



Joins and Maltsev products of congruence permutable varieties

Clifford Bergman

Dedicated to Ralph Freese, Bill Lampe, and J.B. Nation.

Abstract. Let \mathcal{A} and \mathcal{B} be idempotent varieties and suppose that the variety $\mathcal{A} \vee \mathcal{B}$ is congruence permutable. Then the Maltsev product $\mathcal{A} \circ \mathcal{B}$ is also congruence permutable.

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1. Introduction

A group \mathbf{G} is called an extension of \mathbf{A} by \mathbf{B} if there is a normal subgroup \mathbf{N} of \mathbf{G} such that $\mathbf{N} \cong \mathbf{A}$ and $\mathbf{G}/\mathbf{N} \cong \mathbf{B}$. In a series of papers, Bernard and Hanna Neumann explored the properties of the class \mathcal{AB} of groups, each of which is an extension of a member of the class \mathcal{A} by a member of the class \mathcal{B} . They restricted their attention to the case that both \mathcal{A} and \mathcal{B} are varieties of groups (and \mathcal{AB} is defined to consist only of groups). Among other things, they proved that \mathcal{AB} is again a variety, that $(\mathcal{AB})C = \mathcal{A}(\mathcal{BC})$, and that \mathcal{AB} is locally finite if both \mathcal{A} and \mathcal{B} are locally finite. They also described the set of equations that hold in \mathcal{AB} in terms of those that hold in \mathcal{A} and in \mathcal{B} . A full accounting of their results can be found in [9, Chap. 2].

In [7] A. I. Maltsev considered this construction in a very general context. Among his observations he showed that if \mathcal{A} and \mathcal{B} are quasivarieties of finite similarity type then the Maltsev product, which we denote $\mathcal{A} \circ \mathcal{B}$, is again a quasivariety. Moreover, this product contains both \mathcal{A} and \mathcal{B} . Consequently,

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$\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$ (join in the lattice of quasivarieties). We reproduce Maltsev's construction, specifically for quasivarieties, in Definition 2.1 below.

Unfortunately (and in contrast to the situation for groups), it is not the case that the Maltsev product of two varieties is generally closed under homomorphic images. To address this failure Maltsev introduced a further restriction by requiring his algebras to be *polarized*. A class \mathcal{C} is polarized if there is a unary term that is constant on every member of \mathcal{C} and that constant is an idempotent element of the algebra. This constant is called the *pole* of the algebra. Note that the pole of a group is its identity element, the pole of an algebra is unique (if it exists), and a congruence class of a polarized algebra is a subalgebra if and only if it is the congruence class of the pole. Maltsev proved that if \mathcal{C} is a congruence-permutable polarized variety, then for any two subvarieties \mathcal{A} and \mathcal{B} the class $(\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$ is again a variety.

Recently interest in universal algebra has turned in a somewhat different direction, towards idempotent algebras. It is easy to see that the Maltsev product of two idempotent quasivarieties is again idempotent. Freese and McKenzie [4] consider the preservation of various properties under the product. While they show that a number of important Maltsev conditions are preserved, congruence-permutability is not one of them.

In this short paper we provide some context for this failure. The main result shows that for idempotent varieties \mathcal{A} and \mathcal{B} , if $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable then so is $\mathcal{A} \circ \mathcal{B}$. Combining this with Maltsev's argument described above, if $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable, then $\mathcal{A} \circ \mathcal{B}$ is a variety.

2. Maltsev products

Classes of algebras are always assumed to be of some single fixed similarity type and closed under isomorphic image. A *quasivariety* is a class closed under subalgebra, product, and ultraproduct. Equivalently, under subalgebra and reduced product. See [3, Theorem 2.25]. A quasivariety is a *variety* if it is closed under homomorphic images. For an algebra \mathbf{A} , $\text{Sub}(\mathbf{A})$ denotes the set of subuniverses of \mathbf{A} . For all other unfamiliar notions of universal algebra consult [1].

Since we work sometimes in the lattice of varieties and sometimes in the lattice of quasivarieties, we shall use the notation $\mathcal{A} \vee \mathcal{B}$ for the smallest variety containing $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \vee_{\mathbf{Q}} \mathcal{B}$ for the smallest quasivariety. Note that if \mathcal{A} is a quasivariety then its closure under homomorphic images, $\mathbf{H}(\mathcal{A})$, is the variety generated by \mathcal{A} .

Congruence classes play a double role in the context of Maltsev products: as elements of a quotient algebra and as (potential) subalgebras. It may be helpful to use separate notations for the congruence class of an element a modulo the congruence θ to distinguish these roles. We shall write $[a]_{\theta}$ when this congruence class is being treated as a subset, and continue to write a/θ for the corresponding element of the quotient algebra.

Definition 2.1. Let \mathcal{A} and \mathcal{B} be quasivarieties. The *Maltsev product of \mathcal{A} and \mathcal{B}* is

$$\begin{aligned}\mathcal{A} \circ \mathcal{B} = \{ \mathbf{R} : (\exists \theta \in \text{Con}(\mathbf{R})) \mathbf{R}/\theta \in \mathcal{B} \text{ and} \\ (\forall r \in R) [r]_\theta \in \text{Sub}(\mathbf{R}) \implies [r]_\theta \in \mathcal{A} \}.\end{aligned}$$

If \mathcal{A} and \mathcal{B} are subquasivarieties of the quasivariety \mathcal{C} then we write $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B} = (\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$.

An algebra is called *idempotent* if every singleton subset is a subuniverse. A class of algebras is idempotent if every member algebra is idempotent. Observe that in an idempotent algebra every congruence class $[a]_\theta$ is a subalgebra. We summarize the basic properties of the Maltsev product in the following theorem.

Theorem 2.2. Let \mathcal{A} and \mathcal{B} be quasivarieties.

- (1) If the similarity type is finite, or if \mathcal{B} is idempotent, then $\mathcal{A} \circ \mathcal{B}$ is a quasivariety. Moreover $\mathcal{A} \vee_{\mathbf{Q}} \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$.
- (2) If \mathcal{A} and \mathcal{B} are idempotent then $\mathcal{A} \circ \mathcal{B}$ is idempotent.
- (3) If \mathcal{A} and \mathcal{B} are idempotent subvarieties of a congruence-permutable quasivariety \mathcal{C} then $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety.

In [7] Maltsev proved 2.2(3) under the assumption that \mathcal{C} is polarized rather than idempotent. However the proof is essentially the same in the idempotent case. A proof of Theorem 2.2 is also provided in [2].

3. Congruence-permutability of the Maltsev product

In [4, Example 2.1] Freese and McKenzie exhibit idempotent varieties \mathcal{B}_0 and \mathcal{B}_1 , both of which are congruence-permutable, but their join, $\mathcal{B}_0 \vee \mathcal{B}_1$, fails to be congruence-permutable. It follows from Theorem 2.2(1) that $\mathcal{B}_0 \circ \mathcal{B}_1$ can not be congruence-permutable. As we show in Theorem 3.1, this is the only obstacle to permutability of the Maltsev product.

Theorem 3.1. Let \mathcal{A} and \mathcal{B} be idempotent varieties. If $\mathcal{A} \vee \mathcal{B}$ is congruence permutable, then so is $\mathcal{A} \circ \mathcal{B}$.

Recall that a variety (in fact a quasivariety) is congruence-permutable if and only if there is a ternary term $q(x, y, z)$ (a Maltsev term) such that the equations $q(x, x, y) \approx q(y, x, x) \approx y$ hold. The proof of Theorem 3.1 hinges on the observation that if $\mathcal{A} \vee \mathcal{B}$ is congruence permutable then there is a single term q that simultaneously acts as a Maltsev term on \mathcal{A} and on \mathcal{B} . A key role is played by the following result [5, Lemma 2.8] of Kearnes and Tschantz.

Lemma 3.2. Let \mathcal{W} be an idempotent variety that is not congruence permutable. If $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x, y)$ is the 2-generated free algebra in \mathcal{W} , then \mathbf{F} has subuniverses U and V such that

- (1) $x \in U, y \in V$;
- (2) $y \notin U, x \notin V$, and

(3) $S = (U \times F) \cup (F \times V)$ is a subuniverse of $\mathbf{F} \times \mathbf{F}$.

Proof of Theorem 3.1. Let q be a Maltsev term for $\mathcal{A} \vee \mathcal{B}$. Assume $\mathcal{A} \circ \mathcal{B}$ is not congruence permutable. We shall derive a contradiction. Let $\mathcal{W} = \mathbf{H}(\mathcal{A} \circ \mathcal{B})$. Since \mathcal{A} and \mathcal{B} are idempotent so is \mathcal{W} . Certainly \mathcal{W} is not congruence permutable so we can apply Lemma 3.2 to \mathcal{W} .

So set $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x, y)$. Let U and V be the subuniverses provided by the lemma and $S = (U \times F) \cup (F \times V)$. Since \mathbf{F} is free and $\mathcal{W} = \mathbf{H}(\mathcal{A} \circ \mathcal{B})$, we have $\mathbf{F} \in \mathcal{A} \circ \mathcal{B}$. Hence there is a congruence λ on \mathbf{F} such that $\mathbf{G} = \mathbf{F}/\lambda \in \mathcal{B}$, $\mathbf{X} = [x]_{\lambda} \in \mathcal{A}$ and $\mathbf{Y} = [y]_{\lambda} \in \mathcal{A}$. Of course $x \in X$ and $y \in Y$.

Let $a = (x, x)$, $b = (x, y)$, $c = (y, y)$, and $d = (y, x)$. Note that $a, b, c \in S$ while $d \notin S$. We shall derive a contradiction by showing that, in fact, $d \in S$.

Let $d' = q^{\mathbf{F}^2}(a, b, c) = (p_1, p_2)$. Then $a, b, c \in S$ implies $d' \in S$ as well. From the definition of S we must have either $p_1 \in U$ or $p_2 \in V$. Without loss of generality let us assume that

$$p_2 \in V. \quad (3.1)$$

Now from the definitions of a , b , c , and d' , we have $p_1 = q^{\mathbf{F}}(x, x, y)$. But $\mathbf{G} = \mathbf{F}/\lambda \in \mathcal{B}$ and q is a Maltsev term for \mathcal{B} , hence, $p_1/\lambda = q^{\mathbf{G}}(x/\lambda, x/\lambda, y/\lambda) = y/\lambda$, i.e., $p_1 \lambda y$. Thus

$$p_1 \in Y. \quad (3.2)$$

Similarly, $p_2/\lambda = q^{\mathbf{G}}(x/\lambda, y/\lambda, y/\lambda) = x/\lambda$, so

$$p_2 \in X. \quad (3.3)$$

Now let $e = (x, p_2) \in U \times F \subseteq S$. Define $e' = q^{\mathbf{F}^2}(d', e, a) = (p_3, p_4)$. Then e' is a member of S . As before, $p_3/\lambda = q^{\mathbf{G}}(p_1/\lambda, x/\lambda, x/\lambda) = p_1/\lambda$, so

$$p_3 \in Y. \quad (3.4)$$

From (3.3), $p_2, x \in X$, hence $p_4 = q^{\mathbf{F}}(p_2, p_2, x) = q^{\mathbf{X}}(p_2, p_2, x) = x$ since q is a Maltsev term for $\mathbf{X} \in \mathcal{A}$.

Finally, let $f_1 = (y, p_2)$ and $f_2 = (p_3, p_2)$. Then $f_1, f_2 \in F \times V \subseteq S$ by (3.1). Therefore $q^{\mathbf{F}^2}(f_1, f_2, e') \in S$. But

$$q^{\mathbf{F}^2}(f_1, f_2, e') = (q^{\mathbf{Y}}(y, p_3, p_3), q^{\mathbf{X}}(p_2, p_2, x)) = (y, x) = d$$

proving that $d \in S$. Contradiction. \square

Corollary 3.3. *Let \mathcal{A} and \mathcal{B} be idempotent varieties and suppose that $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable. Then $\mathcal{A} \circ \mathcal{B}$ is variety.*

Proof. Let $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$. By Theorem 2.2(1), \mathcal{C} is a quasivariety, and by Theorem 3.1, it is congruence-permutable. Therefore $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety by Theorem 2.2(3). \square

Corollary 3.4. *Let \mathcal{A} be an idempotent, congruence permutable variety. Then $\mathcal{A} \circ \mathcal{A}$ is congruence permutable. Furthermore $\mathcal{A} \circ \mathcal{A}$ is a variety.*

Unfortunately Theorem 3.1 does not provide a recipe for finding a Maltsev term for $\mathcal{A} \circ \mathcal{B}$ given the term for $\mathcal{A} \vee \mathcal{B}$. We have managed this in one case. Let Sq denote the variety of *squags*. This is the variety of binars defined by the identities

$$x \cdot x \approx x, \quad x \cdot y \approx y \cdot x, \quad x \cdot (x \cdot y) \approx y.$$

This variety is obviously idempotent. It is congruence-permutable with Maltsev term $q(x, y, z) = y \cdot (x \cdot z)$. Therefore by Corollary 3.4, $Sq \circ Sq$ must be a congruence-permutable variety. In [6] Li showed that a Maltsev term for $Sq \circ Sq$ is $p(x, y, z) = (x(z(xy))) \cdot (z(x(zy)))$.

Problem. Find an equational base for $Sq \circ Sq$. Is this variety finitely based?

While we have stated Theorem 3.1 for varieties it could just as easily have been stated for quasivarieties.

Corollary 3.5. *Let \mathcal{A} and \mathcal{B} be idempotent quasivarieties. If $\mathcal{A} \vee_{\mathbf{Q}} \mathcal{B}$ is congruence permutable then $\mathcal{A} \circ \mathcal{B}$ is congruence permutable.*

Proof. Suppose that $\mathcal{C} = \mathcal{A} \vee_{\mathbf{Q}} \mathcal{B}$ is a congruence-permutable quasivariety. Then $\mathcal{W} = \mathbf{H}(\mathcal{C})$ is a congruence-permutable variety. But it is easy to check that $\mathcal{W} = \mathbf{H}(\mathcal{A}) \vee \mathbf{H}(\mathcal{B})$. Therefore by Theorem 3.1, $\mathbf{H}(\mathcal{A}) \circ \mathbf{H}(\mathcal{B})$ is congruence-permutable. Consequently $\mathcal{A} \circ \mathcal{B} \subseteq \mathbf{H}(\mathcal{A}) \circ \mathbf{H}(\mathcal{B})$ is congruence-permutable as well. \square

One wonders whether an analog of Theorem 3.1 holds for other conditions that are not preserved by Maltsev products. For example, if \mathcal{A} and \mathcal{B} are idempotent varieties and $\mathcal{A} \vee \mathcal{B}$ has a majority term, must $\mathcal{A} \circ \mathcal{B}$ have a majority term? We do not know the answer, although an affirmative answer is consistent with our current understanding of the subject. Certainly a proof will look quite different from that of Theorem 3.1 since no congruence-distributive analog of the Kearnes–Tschantz lemma is known.

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References

- [1] Bergman, C.: Universal Algebra. Fundamentals and Selected Topics, Pure and Applied Mathematics, vol. 301. CRC Press, Boca Raton (2012)
- [2] Bergman, C.: Notes on quasivarieties and Maltsev products (2014). <https://faculty.sites.iastate.edu/cbergman/files/inline-files/maltsevprods.pdf>. Accessed Feb 2020
- [3] Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra. Springer, New York (1981). <http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html>. Accessed Feb 2020

- [4] Freese, R., McKenzie, R.: Maltsev families of varieties closed under join or Maltsev product. *Algebra Univers.* **77**(1), 29–50 (2017). <https://doi.org/10.1007/s00012-016-0420-1>
- [5] Kearnes, K.A., Tschantsz, S.T.: Automorphism groups of squares and of free algebras. *Int. J. Algebra Comput.* **17**(3), 461–505 (2007). <https://doi.org/10.1142/S0218196707003615>
- [6] Li, J.: Congruence n -permutable varieties. Ph.D. thesis, Iowa State University (2017). Graduate Theses and Dissertations. 15355
- [7] Mal'cev, A.I.: Multiplication of classes of algebraic systems. *Sib. Math. J.* **8**, 254–267 (1967). (Translated in [8])
- [8] Mal'cev, A.I.: The Metamathematics of Algebraic Systems. Collected papers: 1936–1967. North-Holland Publishing Co., Amsterdam (1971). (Translated, edited, and provided with supplementary notes by Benjamin Franklin Wells, III, *Studies in Logic and the Foundations of Mathematics*, vol. 66)
- [9] Neumann, H.: Varieties of Groups. Springer, Berlin (1967)

Clifford Bergman
Department of Mathematics
Iowa State University
Ames IA50011
USA
e-mail: cbergman@iastate.edu
URL: <https://faculty.sites.iastate.edu/cbergman>

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