsilon_{-(i-1)}(s_i s_{i-1}(b)) = 1$, which proves that $\varepsilon_{-i}(b) = 1$. □

The next definition below will be used heavily throughout this section.

Definition 2.8. The *initial k-sequence* of a word $b = b_1 \ldots b_\ell \in \mathcal{B}^{\otimes \ell}$, if it exists, is the sequence of letters $b_{p_k}, b_{p_{k-1}}, \ldots, b_{p_1}$, where b_{p_k} is the leftmost k and b_{p_j} is the leftmost j to the right of $b_{p_{j+1}}$ for all $1 \leqslant j < k$.

Let $i \in I_0$ and $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight with $\mathrm{wt}(b)_{i+1} > 0$, where $\mathrm{wt}(b)_{i+1}$ is the $(i+1)$-st entry in $\mathrm{wt}(b) \in \mathbb{Z}^{n+1}_{\geqslant 0}$. Then note that b has an initial $(i+1)$-

sequence, say $b_{p_{i+1}}, b_{p_i}, \dots, b_{p_1}$. Also let $b_{q_i}, b_{q_{i-1}}, \dots, b_{q_1}$ be the initial i-sequence of b. Note that $p_{i+1} < p_i < \cdots < p_1$ and $q_i < q_{i-1} < \cdots < q_1$ by the definition of initial sequence. Furthermore either $q_j = p_j$ or $q_j < p_{j+1}$ for all $1 \leqslant j \leqslant i$.

Proposition 2.9. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight for $i \in I_0$. Then:*

(a) $\varepsilon_{-i}(b) = 1$ *if and only if* $\mathrm{wt}(b)_{i+1} > 0$ *and* $p_j = q_j$ *for at least one* $j \in \{1, 2, \dots, i\}$.
(b) $\varphi_{-i}(b) = 1$ *if and only if* $\mathrm{wt}(b)_i > 0$ *and either* $\mathrm{wt}(b)_{i+1} = 0$ *or* $p_j \neq q_j$ *for all* $j \in \{1, 2, \dots, i\}$.

Example 2.10. Take $b = 1331242312111$ and $i = 3$. Then $p_4 = 6, p_3 = 8, p_2 = 10, p_1 = 11$ and $q_3 = 2, q_2 = 5, q_1 = 9$. We indicate the chosen letters p_j by underlines and q_j by overlines: $b = 1\overline{3}31\overline{2}\underline{4}2\underline{3}\overline{1}\underline{2}\underline{1}11$. Since no letter has a both an overline and underline (meaning $p_j \neq q_j$ for all j), we have $\varphi_{-3}(b) = 1$.

Proof of Proposition 2.9. Let us first prove claim (a) for $i = 1$. If $\mathrm{wt}(b)_2 = 0$, then certainly $\varepsilon_{-1}(b) = 0$ since by definition e_{-1} changes a 2 into a 1. If $\mathrm{wt}(b)_2 > 0$, then q_1 is the position of the leftmost 1, p_2 is the position of the leftmost 2, and p_1 is the position of the first 1 after this 2. If $p_1 = q_1$, there is no 1 to the left of the leftmost 2. By definition in this case $\varepsilon_{-1}(b) = 1$. If on the other hand $q_1 < p_2$, the leftmost 1 is before the leftmost 2 and hence $\varepsilon_{-1}(b) = 0$. This proves the claim.

Now assume by induction that claim (a) is true for up to $i - 1$. If $\mathrm{wt}(b)_{i+1} = 0$, then $\varepsilon_{-i}(b) = 0$ since e_{-i} changes the weight by the simple root α_i. Otherwise assume that $\mathrm{wt}(b)_{i+1} > 0$.

If $p_i = q_i$, the first letter i or $i + 1$ is the $i + 1$ in position $p_{i+1} < p_i = q_i$. Hence by Lemma 2.7 we have $\varepsilon_{-i}(b) = 1$.

If $q_i < p_i$ (and hence automatically $q_i < p_{i+1}$), recall that by Lemma 2.2 we have $e_{-i} = s_{i-1} s_i e_{-(i-1)} s_i s_{i-1}$. The operator s_{i-1} leaves the letter $i - 1$ in positions q_{i-1} and p_{i-1} unchanged since these letters are bracketed with i in positions q_i and p_i, respectively. All $i - 1$ to the left of position q_{i-1} are unbracketed and since b is $\{1, 2, \dots, i\}$-highest weight, s_{i-1} changes all of these $i - 1$ to i. In $s_{i-1} b$ there are possibly new letters i between positions p_{i+1} and p_i; the $i + 1$ in position p_{i+1} brackets with the leftmost of these in position $p_{i+1} < p'_i \leqslant p_i$. The operator s_i on $s_{i-1} b$ changes all letters i to the left of position p'_i to $i+1$. Hence $\mathrm{wt}(s_i s_{i-1} b)_i > 0$, $s_i s_{i-1} b$ is $\{1, 2, \dots, i-1\}$-highest weight with sequences with respect to $i-1$ given by $p'_i > p_{i-1} > \cdots > p_1$ and $q_{i-1} > q_{i-2} > \cdots > q_1$. Claim (a) now follows by induction on i.

If b is $\{1, 2, \dots, i\}$-highest weight and $\mathrm{wt}(b)_i > 0$, we must have $\varphi_{-i}(b) + \varepsilon_{-i}(b) = 1$. Hence $\varphi_{-i}(b) = 1$ precisely when $\varepsilon_{-i}(b) = 0$, proving (b). □

Recall that in a queer supercrystal B an element $b \in B$ is *highest-weight* if $e_i(b) = 0$ for all $i \in I_0 \cup I_-$, where $I_0 = \{1, 2, \dots, n\}$ and $I_- = \{-1, -2, \dots, -n\}$.

Proposition 2.11. *[9, Proposition 1.13] Let $b \in \mathcal{B}^{\otimes \ell}$ be highest weight. Then* $\mathrm{wt}(b)$ *is a strict partition.*

Proof. Let b be highest weight and suppose that $\mathrm{wt}(b)_i = \mathrm{wt}(b)_{i+1}$ for some i, meaning that b contains the same number of letters i and $i+1$. Since all letters i and $i+1$ must be bracketed in the $(i, i+1)$-bracketing, this means that the first letter in the $(i, i+1)$-subword of b is the letter $i+1$. Then by Lemma 2.7, $\varepsilon_{-i}(b) = 1$, which means that b is not highest weight. Hence $\mathrm{wt}(b)_i > \mathrm{wt}(b)_{i+1}$ for all i, implying that $\mathrm{wt}(b)$ is a strict partition. $\square$

Next, we provide an explicit description of $f_{-i}(b)$ for $i \in I_0$, when b is $\{1, 2, \ldots, i\}$-highest weight. Recall that the sequence $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ is the leftmost sequence of letters $i, i-1, \ldots, 1$ from left to right. Set $r_1 = q_1$ and recursively define $r_j < r_{j-1}$ for $1 < j \leqslant i$ to be maximal such that $b_{r_j} = j$. Note that by definition $q_j \leqslant r_j$. Let $1 \leqslant k \leqslant i$ be maximal such that $q_k = r_k$.

Theorem 2.12. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$ (see Proposition 2.9). Then $f_{-i}(b)$ is obtained from b by changing $b_{q_j} = j$ to $j-1$ for $j = i, i-1, \ldots, k+1$ and $b_{r_j} = j$ to $j+1$ for $j = i, i-1, \ldots, k$.*

Example 2.13. Let us continue Example 2.10 with $b = 1331242312111$ and $i = 3$. We overline b_{q_j} and underline b_{r_j}, so that $b = 1\overline{3}\underline{3}1\overline{2}4\underline{2}3\overline{\underline{1}}2111$. From this we read off $q_3 = 2, q_2 = 5, q_1 = 9$, $r_3 = 3, r_2 = 7, r_1 = 9$, $k = 1$ and $f_{-3}(b) = 1241143322111$.

As another example, take $b = 545423321211$ in the $\mathfrak{q}(6)$-crystal $\mathcal{B}^{\otimes 12}$ and $i = 5$. Again, we overline b_{q_j} and underline b_{r_j}, so that $b = \overline{54}\underline{54}2\overline{3}\underline{3}\overline{\underline{21}}211$. This means that $q_5 = 1$, $q_4 = 2$, $q_3 = 6$, $q_2 = 8$, $q_1 = 9$, $r_5 = 3$, $r_4 = 4$, $r_3 = 7$, $r_2 = 8$, $r_1 = 9$, $k = 2$, and $f_{-5}(b) = 436522431211$.

Proof of Theorem 2.12. We prove the claim by induction on i. For $i = 1$, since by assumption $\varphi_{-1}(b) = 1$, the first letter in the subword of b of letters in $\{1, 2\}$ is a 1. This 1 is in position $q_1 = r_1$ and changes to 2, which proves the claim.

Now assume that the claim is true for $f_{-1}, \ldots, f_{-(i-1)}$. Recall that by Lemma 2.2 we have $f_{-i} = s_{i-1} s_i f_{-(i-1)} s_i s_{i-1}$. Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight. Applying s_{i-1} to b changes all unbracketed $i-1$ in the $(i-1, i)$-bracketing to i. Subsequently applying s_i changes all unbracketed i in the $(i, i+1)$-bracketing to $i+1$. It is not hard to see that the resulting word is $\{1, \ldots, i-1\}$-highest weight, so we can apply the inductive hypothesis in order to apply $f_{-(i-1)}$.

In the notation for Proposition 2.9, we have either $\mathrm{wt}(b)_{i+1} = 0$ or $q_i < p_{i+1}$ and $q_{i-1} < p_i$ since $\varphi_{-i}(b) = 1$. In particular this means that if p_{i+1} is defined and $p_{i+1} < q_{i-1}$, no letter i lies between p_{i+1} and q_{i-1} since otherwise $p_i < q_{i-1}$ contradicting the requirement $q_{i-1} < p_i$. This implies that all $i-1$ and i in the positions to the left of position q_{i-1} become $i+1$ when applying $s_i s_{i-1}$. The letter $i-1$ in position q_{i-1} remains

$i-1$ under $s_i s_{i-1}$ since it is bracketed with an i. Denote the sequences for $f_{-(i-1)}$ in $s_i s_{i-1} b$ by $q'_{i-1}, \dots, q'_1$ and $r'_{i-1}, \dots, r'_1$ and call k' the maximal index such that $q'_{k'} = r'_{k'}$. By the above arguments, we have $q'_{i-1} = q_{i-1}$. We need to distinguish three cases given by $k = i, i-1$ and $k < i-1$.

Case $k = i$: The claim is that the i in position q_i changes to $i+1$. Since $q_i = r_i$ for $k = i$, there is only one i to the left of the $i-1$ in position r_{i-1}. Since $q_{i-1} \leqslant r_{i-1}$, this implies that all $i-1$ between positions q_{i-1} and r_{i-1} (and including r_{i-1}) change to $i+1$ when applying $s_i s_{i-1}$. This means that $k' = i-1$ and by induction $f_{-(i-1)}$ changes the $i-1$ in position q_{i-1} to i. Hence under $s_{i-1} s_i$, the letter in position q_i remains an $i+1$ and all other letters $i+1$ and i return to their original value. This proves the claim.

Case $k = i-1$: In this case, we have at least two i to the left of position $q_{i-1} = r_{i-1}$ and there is no $i-1$ between positions q_{i-1} and $r_{i-2} \geqslant q_{i-2}$. Since $s_i s_{i-1}$ lifts all i to the left of position q_{i-1} to $i+1$, but leaves the $i-1$ in position q_{i-1} and possible $i-2$ in positions q_{i-2} and r_{i-2}, we have $k' = i-1$. Hence by induction $f_{-(i-1)}$ changes the $i-1$ in position $q'_{i-1} = q_{i-1}$ to i. When applying $s_{i-1} s_i$ to $f_{-(i-1)} s_i s_{i-1} b$, the $i+1$ in position r_i remains an $i+1$ since it is now bracketed with the i in position q_{i-1} or an i to its left. In addition, the $i+1$ in position q_i becomes an $i-1$ since the i in position q_{i-1} is now bracketed with the previous bracketing partner of letter in position q_i in b, causing it to drop to $i-1$. This proves the claim for $k = i-1$.

Case $k < i-1$: In this case $q_i < r_i$ and $q_{i-1} < r_{i-1}$, so that there are at least two i to the left of position r_{i-1} and at least two $i-1$ between positions q_i and $r_{i-2} \geqslant q_{i-2}$. By the arguments above, all i to the left of position q_{i-1} become $i+1$ under $s_i s_{i-1}$, the letter $i-1$ in position q_{i-1} remains $i-1$ and $q'_{i-1} = q_{i-1} < r'_{i-1} \leqslant r_{i-1}$. Also, since $s_i s_{i-1}$ leaves all letters $i-2$ and smaller untouched, we have $q'_j = q_j$ and $r'_j = r_j$ for $1 \leqslant j < i-1$. Hence by induction $f_{-(i-1)}$ changes the letter in position $q_{i-1} = q'_{i-1}$ to $i-2$ and the letter in position r'_{i-1} to i, in addition to the letters in positions q_j, r_j for $j < i-1$. Next applying $s_{i-1} s_i$ changes the letter in position r_{i-1} to i since it is now bracketed with the $i-1$ in position r_{i-2}. The letters $i+1$ in positions $r'_{i-1} < p < r_{i-1}$ are changed back to $i-1$ since they are not bracketed. If $r'_{i-1} < r_{i-1}$, then the letter i in position r'_{i-1} changes to $i-1$ since it is also not bracketed. The letter in position $q_{i-1} = q'_{i-1}$ remains $i-2$. The letter $i+1$ in position r_i is bracketed with the i in position r'_{i-1} in $f_{-(i-1)} s_i s_{i-1} b$ and hence remains $i+1$ in $s_{i-1} s_i f_{-(i-1)} s_i s_{i-1} b$. The letters $i+1$ between positions q_i and r_i in $f_{-(i-1)} s_i s_{i-1} b$ return to their original value i under $s_{i-1} s_i$ since they are bracketed with $i-1$ to the right. The letter in position q_i lost its bracketing partner since the $i-1$ in position q_{i-1} became $i-2$. Hence the letter in position q_i becomes $i-1$, proving the claim. $\square$

Corollary 2.14. *Let $b \in \mathcal{B}^{\otimes \ell}$ be J-highest weight for $\{1, 2, \dots, i\} \subseteq J \subseteq I_0$ and $\varphi_{-i}(b) = 1$ for some $i \in I_0$. Then:*

(1) *Either $f_{-i}(b) = f_i(b)$ or $f_{-i}(b)$ is J-highest weight.*

(2) *$f_{-i}(b)$ is I_0-highest weight only if $b = f_{i+1}f_{i+2}\cdots f_{h-1}u$ for some $i < h \leqslant n+1$ and u a I_0-highest weight element.*

Proof. We begin by proving (1). By Theorem 2.12, in $f_{-i}(b)$ the letters b_{q_j} are changed from j to $j-1$ for $j = i, i-1, \ldots, k+1$ and b_{r_j} are changed from j to $j+1$ for $j = i, i-1, \ldots, k$. Hence $f_{-i}(b)$ is not J-highest weight if and only if either there is an $i+1$ to the left of position q_i that is no longer bracketed with an i or the letter $k+1$ in position r_k is no longer bracketed with a k.

First assume that $k < i$. Since k is maximal such that $q_k = r_k$, there must be at least two $k+1$ to the left of position q_k in b, one in position q_{k+1} and one in position r_{k+1}. Since b is J-highest weight, both of these $k+1$ must be bracketed with a k to their right in b, which implies that there is a k to the right of position q_k that is bracketed with the $k+1$ in position q_{k+1} in b. In $f_{-i}(b)$, the letter $k+1$ in position q_{k+1} changes to k, and hence the new $k+1$ in position $q_k = r_k$ is bracketed with the k to its right.

Since by assumption $\varphi_{-i}(b) = 1$, we have by Proposition 2.9 that either $\mathrm{wt}(b)_{i+1} = 0$ (in which case there cannot be an $i+1$ to the left of position q_i in b) or $p_j \neq q_j$ for all $j \in \{1, 2, \ldots, i\}$. The condition $p_i \neq q_i$ implies that $q_i < p_{i+1}$, so that there cannot be a letter $i+1$ to the left of position q_i. This proves that $f_{-i}(b)$ is J-highest weight when $k < i$.

Next assume that $k = i$. In this case $f_{-i}(b)$ differs from b by changing the letter i in position q_i to $i+1$. If there is a letter i to the right of position q_i that is not bracketed with a letter $i+1$, then the new $i+1$ in position q_i will bracket with this i in $f_{-i}(b)$ (or to the left of this i) and hence $f_{-i}(b)$ is J-highest weight. Otherwise, there is no letter i to the right of position q_i in b that is not bracketed with an $i+1$ and therefore $f_i(b) = f_{-i}(b)$. This proves claim (1).

The above arguments also show that $f_{-i}(b)$ can only be I_0-highest weight if either b is I_0-highest weight or $\varepsilon_j(b) = 0$ for $j \in I_0 \setminus \{i+1\}$ and the new letter $i+1$ in position r_i in $f_{-i}(b)$ is bracketed with a letter $i+2$ in b. Such a b is precisely of the form $b = f_{i+1}f_{i+2}\cdots f_{h-1}u$ proving claim (2). □

Next, we describe e_{-i} on a $\{1, 2, \ldots, i\}$-highest weight element b. We again use the initial $(i+1)$-sequence $b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_1}$ in b.

We also need the notion of *cyclically scanning leftwards* for a letter t starting at an entry b_j. By this we mean choosing the rightmost t to the left of b_j, if it exists, or else the rightmost t in the entire word (i.e., "wrapping around" the edge of the word).

We define the *k-bracketed entries* of a word b as follows. Every k in b is k-bracketed, and for $j = k-1, k-2, \ldots, 1$, we recursively determine which j's in b are k-bracketed by considering the subword of only the k-bracketed $(j+1)$'s and all j's, and performing an ordinary crystal bracketing on this subword. The j's that are bracketed in this process are the k-bracketed j's.

Example 2.15. In the word

$$142334122311322111,$$

to obtain the 4-bracketed letters we first mark all 4's as 4-bracketed:

1**4**233**4**122311322111

and then bracket these with 3's and mark the bracketed 3's as being 4-bracketed:

1**4**2**3**3**4**122**3**11322111.

We then consider only the boldface 3's and all the 2's and bracket them to obtain the 4-bracketed 2's:

1**4**2**3**3**4**1**2**2**3**113**2**2111

Finally we bracket these boldface 2's with the 1's to obtain:

1**4**2**3**3**4**1**2**2**31**13**2**2**1**11

The boldface letters above are precisely the 4-bracketed letters in this word.

We now have the tools to describe the application of e_{-i} to an $\{1, 2, \dots, i\}$-highest weight word.

Theorem 2.16. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight for $i \in I_0$ and $\varepsilon_{-i}(b) = 1$ (see Proposition 2.9). Let $b_{p_{i+1}}, \dots, b_{p_1}$ be the initial $(i+1)$-sequence of b. Then $e_{-i}(b)$ is obtained from b by the following algorithm:*

- *Change b_{p_j} from j to $j-1$ for $j = i+1, i, \dots, 3, 2$ to form a word $c^{(1)}$.*
- *Cyclically scan left in $c^{(1)}$ starting just to the left of position p_1 for a 1 that is not i-bracketed in $c^{(1)}$. Change that 1 to 2 to form a word $c^{(2)}$. In $c^{(2)}$, continue cyclically scanning from just to the left of the previously changed entry for a 2 that is not i-bracketed in $c^{(2)}$, and change it to 3. Continue this process until an $i-1$ changes into an i; the resulting word $c^{(i)}$ is $e_{-i}(b)$.*

Proof. We will prove this by induction on i. For $i = 1$ the algorithm simply changes the leftmost 2 to a 1 as required, since the second step is vacuous in this case.

Assume the statement is true for i and let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i+1\}$-highest weight. Recall that $e_{-(i+1)} = s_i s_{i+1} e_{-i} s_{i+1} s_i$ by Lemma 2.2. We will analyze each step of applying $s_i s_{i+1} e_{-i} s_{i+1} s_i$ to b and show that it matches the desired algorithm.

Let $b_{p_{i+2}}, b_{p_{i+1}}, b_{p_i}, \dots, b_{p_2}, b_{p_1}$ be the initial $(i+2)$-sequence of b. Since $e_i b = 0$, applying s_i to b simply changes all unbracketed i entries in the $(i, i+1)$-pairing to $i+1$.

Note that b_{p_i} itself must be bracketed with an $i+1$ in b, for if it is not then $b_{p_{i+1}}$ is paired with an earlier i to its right, contradicting the definition of b_{p_i}. Thus b_{p_i} is still i in s_ib. Note also that s_ib still satisfies $e_{i+1}s_ib = 0$.

Let $b' = s_{i+1}s_ib$. Note that any $i+1$ to the left of $b_{p_{i+2}}$ in s_ib is not bracketed with an $i+2$ since $b_{p_{i+2}}$ is the leftmost $i+2$. Thus every $i+1$ left of $b_{p_{i+2}}$ (including those i's that changed to $i+1$ from b) changes to $i+2$ to form b', along with any other unpaired $i+1$. Let $b_{t_{i+1}}$ be the leftmost $i+1$ between $b_{p_{i+2}}$ and $b_{p_{i+1}}$ in s_ib. Then $b_{t_{i+1}}$ is either equal to $b_{p_{i+1}}$ or was an i in b. Furthermore, $b_{t_{i+1}}$ is still $i+1$ in $b' = s_{i+1}s_ib$ since it must be paired with either $b_{p_{i+2}}$ itself or some $i+2$ to the right of $b_{p_{i+2}}$.

Now consider $e_{-i}b'$. By the induction hypothesis, this can be computed by first lowering the entries of the initial $(i+1)$-sequence $b'_{p'_{i+1}}, b'_{p'_i}, \ldots, b'_{p'_1}$ appropriately to form a word $c'^{(1)}$, then cyclically raising some non-i-bracketed entries $1, 2, 3, \ldots, i-1$ in order to form words $c'^{(2)}, \ldots, c'^{(i)}$. We will show that $p'_j = p_j$ for $j \leqslant i$, and that the same entries $1, 2, \ldots, i-1$ are changed as would be changed in the $e_{-(i+1)}$ algorithm applied to b.

For the first claim, it suffices to show that $p'_i = p_i$. Note that $b'_{p'_{i+1}}$ may be to the left of $b_{p_{i+1}}$, but it is to the right of $b_{p_{i+2}}$ by the above analysis. If $p'_{i+1} = p_{i+1}$ we are done, so suppose $p_{i+2} < p'_{i+1} < p_{i+1}$. Assume by contradiction that there is an entry $b'_a = i$ between positions p'_{i+1} and p_i in b'. Then we further have $p'_{i+1} < a < p_{i+1}$ by the definition of b_{p_i} and b'. It follows that b_a is an i in b that is bracketed with an $i+1$, since applying s_i kept it an i. But then by the definition of p_{i+1}, the entry $b_c = i+1$ that brackets with b_a in b is to the left of position p_{i+2}. Thus $b_{p'_{i+1}}$ itself was a bracketed i in b, a contradiction. Thus $p'_i = p_i$.

Let $c^{(j)}$ be the word in the definition of $e_{-(i+1)}$ acting on b and $c'^{(j)}$ the word in the definition of e_{-i} on b'. Similarly, let t_j (resp. t'_j) be the position of the chosen j in $c^{(j)}$ (resp. $c'^{(j)}$) that is raised to $j+1$. We now wish to show that, for any $j \leqslant i-1$, we have $t'_j = t_j$.

We first show this for $j = 1$. Note that since $p_2 = p'_2$ (assuming $i \geqslant 2$, since otherwise we are done) the same entries are equal to 1 in both $c = c^{(1)}$ and $c' = c'^{(1)}$. Moreover, $p_1 = p'_1$, so we start searching cyclically left for a 1 in the same position in both. It therefore suffices to show that an entry $c_x = 1$ is $(i+1)$-bracketed in c if and only if $c'_x = 1$ is i-bracketed in c'. Note that the i's in c that are bracketed with $i+1$'s are precisely either:

- $c_{p'_{i+1}}$, or
- an i that was bracketed with an $i+1$ in b.

But since c' is formed by applying s_i to b (which changes all unbracketed i's to $i+1$'s), then s_{i+1} (which does not change any i's), then lowering certain entries, where $b_{p'_{i+1}}$ is the only one that becomes a new i, the above characterization gives precisely *all* i's in c'. Since the $1, 2, \ldots, i-1$ entries are the same in both c and c', it follows that an entry is $(i+1)$-bracketed in c if and only if it is i-bracketed in c'.

It now follows that $t_1 = t'_1$, and inductively we can conclude that $t_j = t'_j$ for all $j \leqslant i-1$. Thus if we apply $s_i s_{i+1}$ to $c'^{(i)}$ to obtain $e_{-(i+1)}b$, the entries less than or equal to $i-1$ match those of $c^{(i+1)}$, the result of the algorithm applied to b. Furthermore, since s_i, s_{i+1}, and e_{-i} only change letters less than or equal to $i+2$, the entries larger than $i+2$ also match.

It remains to consider the entries equal to i, $i+1$, and $i+2$. For $i+2$, the application of s_{i+1} to $s_i b$ changes all unbracketed $i+1$ entries in $s_i b$ to $i+2$, and e_{-i} changes the single entry $b'_{p'_{i+1}} = i+1$ to i and otherwise does not affect the $i+1$ or $i+2$ entries. In the $(i+1, i+2)$-bracketing in b', $b'_{p_{i+2}}$ is the leftmost bracketed $i+2$, and $b'_{p'_{i+1}}$ is the first $i+1$ after it, so removing $b'_{p'_{i+1}}$ from the $(i+1, i+2)$-subword leaves the $i+2$ in position p_{i+2} unbracketed, with all other bracketed $(i+2)$'s remaining bracketed. It follows that applying s_{i+1} to $e_{-i}s_{i+1}s_i b$ lowers the $i+2$ in position p_{i+2} to $i+1$, along with any $i+2$ that was raised in the first s_{i+1} step. Therefore, the $i+2$ entries in $s_{i+1}e_{-i}b'$, and hence in $s_i s_{i+1} e_{-i} b' = e_{-(i+1)}b$, match those in the output of the algorithm.

Finally, we consider the $(i, i+1)$-subwords of the words in question. We first analyze how the $(i, i+1)$-subword of $w := s_i b$ differs from that of $w' := s_{i+1}e_{-i}s_{i+1}s_i b$. By inspecting the above analysis, we see that w' differs from w in the following four ways:

- $w'_{p_{i+2}} = i+1$ is a new $i+1$ in the $(i, i+1)$-subword in w' whereas $w_{p_{i+2}} = i+2$ was not in the subword in w.
- $w'_{p'_{i+1}} = i$ whereas $w_{p'_{(i+1)}} = i+1$.
- $w'_{p_i} = i-1$ is no longer in the subword whereas $w_{p_i} = i$ was an i in the subword.
- $w'_{t_{i-1}} = i$ is a new i in the subword, whereas $w_{t_{i-1}} = i-1$.

Note that the last two items above may coincide and cancel each other out if $t_{i-1} = p_i$.

We now apply s_i to both subwords, and analyze how $s_i w' = e_{-(i+1)}b$ differs from $s_i w = b$ in the $(i, i+1)$-subword. In particular, we will show it is the same as how $c^{(i+1)}$ differs from b. Note that the $(i, i+1)$-subword in $c^{(i+1)}$ is formed from that of b by making the following changes:

- A new $i+1$ is inserted in position p_{i+2} ($b_{p_{i+2}} = i+2$ whereas $c^{(i+1)}_{p_{i+2}} = i+1$).
- The $i+1$ in position p_{i+1} is lowered to i.
- The i in position p_i is removed.
- An i is inserted in position t_{i-1}.
- In the current subword, look for the first unbracketed i cyclically left of position t_{i-1}; call this position t_i and change this i to $i+1$.

First, note that there are no $i+1$ entries between $w'_{p_{i+2}} = i+1$ and $w'_{p'_{i+1}} = i$ in w', for if there were, this would contradict the definition of $b_{p_{i+1}}$. It follows that $w'_{p_{i+2}} = i+1$ is bracketed with an i to its right in w', so in $s_i w' = e_{-(i+1)}b$, the entry in position p_{i+2}

remains $i+1$. So this is one position in which it differs from b, since $b_{p_{i+2}} = i+2$, so it matches $c^{(i+1)}$ in this position.

Note also that in w, all i's are bracketed with $(i+1)$'s. Applying s_i to w simply changes the unbracketed $i+1$'s back to i's to form b. We now consider two cases.

Case 1: Suppose $p'_{i+1} \neq p_{i+1}$.

We know that $s_i w$ and $s_i w'$ match b and $c^{(i+1)}$, respectively, in position p_{i+2} by the above analysis. For position p'_{i+1}, note that it is an unbracketed $i+1$ in w, so it changes to i in $s_i w$. It is a bracketed i in w' since it was the first unbracketed $i+1$ to the right of position p_{i+1} in w, so it stays i in $s_i w'$. Thus they are both equal to i in the results, matching b and $c^{(i+1)}$, which do not differ in this entry.

We now wish to show that the $i+1$ in position p_{i+1} is unbracketed in w' unless it is bracketed via the insertion of the i in position t_{i-1}. In other words, if we make all the changes that define w' from w besides the i in position t_{i-1}, we claim that position p_{i+1} is an unbracketed $i+1$. Indeed, before removing i in position p_i, this $i+1$ in position p_{i+1} is the leftmost $i+1$ that is bracketed with an entry weakly right of position p_i, since the position p_{i+2} entry is bracketed with some i weakly left of position p'_{i+1}. It follows that removing the i in position p_i leaves $b_{p_{i+1}}$ unbracketed, and otherwise all other $i+1$'s are bracketed if and only if they are bracketed in w.

Furthermore, the combination of lowering both p_{i+2} and p'_{i+1} to $i+1$ and i and removing the i in position p_i leaves all i's still bracketed, as they are in w.

Finally, when we put back the new i in position t_{i-1} to form w', there are two subcases: first suppose inserting this i makes some unbracketed $i+1$ to its left become bracketed. Then by the above analysis, this must have been the position of the first unbracketed i in $c^{(i)}$ to the left of t_{i-1}, and this is position t_i, which remains $i+1$ in $s_i w'$. Applying s_i to w' then turns the remaining unbracketed $i+1$ entries back to i and matches $c^{(i+1)}$. Otherwise, if inserting the i in position t_{i-1} does not bracket any $i+1$ to the left, it creates an unbracketed i in the word, and so the rightmost unbracketed $i+1$ also will not change under applying s_i to w'. This corresponds to the first unbracketed i cyclically left of position t_{i-1} in $c^{(i)}$, and we are done as before.

Case 2: Suppose $p'_{i+1} = p_{i+1}$.

In this case, the analysis matches the above except for the following steps: first, since position p_{i+1} contains a bracketed $i+1$ in w, lowering it to i may make some i to its right become unbracketed. (The new i in position p_{i+1} itself is bracketed due to the new $i+1$ in position p_{i+2} as before.)

Then, removing the i in position p_i will make all i's bracketed once again, since b_{p_i} was the first i to the right of position p_{i+1} in b and hence in w. So once again, at the step before inserting t_{i-1}, all i's are bracketed, and an $i+1$ in that matches one in w is bracketed if and only if it is bracketed in the modified word. Thus inserting t_{i-1} has the same effect as above, and we are done. $\square$

We now show that the output of e_{-i} on a $\{1, 2, \ldots, i\}$-highest weight element is itself $\{1, 2, \ldots, i\}$-highest weight if and only if there is no "cycling around the edge" in the cycling step of Theorem 2.16.

Proposition 2.17. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$, with $\varepsilon_{-i}(b) = 1$. Let $t_1, \ldots, t_{i-1}$ be the positions of the $1, 2, \ldots, i-1$ that change to $2, 3, \ldots, i$ respectively in the second step of the computation of $e_{-i}(b)$ (see Theorem 2.16). Also define $t_0 = p_1$. Then $e_{-i}(b)$ is $\{1, 2, \ldots, i\}$-highest weight if and only if $t_{i-1} < t_{i-2} < \cdots < t_1 < t_0$.*

Proof. First, suppose that it is not the case that $t_{i-1} < t_{i-2} < \cdots < t_1$; let $1 \leqslant k < i$ be the smallest index for which $t_{k-1} \leqslant t_k$, where $t_0 = p_1$. Then in the algorithm for computing $e_{-i}(b)$, after changing a $k-1$ to k in position t_{k-1}, we search cyclically left for a k that is not i-bracketed to find position t_k. Since $t_{k-1} \leqslant t_k$, we cycle around the end of the word, so t_k is the position of the rightmost k that is not i-bracketed.

Any k to the right of t_k is i-bracketed, and we claim that the $k+1$'s that they bracket with in the i-bracketing are all to the right of position t_k as well. Indeed, if one such $k+1$ was to the left of t_k then it should bracket with the k in position t_k instead, a contradiction. Thus the suffix starting at position $t_k + 1$ has at least as many $k+1$'s as k's.

In particular, just after changing each b_{p_r} to $r-1$ in the first step of the algorithm, the resulting word c is still highest weight. It follows that, just after raising t_{k-1} to k, the resulting word is still $\{k\}$-highest weight. It follows that the suffix starting at position $t_k + 1$ at this step has exactly as many $k+1$'s as k's.

Now, if $t_{k+1} < t_k$, changing t_k to $k+1$ and then changing t_{k+1} to $k+2$ leaves the suffix starting at t_k being not $\{k\}$-highest weight in the final word. Thus we are done in this case.

Otherwise, suppose t_{k+1} also cycles, so that $t_{k+1} \geqslant t_k$ and t_{k+1} is the new position of the rightmost $k+1$ that is not i-bracketed after changing t_k to $k+1$. Changing t_{k+1} to $k+2$ could potentially make the word $\{k\}$-highest weight again. In fact, suppose for contradiction that, just after changing t_{k-1} to k, there were a $k+1$ between position t_{k-1} and t_k that makes its suffix not $\{k\}$-highest weight. Then some entry $k+1$ in position $p < t_k$ brackets with the k in position t_k, and since position t_k is not i-bracketed, this $k+1$ is not i-bracketed either. Thus after changing t_k to $k+1$, the $k+1$ in position p is still not i-bracketed and it would be picked up in the search for t_{k+1}, a contradiction to the assumption that $t_{k+1} \geqslant t_k$.

We now, however, can repeat the argument with t_{k+1} and the $(k+1, k+2)$-subword, and so on until we either reach the last step or a non-cycling step, say with index ℓ. At this point we conclude that the final word $e_{-i}(b)$ is not $\{\ell\}$-highest weight.

It follows that if $t_{k-1} \leqslant t_k$ for some k, then $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight.

For the converse, we wish to show that if $t_{i-1} < t_{i-2} < \cdots < t_1 < t_0$ then $e_{-i}(b)$ remains highest weight. Notice that by construction we must have $t_{k-1} \leqslant p_k$ for all $k \leqslant i$.

We first show that the $(1, 2)$-subword remains highest weight in $e_{-i}(b)$ if $t_2 < t_1$. If $i = 1$, then the first 2 simply changes to a 1 and so it is still $\{1\}$-highest weight. So suppose $i \geqslant 2$.

The changes that affect the $(1, 2)$-subword are that b_{p_3} changes from 3 to 2, b_{p_2} changes from 2 to 1, b_{t_1} changes from 1 to 2, and (if $i \geqslant 3$) b_{t_2} changes from 2 to 3. Note that after the first two of these changes, any suffix of the word starting between positions p_3 and p_2 has at least two more 1's than 2's (due to the change in b_{p_2} starting from a highest weight word) and any suffix starting weakly before position p_3 has at least one more 1 than 2.

If $i = 2$, b_{t_1} is an unbracketed 1, so the suffixes before it must in fact have at least two more 1's than 2's even if $t_1 < p_3$. Thus changing b_{t_1} to 2 leaves the word highest weight, and we are done in this case.

If $i \geqslant 3$, b_{t_1} is a 1 that is not i-bracketed to the left of b_{p_2}, and b_{t_2} is the first 2 that is not i-bracketed to the left of t_1 (and necessarily to the left of b_{p_3}). It follows that, after changing them to 2 and 3 respectively, the suffixes all have at least as many 1's as 2's except possibly those starting between position t_2 and t_1. Assume to the contrary that there is a suffix with more 2's than 1's starting between t_2 and t_1; the rightmost such starts at another entry $b_a = 2$ between t_2 and t_1, and this 2 must be i-bracketed by the definition of t_2. But then since b_{t_1} is not i-bracketed, b_a must be bracketed with a 1 between b_a and b_{t_1}; hence the suffix starting at b_a cannot have a higher difference between 2's and 1's than the suffix starting at b_{t_1} after its change, a contradiction. It follows that the $(1, 2)$-subword remains highest weight.

Now consider the $(k, k + 1)$-subword for some $k \leqslant i - 1$. This is changed by $b_{p_{k+2}}, b_{p_{k+1}}, b_{p_k}$ changing from $k + 2$ to $k + 1$, $k + 1$ to k, and k to $k - 1$ respectively, and then $b_{t_{k-1}}, b_{t_k}, b_{t_{k+1}}$ changing from $k - 1$ to k, k to $k + 1$, $k + 1$ to $k + 2$ respectively.

If we first change b_{p_k} to $k - 1$, then we have removed a k from the subword, but since there are no k entries between $b_{p_{k+1}}$ and b_{p_k}, the rightmost suffix that may become not highest weight for k starts at $b_{p_{k+1}}$ itself. Thus changing $b_{p_{k+1}}$ from $k+1$ to k afterwards keeps the $(k, k + 1)$-subword being $\{k\}$-highest weight, and in fact any suffix starting to the left of $b_{p_{k+1}}$ at this point has at least one more k than $k+1$. Finally if we change $b_{p_{k+2}}$ to $k + 1$, this adds a single $k + 1$ to any suffix starting left of this position, so again the word remains $\{k\}$-highest weight. Next, we change $b_{t_{k-1}}$ from $k - 1$ to k, which means any suffix starting left of t_{k-1} has at least one more k than $k + 1$. The argument for what happens after changing t_k and t_{k+1} now is identical to that of the $(1, 2)$-subword above.

Finally, consider the $(i, i + 1)$-subword. This is only affected by the changes to $b_{p_{i+1}}$, b_{p_i}, and $b_{t_{i-1}}$. The same argument as above shows that it stays $\{i\}$-highest weight after changing $b_{p_{i+1}}$ and b_{p_i}, and then changing $b_{t_{i-1}}$ to i certainly keeps it $\{i\}$-highest weight as well. This completes the proof. $\square$

From the above proof, we immediately obtain the following corollary.

Corollary 2.18. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight for $i \in I_0$, with $\varepsilon_{-i}(b) = 1$. Let $t_1, \dots, t_{i-1}$ be the positions of the $1, 2, \dots, i-1$ that change to $2, 3, \dots, i$ respectively in the second step of the computation of $e_{-i}(b)$ (see Theorem 2.16). Then if $e_{-i}(b)$ is not $\{1, 2, \dots, i\}$-highest weight, the smallest index ℓ for which $e_{-i}(b)$ is not $\{\ell\}$-highest weight is precisely the smallest index for which $t_{\ell-1} \leqslant t_\ell$ and $t_{\ell+1} < t_\ell$ (where the second inequality is assumed to be vacuously true if $\ell = i-1$).*

In other words, ℓ is the smallest index for which one needs to cycle to get from $t_{\ell-1}$ to t_ℓ, but one does not need to cycle to get from t_ℓ to $t_{\ell+1}$.

Proof. The proof of Lemma 2.17 shows that $e_{-i}(b)$ is not $\{\ell\}$-highest weight, and that it is $\{k\}$-highest weight for $k < \ell$ if $t_{k-1} \leqslant t_k \leqslant t_{k+1}$ (i.e., if t_k and t_{k+1} both cycle). □

Remark 2.19. For any word $v \in \mathcal{B}^{\otimes \ell}$, we may combine Proposition 2.9 and Theorem 2.16 in order to algorithmically determine the highest weight element in the connected component of the queer supercrystal containing v. In particular, we may first apply as many e_i operators as possible to obtain an I_0-highest weight word v', then apply Proposition 2.9 to determine whether there is an e_{-i} arrow that we may apply. We can then apply e_{-i} to v' using Theorem 2.16 and repeat this process on the new word, and so on until we have reached a highest weight word w for the queer supercrystal.

Since the operators e_{-i} and e_i determine graphs having unique highest weight elements in each connected component [9, Theorem 1.14], this process will always terminate at the highest weight word in a component. In particular, e_{-1} and e_i for $i \in \{1, 2, \dots, n\}$ were previously the only operators having a known direct combinatorial algorithm, which are not by themselves sufficient to detect the unique highest weight elements. The algorithm in Theorem 2.16 therefore allows us to bypass the computational difficulty of conjugating e_{-1} by s_{w_i}.

2.4. Relation among e_{-i}

The main result of this section is Proposition 2.24, which provides relations between e_{-i} that do and do not yield a $\{1, 2, \dots, i\}$-highest weight element when acting on an I_0-highest weight element. This proposition will be used in Section 4 to deal with “by-pass arrows” in the component graph $G(\mathcal{C})$.

We require several technical lemmas about k-bracketed entries and the e_{-i} operation on highest weight words.

Lemma 2.20. *Suppose $b \in \mathcal{B}^{\otimes \ell}$ is $\{1, 2, \dots, i\}$-highest weight and $1 \leqslant k \leqslant i$. If a letter $b_r = a$ in $b = b_1 b_2 \dots b_\ell$ is k-bracketed, then b_r is j-bracketed for all $a < j \leqslant k$.*

Proof. We first show that if an entry a in b is $(a+2)$-bracketed, then it is $(a+1)$-bracketed; for simplicity we set $a = 1$. Let v be the subword of b consisting of only the 2’s that are bracketed with a 3 along with all the 1’s, and let v' be the subword consisting of all the

1's and 2's. Then v' can be formed from v by inserting some 2 letters. It therefore suffices to show that any 1 that was bracketed in v is still bracketed after inserting a single 2.

Indeed, let $v_s = 2$ and $v_r = 1$ be a bracketed pair in v. Note that by the definition of the ordinary crystal bracketing rule, the subword $v_s \dots v_r$ has exactly the same number of 2's as 1's, all of them bracketed with some other letter in $v_s \dots v_r$. Therefore, if we insert a 2 to the left or right of this pair, then the pair (v_s, v_r) remains bracketed. If instead we insert it between v_s and v_r, then the interval between v_s and v_r contains strictly more 2's than 1's, and so there is some entry v_t between v_s and v_r for which the subword $v_t \cdots v_r$ is tied; in other words, v_r is now bracketed with some 2 to the right of v_s. Thus v_r stays bracketed after inserting a 2, as desired.

Now, if $b_r = a$ is k-bracketed, then by the above reasoning it is also $(k-1)$-bracketed, since there are weakly more $(k-1)$'s available in this bracketing, and hence weakly more $(k-2)$'s available, and so on. The conclusion follows by induction. □

Lemma 2.21. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight and $\varepsilon_{-i}(b) = 1$. Let $b_{p_{i+1}}, \dots, b_{p_1}$ be the initial $(i+1)$-sequence of b and c the word obtained by changing b_{p_j} from j to $j-1$. Let $k \leqslant i' \leqslant i$. If b contains a sequence of letters $k-1, k-2, \dots, 1$ before position p_1 that is not i'-bracketed, then c contains a sequence of letters $k-1, k-2, \dots, 1$ before position p_1 that is not i'-bracketed.*

Proof. Suppose that b contains a sequence S of letters $k-1, k-2, \dots, 1$ in positions $s_{k-1}, \dots, s_1$ respectively, before position p_1, that are not i'-bracketed; take S to be the rightmost such sequence in the sense that it contains the rightmost 1 left of p_1 that is not i'-bracketed, then the rightmost 2 that is not i'-bracketed before that, and so on. Note that $s_1 < p_1$ implies that $s_1 < p_2$ by the definition of p_1. Thus $s_2 < s_1 < p_2$ and so $s_2 < p_3$, and so on, showing that $s_j < p_{j+1}$ for all j. Also note that the initial $(i+1)$-sequence $b_{p_{i+1}}, \dots, b_{p_1}$ is $(i+1)$-bracketed, so that the letters $b_{p_k}, \dots, b_{p_1}$ must also be i'-bracketed by Lemma 2.20. Since $k \leqslant i' \leqslant i$, this means that the initial $(i+1)$-sequence is disjoint from S and hence S remains unchanged in c.

We now form a sequence S' from S that is not i'-bracketed in c as follows. Consider the largest entry $j \leqslant i'$ for which there exists a j between p_{j+2} and p_{j+1}. Then all bracketing with higher letters remains the same in c, but the letter j between positions p_{j+2} and p_{j+1} becomes bracketed with the letter $j+1$ in position p_{j+2} in the i'-bracketing in c, leaving the letter j in position p_{j+1} to be an i'-unbracketed j. If $s_j < p_{j+2}$ (or otherwise c_{s_j} does not become bracketed) we keep it in S', and if $p_{j+2} < s_j < p_{j+1}$ and it becomes bracketed, we replace s_j with the first i'-unbracketed position s'_j of a j in c to the right of s_j, to choose the j for S'.

We now show that we can choose a $j-1$ after this step to be in S'. If the j on the previous step did not change, then we repeat this process for $j-1$. If it did change, from s_j to an index s'_j, note that if $s_{j-1} < s'_j$ then the previous $j-1$ is now i'-bracketed with s_j in c as well, so we also have to choose the next $j-1$ to the right. Either way we replace s_{j-1} with the next i'-unbracketed $j-1$, in position s'_{j-1}, if the $j-1$ became bracketed, and we see that $s'_j < s'_{j-1}$. Furthermore, $s'_{j-1} \leqslant p_j$ since we know that p_j becomes an i'-unbracketed $j-1$ as in the case of j above. Continuing in this manner we can form a sequence S' of elements of c that are not i'-bracketed, all weakly to the left of p_2 (and hence strictly before p_1). □

Lemma 2.22. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight. Let k be the smallest index for which $t_{k-1} \leqslant t_k$, where $t_0 = p_1$ and t_j for $j = 1, \ldots, i-1$ are the indices that are raised in the second step of the computation of $e_{-i}(b)$ (such a k exists by Proposition 2.17). Then we have that $\varepsilon_{-k}(b) = 1$ and $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight.*

Proof. Let $b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_1}$ be the initial $(i+1)$-sequence, $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ be the initial i-sequence, $b_{p'_{k+1}}, \ldots, b_{p'_1}$ the initial $(k+1)$-sequence, and $b_{q'_k}, \ldots, b_{q'_1}$ the initial k-sequence of b. Also define c and c' respectively to be the words formed by lowering the entries in the sequences $\{b_{p_j}\}$ or $\{b_{p'_j}\}$ by one, respectively.

Since $\varepsilon_{-i}(b) > 0$, we have by Proposition 2.9 that $q_a = p_a$ for some $1 \leqslant a \leqslant i$. If a is maximal with this property, then in fact $q_j = p_j$ for all $j \leqslant a$ by the definition of the initial sequences. Assume by contradiction that $\varepsilon_{-k}(b) = 0$. Then again by Proposition 2.9, $q'_j < p'_j$ for all $j \in \{1, \ldots, k\}$. Furthermore, $p'_j \leqslant p_j$ for all $j \leqslant k$ so $q'_j < p_j$ as well.

Suppose that $q'_{a'} = q_{a'}$ for some $1 \leqslant a' \leqslant k$. Then $q'_j = q_j$ for all $j \leqslant a'$ and hence $q'_j = q_j = p_j$ for $j \leqslant \min(a, a')$, contradicting the fact that $q'_j < p_j$ for all j. Hence $q'_j < q_j$ for all $1 \leqslant j \leqslant k$. Thus we also have $q'_j < q_{j+1}$ for all $1 \leqslant j \leqslant k$, for otherwise $b_{q'_j}$ would be the first j after q_{j+1} and we would have $q'_j = q_j$.

The sequence of letters $k, k-1, \ldots, 1$ in positions $q'_k, \ldots, q'_1$ in b is not i-bracketed since the first bracketed $k+1$ in b must be weakly right of position $q_{k+1} > q'_k$. Hence by Lemma 2.21, the word c also contains a sequence $k, k-1, \ldots, 1$ of letters that are not i-bracketed before position p_1, contradicting the fact that $t_{k-1} \leqslant t_k$. It follows that $\varepsilon_{-k}(b) = 1$.

Next we show that $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight. Note that by the definition of the initial sequences $q'_j \leqslant p'_j \leqslant q_j \leqslant p_j$. Since $\varepsilon_{-i}(b) = 1$ and $\varepsilon_{-k}(b) = 1$, we also have $q'_j = p'_j$ for $j \leqslant a'$ and $q_j = p_j$ for $j \leqslant a$ for some a', a. Suppose $p'_j < q_j$ for all j. Then by a similar argument to that above, in the word c there exists a sequence of positions $t_k < t_{k-1} < \cdots < t_1 < t_0 = p_1$ such that $c_{t_j} = j$ which are not i-bracketed in c. This contradicts the fact that $t_{k-1} \leqslant t_k$. Hence we must have $p'_j = q_j$ for some j and hence $q'_j = p'_j = q_j = p_j$ for $j \leqslant x$ for some $x \geqslant 1$. We claim that $t_j < q'_j$ for all $1 \leqslant j < k$. Indeed, t_1 is to the left of position $p_1 = q'_1$, so that $t_1 < q'_1$. By the definition of p_1 we also cannot have $p_2 < t_1 < p_1$ so in fact $t_1 \leqslant p_2$. The letter in position $q'_j = p_j$ for $1 < j \leqslant x$ in c is $j-1$, so that also $t_j < q'_j$ for $1 < j \leqslant x$. For $j > x$, the letter in position $q'_j < p_j$ in c as well as in b is j. It is k-bracketed in c and b since the first letter k in c and b is in position q'_k. If $t_j \geqslant q'_j$ then since the sequence of entries q'_r for $r \geqslant j$ is k-bracketed but not i-bracketed, we would have $t_k < t_{k-1}$, a contradiction. Thus $t_j < q'_j$.

It follows that the t_j entries are not k-bracketed, so b contains a sequence $k-1, k-2, \ldots, 1$ that is not k-bracketed. By Lemma 2.21 this means that c' has a sequence $k-1, \ldots, 1$ in positions $t'_{k-1} < \cdots < t'_1$ that is not k-bracketed, proving that $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight by Proposition 2.17. □

For an element $b \in \mathcal{B}^{\otimes \ell}$, denote by $\uparrow b$ the unique I_0-highest weight element in the same component as b. The next lemma describes the action of $\uparrow$ after an application of e_{-i}.

Lemma 2.23. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight. Let k be as in Lemma 2.22 and let the sequences p_j and t_j be as in Theorem 2.16. Then $\uparrow e_{-i}(b)$ can be obtained from b by changing j in position p_j to $j-1$ for $1 < j \leqslant i+1$ and j in position t_j for $1 \leqslant j < k$ to $j+1$, and lowering some letters larger than $i+1$. In particular, the changes in positions t_j for $j \geqslant k$ in $e_{-i}(b)$ are undone by the application of $\uparrow$.*

Proof. By Corollary 2.18, the smallest index ℓ for which $e_\ell(e_{-i}(b))$ is defined is the first ℓ for which t_ℓ cycled but $t_{\ell+1}$ did not (or does not exist). In particular $\ell \geqslant k$ and all t_j with $k \leqslant j \leqslant \ell$ cycle around the end of the word.

Note that t_ℓ was chosen as the rightmost ℓ that is not i-bracketed (after raising $t_1, \ldots, t_{\ell-1}$). Also recall that the word c formed by lowering the b_{p_j} entries is $\{1, 2, \ldots, i\}$-highest weight, so just before changing t_ℓ the word is still $\{\ell\}$-highest weight. Finally, by assumption t_ℓ is weakly right of $t_{\ell-1}$ (which is the only new ℓ since starting at the word c). Thus, after changing t_ℓ to $\ell+1$, if it bracketed with an ℓ to its right (in the ordinary crystal bracketing) then in fact that ℓ is also not i-bracketed on the previous step, a contradiction since $t_{\ell-1} \leqslant t_\ell$.

Therefore t_ℓ is an unbracketed $\ell+1$ in $e_{-i}(b)$, and since all other $(\ell+1)$'s before it are bracketed with some ℓ, we know that e_ℓ changes it back to an ℓ. After doing so, by the same argument we see that position $t_{\ell-1}$ is now an unbracketed ℓ, so applying $e_{\ell-1}$ changes it back to $\ell-1$, and so on down to t_k. At this point the resulting word

$$w := e_k \cdots e_{\ell-1} e_\ell (e_{-i} b)$$

is $\{1, 2, \ldots, \ell\}$-highest weight, since t_{k-1} did not cycle and so changing t_k back to k leaves w highest weight at that step.

Now suppose $t_{\ell+1}$ exists (that is, $\ell \leqslant i-2$); then $t_{\ell+1} < t_\ell$, and in w the position t_ℓ is changed back to ℓ. We claim that $e_{\ell+1}$ is defined on w and applying it changes $t_{\ell+1}$ from $\ell+2$ back to $\ell+1$. Indeed, if $t_{\ell+1}$ is bracketed with an $\ell+1$ in w then this $\ell+1$ must be to the right of t_ℓ (since otherwise it would have been a preferred non-i-bracketed choice of $t_{\ell+1}$ in the e_{-i} algorithm). But then this $\ell+1$ is bracketed with an ℓ to its right since w is $\{\ell\}$-highest weight, and then this ℓ similarly contradicts the choice of t_ℓ. Thus $t_{\ell+1}$ is an $\ell+2$ that is not bracketed with an $\ell+1$ after lowering t_ℓ back to ℓ. By the weight changes it must be the only such $\ell+2$ and so applying $e_{\ell+1}$ changes $t_{\ell+1}$ back to $\ell+1$. Continuing in this fashion, we can apply $e_{\ell+2}, e_{\ell+3}$, and so on in that order to change the next entries $t_{\ell+2}$, $t_{\ell+3}$, and so on back to their original values, until some $t_{\ell+r}$ cycles again. Let t_m be the next entry for which t_{m+1} does not cycle (the end of the next block of cycling entries); by the same arguments as above we can now apply e_m, then e_{m-1}, and so on down to $e_{\ell+r}$. Repeating this process on every block of cycling and non-cycling entries yields a $\{1,\dots,i\}$-highest weight word formed by changing $t_k,\dots,t_{i-1}$ back to $k, k+1,\dots,i-1$ respectively. Finally, to finish forming $\uparrow e_{-i}(b)$, only entries larger than $i+1$ may be changed, and the conclusion follows. □

The next proposition will be used in Section 4 to deal with "by-pass arrows" in the component graph $G(\mathcal{C})$.

Proposition 2.24. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1,2,\dots,i\}$-highest weight. Then there exists $1 \leqslant k < i$ such that $\varepsilon_{-k}(b) = 1$, $e_{-k}(b)$ is $\{1,2,\dots,k\}$-highest weight and*

$$\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b) \quad \text{or} \quad \uparrow e_{-i}(b) = \uparrow e_{-k}(b). \tag{2.9}$$

Example 2.25. Take $b = 343212211 \in \mathcal{B}^{\otimes 9}$, which satisfies $\varepsilon_{-3}(b) > 0$. Then

$$\uparrow e_{-3}b = e_2 e_1 e_{-3} b = 332112211 = e_2 e_{-3} e_{-1} b = \uparrow e_{-3} \uparrow e_{-1} b.$$

Furthermore, $e_{-1}b = 343112211$ is $\{1\}$-highest weight.

Take $b = 4321321 \in \mathcal{B}^{\otimes 7}$, which satisfies $\varepsilon_{-3}(b) > 0$. Then

$$\uparrow e_{-3}b = e_1 e_2 e_{-3} b = 3211321 = e_{-3} e_2 e_{-1} b = \uparrow e_{-3} \uparrow e_{-1} b.$$

Furthermore, $e_{-1}b = 4311321$ is $\{1\}$-highest weight.

Take $b = 2154321 \in \mathcal{B}^{\otimes 7}$, which satisfies $\varepsilon_{-4}(b) > 0$. Then

$$\uparrow e_{-4}b = e_3 e_{-4} b = 3243211 = e_4 e_{-3} b = \uparrow e_{-3} b.$$

Proof of Proposition 2.24. Let k be as in Lemma 2.22. Then the first statements hold for k by Lemma 2.22 and it only remains to prove (2.9). By Lemma 2.23, $\uparrow e_{-i}b$ changes j in position p_j to $j-1$ for $1 < j \leqslant i+1$ and j in position t_j for $1 \leqslant j < k$ to $j+1$. The changes in positions t_j for $j \geqslant k$ in e_{-i} are undone by $\uparrow$. Some letters bigger than $i+1$ might also be lowered by $\uparrow$.

We use the same notation as in the proof of Lemma 2.22. There we proved that $t_j < q'_j$ for all $1 \leqslant j < k$. Since $q'_j \leqslant p_j$ and there is no letter j between positions p_{j+1} and p_j in b, it follows that $t_j \leqslant p_{j+1}$ for all $1 \leqslant j < k$. Now suppose that $t_j = p_{j+1}$ for some $1 \leqslant j < k$. We claim that then $t_{j-1} = p_j$ as well. Let $d-1$ be maximal such that $t_{d-1} = p_d$. Then there has to be a letter $d-1$ in position p in b with $p_{d+1} < p < p_d$, so that the letter $d-1$ in position p_d in c is not i-bracketed. Suppose that there is no letter $d-2$ between positions p and p_{d-1} in b. In this case the letter $d-2$ in position p_{d-1} in c is i-bracketed, so that $t_{d-2} > p_{d-1}$, which contradicts $t_{d-2} \leqslant p_{d-1}$. Continuing this argument, there has to be a sequence of letters $d-1, d-2, \ldots, 1$ between positions p_{d+1} and p_2 that is not i-bracketed. Moreover, letter j in this sequence has to appear before position p_{j+1}. But this means that the letter j in position p_{j+1} for $1 \leqslant j < d$ is not i-bracketed, so that $t_j = p_{j+1}$ for all $1 \leqslant j < d$.

By the arguments above, we have that $t_j = p_{j+1}$ for $1 \leqslant j < d$ for some d and t_j for $j \geqslant d$ is part of a sequence of non k-bracketed letters in b (by the definition of k and the sequence q'_j). Similarly, we have $t'_j = p'_{j+1}$ for $1 \leqslant j < d'$ for some d' and t'_j for $j \geqslant d'$ is part of the same sequence of non k-bracketed letters in b as t_j. Also, $d' \geqslant d$ since $p'_j \leqslant p_j$ for all $1 \leqslant j \leqslant k+1$. In particular, this implies $t_j = t'_j$ for $d' \leqslant j < k$.

Furthermore, before applying the $\uparrow$ operator the entries that change are:

$$\begin{aligned}
\text{In } \uparrow e_{-i}b: \quad & b_{p_j} : j \mapsto j-1 \quad \text{for } d < j \leqslant i+1 \\
& b_{t_j} : j \mapsto j+1 \quad \text{for } d \leqslant j < i \\
\text{In } \uparrow e_{-k}b: \quad & b_{p'_j} : j \mapsto j-1 \quad \text{for } d' < j \leqslant k+1 \\
& b_{t'_j} : j \mapsto j+1 \quad \text{for } d' \leqslant j < k.
\end{aligned}$$

Recall also that $p'_j = p_j$ for $1 \leqslant j \leqslant x$ for some $x \geqslant 1$. Denote by $\overline{t}_j$ and $\overline{p}_j$ the selected positions by e_{-i} on the element $\uparrow e_{-k}b$.

First assume that $x = k+1$, so that $p'_j = p_j$ for all $1 \leqslant j \leqslant k+1$. In this case $t'_j = t_j$ for $1 \leqslant j < k$. Furthermore, if in $e_{-k}(b)$ the letter $k+2$ in position p_{k+2} is unbracketed, then in $\uparrow e_{-k}(b)$, the letter $k+2$ in position p_{k+2}, then the letter $k+3$ in position p_{k+3} etc. will be lowered. These are the same changes as in $\uparrow e_{-i}(b)$, so that $\uparrow e_{-i}(b) = \uparrow e_{-k}(b)$.

Next assume that $d' < x \leqslant k$ or $x = k+1$ but the letter $k+2$ in position p_{k+2} in $e_{-k}(b)$ is bracketed. We first show that in this case $\overline{p}_j = p_j$ for $x < j \leqslant i+1$. Note that to form $\uparrow e_{-k}(b)$, since $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight, we apply $e_{k+1}, e_{k+2}, \ldots, e_r$ in order for some r, so that we lower a $k+2$ to a $k+1$, $k+3$ to $k+2$, and so on until we reach an I_0-highest weight word. Note also that $b_{p'_{k+1}}$ was the entry that lowered from $k+1$ to k, so the $k+2$ that gets lowered, if it exists, is to the left of $p'_{k+1} < p_{k+1}$. Similarly the $k+3$ that gets lowered is left of $p'_{k+2} < p_{k+2}$, and so on, and hence $r < i$ since p_{i+1} is the leftmost $i+1$. It follows that no $i+1$ lowers to an i, and so $\overline{p}_{i+1} = p_{i+1}$. Since the entries lowered by $\uparrow$ are left of p_j for each $j > x$, it follows that $\overline{p}_j = p_j$ for $x < j \leqslant i+1$.

For the sequence $\overline{t}_j$, note that the entries $\overline{p}_j$ that we lower for $j \leqslant x$ cannot be i-bracketed in $\overline{c}$ due to the condition $\overline{p}_{i+1} = p_{i+1}$ shown above, and because $t_{x-1} = t'_{x-1}$, so that t'_{x-1} cannot be between p_{x+1} and p_x. Furthermore, for $x \leqslant j < k$ the letters in positions $\overline{p}_{j+1}$ are all i-bracketed in $\overline{c}$ and $t_j = t'_j < p'_{j+1} < p_{j+1} = \overline{p}_{j+1}$. Also note that

$d = d'$ since $p_{j+1} = p'_{j+1} = t'_j$ for $d \leqslant j < d' < x$ and the letter j in position $p_{j+1} = p'_{j+1}$ in c' is not k-bracketed and hence not i-bracketed in c' and c. It follows that

$$\bar{t}_j = \begin{cases} \overline{p}_{j+1} & \text{for } 1 \leqslant j < x, \\ p'_{j+1} & \text{for } x \leqslant j \leqslant k, \end{cases}$$

and for $k < j \leqslant r$, we have that $\bar{t}_j$ is equal to the position of letter $j+1$ that is lowered when applying $\uparrow$ to $e_{-k}(b)$. Hence $\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b)$.

Finally, assume that $x \leqslant d'$. In this case, by a similar argument, we have $\overline{p}_j = p_j$ for $1 \leqslant j \leqslant i+1$ and

$$\bar{t}_j = \begin{cases} \overline{p}_{j+1} & \text{for } 1 \leqslant j < d, \\ t_j & \text{for } d \leqslant j < d', \\ p'_{j+1} & \text{for } d' \leqslant j \leqslant k, \end{cases}$$

and for $k < j \leqslant r$, we have that $\bar{t}_j$ is equal to the position of letter $j+1$ that is lowered when applying $\uparrow$ to $e_{-k}(b)$. Again, we have $\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b)$. $\square$

3. Local axioms

In [2, Definition 4.11], Assaf and Oguz give a definition of regular queer supercrystals. In essence, their axioms are rephrased in the following definition, where $\tilde{I} := I_0 \cup \{-1\}$.

Definition 3.1 *(*Local queer axioms*).* *Let $\mathcal{C}$ be a graph with labeled directed edges given by f_i for $i \in I_0$ and f_{-1}. If $b' = f_j b$ for $j \in \tilde{I}$ define e_j by $b = e_j b'$.*

LQ1. The subgraph with all vertices but only edges labeled by $i \in I_0$ is a type A_n Stembridge crystal.

LQ2. $\varphi_{-1}(b), \varepsilon_{-1}(b) \in \{0, 1\}$ for all $b \in \mathcal{C}$.

LQ3. $\varphi_{-1}(b) + \varepsilon_{-1}(b) > 0$ if $\mathrm{wt}(b)_1 + \mathrm{wt}(b)_2 > 0$.

LQ4. Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.

(a) If $\varphi_1(b) > 2$, we have

$$\begin{aligned} f_1 f_{-1}(b) &= f_{-1} f_1(b), \\ \varphi_1(b) &= \varphi_1(f_{-1}(b)) + 2, \\ \varepsilon_1(b) &= \varepsilon_1(f_{-1}(b)). \end{aligned}$$

(b) If $\varphi_1(b) = 1$, we have

$$f_1(b) = f_{-1}(b).$$

LQ5. Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.

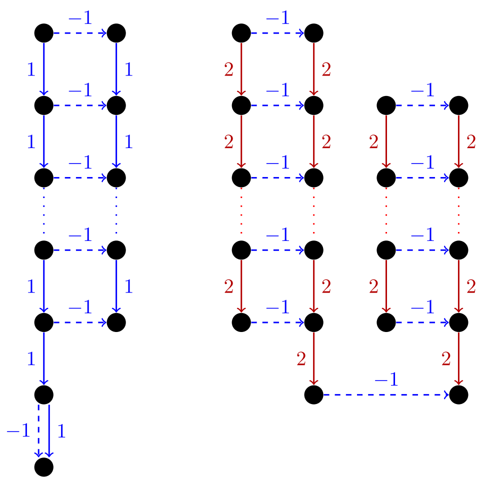


Fig. 2. Illustration of axioms **LQ4** (left) and **LQ5** (right). The (−1)-arrow at the bottom of the right figure might or might not be there.

(a) If $\varphi_2(b) > 0$, we have

$$\begin{aligned} f_2 f_{-1}(b) &= f_{-1} f_2(b), \\ \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)). \end{aligned}$$

(b) If $\varphi_2(b) = 0$, we have

$$\begin{aligned} \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1 = 0, \text{ or} \quad & \varphi_2(b) &= \varphi_2(f_{-1}(b)) = 0, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)), & \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)) + 1. \end{aligned}$$

LQ6. Assume that $\varphi_{-1}(b) = 1$ and $\varphi_i(b) > 0$ with $i \geqslant 3$ for $b \in \mathcal{C}$. Then

$$\begin{aligned} f_i f_{-1}(b) &= f_{-1} f_i(b), \\ \varphi_i(b) &= \varphi_i(f_{-1}(b)), \\ \varepsilon_i(b) &= \varepsilon_i(f_{-1}(b)). \end{aligned}$$

Axioms **LQ4** and **LQ5** are illustrated in Fig. 2.

Proposition 3.2 *([2]). The queer supercrystal of words $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 3.1.*

Proof. **LQ1** follows by definition. **LQ2** and **LQ3** follow from Remark 2.4. **LQ4** follows from Lemma 2.5 and **LQ5** follows from Lemma 2.6. Finally, **LQ6** is **Q4**. □

In [2, Conjecture 4.16], Assaf and Oguz conjecture that every regular queer supercrystal is a normal queer supercrystal. In other words, every connected graph satisfying the local queer axioms of Definition 3.1 is isomorphic to a connected component in some $\mathcal{B}^{\otimes \ell}$. We provide a counterexample to this claim in Fig. 3. In the figure, the I_0-components of the $\mathfrak{q}(3)$-crystal of highest weight $(4,2,0)$ are shown. Some of the f_{-1}-arrows are drawn in green. The remaining arrows can be filled in using the axioms of Fig. 2 in a consistent manner. If the dashed green arrow from 331131 to 332131 and the dashed green arrow from 331132 to 332132 are replaced by the dashed (double-headed) purple arrow from 331131 to 331231 and the dashed (double-headed) purple arrow from 331132 to 332231, respectively, all axioms of Definition 3.1 are still satisfied with the remaining f_{-1}-arrows filled in. However, the I_0-component with highest weight element 132121 has become disconnected and hence the two crystals are not isomorphic.

The problem with Axiom **LQ5** illustrated in Fig. 2 is that the (-1)-arrow at the bottom of the 2-strings is not closed at the top. Hence, as demonstrated by the counterexample in Fig. 3 switching components with the same I_0-highest weights can cause non-uniqueness. In fact, if $f_{-1}b$ is determined for all $b \in \mathcal{C}$ such that

$$\varphi_i(b) = 0 \quad \text{for all } i \in I_0 \setminus \{1\} \text{ and } \quad \varphi_1(b) = 2, \tag{3.1}$$

then, by the relations between f_{-1} and f_i for $i \in I_0$ of Definition 3.1, f_{-1} is determined on all elements in $\mathcal{C}$. Namely, f_i and f_{-1} commute for $i \neq 1, 2$, so that it is enough to consider $f_{-1}b$ when $\varphi_i(b) = 0$. Similarly, by the right picture in Fig. 2, once $f_{-1}b$ is determined for b with $\varphi_2(b) = 0$, which are the elements at the bottom of the 2-strings, then $f_{-1}c$ is determined for all c in this picture. And finally, if $f_{-1}b$ is determined for b with $\varphi_1(b) = 2$, which is the element at height 2 in the left picture of Fig. 2, then f_{-1} is determined on all elements above this b. Furthermore, $f_{-1}(c) = f_1(c)$ when $\varphi_1(c) = 1$. Hence the conditions in (3.1) are indeed enough.

Lemma 3.3. *Let $v \in \mathcal{B}^{\otimes \ell}$ be an I_0-lowest weight element, that is, $\varphi_i(v) = 0$ for all $i \in I_0$. Then every $b \in \mathcal{B}^{\otimes \ell}$ satisfying (3.1) is of the form*

$$g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v \qquad \text{for some } 1 \leqslant j \leqslant k \leqslant n. \tag{3.2}$$

Conversely, every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$ satisfies (3.1).

Proof. The statement of the lemma is a statement about type A_n crystals and hence can be verified by the tableaux model for type A_n crystals (see for example [4]). The element v is I_0-lowest weight and hence as a tableau in French notation contains the letter $n+1$ at the top of each column, the letter n in the second to top box in each column, and in general the letter $n+2-i$ in the i-th box from the top in its column. If there is a letter $k+1$ in the first row of v, then $(e_1 \cdots e_k)$ applies to v and $b' = (e_1 \cdots e_k)v$ satisfies $\varphi_i(b') = 0$ for $i \in I_0 \setminus \{1\}$ and $\varphi_1(b') = 1$. The element b' has several changed entries in the first row, and otherwise the entries above the first row all have letter $n+2-i$

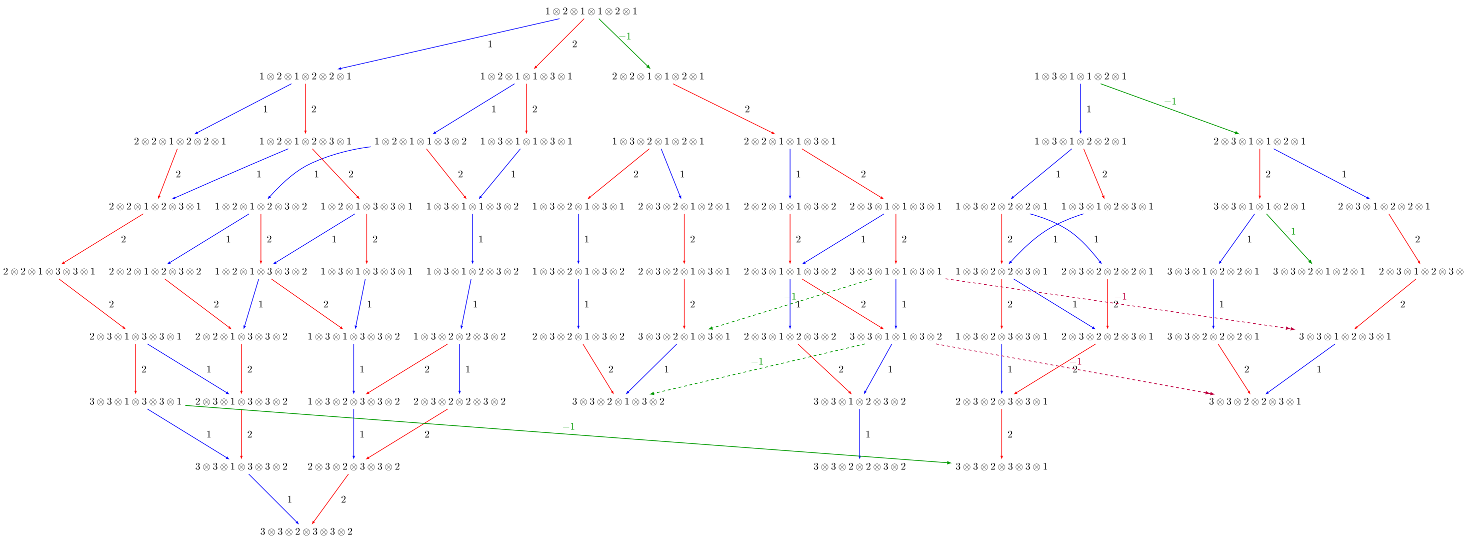


Fig. 3. Counterexample to the unique characterization of the local queer axioms of Definition 3.1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

in the i-th box from the top in their column. If b' has a letter $j+1$ in the first row with $1 \leqslant j \leqslant k$, then $(e_1 \cdots e_j)$ applies to b' and $b = g_{j,k} = (e_1 \cdots e_j)b'$ satisfies (3.1). Note that if $j > k$, then the last e_1 would no longer apply and hence $b = 0$. This proves that $g_{j,k} \neq 0$ as in (3.2) satisfies (3.1). If conversely b satisfies (3.1), then as a tableau it contains two extra 1's in the first row that have a 3 or bigger above them rather than a 2 in their columns, and for entries higher than the first row the i-th box from the top in its column contains $n+2-i$. It is not hard to check that then $(f_k \cdots f_1)(f_j \cdots f_1)b = v$ for some $1 \leqslant j \leqslant k \leqslant n$. Hence b is of the form (3.2). □

In the next section, we introduce a new graph just on I_0-highest weight elements and new connectivity axioms (see Definition 4.4) that uniquely characterizes queer supercrystals (see Theorem 5.1).

4. Graph on type A components

Let $\mathcal{C}$ be an abstract $\mathfrak{q}(n+1)$-crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. In this section, we define a graph for $\mathcal{C}$ labeled by the type A_n components of $\mathcal{C}$. We draw an edge from vertex C_1 to vertex C_2 in this graph if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that $f_{-1}b_1 = b_2$. We provide an easy combinatorial way to describe this graph for a queer supercrystal which is a subcrystal of the crystal of words leveraging the explicit actions of f_{-i} described in Theorem 2.12 and e_{-i} described in Theorem 2.16, respectively (see Theorem 4.9). We also provide new axioms in Definition 4.4 that will be used in Section 5 to provide a unique characterization of queer supercrystals.

Definition 4.1. Let $\mathcal{C}$ be a crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. We define the *component graph* of $\mathcal{C}$, denoted by $G(\mathcal{C})$, as follows. The vertices of $G(\mathcal{C})$ are the type A_n components of $\mathcal{C}$ (typically labeled by their highest weight elements). There is an edge from vertex C_1 to vertex C_2 in this graph, if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that

$$f_{-1}b_1 = b_2.$$

Example 4.2. Let $\mathcal{C}$ be the connected component in the $\mathfrak{q}(3)$-crystal $\mathcal{B}^{\otimes 6}$ with highest weight element $1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1$ of highest weight $(4, 2, 0)$. The graph $G(\mathcal{C})$ is given in Fig. 4 on the left (disregarding the labels on the edges). The graph $G(\mathcal{C}')$ for the counterexample $\mathcal{C}'$ in Fig. 3 is given in Fig. 4 on the right. Since the two graphs are not isomorphic as unlabeled graphs, this confirms that the purple (double-headed) dashed arrows in Fig. 3 do not give the queer supercrystal even though the induced crystal satisfies the axioms in Definition 3.1.

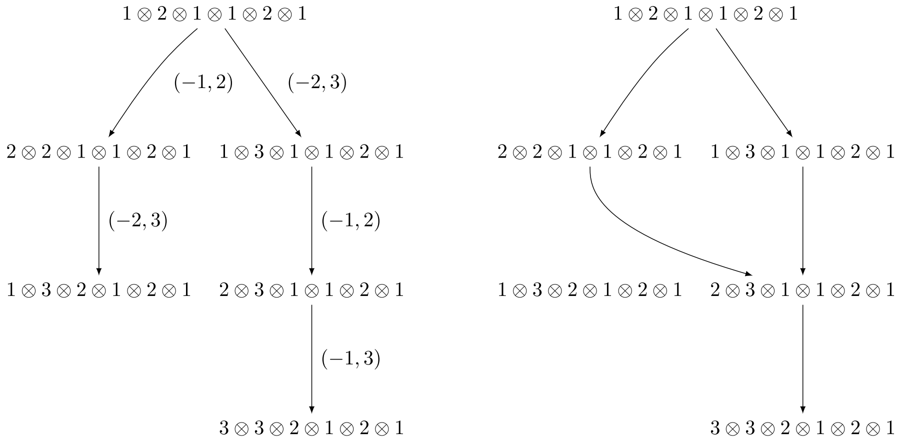


Fig. 4. Left: $\overline{G}(\mathcal{C})$. The graph $G(\mathcal{C})$ is obtained from $\overline{G}(\mathcal{C})$ by removing the labels. **Right:** $G(\mathcal{C}')$ for the crystals of Example 4.2.

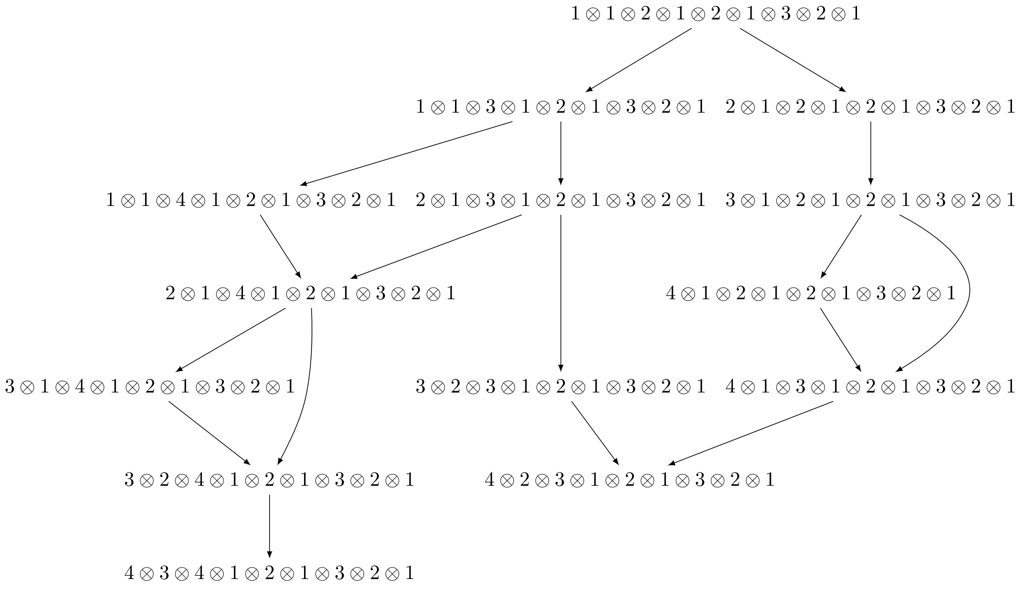


Fig. 5. The graph $G(\mathcal{C})$ for Example 4.3.

Example 4.3. Let $\mathcal{C}$ be the connected component with highest weight element $1 \otimes 1 \otimes 2 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$ in the $\mathfrak{q}(4)$-crystal $\mathcal{B}^{\otimes 9}$. Then the graph $G(\mathcal{C})$ is given in Fig. 5. One may easily check using Theorem 2.12 that all arrows in Fig. 5 are given by the application of f_{-i} for some i except for the arrows that by-pass other arrows, the arrow to the lowest vertex, which is given by $f_{-2}f_3$ (which is also determined by Theorem 2.12), and the arrow going into $3 \otimes 2 \otimes 3 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$, which is given by $f_{-1}f_2$. The result is shown in Fig. 6.

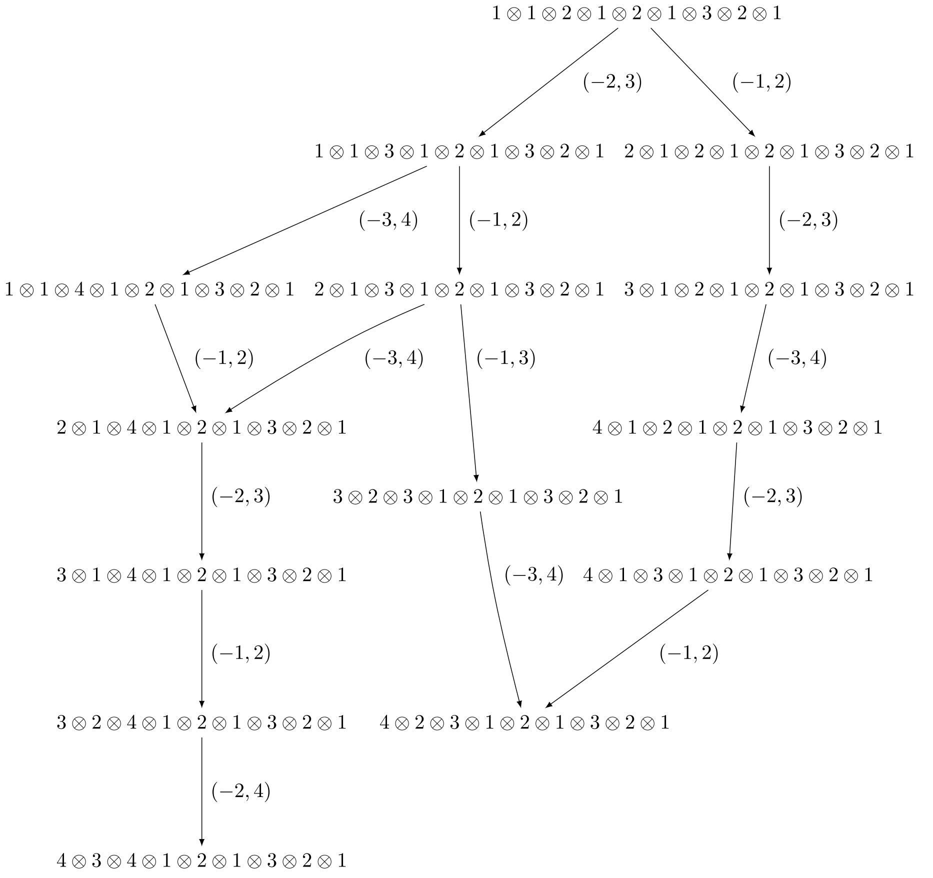


Fig. 6. The graph $\overline{G}(\mathcal{C})$ of Fig. 5 obtained from $G(\mathcal{C})$ by labeling each edge (except for the by-pass edges) by $(-i, h)$ if $f_{(-i,h)}$ applies.

Next we introduce new axioms.

Definition 4.4 *(*Connectivity axioms*).* Let $\mathcal{C}$ be a connected crystal satisfying the local queer axioms of Definition 3.1. Let $v \in \mathcal{C}$ be an I_0-lowest weight element and $u =\uparrow v$. As in (3.2), define $g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v$ for $1 \leqslant j \leqslant k \leqslant n$.

C0. $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

C1. Suppose that $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$. For all $h < j \leqslant k$ such that $g_{j,k} \neq 0$, we require that $f_{-1} g_{j,k} \neq 0$ and

$$f_{-1} g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h)v',$$

where v' is I_0-lowest weight with $\uparrow v' = u'$.

C2. Suppose that either (a) $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$ or (b) no such edge exists in $G(\mathcal{C})$. For all $1 \leqslant j \leqslant h$ in case (a) and all $1 \leqslant j \leqslant k$ in case (b) such that $g_{j,k} \neq 0$ and $f_{-1}g_{j,k} \neq 0$, we require that

$$f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

Remark 4.5. Condition **C0** can be replaced by the following condition:

LQ7. If $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ for $b \in \mathcal{C}$ with $\varepsilon_2(b) > 0$, then $\varphi_{-1}(b) \leqslant \varphi_{-1}(e_1e_2(b))$.

This condition indeed implies **C0**. Suppose $\varphi_{-1}(e_1 \cdots e_k v) = 1$. Then for $b = (e_3 \cdots e_j)(e_1 \cdots e_k)v$, we have $\varphi_{-1}(b) = 1$. However, b satisfies $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$, so the above condition implies that $\varphi_{-1}(e_1e_2(b)) = 1$ as well. But $e_1e_2(b) = g_{j,k}$. Hence $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

Moreover, in $\mathcal{B}^{\otimes \ell}$ the conditions in **LQ7** are satisfied. Namely, the condition $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ implies that $e_2(b) \neq 0$ and $e_1e_2(b) \neq 0$. Moreover, this condition implies that e_1 acts on $e_2(b)$ in a position weakly to the left of where e_2 acts on b. Thus if $\varphi_{-1}(b) = 1$, it immediately follows that $\varphi_{-1}(e_1e_2(b)) = 1$ which proves the statement.

Theorem 4.6. *The $\mathfrak{q}(n+1)$-crystal $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 4.4.*

The proof of Theorem 4.6 is given in Appendix A.

Next we show that the arrows in $G(\mathcal{C})$, where $\mathcal{C}$ is a connected component in $\mathcal{B}^{\otimes \ell}$, can be modeled by e_{-i} on type A highest weight elements.

Proposition 4.7. *Let $\mathcal{C}$ be a connected component in the $\mathfrak{q}(n+1)$-crystal $\mathcal{B}^{\otimes \ell}$. Let C_1 and C_2 be two distinct type A_n components in $\mathcal{C}$ and let u_2 be the I_0-highest weight element in C_2. Then there is an edge from C_1 to C_2 in $G(\mathcal{C})$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$.*

Proof. First note that there is an edge from C_1 to C_2 in $G(\mathcal{C})$ if there exists $b_1 \in C_1$ and $b_2 \in C_2$ such that $e_{-1}b_2 = b_1$. Recall that by (2.4) we have $e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}$. Hence, if $e_{-i}u_2$ is defined and $e_{-i}u_2 \in C_1$, then $b_1 := e_{-1}b_2$ is defined, where $b_2 := s_{w_i}u_2 \in C_2$ and $b_1 \in C_1$. This proves that there is an edge between C_1 and C_2 in $G(\mathcal{C})$.

Conversely assume that $b_1 = e_{-1}b_2$ for some $b_1 \in C_1$ and $b_2 \in C_2$. We want to show that then $e_{-i}u_2 \in C_1$ for some $i \in I_0$. By the discussion before Lemma 3.3, we know that the (-1)-arrow on b_1 is induced (using the local queer axioms of Definition 3.1) by the (-1)-arrow on $g_{j,k} = (e_1 \cdots e_j)(e_1 \cdots e_k)v_1$ for some $j \leqslant k$. By Theorem 4.6 and Condition **C1** of Definition 4.4, we must have

$$f_{-1}g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h)v_2 \qquad \text{for some } h < j \leqslant k,$$

where v_2 is the I_0-lowest weight element in the component C_2. In particular, for the edge $u_1 \to u_2$ in $G(\mathcal{C})$, where u_1 is the I_0-highest weight element in the component C_1, the weight $\mathrm{wt}(u_2)$ differs from $\mathrm{wt}(u_1)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $1 \leqslant h < k \leqslant n$. Furthermore, all $g_{j',k} \neq 0$ with $h < j' \leqslant k$ are mapped to component C_2 under f_{-1}.

Claim. *Set $b := s_{w_{n-h}} u_2$ and $b' := (e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$. If $\mathrm{wt}(b)_2 > 0$, there exist $j_1, \ldots, j_p \in I_0$ such that $b' = f_{j_1} \cdots f_{j_p} b$ and*

$$\varphi_2(f_{j_a} \cdots f_{j_p} b) > 0 \quad \textit{if } j_a = 2. \tag{4.1}$$

The claim is a statement about type A_n crystal operators, hence one may use the tableaux model to verify it. It is straightforward to verify that every column of height $d > n-h$ in the insertion tableau of b contains the letter m in row m; the columns of height $n-h$ contain 1 in the first row and $m+1$ in row $m > 1$; finally the columns of height $d < n-h$ contain the letter $m+2$ in row m. Hence $\mathrm{wt}(b)_2 > 0$ is only satisfied if there is at least one column of height $d > n-h$. Now we start acting with operators f_j on b, where $j \in I_0 \setminus \{2\}$, to make b into a $I_0 \setminus \{2\}$-lowest weight element. This element differs from v_2 only in columns of height $d \geqslant n-h$; columns of height $d > n-h$ contain 1 and 2 in rows 1 and 2, respectively, whereas columns of height $d = n-h$ contain 2 in row 1. Suppose that there are p columns whose height is less than $n+1$ and at least $n-h$. Then we can apply f_2^{p-1} without violating (4.1) since each such column contains an unbracketed 2. Then apply again f_j with $j \in I_0 \setminus \{2\}$ to make the tableau into a $I_0 \setminus \{2\}$-lowest weight element, followed by the maximal number of f_2 satisfying (4.1), followed by making the result $I_0 \setminus \{2\}$-lowest weight. This tableau is exactly $(e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$. This proves the claim.

Now since by assumption $\mathrm{wt}(u_2)$ differs from $\mathrm{wt}(u_1)$ by moving a box from row $n+1-k$ to row $n+1-h$, as a tableau $s_{w_{n-h}} u_2$ indeed has a column of height $d > n-k$, so that $\mathrm{wt}(s_{w_{n-h}} u_2)_2 > 0$. By condition (4.1), the (-1)-arrow coming into $s_{w_{n-h}} u_2$ is induced by the (-1)-arrow coming into $(e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$ by the local queer axioms of Definition 3.1. Hence $e_{-(n-h)} u_2 \in C_1$, which proves the proposition where $i = n-h$. $\square$

Example 4.8. Let us illustrate the claim in the proof of Proposition 4.7. Let $n = 5, h = 2$ and consider the type A_5 component C_2 of weight $(4, 3, 3, 2, 1)$. Then, using the model for type A crystals in terms of semistandard tableaux (see for example [4, Chapter 3]), we have

$$b = s_{w_3} u_2 = \begin{array}{|c|cccc}\cline{1-1} 5 \\ \cline{1-2} 4 & \multicolumn{1}{c|}{4} \\ \cline{1-3} 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \cline{1-3} 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \cline{1-4} 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \cline{1-4}\end{array} \quad . \text{ This becomes } \quad \begin{array}{|c|cccc}\cline{1-1} 6 \\ \cline{1-2} 5 & \multicolumn{1}{c|}{6} \\ \cline{1-3} 4 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} \\ \cline{1-3} 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \cline{1-4} 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \cline{1-4}\end{array}$$

after making it $\{1,3,4,5\}$-lowest weight and applying f_2^2. Making this element $\{1,3,4,5\}$-lowest weight again, no further f_2 are applicable and we obtain

$$\begin{array}{|c|c|c|c}\cline{1-1} 6 & \multicolumn{3}{c}{} \\ \cline{1-2} 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} 4 & 5 & 6 & \\ \cline{1-3} 2 & 3 & 5 & \\ \cline{1-4} 1 & 2 & 4 & \multicolumn{1}{c|}{6} \\ \cline{1-4} \end{array} = (e_2e_3)(e_1e_2)v_2.$$

By Proposition 4.7, there is an edge from component C_1 to component C_2 in $G(\mathcal{C})$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$, where u_2 is the I_0-highest weight element of C_2. We call the arrow *combinatorial* if $e_{-i}u_2$ is $\{1,2,\ldots,i\}$-highest weight. Otherwise the arrow is called a *by-pass arrow*.

Define $f_{(-i,h)} := f_{-i}f_{i+1}f_{i+2}\cdots f_{h-1}$.

Theorem 4.9. *Let $\mathcal{C}$ be a connected component in $\mathcal{B}^{\otimes \ell}$. Then each by-pass arrow is the composition of combinatorial arrows. Furthermore, each combinatorial edge in $G(\mathcal{C})$ can be obtained by $f_{(-i,h)}$ for some $i \in I_0$ and $h > i$ minimal such that $f_{(-i,h)}$ applies.*

Proof. Consider a combinatorial arrow from component C_1 to C_2. This means that $e_{-i}u_2$ is defined for some $i \in I_0$ and $e_{-i}u_2$ is $\{1,2,\ldots,i\}$-highest weight. Then by Theorem 2.12 and Corollary 2.14 we have $f_{(-i,h)}u_1 = u_2$ for some $h > i$.

If the arrow is a by-pass arrow, then $e_{-i}u_2$ is not $\{1,2,\ldots,i\}$-highest weight. By Proposition 2.24 and induction, there exists a sequence of indices $1 \leqslant i_1,\ldots,i_a < i$ such that

$$\uparrow e_{-i}u_2 = \uparrow e_{-i} \uparrow e_{-i_1} \cdots \uparrow e_{-i_a}u_2$$

where each partial sequence $e_{-i_j} \uparrow e_{-i_{j+1}} \cdots \uparrow e_{-i_a}u_2$ is $\{1,2,\ldots,i_j\}$-highest weight. This means that each by-pass arrow is the composition of combinatorial arrows. □

Theorem 4.9 provides a combinatorial description of the graph $G(\mathcal{C})$. Let $\overline{G}(\mathcal{C})$ be the graph $G(\mathcal{C})$ with all by-pass arrows removed and each edge labeled by the tuple $(-i,h)$ for the combinatorial arrow $f_{(-i,h)}u_1 = u_2$, where f_{-i} is given by the combinatorial rules stated in Theorem 2.12. Hence $\overline{G}(\mathcal{C})$ can be constructed from the $\mathfrak{q}(n+1)$-highest weight element u by the application of combinatorial arrows, see for example Fig. 6. In particular, the graph $G(\mathcal{C})$ and the graph $\overline{G}(\mathcal{C})$ have the same vertices.

Next we construct a graph $\widetilde{G}(\mathcal{C})$ from $\overline{G}(\mathcal{C})$ by applying $\uparrow e_{-i}$ to each vertex b in the graph $\overline{G}(\mathcal{C})$ (if applicable). This will add additional labeled edges between the vertices in the graph, see Fig. 7. We would like to emphasize that the construction of $\widetilde{G}(\mathcal{C})$ for a connected component $\mathcal{C}$ of $\mathcal{B}^{\otimes \ell}$ is purely combinatorial, starting with the highest weight element u of a given weight λ, applying $f_{(-i,h)}$ of Theorem 2.12, and then applying $\uparrow e_{-i}$ to all vertices using Theorem 2.16. This provides a combinatorial construction of $G(\mathcal{C})$

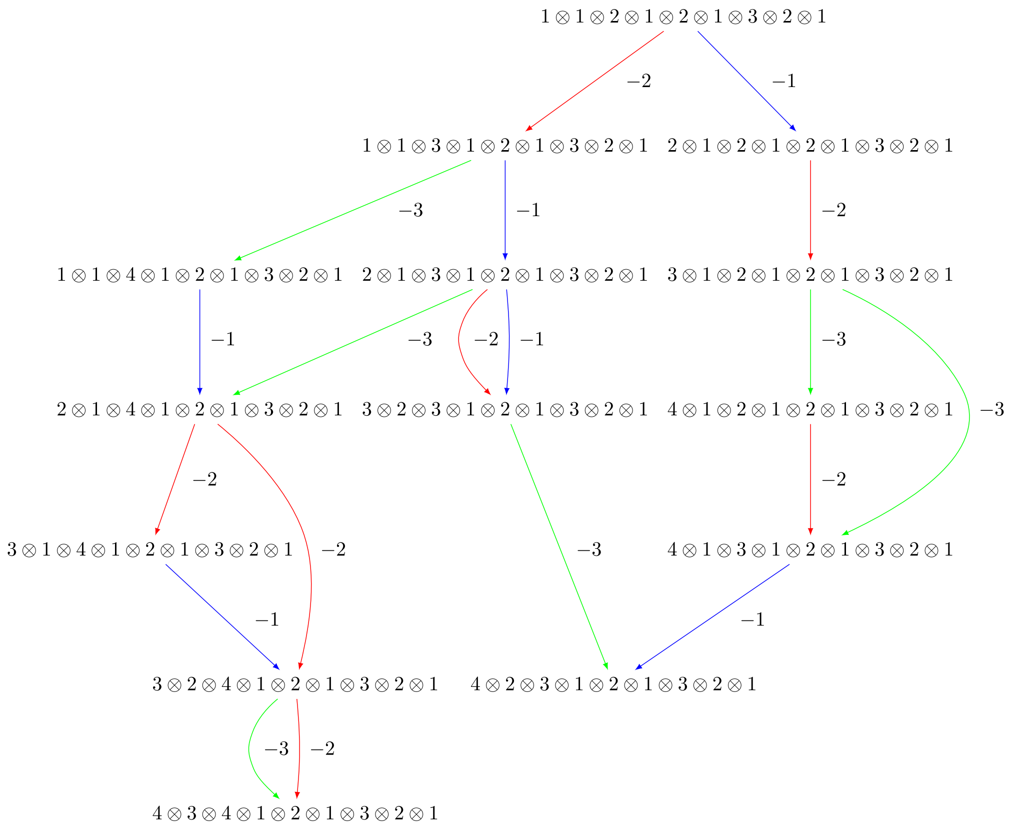


Fig. 7. The graph $\widetilde{G}(\mathcal{C})$ recovered from the graph $\overline{G}(C)$ of Fig. 6.

by dropping the labels in $\widetilde{G}(\mathcal{C})$ (and removing multiple edges between vertices when applicable).

Remark 4.10. The Schur P-polynomial $P_\lambda(x_1, \ldots, x_{n+1})$ in $n+1$ variables is the character of a finite-dimensional irreducible representation of the queer Lie superalgebra $\mathfrak{q}(n+1)$ with highest weight λ (up to a power of 2) [21]. The above combinatorial construction of the component graph of $\mathcal{C}$ with highest weight λ produces a Schur expansion of the Schur P-polynomial $P_\lambda(x_1, \ldots, x_{n+1})$. This expansion is obtained by counting the multiplicities of highest weights for all type A_n components that are present in $G(\mathcal{C})$. For example, the component graph in Example 4.2 yields the expansion $P_{42} = s_{42} + s_{33} + s_{411} + 2s_{321} + s_{222}$. This yields an alternative combinatorial description of the Schur expansion of the Schur P-polynomials compared to those given by Stembridge [22] and by Choi and Kwon [5].

5. Characterization of queer supercrystals

Our main theorem gives a characterization of the queer supercrystals. We say that two component graphs $G(\mathcal{C})$ and $G(\mathcal{D})$ are isomorphic if they are isomorphic as graphs and the weights of the vertices are preserved.

Theorem 5.1. *Let $\mathcal{C}$ be a connected component of a generic abstract queer supercrystal (see Definition 2.1). Suppose that $\mathcal{C}$ satisfies the following conditions:*

(1) *$\mathcal{C}$ satisfies the local queer axioms of Definition 3.1.*
(2) *$\mathcal{C}$ satisfies the connectivity axioms of Definition 4.4.*
(3) *$G(\mathcal{C})$ is isomorphic to $G(\mathcal{D})$, where $\mathcal{D}$ is some connected component of $\mathcal{B}^{\otimes \ell}$.*

Then the queer supercrystals $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

Theorem 5.1 states that the local queer axioms, the connectivity axioms, and the component graph uniquely characterize queer supercrystals.

Remark 5.2. We would like to point out that checking Condition (3) of Theorem 5.1 is algorithmically straightforward. Each component graph has a unique highest weight vertex. For the isomorphism, the weights of these highest weight vertices need to agree. Then one can recursively compare the edges and weights of adjacent vertices. Condition (3) is similar, albeit more complicated, to the condition by Stembridge [23] that for two connected crystal components of a simply-laced crystal to be isomorphic, the highest weights must agree.

Before we give the proof of Theorem 5.1, we need the following statement. Recall that $g_{j,k} = (e_1 \cdots e_j)(e_1 \cdots e_k)v$ was defined in (3.2), where v is an I_0-lowest weight vector.

Lemma 5.3. *In a crystal satisfying the local queer axioms of Definition 3.1 and **C0** of Definition 4.4, we have for any $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k$*

$$\varphi_{-1}(g_{j,k}) = 0 \quad \textit{if and only if} \quad \varphi_{-1}(e_1 \cdots e_k v) = 0.$$

Proof. The condition **C0** requires that $\varphi_{-1}(g_{j,k}) = 0$ implies $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

For the converse direction, note that $\mathrm{wt}(e_1 \cdots e_k v)_1 > 0$. Hence

$$\varphi_{-1}(e_1 \cdots e_k v) = 0 \quad \Leftrightarrow \quad \varepsilon_{-1}(e_1 \cdots e_k v) = 1.$$

By the local queer axioms **LQ6** and **LQ5** of Definition 3.1 (see also Fig. 2), we have

$$\begin{aligned} \varepsilon_{-1}(e_1 \cdots e_k v) = 1 \quad &\Leftrightarrow \quad \varepsilon_{-1}((e_3 \cdots e_j)(e_1 \cdots e_k)v) = 1 \\ &\Rightarrow \quad \varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1. \end{aligned}$$

It can be easily checked that $\varphi_1((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1$ for $j \leqslant k$ (for example using the tableaux model for type A_n crystals). Hence by the local queer axioms

$$\varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1 \quad \Leftrightarrow \quad \varepsilon_{-1}((e_1 \cdots e_j)(e_1 \cdots e_k)v) = 1.$$

This proves that $\varphi_{-1}(e_1 \cdots e_k v) = 0$ implies $\varphi_{-1}(g_{j,k}) = 0$. $\square$

Proof of Theorem 5.1. By Proposition 3.2 and Theorem 4.6, $\mathcal{D}$ satisfies the local queer axioms and the connectivity axioms and hence all conditions of the theorem.

By **LQ1** of the local queer axioms of Definition 3.1, each type A_n-component of $\mathcal{C}$ is a Stembridge crystal and hence is uniquely characterized by [23]. By assumption $G(\mathcal{C}) \cong G(\mathcal{D})$. In particular, the vertices of $G(\mathcal{C})$ and $G(\mathcal{D})$ agree. This proves that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as A_n crystals.

Next we show that all (-1)-arrows also agree on $\mathcal{C}$ and $\mathcal{D}$. As discussed just before Lemma 3.3, given the local queer axioms of Definition 3.1, it suffices to show that f_{-1} acts in the same way in $\mathcal{C}$ and $\mathcal{D}$ on the almost lowest elements satisfying (3.1) or equivalently by Lemma 3.3 on every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$. For the remainder of this proof, fix $g_{j,k} \neq 0$ in the I_0-component u.

Let us first assume that $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ for some $h < k$. If $h < j \leqslant k$, then $f_{-1} g_{j,k}$ is determined by **C1** of Definition 4.4. If $j \leqslant h$, pick $h < j' \leqslant k$ such that $g_{j',k} \neq 0$. Such a j' must exist since there is an edge $u \to u'$ in $G(\mathcal{C})$. By **C1**, we have $\varphi_{-1}(g_{j',k}) = 1$ and hence by Lemma 5.3 also $\varphi_{-1}(g_{j,k}) = 1$. Hence $f_{-1} g_{j,k}$ is determined by **C2**(a).

Next assume that $G(\mathcal{C})$ does not contain an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$.

Claim. *If $g_{k,k} \neq 0$, then $f_{-1} g_{j,k} = 0$.*

Proof. Suppose $f_{-1} g_{k,k} \neq 0$. By **C2**(b), we have $f_{-1} g_{k,k} = (e_2 \cdots e_k)(e_1 \cdots e_k) v = f_1 g_{k,k}$. But this contradicts the local queer axioms of Definition 3.1 since $\varphi_1(g_{k,k}) > 1$. Hence $\varphi_{-1}(g_{k,k}) = 0$ and by Lemma 5.3 also $\varphi_{-1}(g_{j,k}) = 0$, which proves the claim. □

If $g_{k,k} = 0$, we have $j < k$ since by assumption $g_{j,k} \neq 0$.

Claim. *Suppose $g_{k,k} = 0$.*

(1) *Suppose there is an edge $\overline{u} \to u$ in $G(\mathcal{C})$ such that $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$ such that $\overline{h} < k \leqslant \overline{k}$. Then $f_{-1} g_{j,k} = 0$.*
(2) *Suppose $G(\mathcal{C})$ does not contain an edge as in (1). Then $f_{-1} g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j) v$.*

Proof. Suppose that the conditions in (1) are satisfied. Then by **C1** there must exist

$$\overline{g}_{\overline{j},\overline{k}} := (e_1 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{k}}) \overline{v} \neq 0,$$

where $\overline{h} < \overline{j} \leqslant \overline{k}$ and $\overline{v}$ is the I_0-lowest weight element in the component of $\overline{u}$, such that

$$f_{-1} \overline{g}_{\overline{j},\overline{k}} = (e_2 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{h}}) v. \tag{5.1}$$

Since $g_{j,k} \neq 0$, we have in particular that $(e_1 \cdots e_k)v \neq 0$. Since $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$, this hence also implies that $\overline{g}_{k,\overline{k}} = (e_1 \cdots e_k)(e_1 \cdots e_{\overline{k}})\overline{v} \neq 0$. Hence by **C1** Equation (5.1) holds for $\overline{j} = k$.

If $f_{-1}g_{\overline{h},k} = 0$, we also have $f_{-1}g_{j,k} = 0$ by Lemma 5.3 as claimed. Hence we may assume that $f_{-1}g_{\overline{h},k} \neq 0$. Then by **C2**(b) we have

$$f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v.$$

But then $f_{-1}\overline{g}_{k,\overline{k}} = f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v$, which contradicts the fact that the crystal operator f_{-1} has a partial inverse since $\overline{g}_{k,\overline{k}} \neq g_{\overline{h},k}$. This proves (1).

Now suppose that the conditions in (2) are satisfied. Recall that by assumption $g_{j,k} \neq 0$ with $j < k$. This implies that $y := (e_2 \cdots e_k)(e_1 \cdots e_j)v \neq 0$, $\varphi_i(y) = 0$ for $i \in I_0 \setminus \{2\}$ and $\varphi_2(y) = 1$. By the local queer axioms of Definition 3.1, this implies that $x := e_{-1}y \neq 0$ with $\varphi_1(x) \in \{1, 2\}$ and $\varphi_i(x) = 0$ for $i \in I_0 \setminus \{1\}$. Thus we may write $x = (e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v}$, where $0 \leqslant s \leqslant t$ and $\overline{v} \in \mathcal{C}$ is some I_0-lowest weight vector. This yields the equality

$$f_{-1}(e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

If $\overline{v} \neq v$, then by the connectivity axioms of Definition 4.4 this means that $j < k = s \leqslant t$ and there is an edge in $G(\mathcal{C})$ from $\uparrow \overline{v}$ to $u = \uparrow v$, moving a box from row $n+1-t$ to row $n+1-j$. This contradicts the assumptions of (2). Hence we must have $\overline{v} = v$. By **C2**(b) we have $f_{-1}g_{s,t} = (e_2 \cdots e_t)(e_1 \cdots e_s)v$, so that $k = t$ and $j = s$. This implies $f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v$, proving the claim. □

We have now shown that $f_{-1}g_{j,k}$ is determined in all cases, which proves the theorem. □

Remark 5.4. Consider the $\mathfrak{q}(4)$-crystal $\mathcal{B}^{\otimes 4}$. The elements 4114 and 4113 both lie in the same $\{1, 2, 3\}$-component of highest weight $(3, 1)$. The highest (resp. lowest) weight element in this component is $u = 2111$ (resp. $v = 4344$). Both 4114 and 4113 satisfy (3.1). In fact, $4114 = (e_1e_2)(e_1e_2e_3)v = g_{2,3}$ and $4113 = (e_1e_2e_3)(e_1e_2e_3)v = g_{3,3}$. In the component of u there is no sequence of crystal operators that would induce the action of f_{-1} on 4114 from the action of f_{-1} on 4113 using the local queer axioms of Definition 3.1.

This suggests that the connectivity axioms of Definition 4.4 are indeed necessary. However, in this example the graph $G(\mathcal{C})$, where $\mathcal{C}$ is the connected component in $\mathcal{B}^{\otimes 4}$ containing 2111, is linear and hence forces 4114 and 4113 to be mapped to the same $\{1, 2, 3\}$-component by f_{-1}, see Fig. 8.

Remark 5.5. Consider the connected component $\mathcal{C}$ of 111212121 in the $\mathfrak{q}(6)$-crystal $\mathcal{B}^{\otimes 9}$. The $\{1, 2, 3, 4, 5\}$-component containing 321312121 is connected to the components 421312121, 431312121, and 432312121 in $G(\mathcal{C})$. The elements $g_{4,5} = 651615464$ and $g_{3,5} = 651615465$ in the component of 321312121 are mapped to the same component

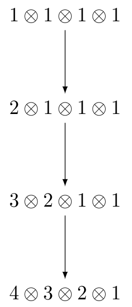


Fig. 8. The graph $G(\mathcal{C})$ for the example in Remark 5.4.

432312121 by **C1** of Definition 4.4. However, the element $g_{4,5}$ is connected to 431413131 in the crystal using only arrows that commute with f_{-1} and the element $g_{3,5}$ is connected to 431413143 in the crystal using only arrows that commute with f_{-1}. However, these two components (containing 431413131 resp. 431413143 using only crystal operators f_i and e_i with $i \in I_0$ that commute with f_{-1}) are disjoint. This suggests that **C1** of Definition 4.4 is necessary for uniqueness.

Appendix A. Proof of Theorem 4.6

In this appendix we prove Theorem 4.6. We use the shorthand notation $e_1^k := e_1 \cdots e_k$, $e_{\bar{1}}^k := e_{-1}e_2 \cdots e_k$, $f_k^1 := f_k \cdots f_1$, and $f_k^{\bar{1}} := f_k \cdots f_2 f_{-1}$.

Lemma A.1. *In $\mathcal{B}^{\otimes \ell}$, condition **C0** of Definition 4.4 holds.*

Proof. This follows from Remark 4.5. □

The connectivity axioms **C1** and **C2** of Definition 4.4 are implied by the following conditions. Here v is an I_0-lowest weight vector in $\mathcal{C}$:

C1'. If $h < k$ and there exists some $j \in (h, k]$ such that $f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ is I_0-lowest weight, then for any $j' \in (h, k]$ with $e_1^{j'} e_1^k(v) \neq 0$ we have $f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v) = f_j^{\bar{1}} e_1^j e_1^k(v)$.

C2'. If $j \leqslant k$ and $f_{-1} e_1^j e_1^k(v) \neq 0$, then either:
(a) $j \neq k$ and $f_j^1 f_k^{\bar{1}} e_1^j e_1^k(v) = v$, or
(b) $f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ is I_0-lowest weight for some $h < j$.

Proposition A.2. *In $\mathcal{B}^{\otimes \ell}$, condition **C2'** holds.*

The proof of Proposition A.2 is given in Section A.1.

Proposition A.3. *In $\mathcal{B}^{\otimes \ell}$, condition **C1'** holds.*

We will prove a seemingly weaker statement:

Lemma A.4. *In $\mathcal{B}^{\otimes \ell}$, condition **C1'** holds for $j = n-1$, $j' = k = n$ and for $j = k = n$, $j' = n-1$.*

The proof of Lemma A.4 is given in Sections A.2 and A.3.

Proposition A.5. *Lemma A.4 implies Proposition A.3.*

Proof. We first assume that $h < j < j' \leqslant k$ and the assumptions in **C1'** hold. Then we have

$$\begin{aligned} f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) &= f_h^1 f_j^{\bar{1}} (f_{j'} \cdots f_{j+2})(e_{j+2} \cdots e_{j'}) e_1^j e_1^k(v) \\ &= (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j (e_{j+2} \cdots e_{j'}) e_1^k(v) \\ &= (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v'), \end{aligned}$$

where $v' = (e_{j+2} \cdots e_{j'})(e_{j+2} \cdots e_k)(v)$. Here we have used Stembridge relations to commute crystal operators and in the last step also that the operators are acting on an I_0-lowest weight element. Note that v' is $\{1, \ldots, j+1\}$-lowest weight. Moreover, $f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v')$ is $\{1, \ldots, j+1\}$-lowest weight. Since $e_1^{j+1} e_1^{j+1}(v') = e_1^{j'} e_1^k(v) \neq 0$, we may apply Lemma A.4 with $n = j+1$. This implies

$$\begin{aligned} (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v') &= (f_{j'} \cdots f_{j+2}) f_h^1 f_{j+1}^{\bar{1}} e_1^{j+1} e_1^{j+1}(v') \\ &= f_h^1 f_{j'}^{\bar{1}} e_1^{j+1} e_1^{j+1} (e_{j+2} \cdots e_{j'})(e_{j+2} \cdots e_k)(v) \\ &= f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v), \end{aligned}$$

which proves the claim.

Next assume that $h < j' < j \leqslant k$. Then

$$\begin{aligned} f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) &= f_h^1 f_j^{\bar{1}} e_1^{j'+1} e_1^{j'+1} (e_{j'+2} \cdots e_j)(e_{j'+2} \cdots e_k)(v) \\ &= (f_j \cdots f_{j'+2}) f_h^1 f_{j'+1}^{\bar{1}} e_1^{j'+1} e_1^{j'+1}(v'), \end{aligned}$$

where $v' = (e_{j'+2} \cdots e_j)(e_{j'+2} \cdots e_k)(v)$. In this case, both v' and $f_h^1 f_{j'+1}^{\bar{1}} e_1^{j'+1} e_1^{j'+1}(v')$ are $\{1, \ldots, j'+1\}$-lowest weight. Since $e_1^{j'} e_1^{j'+1}(v') \neq 0$, we may apply Lemma A.4 with $n = j'+1$ to obtain

$$f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) = (f_j \cdots f_{j'+2}) f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^{j'+1}(v') = f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v),$$

proving the claim. □

A.1. Proof of Proposition A.2

Given a word $w = w_1 \cdots w_\ell$ in the letters $\{1, \ldots, n+1\}$ we write $w^{\#} = \overline{w_\ell} \cdots \overline{w_1}$, where $\overline{w_i} = n + 2 - w_i$. Suppose that $x = g_{j,k} = e_1^j e_1^k(v) \in \mathcal{B}^{\otimes \ell}$, where v is I_0-lowest weight and $1 \leqslant j \leqslant k \leqslant n$, so that by Lemma 3.3 we have $\varphi_1(x) = 2$ and $\varphi_i(x) = 0$ for all $i > 1$. The RSK insertion tableau for $x^{\#}$, denoted by $P(x^{\#})$, can be constructed as follows: Construct the semistandard Young tableau with weight and shape equal to the weight of $v^{\#}$. Change the rightmost $n + 1 - k$ in row $n + 1 - k$ and the rightmost $n + 1 - j$ in row $n + 1 - j$ to $n + 1$.

For instance, suppose $n = 8$ and $x = 198199887766$. Then $x = e_1^6 e_1^8(v)$, where $v = 998799887766$ is I_0-lowest weight and $v^{\#} = 443322113211$ has weight $(4, 3, 3, 2)$. Hence the tableau $P(x^{\#})$ is obtained from the tableau of shape and weight equal to $(4, 3, 3, 2)$ by changing the rightmost 1 in row 1 to 9 and the rightmost 3 in row 3 to 9:

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 4 & 4 \\ \hline 3 & 3 & 9 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 9 \\ \hline \end{array}\,.$$

Below, we consider the entries of a tableau to be linearly ordered in the row reading order. If $f_{-1}(x) \neq 0$ there are two possibilities:

(1) The recording tableau of $x^{\#}$ is the same as the recording tableau of $(f_{-1}(x))^{\#}$. This implies that during the insertion of $x^{\#}$, the final two $(n+1)$'s to be inserted are at no point in the same row. (Note that this is clearly impossible if $j = k$.) This means, that after the insertion of the final two $(n+1)$'s, the rightmost $n+1$ is never inserted into another row containing an $n + 1$, and, moreover, there is never an n being inserted into the row containing the rightmost $n+1$ (since after the insertion of the final two $(n + 1)$'s, the rightmost n or $n + 1$ is always $n + 1$). In this case, $P((f_{-1}(x)^{\#})$ is obtained from $P(x^{\#})$ by changing the $n+1$ in row $n+1-k$ into an n. Since $x^{\#}$ and $(f_{-1}(x))^{\#}$ have the same recording tableau, x and $f_{-1}(x)$ are in the same connected component. Since it is evident from $P((f_{-1}(x)^{\#})$ that $f_j^1 f_k \cdots f_2(f_{-1}(x))$ must be I_0-lowest weight, it follows that $v = f_j^1 f_k^{\bar{1}} e_1^j e_1^k(v)$. This is precisely what happens in the example above; $P((f_{-1}(x)^{\#})$ is obtained from $P(x^{\#})$ by:

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 \\ \hline 3 & 3 & 9 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 4 & 4 \\ \hline 3 & 3 & 9 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 8 \\ \hline \end{array}\,.$$

Hence **C2'**(a) holds.

(2) The recording tableau of $x^{\#}$ differs from the recording tableau of $(f_{-1}(x))^{\#}$. This implies that during the insertion of $x^{\#}$, there is some point at which the final two

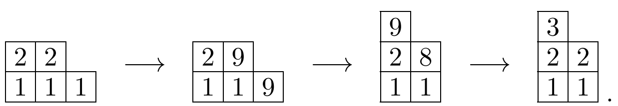
$(n+1)$'s to be inserted are in the same row. Call this row r and suppose that this occurs during the insertion of the i-th letter of $x^{\#}$. Let P_i be the tableau obtained from inserting the first i letters of $x^{\#}$ and let P_i' be the tableau obtained from inserting the first i letters of $(f_{-1}(x))^{\#}$. Then P_i' is obtained from P_i by changing the second to rightmost $n+1$ to n and moving the rightmost $n+1$ from row r to some row $s > r$.

Now continue with the insertion of the $(i+1)$-st letter in each case. Since the $(n, n+1)$-subword of $x^{\#}$ ends with two $(n+1)$'s, and these are the only $(n, n+1)$-unbracketed $(n+1)$'s in this subword, the same is true of the $(n, n+1)$-subword of each of $P_i, P_{i+1}, \ldots, P_\ell$. This implies that at no point in the rest of the insertion of $x^{\#}$ is the second to rightmost $n+1$ inserted into a row containing another $n+1$, and moreover at no point is an n inserted into the row containing the second to rightmost $n+1$ (since after the insertion of the final two $(n+1)$'s, the two rightmost entries which are either n or $n+1$ must both be $n+1$).

It follows that, if we ignore, the rightmost $n+1$ in $P((f_{-1}(x)^{\#})$ and $P(x^{\#})$, then they have the same shape, and the second differs from the first only by changing its rightmost n to $n+1$. Adding back the rightmost $n+1$ to $P(x^{\#})$, we see that it must go somewhere to the right of this position (by definition), and adding back the rightmost $n+1$ to $P(f_{-1}(x^{\#}))$, we see that it must go somewhere to the left of this position (otherwise $P((f_{-1}(x)^{\#})$ would have an $(n, n+1)$-unbracketed $n+1$).

It follows that $P((f_{-1}(x)^{\#})$ is obtained from $P(x^{\#})$ by eliminating the (rightmost) $n+1$ in row $n-k+1$, changing the (leftmost) $n+1$ in row $n-j+1$ to n and adding an $n+1$ to some row $n-h+1$ for $h < j$. It follows that $v' = f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ and v are both (distinct) I_0-lowest weight elements. Hence **C2'**(b) holds.

To see an example of the second case, let $v = 99889$. Then $v^{\#} = 12211$, $(e_1^7 e_1^8(v))^{\#} = 29911$, $(f_{-1}e_1^7 e_1^8(v))^{\#} = 29811$, and $(f_6^1 f_7^{\bar{1}} e_1^7 e_1^8(v))^{\#} = 23211$ have the following insertion tableaux:



A.2. Proof of Lemma A.4 for $j = n-1$ and $j' = n$

Define $X = (e_1 \cdots e_n)v$. For $1 \leqslant i \leqslant n+1$, set $A_i = (e_i \cdots e_n)X$ and $B_i = (e_i \cdots e_{n-1})X$. For $2 \leqslant i \leqslant n+1$, set $A_{-i} = (f_{(i-1)} \cdots f_2 f_{-1})A_1$ and $B_{-i} = (f_{(i-1)} \cdots f_2 f_{-1})B_1$. (So $A_1 = A_{-1}$ and $B_1 = B_{-1}$. Moreover, $B_{n+1} = B_n$.) By assumption $(f_h \cdots f_1)(B_{-n})$ is I_0-lowest weight, so $f_n(f_h \cdots f_1)(B_{-n}) = 0$ and hence $B_{-(n+1)} = 0$.

Let x_i be the integer which represents the position where A_{i+1} and A_i differ, and y_i be the integer which represents the position where B_{i+1} and B_i differ. Also, let x_{-i} be the integer which represents the position where A_{-i} and $A_{-(i+1)}$ differ, and let y_{-i} be the integer which represents the position where B_{-i} and $B_{-(i+1)}$ differ. Note that y_n and y_{-n} are undefined.

Recall that $v \in \mathcal{B}^{\otimes \ell}$. Suppose W is any word of length ℓ in the letters $\{1, \ldots, n+1\}$. If $1 \leqslant p \leqslant \ell$, we define $W(p)$ to be the p-th entry of W. If $1 \leqslant p \leqslant q \leqslant \ell$ are integers, then the notation $W(p : q)$ will be used to refer to the word $W(p)W(p+1) \ldots W(q-1)W(q)$.

If $1 \leqslant i \leqslant n$, we define the $i/(i+1)$-subword of W to be the word composed of the symbols $\{i, i+1, \_\}$ which is obtained from W by changing each entry that is neither i nor $i+1$ to the symbol _. For instance the 2/3-subword of 241432143 is 2____32___3. When we speak of erasing an i or $i+1$, we mean changing that entry to _; similarly, when we speak of adding an i or $i+1$, we mean changing some _ to i or $i+1$. Moving an i or $i+1$ from p to q means erasing an i or $i+1$ from position p and adding an i or $i+1$ to position q. The notation $W(p : q)$ is used in the same way for subwords as it is for words. For instance, if W=3____32___3 then $W(3 : 7) =$ ___32_.

Claim A.6. *For $2 \leqslant i \leqslant n$, we have $x_i \geqslant x_{i-1}$. For $2 \leqslant i \leqslant n-1$, we have $y_i \geqslant y_{i-1}$.*

Proof. If $x_i < x_{i-1}$, then it follows that $f_i A_{i-1} \neq 0$. But this is the statement that

$$f_i(e_{i-1}e_i \cdots e_n)(e_1 \cdots e_n)v \neq 0$$

for some integer $2 \leqslant i \leqslant n$, which is absurd since v is I_0-lowest weight. If $y_i < y_{i-1}$, then it follows that $f_i B_{i-1} \neq 0$. But this is the statement that

$$f_i(e_{i-1}e_i \cdots e_{n-1})(e_1 \cdots e_n)v \neq 0$$

for some integer $2 \leqslant i \leqslant n-1$, which is also absurd. □

Claim A.7. *We have $x_1 > x_{-1}$ and $y_1 > y_{-1}$. (In particular, $f_{-1}(A_1) \neq 0$, so x_{-1} is well-defined.)*

Proof. By the definition of the operator f_{-1} we have $y_1 \geqslant y_{-1}$. Since v and $v^* := f_h^1 f_{n-1}^{\bar{1}} e_1^{n-1} e_1^n v$ are both I_0-lowest weight and have different weights, we cannot have $y_1 = y_{-1}$. Thus $y_1 > y_{-1}$. Now $B_n(1 : y_{-1}) = B_1(1 : y_{-1})$. Therefore, there are no 1's or 2's in $B_n(1 : y_{-1} - 1)$ and we have $B_n(y_{-1}) = 1$ since these statements must be true of B_1. If $x_1 > y_{-1}$, then $A_1(1 : y_{-1}) = B_1(1 : y_{-1})$ and so $A_{-2} \neq 0$ with $x_{-1} = y_{-1}$. If $x_1 < y_{-1}$, then $A_1(1 : x_1 - 1) = B_n(1 : x_1 - 1)$ contains no 1's or 2's and $A_1(x_1) = 1$. Thus $A_{-2} \neq 0$ with $x_{-1} = x_1$. It is clearly impossible for $x_1 = y_{-1}$. Therefore, we have established that $A_{-2} = f_{-1}(A_1) \neq 0$. In the notation of Proposition A.2, we have for $j = k = n$, that $f_{-1} e_1^j e_1^k(v) \neq 0$. Hence we must be in case **C2'**(b) from which we deduce that $f_{-1}(A_1)$ lies in a different I_0-connected component than A_1. From this it follows that $x_1 > x_{-1}$. □

Claim A.8. *For $2 \leqslant i \leqslant n$, we have $x_{-(i-1)} \leqslant x_{-i}$. For $2 \leqslant i \leqslant n$, we have $y_{-(i-1)} \leqslant y_{-i}$. (In particular, $A_{-3}, \ldots, A_{-(n+1)}$ are nonzero, so $x_{-2}, \ldots, x_{-n}$ are well-defined.)*

Proof. Again, case **C2'**(b) applies to $f_{-1}(A_1)$ and so the parenthetical statement is immediate. First, it is clear from the definitions of the f_{-1} and f_2 operators that $x_{-1} \leqslant x_{-2}$ and that $y_{-1} \leqslant y_{-2}$. If $x_{-(i-1)} > x_{-i}$ for $i > 2$, then it follows that $f_i A_{-(i-1)} \neq 0$. But this is the statement that $f_i(e_{i-1}e_i \cdots e_n)(e_1 \cdots e_g)\hat{v} \neq 0$ for some I_0-lowest weight element $\hat{v}$ and integers $3 \leqslant i \leqslant n$ and $0 \leqslant g < n$ which is absurd. If $y_{-(i-1)} > y_{-i}$ for $i > 2$, then it follows that $f_i(B_{-(i-1)}) \neq 0$. But this is the statement that $f_i(e_{i-1}e_i \cdots e_{n-1})(e_1 \cdots e_g)v^* \neq 0$ for some integers $3 \leqslant i \leqslant n$ and $0 \leqslant g < n$ which is equally absurd. □

So far, we have the following situation:

$$x_n \geqslant \cdots \geqslant x_2 \geqslant x_1 > x_{-1} \leqslant x_{-2} \leqslant \cdots \leqslant x_{-n} \qquad \text{and}$$
$$y_{n-1} \geqslant \cdots \geqslant y_2 \geqslant y_1 > y_{-1} \leqslant y_{-2} \leqslant \cdots \leqslant y_{-(n-1)}.$$

Claim A.9. *We have $x_{-1} = y_{-1}$.*

Proof. Since $x_1 = y_{-1}$ is impossible and since $x_1 < y_{-1}$ would imply that $x_{-1} = x_1$, which contradicts $x_1 > x_{-1}$, we may assume $x_1 > y_{-1}$. However, in this case we have $A_1(1 : y_{-1}) = B_1(1 : y_{-1})$. Since f_{-1} acts on B_1 in position y_{-1}, it follows that f_{-1} acts on A_1 in position y_{-1} as well. This implies $x_{-1} = y_{-1}$. □

Claim A.10. *For $1 \leqslant i \leqslant n-1$, we have $x_i \leqslant y_i$.*

Proof. First we show that $x_{n-1} \leqslant y_{n-1}$. Now y_{n-1} represents the position of the leftmost $(n-1,n)$-unbracketed n in B_n. This n is also unbracketed in A_n because the $(n-1)/n$-subword of A_n is obtained from the $(n-1)/n$-subword of B_n by inserting an n. Hence the leftmost $(n-1,n)$-unbracketed n in A_n is weakly to the left of position y_{n-1}, so $x_{n-1} \leqslant y_{n-1}$. Next, suppose that $x_{i+1} \leqslant y_{i+1}$ but $x_i > y_i$. The $i/(i+1)$-subword of A_{i+1} only differs from the $i/(i+1)$-subword of B_{i+1} by moving an $i+1$ to the left from y_{i+1} to x_{i+1}. Since $y_i < x_{i+1}$ by assumption, the $i+1$ which appears in $B_{i+1}(y_i)$ still appears in $A_{i+1}(y_i)$ and is $(i,i+1)$-unbracketed. This implies $x_i \leqslant y_i$. Induction completes the proof. □

Claim A.11. *For $1 \leqslant i \leqslant n$, we have $x_i \geqslant x_{-i}$. For $1 \leqslant i \leqslant n-1$, we have $y_i \geqslant y_{-i}$.*

Proof. We already know that $x_1 \geqslant x_{-1}$. So assume that $x_{i-1} \geqslant x_{-(i-1)}$ but $x_i < x_{-i}$. The $i/(i+1)$-subword of A_i is obtained from the $i/(i+1)$-subword of A_{-i} by moving an i to the right from $x_{-(i-1)}$ to x_{i-1}. Since $A_{-i}(x_{-i})$ contains an $(i,i+1)$-unbracketed i and $x_{i-1} < x_{-i}$, we see that $A_i(x_{-i})$ still contains an $(i,i+1)$-unbracketed i. This implies that $x_i \geqslant x_{-i}$. Induction completes the proof. The second statement is proved in the same way. □

From the previous result, we have the following situation:

$$\begin{array}{cccccccccccccc}
\cdots & \geqslant & x_3 & \geqslant & x_2 & \geqslant & x_1 & > & x_{-1} & \leqslant & x_{-2} & \leqslant & x_{-3} & \leqslant \cdots \\
 & & \text{\rotatebox{90}{$\geqslant$}} & & \text{\rotatebox{90}{$\geqslant$}} & & \text{\rotatebox{90}{$\geqslant$}} & & \| & & & & & \\
\cdots & \geqslant & y_3 & \geqslant & y_2 & \geqslant & y_1 & > & y_{-1} & \leqslant & y_{-2} & \leqslant & y_{-3} & \leqslant \cdots
\end{array}$$

where every entry on the left side of the array is $\geqslant$ to its mirror image on the right side of the array. From now on, let j be minimal such that $x_j < y_j$; if no such j exists, set $j = n$.

Claim A.12. *We have $x_i = y_i$ for all $i < j$ and $x_{i+1} < y_i$ for all $j \leqslant i < n$.*

Proof. The first claim is immediate. Next we note that $x_i < y_i$ for all $i \geqslant j$. (Otherwise $x_i = y_i$ for some $i \geqslant j$. This implies that $x_k = y_k$ for all $k \leqslant i$, and, in particular, $x_j = y_j$.) By definition, we have $B_{i+1}(y_i) = i + 1$ and $A_{i+2}(x_{i+1}) = i + 2$. From the latter, it follows that $B_{i+2}(x_{i+1}) \geqslant i + 2$ and, since $y_{i+1} > x_{i+1}$ (or y_{i+1} is undefined) that $B_{i+1}(x_{i+1}) \geqslant i + 2$. Therefore, we have $x_{i+1} \neq y_i$. If $x_{i+1} > y_i$, we must have $x_i < x_{i+1}$ and $y_i < y_{i+1}$ from which it follows that $A_{i+1}(1 : y_i) = B_{i+1}(1 : y_i)$. But this makes $x_i < y_i$ impossible. By contradiction, we conclude that $x_{i+1} < y_i$. □

Claim A.13. *For $i < j$ we have $x_{-i} = y_{-i}$. Also, $x_j > x_{j-1}$.*

Proof. Since the restrictions of A_{j-1} and B_{j-1} to the alphabet $\{1, 2, \ldots, j - 1\}$ are identical, and since the operators $e_{j-2}, \ldots, e_1, f_{-1}, f_2, \ldots, f_{j-2}$ only depend on and effect these letters, it follows that for $i \leqslant j-2$ we have $x_{-i} = y_{-i}$. Now we must show $x_{-(j-1)} = y_{-(j-1)}$. We have $A_{j+1}(x_j) = j + 1$ and thus $B_{j+1}(x_j) \geqslant j + 1$, and hence by $x_j < y_j$, $B_j(x_j) \geqslant j + 1$. Since $B_j(y_{j-1}) = j$, this yields $x_j \neq y_{j-1}$. In light of $x_{j-1} = y_{j-1}$ this gives $x_j \neq x_{j-1}$. From this it follows that $A_j(1 : x_{j-1}) = B_j(1 : x_{j-1})$. By the minimality of j and by the result for $i \leqslant j-2$ this implies that $A_{-(j-1)}(1 : x_{j-1}) = B_{-(j-1)}(1 : x_{j-1})$. Since we have both $x_{-(j-1)} \leqslant x_{j-1}$ and $y_{-(j-1)} \leqslant y_{j-1}$, the previous equality implies that $x_{-(j-1)} = y_{-(j-1)}$. □

If $1 < i < n$, let $\#(A_{-i}(p : q))$ denote the number of i's minus the number of $(i+1)$'s which appear in $A_{-i}(p : q)$. Define $\#(B_{-i}(p : q))$ analogously. Set $AB_i(p : q) = \#(A_{-i}(p : q)) - \#(B_{-i}(p : q))$.

Claim A.14. *Suppose $1 < i < n$.*

(1) *If $x_{-i} < y_{-i}$, then $AB_i(1 : x_{-i}) > 0$.*
(2) *If $x_{-i} > y_{-i}$, then $AB_i(1 : y_{-i}) < 0$.*
(3) *If $x_{-i} < y_{-i}$, then $AB_i(x_{-i} + 1 : y_{-i}) < 0$.*
(4) *If $x_{-i} < y_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$, then $AB_i(x_{-i} + 1 : y_i) < -1$.*

Proof. Once again, **C2'**(b) applies to $f_{-1}(A_1)$ and so we may write $A_{-i} = e_i \cdots e_n e_1^{h'}(v')$ for some I_0-lowest weight element v' and some $h' < n$. It follows that A_{-i} has exactly one $(i, i+1)$-unbracketed i and it occurs in x_{-i}. In addition, case **C2'**(b) applies to $f_{-1}(B_1)$ by assumption, so $B_{-i} = e_i \cdots e_{n-1} e_1^h(v^*)$ for an I_0-lowest weight element v^*. Hence B_{-i} has exactly one $(i, i+1)$-unbracketed i and it occurs in y_{-i}. Thus we have $\#(A_{-i}(1 : x_{-i})) > 0$ and $\#(B_{-i}(1 : y_{-i})) > 0$. If $x_{-i} < y_{-i}$ then $\#(B_{-i}(1 : x_{-i})) \leqslant 0$, while if $x_{-i} > y_{-i}$ then $\#(A_{-i}(1 : y_{-i})) \leqslant 0$. Together this proves the first two statements. For the third statement we have $\#(A_{-i}(x_{-i} + 1 : y_{-i})) \leqslant 0$ and $\#(B_{-i}(x_{-i} + 1 : y_{-i})) > 0$. For the fourth statement, again, we have $\#(A_{-i}(x_{-i} + 1 : y_{-i})) \leqslant 0$, but now note that $A_{i+1}(x_i) = i + 1$. Since $x_i \neq x_{i+1}$, also, $A_{i+2}(x_i) = i + 1$, whence $B_{i+1}(x_i) = i + 1$, and, by, $x_i \neq y_i$, we have $B_i(x_i) = i + 1$. This now implies that $B_{-i}(x_i) = i + 1$ or $B_{-i}(x_{-i}) = i + 1$. Since the i in $B_{-i}(y_i)$ must be $(i, i + 1)$-unbracketed this implies that $\#(B_{-i}(x_{-i} + 1 : y_{-i})) > 1$. □

Claim A.15. *Fix an interval $[p, q]$. We define the function $[t]$ by $[t] = 1$ if $t \in [p, q]$ and $[t] = 0$ otherwise. With this notation, we have that*

$$AB_i(p : q) = [x_{-(i-1)}] - [x_{i-1}] + 2[x_i] - [x_{i+1}] + [y_{i+1}] - 2[y_i] + [y_{i-1}] - [y_{-(i-1)}].$$

Proof. This is a straightforward computation. □

Claim A.16. *Suppose $j < n$. If either $x_j > x_{-j}$ or $y_j > y_{-j}$, then both $x_j > x_{-j}$ and $y_j > y_{-j}$. In this case we have $x_{-j} = y_{-j}$.*

Proof. If $j = 1$, the conclusions of the claim have already been proven in previous claims. Thus assume $j > 1$. First note that, since $x_{-(j-1)} = y_{-(j-1)}$ and $x_{j-1} = y_{j-1}$, we have $AB_j(p : q) = 2[x_j] - [x_{j+1}] + [y_{j+1}] - 2[y_j]$. To prove the first statement, we will show that both (1) $x_j > x_{-j}$ and $y_j = y_{-j}$ and (2) $y_j > y_{-j}$ and $x_j = x_{-j}$ are impossible.

First suppose that $x_j > x_{-j}$ and that $y_j = y_{-j}$. Since $x_{-j} < x_j < y_j = y_{-j}$, we have by Claim A.14 that $AB_j(1 : x_{-j}) > 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim A.15 we have $AB_j(1 : x_{-j}) = 0$. Hence, $x_j > x_{-j}$ and $y_j = y_{-j}$ is impossible.

Now suppose that $y_j > y_{-j}$ and that $x_j = x_{-j}$.

Case 1: $y_{-j} < x_{-j}$. Since $y_{-j} < x_{-j}$ we have by Claim A.14 that $AB_j(1 : y_{-j}) < 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> y_{-j}$ so by Claim A.15 we have $AB_j(1 : y_{-j}) = 0$.

Case 2: $y_{-j} = x_{-j}$. We have $A_{j+1}(x_j) = j + 1$ and so $B_{j+1}(x_j) \geqslant j + 1$. Hence by $x_j < y_j$ we have $B_j(x_j) \geqslant j + 1$ which gives $B_{-j}(x_j) \geqslant j + 1$. However, by definition $B_{-j}(y_{-j}) = j$ so this makes $x_{-j} = y_{-j}$ impossible in light of $x_j = x_{-j}$.

Case 3a: $y_{-j} > x_{-j}$ and $x_j = x_{j+1}$. Since $y_{-j} > x_{-j}$ we have by Claim A.14 that $AB_j(x_{-j} + 1 : y_{-j}) < 0$. However, x_j, x_{j+1} are each $< x_{-j} + 1$ and y_j, y_{j+1} are each $> y_{-j}$ so by Claim A.15 we have $AB_j(1 : y_{-j}) = 0$.

Case 3b: $y_{-j} > x_{-j}$ and $x_j < x_{j+1}$. Since $y_{-j} > x_{-j} = x_j$, $x_j \neq x_{j+1}$, and $x_j \neq y_j$, we have by Claim A.14 that $AB_j(x_{-j}+1 : y_{-j}) < -1$. However, $x_j < x_{-j}+1$ and y_j, y_{j+1} are each $> y_{-j}$ so by Claim A.15 we have $AB_j(x_{-j}+1 : y_{-j}) \in \{-1, 0\}$.

Hence $y_j > y_{-j}$ and $x_j = x_{-j}$ is impossible. This establishes that if either $x_j > x_{-j}$ or $y_j > y_{-j}$, then both $x_j > x_{-j}$ and $y_j > y_{-j}$.

Now assume that both $x_j > x_{-j}$ and $y_j > y_{-j}$. If $x_{-j} < y_{-j}$, we have by Claim A.14 that $\#_j(A_{-j}(1 : x_{-j})) > 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim A.15 we have $\#_j(A_{-j}(1 : x_{-j})) = 0$. If $x_{-j} > y_{-j}$, we have by Claim A.14 that $\#_j(A_{-j}(1 : y_{-j})) < 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim A.15 we have $\#_j(A_{-j}(1 : y_{-j})) = 0$. Hence $x_{-j} = y_{-j}$. □

Claim A.17. *If $x_j < x_{-j}$ or $y_j < y_{-j}$, then for $j \leqslant i < n$ we have $y_{-i} < y_i$ and $y_{-i} \leqslant x_{-i}$.*

Proof. We proceed by induction. By the first statement of Claim A.16, we can be sure that $y_{-j} < y_j$. By the second statement of Claim A.16 we can be sure that $y_{-j} = x_{-j}$, so in particular, $y_{-j} \leqslant x_{-j}$. Therefore the claim holds for $i = j$. Now let $i > j$ and suppose that the claim holds for $i-1$ so that $y_{-(i-1)} < y_{i-1}$ and $y_{-(i-1)} \leqslant x_{-(i-1)}$. We will show that under this assumption, each of (1) $y_{-i} = y_i$ and $y_{-i} > x_{-i}$, (2) $y_{-i} < y_i$ and $y_{-i} > x_{-i}$, and (3) $y_{-i} = y_i$ and $y_{-i} \leqslant x_{-i}$ is impossible.

First suppose that $y_{-i} = y_i$ and that $y_{-i} > x_{-i}$.

Case 1: $x_{-i} < x_i$. Since $y_{-i} > x_{-i}$ by Claim A.14 we have $AB_i(1 : x_{-i}) > 0$. However, by assumption $x_i, x_{i+1}, y_{i+1}, y_i, y_{i-1}$ are each $> x_{-i}$ and $x_{-(i-1)} = y_{-(i-1)}$ so the only possible relevant change is at x_{i-1}. Thus by Claim A.15 we have $AB_i(1 : y_{-i}) \in \{-1, 0\}$.

Case 2a: $x_{-i} = x_i$ and $x_i = x_{i+1}$. Since $y_{-i} > x_{-i}$ by Claim A.14 we have $AB_i(1 : x_{-i}) > 0$. By assumptions, each of $x_{-(i-1)}, x_{i-1}, x_i, x_{i+1}, y_{-(i-1)}$ are $< x_{-i}+1$. Clearly $y_i = y_{-i} \in [x_{-i}+1 : y_{-i}]$. Moreover, $y_{i-1} \leqslant y_i = y_{-i}$ and $y_{i-1} > x_i = x_{-i}$, so $y_{i-1} \in [x_{-i}+1 : y_{-i}]$. Without computing the value of $[y_{i+1}]$ we may conclude by Claim A.15 that $AB_i(1 : y_{-j}) \in \{-1, 0\}$.

Case 2b: $x_{-i} = x_i$ and $x_i < x_{i+1}$. Since $y_{-i} > x_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$ we have by Claim A.14 that $AB_i(x_{-i}+1 : y_{-i}) < -1$. By assumptions, each of $x_{-(i-1)}, x_{i-1}, x_i, y_{-(i-1)}$ are $< x_{-i}+1$. Again, we know that $y_i, y_{i-1} \in [x_{-i}+1 : y_{-i}]$. Without computing the value of $[y_{i+1}]$ and $[x_{i+1}]$ we may compute by Claim A.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{-1, 0, 1\}$.

Hence it is impossible that $y_{-i} = y_i$ and that $y_{-i} > x_{-i}$. Now suppose that $y_{-i} < y_i$ and that $y_{-i} > x_{-i}$.

Case 1a: $x_{-i} < x_i$ and $x_i \leqslant y_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim A.14 that $AB_i(x_{-i}+1 : y_{-i}) < 0$. We have that $x_{-(i-1)}, y_{-(i-1)}$ are both $< x_{-i}+1$, that $x_i \in [x_{-i}+1 : y_{-i}]$ and that y_i, y_{i+1} are both $> y_{-i}$. Without computing $[x_{i-1}], [x_{i+1}], [y_{i-1}]$ we may determine by Claim A.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{0, 1, 2, 3\}$.

Case 1bi: $x_{-i} < x_i$, $x_i > y_{-i}$, and $x_{i-1} \leqslant x_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim A.14 that $AB_i(x_{-i}+1 : y_{-i}) < 0$. By assumption each of $x_{-(i-1)}, x_{i-1}, y_{-(i-1)}$ are

$< x_{-i}+1$ and $x_{i+1}, x_i, y_i, y_{i+1}$ are $> y_{-i}$. Without computing $[y_{i-1}]$ we may determine by Claim A.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{0,1\}$.

Case 1bii: $x_{-i} < x_i$, $x_i > y_{-i}$, and $x_{i-1} > x_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim A.14 that $AB_i(1 : x_{-i}) < 0$. By assumption $x_{-(i-1)}, y_{-(i-1)}$ are $\leqslant x_{-i}$ whereas each of $x_{i-1}, x_i, x_{i+1}, y_{i-1}, y_i, y_{i+1}$ are $> x_{-i}$. Thus by Claim A.15, we have $AB_i(1 : x_{-i}) = 0$.

Case 2a: $x_{-i} = x_i$ and $x_i = x_{i+1}$. Since $y_{-i} > x_{-i}$ we have by Claim A.14 that $AB_i(x_{-i}+1 : y_{-i}) < 0$. By assumption each of $x_{-(i-1)}, x_{i-1}, x_i, x_{i+1}, y_{-(i-1)}$ are $< x_{-i}+1$ and y_i, y_{i+1} are $> y_{-i}$. Without computing $[y_{i-1}]$ we may determine by Claim A.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{0,1\}$.

Case 2b: $x_{-i} = x_i$ and $x_i < x_{i+1}$. Since $y_{-i} > x_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$ we have by Claim A.14 that $AB_i(x_{-i}+1 : y_{-i}) < -1$. By assumption each of $x_{-(i-1)}, x_{i-1}, x_i, y_{-(i-1)}$ are $< x_{-i}+1$ and y_i, y_{i+1} are $> y_{-i}$. Without computing $[y_{i-1}]$ and $[x_{i-1}]$ we may determine by Claim A.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{-1,0,1\}$.

Hence $y_{-i} < y_i$ and $y_{-i} > x_{-i}$ is impossible. Now suppose $y_{-i} = y_i$ and $y_{-i} \leqslant x_{-i}$. This would imply $y_i = y_{-i} \leqslant x_{-i} \leqslant x_i < y_i$ which is absurd. The three possibilities listed in the beginning of the proof are thus impossible, and the only remaining one is $y_{-i} < y_i$ and $y_{-i} \leqslant x_{-i}$. $\square$

Supposing $j = 3$, and $n = 5$, and $x_j > x_{-j}$ our situation would look as follows:

$$\begin{array}{ccccccccccccccccccc}
x_5 & \geqslant & x_4 & \geqslant & \mathbf{x_3} & > & x_2 & \geqslant & x_1 & > & x_{-1} & \leqslant & x_{-2} & \leqslant & x_{-3} & \leqslant & x_{-4} & \geqslant & x_{-5} \\
\wedge & & \wedge & & & & || & & || & & || & & || & & || & & \not\wedge & & \\
\mathbf{y_4} & \geqslant & \mathbf{y_3} & & \geqslant & & y_2 & \geqslant & y_1 & > & y_{-1} & \leqslant & y_{-2} & \leqslant & y_{-3} & \leqslant & y_{-4} & &
\end{array}$$

where again every entry on the left side of the array is $\geqslant$ its mirror image on the right side of the array, and the bold entries are bigger than their mirror image.

Claim A.18. *If $x_j = x_{-j}$, then $A_{-(n+1)} = B_{-n}$.*

Proof. We have for all $i < j$ that $x_i = y_i$ and $x_{-i} = y_{-i}$. Since by assumption $x_j = x_{-j}$, we have for all $i \geqslant j$, $x_i = x_{-i}$. Moreover, if $j < n$ then by Claim A.16 $y_j = y_{-j}$ and for all $i \geqslant j$, we have $y_i = y_{-i}$. If ℓ is the length of the word v and $1 \leqslant p \leqslant \ell$, define the vector $\vec{p}$ to be the vector of length ℓ, which has a 1 in position p and 0's elsewhere. Then recalling that $A_{n+1} = X = B_n$, we have the equalities:

$$\begin{aligned}
A_{-(n+1)} = X - \sum_{i=1}^{n} \vec{x}_i + \sum_{i=1}^{n} \vec{x}_{-i} = X - \sum_{i=1}^{j-1} \vec{x}_i + \sum_{i=1}^{j-1} \vec{x}_{-i} = X - \sum_{i=1}^{j-1} \vec{y}_i + \sum_{i=1}^{j-1} \vec{y}_{-i} \\
= X - \sum_{i=1}^{n-1} \vec{y}_i + \sum_{i=1}^{n-1} \vec{y}_{-i} = B_{-n}. \quad \square
\end{aligned}$$

Claim A.19. *We have $x_j = x_{-j}$.*

Proof. Suppose $x_j > x_{-j}$.

Case 1: $j = n$. By the definition of j, we have $x_{n-1} = y_{n-1}$ and by Claim A.13 we have $x_{-(n-1)} = y_{-(n-1)}$. Since $x_{-n} < x_n$, this implies $A_{-n}(1 : x_{-n}) = B_{-n}(1 : x_{-n})$. Since A_{-n} contains an $(n, n+1)$-unbracketed n in position x_{-n}, so does B_{-n}. Therefore, $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$.

Case 2a: $j < n$ and $x_{n-1} = x_{-(n-1)}$. We have $y_{-(n-1)} \leqslant x_{-(n-1)} \leqslant x_n$. Since $x_n < y_{n-1}$ this means that we cannot have $y_{-(n-1)} = x_n$, so we must have $y_{-(n-1)} < x_n$. Since $x_{n-1} = x_{-(n-1)}$ and $y_{n-1} > x_n$, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ is obtained from the $n/(n+1)$-subword of $A_n(1 : x_{-n})$ by:

(1) Erasing an n from x_n and adding an n in $y_{-(n-1)}$. (Note $y_{-(n-1)} < x_n$.)
(2) Adding an $n+1$ to x_n.

Therefore, since the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ contains an $(n, n+1)$-unbracketed n and each one of these two steps does not change that property, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ also does. This implies $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$.

Case 2b: $j < n$ and $x_{n-1} > x_{-(n-1)}$. Since, $x_{n-1}, y_{n-1} \in [1 : x_{n-1}]$ and $x_{n-1}, x_n \in [x_{n-1} + 1 : x_n]$ and $y_{n-1} > x_n$, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ is obtained from the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ by:

(1) Erasing an n from $x_{-(n-1)}$ and adding an n in $y_{-(n-1)}$. (Note $y_{-(n-1)} \leqslant x_{-(n-1)}$.)
(2) Adding an n to x_{n-1} and erasing an n from x_n. (Note $x_{n-1} \leqslant x_n$.)
(3) Adding an $n+1$ to x_n.

Therefore, since the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ contains an $(n, n+1)$-unbracketed n and each one of these three steps does not change that property, so does the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$. This implies $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$. □

Since, indeed $x_j = x_{-j}$, we have $A_{-(n+1)} = B_{-n}$ by Claim A.18, which completes the proof of Lemma A.4.

A.3. Proof of Lemma A.4 for $j = n$ and $j' = n - 1$

Lemma A.20. *Suppose v is I_0-lowest weight and $h < n - 1$. Suppose that $(e_2 \cdots e_{n-1})e_1^h(v) \neq 0$ and $e_2 \cdots e_n e_1^h(v) \neq 0$. If $f_n^1 f_n^1 e_{\overline{1}}^n e_1^n(v)$ is I_0-lowest weight, then $f_n^1 f_{n-1}^1 e_{\overline{1}}^{n-1} e_1^n(v)$ is I_0-lowest weight.*

Proof of Lemma A.20. Suppose v and $v' = f_n^1 f_n^1 e_{\overline{1}}^n e_1^h(v)$ are I_0-lowest weight and $(e_2 \cdots e_{n-1})e_1^h(v) \neq 0$. We must show that $f_n^1 f_{n-1}^1 e_{\overline{1}}^{n-1} e_1^h(v)$ is I_0-lowest weight.

Claim A.21. *Given a word W, define $L(W)$ to be the length of the longest weakly increasing subsequence of W. If V is I_0-lowest weight, and W and V are in the same*

I_0-connected component, then the number of $(n+1)$'s which appear in V is equal to $L(W)$.

Proof. This easily follows from analyzing the RSK insertion tableaux of the words. □

Claim A.22. *We have $L(e_{\overline{1}}^{n-1}e_1^h(v)) \geqslant L(e_{\overline{1}}^{n}e_1^h(v))$.*

Proof. Since $Y = e_2 \cdots e_{n-1}e_1^h(v) \neq 0$, by inspection of the insertion tableaux of v and Y we observe that $\varphi_1(Y) = 0$, $\varphi_2(Y) = 1$, and $\varphi_k(Y) = 0$ for all $k > 2$. This implies that Y contains a letter 2 which precedes all letters 1. Hence $e_{\overline{1}}^{n-1}e_1^h(v) = e_{-1}(Y) \neq 0$, so the statement $L(e_{\overline{1}}^{n-1}e_1^h(v)) \geqslant L(e_{\overline{1}}^{n}e_1^h(v))$ is well-defined.

We will now recycle notation from the proof of Section A.2 with slight changes. Let $X = e_1^h(v)$. For $2 \leqslant i \leqslant n+1$, set $A_i = (e_i \cdots e_n)(X)$ and $B_i = (e_i \cdots e_{n-1})(X)$. Set $A_1 = e_{-1}(A_2)$ and $B_1 = e_{-1}(B_2)$. Let x_i be the integer which represents the position, where A_{i+1} and A_i differ and y_i be the integer which represents the position where B_{i+1} and B_i differ.

Suppose that v contains r letters $(n+1)$. It follows from weight considerations that v' contains $(r+1)$ letters $(n+1)$. This implies that $L(e_{\overline{1}}^{n}e_1^h(v)) = r+1$ whereas $L(e_2 \cdots e_n e_1^h(v)) = r$. This is to say $L(A_1) = r+1$ and $L(A_2) = r$. So A_1 contains a weakly increasing subsequence of length $r+1$, specified by the indices $i_1^0, \dots, i_1^r$. We must have that $i_1^0 = x_1$ and that $A_1(i_1^1) = 1$, otherwise the same indices would specify a weakly increasing subsequence of A_2 of length $r+1$. It follows that A_2 has a weakly increasing subsequence given by the indices $i_2^1, \dots, i_2^r$ where $A_2(i_2^1) = 1$. Now suppose $2 \leqslant k \leqslant n$ and A_k has a weakly increasing subsequence given by the indices $i_k^1, \dots, i_k^r$, where $A_k(i_k^1) = 1$. If $x_k \notin \{i_k^1, \dots, i_k^r\}$, then A_{k+1} has such a subsequence specified by the same indices.

Now suppose that $x_k \in \{i_k^1, \dots, i_k^r\}$. Create a list of indices as follows:

(1) If $i_k^j \leqslant x_k$ or $A_k(i_k^j) \neq k$, then $i_{k+1}^j = i_k^j$.
(2) If $i_k^j > x_k$ and $A_k(i_k^j) = k$, then $A_k(i_k^j)$ is $(k, k+1)$-bracketed with some $k+1$ in a position between x_k and i_k^j. Let i_{k+1}^j denote this position.

This creates a set $\{i_{k+1}^1, \dots, i_{k+1}^r\}$, which, after a possible reordering into increasing order, specifies a weakly increasing subsequence of A_{k+1} with $A_{k+1}(i_{k+1}^1) = 1$.

By induction $B_n = A_{n+1} = X$ has a weakly increasing subsequence specified by the indices $\{{i'}_n^1, \dots, {i'}_n^r\}$, with $B_n({i'}_n^1) = 1$. Let $k > 1$ and assume B_{k+1} has a weakly increasing subsequence specified by the indices $\{{i'}_{k+1}^1, \dots, {i'}_{k+1}^r\}$, with $B_{k+1}({i'}_{k+1}^1) = 1$. If $y_k < {i'}_{k+1}^1$, then the same is true of B_k with the same indices. If $y_k > {i'}_{k+1}^1$ then $B_k = e_k(B_{k+1}) = [B_{k+1}(1 : {i'}_{k+1}^1) \;\; e_k(B_{k+1}({i'}_{k+1}^1 + 1 : \ell))]$. Since $B_{k+1}({i'}_{k+1}^1 + 1 : \ell)$ has a weakly increasing subsequence of length $r-1$, $e_k(B_{k+1}({i'}_{k+1}^1 + 1 : \ell))$ does as well. Thus $B_k = [B_{k+1}(1 : {i'}_{k+1}^1) \;\; e_k(B_{k+1}({i'}_{k+1}^1 + 1 : \ell))]$ has a weakly increasing subsequence of length r specified by some indices $\{{i'}_k^1, \dots, {i'}_k^r\}$, with $B_k({i'}_k^1) = 1$ (where ${i'}_k^1 = {i'}_{k+1}^1$).

By induction this is true for $k = 2$. Since $e_{-1}(B_2) = B_1$ is defined and since $B_2(i'^1_2) = 1$, we have $y_1 < i'^1_2$ and so $\{y_1, i'^1_2, \ldots, i'^r_2\}$ is a list of indices which give a weakly increasing subsequence of length $r + 1$ in B_1. □

We want to show that $f^1_n f^1_{n-1} e^{n-1}_{\bar{1}} e^h_1(v)$ is I_0-lowest weight. Now $e_{-1}(Y)$ is obtained from $Y = e_2 \cdots e_{n-1} e^h_1(v)$ by changing its first 2 to 1. As a result $\varphi_1(e_{-1}(Y)) \in \{1, 2\}$ and $\varphi_k(e_{-1}(Y)) = 0$ for all $k > 1$. Therefore, we may write $e_{-1}(Y) = e^s_{\bar{1}} e^t_1(v^*)$ for some I_0-lowest weight element v^*, and $s \geqslant 0$ and $t > 0$ with $t \geqslant s$ (using Lemma 3.3 when $\varphi_1(e_{-1}(Y)) = 2$). This gives $v^* = f^1_t f^1_s e^{n-1}_{\bar{1}} e^h_1(v)$. Since v' contains one more $n + 1$ than v, it follows from Claims A.21 and A.22 that v^* contains at least one more $n + 1$ than v, which means we must have $t = n$. This also means that v and v^* are not in the same connected I_0-component. But if $v = f^1_h f^{\bar{1}}_{n-1} e^s_{\bar{1}} e^n_1(v^*)$ is in a different connected I_0-component than v^*, then **C2'**(b) applies which forces $s = n - 1$. Thus $v^* = f^1_n f^1_{n-1} e^{n-1}_{\bar{1}} e^h_1(v)$.

This concludes the proof of Lemma A.20. □

Proposition A.23. *Lemma A.4 with $j = n - 1$ and $j' = n$ and Lemma A.20 imply Lemma A.4.*

Proof. We need to show that if v is I_0-lowest weight, $e^{n-1}_{\bar{1}} e^n_1(v) \neq 0$, $e^n_{\bar{1}} e^n_1(v) \neq 0$, and $v^* = f^1_h f^{\bar{1}}_n e^n_{\bar{1}} e^n_1(v)$ is I_0-lowest weight, then $f^{\bar{1}}_{n-1} e^{n-1}_{\bar{1}} e^n_1(v) = f^{\bar{1}}_n e^n_{\bar{1}} e^n_1(v)$. Now $v = f^1_n f^1_n e^n_{\bar{1}} e^h_1(v^*)$ is I_0-lowest weight (in particular, $e_2 \cdots e_n e^h_1(v^*) \neq 0$). Now we show that $e_2 \cdots e_{n-1} e^h_1(v^*) \neq 0$. By definition, $e^h_1(v^*) \neq 0$. Either v^* has more n's than $(n - 1)$'s so that $e_2 \cdots e_{n-1} e^h_1(v^*) \neq 0$, or else v^* has the same number of n's as $(n - 1)$'s and $h = n - 2$ in which case also $e_2 \cdots e_{n-1} e^h_1(v^*) \neq 0$. Therefore, by Lemma A.20 $v' = f^1_n f^1_{n-1} e^{n-1}_{\bar{1}} e^h_1(v^*)$ is I_0-lowest weight. Rewriting this as $v^* = f^1_h f^{\bar{1}}_{n-1} e^{n-1}_{\bar{1}} e^n_1(v')$ and noting that $\mathrm{wt}(v) = \mathrm{wt}(v')$ implies $e^n_{\bar{1}} e^n_1(v') \neq 0$ Lemma A.4 with $j = n - 1$ and $j' = n$ gives $v^* = f^1_h f^{\bar{1}}_n e^n_{\bar{1}} e^n_1(v')$. This implies that $v = v'$ and that hence that $f^{\bar{1}}_{n-1} e^{n-1}_{\bar{1}} e^n_1(v) = f^{\bar{1}}_n e^n_{\bar{1}} e^n_1(v)$. □

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