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Abstract
Recently, Peeva and the second author constructed irreducible projective varieties with regularity much larger than their degree, yielding counterexamples to the Eisenbud–Goto Conjecture. Their construction involved two new ideas: Rees-like algebras and step-by-step homogenization. Yet, all of these varieties are singular and the nature of the geometry of these projective varieties was left open. The purpose of this paper is to study the singularities inherent in this process. We compute the codimension of the singular locus of an arbitrary Rees-like algebra over a polynomial ring. We then show that the relative size of the singular locus can increase under step-by-step homogenization. To address this defect, we construct a new process, we call prime standardization, which plays a similar role as step-by-step homogenization but also preserves the codimension of the singular locus. This is derived from ideas of Ananyan and Hochster and we use this to study the regularity of certain smooth hyperplane sections of Rees-like algebras, showing that they all satisfy the Eisenbud–Goto Conjecture, as expected. Along the way, we prove a somewhat surprising characterization of Rees-like algebras of Cohen–Macaulay ideals. In a similar vein, while Rees-like algebras are almost never Cohen–Macaulay and never normal, we fully characterize when they are seminormal, weakly normal, and, in positive characteristic, F-split.

Keywords Rees-like algebra · Regularity · Seminormal · F-split · Singular locus

Mathematics Subject Classification 13D02 · 13A35 · 14J17

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1 Introduction

Given a nondegenerate, embedded projective variety X over an algebraically closed field k corresponding to a homogeneous prime ideal $P \subseteq S = k[x_1, \ldots, x_n]$, the Eisenbud–Goto conjecture predicts an estimate on the Castelnuovo–Mumford regularity of X:

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1,$$

or equivalently

$$\operatorname{reg}(S/P) \leq \deg(S/P) - \operatorname{ht}(P). \tag{1}$$

Equation (1) fails for arbitrary schemes, that is, when P is not prime. A interesting construction introduced by the second author and Peeva [18] produced the first examples of projective varieties failing this bound by producing from a known embedded scheme with large regularity, a new projective variety embedded in a much larger space which also has large regularity. This reinforces the need to control the singularities of X to ensure optimal estimates for its regularity; in particular, the Eisenbud–Goto conjecture remains open for arbitrary smooth projective varieties or even some mildly singular varieties. There are many cases where the conjecture does hold including the case of curves [10], smooth surfaces in characteristic 0 [15,22], and certain 3-folds in characteristic 0 [23]. See also related work of Kwak–Park [14] and Noma [20]. There are also classes of mildly singular surfaces for which Eq. (1) holds, see [19].

The process in [18] of constructing the examples of projective varieties failing Eq. (1) involves two major steps. The first step is the construction of the Rees-like algebra, which defines a subvariety of a weighted projective space. Specifically, given a homogeneous ideal I in a polynomial ring S over a field k, the Rees-like algebra of I is the non-standard graded k-algebra $\mathcal{RL}(I) := S[It, t^2] \subseteq S[t]$.

The second step, which applies to any homogeneous ideal in a non-standard graded polynomial ring, produces an associated ideal in a much larger polynomial ring called its step-by-step homogenization. Unlike the usual homogenization of an ideal which defines the projective closure of an affine variety, the step-by-step homogenization produces a much larger variety; however, it preserves graded Betti numbers and primeness for nondegenerate primes, making it sufficient to produce the counterexamples to Eq. (1).

Thus far, explicit understanding of the geometry of the processes involved in both of these two steps is lacking. It was proved in [18] that Rees-like algebras are not Cohen–Macaulay but further structure of their singularities is not known. Moreover, the step-by-step homogenization used in [18] can increase the relative size of the singular locus. The goal of this paper is to better understand the behavior of the singularities and the size of the singular locus after taking each of these two steps. First, we compute the Jacobian of the Rees-like algebra explicitly leading to a complete description of the reduced subscheme structure of the singular locus of the associated affine variety.

Theorem A (Theorem 2) *Suppose I is a homogeneous ideal in a polynomial ring over a perfect field k with* $\operatorname{char}(k) \neq 2$. *Set* $X = \operatorname{Spec}(\mathcal{RL}(I))$. *Then there is a bijection between the irreducible components of the singular locus of X and those of* $\operatorname{Proj}(S/I)$. *Moreover, its codimension in X is*

$$\operatorname{codim}_X(\operatorname{Sing} X) = \operatorname{ht}(I).$$

One advantage of our explicit understanding of the Jacobian is that it allows one to push Theorem A farther. In particular, we give a sample of how to expand this in Theorem 2.5, where S is replaced by the coordinate ring of a smooth, non-degenerate projective variety. (In this case, further technical conditions on the presentation matrix of I are needed; see also Example 2).

Now we turn our attention to the second major idea in [18], namely step-by-step homogenization. Unfortunately, step-by-step homogenization does not preserve the relative size of the singular locus, see Example 3. We introduce a new notion called *prime standardization*, based on the idea of prime sequences introduced by Ananyan and Hochster in [1]. We show that the codimension of the singular locus of an arbitrary variety is preserved after applying a certain prime standardization.

Theorem B (Corollary 1) *Suppose I is a homogeneous ideal in a polynomial ring over an algebraically closed field k with $\operatorname{char}(k) \neq 2$. There is a prime standardization of the defining prime ideal of the Rees-like algebra $\mathcal{RL}(I)$ such that the irreducible components of the singular locus of the resulting projective variety Y and those of the scheme defined by I are in bijection. Moreover, its codimension in Y is*

$$\operatorname{codim}_Y(\operatorname{Sing} Y) = \operatorname{ht}(I).$$

While the varieties produced in [18] are highly singular, it is natural to consider the possibility of smooth hyperplane sections of those varieties. Using the above results and working over characteristic 0 fields, we exploit Bertini style arguments to show that the resulting smooth varieties satisfy Eq. (1). Along the way, we prove a characterization of Rees-like algebras of Cohen–Macaulay ideals I solely in terms of the singular locus of Rees-like algebras. More precisely, we prove the following:

Theorem C (Theorem 5*, Corollary* 2*) If X is an embedded projective scheme defined by a homogeneous ideal I, then there is a regular sequence of general hyperplane sections of a prime standardization of the Rees-like algebra of I which is smooth if and only if X is arithmetically Cohen-Macaulay. Moreover, all such varieties satisfy Eq.* (1).

Note that the varieties corresponding to Rees-like algebras of Cohen–Macaulay ideals are not themselves Cohen–Macaulay- far from it. In fact, a Rees-like algebra is Cohen–Macaulay if and only if the ideal in question is principal. Thus these examples are not covered by the known Cohen–Macaulay case of the Eisenbud–Goto Conjecture (see [5, Corollary 4.15]).

In Sect. 5,we proceed with our qualitative study of the singularities of Rees-like algebras. Namely, we address weak normality and seminormality of Rees-like algebras. In contrast to the case of Rees algebras, the characterization is simple and somewhat surprising, as $\mathcal{RL}(I)$ being weakly or seminormal is equivalent to the ideal I being radical.

Theorem D (Corollary 4) *Suppose k is a field with $\operatorname{char}(k) \neq 2$ and S is a polynomial ring over k. For a homogeneous S-ideal I, I is radical if and only if its Rees-like algebra $\mathcal{RL}(I)$ is seminormal if and only if $\mathcal{RL}(I)$ is weakly normal.*

The rich source of weakly normal Rees-like algebras indicates that the Rees-like construction should be well-behaved with respect to Frobenius splittings. We prove the following characterization of F-split Rees-algebras.

Theorem E (Theorem 9) *Suppose k is a field with $\operatorname{char}(k) > 2$ and I is a radical ideal in a polynomial ring S over k. The ring S/I is F-split if and only if $\mathcal{RL}(I)$ is F-split.*

2 Singular locus of the Rees-like algebra

We start by establishing our conventions used through the paper. Unless otherwise stated, k is a field and $S = k[x_1, \ldots, x_n]$ is a standard graded polynomial ring. We reserve the type face $\mathsf{A}, \mathsf{M}, \ldots$ for matrices. For a specific matrix M, the notation $I_t(\mathsf{M})$ denotes the ideal of $t \times t$-minors. We reserve bold letters $\mathbf{F}_\bullet, \mathbf{D}_\bullet, \ldots$ for chain complexes of modules with differentials $d_\bullet^{\mathbf{F}}, d_\bullet^{\mathbf{D}}, \ldots$ Whenever there is a specified system of generators $g_1, \ldots, g_t$ for an ideal H, we simply write $\mathsf{Jac}(H)$ for the Jacobian matrix $\mathsf{Jac}(g_1, \ldots, g_t)$ (e.g. in Theorem 1).

Fix a homogeneous S-ideal I with choice of generators $I = (f_1, \ldots, f_m)$. The Rees-like algebra of I is the algebra $S[It, t^2] \subseteq S[t]$, where t is a new variable. We denote the Rees-like algebra by $\mathcal{RL}(I)$. It has an explicit presentation as a quotient of a non-standard graded polynomial ring over S, namely $\mathcal{RL}(I) \cong T/\mathcal{RLP}(I)$ where $T := S[y_1, \ldots, y_m, z]$ has grading defined by $\deg y_i = \deg f_i + 1$ and $\deg z = 2$; $\mathcal{RLP}(I)$ is then a homogeneous ideal of T. The usefulness of Rees-like algebras lies in the detailed understanding of the kernel, $\mathcal{RLP}(I)$, of the map of k-algebras $T \to \mathcal{RL}(I)$ given by $y_i \mapsto f_i t$ and $z \mapsto t^2$ as summarized in the following theorem.

Theorem 1 (McCullough and Peeva [18, Theorem 1.6, Proposition 2.9]) *Let $S = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k and let $I = (f_1, \ldots, f_m)$ be a homogeneous ideal of S. The ideal $\mathcal{RLP}(I)$ is the sum $\mathcal{RLP}(I)_{syz} + \mathcal{RLP}(I)_{gen}$ with generators*

$$\mathcal{RLP}(I)_{syz} = \left\{ r_j := \sum_{i=1}^m c_{ij} y_i \,\middle|\, \sum_{i=1}^m c_{ij} f_i = 0 \right\} \quad \textit{and}$$
$$\mathcal{RLP}(I)_{gen} = \{ y_i y_j - z f_i f_j \mid 1 \le i, j \le m \}.$$

Moreover,

1. *The maximal degree of a minimal generator of P is*
$$\mathit{maxdeg}(P) = \max \left\{ 1 + \mathit{maxdeg}\Big(\mathrm{Syz}_1^S(I)\Big),\ 2\Big(\mathrm{maxdeg}(I) + 1\Big) \right\}.$$
2. *The multiplicity or degree of $T/\mathcal{RLP}(I)$ is*
$$\deg(T/\mathcal{RLP}(I)) = 2 \prod_{i=1}^m \Big(\deg(f_i) + 1 \Big).$$
3. *The Castelnuovo–Mumford regularity, the projective dimension, the depth, the codimension, and the dimension of $T/\mathcal{RLP}(I)$ are:*
   - $\mathrm{reg}(T/\mathcal{RLP}(I)) = \mathrm{reg}(S/I) + 2 + \sum_{i=1}^m \deg(f_i)$
   - $\mathrm{pd}(T/\mathcal{RLP}(I)) = \mathrm{pd}(S/I) + m - 1$
   - $\mathrm{depth}(T/\mathcal{RLP}(I)) = \mathrm{depth}(S/I) + 2$
   - $\mathrm{ht}(\mathcal{RLP}(I)) = m$
   - $\dim(T/\mathcal{RLP}(I)) = n + 1$.

In the previous theorem, $\mathrm{maxdeg}(M)$ denotes the maximal degree of an element in a minimal system of generators of a module M.

Our study of the singular locus of a Rees-like algebra $\mathcal{RL}(I)$ is based on an explicit description of the Jacobian matrix $\mathsf{Jac}(\mathcal{RLP}(I))$ via computing a block decomposition. Some of the blocks will be submatrices of the Jacobians of the ideals $(f_1, \ldots, f_m)^2$ (resp. $(y_1, \ldots, y_m)^2$)

consisting of rows using only partials corresponding to the variables $x_1, \dots, x_n$ (resp. $y_1, \dots, y_m$). Specifically,

- denote by $\mathsf{Jac}_{\underline{x}}\left((f_1, \dots, f_m)^2\right)$ the Jacobian matrix of $(f_1, \dots, f_m)^2$ with respect to $x_1, \dots, x_n$, and
- denote by $\mathsf{Jac}_{\underline{y}}\left((y_1, \dots, y_m)^2\right)$ the Jacobian matrix of $(y_1, \dots, y_m)^2$ with respect to $y_1, \dots y_m$.

Another block is described by a minimal free resolution $\mathbf{F}_\bullet$ of $(f_1, \dots, f_m)$. Specifically, denote by $d_1^{\mathbf{F}} := d_1^{\mathbf{F}}(\underline{f}) = (c_{ij})$ the first differential in $\mathbf{F}$, i.e., the matrix whose columns are the first syzygies of the f_i. Finally, let $\mathsf{A} = (a_{kj})$, where $a_{kj} := \partial_{x_k} r_j = \sum_{i=1}^m \partial_{x_k}(c_{ij})y_k$. With this notation, we may describe the Jacobian $\mathsf{Jac}(\mathcal{RLP}(I))$.

Proposition 1 *Using the notation above, up to reordering of the columns and rows, the Jacobian matrix of $\mathcal{RLP}(I)$ has a block decomposition*

$$
\mathit{Jac}(\mathcal{RLP}(I)) = \begin{array}{c|c|c}
 & \textit{generators in } \mathcal{RLP}(I)_{syz} & \textit{generators in } \mathcal{RLP}(I)_{gen} \\
\hline
\partial_{x_i} & A & -z\mathit{Jac}_{\underline{x}}((f_1, \dots, f_m)^2) \\
\hline
\partial_{y_j} & d_1^{\mathbf{F}} & \mathit{Jac}_{\underline{y}}((y_1, \dots, y_m)^2) \\
\hline
\partial_z & 0 & -f_i f_j
\end{array},
$$

where $\mathsf{A} = \left(\partial_{x_i} r_j\right)$ using the notation above.

Proof We order the rows as follows. The first n rows correspond to ∂_{x_i} for $i = 1, \dots, n$, the next m rows correspond to ∂_{y_i} for $i = 1, \dots, m$, and the last row corresponds to ∂_z. The first $b = \operatorname{rank}(\mathbf{F}_1)$ columns correspond to the minimal generators $r_1, \dots, r_b$ in the set $\mathcal{RLP}(I)_{\mathrm{syz}}$ described in Theorem 1. The following $\binom{m+1}{2}$ columns correspond to the generators in the set $\mathcal{RLP}(I)_{\mathrm{gen}}$.

For the blocks within the first b columns, writing $r_j = \sum_{i=1}^m c_{ij} y_i$ for some $1 \le j \le b$, by linearity we have

$$\partial_{x_k} r_j = \sum_{i=1}^m \partial_{x_k}(c_{ij} y_i) = \sum_{i=1}^m \partial_{x_k}(c_{ij}) y_i,$$

and clearly $\partial_{y_k} r_j = \sum_{i=1}^m \partial_{y_k}(c_{ij} y_i) = c_{kj}$ and $\partial_z r_j = \sum_{i=1}^m \partial_z(c_{ij} y_i) = 0$.

For the blocks concerning the last $\binom{m+1}{2}$ columns, set $b_{ij} := y_i y_j - z f_i f_j$ for $1 \le i \le j \le m$. The following calculations finish the proof:

$$
\begin{aligned}
\partial_{x_k} b_{ij} &= \partial_{x_k}(y_i y_j) - \partial_{x_k}(z f_i f_j) = -z\partial_k(f_i f_j), \\
\partial_{y_k} b_{ij} &= \partial_{y_k}(y_i y_j) - \partial_{y_k}(z f_i f_j) = \partial_{y_k}(y_i y_j), \text{ and} \\
\partial_z b_{ij} &= \partial_z(y_i y_j) - \partial_z(z f_i f_j) = -f_i f_j.
\end{aligned}
$$

Example 1 Let $I = (x_1, x_2) \subseteq k[x_1, x_2]$ and $\mathcal{RL}(I)$ be its Rees-like algebra with defining ideal

$$\mathcal{RLP}(I) = (-y_1 x_2 + x_1 y_2,\ y_1^2 - z x_1^2,\ y_1 y_2 - z x_1 x_2,\ y_2^2 - z x_2^2).$$

The Jacobian $\mathsf{Jac}(\mathcal{RLP}(I))$ is the following matrix with 4 columns and 5 rows.

$$\mathsf{Jac}(\mathcal{RLP}(I)) = \begin{array}{c|cccc} & -y_1x_2+x_1y_2 & y_1^2-zx_1^2 & y_1y_2-zx_1x_2 & y_2^2-zx_2^2 \\ \hline \partial_{x_1} & y_2 & -2x_1z & -x_2z & 0 \\ \partial_{x_2} & -y_1 & 0 & -x_1z & -2x_2z \\ \hline \partial_{y_1} & -x_2 & 2y_1 & y_2 & 0 \\ \partial_{y_2} & x_1 & 0 & y_1 & 2y_2 \\ \hline \partial_z & 0 & -x_1^2 & -x_1x_2 & -x_2^2 \end{array}$$

With this explicit description of the Jacobian in Proposition 1, we determine in Theorem 2 the codimension of the singular locus of the affine cone associated to the Rees-like algebra of any ideal I when 2 is a unit. (See e.g. [4, Theorem 16.19].) Interestingly, this number only depends on the height of I and the minimal number of generators of I. Recall, by Theorem 1, $\mathrm{ht}(\mathcal{RLP}(I)) = \mu(I)$.

Theorem 2 *Let k be a perfect field with $\mathrm{char}(k) \neq 2$. Set $S = k[x_1, \ldots, x_n]$. Let $I = (f_1, \ldots, f_m)$ be a be a non-zero homogeneous, proper ideal with minimal primes $\mathrm{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. The singular locus of $\mathcal{RL}(I)$ is defined by the image of*

$$J = \left(\mathfrak{p}_1 + (\underline{y})\right) \cap \left(\mathfrak{p}_2 + (\underline{y})\right) \cap \cdots \cap \left(\mathfrak{p}_r + (\underline{y})\right) \subset S[\underline{y}, z].$$

In particular,

- *there is a one-to-one correspondence between $\mathrm{Min}(I)$ and $\mathrm{Min}(J)$, and*
- $\mathrm{ht}(J) = \mu(I) + \mathrm{ht}(I)$

Proof Let $T = S[\underline{y}, z]$. By the Jacobian criterion, the image of

$$J := \sqrt{\mathcal{RLP}(I) + I_m(\mathsf{Jac}(\mathcal{RLP}(I))}$$

defines the singular locus in $\mathcal{RL}(I)$.

By Theorem 1,

$$\mathcal{RLP}(I) \subseteq I + (\underline{y}) \subseteq \bigcap_i \left(\mathfrak{p}_i + (\underline{y})\right),$$

thus to prove that $J = \bigcap_i \left(\mathfrak{p}_i + (\underline{y})\right)$ it suffices to show that $\sqrt{I_m(\mathsf{Jac}(\mathcal{RLP}(I))} = \bigcap_i \left(\mathfrak{p}_i + (\underline{y})\right)$, i.e.

$$\mathrm{Min}(I_m(\mathsf{Jac}(\mathcal{RLP}(I)))) = \{\mathfrak{p}_i + (\underline{y}) \mid i = 1, \ldots, r\}.$$

First we handle the case $m = 1$. Suppose $I = (f)$ is principal. We have that $\mathcal{RLP}(I) = (y^2 - zf^2)$. Thus

$$I_1(\mathsf{Jac}(\mathcal{RLP}(I))) = \left(f\partial_{x_1}(f), \ldots, f\partial_{x_n}(f), f^2, 2y\right).$$

The minimal primes of $J = \sqrt{I_1(\mathsf{Jac}(\mathcal{RLP}(I))} = \sqrt{(f, y)} = \sqrt{(f)} + (y)$ are exactly $\{\mathfrak{p}_i + (y)\}$, where $\mathfrak{p}_i$ are the minimal primes of $I = (f)$. For the remainder of the proof we assume that $m \geq 2$.

We use the notation of Proposition 1 for the Jacobian matrix $\mathsf{Jac}(\mathcal{RLP}(I))$. First note that there are no containments among the set of primes of the form $\mathfrak{p}_i + (\underline{y})$ as clearly there are no containments among the ideals $\mathfrak{p}_i$ and these ideals are transversal with the ideal $(\underline{y})$. To prove that $\mathrm{Min}(J) = \{\mathfrak{p}_i + (\underline{y}) \mid i = 1, \ldots, r\}$ we start by showing that $J \subset \mathfrak{p}_i + (\underline{y})$ for all $1 \leq i \leq r$. Fix an arbitrary such i and invoke Proposition 1 to observe that

- all entries of A and $\mathsf{Jac}_{\underline{y}}((y_1, \ldots, y_m)^2)$ lie in $(y_1, \ldots, y_m)$,
- all entries of the block matrices $-z\mathsf{Jac}_{\underline{x}}((f_1, \ldots, f_m)^2)$ and $(-f_i f_j)$ lie in $(f_1, \ldots, f_m)$.

Thus, any m-minor of $\mathsf{Jac}(\mathcal{RLP}(I))$ involving one of the last $\binom{m+1}{2}$ columns or one of the rows corresponding to ∂_{x_j} or ∂_z is contained in $I + (\underline{y}) \subseteq \mathfrak{p}_i + (\underline{y})$. The remaining m-minors generate $I_m(c_{ij})$, which is contained in $\sqrt{I}$ ([3, Theorem 2.1(b)]) and thus in $\mathfrak{p}_i \subseteq \mathfrak{p}_i + (\underline{y})$.

For the converse, let $\mathfrak{q}$ be a prime ideal containing $J = \sqrt{I_m(\mathsf{Jac}(\mathcal{RLP}(I)))}$. By Proposition 1, $\mathfrak{q}$ contains the ideal of m-minors $I_m(\mathsf{Jac}_{\underline{y}}((y_1, \ldots, y_m)^2))$ and so it contains the ideal $(2y_1^m, 2y_2^m, \ldots, 2y_m^m)$. Since $\mathfrak{q}$ is prime and $\mathrm{char}(k) \neq 2$, then $(y_1, \ldots, y_m) \subseteq \mathfrak{q}$.

To finish the proof we show that $\mathfrak{q}$ contains one of the $\mathfrak{p}_i$'s. The ideal $(f_1, f_2, \ldots, f_m)^2 \cdot I_{m-1}(c_{ij})$ is the ideal generated by all m-minors determined by the last row, $(m-1)$ of the m rows corresponding to ∂_{y_k}, $(m-1)$ of the first b columns (corresponding to the generators in $\mathcal{RLP}(I)_{\mathrm{syz}}$), and one column among the last $\binom{m+1}{2}$ (corresponding to one of the generators in $\mathcal{RLP}(I)_{\mathrm{gen}}$). As such, we have

$$(f_1, f_2, \ldots, f_m)^2 \cdot I_{m-1}(c_{ij}) \subseteq I_m(\mathsf{Jac}(\mathcal{RLP}(I))).$$

Since $(c_{ij}) = (d_1^{\mathbf{F}})$, by [3, Theorem 2.1(b)] we have $\sqrt{I} = \sqrt{I_{m-1}(c_{ij})}$, thus taking radical of both sides in the above inclusion, and noticing that the radical of the left-hand side is simply $\sqrt{I}$ and the radical of the right-hand side is J, we finally obtain

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r = \sqrt{I} = \sqrt{(f_1, f_2, \ldots, f_m)^2 \cdot \sqrt{I}} \subseteq J \subseteq \mathfrak{q}.$$

Therefore, $\mathfrak{q}$ contains one of the $\mathfrak{p}_i$. This concludes the proof. □

When S is a polynomial ring over a field, there are more conceptual proofs of Theorem 2. Specifically, K. E. Smith noted in preliminary discussions with us that as $S[It, t^2]$ has a smooth normalization given by $S[t]$, the singularities are relatively mild and defined by the conductor ideal, which can be shown to be $I + It$. However, our explicit approach to the Jacobian also gives similar results for Rees-like algebras of ideals in quotients of polynomial rings. As an example, analogous arguments to those proving Theorem 2 can be used to prove the following result where the ground ring S is not regular.

Theorem 3 *Let $S = \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{p}$, where $\mathfrak{p}$ is a non-degenerate homogeneous prime ideal with* $\mathrm{Proj}(S)$ *smooth. Let I be a homogeneous S-ideal.*

If the presentation matrix of I as an S-ideal contains no linear forms, then the singular locus of $\mathcal{RL}(I) = S[y_1, \ldots, y_m, z]/Q$ is defined by the image of

$$J = (x_1, \ldots, x_n) \cap \left(\mathfrak{p}_1 + (\underline{y})\right) \cap \left(\mathfrak{p}_2 + (\underline{y})\right) \cap \cdots \cap \left(\mathfrak{p}_r + (\underline{y})\right) \subset S[\underline{y}, z].$$

Proof Write $I = (f_1, \ldots, f_m)S$, then $\mathcal{RL}(I) = S[It, t^2] \cong T/Q$ where $T = S[y_1, \ldots, y_m, z]$. By [18], the T-ideal Q is generated by $\mathcal{RLP}(I)_{\mathrm{syz}} + \mathcal{RLP}(I)_{\mathrm{gen}}$ as in the case of the polynomial ring, where

$$\mathcal{RLP}(I)_{\mathrm{syz}} = \left\{ r_j := \sum_{i=1}^{m} c_{ij} y_i \mid \sum_{i=1}^{m} c_{ij} f_i = 0 \text{ in } S \right\} \quad \text{and}$$
$$\mathcal{RLP}(I)_{\mathrm{gen}} = \{ y_i y_j - z f_i f_j \mid 1 \leq i, j \leq m \}.$$

Let $S' = \mathbb{C}[\underline{x}]$ and let $\mathfrak{m}_{S'} = (\underline{x})S'$ be its homogeneous maximal ideal. For every i, j, let c'_{ij} be any lifting of c_{ij} to $S'[y_1, \ldots, y_m] = \mathbb{C}[\underline{x}, \underline{y}]$ and let f'_j be any lifting of f_j to S'. Then

$r'_j = \sum_{i=1}^m c'_{ij} y_i$ is a lifting of r_j to $\mathbb{C}[\underline{x}, \underline{y}]$. We then clearly have $\mathcal{RL}(I) \cong T'/Q'$, where $T' = \mathbb{C}[\underline{x}, \underline{y}, z]$ and

$$Q' = (r'_1, \ldots, r'_b) + \left(y_i y_j - z f'_i f'_j \mid 1 \le i \le j \le m\right) + \mathfrak{p}.$$

Observe that $\operatorname{ht}(Q') = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathcal{RLP}(I)) = \operatorname{ht}(\mathfrak{p}) + m$.

Let d'_1 be the matrix whose entries are the c'_{ij} and let A' be matrix whose (i, j)-entry is $\sum_{k=1}^m (\partial_{x_i} c'_{kj}) y_k$. With the above choice of generators of Q, the Jacobian matrix of Q' is

$$\mathsf{M} = \begin{array}{c} \\ \partial_{x_i} \\ \partial_{y_j} \\ \partial_z \end{array} \begin{array}{c} \begin{array}{ccc} \text{generators of } \mathcal{RLP}(I)_1 & \text{generators of } \mathcal{RLP}(I)_1 & \text{generators of } \mathfrak{p} \end{array} \\ \left(\begin{array}{c|c|c} \mathsf{A}' & -z\mathsf{Jac}_{\underline{x}}((f'_1, \ldots, f'_m)^2) & \mathsf{Jac}_{\underline{x}}(\mathfrak{p}) \\ \hline d'_1 & \mathsf{Jac}_{\underline{y}}((y_1, \ldots, y_m^2) & 0 \\ \hline 0 & -f'_i f'_j & 0 \end{array}\right) \end{array}$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of I in S and $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r$ be the minimal primes of $I + \mathfrak{p}$ in $S' = \mathbb{C}[\underline{x}]$. Let $J' = I_{\operatorname{ht}(\mathfrak{p})+m}(\mathsf{M}) \subseteq T'$ so that $J = J'T$.

To prove the statement we show that $J' = \mathfrak{m}_{S'}T' \cap \bigcap_{i=1}^m (\mathfrak{p}'_i + (\underline{y}))$.

To prove $J' \subset \mathfrak{m}_{S'}T' \cap \bigcap_{i=1}^m (\mathfrak{p}'_i + (\underline{y}))$, we first show that $J' \subseteq \mathfrak{m}_{S'}T'$. Since $\mathfrak{p}$ is non-degenerate, then all entries of $\mathsf{Jac}_{\underline{x}}(\mathfrak{p})$ are contained in $\mathfrak{m}_{S'}T'$. The assumption on the presentation matrix implies that also all entries of A are contained in $\mathfrak{m}_{S'}T'$; then all entries of M are contained in $\mathfrak{m}_{S'}T'$ except the ones in the m rows of the middle block $\mathsf{Jac}_{\underline{y}}((y_1, \ldots, y_m)^2)$. Thus every $(\operatorname{ht}(\mathfrak{p})+m)$-minor of M contains a row whose entries lie in $\mathfrak{m}_{S'}T'$, so it is contained in $\mathfrak{m}_{S'}T'$.

We then prove that $J' \subseteq \mathfrak{p}'_i + (\underline{y})$ for every $i = 1, \ldots, r$. After reducing the entries of M modulo $\mathfrak{p}'_i + (\underline{y})$ the only possibly non-zero blocks left are $\mathsf{Jac}_{\underline{x}}(\mathfrak{p})$ and d'_1, thus $I_{\operatorname{ht}(\mathfrak{p})+m}(\mathsf{M}) = \sum_{i=1}^m I_i(d'_1) I_{\operatorname{ht}(\mathfrak{p})+m-i}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p}))$ modulo $\mathfrak{p}'_i + (y_1, \ldots, y_m)$. By [4, Thm 16.19(b)] $\operatorname{rank}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p})) = \operatorname{ht}(\mathfrak{p})$, thus $I_{\operatorname{ht}(\mathfrak{p})+m-i}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p})) = (0)$ except when $i = m$, thus $I_{m+\operatorname{ht}(\mathfrak{p})}(\mathsf{M}) = I_m(d'_1) I_{\operatorname{ht}(\mathfrak{p})}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p}))$ modulo $\mathfrak{p}'_i + (y_1, \ldots, y_m)$, and then to prove that $I_{\operatorname{ht}(\mathfrak{p})+m}(\mathsf{M}) \subseteq \mathfrak{p}'_i + (\underline{y})$ it suffices to show $I_m(d'_1) \subseteq \mathfrak{p}'_i + (\underline{y})$—this is proved as in Theorem 2.

Now to prove $J' \supset \mathfrak{m}_{S'}T' \cap \bigcap_{i=1}^m (\mathfrak{p}'_i + (\underline{y}))$, observe that there are no irredundant terms in the intersection $\mathfrak{m}_{S'}T' \cap \bigcap_{i=1}^m (\mathfrak{p}'_i + (\underline{y}))$. Since all these ideals are prime and J' is radical, it suffices to show that every prime ideal $\mathfrak{q}'$ containing J' must contain at least an element of $\{\mathfrak{p}'_i + (\underline{y}) \mid i = 1, \ldots, r\} \cup \{\mathfrak{m}_{S'}T'\}$.

Since

$$I_{\operatorname{ht}(\mathfrak{p})}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p})) \cdot I_m(d'_1) \subseteq \sqrt{I_{\operatorname{ht}(\mathfrak{p})+m}(\mathsf{M})} = J',$$

and since $\sqrt{I_{\operatorname{ht}(\mathfrak{p})}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p}))} = \mathfrak{m}_{S'}T'$, then

$$\mathfrak{m}_{S'}T' \cdot \sqrt{I_m(d'_1)} \subseteq \sqrt{\mathfrak{m}_{S'}T' \cdot \sqrt{I_m(d'_1)}} = \sqrt{I_{\operatorname{ht}(\mathfrak{p})}(\mathsf{Jac}_{\underline{x}}(\mathfrak{p})) \cdot I_m(d'_1)} \subseteq \sqrt{J'} = J'$$

If $\mathfrak{m}_{S'}T' \subseteq \mathfrak{q}'$ we are done. Otherwise, since $\mathfrak{q}'$ is prime we have $\sqrt{I_m(d'_1)} \subseteq \mathfrak{q}'$. Moreover, since $\mathfrak{q}'$ contains J', then $\mathfrak{p} \subseteq \mathfrak{q}'$. Thus

$$I + \mathfrak{p} \subseteq \mathfrak{p} + \sqrt{I} = \mathfrak{p} + \sqrt{I_m(d'_1)} \subseteq \mathfrak{q}',$$

and then

$$\bigcap_{i=1}^{r} \mathfrak{p}'_i = \sqrt{I+\mathfrak{p}} \subseteq \sqrt{\mathfrak{q}'} = \mathfrak{q}'.$$

Since $\mathfrak{q}'$ is prime, then one of the $\mathfrak{p}'_i$ is contained in $\mathfrak{q}'$. Moreover, looking at the $(\mathrm{ht}(\mathfrak{p})+m)$-minors of M we see that also

$$I_m(\mathsf{Jac}_{\underline{y}}((y_1,\dots,y_m)^2)) \cdot I_{\mathrm{ht}(\mathfrak{p})}\mathsf{Jac}_{\underline{x}}(\mathfrak{p}) \subseteq J' \subseteq \mathfrak{q}'.$$

As above, since $\mathfrak{m}_{S'}T' \nsubseteq \mathfrak{q}$ we deduce that $I_m(\mathsf{Jac}_{\underline{y}}((y_1,\dots,y_m)^2)) \subseteq \mathfrak{q}'$ and taking radicals we obtain $(y_1,\dots,y_m) \subseteq \mathfrak{q}'$.

We have then showed that either $\mathfrak{m}_{S'}T' \subseteq \mathfrak{q}'$ or $\mathfrak{p}'_i + (y_1,\dots,y_m) \subseteq \mathfrak{q}'$ for some $i = 1,\dots,r$. This concludes the proof. $\square$

In Theorem 3, the assumption on the presentation matrix of I is needed, as the following example illustrates.

Example 2 Assume $\mathrm{char}(k) \neq 2$ and let $S = k[x_1,x_2,x_3]/(x_1^2 - x_2x_3)$ and $I = (x_1,x_2)S$. Observe that I has the linear syzygies $(x_2,-x_1)$ and $(x_1,-x_3)$. The singular locus of $\mathcal{RL}(I)$ has only one minimal prime, which is (x_1,x_2,y_1,y_2).

Proof As in the statement of Theorem 1, one has $\mathcal{RL}(I) \cong k[x_1,x_2,x_3,y_1,y_2,z]/Q$, where

$$Q = (x_1y_2 - x_2y_1, x_1y_1 - x_3y_2, y_1^2 - x_1^2z, y_1y_2 - x_1x_2z, y_2^2 - x_2^2z, x_1^2 - x_2x_3).$$

Then, the Jacobian matrix of Q over $k[x_1,x_2,x_3,y_1,y_2,z]$ is

$$\mathsf{M} = \begin{array}{c} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \\ \partial_{y_1} \\ \partial_{y_2} \\ \partial_z \end{array} \left(\begin{array}{cc|ccc|c} y_2 & y_1 & -2x_1z & -x_2z & 0 & 2x_1 \\ -y_1 & 0 & 0 & -x_1z & -2x_2z & -x_3 \\ 0 & -y_2 & 0 & 0 & 0 & -x_2 \\ \hline -x_2 & x_1 & 2y_1 & y_2 & 0 & 0 \\ x_1 & -x_3 & 0 & y_1 & 2y_2 & 0 \\ \hline 0 & 0 & -x_1^2 & -x_1x_2 & -x_2^2 & 0 \end{array}\right)$$

and set $J = \sqrt{I_3(\mathsf{M})}$. It easy to check that $J \subseteq (x_1,x_2,y_1,y_2)$.
Conversely, let $\mathfrak{q} \in \mathrm{Min}(J)$. Notice that

$$\det\begin{pmatrix} y_2 & -2x_1z & 2x_1 \\ x_1 & 0 & 0 \\ 0 & -x_1^2 & 0 \end{pmatrix} = -2x_1^4 \in J.$$

Similarly, $2x_2^4 \in J$. Since $\mathrm{char}(k) \neq 2$, then $x_1, x_2 \in \sqrt{J} \subseteq q$. Moreover,

$$\det\begin{pmatrix} -2x_1z & -x_2z & 2x_1 \\ 2y_1 & y_2 & 0 \\ 0 & y_1 & 0 \end{pmatrix} = 4x_1y_1^2 \in J.$$

Similarly, $x_2y_1^2, x_3y_1^2, x_2y_1^2, x_2y_2^2$, and $x_3y_2^2$ lie in J. Thus one has an inclusion $(y_1^2, y_2^2)(x_1,x_2,x_3) \subseteq J$.

Now assume by contradiction that $(y_1,y_2) \nsubseteq \mathfrak{q}$. One then has $(x_1,x_2,x_3) \subseteq \mathfrak{q}$. Reducing the entries of M modulo (x_1,x_2,x_3) one sees that $(2y_1^3, 2y_2^3) \subseteq \mathfrak{q}$. This shows that $(y_1,y_2) \subseteq \sqrt{J} \subseteq q$, which is a contradiction. $\square$

3 standardizations

The usual way to homogenize a non-homogeneous prime ideal $I \subset k[x_1, \ldots, x_n]$ is by adjoining a new variable, say w, and homogenizing all terms of all elements of the ideal by multiplying by the appropriate power of w to make the element homogeneous. This corresponds to taking the projective closure of $V(I)$ in $\mathbb{P}^n_k$. Thus the resulting homogeneous ideal is prime but this process does not preserve the structure of free resolution of the corresponding ideal. An alternate method of constructing standard graded analogues of non-standard graded prime ideals, called step-by-step homogenization in [18, Theorem 4.5], preserves primeness for nondegenerate prime ideals and graded Betti numbers at the expense of adding many more variables. For each variable x with $\deg(x) = d > 1$, one appends a new variable u, sets $\deg(u) = \deg(x) = 1$ and replaces every instance of x with xu^{d-1}. As the role of this process is to transform a non-standard graded ring into a standard graded one, we refer to it as a standardization.

Definition 1 Suppose T is a positively graded polynomial ring over a field k. A **standardization** of T is a graded, flat map $(\_)^{\text{std}}: T \to T^{\text{std}}$ of graded k-algebras, where T^{std} is a standard graded polynomial ring over k. For an ideal $I = (f_1, \ldots, f_m) \subseteq T$, write I^{std} for the T^{std}-ideal $(f_1^{\text{std}}, \ldots, f_m^{\text{std}})$.

Thus step-by-step homogenization is a standardization that has the additional property that for any nondegenerate prime ideal Q of T, the ideal Q^{std} is also prime. Any standardization will thus increase the number of variables and thereby increase the dimension of the singular locus of the corresponding varieties. However, it is desirable that the codimension of the singular locus is preserved. Unfortunately, step-by-step homogenization does not preserve it.

Example 3 Let $Q = I_2 \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \subset S = k[u, v, w, x, y, z]$, with the non-standard grading given by setting

$$\deg(x) = \deg(y) = \deg(z) = 2 \text{ and } \deg(u) = \deg(v) = \deg(w) = 1.$$

Consider the step-by-step standardization given by the ring map

$$S \to S^{\text{std}} := k[u, v, w, x_1, x_2, y_1, y_2, z_1, z_2]$$

sending $x \mapsto x_1x_2$, $y \mapsto y_1y_2$, and $z \mapsto z_1z_2$. The image of Q is

$$Q^{\text{std}} = (x_1x_2v - y_1y_2u, x_1x_2w - z_1z_2u, y_1y_2w - z_1z_2v).$$

One may easily verify that $\text{ht}(Q) = 2$, $\text{ht}(Q^{\text{std}}) = 2$, and $\text{ht}(I_2(\text{Jac}(Q))) = 6$ yet $I_2(\text{Jac}(Q^{\text{std}}))$ has height 5. One can also build examples of Rees-like algebras whose singular locus codimension fails to be preserved in a similar fashion.

We adapt work of Ananyan and Hochster to define new standardizations that preserve the relative size of the singular locus. Following [1], we define a sequence of elements $g_1, \ldots, g_t \in S$ to be a **prime sequence** provided $(g_1, \ldots, g_t)$ is a proper ideal, $g_i \notin (g_1, \ldots, g_{i-1})$ for all $1 \le i \le t$, and $S/(g_1, \ldots, g_i)$ is a domain for all $1 \le i \le t$. Clearly any prime sequence is a regular sequence. The following near converse is implicit in their work.

Lemma 1 *Let S be a standard graded polynomial ring and let $g_1, \ldots, g_t$ in S be a homogeneous regular sequence of elements of positive degree. If the ideal $I = (g_1, \ldots, g_t)$ is prime, then $g_1, \ldots, g_t$ is a prime sequence. Moreover, any permutation of $g_1, \ldots, g_t$ is a prime sequence.*

Proof Proceed by contradiction and set $I_i = (g_1, \ldots, g_i)$. Pick i maximal so that I_i is not prime, so $i < t$. Pick homogeneous elements a and b in $S \setminus I_i$ with $ab \in I_i$ and with $\deg(ab)$ minimal. Since I_{i+1} is prime, without loss of generality we may assume $a \in I_{i+1}$. Writing $a = \sum_{j=1}^{i+1} s_j g_j$, we have

$$bs_{i+1}g_{i+1} = b(a - \sum_{j=1}^{i} s_j g_j) \in I_i.$$

Since $g_1, \ldots, g_t$ is a regular sequence, $bs_{i+1} \in I_i$. Also $\deg(bs_{i+1}) < \deg(ab)$. By the minimality assumption, this gives $s_{i+1} \in I_i$ and hence $a \in I_i$, which is a contradiction. $\square$

The usefulness of this idea is contained in the following result, which is essentially the content of [1, Cor. 2.9, Prop. 2.10].

Proposition 2 (Ananyan and Hochster) *Assume k is algebraically closed and let $S = k[x_1, \ldots, x_n]$. Suppose $g_1, \ldots, g_t$ is a homogeneous prime sequence in S and set $R = k[g_1, \ldots, g_t]$. Suppose $I \subset R$ is a homogeneous ideal.*

1. *The ideals I and IS have the same graded Betti numbers.*
2. *For $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{p} \in \mathrm{Ass}(R/I)$ if and only if $\mathfrak{p}S \in \mathrm{Ass}(S/IS)$.*
3. *In particular, if I is prime, then IS is prime.*
4. *If $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is any primary decomposition of I, then $\mathfrak{q}_1 S \cap \cdots \cap \mathfrak{q}_r S$ is a primary decomposition of IS.*

Homogeneous prime sequences give rise to standardizations and we make the following definition.

Definition 2 Suppose T is a positively graded polynomial ring. A **prime standardization** of T is a standardization $(\_)^{\mathrm{std}} \colon T \to T^{\mathrm{std}}$ such that for every prime ideal $P \subseteq T$, P^{std} is prime.

To see the connection with prime sequences, we note the following:

Proposition 3 *Let $T = k[x_1, \ldots, x_n]$ be a positively graded polynomial ring and $(\_)^{std} \colon T \to T^{std}$ a standardization. Let $g_i := x_i^{std}$. Then $(\_)^{std}$ is a prime standardization if and only if $g_1, \ldots, g_n$ is a prime sequence.*

Proof The "if" direction follows from Proposition 2. For the "only if" direction, suppose $(\_)^{\mathrm{std}}$ is a prime standardization. For every $1 \leq i \leq t$, by Proposition 2(3) we have $(g_1, \ldots, g_i) = (x_1, \ldots, x_i)^{\mathrm{std}}$ is a prime ideal; thus, by Lemma 1, $g_1, \ldots, g_n$ is a prime sequence. $\square$

By our definition, step-by-step homogenization is not a prime standardization since non-linear monomials do not form a prime sequence. This is why step-by-step homogenization only works for non-degenerate primes. We now show that there is always a choice of prime standardization that, unlike step-by-step homogenization, preserves the codimension of the singular locus of any ideal.

Construction 1 *Let* $T = k[t_1, \dots, t_n]$ *be a positively graded polynomial ring over an algebraically closed field* k *with* $\deg(t_i) = d_i \in \mathbb{Z}_+$. *Set*

$$W = \{w_{i,j,\ell} \mid 1 \le i \le n, 0 \le j \le n, 1 \le \ell \le d_i\},$$

a set of new variables. Set $T^{std} := k[W]$ *and let* $F_i = \sum_{j=0}^{n} \prod_{\ell=1}^{d_i} w_{i,j,\ell} \in T^{std}$, *where we define* $\deg(w_{i,j,\ell}) = 1$ *for all* i, j, ℓ. *Define the graded map of rings* $(\_)^{std} : T \to T^{std}$ *by setting* $t_i^{std} = F_i$. *Since each* F_i *is irreducible, say by Eisenstein's criterion, and since the variables appearing in* F_i *are disjoint from those of* F_j *for* $i \ne j$, *it follows from Lemma 1 that* $F_1, \dots, F_n$ *form a prime sequence.*

Note that if one merely wants to create a prime standardization, fewer terms will suffice. The extra terms in our chosen prime standardization are necessary for the conclusion of Theorem 4.

Example 4 Let $T = k[x_1, x_2, x_3]$, where $\deg(x_i) = i$ for $i = 1, 2, 3$. The ideal $I = (x_1^2 - x_2, x_1^3 - x_3)$ of T is then homogeneous. The prime standardization I^{std} of I from Construction 1 is then generated by the following two elements

$$\begin{aligned} f = {} & (w_{1,0,1} + w_{1,1,1} + w_{1,2,1} + w_{1,3,1})^2 \\ & - (w_{2,0,1}w_{2,0,2} + w_{2,1,1}w_{2,1,2} + w_{2,2,1}w_{2,2,2} + w_{2,3,1}w_{2,3,2}), \end{aligned}$$

$$\begin{aligned} g = {} & (w_{1,0,1} + w_{1,1,1} + w_{1,2,1} + w_{1,3,1})^3 - (w_{3,0,1}w_{3,0,2}w_{3,0,3} \\ & + w_{3,1,1}w_{3,1,2}w_{3,1,3} + w_{3,2,1}w_{3,2,2}w_{3,2,3} + w_{3,3,1}w_{3,3,2}w_{3,3,3}). \end{aligned}$$

By convention, we set the height of the unit ideal to be $\mathrm{ht}((1)) = \infty$.

Lemma 2 *For the standardization defined in Construction 1, the ideal*

$$\left(\frac{\partial}{w_{i,j,\ell}}(F_i) \mid 0 \le j \le n, 1 \le \ell \le d_i \right)$$

has height at least $n + 1$ *for all* $1 \le i \le m$.

Proof If some $d_i = 1$, then $\partial_{w_{i,j,k}}(F_i) = 1$ and we are done. Else the generators of the form $\frac{\partial}{\partial_{w_{i,j,1}}}(F_i) = \prod_{\ell=2}^{d_i} w_{i,j,k}$ for $0 \le j \le n$ constitute a regular sequence, as they are expressed in disjoint sets of variables. □

We say that an ideal is *unmixed* if all its associated primes have the same height.

Theorem 4 *Let* T *be a positively graded polynomial ring over an algebraically closed field* k, *let* I *be any homogeneous unmixed ideal of* T. *Assume* $\mathrm{char}(k) \ne 2$. *Denote by* $(\_)^{std}$ *the prime standardization in Construction 1.*

For $X = \mathrm{Proj}(T/I)$ *and* $X^{std} = \mathrm{Proj}(T^{std}/I^{std})$,

$$\mathrm{codim}_X(\mathrm{Sing}\, X) = \mathrm{codim}_{X^{std}}(\mathrm{Sing}\, X^{std})$$

and there is a bijection between $\mathrm{Min}(\mathrm{Sing}(I))$ *and the minimal primes of* $\mathrm{Sing}(I^{std})$ *of height at most* $\dim(T)$.

Proof We first prove the case where $\mathrm{char}(k) = 0$. Write $T = k[x_1, \ldots, x_{n-1}, y]$. By induction we may focus on the case where we replace a single variable y of degree d by $F = \sum_{j=0}^{n} \prod_{\ell=1}^{d} w_{j,k}$ and leave all other variables fixed. Let $I = (g_1, \ldots, g_s)$ be a homogeneous ideal of T and let I^{std} denote the ideal generated by the images $G_i = g_i^{\mathrm{std}}$ of the g_i under the map

$$(\_)^{\mathrm{std}} : T \to T^{\mathrm{std}} = k[z_1, \ldots, z_n - 1, w_{0,1}, \ldots, w_{n,d}]$$

defined by $x_i \longmapsto z_i$ and $y \longmapsto F$. Let $c = \mathrm{ht}(I)$. By Lemma 2(3) we know that $c = \mathrm{ht}(I^{\mathrm{std}})$ as well. By the Jacobian criterion, $\mathrm{Sing}(X)$ and $\mathrm{Sing}(X^{\mathrm{std}})$ are defined, up to radical, by $I_c(\mathsf{Jac}(I))$ and $I_c(\mathsf{Jac}(I^{\mathrm{std}}))$, respectively. Write

$$\mathsf{Jac}(g_1, \ldots, g_s) = \begin{array}{c} \\ \partial_{z_1} \\ \partial_{z_2} \\ \vdots \\ \partial_{z_{n-1}} \\ \partial_y \end{array} \begin{array}{c} \begin{array}{cccc} g_1 & g_2 & \cdots & g_s \end{array} \\ \begin{pmatrix} \partial_{z_1}(g_1) & \partial_{z_1}(g_2) & \cdots & \partial_{z_1}(g_s) \\ \partial_{z_2}(g_1) & \partial_{z_2}(g_2) & \cdots & \partial_{z_2}(g_s) \\ \vdots & & & \vdots \\ \partial_{z_{n-1}}(g_1) & \partial_{z_{n-1}}(g_2) & \cdots & \partial_{z_{n-1}}(g_s) \\ \partial_y(g_1) & \partial_y(g_2) & \cdots & \partial_y(g_s) \end{pmatrix} \end{array}$$

Let E be the row vector $(\partial_y(g_1)\ \partial_y(g_2)\ \ldots\ \partial_y(g_s))$ and let D be the $(n-1) \times s$ submatrix of $\mathsf{Jac}(g_1, \ldots, g_s)$ obtained by removing E from $\mathsf{Jac}(g_1, \ldots, g_s)$ so that

$$\mathsf{Jac}(I) = \left(\frac{\mathsf{D}}{\mathsf{E}}\right).$$

By the chain rule, the Jacobian matrix of I^{std} is

$$\mathsf{Jac}(G_1, \ldots, G_s) = \begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \partial_{w_{0,1}}(F) \cdot \mathsf{E}^{\mathrm{std}} \\ \partial_{w_{0,2}}(F) \cdot \mathsf{E}^{\mathrm{std}} \\ \vdots \\ \partial_{w_{n,d}}(F) \cdot \mathsf{E}^{\mathrm{std}} \end{pmatrix},$$

where $\mathsf{D}^{\mathrm{std}}$ and $\mathsf{E}^{\mathrm{std}}$ are obtained by applying $(\_)^{\mathrm{std}}$ to every entry of D and E, and $\partial_{w_{i,j}}(F) \cdot \mathsf{E}^{\mathrm{std}}$ is the scalar product of $\partial_{w_{i,j}}(F)$ and $\mathsf{E}^{\mathrm{std}}$.
Claim. One has

$$I_c(\mathsf{Jac}(I^{\mathrm{std}})) = I_c(\mathsf{D}^{\mathrm{std}}) + (\partial_{w_{0,1}}(F), \ldots, \partial_{w_{n,d}}(F)) \cdot I_c((\mathsf{Jac}(I))^{\mathrm{std}}).$$

Proof of Claim. Write $\mathsf{E}^{\mathrm{std}} = (e_1, \ldots, e_s)$. Let H be a c-minor of $\mathsf{Jac}(I^{\mathrm{std}})$. Observe that H is obtained by taking at least two of the last $(n+1)d$ rows. In particular, we have

$$H = \det \begin{pmatrix} \vdots & & & \vdots \\ \partial_{w_{j,\ell}}(F)e_1 & \partial_{w_{j,\ell}}(F)e_2 & \ldots & \partial_{w_{j,\ell}}(F)e_s \\ \vdots & & & \vdots \\ \partial_{w_{j',\ell'}}(F)e_1 & \partial_{w_{j',\ell'}}(F)e_2 & \ldots & \partial_{w_{j',\ell'}}(F)e_s \\ \vdots & & & \vdots \end{pmatrix}$$

$$= \partial_{w_{j,\ell}}(F)\partial_{w_{j',\ell'}}(F)\det\begin{pmatrix} \vdots & & & \vdots \\ e_1 & e_2 & \dots & e_s \\ \vdots & & & \vdots \\ e_1 & e_2 & \dots & e_s \\ \vdots & & & \vdots \end{pmatrix} = 0.$$

Therefore, every non-zero $c \times c$ minor of $\mathsf{Jac}(I^{\mathrm{std}})$ involves at most one of the last $(n+1)d$ rows, or equivalently,

$$I_c(\mathsf{Jac}(I^{\mathrm{std}})) = \sum_{j=0}^{n}\sum_{\ell=1}^{d} I_c\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \partial_{w_{j,\ell}}(F)\cdot \mathsf{E}^{\mathrm{std}} \end{pmatrix}. \tag{2}$$

Observe that if H is a $c \times c$ minor of $\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \partial_{w_{j,\ell}}(F)\cdot \mathsf{E}^{\mathrm{std}} \end{pmatrix}$ not involving the last row, then $H \in I_c(\mathsf{D}^{\mathrm{std}})$, while if H involves the last row of the above matrix, then $H = \partial_{w_{j,\ell}}(F)\cdot \det(\Theta)$, where Θ is a c by c submatrix of $\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \mathsf{E}^{\mathrm{std}} \end{pmatrix}$ that involves the last row. Since $\partial_{w_{j,\ell}}(F) I_c(\mathsf{D}^{\mathrm{std}}) \subseteq I_c(\mathsf{D}^{\mathrm{std}})$, we can write

$$I_c\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \partial_{w_{j,\ell}}(F)\cdot \mathsf{E}^{\mathrm{std}} \end{pmatrix} = I_c(\mathsf{D}^{\mathrm{std}}) + \partial_{w_{j,\ell}}(F)\cdot I_c\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \mathsf{E}^{\mathrm{std}} \end{pmatrix}.$$

Substituting the above in Eq. (2) for every i, we obtain

$$\begin{aligned} I_c(\mathsf{Jac}(I^{\mathrm{std}})) &= I_c(\mathsf{D}^{\mathrm{std}}) + \textstyle\sum_{j=0}^{n}\sum_{\ell=1}^{d}\left(\partial_{w_{j,\ell}}(F)\cdot I_c\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \mathsf{E}^{\mathrm{std}} \end{pmatrix}\right) \\ &= I_c(\mathsf{D}^{\mathrm{std}}) + (\partial_{w_{0,1}}(F), \dots, \partial_{w_{n,d}}(F))\cdot I_c\begin{pmatrix} \mathsf{D}^{\mathrm{std}} \\ \mathsf{E}^{\mathrm{std}} \end{pmatrix} \\ &= I_c(\mathsf{D}^{\mathrm{std}}) + (\partial_{w_{0,1}}(F), \dots, \partial_{w_{n,d}}(F))\cdot I_c((\mathsf{Jac}(I))^{\mathrm{std}}) \end{aligned}$$

proving the claim. □

Let $\mathrm{Min}(I_c(\mathsf{Jac}(I)) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be the minimal primes in T of $\mathrm{Sing}(I)$. By Lemma 2(2) each $\mathfrak{p}_i^{\mathrm{std}}$ is prime. We claim that $\{\mathfrak{p}_i^{\mathrm{std}} \mid i = 1, \dots, r\}$ are the minimal primes of $I_c(\mathsf{Jac}(I^{\mathrm{std}}))$ of height at most n. To this end, we first observe that $I_c(\mathsf{Jac}(I^{\mathrm{std}})) \subseteq \mathfrak{p}_i^{\mathrm{std}}$—this follows from the claim and the fact that $\mathfrak{p}_i^{\mathrm{std}}$ contains both $I_c(D^{\mathrm{std}})$ and $I_c(\mathsf{Jac}(I)^{\mathrm{std}})$.

Next, we show that any prime containing $I_c(\mathsf{Jac}(I^{\mathrm{std}}))$ has either height at least $n+1$ or it contains one of the $\mathfrak{p}_i^{\mathrm{std}}$. This will conclude the proof.

So, let $\mathfrak{q}$ be a minimal prime ideal with $I_c(\mathsf{Jac}(I^{\mathrm{std}})) \subseteq \mathfrak{q}$. By the claim,

$$(\partial_{w_{0,1}}(F), \dots, \partial_{w_{n,d}}(F))\cdot I_c((\mathsf{Jac}(I))^{\mathrm{std}}) \subseteq \mathfrak{q}.$$

If $(\partial_{w_{0,1}}(F), \dots, \partial_{w_{n,d}}(F)) \subseteq \mathfrak{q}$, by Lemma 2, $\mathrm{ht}(\mathfrak{q}) \geq n+1 > \dim(T)$. If $(\partial_{w_{0,1}}(F), \dots, \partial_{w_{n,d}}(F)) \nsubseteq \mathfrak{q}$, then, since $\mathfrak{q}$ is a prime ideal, the ideal $\mathfrak{q}$ contains $\sqrt{(I_c(\mathsf{Jac}(I)))^{\mathrm{std}}} = \sqrt{I_c((\mathsf{Jac}(g_1, \dots, g_s))^{\mathrm{std}})} = \mathfrak{p}_1^{\mathrm{std}} \cap \mathfrak{p}_2^{\mathrm{std}} \cap \dots \cap \mathfrak{p}_r^{\mathrm{std}}$, where the right-most equality follows by Lemma 2(4). Then $\mathfrak{q}$ contains one of the $\mathfrak{p}_i^{\mathrm{std}}$. It follows that each of the $\mathfrak{p}_i^{\mathrm{std}}$ is a minimal prime of $I_c(\mathsf{Jac}(I^{\mathrm{std}}))$ and these are the only minimal primes of height at most $n = \dim(T)$.

When $\mathrm{char}(k) = p > 0$, the Jacobian criterion states that the singular locus of I is defined, up to radical, by $I + I_c(\mathsf{Jac}(I))$. The proof follows by a similar argument with the following differences: let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be the minimal primes of $I + I_c(\mathsf{Jac}(I))$; in the last part of the proof, we let $\mathfrak{q}$ be a prime ideal containing $I^{\mathrm{std}} + I_c(\mathsf{Jac}(I^{\mathrm{std}}))$, and after finding that $(I_c(\mathsf{Jac}(I)))^{\mathrm{std}} \subseteq \mathfrak{q}$ we have

$$I^{\mathrm{std}} + (I_c(\mathsf{Jac}(I)))^{\mathrm{std}} = (I + I_c(\mathsf{Jac}(I)))^{\mathrm{std}} \subseteq \mathfrak{q},$$

thus $\mathfrak{q}$ contains a minimal prime of $(I + I_c(\mathsf{Jac}(I)))^{\mathrm{std}}$. □

We now apply the preceding theorem to the defining prime ideal of the Rees-like algebra of a homogeneous ideal. Combining it with Theorem 2 we obtain:

Corollary 1 *Let $\mathcal{RLP}(I) \subset T$ be the defining prime ideal of $\mathcal{RL}(I)$ for some homogeneous ideal $I \subset S = k[x_1, \dots, x_n]$ and suppose k is algebraically closed and* $\mathrm{char}(k) \neq 2$.

Using the standardization from Construction 1, $\mathcal{RLP}(I)^{std}$ is a nondegenerate, homogeneous prime ideal in a standard graded polynomial ring T^{std} which defines a projective variety X such that $\mathrm{codim}_X(\mathrm{Sing}\, X) = \mathrm{ht}(I)$.

4 Application: smooth hyperplane sections

It is natural to ask if Rees-like algebras and standardizations are sufficient to give a smooth counterexample to the Eisenbud–Goto conjecture. We exploit the work so far to settle this in the negative. More precisely, we show that a nonzero, homogeneous ideal $I \subset S$ is Cohen–Macaulay if and only if a prime standardization of its Rees-like algebra $\mathcal{RL}(I)$ has a hyperplane section that is both smooth and preserves the original graded Betti numbers. The rest follows by giving a sufficient bound on the regularity of Cohen–Macaulay ideals. For simplicity of exposition, the reader may focus only on the prime standardization from Construction 1.

We note that the defining ideal of a Rees-like algebra (or its prime standardization) is only Cohen–Macaulay when I is principal; thus the majority of the smooth varieties considered in this section are not arithmetically Cohen–Macaulay.

Theorem 5 *Let k be an algebraically closed field with* $\mathrm{char}(k) = 0$, *and let $S = k[x_1, \dots, x_n]$, and let I be a proper homogeneous S-ideal. Let $X \subseteq \mathbb{P}^N$ denote the projective variety corresponding to the prime standardization from Construction 1 applied to the Rees-like algebra of I. The following two conditions are equivalent:*

(i) *There exists a regular sequence of general hyperplane sections of X such that the resulting variety is smooth;*
(ii) *S/I is Cohen–Macaulay.*

Proof Set $I = (f_1, \dots, f_m) \subset S = k[x_1, \dots, x_n]$. Let $\mathcal{RLP}(I)$ be the defining prime ideal of $\mathcal{RL}(I)$ and let $T \to T^{\mathrm{std}}$ be the prime standardization defined in Construction 1. By Bertini's theorem (cf. [9, Chapter 0.H]), we may factor out a regular sequence of $\mathrm{depth}(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}}) - 1$ general linear forms and preserve both the graded Betti numbers of $\mathcal{RLP}(I)^{\mathrm{std}}$ and primeness. Doing so reduces both the dimension of the associated projective variety and that of its singular locus by $\mathrm{depth}(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}}) - 1$. Thus one obtains a smooth variety if and only if one has

$$\mathrm{depth}(R/\mathcal{RLP}(I)^{\mathrm{std}}) - 1 > \dim(\mathrm{Sing}\,\mathrm{Proj}(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}})),$$

or equivalently

$$\dim(T^{\mathrm{std}}) - \mathrm{depth}(R/\mathcal{RLP}(I)^{\mathrm{std}}) + 1 < \mathrm{ht}(J) + 1,$$

where J is the defining ideal of Sing Proj$(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}})$ in T^{std}. By Corollary 1, $\mathrm{ht}(J) = m + \mathrm{ht}(I)$. By the Auslander–Buchsbaum theorem, Theorem 1 and Proposition 2 one has

$$\dim(T^{\mathrm{std}}) - \mathrm{depth}(R/\mathcal{RLP}(I)^{\mathrm{std}}) = \mathrm{pd}(R/\mathcal{RLP}(I)^{\mathrm{std}}) = \mathrm{pd}(S/I) + m - 1.$$

Thus, the above inequality holds if and only if

$$\mathrm{pd}(S/I) + m < \mathrm{ht}(I) + m + 1$$

or equivalently $\mathrm{pd}(S/I) \leq \mathrm{ht}(I)$, which occurs if and only if S/I is Cohen–Macaulay. □

We recall that among all Cohen–Macaulay ideals I generated by forms of fixed degrees, complete intersections have the largest regularity.

Lemma 3 (cf. Huneke et. al. [12, 3.1]) *Let $S = k[x_1, \ldots, x_n]$ and I be a homogeneous S-ideal such that S/I is Cohen-Macaulay. If $d_i = \deg(f_i)$, then $\mathrm{reg}(S/I) \leq \sum_{i=1}^{m}(d_i - 1)$.*

The main result of this section depends on the following elementary lemma whose proof is left to the reader.

Lemma 4 *Let $d_1, \ldots, d_m$ be positive integers,*

$$\sum_{i=1}^{m}(d_i + 1) \leq \prod_{i=1}^{m}(d_i + 1).$$

Here we show that any of the smooth hyperplane sections of Rees-like varieties described above satisfy the Eisenbud–Goto Conjecture [6]. These provide many examples of smooth non-Cohen–Macaulay varieties satisfying the conjecture.

Corollary 2 *Let k be an algebraically closed field with $\mathrm{char}(k) = 0$. Let $I = (f_1, \ldots, f_m) \subset S = k[x_1, \ldots, x_n]$ be a homogeneous ideal such that S/I is Cohen–Macaulay. For any regular sequence of general hyperplane sections that cut out from the prime standardization in Construction 1 of the Rees-like prime of I a smooth variety, Eq. (1) is satisfied.*

Proof Set $d_i = \deg(f_i)$ and set $\overline{T^{\mathrm{std}}}$ to be the quotient of T^{std} by $\mathrm{depth}(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}}) - 1$ general linear forms. Similarly set $\overline{\mathcal{RLP}(I)^{\mathrm{std}}}$ to be $\mathcal{RLP}(I)\overline{T^{\mathrm{std}}}$. If $m = 1$, then $\overline{\mathcal{RLP}(I)^{\mathrm{std}}}$ is a hypersurface and the claim holds. If $m \geq 2$, then

$$\begin{aligned}
&\mathrm{reg}(\overline{T^{\mathrm{std}}}/\overline{\mathcal{RLP}(I)^{\mathrm{std}}}) \\
&\quad = \mathrm{reg}(T^{\mathrm{std}}/\mathcal{RLP}(I)) \\
&\quad = \mathrm{reg}(S/I) + 2 + \sum_{i=1}^{m} d_i && \text{by Theorem 1} \\
&\quad \leq \sum_{i=1}^{m}(d_i - 1) + 2 + \sum_{i=1}^{m} d_i && \text{by Lemma 3} \\
&\quad \leq 2\sum_{i=1}^{m} d_i && \text{since } m \geq 2 \\
&\quad \leq 2\prod_{i=1}^{m}(d_i + 1) - m && \text{by Lemma 4}
\end{aligned}$$

$$\begin{aligned} &= \deg(T^{\mathrm{std}}/\mathcal{RLP}(I)^{\mathrm{std}}) - \mathrm{ht}(\mathcal{RLP}(I)^{\mathrm{std}}) && \text{by Theorem 1} \\ &= \deg(\overline{T^{\mathrm{std}}}/\overline{\mathcal{RLP}(I)^{\mathrm{std}}}) - \mathrm{ht}(\overline{\mathcal{RLP}(I)^{\mathrm{std}}}). \end{aligned}$$

□

5 Seminormality and weak normality

Rees-like algebras are domains, hence they satisfy Serre's conditions (R_0) and (S_1). However, it is easy to check that they are never normal (see Proposition 4 below). When $I = (f)$ is a hypersurface, $\mathcal{RLP}(I) = (y^2 - zf^2)$ fails Serre's condition (R_1), however it satisfies Serre's condition (S_i) for all i.

In contrast, we show that whenever $\mathrm{ht}(I) > 1$, the ideal $\mathcal{RLP}(I)$ satisfies (R_1) but not (S_2). First, let us recall the following equivalent form of Theorem 2.

Theorem 6 *Let k be a perfect field with $\mathrm{char}(k) \neq 2$ and let S be a polynomial ring over k. For any nonzero, proper ideal $I \subset S$, the Rees-like algebra $\mathcal{RL}(I)$ satisfies Serre's condition (R_{h-1}), where $h = \mathrm{ht}(I)$, and does not satisfy Serre's condition (R_h).*

Proposition 4 *For any nonzero, proper ideal $I \subset S$, the Rees-like algebra $\mathcal{RL}(I)$ is not normal.*

Proof Since $\mathcal{RL}(I)$ is a domain, we show that $\mathcal{RL}(I)$ is not integrally closed in its field of fractions. For any $0 \neq f \in I$ we have $t = \frac{ft^2}{ft} \in \mathrm{Frac}(\mathcal{RL}(I))$, and it follows that $\mathrm{Frac}(\mathcal{RL}(I)) = S(t) = \mathrm{Frac}(S[t])$. Clearly $t \notin \mathcal{RL}(I) = S[It, t^2]$ and t satisfies the monic polynomial equation $X^2 - t^2 \in \mathcal{RL}(I)[X]$. Thus $\mathcal{RL}(I)$ is not integrally closed and its integral closure is $S[t]$. □

Corollary 3 *If $\mathrm{ht}(I) > 1$, then $\mathcal{RL}(I)$ does not satisfies Serre's condition (S_2).*

We turn our attention then to alternate forms of normality, namely weak normality and seminormality. We quickly review these notions, but for a more thorough treatment, consult [27].

Definition 3 For a finite extension $A \subset B$ of reduced rings, a subextension $A \subset C \subset B$ is **subintegral** provided it is integral, induces a bijection on spectra, and an isomorphism on residue fields at all points. It is called **weakly subintegral** provided one only asks for purely inseparable extensions of residue fields.

In any extension $A \subset A^{\mathrm{N}}$ of a ring into its normalization, there is a unique largest subextension $A \subset A^{\mathrm{SN}} \subset A^{\mathrm{N}}$ which is subintegral and one says that A is **seminormal** provided that $A = A^{\mathrm{SN}}$. Similarly, there is a unique largest subextension which is weakly subintegral $A \subset A^{\mathrm{WN}} \subset A^{\mathrm{N}}$ and we say that A is **weakly normal** if $A = A^{\mathrm{WN}}$. Consequently all normal rings are seminormal and all seminormal rings are weakly normal; in characteristic 0 weakly normal and seminormal are equivalent properties.

A prototypical example of a seminormal ring which is not normal is the pinch point $k[x, xt, t^2] \cong k[x, y, z]/(y^2 - zx^2)$, where $\mathrm{char}(k) \neq 2$. This ring corresponds to the Rees-like algebra a single linear form. We show that quite often, Rees-like algebras are seminormal and weakly normal. To do this, we exploit the following useful criteria.

Theorem 7 *For a reduced ring A,*

1. *[16, Proposition 1.4] A is seminormal if and only if for a fixed pair of relatively prime integers* $0 < r < s$, *when* $b \in A^N$ *satisfies* $b^r \in A$ *and* $b^s \in A$ *then* $b \in A$,
2. *[28, Theorem 1] if the characteristic of A is* $p > 0$, *then A is weakly normal if for each* $b \in A^N$ *such that* $b^p \in A$ *then* $b \in A$.

For the remainder of this section, set $S = k[x_1, \dots, x_n]$ to be a polynomial ring and I a homogeneous ideal in S. Our next goal illustrates the general theme of characterizing geometric properties of the Rees-like algebra of I in terms of algebraic properties of I. Recall the normalization of $\mathcal{RL}(I)$ is $S[t]$.

Theorem 8 *With the notation as above, the following are equivalent:*

1. I *is radical,*
2. *for every odd integer* $\sigma > 1$, *for every* $b \in S[t]$, *if* $b^\sigma \in \mathcal{RL}(I)$ *then* $b \in \mathcal{RL}(I)$,
3. *there are two coprime integers, r and s both greater than* 1 *such that for every* $b \in S[t]$, *if* $b^r \in \mathcal{RL}(I)$ *and* $b^s \in \mathcal{RL}(I)$, *then* $b \in \mathcal{RL}(I)$.
4. *there is an odd integer* $\sigma > 1$ *such that for every* $b \in S[t]$, *if* $b^\sigma \in \mathcal{RL}(I)$ *then* $b \in \mathcal{RL}(I)$,

Proof The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are clear. We first prove (4) $\Longrightarrow$ (1). Assume $a \in S$ and $a^n \in I$ for some $n \in \mathbb{Z}_+$. If $r \in \mathbb{Z}_+$ with $\sigma^r \geq n$, then $a^{\sigma^r} \in I$. Thus $(at)^{\sigma^r} \in \mathcal{RL}(I)$. By assumption (4) it follows that $at \in \mathcal{RL}(I) = S[It, t^2]$, and so $a \in I$.

The theorem follows by showing that (1) $\Longrightarrow$ (2). Fix an odd integer $\sigma > 1$ and assume I is radical. Let $b \in S[t]$ be an element such that $b^\sigma \in \mathcal{RL}(I)$, we need to show that $b \in \mathcal{RL}(I)$. We consider the grading on $S[t]$ given by $\deg(t) = 1$ and $\deg(f) = 0$ for every $f \in S$. Write $b = \sum_{j=1}^r b_j t^{i_j}$ with $b_j \in S$, for integers $0 \leq i_1 < i_2 < \dots < i_s$ and elements $b_j \in S$.
Claim. We may assume $b_j \notin I$ for any j.

To prove the claim, observe that if $d \in I$, then $dt^k \in \mathcal{RL}(I) = S[It, t^2]$ for every $k \geq 1$. Now, assume $b_j \in I$ for some j. Expand

$$(b - b_j t^{i_j})^\sigma = b^\sigma + \sum_{h=1}^{\sigma} \binom{\sigma}{h} b^{\sigma-h} b_j^h t^{h i_j}.$$

Since $b_j \in I$, each $\binom{\sigma}{h} b^{\sigma-h} b_j^h \in I$ and thus $\binom{\sigma}{h} b^{\sigma-h} b_j^h t^{h i_j} \in S[It, t^2]$ for every $1 \leq h \leq \sigma$. It follows that $b^\sigma \in S[It, t^2]$ if and only if $(b - b_j t^{i_j})^\sigma \in S[It, t^2]$. We may then decompose $b = \widetilde{b} + \widetilde{c}$ where $\widetilde{b} = \sum b_j t^{i_j}$ with each $b_j \notin I$ and $\widetilde{c} = \sum b_j t^{i_j}$ with each $b_j \in I$. By the above, $b^\sigma \in S[It, t^2]$ if and only if $\widetilde{b}^\sigma \in S[It, t^2]$ and $b \in S[It, t^2]$ if and only if $\widetilde{b} \in S[It, t^2]$. By replacing b by $\widetilde{b}$ we can assume $b_j \notin I$ for any j, proving the claim.

It suffices to show that each i_j is even, because then $b \in S[t^2] \subset S[It, t^2]$. We proceed by induction on the number $r \geq 1$ of homogeneous components of b. If $r = 1$, then $b = b_1 t^{i_1}$. Assume by contradiction that i_1 is odd. Since $b^\sigma = b_1^\sigma t^{i_1 \sigma} \in S[It, t^2]$ and $i_1 \sigma$ is odd, then $b_1^\sigma \in I$. Since I is radical, this implies $b_1 \in I$, yielding a contradiction. Therefore $i_1 \in 2\mathbb{Z}$.

Next, assume $r > 1$. Assume by contradiction one of the i_j is odd, we let $u = \min\{j \mid i_j \text{ is odd }\}$. Observe that $b_1^{\sigma-1} b_u t^{i_1(\sigma-1)+i_u}$ is the homogeneous component of smallest odd degree of $b^\sigma \in S[It, t^2]$, thus it lies in $S[It, t^2]$. Since $i_1(\sigma - 1) + i_u$ is odd, then $b_1^{\sigma-1} b_u \in I$ and so $(b_1 b_u)^{\sigma-1} \in I$. Since I is radical, we obtain $b_1 b_u \in I$. Now consider

$$d = b_u b = b_1 b_u t^{i_1} + b_2 b_u t^{i_2} + \dots + b_u^2 t^{i_u} + \dots$$

Set $e := d - b_1 b_u t^{i_1}$. Since $b_1 b_u \in I$, by the proof of the claim it follows that $e^\sigma \in S[It, t^2]$. By induction, it follows that $e \in S[It, t^2]$. Since $b_1 b_u t^{i_1} \in S[It, t^2]$ too, then $d \in S[It, t^2]$, so every homogeneous component of d lies in $S[It, t^2]$. In particular, the homogeneous component of degree i_u, i.e. $b_u^2 t^{i_u}$ lies in $S[It, t^2]$. Since i_u is odd, then $b_u^2 \in I$, since I is radical, $b_u \in I$ which is a contradiction. $\square$

Remark 1 One must work with odd integers in Theorem 8. If σ is even and $b = t \in S[t]$ one has $b^2 \in S[It, t^2]$ but $b \notin S[It, t^2]$.

Combining Theorems 7 and 8, one has the following immediate corollary.

Corollary 4 *Let k be a field with* $\mathrm{char(k)} \neq 2$ *and let S a polynomial algebra over k. A homogeneous ideal I is radical if and only if its Rees-like algebra $\mathcal{RL}(I)$ is seminormal which happens if and only if $\mathcal{RL}(I)$ is weakly normal.*

One should notice that the analogous statement for Rees algebras does not hold. Indeed, the following is an example of a radical ideal I whose Rees Algebra $R[It]$ is not seminormal. This example was found with the help of the Macaulay2 Seminormalization package of Serbinowski and Schwede [11,24].

Example 5 Let k be a field and $S = k[x, y, z]$. Let

$$\mathfrak{p} = (y^4 - x^3 z, xy^3 - z^3, x^4 - yz^2)$$

be the ideal defining the monomial curve $k[v^9, v^{10}, v^{13}]$. Then

$$\mathfrak{p} = I_2 \begin{pmatrix} z & -y & x \\ -y^3 & x^3 & -z^2 \end{pmatrix}.$$

By [25, p. 309], $\mathfrak{p}$ is not normal; that is, not all powers of $\mathfrak{p}$ are integrally closed and thus the Rees algebra $\mathcal{R}(\mathfrak{p}) = S[\mathfrak{p}t]$ is not a normal ring. We show next that $\mathcal{R}(\mathfrak{p})$ is not even seminormal.

Write $p_1 = y^4 - x^3 z$, $p_2 = xy^3 - z^3$, and $p_3 = x^4 - yz^2$. Now set

$$f = \frac{x^2(p_2 t)(p_3 t) + z(p_1 t)^2}{y} = (x^7 y^2 - 3x^3 y^3 z^2 + x^2 z^5 + y^7 z)t^2 \in \mathrm{Frac}(S[\mathfrak{p}t]).$$

Since no product of two monomial terms among the generators of $\mathfrak{p}$ divides $x^2 z^5$, it follows that $f \notin S[\mathfrak{p}t]$. However, we verify below that $f^2, f^3 \in S[\mathfrak{p}t]$. Indeed,

$$f^2 = (-yp_1^3 p_3 + x^2 p_1 p_3^3 + yp_2^4 - zp_2^3 p_3 + xzp_3^4)t^4 \in \mathfrak{p}^4 t^4 \subseteq S[\mathfrak{p}t],$$

and

$$f^3 = (-zp_1^5 p_3 + zp_1^2 p_2^4 + 3xzp_1^2 p_2 p_3^3 + z^2 p_1 p_2 p_3^4 + x^3 p_2^2 p_3^4)t^6 \in \mathfrak{p}^6 t^6 \subseteq S[\mathfrak{p}t].$$

By Theorem 7, we see that $\mathcal{R}(\mathfrak{p})$ is not seminormal. However, since $\mathfrak{p}$ is prime, $\mathcal{RL}(\mathfrak{p}) = S[\mathfrak{p}t, t^2]$ is seminormal by Corollary 4.

In positive characteristic, F-split rings are weakly normal, so in view of Corollary 4 one may hope to find a fairly large class of ideals I for which $\mathcal{RL}(I)$ is F-split. As such from this point forward, for simplicity, we fix a perfect ground field k and all rings and fields considered for the rest of this section are F-finite. We also identify the Frobenius map with the inclusion $S \subset S^{1/p}$ into a choice of p-th roots of elements of S from a fixed algebraic closure.

Theorem 9 *Suppose* $\mathrm{char}(k) = p > 2$ *and I is a radical ideal in S. The ring S/I is F-split if and only if $\mathcal{RL}(I)$ is F-split.*

Proof Assume that S/I is F-split. Every splitting of S/I is induced by a splitting $\varphi\colon S^{1/p} \to S$ of S with $\varphi(I^{1/p}) \subset I$. Next, we consider $\mathcal{RL}(I) = S[It, t^2]$ as a graded subring of $S[t]$. Define $\psi\colon S[t]^{1/p} \to S[t]$ by writing $f \in S[t]^{1/p}$ as $f = \sum a_i^{1/p} t^{i/p}$ and setting

$$\psi(f) = \sum_{i \equiv 0 \pmod p} \varphi(a_i^{1/p}) t^{\frac{i}{p}}.$$

Clearly this is S-linear and $\psi(t \cdot f) = t\psi(f)$ for each $f \in S[t]^{1/p}$. Thus ψ is $S[t]$-linear, whence $\mathcal{RL}(I)$-linear. Moreover ψ is surjective because $\psi(1) = 1$. We show that the $\psi(\mathcal{RL}(I)^{1/p}) \subseteq \mathcal{RL}(I)$. This will show that $\psi|_{\mathcal{RL}(I)}$ is an F-splitting of $\mathcal{RL}(I)$. Let $f = \sum a_i^{1/p} t^{i/p} \in \mathcal{RL}(I)^{1/p}$, so $a_i \in S$ for every even i and $a_i \in I$ for every i odd. To prove $\psi(f) \in \mathcal{RL}(I)$ we need to show that if $\frac{i}{p}$ is an odd integer, then $\varphi(a_i^{1/p}) \in I$. This follows since $\frac{i}{p}$ being odd implies that i is odd. Thus we have $a_i^{1/p} \in I^{1/p}$ and so $\varphi(a_i^{1/p}) \in I$.

Conversely, assume $\mathcal{RL}(I)$ is F-split. We may assume without loss of generality that $\psi\colon \mathcal{RL}(I)^{1/p} \to \mathcal{RL}(I)$ is a splitting which is graded of degree 0. Denote by $\psi_0\colon S^{1/p} \to S$ the restriction of ψ to the degree 0 part of $\mathcal{RL}(I)$. This is clearly S-linear and surjective, so it suffices to see that $\psi_0(I^{1/p}) \subset I$. By $\mathcal{RL}(I)$-linearity, for $a \in I$ we have

$$at^2\psi_0(a^{1/p}) = \psi(a^{1/p} \cdot at^2) = at\psi(a^{1/p}t).$$

As $\mathcal{RL}(I)$ is a domain, we have $\psi_0(a^{1/p})t = \psi(a^{1/p}t)$. Since ψ is graded, $\psi(a^{1/p}t) \in It$, so $\psi_0(a^{1/p})t \in It$ and then $\psi_0(a^{1/p}) \in I$, as desired. $\square$

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