Topology and its Applications 258 (2019) 172–176

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Equivalence of the Rothberger, k-Rothberger, and restricted Menger Games ☆

Logan Crone, Lior Fishman *, Nathaniel Hiers, Stephen Jackson

University of North Texas, Department of Mathematics, 1155 Union Circle #311430, Denton, TX 76203-5017, USA

A R T I C L E   I N F O

Article history:
Received 23 December 2017
Received in revised form 7 June 2018
Accepted 26 July 2018
Available online 1 March 2019

Keywords:
Covering games
Rothberger game
Menger game

A B S T R A C T

We prove that in any Hausdorff space, the Rothberger game is equivalent to the k-Rothberger game, i.e. the game in which player II chooses k open sets in each move. This result follows from a more general theorem in which we show these games are equivalent to a game we call the restricted Menger game. In this game I knows immediately in advance of playing each open cover how many open sets II will choose from that open cover. This result illuminates the relationship between the Rothberger and Menger games in Hausdorff spaces. The equivalence of these games answers a question posed by Aurichi, Bella, and Dias [1], at least in the context of Hausdorff spaces.

1. Introduction

Let X be a topological space. Let $\mathcal{O}$ denote the collection of open covers of X. The *Menger game* [8] on X is the two-player game where at each round $n \in \omega$ of the game player I first plays an open cover $\mathcal{U}_n \in \mathcal{O}$ of X, and player II responds by playing a finite subset $\{U_n^0, \dots, U_n^{k_n-1}\}$ of $\mathcal{U}_n$. Player II wins the run of the game if $X = \bigcup_n \bigcup_{i<k_m} U_n^i$. We denote the Menger game by $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$. The notation reflects the facts that I is playing from $\mathcal{O}$, II is trying to build an element of $\mathcal{O}$, and II is picking a finite subset from I's moves at each round. The *Rothberger game* [2], $G_1(\mathcal{O}, \mathcal{O})$, on X is the game where player I plays at round n an open cover $\mathcal{U}_n \in \mathcal{O}$ and player II plays a single $U_n \in \mathcal{U}_n$. Again, player II wins the run of the game iff $X = \bigcup_n U_n$. The k-Rothberger game $G_k(\mathcal{O}, \mathcal{O})$ is the variation of the Rothberger game where player II plays k sets from I's cover at each round. A natural extension of this is the game $G_f(\mathcal{O}, \mathcal{O})$ where $f : \omega \to \omega \setminus \{0\}$. In this game, at each round n player II plays $f(n)$ sets from player I's move $\mathcal{U}_n$. A still further extension of the

☆ The authors wish to thank the organizers of the conference *Frontiers of Selection Principles* held at Cardinal Stefan Wyszński University in Warsaw in August 2017 where the results of this paper were presented. We also wish to thank Boaz Tsaban and the referee for useful comments and suggestions.

* Corresponding author.
E-mail addresses: logancrone@my.unt.edu (L. Crone), lior.fishman@unt.edu (L. Fishman), nathanielhiers@my.unt.edu (N. Hiers), jackson@unt.edu (S. Jackson).

https://doi.org/10.1016/j.topol.2019.02.051

games is the *restricted Menger game* $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$, which we define precisely below, where player II decides at the start of each round n how many sets he will get to choose from I's play $\mathcal{U}_n$. It is clear that

$$\forall f \text{ II wins } G_f(\mathcal{O},\mathcal{O}) \Leftrightarrow \forall k \text{ II wins } G_k(\mathcal{O},\mathcal{O}) \Leftrightarrow \text{II wins } G_1(\mathcal{O},\mathcal{O}) \Rightarrow$$

$$\Rightarrow \exists k \text{ II wins } G_k(\mathcal{O},\mathcal{O}) \Rightarrow \exists f \text{ II wins } G_f(\mathcal{O},\mathcal{O}) \Rightarrow \text{II wins } G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$$

Our main result, Theorem 2.2, is that for all T_2 spaces X, the above games are all equivalent. Recall two games are said to be equivalent if whenever one of the players has a winning strategy in one of the games, then that same player has a winning strategy in the other game. We note that the equivalence of the above games for arbitrary spaces is no stronger than the equivalence for T_0 spaces (by considering the T_0 quotient of an arbitrary space). On the other hand, it is well known that the full Menger game $G_{\text{fin}}(\mathcal{O},\mathcal{O})$ is not equivalent to the above mentioned games. For example, player II wins the Menger game on $\mathbb{R}$, or any σ-compact space, while I has a winning strategy in $G_1(\mathcal{O},\mathcal{O})$ on $\mathbb{R}$ (I can easily play to ensure that $\lambda(\bigcup U_n) < \varepsilon$, where λ denotes Lebesgue measure, for any given $\varepsilon > 0$).

The games mentioned above are closely related to selection principles on the space X. These types of covering games and selection principles were extensively studied by Scheepers and others, see for example [7], [6]. Recall that X has the *Menger property*, denoted $\mathcal{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$, if whenever $\{\mathcal{U}_n\}_{n\in\omega}$ is a sequence of open covers of X, then there is a sequence $\{\mathcal{F}_n\}_{n\in\omega}$, where each $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$, such that $X = \bigcup_n \cup\mathcal{F}_n$. Similarly, X has the *Rothberger property*, denoted $\mathcal{S}_1(\mathcal{O},\mathcal{O})$, if whenever $\{\mathcal{U}_n\}_{n\in\omega}$ is a sequence of open covers of X, then there is a sequence $U_n \in \mathcal{U}_n$ such that $X = \bigcup_n U_n$. There are two theorems which relate the games with the corresponding selection principles. One theorem, due to Hurewicz [4] (see also [7]), says that for any space X the selection principle $\mathcal{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$ (i.e., X having the Menger property) is equivalent to I not having a winning strategy in $G_{\text{fin}}(\mathcal{O},\mathcal{O})$. Another theorem, due to Pawlikowski [5], says that for any space X the selection property $\mathcal{S}_1(\mathcal{O},\mathcal{O})$ (i.e., X having the Rothberger property) is equivalent to I not having a winning strategy in $G_1(\mathcal{O},\mathcal{O})$. The equivalence of $\mathcal{S}_k(\mathcal{O},\mathcal{O})$ (where $k \in \omega$) and $\mathcal{S}_1(\mathcal{O},\mathcal{O})$ was shown in [3] and noted by the authors of [1].

The Rothberger game $G_1(\mathcal{O},\mathcal{O})$, for any space X, has a dual version called the *point-open game*. In this game, I plays at each round n a point $x_n \in X$, and II then plays an open set U_n with $x_n \in U_n$. Player I wins the run of the game iff $X = \bigcup_n U_n$. A theorem of Galvin [2] says that (for any X) these games are *dual*, that is, one of the players has a winning strategy in one of the games iff the other player has a winning strategy in the other game. A natural variation of the point-open game is the *finite-open game*, where I plays at each round n a finite set $F_n \subseteq X$, and II plays an open set U_n with $F_n \subseteq U_n$. Player I again wins the run iff $X = \bigcup_n U_n$. It is easy to see that for any X that the point-open game is equivalent to the finite-open game.

Using these dual games (specifically the finite-open game) simplifies the presentation of our main result. This observation was noted by R. Dias, whom we thank.

2. Equivalence of restricted Menger and Rothberger games

We define a variation of the Menger game which we call the *restricted Menger game*, denoted by $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ (Fig. 1). The rounds of this game are as in the Menger game except that at the start of round n player II will make an initial move, which must be a positive integer k_n, which is a declaration of how many open sets II intends to select this round. As in the Menger game, I will then play an open cover $\mathcal{U}_n \in \mathcal{O}$, and II will then respond by choosing k_n of the sets from $\mathcal{U}_n$, which we denote $U^0_n, \dots, U^{k_n-1}_n$. Player II wins the run of the game iff $X = \bigcup_n \bigcup_{i<k_n} U^i_n$.

For the remainder of the paper we work in the base theory ZFC.

In general, one does not expect the games $G_1(\mathcal{O},\mathcal{O})$, $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ to be determined as they are not Borel games $A \subseteq Z^\omega$ played on some set Z (with Z having the discrete topology; the determinacy of such Borel

$G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$	I		$\mathcal{U}_0$		$\mathcal{U}_1$		$\mathcal{U}_2$		$\cdots$
	II	k_0		$\{U_0^i\}_{i<k_0}$ k_1		$\{U_1^i\}_{i<k_1}$ k_2		$\{U_2^i\}_{i<k_2}$	

Fig. 1. An illustration of the game $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$.

games is a fundamental theorem of Martin). In fact, even assuming determinacy axioms (which contradict AC) such as AD or the stronger axiom $\mathsf{AD}_{\mathbb{R}}$ of real-game determinacy, these games are not necessarily determined. Galvin [2] showed assuming $\mathsf{ZFC}+\mathsf{CH}$ that there is a subspace of $\mathbb{R}$ for which the game $G_1(\mathcal{O},\mathcal{O})$ is not determined (one can take a Luzin set). Todorčević [9] showed just in ZFC that there is a space (a T_4 space) for which $G_1(\mathcal{O},\mathcal{O})$ is not determined. We refer the reader to [9] and the references therein for a more complete history and discussion.

The assumption of the determinacy of the game $G_1(\mathcal{O},\mathcal{O})$, in fact, essentially trivializes our main result as the next fact shows.

Fact 2.1. *Let X be a topological space and assume the game $G_1(\mathcal{O},\mathcal{O})$ is determined. Then the game $G_1(\mathcal{O},\mathcal{O})$ is equivalent to the restricted Menger game $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$.*

Proof. If II wins $G_1(\mathcal{O},\mathcal{O})$ then clearly II wins $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$. The alternative, since $G_1(\mathcal{O},\mathcal{O})$ is determined, is that I wins $G_1(\mathcal{O},\mathcal{O})$. The first part of the proof of Theorem 2.2 below shows that I then has a winning strategy in $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$. □

The following theorem is our main result.

Theorem 2.2. *Let X be a T_2 space. Then the restricted Menger game $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ is equivalent to the Rothberger game $G_1(\mathcal{O},\mathcal{O})$.*

Proof. It is clear that if I wins $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ then I wins $G_1(\mathcal{O},\mathcal{O})$. It is also clear that if II wins $G_1(\mathcal{O},\mathcal{O})$, then II wins $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$.

If I wins $G_1(\mathcal{O},\mathcal{O})$, then by [5], X does not satisfy the selection principle $\mathcal{S}_1(\mathcal{O},\mathcal{O})$. Thus, there is a sequence $\{\mathcal{V}_n\}$ of open covers of X such that there is no sequence $V_n\in\mathcal{V}_n$ with $X=\bigcup_n V_n$. Then I has a winning strategy in $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ by playing as follows. If II first plays the integer k_0, then I plays the common refinement $\mathcal{U}_0=\mathcal{V}_0\wedge\cdots\wedge\mathcal{V}_{k_0-1}=\{V_0\cap V_1\cap\cdots\cap V_{k_0-1}\,:\,V_0\in\mathcal{V}_0,V_1\in\mathcal{V}_1,\cdots V_{k_0-1}\in\mathcal{V}_{k_0-1}\}$. II will end the round by picking k_0 of the sets $U_0^0,\dots,U_0^{k_0-1}$ from $\mathcal{U}_0$. Player I continues in this manner, i.e. playing for instance in round 1, a refinement $\mathcal{U}_1$ of the covers $\{\mathcal{V}_{k_0},\mathcal{V}_{k_0+1},\dots,\mathcal{V}_{k_0+k_1-1}\}$. Because each of the open covers $\mathcal{U}_n$ refines a block of covers from $\{\mathcal{V}_n\}$, there is a sequence $V_n\in\mathcal{V}_n$ with $\bigcup_n\bigcup_n^i U_n^i\subseteq\bigcup_n V_n$. Since $\bigcup_n V_n\neq X$, I has won this run of $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$.

Assume now that II has a winning strategy τ in $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$. We let $\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)$ denote the response of τ when I plays open covers $\mathcal{U}_0,\dots,\mathcal{U}_n$ (we are suppressing II's moves according to τ in this notation). So, $\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)$ is a finite subset of $\mathcal{U}_n$. We let $\tau'(\mathcal{U}_0,\dots,\mathcal{U}_n)$ denote the integer that τ plays at the start of the next round, immediately after $\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)$ was played. By $\cup\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)$ we mean the union of the (finitely many) open sets in $\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)$. Note that according to this notation $|\tau(\mathcal{U}_0,\dots,\mathcal{U}_n)|=\tau'(\mathcal{U}_0,\dots,\mathcal{U}_{n-1})$.

We define a strategy σ for I in the finite-open game on X. We begin by explicitly describing σ on the first round. Let $k_\varnothing=\tau'(\varnothing)$ be τ's first (integer) move in $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$. Define

$$C_\varnothing=\bigcap_{\mathcal{U}\in\mathcal{O}}\overline{\cup\tau(\mathcal{U})}.$$

The next Lemma is the only point in the proof where we use the assumption that X is T_2.[1]

Lemma 2.3. $|C_\varnothing| \leq k_\varnothing$.

Proof. Suppose towards a contradiction that $x_0, \dots, x_{k_\varnothing}$ are $k_\varnothing + 1$ distinct points in $C_\varnothing$. Since X is T_2, there are open sets $U_0, \dots, U_{k_\varnothing}$ in X with $x_i \in U_i$ for all $i \leq k_\varnothing$ and with the $\{U_i\}$ pairwise disjoint. For each $x \in X \setminus \{x_i\}_{i \leq k_\varnothing}$ let U_x be an open set containing x such that U_x is disjoint from a neighborhood of $\{x_i\}_{i \leq k_\varnothing}$ (using T_2 again). Let $\mathcal{U} = \{U_x : x \notin \{x_i\}_{i \leq k_\varnothing}\} \cup \{U_i\}_{i \leq k_\varnothing}$, so $\mathcal{U}$ is an open cover of X. $\tau(\mathcal{U})$ consists of $k_\varnothing$ of the sets from $\mathcal{U}$. There is an $i \leq k_\varnothing$ such that $U_i \notin \tau(\mathcal{U})$. Then $x_i \notin \overline{\cup\tau(\mathcal{U})}$, a contradiction to $x_i \in C_\varnothing$. ◁

Then let σ's first move in the finite-open game be $C_\varnothing$. Say II responds with V_0. Before we continue, we need to define some auxiliary sets which correspond to the position $(C_\varnothing, V_0)$. If V_0 was legal, then we note that $X \setminus V_0 \subseteq X \setminus C_\varnothing$, and thus for each $x \in X \setminus V_0$, there is some $\mathcal{U} \in \mathcal{O}$ such that $x \in X \setminus \overline{\cup\tau(\mathcal{U})}$. These sets form an open cover of $X \setminus V_0$, which is a closed subspace of X, and thus is Lindelöf, and so we fix $\{\mathcal{U}_{(m)}(V_0)\}_{m\in\omega} = \{\mathcal{U}_{(m)}\}_{m\in\omega}$ such that $\{X \setminus \overline{\cup\tau(\mathcal{U}_{(m)})}\}_{m\in\omega}$ is a cover of $X \setminus V_0$.

To define σ in subsequent rounds, we need to dovetail various moves on subsequences, using the previously defined open covers $\mathcal{U}_s$ for $s \in \omega^{<\omega}$, and for this purpose we fix any bijection $\varphi \colon \omega^{<\omega} \to \omega$ with the property that if $s \subseteq t$ then $\varphi(s) \leq \varphi(t)$. For $s \in \omega^{<\omega}$ we let $\mathrm{lh}(s)$ denote the length of s. Now in general, suppose we are at round n in the finite-open game, and the moves $C_0, V_0, \dots, C_{n-1}, V_{n-1}$ have been played, where $|C_i| = k_i$ for $i < n$. Assume in addition that for each $j < n$ we have defined open covers $\mathcal{U}_{\varphi^{-1}(j)^\frown m}$ for all $m \in \omega$ (which depend on the V_j played thus far). Furthermore, assume that the $C_j, V_j, \mathcal{U}_{\varphi^{-1}(j)^\frown m}$ for $j < n$ satisfy the following. Let $s = \varphi^{-1}(j)$, then:

1. $C_j = \bigcap_{\mathcal{U}\in\mathcal{O}} \overline{\cup\tau(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_{s\restriction \mathrm{lh}(s)}, \mathcal{U})}$.
2. $\{X \setminus \overline{\cup\tau(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_{s\restriction \mathrm{lh}(s)}, \mathcal{U}_{s^\frown m})}\}_{m\in\omega}$ is a cover of $X \setminus \bigcup_{i \leq \mathrm{lh}(s)} V_{\varphi(s\restriction i)}$.

Note that property (2) for j is possible since the space $X \setminus \bigcup_{i\leq \mathrm{lh}(s)} V_{\varphi(s\restriction i)}$ is Lindelöf and $X \setminus \bigcup_{i\leq \mathrm{lh}(s)} V_{\varphi(s\restriction i)} \subseteq X \setminus \bigcup_{i\leq \mathrm{lh}(s)} C_{\varphi(s\restriction i)}$, and using property (1) for the C_i for $i \leq j$.

We define σ's response to this position, and the necessary sets $\mathcal{U}_{t^\frown m}$, in a similar manner to the base step. Let $t = \varphi^{-1}(n)$ and define σ's response to be

$$C_n = \bigcap_{\mathcal{U}\in\mathcal{O}} \overline{\cup\tau(\mathcal{U}_{t\restriction 1}, \mathcal{U}_{t\restriction 2}, \dots, \mathcal{U}_{t\restriction \mathrm{lh}(t)}, \mathcal{U})},$$

which clearly maintains property (1). Note also that C_n is finite, and in fact has size at most $|C_n| \leq \tau'(\mathcal{U}_{t\restriction 1}, \dots, \mathcal{U}_{t\restriction \mathrm{lh}(t)})$, by the same proof of Lemma 2.3.

Similarly to the base step, define $\{\mathcal{U}_{t^\frown m}\}_{m\in\omega}$ to be a countable collection of open covers such that $\{X \setminus \overline{\cup\tau(\mathcal{U}_{t\restriction 1}, \mathcal{U}_{t\restriction 2}, \dots, \mathcal{U}_{t\restriction \mathrm{lh}(t)}, \mathcal{U}_{t^\frown m})}\}_{m\in\omega}$ covers $X \setminus \bigcup_{i\leq \mathrm{lh}(t)} V_{\varphi(t\restriction i)}$. Of course, this uses the fact that $X \setminus \bigcup_{i\leq \mathrm{lh}(t)} V_{\varphi(t\restriction i)}$ is Lindelöf and that it is contained in $X \setminus C_n$. This completes the definition of σ. To show that σ is winning, we suppose that $C_0, V_0, C_1, V_1, \dots$ is a full run of the finite-open game which is consistent with σ. Note that since this run is consistent with σ, we can recover the tree of open covers $\{\mathcal{U}_s\}_{s\in\omega^{<\omega}}$ associated to this run which satisfies the properties (1) and (2) above. Suppose that $X \neq \bigcup_n V_n$, and let $x \in X \setminus \bigcup_n V_n$. In particular, $x \in X \setminus V_0$. Now we use property (2) to obtain i_0 such that $x \notin \overline{\cup\tau(\mathcal{U}_{(i_0)})}$. In general, supposing we have $i_0, i_1, \dots, i_{n-1}$ where $x \notin \overline{\cup\tau(\mathcal{U}_{(i_0)}, \dots \mathcal{U}_{(i_0,\dots,i_k)})}$ for any $k < n$, then use the fact that $x \in X \setminus \bigcup_{s\subseteq\varphi^{-1}(n)} V_{\varphi(s)}$ and property (2) to obtain i_n so that $x \notin \overline{\cup\tau(\mathcal{U}_{(i_0)}, \dots, \mathcal{U}_{(i_0,\dots,i_{n-1})}, \mathcal{U}_{(i_0,\dots,i_n)})}$.

[1] The fact that X is T_2 is necessary for the proof of the following lemma. For instance, consider $\mathbb{N}$ with the co-finite topology, then for any function f which selects k open sets from an open cover, we have that the set $C = \bigcap_{\mathcal{U}\in\mathcal{O}} \overline{\cup f(\mathcal{U})} = \mathbb{N}$ which clearly has more than k elements.

This builds a branch through the tree of open covers $\{\mathcal{U}_s\}_{s\in\omega^{<\omega}}$, associated to this run, which has the property that x is not in any of the closures of τ's moves in response to this branch. This contradicts the assumption that τ was a winning strategy. $\square$

Corollary 2.4. *For any T_2 space X and any $f\colon \omega \to \omega\setminus\{0\}$, the games $G_1(\mathcal{O},\mathcal{O})$ and $G_f(\mathcal{O},\mathcal{O})$ are equivalent.*

In particular, we have the following corollary which answers Problem 4.5 of [1] for T_2 spaces.

Corollary 2.5. *For any T_2 space X and any $n \in \omega$, the games $G_1(\mathcal{O},\mathcal{O})$ and $G_n(\mathcal{O},\mathcal{O})$ are equivalent.*

3. Open questions

A natural question is whether we can drop the assumption that X is T_2 from the hypothesis of Theorem 2.2. In fact, the authors of [1] originally asked if for any topological space the games $G_1(\mathcal{O},\mathcal{O})$ and $G_2(\mathcal{O},\mathcal{O})$ are equivalent. Our Theorem 2.2 shows these games are equivalent for any T_2 space, but the T_2 assumption seems necessary for the argument.

We are not aware of any space (with no assumptions on the space) for which these games are not equivalent. As we noted in Fact 2.1, the determinacy of the game $G_1(\mathcal{O},\mathcal{O})$ gives the equivalence of these games. However, since the determinacy of these games is not guaranteed in ZF, it is possible even that the equivalence for arbitrary spaces is independent of ZF.

Question 3.1. Can we weaken the hypotheses of Theorem 2.2 from T_2 to T_1, or even remove it entirely? That is, can we prove in ZFC that the games $G_1(\mathcal{O},\mathcal{O})$ and $G^*_{\text{fin}}(\mathcal{O},\mathcal{O})$ are equivalent for any space X?

One possibility for a negative answer to Question 3.1 would be to construct in ZFC a space for which the games are not equivalent (in this case the game $G_1(\mathcal{O},\mathcal{O})$ is not determined, and II must win the other game). It is also possible that the existence of a space for which the games are not equivalent is independent of ZFC. So we ask:

Question 3.2. Is it consistent with ZFC that there is a space X for which the games $G_1(\mathcal{O},\mathcal{O})$ and $G_2(\mathcal{O},\mathcal{O})$ are not equivalent? Is the existence of such a space consistent with ZF? In particular are the games equivalent in models of determinacy?

References

[1] Leandro Aurichi, Angelo Bella, Rodrigo Dias, Tightness games with bounded finite selections, Isr. J. Math. 224 (1) (2018) 133–158.
[2] Fred Galvin, Indeterminacy of point-open games, Bull. Acad. Pol. Sci. (1978) 445–448.
[3] Salvador García-Ferreira, Ángel Tamariz-Mascarúa, Some generalizations of rapid ultrafilters in topology and id-fan tightness, Tsukuba J. Math. (1995) 173–185.
[4] Witold Hurewicz, Über die Verallgemeinerung des Borelschen Theorems, Mathematische Zeitschrift, 1925, pp. 401–421.
[5] Janusz Pawlikowski, Undetermined sets of point-open games, Fundam. Math. 144 (1994) 279–285.
[6] Masami Sakai, Marion Scheepers, The combinatorics of open covers, in: K.P. Hart, Jan van Mill, P. Simon (Eds.), Recent Progress in General Topology III, Atlantis Press, 2014, pp. 751–799.
[7] Marion Scheepers, Combinatorics of open covers I: Ramsey theory, Topol. Appl. 69 (1996) 31–62.
[8] Rastislav Telgársky, On games of Topsøe, Math. Scand. 54 (1) (1984) 170–176.
[9] Stevo Todorčević, Aronszajn orderings, Publ. Inst. Math. (N.S.) 57 (71) (1995) 29–46.