A COMPLETE CLASSIFICATION OF HEREDITARILY EQUIVALENT PLANE CONTINUA

L. C. HOEHN AND L. G. OVERSTEEGEN

Abstract. A continuum is hereditarily equivalent if it is homeomorphic to each of its non-degenerate sub-continua. We show in this paper that the arc and the pseudo-arc are the only non-degenerate hereditarily equivalent plane continua.

1. Introduction

By a *continuum*, we mean a compact connected metric space. A continuum is *non-degenerate* if it contains more than one point. We refer to the space $\mathbb{R}^2$, with the Euclidean topology, as *the plane*. The Euclidean distance between two points x, y in $\mathbb{R}^2$ (or $\mathbb{R}^3$) will be denoted $\|x-y\|$. An *arc* is a space which is homeomorphic to the interval $[0,1]$. By a *map* we mean a continuous function.

A continuum X is *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua. This concept was introduced by Mazurkiewicz, who was interested in topological characterizations of the arc. In the second volume of Fundamenta Mathematicae in 1921, Mazurkiewicz [Maz21] asked (Problème 14) whether the arc is the only non-degenerate hereditarily equivalent continuum.

A continuum X is *decomposable* if it is the union of two proper subcontinua, and *indecomposable* otherwise. X is *hereditarily indecomposable* if every subcontinuum of X is indecomposable. X is *arc-like* (respectively, *tree-like*) if for every $\varepsilon > 0$ there exists an ε-map from X to $[0,1]$ (respectively, to a tree), where $f : X \to Y$ is an *ε-map* if for each $y \in Y$ the preimage $f^{-1}(y)$ has diameter less than ε. Henderson [Hen60] showed that the arc is the only decomposable hereditarily equivalent continuum. Cook [Coo70] has shown that every hereditarily equivalent continuum is tree-like.

Problème 14 of Mazurkiewicz was formally answered by Moise [Moi48] in 1948, who constructed another hereditarily equivalent plane continuum which he called the "pseudo-arc", due to this property it has in common with the arc. The pseudo-arc is a one-dimensional fractal-like hereditarily indecomposable arc-like continuum. Such a space was constructed by Knaster [Kna22] in 1922, and another by Bing [Bin48] in 1948 which he proved was topologically homogeneous. Bing [Bin51] proved in 1951 that the pseudo-arc is the only hereditarily indecomposable arc-like continuum. From this characterization it follows that the spaces of Knaster,

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Moise, and Bing are all homeomorphic, and also it can immediately be seen that the pseudo-arc is hereditarily equivalent.

Since Moise's article, the question has been: What are all hereditarily equivalent continua? The main result of this paper is:

Theorem 1. *If X is a non-degenerate hereditarily equivalent plane continuum, then X is homeomorphic to the arc or to the pseudo-arc.*

It remains an open question whether there exists any other hereditarily equivalent continuum in $\mathbb{R}^3$. We remark that by Cook's result [Coo70], all hereditarily equivalent continua are 1-dimensional, hence are embeddable in $\mathbb{R}^3$.

As part of the sequel we will also give a new characterization (Theorem 7) of the pseudo-arc.

2. Plane strips

If a continuum admits an ε-map to an arc then it can be covered by a chain of open sets whose diameters are less than ε (i.e. a set that roughly looks like a tube of small diameter). The notion of an ε-strip (see Definition 3 below), introduced in [OT82] in a slightly different form, conveys a similar feeling. However, it was observed in [OT82, Figure 1] that, for arbitrarily small $\varepsilon > 0$, there exists an ε-strip which does not admit a 1-map to an arc. Nevertheless we show in this paper (Theorem 8 below) that if a hereditarily indecomposable plane continuum is contained in an ε-strip for arbitrarily small $\varepsilon > 0$, then it must in fact be homeomorphic to the pseudo-arc.

Given two points x, y in the plane $\mathbb{R}^2$ we denote by $\overline{xy}$ the straight line segment joining them. Given points $v_1, \ldots, v_n$ in $\mathbb{R}^2$, the *polygonal arc* A with vertices $v_1, \ldots, v_n$ is the union of the straight line segments $\overline{v_1 v_2}, \ldots, \overline{v_{n-1} v_n}$. Denote the *vertex set* of A by $V_A = \{v_1, \ldots, v_n\}$. If $v_n = v_1$, then we call A a *polygonal closed curve*. We will need the following lemma which was proved in [OT82].

Lemma 2 ([OT82], Lemma 2.1)**.** *Let T be a polygonal closed curve in $\mathbb{R}^2$ with vertex set V_T. Given any $z \in \mathbb{R}^2 \setminus T$ we say z is* odd *(respectively,* even*) with respect to T if there exists a polygonal arc A with vertex set V_A from z to a point in the unbounded component of $\mathbb{R}^2 \setminus T$ so that $A \cap V_T = \emptyset = T \cap V_A$ and the number of crossings between A and T is odd (respectively, even). By a* crossing *we mean a pair of segments (between adjacent vertices), one from A and one from T, which intersect.*

Then this notion of odd/even is well-defined, i.e. independent of the choice of A.

Equivalently, z is odd (respectively, even) if the winding number of T around z is odd (respectively, even). Observe that the orientation of T is irrelevant here.

A component U of $\mathbb{R}^2 \setminus T$ is called *odd* (respectively, *even*) if each point of U is odd (respectively, *even*) with respect to T. Clearly the unbounded complementary domain of T is even.

A map $f : [0,1] \to \mathbb{R}^2$ is *piecewise linear* if there are finitely many points $0 = t_1 < t_2 < \ldots < t_n = 1$ such that for each $i = 1, \ldots, n-1$, as t runs from t_i to t_{i+1}, $f(t)$ parameterizes the straight line segment $\overline{f(t_i) f(t_{i+1})}$. If f is a piecewise linear map, then clearly $f([0,1])$ is a polygonal arc.

Definition 3 (ε-strip)**.** Suppose that $f, g : [0,1] \to \mathbb{R}^2$ are two piecewise linear maps into the plane such that $f([0,1]) \cap g([0,1]) = \emptyset$ and for all $t \in [0,1]$, $\|f(t) -$

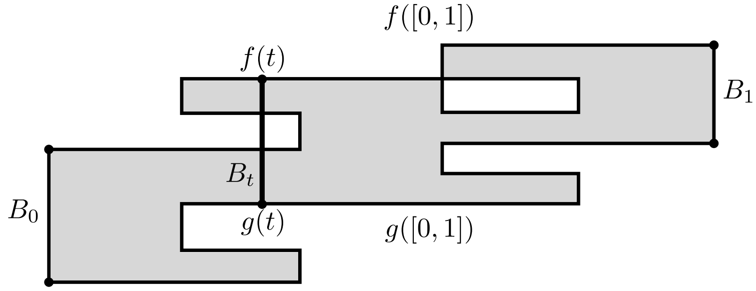

FIGURE 1. An illustration of an ε-strip, with a generic bridge B_t drawn. This strip has one odd domain, which is shaded gray.

$g(t)\| < \varepsilon$. Let $B_t = \overline{f(t)g(t)}$, and let $T_t = B_0 \cup f([0,t]) \cup g([0,t] \cup B_t$. We denote the union of all odd (respectively, even) complementary domains of T_t by S_t^- (respectively, S_t^+). If $B_0 \cap B_1 = \emptyset$, then we say that S_1^- is an *ε-strip with disjoint ends.*

See Figure 1 for an illustration of a simple ε-strip.

We say a continuum X is contained in an ε-strip with disjoint ends if there exist such f, g as in the above definition such that $X \subset S_1^-$. Observe that in this situation, $X \cap B_0 = \emptyset = X \cap B_1$.

If X is an indecomposable and hereditarily equivalent plane continuum, then it contains uncountably many pairwise disjoint copies of itself. In particular it contains a copy of $X \times C$, where C is the Cantor set [vD93]. This is the key observation behind the following result.

Lemma 4 ([OT84], Theorem 15)**.** *Suppose that X is a non-degenerate, indecomposable and hereditarily equivalent plane continuum. Then there exists a non-degenerate subcontinuum Y such that for each $\varepsilon > 0$, Y is contained in an ε-strip with disjoint ends.*

3. Separators

In light of Lemma 4 above, to prove Theorem 1 it suffices to show that any hereditarily indecomposable continuum X contained in arbitrarily small plane strips is homeomorphic to the pseudo-arc (Theorem 8). Our strategy below is to consider a small strip containing the continuum X, and to approximate X by a graph G contained in that strip. If we vary t from 0 to 1, the bridge B_t in the strip sweeps across the graph G. As it does so, it may wander back and forth in G, in a pattern whose essential property is captured in the following result. We will then use the crookedness of the hereditarily indecomposable continuum X to match with that pattern (see Theorems 6 and 7 below) to obtain an ε-map to an arc.

Lemma 5. *Suppose that a graph G is contained in an ε-strip S_1^- with disjoint ends. Let*

$$C = \{(x,t) \in G \times [0,1] : x \in B_t\}.$$

Then C separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$.

Proof. Define the function $\varphi : G \times [0,1] \to \mathbb{R}$ by

$$\varphi(x,t) = \begin{cases} +d(x,B_t) & \text{if } x \in S_t^+ \\ -d(x,B_t) & \text{if } x \in S_t^- \\ 0 & \text{otherwise,} \end{cases}$$

where $d(x,B_t) = \inf\{\|x-b\| : b \in B_t\}$. Then φ is a continuous function (see the proof of Lemma 2.3 in [OT82]). Since $S_0^- = \emptyset = B_0 \cap G$, $\varphi(x,0) > 0$ for each $x \in G$. Similarly, since $G \subset S_1^-$, $\varphi(x,1) < 0$ for each $x \in G$. Hence the set of points C where $\varphi(x,t) = 0$ must separate $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$. □

In [HO16, Theorem 20], the authors gave a characterization of hereditarily indecomposable continua in terms of sets as in Lemma 5 which separate $G \times \{0\}$ from $G \times \{1\}$ in the product $G \times [0,1]$ of a graph G with $[0,1]$. Here we give a simplified version of that theorem, which is more broadly applicable.

Theorem 6. *A continuum X is hereditarily indecomposable if and only if for any map $f : X \to G$ to a graph G, and for any open set $U \subseteq G \times (0,1)$ which separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$, there exists a map $h : X \to U$ such that $f = \pi_1 \circ h$ (where $\pi_1 : G \times [0,1] \to G$ is the first coordinate projection).*

Proof. According to [HO16, Theorem 20], a continuum X is hereditarily indecomposable if and only if for any map $f : X \to G$ to a graph G with metric d, for any set $M \subseteq G \times (0,1)$ which separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$, for any open set $U \subseteq G \times [0,1]$ with $M \subseteq U$, and for any $\varepsilon > 0$, there exists a map $h : X \to U$ such that $d(f(x), \pi_1 \circ h(x)) < \varepsilon$ for all $x \in X$. The condition in the present theorem is clearly stronger than this condition from [HO16, Theorem 20]. Therefore, to prove the present theorem we need only consider the forward implication.

Suppose X is hereditarily indecomposable, let $f : X \to G$ be a map to a graph G with metric d, and let $U \subset G \times (0,1)$ be an open set which separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$. It is well-known (see e.g. [Kur68, Theorem §46.VII.3]) that there exists a closed set $M \subset U$ which also separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$. Let $\varepsilon > 0$ be small enough so that the open set

$$U_1 = \{(g,t) \in G \times [0,1] : \text{there exists } (g',t') \in M \text{ such that } d(g,g') < \varepsilon \text{ and } |t-t'| < \varepsilon\}$$

is contained in U. Let

$$U_2 = \{(g,t) \in G \times [0,1] : \text{there exists } (g',t') \in M \text{ such that } d(g,g') < \tfrac{\varepsilon}{2} \text{ and } |t-t'| < \varepsilon\},$$

and apply [HO16, Theorem 20] to obtain a map $h' : X \to U_2$ such that $d(f(x), \pi_1 \circ h'(x)) < \frac{\varepsilon}{2}$ for all $x \in X$.

Define $h : X \to U$ by $h(x) = (f(x), \pi_2 \circ h'(x))$, where $\pi_2 : G \times [0,1] \to [0,1]$ is the second coordinate projection. Clearly this function h is continuous, and $f = \pi_1 \circ h$. To see that the range of h is really contained in U, let $x \in X$, and denote $h'(x) = (g,t)$, so that $h(x) = (f(x),t)$. Because $h'(x) \in U_2$, there exists $(g',t') \in M$ such that $d(g,g') < \frac{\varepsilon}{2}$ and $|t-t'| < \varepsilon$. Moreover, by choice of h' we have $d(f(x),g) < \frac{\varepsilon}{2}$. So by the triangle inequality, we have $d(f(x),g') < \varepsilon$, which means $h(x) \in U_1 \subseteq U$, as desired. □

By Bing's [Bin51] result a hereditarily indecomposable continuum is homeomorphic to the pseudo-arc if and only if it is arc-like. A new characterization of the pseudo-arc, involving the notion of *span zero* (see [Lel64]), was obtained in [HO16]. It states that a hereditarily indecomposable continuum is a pseudo-arc if and only if it has span zero. The more technical characterization of the pseudo-arc in Theorem 7 below is useful in cases when (like in the case of hereditarily equivalent plane continua) it is not a priori known that X has span zero.

In the statement below we assume that all spaces (i.e., X, G, and I) are contained in Euclidean space $\mathbb{R}^3$. One could just as well use the Hilbert cube $[0,1]^{\mathbb{N}}$, depending on the intended application.

Theorem 7. *Suppose that $X \subset \mathbb{R}^3$ is a hereditarily indecomposable continuum. Then the following are equivalent:*

(1) X is homeomorphic to the pseudo-arc;

(2) For each $\varepsilon > 0$ there exist a graph $G \subset \mathbb{R}^3$, a map $f : X \to G$ with $\|x - f(x)\| < \varepsilon$ for each $x \in X$, and an arc $I \subset \mathbb{R}^3$ with endpoints a and b, such that the set

$$U = \{(x,t) \in G \times (I \setminus \{a,b\}) : \|x - t\| < \varepsilon\}$$

separates $G \times \{a\}$ from $G \times \{b\}$ in $G \times I$.

Proof. Suppose X is homeomorphic to the pseudo-arc, and fix $\varepsilon > 0$. Note X is arc-like and, hence [Lel64], X has span zero. Therefore, according to Theorem 4 of [HO16], there exists $\delta > 0$ such that for any graph $G \subset \mathbb{R}^3$ and arc $I \subset \mathbb{R}^3$ both within Hausdorff distance δ from X, the set $U = \{(x,t) \in G \times I : \|x - y\| < \varepsilon\}$ separates $G \times \{a\}$ from $G \times \{b\}$ in $G \times I$, where a, b are the endpoints of I. We may assume that $\delta < \varepsilon$. Since X is arc-like, we may choose an arc $G \subset \mathbb{R}^3$ within Hausdorff distance δ of X and a map $f : X \to G$ such that $\|x - f(x)\| < \varepsilon$ for all $x \in X$. Choose any arc I within Hausdorff distance δ from X. Then G, f, and I satisfy the conditions of statement (2), as desired.

Conversely, suppose statement (2) holds. To prove that X is homeomorphic to the pseudo-arc, by [Bin51] it suffices to show that for each $\varepsilon > 0$ there exists an ε-map from X to an arc. Fix $\varepsilon > 0$. Suppose that $G \subset \mathbb{R}^3$ is a graph, $f : X \to G$ is a map such that $\|x - f(x)\| < \frac{\varepsilon}{4}$ for all $x \in X$, $I \subset \mathbb{R}^3$ is an arc with endpoints a and b, and

$$U = \left\{(x,t) \in G \times (I \setminus \{a,b\}) : \|x - t\| < \frac{\varepsilon}{4}\right\}$$

separates $G \times \{a\}$ from $G \times \{b\}$ in $G \times I$. Denote by $\pi_1 : G \times I \to G$ the first coordinate projection and by $\pi_2 : G \times I \to I$ the second coordinate projection. By Theorem 6 there exists a map $h : X \to U$ such that $f = \pi_1 \circ h$. We claim that $\pi_2 \circ h(x) : X \to I$ is an ε-map. To see this suppose that $\pi_2 \circ h(x_1) = \pi_2 \circ h(x_2)$. Then

$$\begin{aligned} \|x_1 - x_2\| \leq \|x_1 - f(x_1)\| + \|\pi_1 \circ h(x_1) - \pi_2 \circ h(x_1)\| + \\ + \|\pi_2 \circ h(x_2) - \pi_1 \circ h(x_2)\| + \|f(x_2) - x_2\| < \varepsilon. \end{aligned}$$

□

4. Proof of main result

We now apply the results established above to prove the following key theorem.

Theorem 8. *Let $X \subset \mathbb{R}^2$ be a hereditarily indecomposable plane continuum such that for each $\varepsilon > 0$, there is an ε-strip with disjoint ends containing X. Then X is homeomorphic to the pseudo-arc.*

Proof. Let $\varepsilon > 0$, and consider an $\frac{\varepsilon}{2}$-strip with disjoint ends containing X. That is, consider piecewise linear maps $f, g : [0,1] \to \mathbb{R}^2$ such that $f([0,1]) \cap g([0,1]) = \emptyset$, $\|f(t) - g(t)\| < \frac{\varepsilon}{2}$ for each $t \in [0,1]$, and $X \subset S_1^-$. Identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, and adjust f slightly to obtain a map $f' : [0,1] \to \mathbb{R}^3$ which is one-to-one (so that $f'([0,1])$ is an arc) and $\|f(t) - f'(t)\| < \frac{\varepsilon}{2}$ for all $t \in [0,1]$.

Clearly X is 1-dimensional, so there exists a graph $G \subset S_1^-$ and a map $h : X \to G$ such that $\|x - h(x)\| < \varepsilon$ for all $x \in X$. By Lemma 5, the set

$$C = \{(x,t) \in G \times [0,1] : x \in B_t\}$$

separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$. Clearly this set C is contained in

$$U = \left\{(x,t) \in G \times (0,1) : \|x - f(t)\| < \frac{\varepsilon}{2}\right\},$$

and the image of this set U under the homeomorphism $\mathrm{id} \times f' : G \times [0,1] \to G \times f'([0,1])$ is contained in

$$U' = \{(x,y) \in G \times f'((0,1)) : \|x - y\| < \varepsilon\}.$$

Therefore U' separates separates $G \times \{f'(0)\}$ from $G \times \{f'(1)\}$ in $G \times f'([0,1])$. Hence, by Theorem 7, X is homeomorphic to the pseudo-arc. □

We are now ready to prove our main result, Theorem 1.

Proof of Theorem 1. Let X be a non-degenerate hereditarily equivalent plane continuum. If X is decomposable, then X is an arc by [Hen60]. Suppose then that X is indecomposable, and hence hereditarily indecomposable. By Lemma 4, we may assume that X is embedded in the plane so that for each $\varepsilon > 0$, X is contained in an ε-strip with disjoint ends. It then follows from Theorem 8 that X is homeomorphic to the pseudo-arc. □

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(L. C. Hoehn) Nipissing University, Department of Computer Science & Mathematics, 100 College Drive, Box 5002, North Bay, Ontario, Canada, P1B 8L7

Email address: `loganh@nipissingu.ca`

(L. G. Oversteegen) University of Alabama at Birmingham, Department of Mathematics, Birmingham, AL 35294, USA

Email address: `overstee@uab.edu`