STRATEGIES USED BY STUDENTS WITH LEARNING DISABILITIES FOR REASONING ABOUT SLOPE

Anna F. DeJarnette
University of Cincinnati
anna.dejarnette@uc.edu

Casey Hord
University of Cincinnati
casey.hord@uc.edu

Students with learning disabilities (LD), like other learners, show a range of resources and strategies for reasoning about complex concepts in mathematics. This study comes from a project in which a group of five ninth-grade students with LD participated in a once-weekly tutoring program with university pre-service teachers. We asked, what strategies did students use to reason about slope? Students drew upon knowledge of concepts related to constant covariation when given the opportunity. This study suggests that students with LD have rich conceptual knowledge that can be leveraged to improve their success in Algebra.

Keywords: Algebra and algebraic thinking, Classroom discourse, Students with learning disabilities

The concept of slope presents a significant challenge in the teaching and learning of linear functions. On one hand, slope is a complex concept, referring to a single quantity that describes a pattern of constant covariation among two distinct, but related, quantities in a linear relationship (Lamon, 1995). On the other hand, it is possible for students to work somewhat fluently with slope on "rise over run" procedural tasks without attending to the conceptual underpinnings of slope (DeJarnette, Marita, & Hord, 2019; Zahner, 2015). Students with learning disabilities (LD) may be more likely to receive instruction that gives more focus to the correct application of procedures and less focus to underlying concepts (Foegen & Dougherty, 2017). Given the high stakes of passing high school algebra, such efforts seem well-intentioned towards supporting struggling students to be successful. However, there is an opportunity to create more meaningful learning opportunities for all students, and students with LD in particular, by attending to and building upon the conceptual knowledge that they apply to tasks about linear functions.

With this study we posed the question, what strategies do students use to reason about slope across different types of tasks related to linear functions? The purpose of this study is to use knowledge of students' strengths to suggest ways in which future instruction can privilege those strengths. In particular, this study supports the claim that procedural fluency is not a pre-requisite for building upon students' conceptual understanding of slope, and in fact making connections to concepts can support students in completing procedural tasks.

Research Perspectives

Attention to slope as a composite quantity is part of a broader perspective known as a covariational approach to the teaching and learning of functions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994; Saldanha & Thompson, 1998). A covariational approach to functions is a contrast to a "correspondence" approach, through which functions are treated as relationships between individual inputs and outputs (Confrey & Smith, 1994). More generally, the covariational approach is part of an effort to develop students' understanding of foundational concepts such as variable, function, and rate of change in ways that can be applied through increasingly advanced levels of mathematics (Thompson & Carlson, 2017).

Many recent efforts to support students' conceptual understanding of slope align with a covariational perspective. Slope as a ratio is a "complex composite unit" (Lamon, 1995), a single quantity representing a multiplicative relationship between two distinct quantities. To work fluently with linear functions, including making interpretations or predictions about what linear relationships represent, students need to act on this ratio as a single quantity (Ellis, 2007). In practice, however, students often treat numerical representations of slope as two distinct quantities representing horizontal and vertical change along a graph (DeJarnette et al., 2019; Lobato, Ellis, & Muñoz, 2003; Zahner, 2015). Such conceptions can be sufficient for solving tasks related to calculating slope given a pair of points or a graph of a line. However, these conceptions tend to leave out foundational ideas, such as how slope describes the steepness or direction of a graph. Knowledge of the concept of slope can serve as a resource for students on a variety of tasks, including correcting errors when procedures are mis-applied.

Methods

This study comes from a broader research effort to train university pre-service teachers to be effective tutors for students with LD taking Algebra 1. Five ninth-grade students from a large suburban high school agreed to participate in the project, after being selected by their math teacher as students who would likely appreciate and benefit from 1-1 tutoring. The students all were identified by their schools as having LD. The students were enrolled in the first year of a two-year Algebra 1 sequence at the school. The tutors were recruited for the project from a mathematics teaching methods course taken by pre-service teachers in the special education program and the middle childhood program at the university. Four of the tutors were pre-service special education teachers; one tutor was a pre-service middle-grades teacher. The tutors were recruited based on high achievement and engagement in their methods course.

Beginning in December 2018, students met with the tutors for 1-1 tutoring one time per week during their usual math class. There were two tutoring sessions from which we drew data for this study. The first was a day on which students had an assignment where they needed to solve systems of linear equations by graphing (Figure 1). The second was a day on which students worked on a task for which they compared slopes of six different lines. We selected these two sessions for analysis because the tasks elicited the use of slope concepts and procedures.

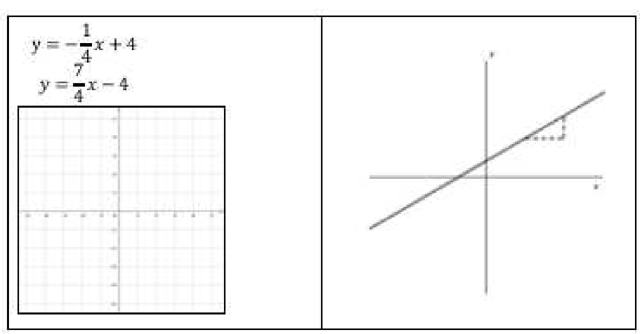

Figure 1: Tasks from the Sessions, Systems of Equations (left) and Comparing Slope (right)

All of the tutoring sessions were recorded with document cameras that captured students' papers as well as the students' and tutors' hands as they worked. We analyzed seven tutoring videos—four videos from the "systems of equations" session and three videos from the "comparing slopes" session. We applied a constant comparison analysis (Strauss & Corbin, 1998), using the video recordings and partial transcripts of the sessions, to look for patterns in the strategies that students used for calculating, graphing, or interpreting slope. The first author viewed each video in its entirety, separating the tutoring sessions into "segments" to highlight key moments in students' and tutors' interactions and constructing transcripts of notable excerpts. After writing a descriptive narrative of each of the relevant segments, the first author used these descriptions to identify emerging themes to characterize students' strategies. The first author shared these emerging themes, as well as the relevant video data, with the second author as well as with a research assistant to refine our interpretation of students' work.

Results

We identified four different types of strategies that students applied to reason about slope.

Strategy 1: Flexible Counting "Rise Over Run"

For graphing lines that had been written in the form $y=mx+b$, students had learned a process of first graphing the y-intercept, followed by one or two more points on the line. One student in particular, Ben, showed a great deal of flexibility in how he applied the "rise over run" technique to identify points on the line. For example, one task on the worksheet required him to plot a line with a slope of 1/2; after plotting the y-intercept of the line Ben audibly counted, "one, one-two; one, one-two" while moving his pencil sequentially up and to the right. With this counting Ben first accounted for the vertical change in the graph and then accounted for horizontal change. The next task required him to graph a line with a slope of two. In this case Ben used the same audible counting pattern, "one, one-two; one, one-two," but this time he accounted for the horizontal change between points first, and then the vertical change. Moreover, although his audible counting was identical to the previous task, Ben moved to the right and down—and, then, to the left and up—on this task to account for the negative slope. Throughout the worksheet, Ben switched back and forth between whether he counted vertical change or horizontal change first, and he did so with few errors.

Ben's flexibility in how he counted "rise over run" is significant because it challenges the assumption that students should apply the same procedure in the same way for every task in order to avoid errors. Our tutors recalled mnemonics that teachers had used (e.g., "place the ladder before you climb it") to reinforce a specific order of graphing procedures. Cases like Ben's illustrate students' flexibility in applying procedures to represent constant covariation.

Strategy 2: Extending Slope Patterns

Another area of strength that students showed in applying linear reasoning was to establish a pattern for the distance between the y-intercept and another point on the graph, and then to extend that pattern to locate other points on the line. For example, Katy needed to graph the line $y=(-1/2)x+3$. After plotting a point at (0, 3), she moved up and to the right to plot a second point at (1,5). She then seemed to interpret the visual relationship between the two points she had plotted, and she transposed that relationship to plot a third point at (-1,-2) to create a line with constant slope. Katy solution here began with an error stemming from apparent confusion about the fraction of -1/2 should be used to determine the slope of the line. However, once she defined a line by plotting two points, Katy applied knowledge of constant rate of change to condense the procedure for locating points on the graph. Although students had spent substantial time developing procedures for counting over and up (or down) between points, examples like this illustrated how students applied broader conceptual knowledge of the shapes of linear graphs.

Strategy 3: Distinguishing Between Positive and Negative Slope

When students worked on the "comparing slopes" tasks (without numerical information), all three of the students from whom we collected data correctly identified which of the six lines had positive slope and which had negative slope. But students often made errors in the directions they needed to move horizontally and vertically to construct a graph with negative slope, even when they had reminders in place. One example of this came from the work of Mia. When working on the "systems of equations" tasks, she consistently wrote each case of a negative slope in two ways on her paper. For example, if given the line $y=(-2/3)x+b$, Mia made a note of $m=-2/3=2/-3$, to remind herself that she could move two places down and three to the right or two places up and three to the left. However, even with this notation and the explanation of why she used it, Mia made several errors in the direction she needed to move along the graph.

Around halfway into the lesson, Mia's tutor had a brief conversation about how lines with positive slope would increase from left to right, and lines with negative slope would decrease from left to right. This concept was familiar to the student, and it seemed to spark her memory that she could use knowledge of the general direction of a graph to construct lines, rather than relying on instructions for what direction to move and when. Several minutes later in the session Mia needed to graph the line $y=(-1/7)x+7$, and her tutor briefly noted the negative slope. Mia reacted with some excitement, noting, "so it's gonna go this way," indicating with her pencil that the graph would move down and to the right, and she made no errors when graphing the line.

Strategy 4: Attending to Relative Horizontal and Vertical Change

When students worked on the "comparing slopes" activity, they attended to the relationship between the horizontal and vertical change without any numbers or scale provided (except to note that the axes were scaled in a 1:1 ratio). When Ben encountered the question of whether the graph include in Figure 1 had slope greater than, less than, or equal to, one, he observed the triangle included with the graph and noted that the horizontal leg of the triangle looked twice as long as the vertical leg. In the next graph he noted that the horizontal and vertical legs of the triangle were equal in length. Ben showed a sophisticated level of reasoning here in attending to the multiplicative relationship between the horizontal and vertical change of the line. This multiplicative reasoning is essential towards conceiving of slope as a composite unit.

Discussion and Conclusion

Prior research has recorded how challenging it is for all students to move beyond procedures for calculating slope towards understanding of slope as a single quantity representing constant covariation (Ellis, 2007; DeJarnette et al., 2019; Lobato et al., 2003; Zahner, 2015). We observed several instances of students with LD applying sophisticated aspects of covariational reasoning—including knowledge of constant covariation and the multiplicative relationship between horizontal and vertical change—in addition to flexible application of procedures. It is illuminating that procedural fluency is not a pre-requisite for the development and application of conceptual understanding, even for students with LD who are struggling in Algebra. Students' knowledge of constant rate of change—and how they interpreted this concept visually—was a resource that they could draw upon to avoid or correct procedural errors. From this preliminary study it is clear that students might sometimes compartmentalize their understanding of concepts away from their use of procedures. But when carefully examining the strategies students draw upon, it becomes clear that students do a better job of bridging concepts with procedures than is immediately apparent. Students may benefit from instruction that makes these connections—and, especially, the strengths of students' reasoning—more explicit.

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