Large Cardinals Beyond Choice

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Set theory is presently at a critical crossroads, one in which we are faced with two alternative, radically different possible pictures of V. The story begins with the inner models L and HOD.

In many respects the inner models L and HOD are at opposite ends of the inner model spectrum. L is the most slender of inner models, while, in some sense, HOD is the broadest; L is given by iterating a local notion of definability, while HOD is given by a notion of definability that is coupled into V; L cannot accomodate modest large cardinals, while HOD can accomodate all traditional large cardinals; and, finally, there are simple sets, like $0^\#$, which are not set-generic over L, while *every* set is set-generic over HOD.

These last two characteristics of L turn on the existence of $0^\#$. They are part of the picture revealed by $0^\#$. For example, by a result of Silver, if $0^\#$ exists, then every uncountable cardinal is an inaccessible cardinal in L. On the other hand, if $0^\#$ doesn't exist, then the picture is quite different. For Jensen showed that if $0^\#$ doesn't exist, then L "covers" V and, in particular, singular cardinals are singular in L, and L correctly computes successors of singular cardinals. Combining these two results we have the following theorem—the L Dichotomy Theorem—which presents us with two alternative, radically different possibilities, with $0^\#$ being the "switch" that determines which alternative holds:[1]

[1]See Silver (1966) and Devlin & Jensen (1975).

Theorem 1 (Silver, Jensen). *Exactly one of the following hold*:

(1) *For every singular cardinal γ, γ is singular in L and $(\gamma^+)^L = \gamma^+$.*

(2) *Every uncountable cardinal is an inaccessible cardinal in L.*

The first alternative is one in which L is "close" to V, in that it correctly computes much of the cardinal structure of V. The second alternative is one in which L is "far" from V, in that it radically fails to capture the cardinal structure of V, thinking, for example, that ω_1 is an inaccessible cardinal.

The third author proved a similar dichotomy theorem for HOD—the HOD *Dichotomy Theorem*—a weak version of which is the following.[2]

Theorem 2 (Woodin). *Suppose that κ is an extendible cardinal. Then exactly one of the following hold*:

(1) *For every singular cardinal $\gamma > \kappa$, γ is singular in* HOD *and* $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.

(2) *Every regular cardinal $\gamma \geqslant \kappa$ is a measurable cardinal in* HOD.

In the first alternative HOD is "close" to V, and in the second alternative HOD is "far" from V.

There is an important foundational difference between the two dichotomies. In the case of the L Dichotomy, granting modest large cardinals, we know which side of the dichotomy we are on; in particular, if $0^\#$ exists then we are on the "far" side of the dichotomy. But in the case of the HOD Dichotomy, *no* traditional large cardinal axiom can force us into the "far" side of the dichotomy (since every traditional large cardinal axiom is compatible with $V = \mathrm{HOD}$). So perhaps we are on the "close" side of the HOD Dichotomy and perhaps this is even provable assuming traditional large cardinals. Or perhaps there are new large cardinals, including a higher analogue of $0^\#$, one which does for HOD what $0^\#$ does for L, forcing us into "far" side of the HOD Dichotomy.

[2]See Woodin (2010), §7.1 and Woodin (2017), §3.

There is a program aimed at establishing the first alternative—the "close" side of the HOD Dichotomy. This is the program of *inner model theory*. Recent work of the third author has shown that if inner model theory reaches the level of one supercompact cardinal then it "goes all the way." There is a candidate for this "ultimate inner model" (Ultimate-L) and one can formulate, in relatively simple terms, the axiom that characterizes this model. Moreover, there is a natural conjecture (the Ultimate-L Conjecture) concerning the existence of Ultimate-L. The point, for our present purposes, is that if the Ultimate-L Conjecture holds, then (assuming that there is an extendible cardinal with a huge cardinal above it) the first side of the HOD Dichotomy *must* hold. In this future, HOD is "close" to V and there is no higher analogue of $0^{\#}$.

In this paper we shall introduce a very different program, one aimed at the second alternative—the "far" side of the HOD Dichotomy. This is the program of *large cardinals beyond choice*.

In order to motivate this program it will be useful to say a few words about the traditional large cardinal hierarchy.

Recall that a natural template for formulating large cardinal axioms is to assert that there is a non-trivial elementary embedding $j : V \to M$, where M is a transitive class. The critical point, $\text{crit}(j)$, of the embedding is the first ordinal moved by the embedding, and it is generally the large cardinal associated with the embedding. It follows immediately that for any such embedding, if κ is the critical point, then M resembles V to the extent that $(V_{\kappa+1})^M = V_{\kappa+1}$. It is this degree of resemblance which is responsible for the strong reflection properties that hold at κ. For example, it readily implies that there are many inaccessible cardinals below κ, many Mahlo cardinals below κ, and so on. To obtain embeddings with greater strength one demands that M resemble V to a higher degree. For example, if one demands that $(V_{\kappa+2})^M = V_{\kappa+2}$ then it follows that there are many measurable cardinals below κ. In the limit, it is natural to consider, as Reinhardt did in his dissertation,[3] the "ultimate axiom," where one demands full resemblance, by positing a non-trivial elementary embedding $j : V \to V$. Let us call the critical point of such an embedding a *Reinhardt cardinal.*

Kunen famously showed that if Reinhardt cardinals exist then AC fails. Since ZFC is generally the accepted background theory, this was taken to show that Reinhardt cardinals simply do not exist.[4] But it has remained a longstanding open question whether Reinhardt cardinals are inconsistent in the context of ZF alone. In this paper we will investigate the hierarchy of such "choiceless" large cardinal axioms, a hierarchy that starts with a Reinhardt cardinal, passes upward through strong forms of Reinhardt cardinals, and passes further still through strong forms of Berkeley cardinals.

The above large cardinals are, of course, inconsistent with AC. However, each of the choiceless large cardinals has a "HOD-analogue" that is formulated in the context of ZFC. The relevance of all of this to the HOD Dichotomy is the following: If the choiceless large cardinals are consistent, then the Ultimate-L Conjecture must fail, and so we will have lost our main reason for believing that the "close" side of the HOD Dichotomy must hold. Moreover, if the HOD-analogues of the choiceless large cardinals exist, then there is indeed a higher analogue of $0^\#$ and the "far" side of the HOD Dichotomy must hold.

We are thus faced with two radically different, mutually incompatible ways in which future might unfold, granting traditional large cardinals. In the first future the Ultimate-L Conjecture holds, Ultimate-L exists, inner model theory succeeds, HOD is "close" to V, and large cardinals beyond choice are inconsistent.[5] In the second future, large cardinals beyond choice are consistent, their HOD-analogues exist, HOD is "far" from V, Ultimate-L does not exist, and inner model theory as we know it fails at the level of

[3] Reinhardt (1967).

[4] See Kunen (1971) for the original proof, and see Kanamori (2003) for two alternative proofs.

[5] More precisely, *most* of the large cardinals beyond choice are inconsistent.

supercompact cardinals.

Here is an overview of the paper: In §1 we discuss some metamathematical preliminaries; in § 2 we introduce the Reinhardt hierarchy and establish some results concerning it; in §3 we introduce the Berkeley hierarchy and establish some results concerning it and its relationship to the Reinhardt hierarchy; in §4 we summarize what is known about the hierarchy of large cardinals beyond choice; in §5 we show that the degree of the failure of AC is connected to the question of the cofinality of the least Berkeley cardinal; in §6 we show that the question of the cofinality of the least Berkeley cardinal is independent of ZF and our large cardinal assumptions; in §7 we review the recent advances in inner model theory; finally, in §8 we return to the above discussion of the HOD Dichotomy and we describe in more detail the two possible futures that lie before us.[6]

1 Metamathematical Preliminaries

Throughout this paper we work in ZF, unless otherwise noted. In this section we shall discuss some metamathematical subtleties that arise in the choiceless setting and we will introduce some of the basic notions that figure in what follows.

1.1 Traditional Large Cardinals

In the choiceless setting some care is needed when formulating traditional large cardinal notions. To begin with, some of the traditional formulations do not even make sense without AC. Moreover, when one reformulates a traditional large cardinal notion in the choiceless setting, there are often several options, and formulations that are equivalent under AC can come

[6] Acknowledgements. We are grateful to Raffaella Cutolo, Gabriel Goldberg, and an anonymous referee for helpful comments.

apart, and even split into notions of different strength. For this reason it will be useful at the outset to discuss some of the subtleties and state our official definitions.

The need for reformulation is apparent right at the start, with strongly inaccessible cardinals. In the context of AC, a cardinal κ is strongly inaccessible iff it is regular and for all $\alpha < \kappa$, $|V_\alpha| < \kappa$. But this does not make sense without AC since without AC there is no guarantee that V_α admits a well-ordering. However, in the context of AC, a cardinal κ is strongly inaccessible (as we have just defined it) iff for all $\alpha < \kappa$ there does not exist a function $f : V_\alpha \to \kappa$ with range unbounded in κ. And this formulation does make sense without AC. We shall we take this as our official definition.

Definition 1.1. A cardinal κ is *strongly inaccessible* if for all $\alpha < \kappa$ there does not exist a function $f : V_\alpha \to \kappa$ with range unbounded in κ.

Notice that in our present choiceless setting, if κ is strongly inaccessible, we still have that $\langle V_\kappa, V_{\kappa+1}\rangle \models \mathrm{ZF}_2$, where here ZF_2 is the second-order version of ZF (with second-order Separation, Collection, and Replacement) and, in writing '$\langle V_\kappa, V_{\kappa+1}\rangle \models \mathrm{ZF}_2$' we are interpreting the second-order variables as ranging over the full powerset of V_κ, that is, $V_{\kappa+1}$.

New subtleties arise with some of the stronger large cardinal notions that we shall employ, most notably with supercompact and extendible cardinals. For these we need the following technical notion.[7]

Definition 1.2. Suppose $V_\gamma \prec_{\Sigma_1} V$. Then $V_\gamma \prec_{\Sigma_1^*} V$ if for all $\alpha < \gamma$, for all $a \in V_\gamma$, and for all Σ_0-formulas $\varphi(x, y)$, if there exists $b \in V$ such that

$$^{V_\alpha}b \subseteq b \text{ and } V \models \varphi[a, b],$$

then there exists $b \in V_\gamma$ such that

$$^{V_\alpha}b \subseteq b \text{ and } V \models \varphi[a, b].$$

Remark 1.3. Under AC, $V_\gamma \prec_{\Sigma_1^*} V$ if and only if $V_\gamma \prec_{\Sigma_1} V$.

[7]For the motivation behind this notion see Woodin (2010).

The definition of a supercompact cardinal that we will use in the choiceless setting will be modeled on Magidor's formulation of supercompact cardinals in the AC setting. In the choiceless setting, the large cardinals given by Magidor's formulation will be called 'weakly supercompact' cardinals:

Definition 1.4. A cardinal κ is *weakly supercompact* if for all $\gamma > \kappa$ and for all $a \in V_\gamma$, there exists $\bar{\gamma} < \kappa$ and $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with $\mathrm{crit}(j) = \bar{\kappa}$ such that $\bar{\kappa} < \kappa$, $j(\bar{\kappa}) = \kappa$ and $j(\bar{a}) = a$.

In the context of AC this notion is equivalent to the following notion, which is the one we will need in what follows:

Definition 1.5. A cardinal κ is *supercompact* if for all $\gamma > \kappa$ such that $V_\gamma \prec_{\Sigma_1^*} V$, for all $a \in V_\gamma$, there exists $\bar{\gamma} < \kappa$ and $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

with $\mathrm{crit}(j) = \bar{\kappa}$ such that $\bar{\kappa} < \kappa$, $j(\bar{\kappa}) = \kappa$, $j(\bar{a}) = a$, and $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.

Similarly, the definition of an extendible cardinal in the choiceless setting will be modeled on the standard formulation. In the choiceless setting, the large cardinals given by the standard formulation will be called 'weakly extendible' cardinals:

Definition 1.6. A cardinal κ is *weakly extendible* if for all α there exists and α' and an elementary embedding

$$j : V_{\kappa+\alpha} \to V_{j(\kappa)+\alpha'}$$

such that $\mathrm{crit}(j) = \kappa$ and $\alpha < j(\kappa)$.

In the context of AC this notion is equivalent to the following notion, which is the one we shall need in what follows:

Definition 1.7. Let A be the class of γ such that $V_\gamma \prec_{\Sigma_1^*} V$. Then κ is *extendible* if κ is A-extendible, that is, for all α there exists an α' and an elementary embedding

$$j : V_{\kappa+\alpha} \to V_{j(\kappa)+\alpha'}$$

such that $\mathrm{crit}(j) = \kappa$, $\alpha < j(\kappa)$, and for all $\beta < \alpha$, $j(A \cap V_{\kappa+\beta}) = A \cap V_{j(\kappa)+j(\beta)}$.

1.2 Large Cardinals Beyond Choice

The above large cardinal notions are simply the traditional large cardinal notions reconfigured for the choiceless setting. Ultimately, however, our real interest is in large cardinal notions that actually imply the failure of AC.

We shall consider two stretches of the hierarchy of large cardinals beyond choice. The first stretch is the *Reinhardt hierarchy*, consisting of Reinhardt cardinals, super Reinhardt cardinals, and totally Reinhardt cardinals. The second stretch is the *Berkeley hierarchy*, consisting of Berkeley cardinals, club Berkeley cardinals, and limit club Berkeley cardinals. We will give the definitions in subsequent sections but here it will be useful to say something about our background theory.

The large cardinals in the Berkeley hierarchy all admit of a first-order definition and so their formulation takes place in ZF. This is also true of totally Reinhardt cardinals. However, the notion of a Reinhardt cardinal and the notion of a super Reinhardt cardinal do not admit a first-order definition in the language of ZF since they involve proper class embeddings in an essential way. There are standard ways of dealing with this. One approach is to work in the conservative extension NBG (without choice) of ZF. Here one has recourse to proper classes and although one does not have recourse to the full definition of truth (which would be required to assert the statement that j is elementary) one can capture the scheme expressing that "j is elementary" in terms of the statement asserting that "j is a cofinal Δ_0-elementary embedding."

In what follows we shall work in ZF, except for the cases where we employ notions like that of a Reinhardt cardinal or that of a super Reinhardt cardinal, in which case the reader should understand us to be working in the conservative extension NBG (without choice).

1.3 Three Grades of Reflection

We are interested in the relative strength of these notions. In the first instance one is interested in consistency strength. But in general when one shows that one large cardinal axiom is stronger than another, one gets much more than greater consistency strength; one actually gets that the one large cardinal axiom implies that there are rank initial segments of the universe that satisfy the other large cardinal axiom.

The following three grades of "reflection" will figure in what follows:

Definition 1.8. Suppose that Φ_1 and Φ_2 are large cardinal notions.

- Φ_1 *reflects* Φ_2 if for all κ such that $\Phi_1(\kappa)$ there exists $\bar{\kappa} < \kappa$ such that $\Phi_2(\bar{\kappa})$.
- Φ_1 *rank-reflects* Φ_2 if for all κ such that $\Phi_1(\kappa)$ there are $\bar{\kappa} < \gamma \leqslant \kappa$ such that $\langle V_\gamma, V_{\gamma+1}\rangle \models \mathrm{ZF}_2 + \Phi_2(\bar{\kappa})$.
- Φ_1 *strongly rank-reflects* Φ_2 if for all κ such that $\Phi_1(\kappa)$ there are $\bar{\kappa} < \gamma < \kappa$ such that $\langle V_\gamma, V_{\gamma+1}\rangle \models \mathrm{ZF}_2 + \Phi_2(\bar{\kappa})$.

2 The Reinhardt Hierarchy

The first large cardinal beyond choice that we shall discuss is the one that Reinhardt introduced.

Definition 2.1. A cardinal κ is *Reinhardt* if there exists a non-trivial elementary embedding $j : V \to V$ such that $\mathrm{crit}(j) = \kappa$.

This large cardinal notion can be strengthened in a natural way, by employing the template used to define strong cardinals.

Definition 2.2. A cardinal κ is *super Reinhardt* if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

This large cardinal notion can in turn be strengthened by employing the template used to define Woodin cardinals.

Definition 2.3. Let A be a proper class. A cardinal κ is *A-super Reinhardt* if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j(A) = A$, where $j(A) = \bigcup_{\alpha \in \mathrm{On}} j(A \cap V_\alpha)$. A cardinal κ is *totally Reinhardt* if for each $A \in V_{\kappa+1}$,

$$\langle V_\kappa, V_{\kappa+1} \rangle \models \mathrm{ZF}_2 + \text{"There is an } A\text{-super Reinhardt cardinal."}$$

It is immediate that totally Reinhardt cardinals rank-reflect super Reinhardt cardinals. It turns out that super Reinhardt cardinals strongly rank-reflect Reinhardt cardinals.

Theorem 2.1. *Suppose that κ is a super Reinhardt cardinal. Then there exists $\gamma < \kappa$ such that*

$$\langle V_\gamma, V_{\gamma+1} \rangle \models \mathrm{ZF}_2 + \text{"There is a Reinhardt cardinal."}$$

Proof. Let $j : V \to V$ be a non-trivial elementary embedding with $\mathrm{crit}(j) = \kappa$. Let $\kappa_0 = \kappa$ and, for $n < \omega$, let $\kappa_{n+1} = j(\kappa_n)$. Let

$$\lambda = \sup\{\kappa_n : n < \omega\}.$$

Notice that $j(\lambda) = \lambda$.

Since κ is super Reinhardt, there exists a non-trivial elementary embedding $j' : V \to V$ with $\mathrm{crit}(j') = \kappa$ and $j'(\kappa) > \lambda$. Notice that $j'(\kappa)$ is a limit of inaccessible cardinals (since, by reflection via j', κ is a limit of inaccessible cardinals). Let γ_0 be the least inaccessible above λ. Since γ_0 is definable from λ and since $j(\lambda) = \lambda$, we have that $j(\gamma_0) = \gamma_0$. So

$$\langle V_{\gamma_0}, V_{\gamma_0+1} \rangle \models \mathrm{ZF}_2 + \text{"}\kappa \text{ is a Reinhardt cardinal,"}$$

as witnessed by $j \restriction V_{\gamma_0}$. Finally, since $\gamma_0 < j'(\kappa)$, by applying $(j')^{-1}$, there exists $\gamma < \kappa$ such that

$$\langle V_\gamma, V_{\gamma+1} \rangle \models \mathrm{ZF}_2 + \text{“there is a Reinhardt cardinal,”}$$

as desired. □

Thus, we have a proper hierarchy: Totally Reinhardt cardinals rank reflect super Reinhart cardinals, and super Reinhardt cardinals strongly rank-reflect Reinhardt cardinals.

Moreover, it should be noted that (in terms of consistency strength) this hierarchy starts above the traditional large cardinal hierarchy:

Theorem 2.2 (Goldberg)**.** *Assume* DC *and that κ is a Reinhardt cardinal. Then there is a forcing extension $V[G]$ such that $V[G]_\kappa \models \mathrm{ZFC} + I_0$.*[8]

Question 1. Do totally Reinhardt cardinals strongly rank-reflect super Reinhardt cardinals?

Question 2. Do super Reinhardt cardinals reflect Reinhardt cardinals?

3 The Berkeley Hierarchy

Let us now proceed onward and upward.

3.1 Proto-Berkeley Cardinals

Definition 3.1. For a transitive set M, let $\mathscr{E}(M)$ be the set of all non-trivial elementary embeddings $j : M \to M$.

Definition 3.2. An ordinal δ is a *proto-Berkeley cardinal* if for all transitive sets M such that $\delta \in M$ there exists $j \in \mathscr{E}(M)$ with $\mathrm{crit}(j) < \delta$.

[8]See Goldberg (2017).

The idea is that a proto-Berkeley cardinal δ is so large that it "shatters" any transitive set that contains it; more precisely, it forces any such set to be non-rigid, as witnessed by an elementary embedding with critical point less than δ. For example, if δ is a proto-Berkeley cardinal then, as a special case, for any $\lambda > \delta$ there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) < \delta$. But we have much more. For example, now we can fold j itself into a new transitive set to obtain $j' : \langle V_\lambda, j\rangle \to \langle V_\lambda, j\rangle$ with $\text{crit}(j') < \delta$, and so on.

Notice that if δ_0 is the least proto-Berkeley cardinal then every ordinal greater than δ_0 is also trivially a proto-Berkeley cardinal. These other proto-Berkeley cardinals merely inherit their proto-Berkeleyness from δ_0, and—for example, in the case of $\delta_0 + 1$—need not be significantly larger than δ_0. To isolate a non-degenerate notion it is instructive to examine the first non-degenerate case, δ_0, and isolate a key feature that distinguishes it from the degenerate cases.

We will need the following trivial lemma.

Lemma 3.1. *For any set a there exists a transitive set M such that $a \in M$ and a is definable (without parameters) in M.*

Proof. Let λ be such that $a \in V_\lambda$, and let

$$M = V_\lambda \cup \big\{\{\langle a, x\rangle : x \in V_\lambda\}\big\}.$$

It is straightforward to see that M is transitive. Moreover, a is definable (without parameters) in M as the first element in each of the pairs belonging to the set of highest rank. $\square$

This lemma enables us to give the following reformulation of the notion of a proto-Berkeley cardinal, something that will be useful later.

Lemma 3.2. *The following are equivalent:*

(1) *δ is a proto-Berkeley cardinal.*

(2) *For all sets a, for all transitive M such that $a, \delta \in M$, there exists $j \in \mathscr{E}(M)$ such that $j(a) = a$ and $\text{crit}(j) < \delta$.*

Proof. (2) → (1). Immediate: Take $a = \varnothing$.

(1) → (2). Let a be a set and let M be a transitive set such that $a, \delta \in M$. By Lemma 3.1 let M' be such that a and M are definable in M'. Since δ is a proto-Berkeley cardinal, there exists $j' : M' \to M'$ with $\mathrm{crit}(j') < \delta$. Since a and M are definable in M', $j'(a) = a$ and $j'(M) = M$. So $j' \restriction M \in \mathscr{E}(M)$ is our desired embedding. □

But our real interest in Lemma 3.1 is that it enables one to prove the following:

Theorem 3.3. *Let δ_0 be the least proto-Berkeley cardinal. Then for all transitive sets M such that $\delta_0 \in M$, and for all $\eta < \delta_0$, there exists $j \in \mathscr{E}(M)$ such that*

$$\eta < \mathrm{crit}(j) < \delta_0.$$

Proof. Suppose for contradiction that the statement of the theorem fails. So there is a transitive set M such that $\delta_0 \in M$ and there are ordinals $\eta < \delta_0$ such that there does *not* exist $j \in \mathscr{E}(M)$ with $\eta < \mathrm{crit}(j) < \delta_0$. Ranging over all such M let $\eta_0 < \delta_0$ be the least ordinal such that there exists a transitive M such that $\delta_0 \in M$ and there does not exist $j \in \mathscr{E}(M)$ with

$$\eta_0 < \mathrm{crit}(j) < \delta_0.$$

Let M_0 be one such M. We claim that η_0 is a proto-Berkeley cardinal, which implies that $\eta_0 = \delta_0$, a contradiction.

Let M be any transitive set such that $\eta_0 \in M$. Let M' be such that

$$\langle M_0, M, \eta_0 \rangle \text{ is definable in } M'.$$

Since $\delta_0 \in M'$ there exists $j' \in \mathscr{E}(M')$ with $\mathrm{crit}(j') < \delta_0$. Since M_0, M, and η_0 are definable in M', j' fixes M_0, M, and η_0. Since j' fixes M_0, it follows that $j' \restriction M_0 \in \mathscr{E}(M_0)$. Moreover, $\mathrm{crit}(j' \restriction M_0) \leqslant \eta_0$ (by the definition of η_0). But $j'(\eta_0) = \eta_0$. So,

$$\mathrm{crit}(j' \restriction M_0) < \eta_0.$$

Similarly, since j' fixes M, it follows that $j' \restriction M \in \mathscr{E}(M)$. And we have just shown that $\mathrm{crit}(j' \restriction M) < \eta_0$. But M was *any* transitive set such that $\eta_0 \in M$. So we have shown that η_0 is a proto-Berkeley cardinal, which is a contradiction. □

This theorem is really just the first instance of something more general.

Definition 3.3. Suppose $\alpha \in \mathrm{On}$. An ordinal δ is an *α-proto-Berkeley cardinal* if for all transitive sets M such that $\delta \in M$ there exists $j \in \mathscr{E}(M)$ with $\alpha < \mathrm{crit}(j) < \delta$. Let δ_α be the least α-proto-Berkeley cardinal.

Theorem 3.4. *Suppose $\alpha \in \mathrm{On}$. Then for all transitive sets M such that $\delta_\alpha \in M$ and for all $\eta < \delta_\alpha$ there exists an elementary embedding $j : M \to M$ such that*

$$\eta < \mathrm{crit}(j) < \delta_\alpha.$$

Proof. The proof is almost exactly the same as the proof of Theorem 3.3. □

To summarize: If δ_0 is the least proto-Berkeley cardinal then every ordinal $\delta > \delta_0$ is a proto-Berkeley cardinal. But the key feature distinguishing δ_0 from most of these other degenerate proto-Berkeley cardinals is that in the case of δ_0 the critical points of the witnessing embeddings are cofinal in δ_0. The next proto-Berkeley cardinal sharing this feature is the least δ_0-proto-Berkeley cardinal, and so on. This motivates the definition of a Berkeley cardinal.

3.2 Berkeley Cardinals

Definition 3.4. A cardinal δ is a *Berkeley cardinal* if for every transitive set M such that $\delta \in M$, and for every ordinal $\eta < \delta$, there exists $j \in \mathscr{E}(M)$ with $\eta < \mathrm{crit}(j) < \delta$.

Remark 3.5. Notice that:

(1) For all $\alpha \in \mathrm{On}$, the least α-proto-Berkeley cardinal, δ_α, is a Berkeley cardinal.

(2) If δ is a limit of Berkeley cardinals, then δ is a Berkeley cardinal. In other words, the class of Berkeley cardinals is closed. (Note that the limit Berkeley cardinals are not among the δ_α.)

(3) If δ is a Berkeley cardinal, then for all limit ordinals $\lambda > \delta$, V_λ thinks that δ is a Berkeley cardinal.

(4) The property "δ is a Berkeley cardinal" is a Π_2 property. Hence, if λ is a limit ordinal such that $V_\lambda \prec_{\Sigma_2} V$, then V_λ is correct in its identification of the Berkeley cardinals below λ.

Let us now turn to the question of how, in terms of strength, Berkeley cardinals are related to the large cardinals in the Reinhardt hierarchy. We begin with a simple observation involving extendible cardinals.

Theorem 3.5. *Suppose δ_0 is the least Berkeley cardinal. Then there are no extendible cardinals $\leqslant \delta_0$.*

Proof. Assume for a contradiction that $\delta \leqslant \delta_0$ is extendible. It is a standard result that $V_\delta \prec_{\Sigma_3} V$ and this result carries over to the choiceless setting. Since V satisfies that there is a Berkeley cardinal, namely δ_0, and since the notion of being a Berkeley cardinal is Π_2, it follows by Σ_3-elementarity that V_δ also satisfies that there is a Berkeley cardinal. But V_δ is correct in its computation of Berkeley cardinals. Thus, there is a Berkeley cardinal below δ, which is a contradiction. □

So Berkeley cardinals do not reflect extendible cardinals. It follows that they do not reflect super Reinhardt cardinals. Nevertheless, as we will now show, they strongly rank-reflect both extendible cardinals and Reinhardt cardinals. We will need the following lemma.

Lemma 3.6. *Let δ_0 be the least Berkeley cardinal. Then, for a tail of limit ordinals λ, if $j \in \mathscr{E}(V_\lambda)$ is an elementary embedding with $\mathrm{crit}(j) < \delta_0$, then*

(1) *$j(\delta_0) = \delta_0$, and*

(2) $\{\alpha < \delta_0 : j(\alpha) = \alpha\}$ *is cofinal in* δ_0.

Proof. (1). The key point is that for a tail of limit ordinals λ, V_λ recognizes that δ_0 is the least Berkeley cardinal: To see this, note first that the least Berkeley cardinal is also characterized as the least proto-Berkeley cardinal. Now, for each $\delta < \delta_0$, since δ is not a proto-Berkeley cardinal there exists M_δ such that $\delta \in M_\delta$ and there does *not* exist $j \in \mathscr{E}(M_\delta)$ with $\mathrm{crit}(j) < \delta$. Let β_δ be least such that V_{β_δ} contains such a counter-example, M_δ. Let λ be a limit ordinal such that $\lambda > \delta_0$ and $\lambda > \beta_\delta$, for all $\delta < \delta_0$. Since λ is a limit ordinal greater than δ_0, V_λ thinks that δ_0 is a Berkeley cardinal (by Remark 3.5(3)). Moreover, V_λ thinks that any $\delta < \delta_0$ is not a proto-Berkeley cardinal since it has all of the counter-examples M_δ. In other words, for any such λ, V_λ recognizes that δ_0 is the *least* proto-Berkeley cardinal, and hence that it is the least Berkeley cardinal.

Finally, if λ is in the above tail, then for any $j \in \mathscr{E}(V_\lambda)$ with $\mathrm{crit}(j) < \delta_0$ we must have $j(\delta_0) = \delta_0$, as δ_0 is definable in V_λ as the least Berkeley cardinal.

(2). Let λ be in the above tail and suppose that $j \in \mathscr{E}(V_\lambda)$ is such that $\mathrm{crit}(j) < \delta_0$. Assume, for a contradiction, that (2) fails. Let

$$\eta_0 = \sup\{\alpha < \delta_0 : j(\alpha) = \alpha\}$$

and, for $i < \omega$, let $\eta_{i+1} = j(\eta_i)$. It follows that

$$\delta_0 = \sup\{\eta_i : i < \omega\}.$$

Let M_0 be a witness in V_λ to the fact that η_0 is not a proto-Berkeley cardinal; that is, let M_0 be a transitive set such that $\eta_0 \in M_0$ and there is no $j \in \mathscr{E}(M_0)$ with $\mathrm{crit}(j) < \eta_0$. For $i < \omega$, let $M_{i+1} = j(M_i)$. Notice that by elementarity M_{i+1} is a witness that η_{i+1} is not a proto-Berkeley cardinal.

Now, for a tail of limit ordinals λ,

$$\langle M_i : i < \omega \rangle \in V_\lambda.$$

It follows (from Lemma 3.2) that for some such λ there exists $j' \in \mathscr{E}(V_\lambda)$ such that

$$j'(\langle M_i : i < \omega \rangle) = \langle M_i : i < \omega \rangle$$

and $\operatorname{crit}(j') < \delta_0$. Let i be such that $\operatorname{crit}(j') < \eta_i$. We have $j'(M_i) = M_i$ and so $j' \upharpoonright M_i \in \mathscr{E}(M_i)$ and is such that $\operatorname{crit}(j' \upharpoonright M_i) < \eta_i$. But this contradicts the fact that M_i is a witness that η_i is not a proto-Berkeley cardinal. $\square$

Remark 3.6. The same proof shows that the result holds for the second Berkeley cardinal, the third Berkeley cardinal, and so on. More generally, if α is such that for a tail of limit ordinals λ, α is definable in V_λ, then the result holds for δ_α, where that δ_α is the least α-proto Berkeley cardinal.

We shall now show that Berkeley cardinals strongly rank-reflect Reinhardt cardinals and, moreover, that they strongly rank-reflect Reinhardt cardinals in conjunction with the large cardinals in the traditional large cardinal hierarchy. In order to be specific we shall take as our traditional large cardinal notion that of an ω-huge cardinal, but the proof readily generalizes to other traditional large cardinal notions.

Definition 3.7. A cardinal κ is *ω-huge* if there exists a $\lambda > \kappa$ and a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$ such that $\kappa = \operatorname{crit}(j)$ and $\lambda = \kappa_\omega(j)$, where

$$\kappa_\omega(j) = \sup_{n<\omega} \kappa_n,$$

where $\kappa_0 = \operatorname{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$, for all $n < \omega$.

Theorem 3.7. *Suppose that δ_0 is the least Berkeley cardinal. Then there exists $\gamma < \delta_0$ such that*

$$\langle V_\gamma, V_{\gamma+1}\rangle \models \mathrm{ZF}_2 + \text{"}\textit{there exists a Reinhardt cardinal, as witnessed by } j, \textit{ and there is an } \omega\textit{-huge cardinal above } \kappa_\omega(j)\text{."}$$

Proof. By Lemma 3.6, for a tail of limit ordinals β, if $j \in \mathscr{E}(V_\beta)$ is such that $\operatorname{crit}(j) < \delta_0$, then $j(\delta_0) = \delta_0$ and $\{\alpha < \delta_0 : j(\alpha) = \alpha\}$ is cofinal in δ_0. Fix such a β in this tail and, using the fact that δ_0 is a Berkeley cardinal, let $j \in \mathscr{E}(V_\beta)$ be such that $\operatorname{crit}(j) < \delta_0$. Let $\kappa = \operatorname{crit}(j)$ and $\lambda = \kappa_\omega(j)$.

The restriction of j to some suitable V_γ (for $\gamma < \delta_0$) will be our witness that $\langle V_\gamma, V_{\gamma+1}\rangle$ thinks that κ is a Reinhardt cardinal. We seek such a local environment $\langle V_\gamma, V_{\gamma+1}\rangle$ and an ω-huge cardinal in this environment which is above λ.

Since δ_0 is a Berkeley cardinal there are embeddings $j'' \in \mathscr{E}(V_\beta)$ with $\lambda < \mathrm{crit}(j'') < \delta_0$ and, since β is in the tail chosen above, for any such j'' we also have that $\lambda'' =_{\mathrm{df}} \kappa_\omega(j'') < \delta_0$. Thus there are plenty of ω-huge cardinals above λ and below δ_0.

We now isolate the appropriate local environment $\langle V_\gamma, V_{\gamma+1}\rangle$. We have shown (in V_β) that there are λ'' with the property that there exists $k \in \mathscr{E}(V_{\lambda''})$ such that $\lambda < \mathrm{crit}(k) < \delta_0$ and $\kappa_\omega(k) = \lambda'' < \delta_0$. (Just take λ'' from the previous paragraph and set $k = j''\restriction V_{\lambda''}$.) Working in V_β, let λ' be the least such λ'', let $j' \in \mathscr{E}(V_\lambda)$ be one of the embeddings k associated with λ', and let γ be the least strongly inaccessible above λ'. Notice that λ' and γ are definable (in V_β) from λ and δ_0. Thus, since $j(\lambda) = \lambda$ and $j(\delta_0) = \delta_0$, we have $j(\lambda') = \lambda'$ and $j(\gamma) = \gamma$, so $j\restriction V_\gamma \in \mathscr{E}(V_\gamma)$. Thus, the embeddings $j\restriction V_\gamma \in \mathscr{E}(V_\gamma)$ and $j' \in \mathscr{E}(V_{\lambda'})$ witness that

$$\langle V_\gamma, V_{\gamma+1}\rangle \models \mathrm{ZF}_2 + \text{“there is a Reinhardt cardinal, as witnessed by } j\restriction V_\gamma, \\ \text{and there is an } \omega\text{-huge cardinal above } \kappa_\omega(j\restriction V_\gamma).\text{”}$$

□

Question 3. Do Berkeley cardinals strongly rank-reflect super Reinhardt cardinals?

3.3 Club Berkeley Cardinals

Let us now introduce a stronger version of a Berkeley cardinal, one that rank-reflects super Reinhardt cardinals.

Definition 3.8. A cardinal δ is a *club Berkeley cardinal* if δ is regular and for all clubs $C \subseteq \delta$ and for all transitive M with $\delta \in M$ there exists $j \in \mathscr{E}(M)$ with $\mathrm{crit}(j) \in C$.

Theorem 3.8. *Suppose δ is a club Berkeley cardinal. Then δ is a totally Reinhardt cardinal.*

Proof. Fix $A \subseteq V_\delta$. We have to show that

$$\langle V_\delta, V_{\delta+1}\rangle \models \mathrm{ZF}_2 + \text{“There is an } A\text{-super Reinhardt cardinal.”}$$

Recall that (by Lemma 3.1) for arbitrary $A \subseteq V_\delta$ there are transitive sets M such that $V_{\delta+1} \in M$ and A is definable in M. The main ingredient is the following claim, which concerns such M.

Claim. *For all transitive sets M such that $V_{\delta+1} \in M$ and A is definable in M, there exists $\kappa < \delta$ such that for all $\alpha < \delta$ there exists $j \in \mathscr{E}(M)$ such that*

(1) $\mathrm{crit}(j) = \kappa$,

(2) $j(\kappa) > \alpha$, *and*

(3) $j(A) = A$.

Proof. Suppose that the claim is false. Let M be a transitive set such that $V_{\delta+1} \in M$, A is definable in M and such that for all $\kappa < \delta$ there exists $\alpha < \delta$ such that there does *not* exist $j \in \mathscr{E}(M)$ with

(1) $\mathrm{crit}(j) = \kappa$,

(2) $j(\kappa) > \alpha$, and

(3) $j(A) = A$.

For each $\kappa < \delta$, let α_κ be the least such α. Let

$$C = \{\gamma < \delta : \forall \kappa < \gamma\, (\alpha_\kappa < \gamma)\}.$$

The set C is the set of “no crossover points.” Notice that since δ is regular, C is club in δ. Since δ is a club Berkeley cardinal there exists $j \in \mathscr{E}(M)$ such that

(1) $\mathrm{crit}(j) \in C$,

(2) $j(C) = C$, and

(3) $j(A) = A$.

(To arrange (2) one uses the fact that $C \in M$ (as $V_{\delta+1} \in M$) and applies the standard trick involving Lemma 3.2, where one passes to a transitive set M' in which C and M are definable.) Now, $\kappa = \mathrm{crit}(j) \in C$ and so $j(\kappa) \in j(C) = C$. But since $j(\kappa) \in C$ and $\kappa < j(\kappa)$ we have (by the definition of C) that $\alpha_\kappa < j(\kappa)$, which contradicts the definition of α_κ. □

Take any transitive set M as in the claim. It follows that

$$\langle V_\delta, V_{\delta+1}\rangle \models \mathrm{ZF}_2 + \text{“There is an } A\text{-super Reinhardt cardinal,”}$$

as witnessed by κ and the embeddings $j{\restriction}V_\delta$ in the claim. □

Question 4. Does a club Berkeley cardinal rank-reflect a Berkeley cardinal?

3.4 Limit Club Berkeley Cardinals

We know that the least Berkeley cardinal cannot be super Reinhardt. But it is of interest to ask whether some Berkeley cardinal can be super Reinhardt. We now introduce an even stronger notion of Berkeley cardinal, one which rank-reflects a Berkeley cardinal that is also super Reinhardt.

Definition 3.9. A cardinal δ is a *limit club Berkeley cardinal* if δ is a club Berkeley cardinal which is a limit of Berkeley cardinals.

Theorem 3.9. *Suppose δ is a limit club Berkeley cardinal. Then*

$$\langle V_\delta, V_{\delta+1}\rangle \models \mathrm{ZF}_2 + \text{“\textit{there is a Berkeley cardinal that is super Reinhardt.}”}$$

Proof. The proof is a variation of the proof of Theorem 3.8.

Claim. *For all transitive sets M such that $V_{\delta+1} \in M$, and for all $D \subseteq \delta$ which are club in δ there exists $\kappa \in D$ such that for all $\alpha < \delta$ there exists $j \in \mathscr{E}(M)$ such that*

(1) $\text{crit}(j) = \kappa$ *and*

(2) $j(\kappa) > \alpha$.

Proof. Suppose that the claim is false. Let M be a transitive set such that $V_{\delta+1} \in M$ and let $D \subseteq \delta$ be club in δ such that for all $\kappa \in D$ there exists $\alpha < \delta$ such that there does not exist $j \in \mathscr{E}(M)$ with

(1) $\text{crit}(j) = \kappa$ and

(2) $j(\kappa) > \alpha$.

For each $\kappa \in D$, let α_κ be the least such α. Let

$$C = \{\gamma < \delta : \forall \kappa \in D \cap \gamma \, (\alpha_\kappa < \gamma)\}$$

be the set of "no crossover points". Notice that since δ is regular, C is club in δ. Since δ is a club Berkeley cardinal there exists $j \in \mathscr{E}(M)$ such that

(1) $\text{crit}(j) \in C \cap D$,

(2) $j(C) = C$, and

(3) $j(D) = D$.

(To arrange (2) and (3) one uses the fact that $C, D \in M$ (as $V_{\delta+1} \in M$) and applies the standard trick involving Lemma 3.2, where one passes to a transitive set M' in which C, D, and M are definable.) Now, $\kappa \in C \cap D$ and so $j(\kappa) \in j(C) \cap j(D) = C \cap D$. This is a contradiction since by the definition of C, for all $\kappa \in j(\kappa) \cap D$, $\alpha_\kappa < j(\kappa)$. □

It follows that

$$\langle V_\delta, V_{\delta+1} \rangle \models \text{ZF}_2 + \text{"there are stationarily-many super Reinhardt cardinals."}$$

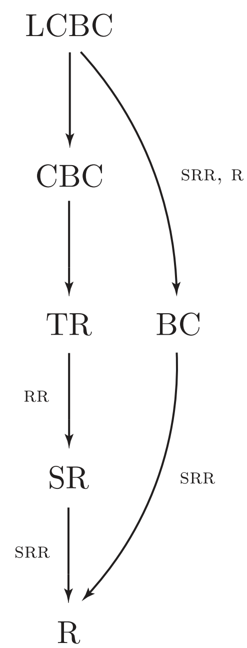
Since the Berkeley cardinals below δ are club in δ (by Remark 3.5(2)), it follows that

$$\langle V_\delta, V_{\delta+1} \rangle \models \text{ZF}_2 + \text{"there is Berkeley cardinal which is super Reinhardt,"}$$

which completes the proof. □

4 The Choiceless Hierarchy

The situation thus far looks like this:



Here 'LCBC' stands for 'limit club Berkeley cardinal', 'CBC' stands for 'club Berkeley cardinal', 'BC' stands for 'Berkeley cardinal', 'TR' stands for 'totally Reinhardt', 'SR' stands for 'super Reinhardt', and 'R' stands for 'Reinhardt'. The arrows are to be interpreted as follows: 'SRR' stands for 'strongly rank-reflects', 'RR' stands for 'rank-reflects', 'R' stands for 'reflects', and an unlabeled arrow between X and Y means that 'X is a Y'.

5 Choice

The choiceless large cardinals are, of course, inconsistent with AC. In the case of the weakest large cardinal in the choiceless hierarchy—a Reinhardt cardinal—the conflict with AC is most transparently seen through a violation of Solovay's theorem on splitting stationary sets. It is reasonable to expect that as one strengths the choiceless large cardinal notion the violation of AC becomes more transparent. It turns out that this expectation is borne out in the case of Berkeley cardinals.

Definition 5.1. Suppose γ is an ordinal. Then γ-DC is the statement that for every non-empty set X and for every function $F : {}^{<\gamma}X \to P(X) \smallsetminus \{\varnothing\}$ there exists a function $f : \gamma \to X$ such that for all $\alpha < \gamma$, $f(\alpha) \in F(f \restriction \alpha)$.

It is well known that AC is equivalent to the statement that γ-DC holds for all γ. So the principles γ-DC provide us with a nice stratification of AC into stronger and stronger fragments.

Definition 5.2. Let δ_0 be the least Berkeley cardinal. For any transitive set M such that $\delta_0 \in M$, let $\kappa_M = \min\{\mathrm{crit}(j) : j \in \mathscr{E}(M)\}$.

Lemma 5.1. *Let δ_0 be the least Berkeley cardinal. Then for all $\eta < \delta_0$ there exists a transitive set M_η such that $\delta_0 \in M_\eta$ and $\kappa_{M_\eta} > \eta$, (that is,* $\mathrm{crit}(j) > \eta$ *for all $j \in \mathscr{E}(M_\eta)$).*

Proof. Suppose for contradiction that the lemma fails. Let $\eta_0 < \delta_0$ be the least η such that $\kappa_M \leqslant \eta$ for all transitive sets M with $\delta_0 \in M$. We show that η_0 is a proto-Berkeley cardinal, which is a contradiction: Let M be a transitive set such that $\eta_0 \in M$. By Lemma 3.1, let $\hat{M}$ be a transitive set such that $\delta_0 \in \hat{M}$ and η_0 and M are definable in $\hat{M}$. Let $j \in \mathscr{E}(\hat{M})$ be such that $\mathrm{crit}(j) \leqslant \eta_0$. Since η_0 is definable in $\hat{M}$, $j(\eta_0) = \eta_0$ and so $\mathrm{crit}(j) < \eta_0$. Since M is definable in $\hat{M}$, $j \restriction M \in \mathscr{E}(M)$. Finally, since M was an arbitrary transitive set containing η_0, this shows that η_0 is a proto-Berkeley cardinal, which is a contradiction since δ_0 is the least proto-Berkeley cardinal. □

Theorem 5.2. *Suppose that δ_0 is the least Berkeley cardinal. Let $\gamma = \mathrm{cof}(\delta_0)$. Then γ-DC fails.*

Proof. Let $f : \gamma \to \delta_0$ be cofinal. For each $\xi < \gamma$, let β_ξ be least such that V_{β_ξ} contains a transitive set M such that $\delta_0 \in M$ and $\kappa_M > f(\xi)$. (This ordinal exists, by Lemma 5.1.)

For each $\xi < \gamma$, let

$$\hat{M}_\xi = \{M \in V_{\beta_\xi} : M \text{ is transitive, } \delta_0 \in M, \text{ and } \kappa_M > f(\xi)\}.$$

Assume for contradiction that γ-DC holds. Using γ-DC, let

$$\langle M_\xi : \xi < \gamma \rangle$$

be such that $M_\xi \in \hat{M}_\xi$, for all $\xi < \gamma$.

Now, by Lemma 3.1, let M' be a transitive set such that $\langle M_\xi : \xi < \gamma \rangle$ is definable in M'. Let $j' \in \mathscr{E}(M')$ be such that $\text{crit}(j') < \delta_0$. So

$$j'(\langle M_\xi : \xi < \gamma \rangle) = \langle M_\xi : \xi < \gamma \rangle.$$

It follows that if $j'(\xi) = \xi$, then $j'(M_\xi) = M_\xi$ and so $j' \restriction M_\xi \in \mathscr{E}(M_\xi)$, which means $\text{crit}(j') \geqslant \kappa_{M_\xi} > f(\xi)$. Thus, there cannot be cofinally many $\xi < \gamma$ such that $j'(\xi) = \xi$, as this would imply $\text{crit}(j') \geqslant \delta_0$.

It follows that there exists

$$\langle \eta_i : i < \omega \rangle$$

such that $\eta_{i+1} = j'(\eta_i)$ and $\sup\{\eta_i : i < \omega\} = \gamma$. Now, consider

$$\langle M_{\eta_i} : i < \omega \rangle.$$

Let M'' be such that $\langle M_{\eta_i} : i < \omega \rangle$ is definable in M'' and let $j'' \in \mathscr{E}(M'')$ be such that $\text{crit}(j'') < \delta_0$. We have

$$j''(\langle M_{\eta_i} : i < \omega \rangle) = \langle M_{\eta_i} : < \omega \rangle.$$

Thus, for each $i < \omega$,

$$j''(M_{\eta_i}) = M_{\eta_i}$$

and so

$$j''\restriction M_{\eta_i} \in \mathscr{E}(M_{\eta_i}).$$

It follows that for each $i < \omega$, $\mathrm{crit}(j'') \geqslant \kappa_{M_{\eta_i}}$. Hence $\mathrm{crit}(j'') \geqslant \delta_0$, which is a contradiction. □

Remark 5.3. The proof actually shows that γ-AC fails. We have chosen to phrase matters (here and below) in terms of γ-DC in part because in contrast to γ-AC it provides a stratification of AC. (AC is not equivalent to the statement that γ-AC holds for all γ.)

6 Cofinality

The previous theorem shows that there is an intimate connection between the cofinality of the least Berkeley cardinal and the degree of the conflict AC—the smaller the cofinality, the greater the conflict. This raises the question: What is the cofinality of the least Berkeley cardinal?

This looks like the sort of question that should have an easy answer. Moreover, since the assumption that δ_0 has countable cofinality implies a drastic failure of AC—the failure of countable choice—one might think that it should be possible to simply rule out this possibility.

It turns out however that the question of the cofinality of the least Berkeley cardinal is independent of our background assumptions. Moreover, if Berkeley cardinals are consistent, then it is also consistent that the least Berkeley cardinal have countable cofinality, and so enforce the failure of countable choice.

In an early version of this paper we showed that if there is a club Berkeley cardinal then there is a forcing extension where (in a rank initial segment) the least Berkeley cardinal has countable cofinality, and there is a forcing extension where (in a rank initial segment) the least Berkeley cardinal has uncountable cofinality. The forcing construction involved Prikry forcing in the choiceless setting and the proof was rather involved. Recently, Raffaella

Cutolo (a student of the second and third authors) found simpler proofs of sharper results. We include an account of her refinement.

Theorem 6.1. *Assume* ZF + DC + BC. *Then there is a forcing extension* $V[G]$ *such that*

$$V[G] \models \text{“cof}(\gamma_0) = \omega_1\text{”}$$

where γ_0 *is the least Berkeley cardinal as computed in* $V[G]$.

Proof. Work in V. Let $\gamma_0 = (\delta_0)^V$ be the least Berkeley cardinal in V. Since we are assuming DC, by Theorem 5.2 we have that $\text{cof}(\gamma_0) \geqslant \omega_1$. If $\text{cof}(\gamma_0) = \omega_1$ then we are done. So assume $\text{cof}(\gamma_0) > \omega_1$.

THE FORCING: $(\mathbb{P}_{\gamma_0}, \leqslant_{\mathbb{P}_{\gamma_0}})$

Let

$$S^{\gamma_0}_{\omega} = \{\alpha < \gamma_0 : \text{cof}(\alpha) = \omega\}$$

and

$$[S^{\gamma_0}_{\omega}]^{\aleph_0} = \{\sigma \subseteq S^{\gamma_0}_{\omega} : |\sigma| = \aleph_0\}.$$

The conditions of the forcing are of the form $\langle \sigma, C\rangle$ where $\sigma \in [S^{\gamma_0}_{\omega}]^{\aleph_0}$ is closed and C is an ω-club in γ_0. The ordering of conditions is given by

$$\begin{aligned} \langle \sigma_2, C_2\rangle \leqslant_{\mathbb{P}_{\gamma_0}} \langle \sigma_1, C_1\rangle \leftrightarrow\ &(1)\ \ C_2 \subseteq C_1,\\ &(2)\ \ \sigma_2 \text{ end-extends } \sigma_1,\\ &\qquad \text{i.e. } \sigma_1 \subseteq \sigma_2 \text{ and } \sigma_2 \cap \sup(\sigma_1) = \sigma_1, \text{ and,}\\ &(3)\ \ \sigma_2 \smallsetminus \sigma_1 \subseteq C_1. \end{aligned}$$

This relation is clearly transitive and, by DC, it is easy to see that it is countably closed (and hence that $(\omega_1)^{V[G]} = (\omega_1)^V$).

Let $G \subseteq \mathbb{P}_{\gamma_0}$ be V-generic. Let

$$\sigma_G = \bigcup \{\sigma : \exists C\, \langle \sigma, C\rangle \in G\}.$$

Claim 1. *The following hold in* $V[G]$.

(1) $\mathrm{ot}(\sigma_G) = \omega_1$.

(2) *For all $C \in V$, if C is an ω-club in γ_0, then $\sigma_G \smallsetminus C$ is bounded.*

Proof. (1). Since each σ is countable, we have $\mathrm{ot}(\sigma_G) \leqslant \omega_1$. For the other direction note that for each $\alpha < \omega_1$, each condition $\langle \sigma, C \rangle$ can be extended to a condition $\langle \sigma', C' \rangle$ such that $\mathrm{ot}(\sigma') > \alpha$; so, by genericity, $\mathrm{ot}(\sigma_G) \geqslant \omega_1$.

(2). Suppose $C \in V$ is an ω-club in γ_0. The set $\{\langle \sigma, D \rangle \in \mathbb{P}_{\gamma_0} : D \subseteq C\}$ is dense in $\mathbb{P}_{\gamma_0}$. If $\langle \sigma, D \rangle \in G$ is in this set, then all further extensions $\langle \sigma'', C'' \rangle$ are such that $\sigma'' \smallsetminus \sigma \subseteq C$. So $\sigma_G \smallsetminus C$ is bounded. □

It follows that in $V[G]$, σ_G is a club in γ_0 of ordertype ω_1. So $(\mathrm{cof}(\gamma_0))^{V[G]} \leqslant \omega_1$. The question is whether γ_0 is a Berkeley cardinal in $V[G]$.

Claim 2. *Suppose φ is a statement in the language of forcing with only check names. Then either $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \varphi$ or $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi$.*

Proof. The key claim is the following:

Subclaim. *Suppose $\langle \sigma_0, C_0 \rangle$ is a condition. Suppose G is a generic through $\langle \varnothing, C_0 \rangle$. Then there is a generic G^{σ_0} through $\langle \sigma_0, C_0 \rangle$ such that $V[G^{\sigma_0}] = V[G]$.*

Proof. The point is that if D is dense below $\langle \sigma_0, C_0 \rangle$, then

$$\begin{aligned} D_{\sigma_0} = \big\{ \langle \sigma' \cup \sigma'', C \rangle : \exists \sigma \text{ such that } &\langle \sigma, C \rangle \in D, \\ &\sigma \smallsetminus \sup(\sigma_0) = \sigma'',\ \sigma' \subseteq \sup(\sigma_0),\ \text{and } \sigma' \subseteq C \big\} \end{aligned}$$

is dense below $\langle \varnothing, C_0 \rangle$. Now, for G the generic in the statement of the claim, let

$$\begin{aligned} G^{\sigma_0} = \big\{ \langle \sigma_0 \cup \sigma'', C \rangle : \exists \sigma \text{ such that } &\langle \sigma, C \rangle \in G \\ &\text{and } \sigma \smallsetminus \sup(\sigma_0) = \sigma'' \big\}. \end{aligned}$$

We claim that G^{σ_0} is a generic through $\langle \varnothing, C_0 \rangle$. For suppose D is dense below $\langle \sigma_0, C_0 \rangle$. Then, since D_{σ_0} is dense below $\langle \varnothing, C_0 \rangle$, G must hit D_{σ_0}. Let

$\langle \sigma' \cup \sigma'', C\rangle \in G \cap D_{\sigma_0}$. So there exists σ such that $\langle \sigma, C\rangle \in D$, $\sigma \setminus \sup(\sigma_0) = \sigma''$, and $\sigma' \subseteq (\sup(\sigma_0) \cap C)$. So $\langle \sigma_0 \cup \sigma'', C\rangle \in G^{\sigma_0} \cap D$. So G^{σ_0} hits D.

Finally, since $\sigma_0 \in V$, we have that $V[G^{\sigma_0}] = V[G]$. □

Either there exists a condition $\langle \sigma_0, C_0\rangle$ such that

$$\langle \sigma_0, C_0\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \varphi$$

or there exists a condition $\langle \sigma_1, C_1\rangle$ such that

$$\langle \sigma_1, C_1\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \varphi.$$

In the first case we have that $\langle \varnothing, C_0\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \varphi$, since for every generic through $\langle \sigma_0, C_0\rangle$ there is an "equivalent" generic through $\langle \varnothing, C_0\rangle$. In the second case, $\langle \varnothing, C_1\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi$. But $\langle \varnothing, C_0\rangle$ and $\langle \varnothing, C_1\rangle$ are compatible conditions. So *every* condition that decides φ must decide it in the same way. This implies that $1_{\mathbb{P}_{\gamma_0}}$ decides φ. (The reason is that the set $\{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi \text{ or } p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi\}$ is dense. So in our case $\{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \varphi\}$ is dense or $\{p \in \mathbb{P}_{\gamma_0} : p \Vdash_{\mathbb{P}_{\gamma_0}} \neg\varphi\}$ is dense.) □

Claim 3. $1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}}$ "*γ_0 is a Berkeley cardinal.*"

Proof. Suppose for contradiction that the claim fails. By Claim 2

$$1_{\mathbb{P}_{\gamma_0}} \Vdash_{\mathbb{P}_{\gamma_0}} \text{“}\gamma_0 \text{ is not a Berkeley cardinal.”}$$

Let $\langle \sigma, C\rangle \in \mathbb{P}_{\gamma_0}$ and $\tau \in \mathrm{Name}_{\mathbb{P}_{\gamma_0}}$ be such that

$$\begin{aligned}\langle \sigma, C\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \text{“}\tau &\text{ is a transitive set such that}\\ &\gamma_0 \in \tau \text{ and } \exists \eta < \gamma_0 \text{ such that}\\ &\neg\exists j \in \mathscr{E}(\tau)\,(\eta < \mathrm{crit}(j) < \gamma_0)\text{.”}\end{aligned}$$

Let $\eta < \gamma_0$ be least such that there exists $\langle \sigma_0, C_0\rangle \leq \langle \sigma, C\rangle$ such that

$$\langle \sigma_0, C_0\rangle \Vdash_{\mathbb{P}_{\gamma_0}} \text{“}\neg\exists j \in \mathscr{E}(\tau)\ (\eta < \mathrm{crit}(j) < \gamma_0)\text{.”}$$

Let λ be a limit ordinal much larger than γ_0 such that $\tau \in V_\lambda$, V_λ recognizes that γ_0 is the least Berkeley cardinals, and $V_\lambda \models \mathrm{ZF}^*$, where ZF^* is a sufficiently large fragment of ZF to implement the "lifting of embeddings" argument that we give below. Since γ_0 is a Berkeley cardinal in V, there exists $j \in \mathscr{E}(V_\lambda)$ such that

(1) $j(\gamma_0) = \gamma_0$, $j(\eta) = \eta$, $j(\tau) = \tau$,

(2) $j([S^{\gamma_0}_\omega]^{\aleph_0}) = [S^{\gamma_0}_\omega]^{\aleph_0}$, $j(\mathbb{P}_{\gamma_0}) = \mathbb{P}_{\gamma_0}$,

(3) $j(\langle \sigma_0, C_0 \rangle) = \langle \sigma_0, C_0 \rangle$, and

(4) $\eta < \mathrm{crit}(j) < \gamma_0$.

Let

$$C_j = \{\alpha \in S^{\gamma_0}_\omega : j(\alpha) = \alpha\}.$$

By the proof of Lemma 3.6(2), C_j is cofinal in γ_0. Moreover, C_j is an ω-club in γ_0. So have

$$\langle \sigma_0, C_0 \cap C_j \rangle \leqslant_{\mathbb{P}_{\gamma_0}} \langle \sigma_0, C_0 \rangle.$$

Let $G \subseteq \mathbb{P}_{\gamma_0}$ be a V-generic filter such that $\langle \sigma_0, C_0 \cap C_j \rangle \in G$. We wish to show that $j : V_\lambda \to V_\lambda$ lifts to an elementary embedding $j^+ : V_\lambda[G] \to V_\lambda[G]$. The restriction of this embedding to τ^G will then give the desired embedding from τ^G into τ^G with $\eta < \mathrm{crit}(j^+) < \gamma_0$.

Recall the following "lifting criterion" for elementary embeddings: Suppose $j : M \to N$ is an elementary embedding of two transitive models of (a sufficiently large fragment of) ZF. Let $\mathbb{P} \in M$ and suppose $G \subseteq \mathbb{P}$ is M-generic and $H \subseteq j(\mathbb{P})$ is N-generic. Then j lifts to and elementary embedding $j^+ : M[G] \to N[H]$ (with $j^+(G) = H$) iff $j``G \subseteq H$.[9] In our present case we have that $j(\mathbb{P}_{\gamma_0}) = \mathbb{P}_{\gamma_0}$ and we will show that in fact $j``G \subseteq G$ and hence that the embedding lifts to $j^+ : V_\lambda[G] \to V_\lambda[G]$.

[9] The lifting criterion was used in Silver's consistency proof of the failure of GCH at a measurable. See Cummings (2010, §9) for a modern account.

It remains to show that $j``G \subseteq G$: Notice that for any $\langle \sigma, C\rangle \in \mathbb{P}_{\gamma_0}$ if $\langle \sigma, C\rangle \leqslant \langle \sigma_0, C_0 \cap C_j\rangle$ then $j(\sigma) = \sigma$. (Let $\sigma = \sigma_0{}^\frown\bar{\sigma}$. We have $j(\sigma_0) = \sigma_0$ by (3) above, and we have $j(\bar{\sigma}) = \bar{\sigma}$ since $\bar{\sigma} \subseteq C_j$ and j is the identity on $[C_j]^{\aleph_0}$.) Notice also that for all $C \subseteq C_j$, $C \subseteq j(C)$. So for any $\langle \sigma, C\rangle \in \mathbb{P}_{\gamma_0}$ if $\langle \sigma, C\rangle \leqslant \langle \sigma_0, C_0 \cap C_j\rangle$, then $\langle \sigma, C\rangle \leqslant j(\langle \sigma, C\rangle)$.

Now suppose $\langle \sigma, C\rangle \in G$. Let $\langle \sigma', C'\rangle \in G$ extend both $\langle \sigma, C\rangle$ and $\langle \sigma_0, C_0 \cap C_j\rangle$. Since $\langle \sigma', C'\rangle \leqslant \langle \sigma_0, C_0 \cap C_j\rangle$, we have (by the above) $\langle \sigma', C'\rangle \leqslant j(\langle \sigma', C'\rangle)$ and hence $j(\langle \sigma', C'\rangle) \in G$. Since $\langle \sigma', C'\rangle \leqslant \langle \sigma, C\rangle$, by elementarity $j(\langle \sigma', C'\rangle) \leqslant j(\langle \sigma, C\rangle)$, and so $j(\langle \sigma, C\rangle) \in G$. Thus, $j``G \subseteq G$.

By the "lifting criterion" we have that $j : V_\lambda \to V_\lambda$ lifts to an elementary embedding $j^+ : V_\lambda[G] \to V_\lambda[G]$. Since $j^+(G) = G$ and $j^+(\tau) = \tau$, we have $j^+(\tau_G) = \tau_G$. Moreover, $\eta < \mathrm{crit}(j^+) < \gamma_0$. Therefore, $j^+ \restriction \tau_G : \tau_G \to \tau_G$ is an elementary embedding which contradicts the choice of $\langle \sigma_0, C_0\rangle$ and τ as a counterexample to γ_0 being a Berkeley cardinal in $V[G]$. □

To summarize, we have thus shown that if $G \subseteq \mathbb{P}$ is V-generic then $V[G]$ satisfies $\mathrm{ZF} + \mathrm{DC} + \mathrm{BC}$ and "γ_0 is a Berkeley cardinal with $\mathrm{cof}(\gamma_0) \leqslant \omega_1$". If γ_0 is the *least* Berkeley cardinal of $V[G]$ then we also have $\mathrm{cof}(\gamma_0)^{V[G]} \geqslant \omega_1$ (since the forcing is countably closed and so DC is preserved) and so we are done.

However, we have not shown that γ_0 is the *least* Berkeley cardinal of $V[G]$ and we do not know whether this is true. In any case, the difficulty is easily handled by iterating the procedure. Let $G_0 = G$. If γ_0 *is* the least Berkeley cardinal of $V[G_0]$ then we are done. If γ_0 is not the least Berkeley cardinal in $V[G_0]$ then let $\gamma_1 < \gamma_0$ be the least Berkeley cardinal in $V[G_0]$. We now force over $V[G_1]$ with the partial order $\mathbb{P}_{\gamma_1}$. If G_1 is $\mathbb{P}_{\gamma_1}$-generic then $V[G_0][G_1]$ satisfies that γ_1 is a Berkeley cardinal and $\mathrm{cof}(\gamma_1) \leqslant \omega_1$. But again, we have no guarantee that γ_1 is the *least* Berkeley cardinal of $V[G_0][G_1]$. So we continue, defining $\gamma_0 > \gamma_1 > \gamma_2 > \dots$. The point is that at some finite stage n this must stop. Letting n be the stage at which it stops, we have a model $V[G_0]\cdots[G_n]$ that satisfies ZF+DC+BC and where the least Berkeley cardinal γ_n is such that $\mathrm{cof}(\gamma_n) = \omega_1$. □

Remark 6.1. It is worth noting that the lifting argument in the above proof could never succeed in the context of AC. The reason is that for all we have said the least Berkeley cardinal with which we started could have been regular, but under AC if a countably closed (more generally, proper) forcing changes the cofinality of a regular cardinal to ω_1 then it must collapse it to ω_1.[10] Yet, in our choiceless context, as we have shown in Claim 3, we have changed the cofinality of γ_0 to ω_1 while preserving the fact that it is a Berkeley cardinal (and hence without collapsing it to ω_1).

The key choiceless feature that is being leveraged in the proof of Claim 3 is the following: If there is a Berkeley cardinal δ, then given *any* set a and any λ such that $\delta, a \in V_\lambda$, one can find elementary embeddings $j \in \mathscr{E}(V_\lambda)$ which *fix* a. (We applied this feature in the proof of Claim 3 when we obtained $j \in \mathscr{E}(V_\lambda)$ satisfying (1)–(3).) This is an extremely powerful large cardinal feature, one that cannot be had in the context of AC.

Theorem 6.2. *Assume* ZF + BC. *Then there is a forcing extension* $V[G]$ *such that*

$$V[G] \models \text{“cof}(\delta_0) = \omega.\text{”}$$

Proof. The proof is almost exactly the same. The only change is that now in the definition of $\mathbb{P}_{\gamma_0}$ we use finite subsets of $S^{\gamma_0}_\omega$. The point is that in this case we do not need DC. □

Remark 6.2. The same proof also works for other cofinalities.

Thus, ‘ZF + BC’ is equiconsistent with ‘ZF + BC + $\text{cof}(\delta_0) = \omega$’; and ‘ZF + DC + BC’ is equiconsistent with ‘ZF + DC + BC + $\text{cof}(\delta_0) = \omega_1$’. So the question arises: What is the relative consistency strength of ‘ZF + BC’ and ‘ZF + DC + BC’?

Here is an intriguing idea: Large cardinal axioms are in tension with axioms asserting that the universe is “orderly.” For example, if there is a measurable cardinal, then $V = L$ fails; if there is a supercompact cardinal κ,

[10] See Hayut & Karagila (2016).

then $\square_\lambda$ fails for all $\lambda \geqslant \kappa$; and, continuing this theme, if there is a Reinhardt cardinal, then $<\lambda^+$-DC fails, where λ is the limit of the critical sequence.[11] Of course, in all of these cases the tension is too extreme—the principle of order ($V = L$, $\square$, $<\lambda^+$-DC) is outright inconsistent with the corresponding large cardinal (measurable, supercompact, Reinhardt). But perhaps, one can leverage the tension to obtain strength, without going over the edge into inconsistency; that is, perhaps as one folds in more and more choice, one can get stronger and stronger large cardinal principles, approximating, but not reaching, inconsistency. In particular, perhaps the following sequence of theories is strictly increasing in terms of consistency strength.

- $\mathrm{ZF} + \mathrm{BC} + \mathrm{cof}(\delta_0) = \omega$
- $\mathrm{ZF} + \mathrm{BC} + \mathrm{cof}(\delta_0) = \omega_1$
- $\mathrm{ZF} + \mathrm{BC} + \mathrm{cof}(\delta_0) = \omega_2$

$$\vdots$$

- $\mathrm{ZF} + \mathrm{BC} + \mathrm{DC} + \mathrm{cof}(\delta_0) = \omega_1$
- $\mathrm{ZF} + \mathrm{BC} + \mathrm{DC} + \mathrm{cof}(\delta_0) = \omega_2$
- $\mathrm{ZF} + \mathrm{BC} + \mathrm{DC} + \mathrm{cof}(\delta_0) = \omega_3$

$$\vdots$$

- $\mathrm{ZF} + \mathrm{BC} + \omega_1\text{-DC} + \mathrm{cof}(\delta_0) = \omega_2$
- $\mathrm{ZF} + \mathrm{BC} + \omega_1\text{-DC} + \mathrm{cof}(\delta_0) = \omega_3$
- $\mathrm{ZF} + \mathrm{BC} + \omega_1\text{-DC} + \mathrm{cof}(\delta_0) = \omega_4$

$$\vdots$$

[11] If there is a super Reinhardt cardinal κ, then there is a forcing notion which forces λ-DC and preserves the fact that κ is a Reinhardt cardinal. (The forcing notion is that used in Theorem 226 of Woodin (2010).) But $<\lambda^+$-DC must of necessity fail.

- $\mathrm{ZF} + \mathrm{BC} + \mathrm{DC} + \mathrm{cof}(\delta_0) = \delta_0$
- $\mathrm{ZF} + \mathrm{BC} + \omega_1\text{-DC} + \mathrm{cof}(\delta_0) = \delta_0$
- $\mathrm{ZF} + \mathrm{BC} + \omega_2\text{-DC} + \mathrm{cof}(\delta_0) = \delta_0$

$$\vdots$$

Question 5. Is the above hierarchy of statements increasing in consistency strength?

Question 6. Is it possible for the least Berkeley cardinal to be regular?

The trouble is that these questions appear to be completely beyond the reach current technology.

7 Inner Model Theory

The key question is whether Berkeley cardinals are even consistent. In fact, there are results which provide some reason for thinking that Berkeley cardinals are *not* consistent. In order to describe these results, and in order to situate our discussion of the two futures, we need to say something about recent developments in inner model theory.[12]

It will be helpful to return to our discussion of the HOD Dichotomy.

7.1 The HOD Dichotomy

The version of the HOD Dichotomy that we stated in the introduction was a weak version of the official version. The official version involves the following notion:

[12]The results discussed in this section are due to the third author and are discussed in further detail in Woodin (2010) and Woodin (2017).

Definition 7.1. Let γ be an uncountable regular cardinal. Let $S^\gamma_\omega = \{\alpha < \gamma : \text{cof}(\alpha) = \omega\}$. Then γ is *ω-strongly measurable in* HOD if there exists $\kappa < \gamma$ such that

(1) $(2^\kappa)^{\text{HOD}} < \gamma$ and

(2) There is no partition $\langle S_\alpha : \alpha < \kappa \rangle$ of S^γ_ω into stationary sets such that $\langle S_\alpha : \alpha < \kappa \rangle \in \text{HOD}$.

The official version of the HOD Dichotomy is obtained by simply replacing 'measurable in HOD' with 'ω-strongly measurable in HOD' in the weak version:

Theorem 7.1 (The HOD Dichotomy Theorem)**.** *Suppose that κ is an extendible cardinal. Then exactly one of the following hold.*

(1) *For every singular cardinal $\gamma > \kappa$, γ is singular in* HOD *and* $(\gamma^+)^{\text{HOD}} = \gamma^+$.

(2) *Every regular cardinal $\gamma \geqslant \kappa$ is ω-strongly measurable in* HOD.

As before, the first alternative is one in which HOD is "close" to V, in that it correctly computes much of the cardinal structure of V, while the second alternative is one in which HOD is "far" from V, in that it radically fails to capture the cardinal structure of V, thinking, for example, that κ^+ is ω-strongly measurable.

The connection between the weak version and the official version of the HOD Dichotomy is given by the following easy lemma:

Lemma 7.2. *Assume that γ is ω-strongly measurable in* HOD. *Then*

$$\text{HOD} \models \gamma \textit{ is a measurable cardinal.}$$

So the weak version of the HOD Dichotomy follows from the official version.

The importance of the official version is it makes it clear that the issue in deciding which side of the dichotomy holds is whether or not there can be "definable" partitions of S^γ_ω into stationary sets, as we shall now explain.

Definition 7.2 (The HOD Hypothesis)**.** There exists a proper class of regular cardinals γ which are not ω-strongly measurable in HOD.

Unpacking the definitions, the HOD Hypothesis asserts that there are arbitrarily large regular cardinals γ such that for every $\kappa < \gamma$ such that $(2^\kappa)^{\text{HOD}} < \gamma$ there is a partition $\langle S_\alpha : \alpha < \kappa \rangle \in \text{HOD}$ of S^γ_ω into sets which are stationary in V. We know, by Solovay's theorem on stationary splitting that there are always such partitions in V, but what the HOD Hypothesis is asserting is that such splittings can be done "definably," in the sense that they exist in HOD.

There is a series of conjectures to the effect that the HOD Hypothesis is provable from ZFC, or from ZFC plus various large cardinal axioms. The *Strong HOD Conjecture* asserts that the HOD Hypothesis is provable from ZFC. The *HOD Conjecture* asserts that the HOD Hypothesis is provable from ZFC+"There is an extendible cardinal." Notice that these conjectures become more plausible as one strengthens the large cardinal assumption. In what follows we shall focus on the following conjecture, which is more plausible than the two we have just mentioned.

Definition 7.3 (The Weak HOD Conjecture)**.** The *Weak HOD Conjecture* is the conjecture that

ZFC + "There is an extendible cardinal with a huge cardinal above"

proves the HOD Hypothesis.

It is important to note that the Weak HOD Conjecture is a Σ^0_1-statement, and so it is not going to run up against the rock of undecidability.[13] If this Σ^0_1-statement holds, then, assuming large cardinals—in particular, an extendible cardinal with a huge cardinal above—we must be in the first half of the HOD Dichotomy, where HOD is "close" to V.

It is natural to ask why one might make such a conjecture. It posits that in the presence of an extendible cardinal with a huge cardinal above

[13] If it is independent, then it is false.

one can actually *prove* that there are arbitrarily large regular cardinals γ such that for every $\kappa < \gamma$ such that $(2^\kappa)^{\mathrm{HOD}} < \gamma$ there is a partition $\langle S_\alpha : \alpha < \kappa\rangle \in \mathrm{HOD}$ of S^γ_ω into sets which are stationary in V. It is really quite a surprising conjecture. In fact, when the second author was a graduate student this conjecture was known by a different name. It was known as "the silly conjecture."

But it is not so silly anymore. The reason has to do with recent developments in inner model theory.

7.2 Weak Extender Models

Inner model theory began, of course, with Gödel's L, but it entered the large cardinal hierarchy with Solovay's $L[U]$. Since then the holy grail of inner model theory has been an inner model of a supercompact cardinal.

Prior to the actual construction of such a model it is hard to know what it will look like. But one can isolate certain basic features that such a model is expected to have. For example, an inner model N targeting a supercompact cardinal κ should be such that the measures in N witnessing the supercompactness of κ in N are "inherited" from the measures in V witnessing the supercompactness of κ in V.

Definition 7.4. A transtive class $N \models \mathrm{ZFC}$ is a *weak extender model of the supercompactness of* κ if for every $\lambda > \kappa$ there exists a κ-complete normal fine measure U on $P_\kappa(\lambda)$ such that

(1) $N \cap P_\kappa(\lambda) \in U$ and

(2) $U \cap N \in N$.

Each of these conditions is motivated by the case of $L[U]$ and the various other inner models in a long progression. One would expect that an inner model targeting a supercompact cardinal κ would have these features as well, and hence be a weak extender model of the supercompactness of κ.

Weak extender models of the supercompactness of κ have features that are quite remarkable. Recall that in general, when one constructs an inner model for a given large cardinal, the existence of stronger large cardinals implies that the model is "far" from V. (One constructs an L-like paradise of understanding for a given large cardinal, only to have it revealed as illusory—in the sense of being "far" from V—by the presence of stronger large cardinals.) Remarkably, at the level of a supercompact cardinal the situation completely changes.

Theorem 7.3. *Suppose that N is a weak extender model the supercompactness of κ. Then for every singular cardinal $\gamma > \kappa$, γ is singular in N and $(\gamma^+)^N = \gamma^+$.*

In other words N is "close" to V, *regardless of which additional large cardinals live in V.* (So if there is an L-like paradise of understanding at this level, then *no* large cardinal can reveal it to be illusory.)

It turns out that such an N actually *absorbs* all of the large cardinal structure of V.

Theorem 7.4 (Universality)**.** *Suppose that N is a weak extender model of the supercompactness of κ. Suppose that $\alpha > \kappa$ is an ordinal and*

$$j : N \cap V_{\alpha+1} \to N \cap V_{j(\alpha)+1}$$

is an elementary embedding such that $\kappa \leqslant \operatorname{crit}(j)$. Then $j \in N$.

This theorem lies at the heart of a cluster of results which collectively show that N captures the large cardinal structure of V. For example:

Theorem 7.5. *Suppose that N is a weak extender model of the supercompactness of κ and suppose that $\delta > \kappa$ is supercompact. Then N is a weak extender model of the supercompactness of δ.*

Theorem 7.6. *Suppose that N is a weak extender model of the supercompactness of κ and suppose that $\delta > \kappa$ is extendible. Then δ is extendible in N.*

And so on, up through the traditional large cardinal hierarchy. One gets direct transference to N from V of each large cardinal, no matter how strong.[14]

This was entirely unexpected. In general, in inner model theory when one targets a given large cardinal, the resulting model captures that large cardinal but fails to capture stronger large cardinals; indeed the existence of stronger large cardinals typically implies that the model is "far" from V (in parallel to the manner in which the existence of $0^{\#}$ implies that L is "far" from V). But in the case of a weak extender model of the supercompactness of κ is completely different one is just targeting a single supercompact cardinal and the resulting model is not only "close" to V with regard to its computation of cardinal structure (above κ) but is also "close" to V in that (above κ) it inherits *all* of the traditional large cardinals existing in V. This includes large cardinals (like n-huge cardinals) which are far beyond the level of supercompactness. In short, in the case of supercompact cardinals there is an "ignition point" and the model "goes all the way." This suggests that the problem of inner model theory is reduced to the problem of finding an inner model of a supercompact cardinal.

7.3 The HOD Dichotomy and Weak Extender Models

The relevance of weak extender models to the HOD Dichotomy is contained in the following theorem:

Theorem 7.7. *Suppose that κ is an extendible cardinal. Then the following are equivalent.*

(1) *The* HOD *Hypothesis holds.*

(2) *There is a regular cardinal $\gamma \geqslant \kappa$ which is not ω-strongly measurable in* HOD.

[14] Notice that these results are stronger than the ones stated Woodin (2010). Here one obtains direct transference of a large cardinal notion, whereas with the earlier results one had to assume a (slightly) stronger larger cardinal property to transfer a given one.

(3) *No regular cardinal* $\gamma \geq \kappa$ *is* ω*-strongly measurable in* HOD.

(4) *There is a cardinal* $\gamma \geqslant \kappa$ *such that* $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.

(5) HOD *is a weak extender model of the supercompactness of* κ.

(6) *There is a weak extender model* N *for the supercompactness of* κ *such that* $N \subseteq \mathrm{HOD}$.

(7) (*Goldberg*) *There is a weak extender model* N *for the supercompactness of* κ *such that* $N \models$ "$V = \mathrm{HOD}$."[15]

It is this last equivalence which leads to the expectation that the HOD Hypothesis actually holds. For, assuming large cardinals, it is natural to expect that there is a weak extender model of the supercompactness of κ and, given the course of inner model theory, it is natural to expect that such a model satisfies $V = \mathrm{HOD}$. The Weak HOD Conjecture is thus really just that conjecture that 'ZFC+"There is an extendible cardinal κ with a huge cardinal above κ"' proves that there is a weak extender model N of the supercompactness of κthat satisfies $V = \mathrm{HOD}$. That is why the Weak HOD Conjecture is not silly anymore.

7.4 The Ultimate-L Conjecture

The notion of a weak extender model is a very general notion. One would like a more specific target and hence a more specific conjecture than the Weak HOD Conjecture.[16]

[15] If one strengthens the background assumption of the theorem to 'Suppose that there is a proper class of extendible cardinals,' then one can add 'There is a weak extender model N for the supercompactness of κ such that $N \models$ "The HOD Hypothesis"' to the above sequence of equivalences.

[16] In this subsection we will have to invoke some notions that are beyond the scope of this paper, but in the interest of providing the reader with a broad picture we will give a brief account. For further details see Woodin (2017).

The trouble is that in the case of the canonical inner models M that have been built to date, one is generally not in a position to even state the axiom "$V = M$" prior to the actual construction of M, and, furthermore, the construction is generally so complex that the associated axiom is not expressible in simple, readily understood terms. There is, however, one exception, namely, the case of L.

For each ordinal α, let

$$N_\alpha = \bigcap \big\{ M : M \text{ is transitive, } M \models \text{ZFC} - \text{Powerset, and } \text{On}^M = \alpha \big\}.$$

Lemma 7.8. *The following are equivalent*:

(1) $V = L$.

(2) *For each* Σ_2*-sentence* φ*, if* φ *holds in* V*, then there exists a countable ordinal* α *such that* $N_\alpha \models \varphi$.

So the axiom $V = L$ could have been stated prior to actual the construction of L, in terms of reflection of Σ_2-truth into models of the form N_α for countable α.

Curiously, at the opposite end of the spectrum, in the case of the candidate for the ultimate inner model, one can also state the axiom prior to the actual construction. The definition is motivated by the discovery that the HODs of determinacy models turn out to be canonical (strategic) inner models. Instead of the reflection of Σ_2-truth into models of the form N_α, the defintion of Ultimate-L involves reflection of Σ_2-truth into the HODs of determinacy models.

Definition 7.5. "$V = $ Ultimate-L" is the conjunction of the following two statements:

(1) There is a proper class of Woodin cardinals.

(2) For each Σ_2-sentence φ, if φ holds in V, then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\text{HOD}^{L(A,\mathbb{R})} \models \varphi.$$

The virtue of knowing the axiom "$V = \text{Ultimate-}L$" prior to the actual construction of Ultimate-L is that one can start to mine the consequences of the axiom before the construction is completed.

Theorem 7.9. *Assume* $V = \text{Ultimate-}L$. *Then*

(1) CH *holds.*

(2) *The* Ω *Conjecture holds.*

(3) $V = \text{HOD}$.

(4) V *is the minimum universe of the Generic-Multiverse.*[17]

The key question is whether (granting large cardinals) there are models of "$V = \text{Ultimate-}L$" that are weak extender models for the supercompactness of κ."

Definition 7.6 (The Weak Ultimate-L Conjecture)**.** The *Weak Ultimate-L Conjecture*[18] is the conjecture that

ZFC + "There is an extendible cardinal with a huge cardinal above"

proves: If κ is extendible and there is a huge cardinal above κ, then there exists a weak extender model N of the supercompactness of κ such that

$$N \models \text{"}V = \text{Ultimate-}L.\text{"}$$

Notice that the Weak Ultimate-L Conjecture is a Σ^0_1-statement, and so it is not going to run up against the rock of undecidability. The point, for our present purposes, is that the Weak Ultimate-L Conjecture implies the Weak HOD Conjecture. (This is immediate from the definitions, Theorem 7.9(3),

[17]See Woodin (2011) for a definition of 'Generic-Multiverse'.

[18]The term 'The Weak Ultimate-L Conjecture' is used in Woodin (2014) for a different conjecture. It seemed appropriate to reclaim the terminology for the above conjecture, which is just the Ultimate-L Conjecture, except with the 'extendible with a huge cardinal above' instead of 'extendible.'

and the equivalence of (1) and (7) in Theorem 7.7.) A great deal of work has been done toward the construction of Ultimate-L and toward proving the Weak Ultimate-L Conjecture. All of this points toward the first half of the HOD Dichotomy, where HOD (and, in fact, Ultimate-L) is "close" to V.

8 Two Futures

One virtue of the HOD Dichotomy Theorem is that it presents us with a discrete set of possibilities for the universe of sets. There are two, radically distinct, possibilities. Either HOD is "close" to V, or HOD is "far" from V. Moreover, this is a theorem. The fact that there are just these two possibilities is *provable* in ZFC+"there is an extendible cardinal."[19] One would, of course, like to know which of the two possibilities actually holds.

Now, as we noted in the introduction, in contrast to the case of the L Dichotomy, *no* traditional large cardinal axiom can force us into the second side of the HOD Dichotomy, where HOD is "far" from V. So perhaps the first side of the HOD Dichotomy holds. Perhaps HOD is "close" to V and perhaps one can prove this assuming traditional large cardinal axioms. This is what the Weak Ultimate-L Conjecture (through implying the Weak HOD Conjecture) asserts. The virtue of this conjecture is that it is a definite, Σ^0_1-statement. If this Σ^0_1-statement holds, then, assuming ZFC+"there is an extendible with a huge cardinal above," we are thrown into the first side of the HOD Dichotomy, where HOD is "close" to V.

What about the second side of the HOD Dichotomy? We noted that there is no traditional large cardinal axiom A such that ZFC + A proves that HOD is "far" from V. In other words, there can be no Σ^0_1-sentence of the above form that forces us into the second side of the HOD Dichotomy, where HOD

[19]There is of course a third possibility. It could be that ZFC+"there is an extendible cardinal" is inconsistent—or, if consistent, at least something that is not justifiable, or even something which is overturned by other considerations. In what follows we shall have nothing more to say about this third possibility. Instead we will adopt ZFC along with traditional large cardinal axioms.

is "far" from V. But as we shall now show, if large cardinals beyond choice are *consistent* (Π^0_1-facts), then the Weak Ultimate-L Conjecture must fail, and so we will have lost our best candidate for a Σ^0_1-truth that throws us into the first side of the HOD Dichotomy. Moreover, if Berkeley cardinals are consistent, then we will be on the road to new large cardinal axioms (namely, the HOD analogues we shall describe below) which are consistent with AC and which *do* force us into the "far" side of the HOD Dichotomy, and thus do for HOD what $0^\#$ does for L.

We will thus be describing two tangible futures. In the first future the Weak Ultimate-L Conjecture holds. In the second future Berkeley cardinals are consistent, the Weak Ultimate-L Conjecture fails, and we are on the road to new axioms that force us into the second side of the HOD Dichotomy, where HOD is "far" from V.

In the remainder of this section we will describe, in a speculative fashion, how these two futures might unfold.

8.1 The First Future: The Road to Ultimate-*L*

The first future is the future where, to begin with, the Weak Ultimate-L Conjecture holds.

This is a definite Σ^0_1-statement. Assuming ZFC+"there is an extendible with a huge cardinal above" it implies that we are on the first side of the HOD Dichotomy, where HOD is "close" to V. But it implies much more.

Let us begin with the implications for large cardinals beyond choice. It turns out that the Weak Ultimate-L Conjecture wipes out almost all of the large cardinals that we have been investigating. In particular, it wipes out the existence of a Reinhardt cardinal with an ω-huge cardinal above $\kappa_\omega(j)$, where j is the witness to the Reinhardt cardinal, and so, as a corollary, it wipes out super Reinhardt cardinals and (by Theorem 3.7) Berkeley cardinals.

We will now give a sketch of this result. The proof requires three theorems from Woodin (2010), namely, Theorems 226, 228, and 229. In some cases we will need slight variants of the theorems.

We will need the following slight variant of Theorem 226, which the proof actually establishes.

Theorem 8.1 (ZF)**.** *Suppose that there exists $j \in \mathscr{E}(V_\lambda)$ with $\lambda = \kappa_\omega(j)$. Then there is a homogeneous partial order $\mathbb{Q}$ which is a Σ_3-definable class in V_λ and such that if $G \subseteq \mathbb{Q}$ is V_λ-generic, then*

$$V_\lambda[G] \models \text{ZFC} + \text{“}\textit{there is a proper class of extendible cardinals and there is a proper class of huge cardinals.}\text{”}$$

We will use Theorem 228 as it stands, and so it will be convenient to restate it here.

Theorem 8.2 (ZF)**.** *Suppose that* $\operatorname{cof}(\lambda) = \omega$*,* λ *is a limit of supercompact cardinals and that for some set* $A \subseteq \text{On}$*,*

$$\lambda^+ = (\lambda^+)^{L[A]}.$$

Then there is no non-trivial elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Finally, we will need the following variant of Theorem 229.

Theorem 8.3 (ZF)**.** *Suppose that the Weak* HOD *Conjecture holds. Suppose that there exists $j \in \mathscr{E}(V_\lambda)$ with $\lambda = \kappa_\omega(j)$. Then there is a transitive class $N \subseteq V_\lambda$ and a parameter $X \in V_\lambda$ such that the following hold*:

(1) $N \models \text{ZFC}$.

(2) *N is Σ_2-definable in V_λ from X.*

(3) *There is a partial order $\mathbb{P} \in N$ such that for all sets of ordinals $A \in V_\lambda$ there is an N-generic filter $G \subseteq \mathbb{P}$ such that $A \in N[G]$.*

Proof. The proof is a minor modification of the proof of Theorem 229. Let $\mathbb{Q}$ be the homogeneous partial order from Theorem 8.1. Let $G \subseteq \mathbb{Q}$ be a V_λ-generic filter. So

$$V_\lambda[G] \models \text{ZFC} + \text{“there is a proper class of extendible cardinals and there is a proper class of huge cardinals.”}$$

So, by the Weak HOD Conjecture

$$V_\lambda[G] \models \text{The HOD Hypothesis.}$$

The rest of the proof is the same. □

With these theorems at hand we are now in a position to prove the main result.

Theorem 8.4. *Suppose that the Weak* HOD *Conjecture holds. Then there cannot be a non-trivial elementary embedding $j : V \to V$ along with an ω-huge cardinal that is above $\kappa_\omega(j)$.*

Proof. Suppose for contradiction that there is a Reinhardt cardinal κ_0 as witnessed by j, and that there is an ω-huge cardinal above $\kappa_\omega(j)$. Let $\lambda_0 = \kappa_\omega(j)$. Let κ_1 be the least ω-huge cardinal above λ_0 and let λ_1 be the least ordinal such that there exists $k \in \mathscr{E}(V_{\lambda_1})$ witnessing that κ_1 is ω-huge (so $\lambda_1 = \kappa_\omega(k)$). Since κ_1 and λ_1 are definable from λ_0, and since $j(\lambda_0) = \lambda_0$, we have that $j(\kappa_1) = \kappa_1$ and $j(\lambda_1) = \lambda_1$.

By Theorem 8.3 there is a transitive class $N \subseteq V_{\lambda_1}$ and a parameter $X \in V_{\lambda_1}$ such that

(1) $N \models \text{ZFC}$,

(2) N is Σ_2-definable in V_{λ_1} from X, and

(3) There is a partial order $\mathbb{P} \in N$ such that for all sets of ordinals $A \in V_{\lambda_1}$ there is an N-generic filter $G \subseteq \mathbb{P}$ such that $A \in N[G]$.

Let φ be a Σ_2-formula defining some such N from some such parameter X. Let $\alpha \geqslant \lambda_0$ be the least ordinal such that there exists a parameter $X \in V_\alpha$ such that φ defines an N over V_{λ_1} from X which meets conditions (1)–(3) where condition (3) is met for some $\mathbb{P} \in V_\alpha$. Pick such an $X \in V_\alpha$ and let N be the model defined over V_{λ_1} with φ using X. The ordinal α is definable from λ_0 and so $j(\alpha) = \alpha$. Now, in V_{λ_1} there is a proper class of supercompact cardinals, namely, the cardinals of the critical sequence $\langle k^n(\kappa_1) : n < \omega \rangle$. In V_{λ_1} let λ be the limit of the first ω-many supercompact cardinals above α. Notice $\lambda < \lambda_1$ since $V_{\lambda_1} \models \mathrm{ZF}$. And notice that since λ is definable from λ_1 and α, we have that $j(\lambda) = \lambda$.

In V_{λ_1} we have our Σ_2-definable inner model N satisfying ZFC and (3) with $\mathbb{P} \in V_\alpha$. Notice that by this version of (3) we have

$$\lambda^+ = (\lambda^+)^N.$$

Since N satisfies ZFC, there exists a set of ordinals $A \in N$ such that $L[A]$ codes up $(V_{\lambda+1})^N$. It follows that

$$\lambda^+ = (\lambda^+)^{(L[A])^{V_{\lambda_1}}}.$$

We can now apply Theorem 8.2 inside V_{λ_1} to get that there does not exist a $j \in \mathscr{E}(V_{\lambda+2})$. On the other hand, since $j(\lambda) = \lambda$ we have $j \restriction V_{\lambda+2} \in \mathscr{E}(V_{\lambda+2})$, which is a contradiction. □

Corollary 8.1. *Assume that the Weak* HOD *Conjecture holds. Then there cannot be a Berkeley cardinal.*

Proof. This is an immediate consequence of Theorem 3.7 and Theorem 8.4. □

In short, the Weak Ultimate-L Conjecture (and even the Weak HOD Conjecture) wipes out almost all of the hierarchy of large cardinals beyond choice, by provably showing that they are inconsistent. Although it is not presently known that it rules out Reinhardt cardinals alone, as we have just seen it rules out Reinhardt cardinals in the presence of standard large cardinals (such as a proper class of ω-huge cardinals) and so it arguably rules out Reinhardt cardinals as genuine large cardinals. In any case, it rules out super Reinhardt cardinals, Berkeley cardinals, and everything beyond.

For this reason, someone who is convinced that the Weak Ultimate-L Conjecture will hold—say, on the basis of developments in inner model theory—might wonder about the utility of our project. For if the first future holds then we have been investigating a series of large cardinal axioms that are inconsistent. Nevertheless, even if the choiceless large cardinals turn out to be inconsistent, we still think that there is some utility in investigating them since it raises the prospect of obtaining "deeper" inconsistency proofs, something we shall now describe.

There have been many purported proofs of the inconsistency of large cardinal axioms. For example, quite frequently one finds purported proofs of the inconsistency of measurable cardinals. And, lower down, it has even been claimed—for example, by Edward Nelson—that PA (and even the weaker PRA) is inconsistent. These proofs have not stood the test of time. The only inconsistency proofs that have stood the test of time are rather simple ones, like Kunen's proof that Reinhardt cardinals are inconsistent (with AC). But this raises an interesting question: Are there "deeper" inconsistency proofs?[20]

[20]We would like to stress the need for the scare quotes around "deeper." We do not wish to give the impression that we think that the notion of one proof being "deeper" than another is a completely clear one. And we do not pretend to have an analysis of this notion, one that provides clear and precise, necessary and sufficient conditions. (It is rarely the case that such an analysis can be given for a non-mathematical notion—even for such everyday notions as the notion of a "chair"—but nevertheless such notions have utility and can even admit of clear cases.) We are relying on the reader's understanding of what it would take to have a relatively clear case of an inconsistency proof that is "deeper" than the ones that we presently have—just as, for example, it seems clear the proof of Fermat's last theorem is "deeper" than the proof that there are infinitely many primes (something that amounts to more than its merely being a *longer* proof.)

Now, since the proof of an inconsistency is going to be easier to find the stronger the assumption, instead of trying to show that PA is inconsistent or that measurable cardinals are inconsistent (which we do not believe) it makes sense to start much higher, with some outlandishly strong hypothesis, show that it is inconsistent, and then work one's way down.

The first future holds the promise of unearthing "deeper" inconsistency proofs since at the moment it seems likely that the surest way to show that Berkeley cardinals are inconsistent is to prove the Weak Ultimate-L Conjecture or the Weak HOD Conjecture, and the proofs of these conjectures are very likely going to be much "deeper" than Kunen's inconsistency proof.

But instead of waiting for the Weak Ultimate-L Conjecture of the Weak HOD Conjecture to be proved, it is of interest to formulate even stronger large cardinal notions and see whether they can be shown to be inconsistent in a relatively straightforward manner. Perhaps in doing this one will gain insight into the weaker large cardinal notions and find more involved proofs of their inconsistency. And, in the end, one might gain insight into the Weak HOD Conjecture. What has happened so far is that the hierarchy of large cardinals beyond choice has *not* led to notions that are readily shown to be inconsistent. Indeed the only plausible scenario we have for the inconsistency of *any* large cardinal beyond choice is through the Weak Ultimate-L Conjecture. This is not to say that it won't happen. It is just to say that it hasn't happened yet.[21]

In any case, the hope in the first future is this: The study of large cardinals beyond choice leads to significant, new inconsistency results, and this in turn leads to a deeper study of inconsistency, certifying consistency from below, via more and more involved supporting considerations, and falsifying consistency from above, via more and more involved considerations, thereby probing the border between the consistent and the inconsistent large cardinal

[21] There are ways in which it could happen. Here is one: Suppose that one could show in ZF that the least Berkeley cardinal must be regular. Then, in light of the results on the possible cofinalities of the least Berkeley cardinal, this would show that Berkeley cardinals are inconsistent.

axioms.

To summarize: The first future is the future where, to begin with, the Weak Ultimate-L Conjecture holds. Assume that traditional large cardinal axioms hold; in particular, assume that there is an extendible cardinal κ with a huge cardinal above. In this future:

(1) The Weak HOD Conjecture holds and so we are on the first side of the HOD Dichotomy, where HOD is "close" to V.

(2) There is a weak extender model N of the supercompactness of κ such that
$$N \models \text{“}V = \text{Ultimate-}L\text{.”}$$

(3) This model, Ultimate-L, is itself "close" to V. It correctly computes successors of singular cardinals above κ and it absorbs all traditional large cardinals above κ.

(4) The model Ultimate-L satisfies

   (a) CH,

   (b) The Ω Conjecture,

   (c) $V = \text{HOD}$, and

   (d) V is the minimum of the Generic Multiverse.

(5) Reinhardt cardinals are inconsistent (assuming there is a proper class of ω-huge cardinals) and Berkeley cardinals are inconsistent.

(6) Assuming that the proof of the Weak Ultimate-L Conjecture is (as seems virtually certain) "deeper" than the proof of Kunen's theorem, we will have obtained a "deeper" inconsistency result.

This is the future in which *pattern* prevails.

The above consequences follow from the truth of this Σ^0_1-statement along with traditional large cardinal axioms. But one can speculate further about how this future might unfold.

The inner model Ultimate-L will quite likely, just like L, admit of a complete analysis, and, given that Ultimate-L is provably "close" to V, this analysis will give us great insight into V itself. But one could go further. For we will have reached an L-like "paradise of understanding" which resembles L in being completely understood but which differs from L in that no traditional large cardinal axiom can reveal it to be illusory, since it will be "close" to V regardless of which traditional large cardinals exist in V. One might use this feature as a basis for a case that V is actually equal to Ultimate-L. To strengthen that case one might begin to "verify" certain consequences of $V =$ Ultimate-L by proving them from traditional large cardinal axioms. And, going further, one might strengthen the case further by showing that $V =$ Ultimate-L is recoverable from some of its more intrinsically plausible consequences (in parallel to the situation with $\mathrm{AD}^{L(\mathbb{R})}$ and its intrinsically plausible consequences). It would take us too far afield to explore the details of this scenario. The point is that although this future is already known to have profound consequences for our understanding of V, there is much more potential beyond the six points above.

Suppose that this development of the first future leads to a justification of $V =$ Ultimate-L. Let "LCA" stand for the non-precisely specifiable, open-ended sequence of large cardinal axioms, and suppose that these axioms are justified. Then:

(7) The theory 'ZFC + $V =$ Ultimate-L + "LCA"' would arguably provide us with a correct notion of "absolute provability."

The axiom $V =$ Ultimate-L would erase instances of "horizontal independence" (that is, statements like CH which are shown to be independent via the method of forcing and do not involve an increase in consistency strength) and the open-ended sequence "LCA" would capture the instances of "vertical independence" (that is, sentences statements that are independent because they have high consistency strength). In this further unfolding of the first future, in addition to having *pattern*, the search for new axioms would be reduced to discovering stronger and stronger true large cardinal axioms and, through this process, every undecided statement would in principle be resolvable.

8.2 The Second Future: The Road to a Higher Analogue of $0^{\#}$

The second future is the future where, to begin with, Berkeley cardinals are consistent (a Π^0_1 fact).

It might appear that there is an asymmetry between the conditions on which each future is conditioned. For the first future is conditioned on a Σ^0_1-statement (the Weak Ultimate-L Conjecture), while the second is conditioned on a Π^0_1-statement (the consistency of a Berkeley cardinal). This apparent asymmetry seems significant since the virtue of a Σ^0_1-statement—in contrast to a Π^0_1-statement of high consistency strength—is that it can admit of a definitive verification. But the asymmetry is merely apparent since the Weak Ultimate-L Conjecture is vacuous if ZFC+ "there is an extendible cardinal with a huge cardinal above" is inconsistent, and so the first future is implicitly conditioned on a Π^0_1-statement of high consistency strength.

If large cardinals beyond choice are *consistent* then, as we have seen, the Weak HOD Conjecture fails, and hence the Ultimate-L Conjecture fails, and this constitutes an anti-inner model theorem. We should stress that the result isn't about just a single inner model—Ultimate-L. It is much more general. Inner model theory proceeds in a general setting, by assuming a large cardinal hypothesis and then showing that one can build a canonical inner model for that large cardinal. The failure of the Weak HOD Conjecture would show that even if one assumes that there is an extendible cardinal κ with a huge cardinal above it (or even much stronger large cardinal hypotheses) then one cannot show that there is a weak extender model N of the supercompactness of κ such that $N \models$ "$V = \mathrm{HOD}$". It is hard to see what meaning there could be to inner model theory if it cannot be executed even at this level of generality. So, in this scenario, not only would the Weak Ultimate-L Conjecture be false, but we would also very likely be faced the failure of inner model theory and the failure of the prospect of ever having detailed, fine-structural insight into V. This is the future in which *chaos* prevails.

Finally, if large cardinals beyond are consistent then we are led to a whole host of very basic questions that appear to be intractable. We have already mentioned some such questions, which concern the cofinality of the least Berkeley cardinal. But there are many others. For example:

Question 7. Which of the following propositions is stronger:

(1) There is a non-trivial elementary embedding $j : V \to V$ and an extendible cardinal above $\kappa_\omega(j)$.

(2) There is a non-trivial elementary embedding $j : V \to V$ and an extendible cardinal below $\mathrm{crit}(\kappa)$.

This is the kind question about large cardinals that is typically readily answered but in the choiceless setting such answers are not forthcoming; indeed, the answers seem to be beyond the reach of current technology. And so, in the second future, there is a whole host of statements that could very well be "absolutely undecidable."

The consistency of large cardinals beyond choice does not strictly speaking imply that we are on the second side of the HOD Dichotomy, but it opens the way to a higher analogue of $0^\#$, which *does* imply that we are on the second side of the HOD Dichotomy, where HOD is "far" from V.

For a given inner model N of ZFC one can consider relativized versions of the choiceless large cardinals in the context of ZFC. For example, a cardinal κ is *N-Reinhardt* if there exists a non-trivial elementary embedding $j : N \to N$

with $\text{crit}(j) = \kappa$; a cardinal κ is *N-super Reinhardt* if for all γ there exists a non-trivial elementary embedding $j : N \to N$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \gamma$; a cardinal δ is an *N-Berkeley cardinal* if for all transitive sets $M \in N$ such that $\delta \in M$, and for every ordinal $\eta < \delta$, there exists $j \in \mathscr{E}(M)$ with $\eta < \text{crit}(j) < \delta$; and so on.

In the case where $N = L$ all of the notions collapse—they are all equivalent to the statement "$0^\#$ exists"; in particular, the existence of an L-Berkeley cardinal is equivalent to the existence of $0^\#$. Let us focus on the other extreme, the case where $N = \text{HOD}$. These are the "HOD-analogues" of the choiceless large cardinals. The HOD-analogues, if consistent, would constitute a whole new hierarchy of traditional large cardinal axioms, all formulated in the context of ZFC. The point is that one can use the choiceless large cardinals to obtain the consistency of the HOD-analogues by applying the forcing construction used to prove Theorem 8.1.[22]

The relevance of this to our present discussion is that these new large cardinals would provide us with a higher analogue of $0^\#$. For just as the existence of an L-Berkeley cardinal (equivalently, the existence of $0^\#$) implies that we are on the second side of the L Dichotomy, the existence of a HOD-Berkeley cardinal provides implies that we are on the second side of the HOD Dichotomy:

Theorem 8.5. *Assume* ZFC *and that there is an extendible cardinal. Suppose that there is a* HOD*-Berkeley cardinal. Then the second side of the* HOD *Dichotomy holds, where* HOD *is "far" from V.*

In fact, one does not have to appeal to the HOD Dichotomy Theorem or the existence of an extendible to get this conclusion.

[22] It is of interest to ask whether this can be reversed—that is, whether one can obtain models of the choiceless large cardinals from the HOD-analogues. (Perhaps the hierarchy of HOD-analogues is cofinal (in terms of consistency strength) in the choiceless hierarchy, and perhaps it provides a vantage point from which one might gain insight into the choiceless hierarchy.)

Theorem 8.6. *Assume* ZFC*. Suppose that* δ *is a* HOD*-Berkeley cardinal. Then every regular cardinal* $\gamma > \delta$ *is* ω*-strongly measurable in* HOD*.*

In other words, if there is a HOD-Berkeley cardinal then HOD is "far" from V.

In this development of the second future "$V =$ Ultimate-L" would be a higher analogue of "$V = L$" in the following sense: Just as the axiom "$V = L$" is a limiting principle in that while it is consistent with "small" large cardinals, it is obliterated by large cardinal axioms at the level of L-Berkeley cardinals and beyond, so too the axiom "$V =$ Ultimate-L" would be a limiting principle in that while it is consistent with traditional large cardinal axioms, it is obliterated by large cardinal axioms at the level of HOD-Berkeley cardinals and beyond.

There is an even further, more radical way in which the second future might unfold, though we admit that it is rather farfetched. One might go beyond accepting the HOD-analogues and actually accept the corresponding choiceless large cardinals and thereby reject AC. The advocate of this development would view AC as a limiting principle, on part with the view of "$V = L$" as a limiting principle much like $V = L$—just as "$V = L$" is violated by modest large cardinals, so too, the view would maintain, AC is violated by much stronger large cardinals. It is hard to see how one could make a case for such a radical shift. There would have to be an extrinsic case for choiceless large cardinals that was so strong it outweighed the case for AC. It should be stressed that we are not endorsing this development as one that is plausible. We mention it only for completeness.

To summarize: The second future is the future where, to begin with, large cardinals beyond choice—in particular, Berkeley cardinals—are consistent. In this future:

(1) The Weak HOD Conjecture fails.

(2) The Weak Ultimate-L Conjecture fails.

(3) Inner model theory as we know it fails. And it is very likely that inner model theory in general fails and there can be no fine-structural insight into V.

(4) There are basic statements concerning large cardinals beyond choice that are completely out of reach of current technology and are good candidates for "absolutely undecidable" statements.

If one assumes further that the HOD-analogues exist, then:

(5) The second side of the HOD Dichotomy holds: HOD is "far" from V.

This is the future where *chaos* prevails.

We don't know which future will unfold—whether pattern or chaos will prevail. But either way, it's going to be interesting.

References

Cummings, J. (2010). Iterated forcing and elementary embeddings, *in* A. Kanamori & M. Foreman (eds), *Handbook of Set Theory*, Vol. 2, Springer, pp. 775–884.

Devlin, K. J. & Jensen, R. (1975). Marginalia to a theorem of Silver, *Proceedings of the ISILC logic conference (Kiel 1974)*, Vol. 499 of *Lecture Notes in Mathematics*, Springer, Berlin, pp. 115–142.

Goldberg, G. (2017). On the consistency strength of Reinhardt cardinals. Unpublished.

Hayut, Y. & Karagila, A. (2016). Restrictions on forcings that change cofinalities, *Archive for Mathematical Logic* **55**: 373–84.

Kanamori, A. (2003). *The Higher Infinite: Large Cardinals in Set Theory from their Beginnings*, Springer Monographs in Mathematics, second edn, Springer, Berlin.

Kunen, K. (1971). Elementary embeddings and infinitary combinatorics, *Journal of Symbolic Logic* **36**: 407–413.

Reinhardt, W. (1967). *Topics in the Metamathematics of Set Theory*, PhD thesis, University of California, Berkeley.

Silver, J. H. (1966). *Some applications of model theory to set theory*, PhD thesis, University of California at Berkeley.

Woodin, H. (2014). The Weak Ultimate-L Conjecture, *in* S. Geschke, B. Loewe & P. Schlicht (eds), *Infinity, Computability, and Metamathematics*, Vol. 23 of *Tributes*, College Publications, London, pp. 309–329.

Woodin, W. H. (2010). Suitable extender models I, *Journal of Mathematical Logic* **10**(1–2): 101–339.

Woodin, W. H. (2011). The Continuum Hypothesis, the generic-multiverse of sets, and the Ω Conjecture, *in* J. Kennedy & R. Kossak (eds), *Set Theory, Arithmetic and the Foundations of Mathematics: Theorems, Philosophies*, Vol. 36 of *Lecture Notes in Logic*, Cambridge University Press, pp. 13–42.

Woodin, W. H. (2017). In search of Ultimate-L: The 19th Midrasha Mathematicae lectures, *Bulletin of Symbolic Logic* **23**(1).