Jacobians with prescribed eigenvectors

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathfrak{R}$ be a partial frame on Ω; that is, a set of m linearly independent vector fields prescribed on Ω ($m \leq n$). We consider the issue of describing the set of all maps $F : \Omega \to \mathbb{R}^n$ with the property that each of the given vector fields is an eigenvector of the Jacobian matrix of F. By introducing a coordinate independent definition of the Jacobian, we obtain an intrinsic formulation of the problem, which leads to an overdetermined PDE system, whose compatibility conditions can be expressed in an intrinsic, coordinate independent manner. To analyze this system we use Darboux and generalized Frobenius integrability theorems. The size and structure of the solution set of this system depends on the properties of the partial frame; in particular, whether or not it is in involution. A particularly nice subclass of involutive partial frames, called *rich* frames, can be completely analyzed. The involutive, non-rich case is somewhat harder to handle. We provide a complete answer in the case of $m = 3$ and arbitrary n, as well as some general results for arbitrary m. The non-involutive case is far more challenging, and we only obtain a comprehensive analysis in the case $n = 3$, $m = 2$. Finally, we provide explicit examples illustrating the various possibilities.

Keywords: Jacobian matrix and map; affine connections; prescribed eigenvectors; integrability theorems; conservative systems; hyperbolic fluxes.
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1 Introduction

The present work deals with the construction of maps $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ whose Jacobian matrix has a partially prescribed set of eigenvector fields on Ω. We consider this problem locally, i.e., in a sufficiently small neighborhood of a given point in Ω. The case when the full frame of n independent eigenvectors is prescribed has been considered in [7]. The generalization to a partially prescribed set of eigenvector fields allows a greater degree of flexibility in constructing such maps F and, in particular, it allows us to include maps F whose Jacobian matrix is not diagonalizable. Another difference from the previous work is that all the overdetermined systems of PDEs arising in the current paper are analyzed using smooth[1] integrability theorems, including a recently proved generalization of the Frobenius theorem (see Section 3.3). This theorem allows us to remain in the smooth category, while in [7] we appealed in some cases to the Cartan-Kähler theorem, which requires analyticity.

Our motivation stems from the study of initial value problems for one dimensional conservative systems of the form

$$u_t + F(u)_x = 0, \qquad u(0, x) = u_0(x), \tag{1}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}$ are the independent variables, $u = u(t, x) \in \mathbb{R}^n$ is a vector of unknowns, and the flux function F is defined on some open set in $\mathbb{R}^n$ and takes values in $\mathbb{R}^n$. One approach

[1]We employ C^1 integrability theorems, but to avoid technicalities C^∞-smoothness is assumed throughout.

for constructing solutions to (1) exploits Riemann problems. These are initial value problems for (1) with data of the form

$$u_0(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0, \end{cases} \tag{2}$$

where $u^\pm$ are constant vectors. Lax [8] provided the general form of such solutions. The seminal work of Glimm [6] used Riemann solutions as local building blocks to prove global-in-time existence of weak solutions, provided the initial data $u_0(x)$ have sufficiently small total variation. For detailed accounts of this theory see [14, 3, 4]. It is largely an open problem to extend the Glimm theory to systems of physical interest beyond the regime of small variation solutions.

Solutions to Riemann problems depend essentially on the eigen-structure (i.e., eigenvalues and eigenvectors) of the Jacobian $DF(u)$. It is then a basic question to what extent one can prescribe some or all of the eigenvectors of the flux F, regarding the eigenvalues as unknowns. The present work is concerned with this last, purely geometric, problem.

A precise formulation of the problem is provided in Section 2. This first formulation, "Problem 1," makes use of a chosen coordinate system. Section 3 provides the geometric framework required to obtain a coordinate-free formulation. We also state an integrability theorem due to Darboux and a generalization of the Frobenius integrability theorem, which we use in this paper. In Section 4, we give an intrinsic (coordinate independent) definition of the Jacobian, and use it to reformulate Problem 1 in an intrinsic manner (see Problem 2). Exploiting the coordinate independent formulation, Section 5 treats the case when the prescribed vector fields are in involution, and, in particular, the case of rich partial frames. In the involutive case, the relevant integrability conditions lead to a closed algebro-differential system for the unknown eigenvalues. Section 6 analyzes the simplest non-involutive case of two prescribed vector fields in $\mathbb{R}^3$. Finally, Section 7 provides a list of examples that illustrate the results from the earlier sections.

2 Problem formulation

Let $[D_u\Psi]$ denote the Jacobian matrix of a map Ψ from an open subset $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$, relative to coordinates u, i.e.

$$[D_u\Psi] = \left[\frac{\partial \Psi^i}{\partial u^j}\right]_{i,j=1,\dots,n}.$$

We use the notation $[D_u\Psi]|_{u=\bar{u}}$, or simply $[D_u\Psi]|_{\bar{u}}$, when the matrix is evaluated at a point $\bar{u}$. We consider the following problem:

Problem 1. *Given an open set $\Omega \subset \mathbb{R}^n$ on which we fix a coordinate system $u = (u^1, \dots, u^n)$ and a point $\bar{u} \in \Omega$. Let $\mathfrak{R} = \{R_1, \dots, R_m\}$ be a set of $m \leq n$ smooth vector valued functions $R_i : \Omega \to \mathbb{R}^n$ which are linearly independent at $\bar{u}$. Then: describe the set $\mathcal{F}(\mathfrak{R})$ of all smooth vector-valued functions*

$$F(u) = [F^1(u), \dots, F^n(u)]^T$$

defined near $\bar{u}$ and with the property that $R_1(u), \dots, R_m(u)$ are right eigenvectors of the Jacobian matrix $[D_uF]|_u$ throughout a neighborhood of $\bar{u}$. In other words, we ask that there exist smooth, scalar functions λ^i such that

$$[D_uF]|_u R_i(u) = \lambda^i(u) R_i(u), \qquad i = 1, \dots, m, \tag{3}$$

holds on a neighborhood of $\bar{u}$.

As noted above, we are motivated by the construction of flux functions F in systems of conservation laws of the form (1). The system (1) is *hyperbolic* on Ω provided the Jacobian matrix $[D_uF]$ has a basis of real eigenvectors at each $u \in \Omega$, and it is *strictly hyperbolic* if all its eigenvalues are distinct at each $u \in \Omega$. We adopt the term *flux* for a vector-function satisfying (3), with adjectives *hyperbolic, strictly hyperbolic or non-hyperbolic* depending on the structure of eigenvectors and eigenvalues of $[D_uF]$, as described above. $\mathcal{F}(\mathfrak{R})$ will be called the *flux space.*

Next, we clarify the meaning of "describe" in Problem 1 and make some basic observations.

1. (*PDE system*) Equations (3) comprise a system of mn first order PDEs for $n+m$ unknown functions λ^i and F^j:

$$\sum_{k=1}^{n} R_i^k \frac{\partial F^j}{\partial u^k} = \lambda^i R_i^j \qquad \text{for } i = 1, \dots, m,\ j = 1, \dots, n, \tag{4}$$

   where $R_i(u) = [R_i^1(u), \dots, R_i^n(u)]^T$, $i = 1, \dots, m$. For all $n \geq m$, such that $n > 2$ and $m \geq 2$, this system is overdetermined with (as we show below) non-trivial compatibility conditions. Although derivatives of λ^i do not appear in the equations, these functions are not arbitrary parameters, but must, in turn, satisfy certain differential equations arising as differential consequence of (4).

2. (*Vector space structure*) Let $F_1, F_2 \in \mathcal{F}(\mathfrak{R})$ have domains of definitions Ω_1 and Ω_2, respectively. As $\bar{u}$ belongs to both Ω_1 and Ω_2, then $\Omega_1 \cap \Omega_2$ is a non-empty open neighborhood of $\bar{u}$. It is easy to check that for any real numbers a, b, the linear combination $aF_1 + bF_2$, defined on $\Omega_1 \cap \Omega_2$, belongs $\mathcal{F}(\mathfrak{R})$. Thus $\mathcal{F}(\mathfrak{R})$ is a *vector space* over $\mathbb{R}$. We will see below that in some instances this is a finite dimensional vector space, while in others it is an infinite dimensional space. In the latter case, we describe the "size" of $\mathcal{F}(\mathfrak{R})$ in terms of the number of arbitrary functions of a certain number of variables appearing in the general solution of (4). These arbitrary functions prescribe the values of F and λ's along certain submanifolds of Ω. To obtain these results we use the integrability theorem stated in Section 3.3.

3. (*Trivial solutions*) For any choice of $\bar{\lambda} \in \mathbb{R}$ and $\bar{f} \in \mathbb{R}^n$, the "trivial" flux

$$F(u) = \bar{\lambda} u + \bar{f} \tag{5}$$

   satisfies (3). The set of such trivial solutions, denoted by $\mathcal{F}^{\text{triv}}$, is an $(n+1)$-dimensional vector subspace of $\mathcal{F}(\mathfrak{R})$.

4. (*Triviality is generic*) For $n > 2$ and $m \geq 2$, the compatibility conditions for system (4) imply that the non-trivial fluxes exist only for partial frames that satisfy some non-trivial algebra-differential equations and, therefore, a generic frame admits only trivial fluxes. One of the goals of the paper is to determine the properties of the frames that allow them to possess non-trivial, and, in particular, strictly hyperbolic, fluxes.

5. (*Scaling invariance*) Since eigenvectors are defined up to *scaling*, it is clear that

$$\mathcal{F}(R_1, \dots, R_m) = \mathcal{F}(\alpha^1 R_1, \dots, \alpha^m R_m) \tag{6}$$

   for any nowhere zero smooth functions α^i on Ω.

The next remarks address the coordinate dependence of Problem 1.

Remark 2.1 (Coordinate dependence of the problem formulation)**.** *Assume $F(u) \in \mathcal{F}(\mathfrak{R})$ for $\mathfrak{R} = \{R_1, \dots, R_m\}$, i.e., there exist $\lambda^1(u), \dots, \lambda^m(u)$, such that system (3) is satisfied. Let a change of variables be given by a local diffeomorphism $u = \Phi(w)$. It is then not true, in general, that $\tilde{F}(w) = F(\Phi(w))$ belongs to $\mathcal{F}(\tilde{\mathfrak{R}})$, where $\tilde{\mathfrak{R}} = \{\tilde{R}_1(w), \dots, \tilde{R}_m(w)\}$, with $\tilde{R}_i(w) = R_i(\Phi(w))$. Indeed,*

$$[D_w (F \circ \Phi)]\, \tilde{R}_i = [D_uF]|_{u=\Phi(w)} [D_w\Phi]\, R_i(\Phi(w)).$$

In general, $R_i(\Phi(w))$ is not an eigenvector of $[D_uF]|_{u=\Phi(w)}[D_w\Phi]$. Furthermore, if we transform each $R_i(u)$, by treating them, more appropriately, as vector fields, viz.

$$R_i^*(w) = [D_w\Phi]^{-1}\, R_i(\Phi(w)),$$

then

$$\begin{aligned}[D_w\,(F\circ\Phi)]\,R_i^* &= [D_uF]|_{u=\Phi(w)}[D_w\Phi][D_w\Phi]^{-1}\,R_i(\Phi(w))\\ &= [D_uF]|_{u=\Phi(w)}R_i(\Phi(w))\\ &= \lambda^i(\Phi(w))R_i(\Phi(w)) = \lambda^i(\Phi(w))\,[D_w\Phi]R_i^*(w),\end{aligned}$$

and we see that $R_i^*(w)$ is not an eigenvector of $[D_w\,(F\circ\Phi)]$, unless it is an eigenvector of $[D_w\Phi]$.

Remark 2.2 (Coordinate dependence of the property of a matrix being a Jacobian matrix)**.** *Assume $A(u) = [D_uF]$ for some smooth map $F:\Omega\to\mathbb{R}^n$, and let a change of coordinates to be given by a diffeomorphism $u = \Phi(w)$. Then it is not necessarily the case that the matrix $A(\Phi(w))$ is a Jacobian matrix of any map in w coordinates.*

On the other hand, it is still possible to give a coordinate independent definition of the Jacobian linear map, as we do in Section 4.1. This coordinate independent definition is used to formulate a coordinate independent version (Problem 2), which, when expressed in an *affine system of coordinates* (see Section 3.2), coincides with Problem 1. This intrinsic formulation allows us to apply a geometric approach to analyze the solution set of the PDE system (4). We exploit this by working in frames that are adapted to the problem at hand, and we use the following geometric preliminaries.

3 Geometric preliminaries

Most of the material in Sections 3.1 and 3.2 can be found in standard differential geometry textbooks. We include it to set up notation and to make the paper self-contained. In Section 3.3, we state an integrability theorem due to Darboux and a generalization of the Frobenius integrability theorem, which are repeatedly useed in the paper to analyze overdetermined systems of PDEs.

3.1 Partial frames, involutivity, richness

As usual we identify a smooth vector field $\mathbf{r}$ on an open subset Ω pf $\mathbb{R}^n$ with a derivation, i.e. with $\mathbb{R}$-linear map from the set of smooth functions $C^\infty(\Omega)$ to itself that satisfies the product rule. The set of all smooth vector fields is denoted as $\mathcal{X}(\Omega)$.

Definition 1 (Partial frame)**.** *A set of smooth vector fields $\{\mathbf{r}_1,\dots,\mathbf{r}_m\}$ on $\Omega\subset\mathbb{R}^n$, with $m\le n$, is a* partial frame *on Ω if they are independent at all $\bar u\in\Omega$. If $m = n$, the set is called a* frame*.*

For a fixed system $u^1,\dots,u^n$ of coordinate functions on Ω, the frame $\left\{\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n}\right\}$ of partial derivatives is called a *coordinate frame*. We shall see below that using non-coordinate frames simplifies our problem. Non-coordinate frames do not commute. The commutator of two vector fields is called a Lie bracket: for $\mathbf{r}_1,\mathbf{r}_2\in\mathcal{X}(\Omega)$, their Lie bracket is the map $C^\infty(\Omega)\to C^\infty(\Omega)$ defined by

$$[\mathbf{r}_1,\mathbf{r}_2]\phi = \mathbf{r}_1(\mathbf{r}_2(\phi)) - \mathbf{r}_2(\mathbf{r}_1(\phi)).$$

Straightforward calculations show that $[\mathbf{r}_1,\mathbf{r}_2]$ is a vector field, obviously the Lie bracket is skew-symmetric, and one can check that Jacoby identity is satisfied. Therefore, $\mathcal{X}(\Omega)$ has the structure of an infinite-dimensional real Lie algebra.

For a frame $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$, one can express the Lie bracket of every pair of vector fields from $\mathfrak{R}$ as a linear combination:

$$[\mathbf{r}_j, \mathbf{r}_k] = \sum_{l=1}^{n} c^l_{jk} \mathbf{r}_l, \tag{7}$$

where c^l_{jk}, satisfying $c^l_{jk} = -c^l_{kj}$, are some smooth functions on Ω, called *structure coefficients*. In the conservation laws literature, these functions are called *interaction coefficients* because of their role in wave interaction formulas [6]. Equations (7) are called *structure equations*. The Jacobi identity implies the relationships, that will be used later

$$\mathbf{r}_l\big(c^i_{jk}\big) + \mathbf{r}_k\big(c^i_{lj}\big) + \mathbf{r}_j\big(c^i_{kl}\big) + \sum_{s=1}^{n} \big(c^s_{jk} c^i_{ls} + c^s_{lj} c^i_{ks} + c^s_{kl} c^i_{js}\big) = 0 \qquad 1 \le i, j, k, l \le n. \tag{8}$$

Below we give definitions of partial frames with especially nice properties.

Definition 2 (Commutative partial frame)**.** *A partial frame* $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ *is* commutative *if* $[\mathbf{r}_i, \mathbf{r}_j] = 0$ *for all* $\mathbf{r}_i, \mathbf{r}_j \in \mathfrak{R}$.

Definition 3 (Involutive partial frame)**.** *A partial frame* $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ *is* in involution *if* $[\mathbf{r}_i, \mathbf{r}_j] \in \operatorname{span}_{C^\infty(\Omega)} \mathfrak{R}$ *for all* $\mathbf{r}_i, \mathbf{r}_j \in \mathfrak{R}$.

The proof of the following proposition is contained in the proof of Theorem 6.5 in [15].

Proposition 3.1. *If* $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ *is a partial frame in involution on* Ω*, then there is a commutative partial frame* $\{\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_m\}$ *on some open* $\Omega' \subset \Omega$ *with*

$$\operatorname{span}_{\mathbb{R}}\{\mathbf{r}_1|_u, \ldots, \mathbf{r}_m|_u\} = \operatorname{span}_{\mathbb{R}}\{\tilde{\mathbf{r}}_1|_u, \ldots, \tilde{\mathbf{r}}_m|_u\} \text{ for all } u \in \Omega'.$$

Proposition 3.2. (Theorem 5.14 in [15]) *If* $\mathbf{r}_1, \ldots, \mathbf{r}_m$ *is a commutative partial frame on* Ω*, then in a neighborhood of each point* $\bar{u} \in \Omega$ *there exist coordinate functions* $v^1, \ldots, v^n$*, such that*

$$\mathbf{r}_i = \frac{\partial}{\partial v^i}, \quad i = 1, \ldots, m.$$

Definition 4 (Rich frame)**.** *A partial frame* $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ *is* rich *if every pair of its vector fields is in involution, i.e.,* $[\mathbf{r}_i, \mathbf{r}_j] \in \operatorname{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\}$ *for all* $i, j = 1, \ldots, m$.

Lemma 5.6 below shows that every rich partial frame $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ can be scaled to become a commutative frame. Thus, near each point there are coordinates $w^1, \ldots, w^n$ and non-zero functions $\alpha^1, \ldots, \alpha^n$, such that $\alpha^i \mathbf{r}_i = \frac{\partial}{\partial w^i}$, $i = 1, \ldots, m$. A conservative system (1) is called rich if there are coordinate functions, called *Riemann invariants*, in which the system is diagonalizable. For definitions, and the fact that richness of a conservative system is equivalent to the richness of its eigenframe in the sense above, we refer to [12], and Section 7.3 in [4]. Riemann invariants are exactly the coordinates appearing in Lemma 5.6 in the case of full frame ($n = m$). The term *rich* refers to a large family of extensions (companion conservation laws) that strictly hyperbolic diagonalizable systems possess [4, 12].

3.2 Connections, symmetry, flatness, affine coordinates

To give a coordinate free definition of the Jacobian, we will use the notion of connection, which we briefly recall here. A *connection* ∇ on Ω is an $\mathbb{R}$-bilinear map

$$\nabla\colon \mathcal{X}(\Omega) \times \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \qquad\qquad (\mathbf{r}, \mathbf{s}) \mapsto \nabla_{\mathbf{r}} \mathbf{s}$$

such that for any smooth function ϕ on Ω

$$\nabla_{\phi\,\mathbf{r}} \mathbf{s} = \phi \nabla_{\mathbf{r}} \mathbf{s} \;\text{ and }\; \nabla_{\mathbf{r}}(\phi\,\mathbf{s}) = \mathbf{r}(\phi)\,\mathbf{s} + \phi \nabla_{\mathbf{r}} \mathbf{s}\,. \tag{9}$$

The vector field $\nabla_{\mathbf{r}}\mathbf{s}$ is called the covariant derivative of $\mathbf{s}$ in the direction of $\mathbf{r}$. Given a connection ∇ and a frame $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$, we write

$$\nabla_{\mathbf{r}_i}\mathbf{r}_j = \sum_{k=1}^{n} \Gamma_{ij}^{k}\mathbf{r}_k, \qquad 1 \le i, j \le n, \tag{10}$$

where the smooth functions Γ_{ij}^k are called *Christoffel symbols*. Conversely, by $\mathbb{R}$-bilinearity and (9), any choice of a frame and n^3 functions Γ_{ij}^k, $1 \le i, j, k \le n$, defines a connection via (10).

A connection ∇ is *symmetric* if for all $\mathbf{r}, \mathbf{s} \in \mathcal{X}(\Omega)$:

$$\nabla_{\mathbf{r}}\mathbf{s} - \nabla_{\mathbf{s}}\mathbf{r} = [\mathbf{r}, \mathbf{s}]. \tag{11}$$

A connection ∇ is *flat* if for all $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \mathcal{X}(\Omega)$:

$$\nabla_{\mathbf{r}}\nabla_{\mathbf{s}}\,\mathbf{t} - \nabla_{\mathbf{s}}\nabla_{\mathbf{r}}\mathbf{t} = \nabla_{[\mathbf{r},\mathbf{s}]}\mathbf{t}. \tag{12}$$

The above conditions are equivalent to the following relationships among the structure coefficients and Christoffel symbols relative to an *arbitrary* frame: for all $i, j, k, s = 1, \ldots, n$,

$$\Gamma_{ij}^k - \Gamma_{ji}^k = c_{ij}^k \qquad \text{Symmetry} \tag{13}$$

$$\mathbf{r}_s\left(\Gamma_{ki}^j\right) - \mathbf{r}_k\left(\Gamma_{si}^j\right) = \sum_{l=1}^{n}\left(\Gamma_{kl}^j\Gamma_{si}^l - \Gamma_{sl}^j\Gamma_{ki}^l - c_{ks}^l\Gamma_{li}^j\right) \qquad \text{Flatness.} \tag{14}$$

Given a connection ∇, a coordinate system such that all Christoffel symbols of ∇ relative to the corresponding coordinate frame vanish, is called *affine*. The following is a well known result (compare, for instance, with Proposition 1.1 of [13]):

Proposition 3.3. *A connection ∇ on an n-dimensional manifold M is symmetric and flat if and only if M can be covered with an atlas of affine coordinate systems. Let Ω be an open subset of a manifold M with a flat and symmetric connection ∇ and an affine coordinate system $v = (v^1, \ldots, v^n)$. Then $w = (w^1, \ldots, w^n)$ is another affine coordinate system on Ω if and only if $[w^1, \ldots, w^n]^T = C\,[v^1, \ldots, v^n]^T + \bar{b}$, where u and w are treated as column vectors, $C \in \mathbb{R}^{n\times n}$ is an $n \times n$ invertible matrix and $\bar{b} \in \mathbb{R}^n$ is a constant vector.*

Throughout the paper, we will use a particular connection, denoted $\widetilde{\nabla}$, defined by setting all Christoffel symbols to be zero, relative to the coordinate frame corresponding to the coordinate system $u^1, \ldots, u^n$ fixed in Problem 1:

$$\widetilde{\nabla}_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = 0 \qquad \text{for all } i, j = 1, \ldots, n. \tag{15}$$

However, our coordinate-free formulation of the problem in Section 4.1, makes sense for general connections.

3.3 Integrability theorems

To analyze the "size" of the flux space $\mathcal{F}(\mathfrak{R})$ in Problem 1, we shall use two integrability results: a theorem due to Darboux and a generalized Frobenius Theorem.

In his monograph "Systèmes Orthogonaux" [5], Darboux stated three theorems concerning local existence and uniqueness of solutions to first order systems of PDEs of a certain type. The most general of those is Theorem III in Book III, Chapter I. This theorem considers a system for p unknown functions of n variables, where a subset of partial derivatives is prescribed for each unknown function. The subset of derivatives prescribed for one of the unknowns may differ from the subset prescribed for another. We refer to such systems as *Darboux systems*. The theorem states that provided the natural integrability conditions are satisfied, there is a unique solution for appropriately prescribed initial data. Below the theorem is stated in the smooth case considered in this paper. However, the result is true in the C^1 case as well.

Theorem 3.4. *(Darboux [5]) Let $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^p$ be open sets, and let $\bar{u} = (\bar{u}^1, \dots, \bar{u}^n) \in \Omega$. For each $i = 1, \dots, p$, let $I_i \subseteq \{1, \dots, n\}$, and assume h^i_j, for $j \in I_i$, are given smooth functions on $\Omega \times \Theta$. Consider the following system of differential equations for unknown functions $(\phi^1, \dots \phi^p)$ defined on Ω:*

$$\frac{\partial \phi^i}{\partial u^j} = h^i_j(u, \phi), \qquad j \in I_i,\, i = 1, \dots, p. \tag{16}$$

Assume (16) prescribes compatible second order mixed derivatives in the following sense:

- *(C) Whenever two distinct derivatives $\frac{\partial \phi^i}{\partial u^j}$ and $\frac{\partial \phi^i}{\partial u^k}$ of the same unknown ϕ^i are present on the left hand side of (16), the equation*

$$\frac{\partial}{\partial u^k}\left[h^i_j(u, \phi(u))\right] = \frac{\partial}{\partial u^j}\left[h^i_k(u, \phi(u))\right]$$

  contains (after expanding each side using the chain rule) only first order derivatives which appear in (16), and substitution from (16) for these first derivatives results in an identity in u and ϕ.

Next, to describe the data, suppose a dependent variable ϕ^i appears differentiated in (16) with respect to $u^{j_1}, \dots, u^{j_s}$. Then, letting $\tilde{u}$ denote the remaining independent variables, we prescribe a smooth function $g^i(\tilde{u})$ and require that

$$\phi^i(u^1, \dots, u^n)\big|_{u^{j_1} = \bar{u}^{j_1}, \dots,\, u^{j_s} = \bar{u}^{j_s}} = g^i(\tilde{u})\,. \tag{17}$$

We make such an assignment for each ϕ^i that appears differentiated in (16). Then, under the compatibility condition (C), the problem (16)-(17) has a unique, local smooth solution near $\bar{u}$.

In [5], Darboux proved the theorem for the case of $n = 2$ and $n = 3$ only. In [2], we formulated and proved a generalized version of the Darboux theorem for an arbitrary number n of independent variables. Our result in [2] generalizes Darboux's theorem in two ways:

(i) The unknown functions may be differentiated along vector fields in a fixed frame $\mathcal{R} = \{\mathbf{r}_i\}_{i=1}^n$ defined near $\bar{u}$. That is, for each $i = 1, \dots, m$, there is an index set $I_i \subseteq \{1, \dots, n\}$ such that the system contains the equations

$$\mathbf{r}_j(\phi^i)\big|_u = f^i_j(u, \phi(u)) \qquad \text{for each } j \in I_i. \tag{18}$$

As in the original Darboux's theorem, the elements and cardinality of the index sets I_i may vary with i.

(ii) The prescribed data g^i for the unknown ϕ^i may be given along a manifold Ξ_α through the point $\bar{u}$ which is transverse to the vector fields $\mathbf{r}_j$ with $j \in I_i$.

In [2], we show that under the appropriate conditions the PDE system (18) has a unique local solution which takes on the assigned data. For the current paper we only need a specific case of this generalized Darboux theorem, where a PDE system on p functions of n variables prescribes derivatives of each unknown function in the directions of *the same set* of $m \leq n$ vector fields comprising an involutive partial frame. We refer to such systems as *generalized Frobenius systems*, because when $m = n$, this theorem is equivalent to the PDE version of the well known Frobenius theorem (Theorem 6.1 in [15]). The generalized Frobenius theorem stated below claims that under the natural integrability conditions, there is a unique local solution to a generalized Frobenius systems with initial data prescribed along any m-dimensional manifold transversal to the given partial frame.

Theorem 3.5 (Generalized Frobenius: PDE version)**.** *Let $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be a partial frame in involution on an open subset $\Omega \subset \mathbb{R}^n$ with coordinates $(u^1, \ldots, u^n)$. Let $\Theta \subset \mathbb{R}^p$ be an open subset with coordinates $(\phi^1, \ldots, \phi^p)$. Let h^i_j, $i = 1, \ldots, p$, $j = 1, \ldots, m$, be given smooth functions on $\Omega \times \Theta$. Consider the system of differential equations*

$$\mathbf{r}_j(\phi^i(u)) = h^i_j(u, \phi(u)) \qquad i = 1, \ldots, p;\ j = 1, \ldots, m. \tag{19}$$

Assume the integrability conditions

$$\mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) - \mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) = \sum_{l=1}^{m} c^l_{jk}\mathbf{r}_l(\phi^i) \qquad i = 1, \ldots, p;\ j, k = 1, \ldots, m, \tag{20}$$

where the structure coefficients c^l_{jk} are as in (7) (with n replaced by m), are satisfied identically on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\mathbf{r}_j(\phi^i(u))$ as prescribed by (19)[2]*. Then for any point $\bar{u} \in \Omega$ and for any smooth initial data prescribed along any embedded submanifold $\Xi \subset \Omega$ of codimension m containing $\bar{u}$ and transversal*[3] *to $\mathfrak{R}$, there is a unique smooth local solution of* (19)*.*

For the future use we expand conditions (20). After the first substitution of the derivatives of ϕ as prescribed by (19) into (20), we get for $i = 1, \ldots, p$ and $j, k = 1, \ldots, m$ that

$$\mathbf{r}_j\left(h^i_k(u, \phi(u))\right) - \mathbf{r}_k\left(h^i_j(u, \phi(u))\right) = \sum_{l=1}^{m} c^l_{jk}\, h^i_l\left(u, \phi(u)\right). \tag{21}$$

Applying the chain rule and again substituting according to (19), we obtain

$$\sum_{l=1}^{n}\Big(\frac{\partial h^i_k(u,\phi)}{\partial u^l}\mathbf{r}_j(u^l) - \frac{\partial h^i_j(u,\phi)}{\partial u^l}\mathbf{r}_k(u^l)\Big) + \sum_{s=1}^{p}\Big(\frac{\partial h^i_k(u,\phi)}{\partial \phi^s}\, h^s_j(u,\phi) - \frac{\partial h^i_j(u,\phi)}{\partial \phi^s}\, h^s_k(u,\phi)\Big)$$
$$= \sum_{l=1}^{m} c^l_{jk}(u)\, h^i_l(u,\phi). \tag{22}$$

As we mentioned above, the generalized Frobenius theorem is a particular case of the generalized Darboux Theorem proven in [2] [4]. A direct proof via Picard iteration can be found in thesis [1] of the first author. A weaker version of Theorem 3.5 (with right hand-sides of (19) independent of ϕ's) appears in Lee [9], Theorem 19.27.

Remark 3.6. *If the same partial derivatives are prescribed for all unknowns (i.e., $I_1 = \cdots = I_p$), the Darboux system (16) is a generalized Frobenius system. Conversely, using Propositions 3.1 and 3.2, one can show that for any generalized Frobenius system there is an equivalent Darboux system, with all partial derivatives of all unknown functions prescribed for the same set of coordinate directions. In this case, the integrability conditions (C) of Theorem 3.4 are equivalent to the integrability conditions in Theorem 3.5. However, the manifold Ξ along which the initial data is allowed to be prescribed in Theorem 3.5 is more general than the coordinate subspace for which the data is prescribed in Theorem 3.4.*

[2] The resulting equations, explicitly written down as (22), involve no derivatives of ϕ.

[3] Here transversality means that $\mathrm{span}_{\mathbb{R}}\{\mathbf{r}_1|_{\bar{u}}, \ldots, \mathbf{r}_m|_{\bar{u}}\} \oplus T_{\bar{u}}\Xi = \mathbb{R}^n$ at very point $\bar{u} \in \Xi$, where $T_{\bar{u}}\Xi$ denotes the tangent space to Ξ at $\bar{u}$.

[4] An additional geometric condition on the data manifolds and the partial frame, called the stable configuration condition (SCC) in the hypothesis of the generalized Darboux theorem in [2], is trivially satisfied in the particular case of Theorem 3.5.

4 Intrinsic formulation of the problem

In this section, we give an intrinsic (coordinate independent) formulation of Problem 1, which leads to a system of differential equations written in terms of the frame adapted to the problem. We derive some differential consequences of this system, which, in particular, lead to a set of necessary conditions for the existence of strictly hyperbolic fluxes.

4.1 Intrinsic definition of the Jacobian and the $\mathcal{F}(\mathfrak{R})$-system

Definition 5. *Given a connection ∇ on a smooth manifold M, the* ∇-Jacobian *of a vector field $\mathbf{f} \in \mathcal{X}(M)$ is the $C^\infty(M)$-linear map $J\mathbf{f}\colon \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ defined by:*

$$J\mathbf{f}(\mathbf{s}) = \nabla_{\mathbf{s}}\mathbf{f}, \quad \forall \mathbf{s} \in \mathcal{X}(\Omega). \tag{23}$$

If $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ is a frame with Christoffel symbols Γ^k_{ij} and $\mathbf{f} = \sum_{i=1}^n F^i\,\mathbf{r}_i$, then (23) implies

$$J\mathbf{f}(\mathbf{r}_j) = \sum_{i=1}^{n} \Big(\mathbf{r}_j(F^i) + \sum_{k=1}^{n} \Gamma^i_{jk}\, F^k\Big)\, \mathbf{r}_i. \tag{24}$$

To arrive to a coordinate independent formulation of Problem 1, let us return to the coordinate system $(u^1, \ldots, u^n)$, fixed in this problem, and consider the flat and symmetric connection $\widetilde{\nabla}$ defined by (15). Write out $\mathbf{f} = \sum_{i=1}^n \tilde{F}^i(u)\frac{\partial}{\partial u^i}$ in the coordinate frame and let $\tilde{J}\mathbf{f}$ denote the $\widetilde{\nabla}$-Jacobian of $\mathbf{f}$. Then a direct computation shows that

$$\tilde{J}\mathbf{f}\Big(\frac{\partial}{\partial u^j}\Big) = \sum_{i=1}^{n} \frac{\partial \tilde{F}^i}{\partial u^j}\, \frac{\partial}{\partial u^i},$$

which corresponds to the j-th column vector of the usual Jacobian matrix $[D_u\,\tilde{F}]$ of the vector valued function $\tilde{F}(u) = [\tilde{F}^1(u), \ldots, \tilde{F}^n(u)]^T$. Let R_i be the column vectors of the components of $\mathbf{r}_i$ in the u-coordinates. Observing that the right-hand side of (3) in Problem 1 can be rewritten as $\Big[D_u\tilde{F}\Big]\Big|_u R_i(u) = \tilde{J}\mathbf{f}(\mathbf{r}_i) = \widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f}$, we formulate the following problem:

Problem 2. *Assume $\widetilde{\nabla}$ is a flat and symmetric connection on $\Omega \subset \mathbb{R}^n$ with $\bar{u} \in \Omega$. Given a partial frame $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ on Ω, describe the set $\mathcal{F}(\mathfrak{R})$ of smooth vector fields $\mathbf{f}$ for which there exist an open neighborhood $\Omega' \subset \Omega$ of $\bar{u}$ and smooth functions $\lambda^i\colon \Omega' \to \mathbb{R}$ such that*

$$\widetilde{\nabla}_{\mathbf{r}_i}\,\mathbf{f} = \lambda^i\,\mathbf{r}_i \qquad \text{on } \Omega' \text{ for } i = 1, \ldots, m. \tag{25}$$

From Proposition 3.3, we know that any flat and symmetric connection admits an affine system of coordinates. If $F^1, \ldots, F^n$ are the components of $\mathbf{f}$, and $R^1_i, \ldots, R^n_i$ are the components of $\mathbf{r}_i$ in an affine system of coordinates, then (25) turns into a system of mn first order PDE's on $n + m$ unknown functions F's and λ's:

$$\mathbf{r}_i(F^j) = \lambda^i\,R^j_i, \quad \text{for } i = 1, \ldots, m,\ j = 1, \ldots, n, \tag{26}$$

which is equivalent to (3). Therefore, Problem 2 is equivalent to Problem 1.

We call system (25) the $\mathcal{F}(\mathfrak{R})$-system. The set of vector fields $\mathbf{f}$ satisfying (25) is called the *flux set* and is denoted $\mathcal{F}(\mathfrak{R})$; its elements are called *fluxes* for $\mathfrak{R}$. The flux set always includes the set of *identity* fluxes $\mathcal{F}^{\text{id}}$ defined by the property that

$$\widetilde{\nabla}_{\mathbf{r}}\mathbf{f} = \mathbf{r} \qquad \text{for all vector fields } \mathbf{r} \in \mathcal{X}(\Omega). \tag{27}$$

It is easy to show that $\hat{\mathbf{f}} \in \mathcal{F}^{\text{id}}$ if and only if relative to any affine coordinates system $(u^1, \ldots, u^n)$

$$\hat{\mathbf{f}} = [u^1, \ldots, u^n]^T + \bar{b}, \text{ for some } \bar{b} \in \mathbb{R}^n.$$

The previously defined vector space of trivial fluxes (5), in this more abstract setting, corresponds to the vector space

$$\mathcal{F}^{\mathrm{triv}} = \{\mathbf{f} \in \mathcal{X}(\Omega) \,|\, \forall \mathbf{r} \in \mathcal{X}(\Omega)\, \exists \bar{\lambda} \in \mathbb{R} \text{ such that } \widetilde{\nabla}_{\mathbf{r}}\mathbf{f} = \bar{\lambda}\mathbf{r}\}. \tag{28}$$

Equivalently, we have $\mathcal{F}^{\mathrm{triv}} = \{\bar{\lambda}\,\hat{\mathbf{f}} \,|\, \bar{\lambda} \in \mathbb{R},\, \hat{\mathbf{f}} \in \mathcal{F}^{\mathrm{id}}\}$. Clearly, $\mathcal{F}^{\mathrm{id}} \subset \mathcal{F}^{\mathrm{triv}} \subset \mathcal{F}(\mathfrak{R})$ for any partial frame $\mathfrak{R}$.

Remark 4.1. *Both equations* (25) *and Problem 2 makes sense if we replace* $\mathbb{R}^n$ *with an arbitrary manifold* M*, and replace* $\widetilde{\nabla}$ *with an arbitrary connection on the tangent bundle of* M*. In particular, it would be of interest to consider this problem on a Riemannian manifold with the Riemannian connection. These generalizations, however, fall outside of the scope of the present paper.*

4.2 Differential consequences of the $\mathcal{F}(\mathfrak{R})$-system

We next derive the differential consequences of (25) implied by the flatness of the connection.

Proposition 4.2. *Given a partial frame* $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$*, assume that* $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$ *is a flux, and* $\mathbf{s}_1, \ldots, \mathbf{s}_{n-m}$ *is any completion of* $\mathfrak{R}$ *to a full frame. Let the functions* a_k^l *be defined by*

$$\widetilde{\nabla}_{\mathbf{s}_l}\mathbf{f} = \sum_{k=1}^{m} a_l^k \mathbf{r}_k + \sum_{k=m+1}^{n} a_l^k\, \mathbf{s}_t, \qquad l = 1, \ldots, n-m. \tag{29}$$

Then the functions λ^i*,* $i = 1, \ldots, m$*, prescribed by (25), and the functions* a_k^l*,* $l = 1, \ldots, n-m$*,* $k = 1, \ldots, n$ *satisfy the following system of differential and algebraic equations:*

$$\mathbf{r}_i(\lambda^j) = \Gamma_{ji}^j\,(\lambda^i - \lambda^j) + \sum_{l=m+1}^{n} a_l^j\, c_{ij}^l \qquad \textit{for all } 1 \le i \ne j \le m \tag{30}$$

$$\lambda^j\,\Gamma_{ij}^k - \lambda^i\,\Gamma_{ji}^k - c_{ij}^k \lambda^k = \sum_{l=m+1}^{n} a_l^k\, c_{ij}^l \qquad \textit{for all distinct triples } i, j, k \in \{1, \ldots, m\} \tag{31}$$

$$\lambda^j\,\Gamma_{ij}^l - \lambda^i\,\Gamma_{ji}^l = \sum_{t=m+1}^{n} a_t^l\, c_{ij}^t \qquad \textit{for all } 1 \le i \ne j \le m \textit{ and } l = m+1, \ldots, n. \tag{32}$$

In the above equations, the functions c_{ij}^k *and* Γ_{ij}^k *are the structure functions and the Christoffel symbols for the full frame* $\mathfrak{R} \cup \{\mathbf{s}_1, \ldots, \mathbf{s}_{n-m}\}$*.*

$$[\mathbf{r}_i, \mathbf{r}_j] \;=\; \sum_{k=1}^{m} c_{ij}^k \mathbf{r}_k + \sum_{l=m+1}^{n} c_{ij}^l\, \mathbf{s}_l \tag{33}$$

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \;=\; \sum_{k=1}^{m} \Gamma_{ij}^k \mathbf{r}_k + \sum_{l=m+1}^{n} \Gamma_{ij}^l\, \mathbf{s}_l. \tag{34}$$

Proof. The flatness condition (12) implies that

$$\widetilde{\nabla}_{\mathbf{r}_i}\widetilde{\nabla}_{\mathbf{r}_j}\mathbf{f} - \widetilde{\nabla}_{\mathbf{r}_j}\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f} = \widetilde{\nabla}_{[\mathbf{r}_i,\mathbf{r}_j]}\mathbf{f} \quad \text{ for all } i, j = 1, \ldots, m \tag{35}$$

must hold for the solutions of (25). Therefore,

$$\mathbf{r}_i(\lambda^j)\,\mathbf{r}_j + \lambda^j\,\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j - \mathbf{r}_j(\lambda^i)\,\mathbf{r}_i - \lambda^i\,\widetilde{\nabla}_{\mathbf{r}_j}\mathbf{r}_i = \widetilde{\nabla}_{[\mathbf{r}_i,\mathbf{r}_j]}\mathbf{f}. \tag{36}$$

According to (33) and (34), (36) is equivalent to:

$$\mathbf{r}_i(\lambda^j)\,\mathbf{r}_j + \sum_{k=1}^{m} \lambda^j\,\Gamma_{ij}^k \mathbf{r}_k + \sum_{l=m+1}^{n} \lambda^j \Gamma_{ij}^l\,\mathbf{s}_l - \mathbf{r}_j(\lambda^i)\,\mathbf{r}_i - \sum_{k=1}^{m} \lambda^i\,\Gamma_{ji}^k \mathbf{r}_k - \sum_{l=m+1}^{n} \lambda^i \Gamma_{ji}^l\,\mathbf{s}_l$$
$$= \sum_{k=1}^{m} c_{ij}^k \widetilde{\nabla}_{\mathbf{r}_k}\,\mathbf{f} + \sum_{l=m+1}^{n} c_{ij}^l\;\widetilde{\nabla}_{\mathbf{s}_l}\,\mathbf{f}. \tag{37}$$

It remains to rewrite the right-hand side of (37) in terms of the frame using (25) for the first sum and (29) for the second sum:

$$\mathbf{r}_i(\lambda^j)\,\mathbf{r}_j + \sum_{k=1}^{m} \lambda^j\,\Gamma_{ij}^k \mathbf{r}_k + \sum_{l=m+1}^{n} \lambda^j \Gamma_{ij}^l\,\mathbf{s}_l - \mathbf{r}_j(\lambda^i)\,\mathbf{r}_i - \sum_{k=1}^{m} \lambda^i\,\Gamma_{ji}^k \mathbf{r}_k - \sum_{l=m+1}^{n} \lambda^i \Gamma_{ji}^l\,\mathbf{s}_l$$
$$= \sum_{k=1}^{m} c_{ij}^k \lambda^k\,\mathbf{r}^k + \sum_{k=1}^{m}\sum_{l=m+1}^{n} a_l^k\,c_{ij}^l \mathbf{r}_k + \sum_{l,t=m+1}^{n} a_l^t\,c_{ij}^l \mathbf{s}_t.$$

Collecting coefficients of the frame vector fields, we obtain equations (30)–(32). □

We emphasize that, in general, the structure functions c_{ij}^k and the Christoffel symbols Γ_{ij}^k appearing in (30)–(32), depend on the completion of $\mathfrak{R}$ to a full frame.

Remark 4.3. *We note that (30)–(32) do not provide a complete set of integrability conditions for the Frobenius system* (25), (29)*, because they do not include conditions derived from* $\widetilde{\nabla}_{\mathbf{r}_i}\widetilde{\nabla}_{\mathbf{s}_j}\mathbf{f} - \widetilde{\nabla}_{\mathbf{s}_j}\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f} = \widetilde{\nabla}_{[\mathbf{r}_i,\mathbf{s}_j]}$ *and* $\widetilde{\nabla}_{\mathbf{s}_i}\widetilde{\nabla}_{\mathbf{s}_j}\mathbf{f} - \widetilde{\nabla}_{\mathbf{s}_j}\widetilde{\nabla}_{\mathbf{s}_i}\mathbf{f} = \widetilde{\nabla}_{[\mathbf{s}_i,\mathbf{s}_j]}$. *We will derive these additional conditions in Section 6 for the case $m = 2$, $n = 3$ only, and we shall observe how involved they are already in this case. On the other hand, we will see in Section 5, that if $\mathfrak{R}$ is an* involutive *partial frame, then (30)-(32) simplify to a system which involves only the unknown functions λ^i, and this system does provide a complete set of integrability conditions for* (25)*. In the case of the full frame ($m = n$), equations (30)-(32) reduce to the λ-system studied in [7].*

We can use (30)-(32) to get necessary conditions for $\mathcal{F}(\mathfrak{R})$ to contain a strictly hyperbolic flux. We shall see below that these conditions are not sufficient except for rich, partial frames.

Proposition 4.4. *Let $\mathfrak{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ be a partial frame on $\Omega \subset \mathbb{R}^n$ containing $\bar{u}$. If there is a strictly hyperbolic flux $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$ on some open neighborhood Ω' of $\bar{u}$, then for each pair of distinct indices $i, j \in \{1, \dots, m\}$ the following equivalence condition holds*

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \in \mathrm{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \qquad \textit{if and only if} \qquad [\mathbf{r}_i, \mathbf{r}_j] \in \mathrm{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \tag{38}$$

Proof. If $\mathbf{f}$ is strictly hyperbolic on Ω', then $\mathfrak{R}$ can be completed to a frame of eigenvectors $\mathbf{r}_1, \dots, \mathbf{r}_m$, $\mathbf{r}_{m+1}, \dots, \mathbf{r}_n$, such that there exist functions $\lambda^1, \dots, \lambda^n \colon \Omega' \to \mathbb{R}$, with pairwise distinct values at each point of Ω', and

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f} = \lambda^i \mathbf{r}_i, \quad i = 1, \dots, n.$$

In the statement of Proposition 4.2, let $\mathbf{s}_l = \mathbf{r}_l$ for $l = m+1, \dots, n$. Then $a_l^i = \delta_l^i \lambda^l$, where δ_l^i is the Kronecker delta, and the algebraic conditions (31), (32) become

$$\Gamma_{ij}^k\,\lambda^j - \Gamma_{ji}^k \lambda^i\; - c_{ij}^k \lambda^k = 0 \text{ for all } 1 \le i \ne j \le m \text{ and } 1 \le k \le n, \text{ with } k \ne i \text{ and } k \ne j. \tag{39}$$

Let us first assume that for some i, j, such that $1 \le i \ne j \le m$, we have $\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \in \mathrm{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$ and $[\mathbf{r}_i, \mathbf{r}_j] \notin \mathrm{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$. Then, from the latter condition, there exists $k \in \{1, \dots, n\}$, such that $k \neq i$ and $k \neq j$ and $c_{ij}^k \not\equiv 0$, while the former condition implies that $\Gamma_{ij}^k \equiv 0$. Symmetry of $\widetilde{\nabla}$ implies that $c_{ij}^k = -\Gamma_{ji}^k \not\equiv 0$, and then from (39) we have

$$c_{ij}^k\,(\lambda^i - \lambda^k) \equiv 0.$$

We then have $\lambda^i = \lambda^k$ at least somewhere in Ω', which contradicts strict hyperbolicity.

Let us now assume that for some i, j such that $1 \leq i \neq j \leq m$, we have $\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \notin \operatorname{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$ and $[\mathbf{r}_i, \mathbf{r}_j] \in \operatorname{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$. Then, from the former condition, there exists $k \in \{1, \dots, n\}$, such that $k \neq i$ and $k \neq j$ and $\Gamma_{ij}^k \not\equiv 0$; from the latter condition we have $c_{ij}^k \equiv 0$. Symmetry of $\widetilde{\nabla}$ implies that $\Gamma_{ij}^k = \Gamma_{ji}^k$, and then from (39) we have

$$\Gamma_{ij}^k\,(\lambda^j - \lambda^i) \equiv 0.$$

We then have $\lambda^i = \lambda^j$ at least somewhere in Ω', which contradicts strict hyperbolicity. □

5 Involutive partial frame

As noted in Remark 4.3, the analysis of the $\mathcal{F}(\mathfrak{R})$-system is much simpler when the frame $\mathfrak{R}$ is in involution. The two extreme cases $m = 1$ and $m = n$ fall into this category. In the former case $\mathfrak{R}$ also trivially satisfies the definition of a rich partial frame, see Remark 5.4. The latter case of a full frame was considered in [7], and some of the theorems of the present paper are natural generalizations of those results.

5.1 General involutive partial frames

If the partial frame $\mathfrak{R}$ is in involution, then for any completion of $\mathfrak{R}$ to a full frame $\{\mathbf{r}_1, \dots, \mathbf{r}_m\} \cup \{\mathbf{s}_{m+1}, \dots, \mathbf{s}_n\}$, we have $c_{ij}^l = 0$ for all $i, j = 1, \dots, m$, $l = m+1, \dots, n$ and, therefore, $\Gamma_{ij}^l = \Gamma_{ji}^l$ due to the symmetry of the connection (11). In this case (30)-(32) simplify to

$$\begin{aligned} \mathbf{r}_i(\lambda^j) &= \Gamma_{ji}^j\,(\lambda^i - \lambda^j) \text{ for all } 1 \leq i \neq j \leq m && (40)\\ \lambda^j\,\Gamma_{ij}^k - \lambda^i\,\Gamma_{ji}^k - c_{ij}^k \lambda^k &= 0 \text{ for all distinct triples } i, j, k \in \{1, \dots, m\} && (41)\\ (\lambda^j - \lambda^i)\,\Gamma_{ji}^l &= 0 \text{ for all } 1 \leq i \neq j \leq m \text{ and } l = m+1, \dots, n. && (42) \end{aligned}$$

where the functions c_{ij}^k and Γ_{ij}^k are given by (33) and (34). Note that, due to involutivity of $\mathfrak{R}$, the functions c_{ij}^k, $i, j, k \in \{1, \dots, m\}$ do not depend on the choice of completion of $\mathfrak{R}$ to a frame, while the Christoffel symbols, in general, do depend on the choice of such completion. We call (40)-(42) the λ-*system*, generalizing the terminology of [7] to partial involutive frames.

The following proposition allows us, in the involutive case, to solve Problems 2 (and 1) in two steps: first find all solutions λ^i of system (40)-(42), and then determine all solutions $\mathbf{f}$ of (25) with these functions λ^i. This is possible because (40)-(42) provide a complete set of the integrability conditions for the $\mathcal{F}(\mathfrak{R})$-system (25) in this case, as the proof of the following proposition shows.

Proposition 5.1. *If a partial frame $\mathfrak{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ is in involution, then*

(1) For every $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$, the functions $\lambda^1, \dots, \lambda^m$ prescribed by (25) *satisfy* (40)-(42)*.*

(2) For every solution $\lambda^1, \dots, \lambda^m$ of (40)-(42)*, and any smooth data $\tilde{\mathbf{f}}$ for $\mathbf{f}$ prescribed along any embedded submanifold $\Xi \subset \Omega$ of codimension m and transverse to $\mathfrak{R}$, there is a unique smooth local solution of $\mathcal{F}(\mathfrak{R})$-system* (25) *taking on the given data.*

Proof. (1) Equations (40)-(42) are differential consequences of (25). Therefore, for every $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$, the functions $\lambda^1, \ldots, \lambda^m$ prescribed by (25) satisfy (40)-(42).

(2) Assume $\lambda^1, \ldots, \lambda^m$ are solutions of (40)-(42). In an affine system of coordinates $u = (u^1, \ldots, u^n)$, equations (25) turn into (26). To simplify the notation we write these as

$$r_i(F)|_u = \lambda^i(u)\, R_i(u) \qquad \text{for } i = 1, \ldots, m, \tag{43}$$

where F and R_i are the column vectors of the components of the vector fields $\mathbf{f}$ and $\mathbf{r}_i$, respectively, relative to the coordinate frame $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}$. The system (43) is of the form (19); the integrability conditions are

$$\mathbf{r}_i(\lambda^j)\,(R_j) + \lambda^j\,\mathbf{r}_i(R_j) - \mathbf{r}_j(\lambda^i)\,(R_i) - \lambda^i\,\mathbf{r}_j(R_i) = \sum_{k=1}^{m} c_{ij}^k \lambda^k R_k. \tag{44}$$

Since in an affine coordinate system, the components of $\tilde{\nabla}_{\mathbf{r}_s}\mathbf{r}_l$ are given by the column vector $\mathbf{r}_s(R_l)$, we have that (44) is equivalent to

$$\mathbf{r}_i(\lambda^j)\,\mathbf{r}_j + \lambda^j\,\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j - \mathbf{r}_j(\lambda^i)\,\mathbf{r}_i - \lambda^i\,\widetilde{\nabla}_{\mathbf{r}_j}\mathbf{r}_i = \sum_{k=1}^{m} c_{ij}^k \lambda^k \mathbf{r}_k, \tag{45}$$

which, when written out in components relative to a completion of $\mathfrak{R}$ to a frame $\mathbf{r}_1, \ldots, \mathbf{r}_m$, $\mathbf{s}_{m+1}, \ldots, \mathbf{s}_n$, is equivalent to (40)-(42). The components of the vector field $\tilde{\mathbf{f}}$ provide the data for F and are of the type described in Theorem 3.5. This theorem guarantees the existence of a locally unique solution of (43) with this data. □

The system (40)-(42) always has the trivial solution $\lambda^1 = \cdots = \lambda^m$. However, the existence of other solutions of (40)-(42) is a subtle issue. Furthermore, even when non-trivial solutions of (40)-(42) exist, their (strict) hyperbolicity requires further analysis. We note that conditions (41) and (42) provide us with *necessary* conditions for the existence of strictly hyperbolic solutions for Problem 1, in the case of involutive partial frames.

Proposition 5.2. *If a partial frame $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is in involution on Ω, then the following conditions must be satisfied for all $1 \leq i \neq j \leq m$ on some open neighborhood $\Omega' \subset \Omega$ of $\bar{u}$, in order for the flux set $\mathcal{F}(\mathfrak{R})$ to contain a strictly hyperbolic flux:*

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \in \operatorname{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \Longleftrightarrow [\mathbf{r}_i, \mathbf{r}_j] \in \operatorname{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \tag{46}$$

and

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \in \operatorname{span}_{C^\infty(\Omega')}\mathfrak{R}. \tag{47}$$

Proof. Condition (46) is the same as (38) proved earlier. If for all open subsets $\Omega' \subset \Omega$, there are $1 \leq i \neq j \leq m$, such that $\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \notin \operatorname{span}_{C^\infty(\Omega')}\mathfrak{R}$, then there exists $m+1 \leq l \leq n$, such that $\Gamma_{ij}^l \not\equiv 0$ on Ω'. From (42), it then follows that $\lambda^i = \lambda^j$ at least somewhere on Ω' and therefore $\mathcal{F}(\mathfrak{R})$ contains no strictly hyperbolic fluxes. □

We observe that involutivity implies that $[\mathbf{r}_i, \mathbf{r}_j] \in \operatorname{span}_{C^\infty(\Omega')}\mathfrak{R}$. Thus, due to the symmetry condition (11), we can replace the condition $1 \leq i \neq j \leq m$ in (47) with $1 \leq i < j \leq m$. The above conditions are not sufficient as will be illustrated by Example 5.3 in [7]. However, we can prove the following condition is sufficient.

Proposition 5.3. *Assume that the functions $\lambda^1, \ldots, \lambda^m$ satisfying (40)-(42) are such that for some $\bar{u} \in \Omega$, $\lambda^1(\bar{u}), \ldots, \lambda^m(\bar{u})$ are distinct. Then on an open neighborhood of $\bar{u}$ there exists a strictly hyperbolic flux $\mathbf{f}$, such that*

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f} = \lambda^i\mathbf{r}_i, \quad i = 1, \ldots, m.$$

Proof. Let R_i be the column vector of components of $\mathbf{r}_i$ in an affine system of coordinates $u^1, \dots, u^n$, and let $R = [R_1|\dots|R_m]$ be an $n \times m$ matrix comprised of these column vectors. Since $\mathbf{r}_i$, $i = 1, \dots, m$ are independent at $\bar{u}$, there is a non-zero $m \times m$ minor of $R(\bar{u})$. Due to continuity the same minor is non-zero on some open neighborhood of $\bar{u}$. Let $\{i_1, \dots, i_m\}$ be the row indices of the submatrix corresponding to this minor. Up to permuting coordinate functions $u^1, \dots, u^n$ we may, in order to simplify the notation, assume that $i_j = j$. Then the set of vector fields $\mathbf{r}_1, \dots, \mathbf{r}_m, \frac{\partial}{\partial u^{m+1}}, \dots, \frac{\partial}{\partial u^n}$ are independent and, therefore, a submanifold Ξ defined by $u^i = \bar{u}^i$ for $i = 1, \dots, m$ is transversal to $\mathfrak{R}$.

For $l \in \{m+1, \dots, n\}$, choose arbitrary constants $\bar{\lambda}^l$, such that all n real numbers $\lambda^1(\bar{u}), \dots, \lambda^m(\bar{u}), \bar{\lambda}^{m+1}, \dots, \bar{\lambda}^n$ are distinct. Define

$$\tilde{F}(\bar{u}^1, \dots, \bar{u}^m, u^{m+1}, \dots, u^n) = \left[0, \dots, 0, \bar{\lambda}^{m+1} u^{m+1}, \dots, \bar{\lambda}^n u^n\right]^T$$

and let F be an extension of $\tilde{F}$ such that $[D_u F] \, R_i(u) = \lambda^i \, R_i(u)$, for $i = 1, \dots, m$; the existence of such an extension is guaranteed by Proposition 5.1. Then

$$[D_u F](\bar{u}) = \begin{bmatrix} \frac{\partial F^1}{\partial u^1}(\bar{u}) & \dots & \frac{\partial F^1}{\partial u^1}(\bar{u}) & & \dots & \\ \vdots & \dots & \vdots & & \dots & \\ \frac{\partial F^m}{\partial u^1}(\bar{u}) & \dots & \frac{\partial F^m}{\partial u^m}(\bar{u}) & & \dots & \\ \frac{\partial F^{m+1}}{\partial u^1}(\bar{u}) & \dots & \frac{\partial F^{m+1}}{\partial u^m}(\bar{u}) & \bar{\lambda}^{m+1} & & \\ \vdots & \vdots & \vdots & & \ddots & \\ \frac{\partial F^n}{\partial u^1}(\bar{u}) & \dots & \frac{\partial F^n}{\partial u^m}(\bar{u}) & & & \bar{\lambda}^n \end{bmatrix},$$

where empty spaces are filled with zeros. The matrix $[D_u F]|_{\bar{u}}$ has n distinct real eigenvalues $\lambda^1(\bar{u}), \dots, \lambda^m(\bar{u}), \bar{\lambda}^{m+1}, \dots, \bar{\lambda}^n$. Since the entries of $[D_u F]$ are smooth real functions, a standard argument, involving the implicit function theorem, implies that there is an open neighborhood $\Omega' \subset \Omega$ of $\bar{u}$, such that at every point of Ω' the matrix $[D_u F]$ has n distinct real eigenvalues, and, therefore, F is strictly hyperbolic on Ω'. □

Remark 5.4 (Single vector field case)**.** *When $\mathfrak{R} = \{\mathbf{r}_1\}$, all three conditions* (40)-(42) *trivially hold. Therefore, we can assign λ^1 to be any function on Ω. Then, by Proposition 5.1, for every assignment of the vector field $\tilde{\mathbf{f}}$ on an $(n-1)$-dimensional manifold Ξ, transverse to $\mathbf{r}_1$, there exists a unique local vector field $\mathbf{f}$ such that $\widetilde{\nabla}_{\mathbf{r}_1}\mathbf{f} = \lambda^1 \, \mathbf{r}_1$ and $\mathbf{f}|_\Xi = \tilde{\mathbf{f}}|_\Xi$. Thus, the general solution of the $\mathcal{F}(\mathfrak{R})$-system* (25) *depends on one arbitrary function of n variables (the function λ^1) and n functions of $n-1$ variables, that locally describe the initial data for the vector field $\mathbf{f}$. Due to Proposition 5.3, the $\mathcal{F}(\mathfrak{R})$-set contains strictly hyperbolic fluxes.*

Remark 5.5 (Full frame)**.** *If $\mathfrak{R}$ is a full frame, then* (42) *trivially holds while the remaining equations* (40)-(41) *form the λ-system analyzed in detail in [7]. According to Proposition 5.1, for every solution of the λ-system and for every assignment of the vector $\tilde{\mathbf{f}}$ at a point $\bar{u} \in \Omega$, there exists a locally unique solution $\mathbf{f}$ of* (25) *with a prescribed value for $\mathbf{f}|_{\bar{u}}$.*

5.2 Rich partial frames

Rich frames (see Definition 4) comprise a particularly nice subclass of involutive frames. This case trivially includes all partial frames consisting of a single vector field. It also includes all involutive partial frames consisting of two vector fields.

Let $\{\mathbf{r}_1, \dots, \mathbf{r}_m, \mathbf{s}_{m+1}, \dots, \mathbf{s}_n\}$ be any completion of $\mathfrak{R}$ to a frame. With the same notation as above, since $\mathfrak{R}$ is rich, the symmetry of the connection $\widetilde{\nabla}$ yields

$$c^l_{ij} = 0 \text{ and } \Gamma^l_{ij} = \Gamma^l_{ji} \text{ for all distinct triples } i, j, l, \text{ such that } 1 \le i, j \le m,\ 1 \le l \le n. \qquad (48)$$

The $\mathcal{F}(\mathfrak{R})$-system (40)-(42) thus reduces to

$$\mathbf{r}_i(\lambda^j) = \Gamma^j_{ji}\,(\lambda^i - \lambda^j) \qquad \text{for all } 1 \le i \ne j \le m, \tag{49}$$

$$\Gamma^l_{ij}\,(\lambda^i - \lambda^j) = 0 \qquad \text{for all } 1 \le i < j \le m,\ 1 \le l \le n \text{ with } l \ne i \text{ and } l \ne j. \tag{50}$$

In the rich case, the necessary conditions recorded in Proposition 5.2 for the flux set $\mathcal{F}(\mathfrak{R})$ to contain strictly hyperbolic fluxes, become

$$\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \in \operatorname{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \text{ for all } 1 \le i \ne j \le m. \tag{51}$$

Theorem 5.7 below shows that, for a rich partial frame, these necessary conditions are also sufficient. Moreover, for frames satisfying (51), the theorem describes the size of the set $\mathcal{F}(\mathfrak{R})$. Theorem 5.8 describes the size of the set $\mathcal{F}(\mathfrak{R})$ for partial frames that do not satisfy (51), and therefore, do not admit strictly hyperbolic fluxes.

The following lemma allows us to introduce a coordinate system adapted to a given rich partial frame, and subsequently to invoke Darboux's theorem to describe the flux set $\mathcal{F}(\mathfrak{R})$.

Lemma 5.6. *If a partial frame $\mathfrak{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ on Ω is rich, then in a neighborhood of every point $\bar{u} \in \Omega$ there exist*

(1) strictly positive scalar functions $\alpha^1, \dots, \alpha^m$, such that the vector fields $\tilde{\mathbf{r}}_i = \alpha^i\,\mathbf{r}_i$, $i = 1, \dots, n$ commute, i.e., $[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = 0$ for all $i, j \in \{1, \dots, m\}$;

(2) local coordinate functions $(w^1, \dots, w^n)$ such that $\tilde{\mathbf{r}}_i = \frac{\partial}{\partial w^i}$, $i = 1, \dots, m$.

Proof. For a rich partial frame $\mathfrak{R}$ the following structure equations hold:

$$[\mathbf{r}_i, \mathbf{r}_j] = c^i_{ij}\mathbf{r}_i + c^j_{ij}\mathbf{r}_j \qquad i, j = 1, \dots, m,$$

where the structure functions c^k_{ij} are independent of the completion of $\mathfrak{R}$ to a frame. We will show that the conditions $[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = 0$ lead to a generalized Frobenius system for the α^i. Indeed,

$$\begin{aligned}[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] &= [\alpha^i\mathbf{r}_i, \alpha^j\mathbf{r}_j] = \alpha^i\,\alpha^j\,[\mathbf{r}_i, \mathbf{r}_j] + \alpha^i\,\mathbf{r}_i(\alpha^j)\mathbf{r}_j - \alpha^j\,\mathbf{r}_j(\alpha^i)\mathbf{r}_i \\ &= \alpha^j\left(\alpha^i c^i_{ij} - \mathbf{r}_j(\alpha^i)\right)\mathbf{r}_i - \alpha^i\left(\alpha^j c^j_{ji} - \mathbf{r}_i(\alpha^j)\right)\mathbf{r}_j. \end{aligned} \tag{52}$$

Then $[\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j] = 0$ if and only if $\beta^i = \ln(\alpha^i)$ satisfies the PDE system.

$$\mathbf{r}_j(\beta^i) = c^i_{ij}(u) \text{ for all } 1 \le i \ne j \le m \tag{53}$$

To this system we add the equations

$$\mathbf{r}_j(\beta^j) = 0 \text{ for all } 1 \le j \le m, \tag{54}$$

making an additional requirement that, for each $i = 1, \dots, m$, β^i is constant along the integral curve of $\mathbf{r}_i$. As $c^j_{jj} = 0$, we can combine (53) and (54) into one system of m^2 equations for m unknown functions β of n variables of generalized Frobenius type:

$$\mathbf{r}_j(\beta^i) = c^i_{ij}(u) \qquad \text{for all } 1 \le i, j \le m. \tag{55}$$

We now write out the integrability conditions (20), given in Theorem 3.5:

$$\mathbf{r}_j(c^i_{ik}) - \mathbf{r}_k(c^i_{ij}) = c^j_{jk}\,c^i_{ij} + c^k_{jk}\,c^i_{ik} \qquad \text{for all } 1 \le i, j, k \le m, \tag{56}$$

and note that these are satisfied due to Jacobi identities (8). By Theorem 3.5, we can prescribe data for β along a submanifold transversal to $\mathfrak{R}$ and obtain a solution of (55) near $\bar{u}$ taking on these data. Then the positive functions $\alpha^i := e^{\beta^i}$ satisfy the requirements in (1).

Part (2) is a direct consequence of Proposition 3.2. ☐

Due to Lemma 5.6 and thanks to the scaling invariance of Problems 1 and 2, we may assume that the given rich partial frame is commutative. We then can use a local coordinate system $w^1, \dots, w^n$, such that $\mathbf{r}_i = \frac{\partial}{\partial w^i}$, for $i = 1, \dots, m$. We complete $\mathfrak{R}$ to a frame $\{\mathbf{r}_1, \dots, \mathbf{r}_m, \mathbf{s}_{m+1}, \dots, \mathbf{s}_n\}$, where $\mathbf{s}_l = \frac{\partial}{\partial w^l}$, for $l = m+1, \dots, n$. Commutativity of the frame together with symmetry of the connection $\widetilde{\nabla}$, imply the following conditions on the structure coefficients (33) and Christoffel symbols (34) for this frame:

$$c^l_{rs} = 0 \quad \text{and} \quad \Gamma^l_{rs} = \Gamma^l_{sr} \qquad \text{for all } l, s, r \in \{1, \dots, n\}. \tag{57}$$

Then equations (40)-(42) reduce to:

$$\frac{\partial \lambda^j}{\partial w_i} = \Gamma^j_{ji}\,(\lambda^i - \lambda^j) \qquad \text{for all } 1 \le i \ne j \le m \tag{58}$$

$$\Gamma^l_{ij}\,(\lambda^i - \lambda^j) = 0 \qquad \text{for all } 1 \le i < j \le m,\ 1 \le l \le n \text{ such that } l \ne i \text{ and } l \ne j. \tag{59}$$

Assuming that the Γ^k_{ij} and unknowns λ^i are expressed in w-coordinates, we can treat (58)-(59) as a system of PDEs with simple linear constraints on the unknowns.

Theorem 5.7. *If a partial frame $\mathfrak{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ is rich and satisfies the conditions* (51)*, then the set $\mathcal{F}(\mathfrak{R})$ of all local solutions of* (25) *near $\bar{u}$ depends on:*

(1) m arbitrary functions of $n - m + 1$ variables, prescribing for each $j = 1, \dots, m$ a function λ^j along an arbitrary $(n-m+1)$ dimensional manifold Ξ_j containing $\bar{u}$ and transverse to the partial frame $\{\mathbf{r}_1, \dots, \mathbf{r}_{j-1}, \mathbf{r}_{j+1}, \dots, \mathbf{r}_m\}$;

(2) n functions of $n - m$ variables[5]*, prescribing the components of a vector field $\mathbf{f}$ along an arbitrary $(n-m)$-dimensional manifold Ξ transverse to the partial frame $\mathfrak{R}$.*

The above data uniquely determines $\mathbf{f}$ in an open neighborhood of $\bar{u}$. Finally, the flux set $\mathcal{F}(\mathfrak{R})$ always contains strictly hyperbolic fluxes.

Proof. First, as discussed above, after rescaling we may assume that $\mathfrak{R}$ is a commutative frame and we choose a coordinate system such that $\mathbf{r}_i = \frac{\partial}{\partial w^i}$, $i = 1, \dots, m$. Conditions (51) are invariant under rescaling of $\mathfrak{R}$ and imply that

$$\Gamma^l_{ij} \equiv 0 \qquad \text{for all } 1 \le i \ne j \le m,\ 1 \le l \le n, \text{ with } l \ne i \text{ and } l \ne j. \tag{60}$$

It follows that (59) trivially hold. Next, (58) is a Darboux system and we proceed to verify the integrability conditions (C) stated in Theorem 3.4. For this purpose we substitute partial derivatives prescribed by (58) into the equality of mixed partials (writing $\partial_i = \frac{\partial}{\partial w^i}$):

$$\partial_k(\partial_i \lambda^j) \equiv \partial_i(\partial_k \lambda^j) \qquad \text{for all distinct triples } i, j, k \in \{1, \dots, m\}.$$

The first substitution leads to

$$\partial_k(\Gamma^j_{ji}\,(\lambda^i - \lambda^j)) \equiv \partial_i(\Gamma^j_{jk}\,(\lambda^k - \lambda^j)) \qquad \text{for all distinct triples } i, j, k \in \{1, \dots, m\},$$

and the subsequent substitution leads to the condition:

$$\begin{aligned}(\partial_i\Gamma^j_{jk} - \partial_k\Gamma^j_{ji})\lambda^j &+ (\Gamma^j_{ji}\Gamma^i_{ik} + \Gamma^j_{jk}\Gamma^k_{ki} - \Gamma^j_{ji}\Gamma^j_{jk} - \partial_i\Gamma^j_{jk})\lambda^k \\ &- (\Gamma^j_{ji}\Gamma^i_{ik} + \Gamma^j_{jk}\Gamma^k_{ki} - \Gamma^j_{jk}\Gamma^j_{ji} - \partial_k\Gamma^j_{ji})\lambda^i \equiv 0,\end{aligned} \tag{61}$$

[5]Example 7.1 demonstrates that, when a general solution of an $\mathcal{F}(\mathfrak{R})$-system is explicitly written out, some of the arbitrary functions of $n - m$ variables may be absorbed into arbitrary functions of $n - m + 1$ variables (a larger number of variables). This is a standard phenomenon arising in applications of integrability theorems.

which must hold for all triples of pairwise distinct indices $i, j, k \in \{1, \ldots, m\}$. We will use the flatness condition (14) to show that all functions λ^i appearing in (61) have vanishing coefficients.

We first substitute $s = j$ in (14) and we assume that $i, j, k \in \{1, \ldots, m\}$ are pairwise distinct indices. Then using (57) and (60), we obtain that for all triples of pairwise distinct indices $i, j, k \in \{1, \ldots, m\}$, we have

$$-\partial_k \Gamma^j_{ji} = \Gamma^j_{jk}\Gamma^j_{ji} - \Gamma^j_{ji}\Gamma^i_{ik} - \Gamma^j_{jk}\Gamma^k_{ki}. \tag{62}$$

This immediately implies that the coefficient $(\Gamma^j_{ji}\Gamma^i_{ik} + \Gamma^j_{jk}\Gamma^k_{ki} - \Gamma^j_{jk}\Gamma^j_{ji} - \partial_k\Gamma^j_{ji})$ of λ^i in (61) vanishes. Interchanging k and i in (62), we obtain:

$$-\partial_i \Gamma^j_{jk} = \Gamma^j_{ji}\Gamma^j_{jk} - \Gamma^j_{jk}\Gamma^k_{ki} - \Gamma^j_{ji}\Gamma^i_{ik}. \tag{63}$$

so that the coefficient of λ^k in (61) vanishes. We note that the right hand sides of the identities (62) and (63) are equal. Therefore, the coefficient $(\partial_i\Gamma^j_{jk} - \partial_k\Gamma^j_{ji})$ of λ^j in (61) also vanishes.

We have thus verified that the integrability conditions (C) stated in Theorem 3.4 hold for the PDE system (58). Hence, for any fixed point $\bar{u} \in \Omega$ whose w-coordinates are $(\bar{w}^1, \ldots, \bar{w}^n)$, and any assignment of m arbitrary functions of $n - m + 1$ variables,

$$\tilde{\lambda}^i(\bar{w}^1, \ldots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1}, \ldots, \bar{w}^m, w^{m+1}, \ldots, w^n),\ i = 1, \ldots, m$$

on the subsets $\Xi_i \subset \Omega$, where $w^j = \bar{w}^j$, for $1 \le j \le m$, $j \ne i$, there is a unique local solution $\lambda^1, \ldots, \lambda^m$ of (58) such that $\lambda^i|_{\Xi_i \cap \Omega'} = \tilde{\lambda}^i|_{\Xi_i \cap \Omega'}$ on some open subset $\Omega' \subset \Omega$ containing $\bar{u}$. Thus the general solution λ of (58) depends on m arbitrary functions of $n - m + 1$ variables.

Next, recalling that for a rich frame satisfying (51), the system (49) is equivalent to the λ-system (40)-(42), we use Proposition 5.1 to conclude that for any solution λ of (49) and any smooth data for $\mathbf{f}$ prescribed along any embedded submanifold $\Xi \subset \Omega$ of codimension m transversal to $\mathfrak{R}$, there is a unique smooth local solution of the $\mathcal{F}(\mathfrak{R})$-system (25). In local coordinates the data can be defined by n functions (components of $\mathbf{f}$) of $n - m$ variables (local coordinates on Ξ). Therefore, for a given solution λ of (49), the general solution $\mathbf{f}$ of the $\mathcal{F}(\mathfrak{R})$-system (25) depends on n arbitrary functions of $n - m$ variables.

Finally, we may choose $\tilde{\lambda}^1, \ldots, \tilde{\lambda}^m$ in the first part of the proof such that $\tilde{\lambda}^1(\bar{u}), \ldots, \tilde{\lambda}^m(\bar{u})$ are all distinct. Let $\lambda^1, \ldots, \lambda^m$ be the corresponding solutions of (49). The existence of strictly hyperbolic fluxes in the flux set $\mathcal{F}(\mathfrak{R})$ then follows from Proposition 5.3. □

We note that in the case of a single vector field ($m = 1$), the conclusion of Theorem 5.7 is consistent with the observation made in Remark 5.4. The first part of the proof of Theorem 5.7 is a rather straightforward generalization of the proof of Theorem 4.3 in [7], where the λ system (49) was considered in the case of the full frame ($m = n$). In a similar way, we can generalize Theorem 4.4 in [7] to treat the case when necessary conditions (51) for strict hyperbolicity are not satisfied. In this case, the algebraic relationship (50) implies that there exist distinct $i, j \in \{1, \ldots, m\}$ such that $\lambda^i \equiv \lambda^j$, and therefore, there are no strictly hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$. The next theorem gives a somewhat involved description of $\mathcal{F}(\mathfrak{R})$; a proof (omitted) may be obtained by combining the arguments in the proofs of Theorem 4.4 in [7] and Theorem 5.7.

Theorem 5.8. *Let $\mathfrak{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be a rich partial frame that* does not *satisfy conditions* (51)*. Then the system (49)-(50) imposes multiplicity conditions*[6] *on λ^i in the following sense. There are disjoint subsets $A_1, \ldots, A_{s_0} \subset \{1, \ldots, m\}$ ($s_0 \ge 1$) of cardinality two or more, and such that (49)-(50) impose the equality $\lambda^i = \lambda^j$ if and only if $i, j \in A_\alpha$ for some $\alpha \in \{1, \ldots, s_0\}$. Let $l = \sum_{\alpha=1}^{s_0} |A_\alpha| \le m$ and $s_1 = m - l$. By relabeling indices we may assume that $\{1, \ldots, m\} \setminus \bigcup_{\alpha=1}^{s_0} A_\alpha = \{1, \ldots, s_1\}$. Then the set $\mathcal{F}(\mathfrak{R})$ of all local solutions of* (25) *near $\bar{u}$ depends on:*

[6] Clearly, for all $i \ne j$, with $\tilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j \notin \mathrm{span}\{\mathbf{r}_i, \mathbf{r}_j\}$, (50) implies a multiplicity condition $\lambda^i = \lambda^j$. Less obviously, (49) may impose additional multiplicity conditions on λ^i. See the proof of Lemma 4.5 in [7] for more details.

- *s_1 arbitrary functions $\tilde{\lambda}^1, \ldots, \tilde{\lambda}^{s_1}$ of $n-m+1$ variables prescribing, for $j = 1, \ldots, s_1$, data for the functions λ^j, so that $\lambda^j|_{\Xi_j} = \tilde{\lambda}^j$, where Ξ_j is an arbitrary $(n-m+1)$-dimensional manifold Ξ_j containing $\bar{u}$ and transverse to the vector fields $\mathbf{r}_1, \ldots, \mathbf{r}_{j-1}, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_m$;*
- *s_0 arbitrary functions $\kappa^1, \ldots, \kappa^{s_0}$ of $m-n$ variables prescribing, for $j = s_1+1, \ldots, m$, data for the functions λ^j, so that when $j \in A_\alpha$ for some $\alpha = 1, \ldots, s_0$ when $j \in A_\alpha$ for some $\alpha = 1, \ldots, s_0$, then $\lambda^j|_{\Xi_j} = \kappa^\alpha$, where Ξ_j is an $(n-m)$-dimensional manifold containing $\bar{u}$ and transverse to $\mathfrak{R}$;*
- *n functions of $n-m$ variables prescribing the components of a vector field $\mathbf{f}$ along an arbitrary $(n-m)$-dimensional manifold Ξ transverse to the partial frame $\mathfrak{R}$.*

The above data uniquely determines $\mathbf{f}$ in an open neighborhood of $\bar{u}$. The flux set $\mathcal{F}(\mathfrak{R})$ never contains strictly hyperbolic fluxes.

5.3 Non-rich involutive frames consisting of three vector fields

The lowest cardinality of a partial frame for which the involutive, non-rich scenario may occur is $m = 3$. The full frame case $m = n = 3$ was treated in [7]. We now generalize these results to $n \geq 3$. Generalization to $m > 3$ would involve a large number of cases and is not pursued here.

We first treat the case when $\mathfrak{R}$ satisfies the necessary conditions of Proposition 5.2 for the existence of strictly hyperbolic fluxes. We choose an arbitrary completion of $\mathfrak{R}$ to a frame and write out the λ-system (40)-(42). The differential part (40) becomes

$$\begin{aligned} &\mathbf{r}_2(\lambda^1) = \Gamma^1_{12}(\lambda^2 - \lambda^1) && \mathbf{r}_3(\lambda^1) = \Gamma^1_{13}(\lambda^3 - \lambda^1) && \mathbf{r}_1(\lambda^2) = \Gamma^2_{21}(\lambda^1 - \lambda^2) \\ &\mathbf{r}_3(\lambda^2) = \Gamma^2_{23}(\lambda^3 - \lambda^2) && \mathbf{r}_1(\lambda^3) = \Gamma^3_{31}(\lambda^1 - \lambda^3) && \mathbf{r}_2(\lambda^3) = \Gamma^3_{32}(\lambda^2 - \lambda^3), \end{aligned} \tag{64}$$

while the algebraic equations (41) may be written as

$$A_\lambda \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{bmatrix} = 0\,, \qquad \text{where} \qquad A_\lambda = \begin{bmatrix} c^1_{23} & \Gamma^1_{32} & -\Gamma^1_{23} \\ \Gamma^2_{31} & c^2_{13} & -\Gamma^2_{13} \\ \Gamma^3_{21} & -\Gamma^3_{12} & c^3_{12} \end{bmatrix}. \tag{65}$$

Condition (47) in Proposition 5.2 implies that (42) is trivial. We also note that, since $\mathfrak{R}$ is involutive and satisfies the conditions in Proposition 5.2, for all $i, j, k \in \{1, 2, 3\}$ the structure coefficients c^k_{ij} and Christoffel symbols Γ^k_{ij} are independent of the completion of $\mathfrak{R}$ to a frame. Thus, the system (64)-(65) can be written out without specifying a completion to a full frame. Our goal is to describe the solution set of (64)-(65). We observe that:

- From the symmetry of the connection it follows that the last column of A_λ is the sum of the first two columns; thus $\operatorname{rank} A_\lambda \leq 2$.
- Non-richness of $\mathfrak{R}$ implies that at least one of the c^k_{ij} in A_λ is nonzero; thus $\operatorname{rank} A_\lambda \geq 1$.
- Condition (46) in Proposition 5.2 implies that, for each row in A_λ, either all three entries are zero, or all three entries are non-zero.

Following the same argument as in Section 3 of [7], one can show that if $\operatorname{rank} A_\lambda = 2$ at $\bar{u}$, then the three eigenfunctions must coincide in a neighborhood of $\bar{u}$; i.e., $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$ for some functions λ, and, therefore, $\mathcal{F}(\mathfrak{R})$ does not contain strictly hyperbolic fluxes. Moreover, (64) imply that λ is constant along the integral manifolds of the involutive frame $\mathfrak{R}$, and we can prescribe an arbitrary value of λ along a manifold Ξ transverse to $\mathfrak{R}$. Otherwise, $\operatorname{rank} A_\lambda = 1$, and we may assume without loss of generality that $c^1_{23} \neq 0$. The first equation in (65) can be solved for λ^1 and substituted in (64). This yields a system of six equations specifying the derivatives of the two unknown functions λ^2 and λ^3 along a partial, involutive frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$:

$$\begin{aligned}
\mathbf{r}_1(\lambda^2) &= \frac{\Gamma^2_{21}\Gamma^1_{23}}{c^1_{32}}(\lambda^2-\lambda^3)\,,\\
\mathbf{r}_2(\lambda^2) &= \left[\frac{\Gamma^1_{23}}{\Gamma^1_{32}}(\Gamma^3_{32}-\Gamma^1_{12}) - \frac{c^1_{32}}{\Gamma^1_{32}}\, r_2\left(\frac{\Gamma^1_{32}}{c^1_{32}}\right)\right](\lambda^2-\lambda^3)\,,\\
\mathbf{r}_3(\lambda^2) &= -\Gamma^2_{23}(\lambda^2-\lambda^3)\,,
\end{aligned} \tag{66}$$

$$\begin{aligned}
\mathbf{r}_1(\lambda^3) &= \frac{\Gamma^3_{31}\Gamma^1_{32}}{c^1_{32}}(\lambda^2-\lambda^3)\,,\\
\mathbf{r}_2(\lambda^3) &= \Gamma^3_{32}(\lambda^2-\lambda^3)\,,\\
\mathbf{r}_3(\lambda^3) &= \left[\frac{\Gamma^1_{32}}{\Gamma^1_{23}}(\Gamma^1_{13}-\Gamma^2_{23}) + \frac{c^1_{32}}{\Gamma^1_{23}}\, r_3\left(\frac{\Gamma^1_{23}}{c^1_{32}}\right)\right](\lambda^2-\lambda^3)\,,
\end{aligned}$$

A similar system, for $n = 3$, was analyzed in [7] via the classical Frobenius theorem. For the present setting with $n \geq 3$ we need to use the generalized Frobenius Theorem 3.5. To verify the integrability conditions, we rewrite (66) as

$$r_i(\lambda^s) = \phi^s_i(u)(\lambda^2-\lambda^3) \qquad \text{for } i = 1,2,3 \text{ and } s = 2,3, \tag{67}$$

where ϕ^s_i are given functions of the Γ^k_{ij}. The integrability conditions amount to

$$\Big[r_i(\phi^s_j) - r_j(\phi^s_i) + \phi^s_j(\phi^2_i - \phi^3_i) - \phi^s_i(\phi^2_j - \phi^3_j)\Big](\lambda^2-\lambda^3) = \left[\sum_{k=1}^{3} c^k_{ij}\phi^s_k\right](\lambda^2-\lambda^3)\,, \tag{68}$$

where $1 \leq i < j \leq 3$, $s = 2,3$ and $c^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$.

These conditions are satisfied if $\lambda^2 = \lambda^3$ in a neighborhood of $\bar{u}$, in which case the first equation in (65) implies $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$, and, as above, the functions λ^i must be constant along the integral manifolds of the involutive frame $\mathfrak{R}$, and we can prescribe an arbitrary value of the λ^i along a manifold Ξ transverse to $\mathfrak{R}$. For a strictly hyperbolic flux to exist the following six conditions must hold:

$$\mathbf{r}_i(\phi^2_j) - \mathbf{r}_j(\phi^2_i) \;=\; \phi^2_j\phi^3_i - \phi^2_i\phi^3_j + \sum_{k=1}^{3} c^k_{ij}\phi^2_k \qquad 1 \leq i < j \leq 3, \tag{69}$$

$$\mathbf{r}_i(\phi^3_j) - \mathbf{r}_j(\phi^3_i) \;=\; \phi^2_j\phi^3_i - \phi^2_i\phi^3_j + \sum_{k=1}^{3} c^k_{ij}\phi^3_k \qquad 1 \leq i < j \leq 3. \tag{70}$$

Conditions (69)-(70) were derived in [7] in the case of full frames in $\mathbb{R}^3$, and Examples 5.1 and 5.3 in [7] show that these compatibility conditions may or may not be satisfied; they must be checked for each case individually. If these integrability conditions are met, then by Theorem 3.5, the general solution of the λ-system depends on two functions of $n-3$ variables prescribing the values of λ^2 and λ^3 along any two $n-3$ dimensional manifolds passing $\bar{u}$ and transverse to $\mathfrak{R}$. The function λ^1 is then determined by the first equation in (65). Combining the above argument with Propositions 5.1-5.2 we arrive at the following theorem.

Theorem 5.9. *Assume $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a non-rich partial frame in involution on a neighborhood Ω of $\bar{u}$ satisfying conditions (46) and (47) in Proposition 5.2. For $i, j, k \in \{1,2,3\}$, let c^k_{ij} and Γ^k_{ij} be defined by*

$$[\mathbf{r}_i, \mathbf{r}_j] = \sum_{k=1}^{3} c^k_{ij}\mathbf{r}_k \qquad \nabla_{\mathbf{r}_i}\mathbf{r}_j = \sum_{k=1}^{3} \Gamma^k_{ij}\mathbf{r}_k.$$

Up to permutation of indices and by shrinking Ω we may assume c^1_{23} is nowhere zero on Ω.

- *If the matrix A_λ defined in* (65) *has rank 1 and (69)-(70) are satisfied in a neighborhood of $\bar{u}$, then the flux set $\mathcal{F}(\mathfrak{R})$ of system* (25) *depends on $n+2$ arbitrary functions of $n-3$ variables (2 of those determine λ^2 and λ^3, while n of those determine $\mathbf{f}$ along an $(n-3)$-dimensional manifold passing through $\bar{u}$ and transverse to $\mathfrak{R}$). The set $\mathcal{F}(\mathfrak{R})$ contains strictly hyperbolic fluxes.*
- *If the matrix A_λ defined in* (65) *has rank 2 at $\bar{u}$ or if (69)-(70) are not satisfied at $\bar{u}$, then the three eigenfunctions must coincide in a neighborhood of $\bar{u}$, i.e. $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$ for some function λ which is constant along the integral manifolds of the involutive frame $\mathfrak{R}$, and can take arbitrary values along a manifold Ξ transverse to $\mathfrak{R}$. The flux set $\mathcal{F}(\mathfrak{R})$ depends on $n+1$ arbitrary functions of $n-3$ variables (1 of those determine λ and n of those determine $\mathbf{f}$ along an $(n-3)$-dimensional manifold passing through $\bar{u}$ and transverse to $\mathfrak{R}$). The set $\mathcal{F}(\mathfrak{R})$ does not contain strictly hyperbolic fluxes.*

When the partial frame $\mathfrak{R}$ does *not* satisfy the necessary conditions of Proposition 5.2 for the existence of strictly hyperbolic fluxes, then the algebraic conditions (41) and (42) force two or more of eigenfunctions to be equal to each other, and we can prove the following result:

Theorem 5.10. *Assume $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a non-rich partial frame in involution on a neighborhood Ω of $\bar{u}$, such that $\mathfrak{R}$ does* not *satisfy condition (46) or condition (47) in Proposition 5.2. Then there are exactly two possibilities: either*

- *the λ-system (40)-(42) implies that $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$, in a neighborhood of $\bar{u}$, where the function λ is constant along the integral manifolds of the frame $\mathfrak{R}$ and may take arbitrary values on an $(n-3)$-dimensional manifold Ξ_0 passing through $\bar{u}$ and transverse to $\mathfrak{R}$; or*
- *up to permutation of indices, the λ-system (40)-(42) implies that $\lambda^1 = \lambda^2 = \lambda$, but allows the possibility that $\lambda \neq \lambda^3$ in a neighborhood of $\bar{u}$. The function λ^3 is uniquely determined by its values on an $(n-2)$-dimensional manifold Ξ_1 passing through $\bar{u}$ and transverse to $\{\mathbf{r}_1, \mathbf{r}_2\}$, while the function λ is uniquely determined by its values on an $(n-3)$-dimensional manifold Ξ_2 passing through $\bar{u}$ and transverse to $\mathfrak{R}$.*

In either case the λ-system (40)-(42) has a locally unique solution with data as described above; for each such solution the $\mathcal{F}(\mathfrak{R})$-system (25) *has a locally unique solution determined by the values of $\mathbf{f}$ on an $(n-3)$-dimensional manifold Ξ passing through $\bar{u}$ and transverse to $\mathfrak{R}$. The set $\mathcal{F}(\mathfrak{R})$ contains* no *strictly hyperbolic fluxes.*

Proof. If (46) fails, then (41) implies that at least two functions among λ^1, λ^2 and λ^3 coincide on a neighborhood of $\bar{u}$. If (47) fails, then (42) yields the same conclusion. In either case the set $\mathcal{F}(\mathfrak{R})$ does not contain strictly hyperbolic fluxes.

If (41) and (42) imply that all three are equal, i.e., $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$, then (40) implies that the function λ is constant along the integral manifolds of the involutive frame $\mathfrak{R}$. In this case, the system (40) trivially satisfies the assumptions of Theorem 3.5. Consequently, for any assignment of λ along an $(n-3)$-dimensional manifold Ξ_0 passing through $\bar{u}$ and transverse to $\mathfrak{R}$, there is unique such function in a neighborhood of $\bar{u}$.

If (41) and (42) imply that only two of the λ^i coincide, e.g. $\lambda^1 = \lambda^2 = \lambda$, but not that they are equal to λ^3, then one can argue that the c_{ij}^k and Γ_{ij}^k satisfy

$$c_{12}^3 = 0,\ \Gamma_{13}^2 = 0 \text{ and } \Gamma_{23}^1 = 0, \tag{71}$$

in which case the λ-system (40)-(42) becomes:

$$\begin{aligned}
&\mathbf{r}_2(\lambda) = 0 && \mathbf{r}_1(\lambda^3) = \Gamma_{31}^3(\lambda - \lambda^3)\\
&\mathbf{r}_3(\lambda) = \Gamma_{13}^1(\lambda^3 - \lambda) && \mathbf{r}_2(\lambda^3) = \Gamma_{32}^3(\lambda - \lambda^3).\\
&\mathbf{r}_1(\lambda) = 0 &&\\
&\mathbf{r}_3(\lambda) = \Gamma_{23}^2(\lambda^3 - \lambda) &&
\end{aligned}$$

If $\Gamma^2_{23} \neq \Gamma^1_{13}$, then the second and the fourth equations imply $\lambda = \lambda^3$, so that $\lambda^1 = \lambda^2 = \lambda^3 = \lambda$, and we are in the situation considered above. If instead

$$\Gamma^2_{23} = \Gamma^1_{13} \tag{72}$$

we obtain the system

$$\mathbf{r}_1(\lambda) = 0 \tag{73}$$
$$\mathbf{r}_2(\lambda) = 0 \tag{74}$$
$$\mathbf{r}_3(\lambda) = \Gamma^1_{13}(\lambda^3 - \lambda) \tag{75}$$
$$\mathbf{r}_1(\lambda^3) = \Gamma^3_{31}(\lambda - \lambda^3) \tag{76}$$
$$\mathbf{r}_2(\lambda^3) = \Gamma^3_{32}(\lambda - \lambda^3). \tag{77}$$

Subtracting (73) from (76), (74) from (77), and introducing the unknown $\mu = \lambda^3 - \lambda$, yield

$$\mathbf{r}_1(\lambda) = 0 \tag{78}$$
$$\mathbf{r}_2(\lambda) = 0 \tag{79}$$
$$\mathbf{r}_3(\lambda) = \Gamma^1_{13}\,\mu \tag{80}$$
$$\mathbf{r}_1(\mu) = -\Gamma^3_{31}\,\mu \tag{81}$$
$$\mathbf{r}_2(\mu) = -\Gamma^3_{32}\,\mu. \tag{82}$$

By assumption $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are in involution and the first condition in (71) implies that the vector fields $\mathbf{r}_1$ and $\mathbf{r}_2$ are in involution. Thus we can first apply Theorem 3.5 to the subsystem (81)-(82), whose integrability condition

$$\mathbf{r}_2(\Gamma^3_{31}) - \mathbf{r}_1(\Gamma^3_{32}) = c^2_{21}\,\Gamma^3_{32} + c^1_{21}\,\Gamma^3_{31} \tag{83}$$

is satisfied as shown in Lemma 3.6 of [7], due to the flatness and symmetry property of the connection, combined with conditions (71) and (72). Thus there is a unique solution μ for the subsystem (81)-(82) with any data prescribed along an $(n-2)$-dimensional manifold Ξ_1 passing through $\bar{u}$ and transversal to $\mathbf{r}_1, \mathbf{r}_2$. Substituting the solution μ into (80) we obtain a subsystem (78)-(80) whose integrability condition is

$$\begin{aligned} \mathbf{r}_2(\Gamma^1_{13}) &= \Gamma^3_{23}\,\Gamma^1_{13} \\ \mathbf{r}_1(\Gamma^1_{13}) &= \Gamma^3_{13}\,\Gamma^1_{13}. \end{aligned} \tag{84}$$

As shown in Lemma 3.6 of [7], conditions (84) hold identically on Ω due to the flatness and symmetry property of the connection, combined with conditions (71) and (72). Theorem 3.5 now guarantees the existence of a locally unique solution of the subsystem (78)-(80), with the values of function λ prescribed along an $(n-3)$ dimensional manifold Ξ_2 passing through $\bar{u}$ and transverse to $\mathfrak{R}$. Recalling that $\mu = \lambda^3 - \lambda$, we conclude that λ is uniquely determined by its values on an $(n-3)$ dimensional manifold Ξ_1 passing through $\bar{u}$ and transverse to $\mathfrak{R}$, and the function λ^3 is uniquely determined by its values on an $(n-2)$ dimensional manifold Ξ_2 passing through $\bar{u}$ and transverse to $\{\mathbf{r}_1, \mathbf{r}_2\}$.

Finally, it follows from Proposition 5.1 that for each solution of the λ system, the $\mathcal{F}(\mathfrak{R})$-system (25) has a locally unique solution determined by the values of $\mathbf{f}$ on an $(n-3)$-dimensional manifold Ξ passing through $\bar{u}$ and transverse to $\mathfrak{R}$. □

6 Non-involutive partial frames of two vector fields in $\mathbb{R}^3$.

In the non-involutive case, the differential consequences (30)-(32) of the $\mathcal{F}(\mathfrak{R})$-system (25) involve the unknowns a^j_i. Thus, instead of a "λ-system" we now have a "(λ, a)-system." Moreover, (30)-(32) do not provide a complete set of integrability conditions for the $\mathcal{F}(\mathfrak{R})$-system. This makes the non-involutive case much harder than the involutive case, and we can only treat the lowest dimensional case where $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ is a partial frame in $\mathbb{R}^3$ with

$$[\mathbf{r}_1, \mathbf{r}_2]_{\bar{u}} \notin \text{span}_{\mathbb{R}}\{\mathbf{r}_1|_{\bar{u}}, \mathbf{r}_2|_{\bar{u}}\}. \tag{85}$$

The $\mathcal{F}(\mathfrak{R})$-system then consists of the two equations

$$\widetilde{\nabla}_{\mathbf{r}_1} \mathbf{f} = \lambda^1 \mathbf{r}_1 \qquad \text{and} \qquad \widetilde{\nabla}_{\mathbf{r}_2} \mathbf{f} = \lambda^2 \mathbf{r}_2. \tag{86}$$

The necessary conditions (38) for strict hyperbolicity become

$$\widetilde{\nabla}_{\mathbf{r}_1} \mathbf{r}_2|_{\bar{u}} \notin \text{span}_{\mathbb{R}}\{\mathbf{r}_1|_{\bar{u}}, \mathbf{r}_2|_{\bar{u}}\} \text{ and } \widetilde{\nabla}_{\mathbf{r}_2} \mathbf{r}_1|_{\bar{u}} \notin \text{span}_{\mathbb{R}}\{\mathbf{r}_1|_{\bar{u}}, \mathbf{r}_2|_{\bar{u}}\}. \tag{87}$$

We next state two theorems describing the size and structure of the flux space $\mathcal{F}(\mathfrak{R})$ for partial frames $\mathfrak{R}$ satisfying (87). The proofs of the theorems rely on the sequence of lemmas listed below. We remind the reader that $\mathcal{F}^{\text{triv}}$ denotes the 4-dimensional space of trivial fluxes.

Theorem 6.1. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a non-involutive partial frame on an open neighborhood of $\bar{u} \in \mathbb{R}^3$ satisfying conditions* (87)*. Then*

1. *A non-zero flux $\mathbf{f} \in \mathcal{F}(\mathfrak{R})/\mathcal{F}^{triv}$ is either strictly hyperbolic or non-hyperbolic.*
2. *If $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{triv} > 1$, then $\mathcal{F}(\mathfrak{R})$ contains strictly hyperbolic fluxes.*
3. *If $\mathcal{F}(\mathfrak{R})$ contains a non-hyperbolic flux, then for any vector field $\mathbf{s}$ completing $\mathfrak{R}$ to a local frame, the following identity holds on an open neighborhood of $\bar{u}$:*

$$\Gamma^3_{12}\, \Gamma^3_{21} - 2\,(c^3_{12})^2 = \Gamma^3_{11}\, \Gamma^3_{22}, \tag{88}$$

where the c^k_{ij} and Γ^k_{ij} are the structure components and Christoffel symbols, respectively, of the connection $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}\}$.

Although identity (88) is a restrictive condition, Examples 7.7 and 7.9 show that there are partial frames with non-hyperbolic fluxes. On the other hand, Examples 7.5, 7.6, 7.8 and 7.11 show that there are partial frames for which all non-trivial fluxes are strictly hyperbolic.

Theorem 6.2. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a non-involutive partial frame on an open neighborhood of $\bar{u} \in \mathbb{R}^3$ satisfying conditions* (87)*. Let $\mathbf{s}$ be any completion of $\mathfrak{R}$ to a local frame near $\bar{u}$ and let Γ^k_{ij} be the Christoffel symbols for connection $\widetilde{\nabla}$ relative to this frame. Assume further that the following condition is satisfied:*

$$\Gamma^3_{22}(\bar{u})\, \Gamma^3_{11}(\bar{u}) - 9\, \Gamma^3_{12}(\bar{u})\, \Gamma^3_{21}(\bar{u}) \neq 0. \tag{89}$$

Then:

1. $0 \leq \dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{triv} \leq 4$.
2. *For each $k = 0, \ldots, 4$ there exists a partial frame $\mathfrak{R}$ satisfying the assumptions of the theorem and such that $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{triv} = k$.*

Lemma 6.3. *Conditions* (88)-(89) *are independent of the vector field $\mathbf{s}$ completing $\mathfrak{R}$ to a frame.*

Proof. Consider two completions of $\mathfrak{R}$ to a local frame in a neighborhood Ω of $\bar{u}$, by vector fields $\mathbf{s}$ and $\bar{\mathbf{s}}$, respectively. Let

$$\bar{\mathbf{s}} = \alpha\, \mathbf{r}_1 + \beta\, \mathbf{r}_2 + \gamma\, \mathbf{s},$$

for some smooth functions α, β and γ, with γ non-vanishing on Ω. Let c_{ij}^k and $\bar{c}_{ij}^k$ (Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$) be the structure coefficients (Christoffel symbols) for the connection $\widetilde{\nabla}$ relative to the frames $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}\}$ and $\{\mathbf{r}_1, \mathbf{r}_2, \bar{\mathbf{s}}\}$, respectively. Then for $i, j = 1, 2$:

$$\begin{aligned}\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{r}_j &= \Gamma_{ij}^1\,\mathbf{r}_1 + \Gamma_{ij}^2\,\mathbf{r}_2 + \Gamma_{ij}^3\,\mathbf{s} = \Gamma_{ij}^1\,\mathbf{r}_1 + \Gamma_{ij}^2\,\mathbf{r}_2 + \Gamma_{ij}^3\,\gamma^{-1}(\bar{\mathbf{s}} - \alpha\mathbf{r}_1 - \beta\,\mathbf{r}_2)\\ &= (\Gamma_{ij}^1 - \gamma^{-1}\alpha\Gamma_{ij}^3)\,\mathbf{r}_1 + (\Gamma_{ij}^2 - \gamma^{-1}\beta\,\Gamma_{ij}^3)\,\mathbf{r}_2 + \gamma^{-1}\Gamma_{ij}^3\,\bar{\mathbf{s}}.\end{aligned}$$

Hence, $\bar{\Gamma}_{ij}^3 = \gamma^{-1}\Gamma_{ij}^3$ and $\bar{c}_{ij}^3 = \gamma^{-1}c_{ij}^3$ for $i, j = 1, 2$. It follows that (88) and (89) hold for $\bar{c}_{ij}^k$ and $\bar{\Gamma}_{ij}^k$ if and only if they hold for c_{ij}^k and Γ_{ij}^k. □

Condition (89) arises in the proof of Lemma 6.5 and Example 7.11 illustrates that there are partial frames, with non-trivial fluxes, for which (89) does not hold. Beyond this, we shall not pursue further this non-generic situation.

Lemma 6.4. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a non-involutive partial frame satisfying* (87). *Set $\mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2]$ and denote by c_{ij}^k, Γ_{ij}^k the structure coefficients and Christoffel symbols of the connection $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}\}$. Then the following conditions are equivalent for functions λ^1 and λ^2 defined near $\bar{u}$:*

1. *There is a solution $\mathbf{f}$ of the $\mathcal{F}(\mathfrak{R})$-system (86) for the* prescribed *functions λ^1 and λ^2.*
2. *The functions λ^1 and λ^2, together with the functions a^1 and a^2 defined by*

$$a^1 = -\mathbf{r}_2(\lambda^1) - \Gamma_{12}^1\,(\lambda^1 - \lambda^2) \tag{90}$$

$$a^2 = \quad \mathbf{r}_1(\lambda^2) - \Gamma_{21}^2\,(\lambda^1 - \lambda^2) \tag{91}$$

satisfy the following system of 6 *equations:*

$$\mathbf{r}_1(\lambda^1) = \frac{1}{\Gamma_{21}^3}\Big(\Upsilon_1\,(\lambda^1 - \lambda^2) + \Gamma_{11}^3\,a^1 + 2\,\Gamma_{12}^3\,a^2\Big) \tag{92}$$

$$\mathbf{r}_2(\lambda^2) = \frac{1}{\Gamma_{12}^3}\Big(\Upsilon_2\,(\lambda^1 - \lambda^2) - 2\,\Gamma_{21}^3\,a^1 - \Gamma_{22}^3\,a^2\Big) \tag{93}$$

$$\mathbf{r}_2(a^1) = (\Gamma_{23}^1\,\Gamma_{12}^3 - \Gamma_{32}^1)\,(\lambda^1 - \lambda^2) + (c_{23}^3 - \Gamma_{21}^1)\,a^1 - \Gamma_{22}^1\,a^2 \tag{94}$$

$$\mathbf{r}_1(a^2) = (\Gamma_{13}^2\,\Gamma_{21}^3 + \Gamma_{31}^2)\,(\lambda^1 - \lambda^2) - \Gamma_{11}^2\,a^1 + (c_{13}^3 - \Gamma_{12}^2)\,a^2 \tag{95}$$

$$\mathbf{r}_1(a^1) - \mathbf{s}(\lambda^1) = \Gamma_{13}^1\Gamma_{12}^3\,(\lambda^1 - \lambda^2) - (\Gamma_{11}^1 - c_{13}^3)\,a^1 - \Gamma_{12}^1\,a^2 \tag{96}$$

$$\mathbf{r}_2(a^2) - \mathbf{s}(\lambda^2) = \Gamma_{23}^2\,\Gamma_{21}^3\,(\lambda^1 - \lambda^2) - \Gamma_{21}^2\,a^1 + (c_{23}^3 - \Gamma_{22}^2)a^2 \tag{97}$$

where

$$\Upsilon_1 = \Gamma_{12}^3\,(\Gamma_{21}^2 - \Gamma_{31}^3) - \mathbf{r}_1(\Gamma_{12}^3) \qquad \textit{and} \qquad \Upsilon_2 = \Gamma_{21}^3\,(\Gamma_{32}^3 - \Gamma_{12}^1) + \mathbf{r}_2(\Gamma_{12}^3). \tag{98}$$

Moreover, for every pair of functions λ^1 and λ^2 satisfying these conditions, there exists a unique (up to a constant term) flux $\mathbf{f}$ satisfying (86).

Proof. We first note that, due to the symmetry of $\widetilde{\nabla}$ and our definition of $\mathbf{s}$ we have

$$\Gamma_{12}^1 - \Gamma_{21}^1 = c_{12}^1 = 0, \quad \Gamma_{12}^2 - \Gamma_{21}^2 = c_{12}^2 = 0, \quad \Gamma_{12}^3 - \Gamma_{21}^3 = c_{12}^3 = 1. \tag{99}$$

Assume λ^1 and λ^2 are given functions and that there exists $\mathbf{f}$ such that (86) holds. The flatness condition (12) implies that

$$\widetilde{\nabla}_{[\mathbf{r}_1,\mathbf{r}_2]}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_1}\widetilde{\nabla}_{\mathbf{r}_2}\mathbf{f} - \widetilde{\nabla}_{\mathbf{r}_2}\widetilde{\nabla}_{\mathbf{r}_1}\mathbf{f}. \tag{100}$$

As $\mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2]$, expanding the right hand side, substituting (86), and using (99) give

$$\widetilde{\nabla}_{\mathbf{s}}\, \mathbf{f} = a^1\, \mathbf{r}_1 + a^2\, \mathbf{r}_2 + a^3\, \mathbf{s}, \tag{101}$$

where a^1, a^2 are given by (90),(91), and

$$a^3 = \Gamma^3_{12}\, \lambda^2 - \Gamma^3_{21}\, \lambda^1. \tag{102}$$

The following consequence of (102) and the last equation in (99) is repeatedly used below:

$$\lambda^1 - a^3 = \Gamma^3_{12}\, (\lambda^1 - \lambda^2) \text{ and } \lambda^2 - a^3 = \Gamma^3_{21}\, (\lambda^1 - \lambda^2) \tag{103}$$

By expanding the identity

$$\widetilde{\nabla}_{[\mathbf{r}_1, \mathbf{s}]}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_1} \widetilde{\nabla}_{\mathbf{s}}\mathbf{f} - \widetilde{\nabla}_{\mathbf{s}} \widetilde{\nabla}_{\mathbf{r}_1}\mathbf{f}, \tag{104}$$

and eliminating any occurrences of a^3 using (103), we obtain

$$\mathbf{r}_1(a^1) = \mathbf{s}(\lambda^1) + \Gamma^1_{13}\Gamma^3_{12}\, (\lambda^1 - \lambda^2) - (\Gamma^1_{11} - c^3_{13})\, a^1 - \Gamma^1_{12}\, a^2 \qquad \text{(coefficient of } \mathbf{r}_1\text{)} \tag{105}$$

$$\mathbf{r}_1(a^2) = (\Gamma^2_{13}\, \Gamma^3_{21} + \Gamma^2_{31})\, (\lambda^1 - \lambda^2) - \Gamma^2_{11}\, a^1 + (c^3_{13} - \Gamma^2_{12})\, a^2 \qquad \text{(coefficient of } \mathbf{r}_2\text{)} \tag{106}$$

$$\mathbf{r}_1(a^3) = \Gamma^3_{31}\, \Gamma^3_{12}(\lambda^1 - \lambda^2) - \Gamma^3_{11}\, a^1 - \Gamma^3_{12}\, a^2. \qquad \text{(coefficient of } \mathbf{r}_3\text{)} \tag{107}$$

Similarly, the identity

$$\widetilde{\nabla}_{[\mathbf{r}_2, \mathbf{s}]}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_2} \widetilde{\nabla}_{\mathbf{s}}\mathbf{f} - \widetilde{\nabla}_{\mathbf{s}} \widetilde{\nabla}_{\mathbf{r}_2}\mathbf{f} \tag{108}$$

leads to

$$\mathbf{r}_2(a^1) = (\Gamma^1_{23}\, \Gamma^3_{12} - \Gamma^1_{32})\, (\lambda^1 - \lambda^2) + (c^3_{23} - \Gamma^1_{21})\, a^1 - \Gamma^1_{22}\, a^2 \qquad \text{(coefficient of } \mathbf{r}_1\text{)} \tag{109}$$

$$\mathbf{r}_2(a^2) = \mathbf{s}(\lambda^2) + \Gamma^2_{23}\, \Gamma^3_{21}\, (\lambda^1 - \lambda^2) - \Gamma^2_{21}\, a^1 + (c^3_{23} - \Gamma^2_{22})a^2 \qquad \text{(coefficient of } \mathbf{r}_2\text{)} \tag{110}$$

$$\mathbf{r}_2(a^3) = \Gamma^3_{32}\, \Gamma^3_{21}\, (\lambda^1 - \lambda^2) - \Gamma^3_{21}\, a^1 - \Gamma^3_{22}\, a^2. \qquad \text{(coefficient of } \mathbf{s}\text{)} \tag{111}$$

Note that a^3 was eliminated from the right-hand sides of the above equations using (103). We note that (109), (106), (105), (110) coincide with (94), (95), (96), and (97), respectively. To show that the remaining two equations, (92) and (93), hold, we note that equations (107) and (111) express the derivatives of a_3 in the $\mathbf{r}_1$ and $\mathbf{r}_2$ directions, respectively. However, these derivatives can be also obtained by differentiating (102) and substituting (90) and (91):

$$\begin{aligned} \mathbf{r}_1(a_3) &= \Gamma^3_{12}\, \mathbf{r}_1(\lambda^2) - \Gamma^3_{21}\, \mathbf{r}_1(\lambda^1) + \mathbf{r}_1(\Gamma^3_{12})\, (\lambda^2 - \lambda^1) & (112)\\ &= \left(\Gamma^3_{12}\, \Gamma^2_{21} - \mathbf{r}_1(\Gamma^3_{12})\right)\, (\lambda^1 - \lambda^2) + \Gamma^3_{12} a^2 - \Gamma^3_{21}\, \mathbf{r}_1(\lambda^1), \\ \mathbf{r}_2(a_3) &= \Gamma^3_{12}\, \mathbf{r}_2(\lambda^2) - \Gamma^3_{21}\, \mathbf{r}_2(\lambda^1) + \mathbf{r}_2(\Gamma^3_{12})\, (\lambda^2 - \lambda^1) & (113)\\ &= \Gamma^3_{12}\, \mathbf{r}_2(\lambda^2) + \left(\mathbf{r}_2(\Gamma^3_{12}) - \Gamma^3_{21}\, \Gamma^1_{12}\right)\, (\lambda^2 - \lambda^1) + \Gamma^3_{21}\, a^1, \end{aligned}$$

where we used the fact that, due to the last equation in (99), derivatives of Γ^3_{12} and Γ^3_{21} are equal. From (107) and (112) we obtain:

$$\Gamma^3_{21}\, \mathbf{r}_1(\lambda^1) = \left(\Gamma^3_{12}\, (\Gamma^2_{21} - \Gamma^3_{31}) - \mathbf{r}_1(\Gamma^3_{12})\right)\, (\lambda^1 - \lambda^2) + \Gamma^3_{11}\, a^1 + 2\, \Gamma^3_{12}\, a^2 \tag{114}$$

Similarly, from (111) and (113) we obtain:

$$\Gamma^3_{12}\, \mathbf{r}_2(\lambda^2) = \left(\mathbf{r}_2(\Gamma^3_{12}) + \Gamma^3_{21}\, (\Gamma^3_{32} - \Gamma^1_{12}\right)\, (\lambda^1 - \lambda^2) - 2\, \Gamma^3_{21}\, a^1 - \Gamma^3_{22}\, a^2. \tag{115}$$

Condition (38) implies that $\Gamma^3_{21} \neq 0$ and $\Gamma^3_{12} \neq 0$ and, therefore, we can solve (114) and (115) for $\mathbf{r}_1(\lambda^1)$ and $\mathbf{r}_2(\lambda^2)$, establishing (92) and (93).

Conversely, given functions λ^1 and λ^2, let a^1, a^2 and a^3 be defined by (90), (91), (102) respectively. Then equations (86) and (101) constitute a Frobenius system for the three unknown components of the flux $\mathbf{f}$. It is straightforward to check that the integrability conditions for this system coincide with the flatness conditions (100), (104) and (108). Reversing the proof of the first part, we see that they are satisfied provided λ^1 and λ^2 satisfy condition 2. Thus, if λ^1 and λ^2 satisfy condition 2, then for any prescription of the initial value $\mathbf{f}(\bar{u})$, there exists a unique $\mathbf{f}$ satisfying (86) and (101). Moreover, since (101) is a consequence of (86), there is a unique $\mathbf{f}$ satisfying (86) for any prescription of the initial value $\mathbf{f}(\bar{u})$. Thus, the generic solution to (86) depends on three constants. We finally note that if $\mathbf{f}$ satisfies (86), then so does $\mathbf{f}$ + (a constant vector in $\mathbb{R}^3$). Therefore, the three arbitrary constants in the generic solution correspond to the components of an arbitrary constant vector. Thus, for the given pair of functions λ^1 and λ^2, the solution of the $\mathcal{F}(\mathfrak{R})$-system (86) is unique up to a constant vector. □

Lemma 6.5. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a partial frame satisfying the assumptions of Theorem 6.2. Then the set of pairs of functions $\lambda(\mathfrak{R}) = \{(\lambda^1, \lambda^2)\}$ satisfying condition 2 of Lemma 6.4 is a real vector space of dimension at most 5.*

Proof. It is straightforward to check that $\lambda(\mathfrak{R})$ is a vector space. To prove the bound on its dimension, we prolong the system of equations (90)-(97) listed in condition 2 of Lemma 6.4 to a Frobenius system for 5 unknown functions λ^1, λ^2, a^1, a^2, and τ, where we define

$$\tau = \mathbf{s}(\lambda^2) \qquad \text{for} \qquad \mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2]. \tag{116}$$

This is done by the following steps.

(1) By expanding the right-hand side of the commutator relationship

$$\mathbf{s}(\lambda^1) = [\mathbf{r}_1, \mathbf{r}_2](\lambda^1)$$

and substituting the expressions for $\mathbf{r}_1(\lambda^1)$, $\mathbf{r}_2(\lambda^1)$, $\mathbf{r}_1(\lambda^2)$, $\mathbf{r}_2(\lambda^2)$, $\mathbf{r}_1(a^1)$, $\mathbf{r}_2(a^1)$, $\mathbf{r}_1(a^2)$, $\mathbf{r}_2(a^2)$ from (90)-(97), we obtain

$$2\,\Gamma^3_{21}\,\mathbf{s}(\lambda^1) + 2\,\Gamma^3_{12}\,\mathbf{s}(\lambda^2) = \Gamma^3_{21}\,\left(A_1\,(\lambda^1 - \lambda^2) + B_1\,a^1 + C_1\,a^2\right), \tag{117}$$

where

$$A_1 = -\mathbf{r}_2\left(\frac{\Upsilon_1}{\Gamma^3_{21}}\right) - \mathbf{r}_1(\Gamma^1_{12}) + \frac{\Upsilon_1\,\Upsilon_2}{\Gamma^3_{21}\,\Gamma^3_{12}} - \Gamma^3_{12}\,(\Gamma^1_{13} + 2\,\Gamma^2_{23}) - \frac{\Gamma^3_{11}}{\Gamma^3_{21}}\,(\Gamma^1_{23}\,\Gamma^3_{12} - \Gamma^1_{32}) + \Gamma^1_{12}\,\Gamma^2_{21} \tag{118}$$

$$B_1 = -\mathbf{r}_2\left(\frac{\Gamma^3_{11}}{\Gamma^3_{21}}\right) - \frac{\Upsilon_1\,(\Gamma^3_{21} - 1)}{\Gamma^3_{21}\,\Gamma^3_{12}} - \frac{\Gamma^3_{11}\,c^3_{23}}{\Gamma^3_{21}} + 2\,\frac{\Gamma^3_{12}\,\Gamma^2_{21}}{\Gamma^3_{21}} + \Gamma^1_{11} - c^3_{13} \tag{119}$$

$$C_1 = 2\,\frac{\mathbf{r}_2(\Gamma^3_{12})}{(\Gamma^3_{21})^2} - \frac{\Upsilon_1\,\Gamma^3_{22}}{\Gamma^3_{21}\,\Gamma^3_{12}} - 2\,\frac{\Gamma^1_{12}}{\Gamma^3_{21}} + \frac{\Gamma^3_{11}\,\Gamma^1_{22}}{\Gamma^3_{21}} + 2\,\frac{\Gamma^3_{12}}{\Gamma^3_{21}}\,(\Gamma^2_{22} - c^3_{23}). \tag{120}$$

(2) By expanding the right-hand side of the commutator relationship

$$\mathbf{s}(\lambda^2) = [\mathbf{r}_1, \mathbf{r}_2](\lambda^2)$$

and substituting the expressions for $\mathbf{r}_1(\lambda^1)$, $\mathbf{r}_2(\lambda^1)$, $\mathbf{r}_1(\lambda^2)$, $\mathbf{r}_2(\lambda^2)$, $\mathbf{r}_1(a^1)$, $\mathbf{r}_2(a^1)$, $\mathbf{r}_1(a^2)$, $\mathbf{r}_2(a^2)$ from (90)-(97), we obtain

$$2\,\Gamma^3_{21}\,\mathbf{s}(\lambda^1) + 2\,\Gamma^3_{12}\,\mathbf{s}(\lambda^2) = \Gamma^3_{12}\left(A_2(\lambda^1 - \lambda^2) + B_2\,a^1 + C_2\,a^2\right), \tag{121}$$

where

$$A_2 = \mathbf{r}_1\left(\frac{\Upsilon_2}{\Gamma^3_{12}}\right) - \mathbf{r}_2\,\left(\Gamma^2_{21}\right) + \frac{\Upsilon_2\,\Upsilon_1}{\Gamma^3_{12}\,\Gamma^3_{21}} - \Gamma^3_{21}(\Gamma^2_{23} + 2\,\Gamma^1_{13}) - \frac{\Gamma^3_{22}}{\Gamma^3_{12}}\,(\Gamma^2_{13}\,\Gamma^3_{21} + \Gamma^2_{31}) + \Gamma^2_{21}\,\Gamma^1_{12} \tag{122}$$

$$B_2 = -2\,\frac{\mathbf{r}_1\,(\Gamma^3_{21})}{(\Gamma^3_{12})^2} + \frac{\Upsilon_2\,\Gamma^3_{11}}{\Gamma^3_{12}\,\Gamma^3_{21}} + 2\,\frac{\Gamma^2_{21}}{\Gamma^3_{12}} + \frac{\Gamma^3_{22}\,\Gamma^2_{11}}{\Gamma^3_{12}} + 2\,\frac{\Gamma^3_{21}}{\Gamma^3_{12}}\,(\Gamma^1_{11} - c^3_{13}) \tag{123}$$

$$C_2 = -\mathbf{r}_1\left(\frac{\Gamma^3_{22}}{\Gamma^3_{12}}\right) + \frac{\Upsilon_2\,(\Gamma^3_{12} + 1)}{\Gamma^3_{12}\,\Gamma^3_{21}} - \frac{\Gamma^3_{22}\,c^3_{13}}{\Gamma^3_{12}} + \frac{2\,\Gamma^3_{21}\,\Gamma^1_{12}}{\Gamma^3_{12}} + \Gamma^2_{22} - c^3_{23}. \tag{124}$$

(3) As the left hand sides of (117) and (121) agree, so do their right hand sides. In fact,

$$\Gamma^3_{21}A_1 \equiv \Gamma^3_{12}A_2, \quad \Gamma^3_{21}B_1 \equiv \Gamma^3_{12}B_2, \qquad \text{and} \qquad \Gamma^3_{21}C_1 \equiv \Gamma^3_{12}C_2 \tag{125}$$

due to the flatness condition (12). To show the A identity in (125), we first compute $\Gamma^3_{21}A_1 - \Gamma^3_{12}A_2$ by substituting Υ_1 and Υ_2 into (118) and (122) and making various simplifications. We obtain

$$\begin{aligned}\Gamma^3_{21}A_1 - \Gamma^3_{12}A_2 = &-\mathbf{s}(\Gamma^3_{12}) + \Gamma^3_{12}\,\mathbf{r}_2(\Gamma^3_{31}) - \Gamma^3_{21}\,\mathbf{r}_1(\Gamma^3_{32}) - \Gamma^2_{21}\,\Gamma^3_{32} + \Gamma^3_{31}\,\Gamma^3_{32} - \Gamma^3_{31}\Gamma^1_{12}\\ &+ \Gamma^3_{21}\,\Gamma^3_{12}\,(\Gamma^1_{13} - \Gamma^2_{23}) - \Gamma^3_{11}\,(\Gamma^1_{23}\Gamma^3_{12} - \Gamma^1_{32}) + \Gamma^3_{22}\,(\Gamma^2_{13}\,\Gamma^3_{21} + \Gamma^2_{31}).\end{aligned} \tag{126}$$

We then expand the identity

$$\Gamma^3_{12}\left(\nabla_{\mathbf{r}_2}\nabla_{\mathbf{s}}\mathbf{r}_1 - \nabla_{\mathbf{s}}\nabla_{\mathbf{r}_2}\mathbf{r}_1 - \nabla_{[\mathbf{r}_2,\mathbf{s}]}\,\mathbf{r}_1\right) - \Gamma^3_{21}\left(\nabla_{\mathbf{r}_1}\nabla_{\mathbf{s}}\mathbf{r}_2 - \nabla_{\mathbf{s}}\nabla_{\mathbf{r}_1}\mathbf{r}_2 - \nabla_{[\mathbf{r}_1,\mathbf{s}]}\,\mathbf{r}_2\right) \equiv 0. \tag{127}$$

and observe that the coefficient of $\mathbf{s}$ in (127) equals to the left hand side of (126). Similarly, we use the $\mathbf{s}$ coefficient of the expanded identity $\nabla_{\mathbf{r}_1}\nabla_{\mathbf{r}_2}\mathbf{r}_1 - \nabla_{\mathbf{r}_2}\nabla_{\mathbf{r}_1}\mathbf{r}_1 \equiv \nabla_{\mathbf{s}}\mathbf{r}_2$ to show the B-identity of (125), and the $\mathbf{s}$ coefficient of the expanded identity $\nabla_{\mathbf{r}_1}\nabla_{\mathbf{r}_2}\mathbf{r}_2 - \nabla_{\mathbf{r}_2}\nabla_{\mathbf{r}_1}\mathbf{r}_2 \equiv \nabla_{\mathbf{s}}\mathbf{r}_2$ to show the C-identity of (125).

(4) Introducing a new unknown function τ, defined by (116), we solve (117) for $\mathbf{s}(\lambda^1)$:

$$\mathbf{s}(\lambda^1) = -\frac{\Gamma^3_{12}}{\Gamma^3_{21}}\,\tau + \frac{1}{2}A_1\,(\lambda^1 - \lambda^2) + \frac{1}{2}\,B_1\,a^1 + \frac{1}{2}C_1\,a^2 \tag{128}$$

and rewrite (96) and (97) as

$$\mathbf{r}_1(a^1) = \Big(\frac{1}{2}A_1 + \Gamma^1_{13}\Gamma^3_{12}\Big)(\lambda^1 - \lambda^2) + \Big(\frac{1}{2}\,B_1 - \Gamma^1_{11} + c^3_{13}\Big)a^1 + \Big(\frac{1}{2}C_1 - \Gamma^1_{12}\Big)a^2 - \frac{\Gamma^3_{12}}{\Gamma^3_{21}}\,\tau \tag{129}$$

$$\mathbf{r}_2(a^2) = \Gamma^2_{23}\,\Gamma^3_{21}\,(\lambda^1 - \lambda^2) - \Gamma^2_{21}\,a^1 + (c^3_{23} - \Gamma^2_{22})a^2 + \tau. \tag{130}$$

(5) To complete the system (90)-(95), (116), (128), (129), and (130) to a Frobenius system we need to express the remaining derivatives $\mathbf{s}(a^1)$, $\mathbf{s}(a^2)$, $\mathbf{r}_1(\tau)$, $\mathbf{r}_2(\tau)$ and $\mathbf{s}(\tau)$ as functions of $\lambda^1, \lambda^2, a^1, a^2$ and τ. For this purpose, we consider further commutator relationships. Expanding the left hand side of the relation $[\mathbf{r}_1, \mathbf{s}](\lambda^2) = \mathbf{r}_1(\mathbf{s}(\lambda^2)) - \mathbf{s}(\mathbf{r}_1(\lambda^2))$, and substituting (116) and (91) into the right hand side, we obtain

$$c^1_{13}\,\mathbf{r}_1(\lambda^2) + c^2_{13}\,\mathbf{r}_2(\lambda^2) + c^3_{13}\,\mathbf{s}(\lambda^2) = \mathbf{r}_1(\tau) - \mathbf{s}(\Gamma^2_{21}\,(\lambda^1 - \lambda^2) + a^2). \tag{131}$$

By substituting the already known expressions for $\mathbf{r}_1(\lambda^2)$, $\mathbf{r}_2(\lambda^2)$, $\mathbf{s}(\lambda^2)$, $\mathbf{s}(\lambda^1)$, given by (91), (93), (116), (128), respectively, into (131), we obtain:

$$\mathbf{r}_1(\tau) - \mathbf{s}(a^2) = \mathcal{L}_1(\lambda^1 - \lambda^2, a^1, a^2, \tau), \tag{132}$$

where $\mathcal{L}_1$ is a linear function with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives. Applying the same procedure to the relation $[\mathbf{r}_2, \mathbf{s}](\lambda^2) = \mathbf{r}_2(\mathbf{s}(\lambda^2)) - \mathbf{s}(\mathbf{r}_2(\lambda^2))$, yields

$$c_{23}^1\,\mathbf{r}_1(\lambda^2) + c_{23}^2\,\mathbf{r}_2(\lambda^2) + c_{23}^3\,\mathbf{s}(\lambda^2) = \mathbf{r}_2(\tau) - \mathbf{s}\big(\tfrac{1}{\Gamma_{12}^3}(\Upsilon_2\,(\lambda^1 - \lambda^2) - 2\,\Gamma_{21}^3\,a^1 - \Gamma_{22}^3\,a^2)\big), \quad (133)$$

and the same substitutions as above yields

$$\Gamma_{12}^3\,\mathbf{r}_2(\tau) + 2\,\Gamma_{21}^3\,\mathbf{s}(a^1) + \Gamma_{22}^3\,\mathbf{s}(a^2) = \mathcal{L}_2(\lambda^1 - \lambda^2, a^1, a^2, \tau), \quad (134)$$

where $\mathcal{L}_2$ is linear with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives. Performing similar calculations for the relations $[\mathbf{r}_2, \mathbf{s}](\lambda^1) = \mathbf{r}_2(\mathbf{s}(\lambda^1)) - \mathbf{s}(\mathbf{r}_2(\lambda^1))$ and $[\mathbf{r}_1, \mathbf{s}](\lambda^1) = \mathbf{r}_1(\mathbf{s}(\lambda^1)) - \mathbf{s}(\mathbf{r}_1(\lambda^1))$, we obtain

$$-\,\Gamma_{12}^3\,\mathbf{r}_2(\tau) + \Gamma_{21}^3\,\mathbf{s}(a^1) = \mathcal{L}_3(\lambda^1 - \lambda^2, a^1, a^2, \tau), \quad (135)$$

and

$$\Gamma_{12}^3\,\mathbf{r}_1(\tau) + \Gamma_{11}^3\,\mathbf{s}(a^1) + 2\,\Gamma_{12}^3\,\mathbf{s}(a^2) = \mathcal{L}_4(\lambda^1 - \lambda^2, a^1, a^2, \tau), \quad (136)$$

where $\mathcal{L}_3$ and $\mathcal{L}_4$ are linear with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives.

Equations (132), (134), (135) and (136) can be viewed as a linear inhomogeneous system of four equations for the four unknowns $\mathbf{s}(a^1)$, $\mathbf{s}(a^2)$, $\mathbf{r}_1(\tau)$ and $\mathbf{r}_2(\tau)$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \Gamma_{12}^3 & 2\,\Gamma_{21}^3 & \Gamma_{22}^3 \\ 0 & -\Gamma_{12}^3 & \Gamma_{21}^3 & 0 \\ \Gamma_{12}^3 & 0 & \Gamma_{11}^3 & 2\,\Gamma_{12}^3 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(\tau) \\ \mathbf{r}_2(\tau) \\ \mathbf{s}(a^1) \\ \mathbf{s}(a^2) \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \end{bmatrix}. \quad (137)$$

Let M denote the 4×4 matrix on the left of (137). The upper left 3×3 minor of M equals to $3\,\Gamma_{12}^3\Gamma_{21}^3$ and is non-zero under our assumptions. Thus the rank of M is at least 3. We find that

$$\det(M) = \Gamma_{12}^3\,(9\Gamma_{12}^3\,\Gamma_{21}^3 - \Gamma_{11}^3\,\Gamma_{22}^3),$$

where the expression in the parentheses is non-zero near $\bar{u}$ due to the assumption (89) in Theorem 6.2. Solving (137) we obtain expressions for $\mathbf{s}(a^1)$, $\mathbf{s}(a^2)$, $\mathbf{r}_1(\tau)$, and $\mathbf{r}_2(\tau)$ as linear functions of $\lambda^1 - \lambda^2, a^1, a^2, \tau$, with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives. Finally,

$$\mathbf{s}(\tau) = [\mathbf{r}_1, \mathbf{r}_2](\tau) = \mathbf{r}_1(\mathbf{r}_2(\tau)) - \mathbf{r}_2(\mathbf{r}_1(\tau)), \quad (138)$$

and substitution of the known expressions of the derivatives, $\mathbf{r}_1(\tau)$, $\mathbf{r}_2(\tau)$, $\mathbf{r}_1(\lambda^1)$, $\mathbf{r}_2(\lambda^1)$, $\mathbf{r}_1(\lambda^2)$, $\mathbf{r}_2(\lambda^2)$, $\mathbf{r}_1(a^1)$, $\mathbf{r}_2(a^1)$, $\mathbf{r}_1(a^2)$, and $\mathbf{r}_2(a^2)$, yields a linear function of $\lambda^1 - \lambda^2, a^1, a^2, \tau$, with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives.

(6) The fifteen equations (90)-(95), (116), (128), (129), (130), (137) and (138) can be used to express all directional derivatives of the functions λ^1, λ^2, a^1, a^2 and τ as linear combinations of $\lambda^1 - \lambda^2$, a^1, a^2, and τ, with coefficients depending on c_{ij}^k, Γ_{ij}^k, and their derivatives. These expressions provide a Frobenius system. If its integrability conditions are satisfied, the generic solution depends on 5 constants, the prescribed values of these functions at $\bar{u}$. If the integrability conditions for this system are not identically satisfied, they will impose additional relationships on λ^1, λ^2, a^1, a^2 and τ, thus reducing the size of the solution set.

(7) The Frobenius type system (90)-(95), (116), (128), (129), (130), (137) and (138) was obtained as a consequence of condition 2 of Lemma 6.4. Therefore, the vector space of pairs of functions $\lambda(\mathfrak{R}) = \{(\lambda^1, \lambda^2)\}$ satisfying this condition is of dimension at most 5. □

Lemma 6.6. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a non-involutive partial frame satisfying condition* (87)*. Assume $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$ is a non-hyperbolic flux. Then the corresponding eigenfunctions λ^1 and λ^2, appearing in* (86)*, coincide and are non-constant: $\lambda^1 = \lambda^2 = \lambda$ with λ non-constant.*

Proof. We recall that $\mathbf{f}$ being non-hyperbolic means that the operator $\widetilde{\nabla}_{(\cdot)}\mathbf{f}$, does not posses three real eigenfunctions. However, (86) holds by assumption, so that it possesses two real eigenfunctions λ^1 and λ^2. As complex eigenfunctions come in conjugate pairs and $n = 3$, the possibility of a third eigenfunction being complex is excluded. Therefore, $\mathbf{f}$ must have a generalized eigenvector field, which we denote $\mathbf{s}$. Let c_{ij}^k and Γ_{ij}^k denote structure coefficients and Christoffel symbols for $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}\}$. First, assume for contradiction that $\lambda^1 \neq \lambda^2$. Then either

$$\widetilde{\nabla}_{\mathbf{s}}\mathbf{f} = \mathbf{r}_1 + \lambda^1\,\mathbf{s} \qquad \text{or} \qquad \widetilde{\nabla}_{\mathbf{s}}\mathbf{f} = \mathbf{r}_2 + \lambda^2\,\mathbf{s}. \tag{139}$$

Without loss of generality we assume that the second equality holds (if not, relabel $\mathbf{r}_1$ and $\mathbf{r}_2$). Then (100) together with (86) and the second equation in (139) imply

$$c_{12}^1\lambda^1\mathbf{r}_1 + c_{12}^2\lambda^2\mathbf{r}_2 + c_{12}^3(\mathbf{r}_2 + \lambda^2\,\mathbf{s}) = \widetilde{\nabla}_{\mathbf{r}_1}(\lambda^2\,\mathbf{r}_2) - \widetilde{\nabla}_{\mathbf{r}_2}(\lambda^1\,\mathbf{r}_1). \tag{140}$$

Using (86) again and collecting the coefficients for $\mathbf{s}$, we obtain $\Gamma_{21}^3(\lambda^2 - \lambda^1) = 0$. Condition (87) implies $\Gamma_{21}^3 \neq 0$ on an open neighborhood of $\bar{u}$, and, therefore, $\lambda^1 = \lambda^2$ on this neighborhood. Next, let $\lambda^1 = \lambda^2 = \lambda$. Then, since $\mathbf{s}$ is a generalized eigenvector field, we must have

$$\widetilde{\nabla}_{\mathbf{s}}\mathbf{f} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2 + \lambda\,\mathbf{s}, \tag{141}$$

where α and β are functions with $\alpha(\bar{u})$ or $\beta(\bar{u})$ non-zero. Assume for contradiction that λ is a constant function on a neighborhood of $\bar{u}$. Then (100) together with (86) and (141) imply that

$$c_{12}^1\lambda\mathbf{r}_1 + c_{12}^2\lambda\mathbf{r}_2 + c_{12}^3(\alpha\,\mathbf{r}_1 + \beta\,\mathbf{r}_2 + \lambda\,\mathbf{s}) = \lambda\widetilde{\nabla}_{\mathbf{r}_1}\mathbf{r}_2 - \lambda\widetilde{\nabla}_{\mathbf{r}_2}\mathbf{r}_1. \tag{142}$$

On the left hand side of (142), we notice that $c_{12}^1\lambda\mathbf{r}_1 + c_{12}^2\lambda\mathbf{r}_2 + c_{12}^3\lambda\,\mathbf{s} = \lambda\,[\mathbf{r}_1, \mathbf{r}_2]$. At the same time, the right hand side of (142) equals to $\lambda\,[\mathbf{r}_1, \mathbf{r}_2]$ due to the symmetry condition (11). Then $\alpha\,\mathbf{r}_1 + \beta\,\mathbf{r}_2 = 0$, which contradicts our assumption that vectors $\mathbf{r}_1|_{\bar{u}}$ and $\mathbf{r}_2|_{\bar{u}}$ are independent and α and β are functions with $\alpha(\bar{u})$ or $\beta(\bar{u})$ non zero. Thus λ is a non-constant function. $\square$

Lemma 6.7. *Let $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ be a non-involutive partial frame satisfying condition* (87). *Assume $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$ is a non-hyperbolic flux. Then all other non-hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are of the form $c\,\mathbf{f}$ + (trivial flux) where $c \neq 0 \in \mathbb{R}$.*

Proof. (1) Let $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$ be a non-hyperbolic flux. By Lemma 6.6 there exists a non-constant function λ in a neighborhood of $\bar{u}$, such that $\mathbf{f}$ and $\lambda^1 = \lambda^2 = \lambda$ satisfy (86). A calculation shows that if $\bar{\lambda} \in \mathbb{R}$ and $\bar{f} \in \mathcal{F}^{\text{id}}$ (see (27)), then $c\,\mathbf{f} + \bar{\lambda}\bar{\mathbf{f}}$ is a non-hyperbolic flux which verifies (86) (with $\lambda^1 = \lambda^2 = c\,\lambda + \bar{\lambda}$). Recalling (28), we conclude that $c\,\mathbf{f}$ + (a trivial flux) belongs to $\mathcal{F}(\mathfrak{R})$; clearly these fluxes are non-hyperbolic. It remains to show that any non-hyperbolic flux in $\mathcal{F}(\mathfrak{R})$ is of this form.

(2) Lemma 6.4 implies that the function λ together with the functions

$$a^1 := -\mathbf{r}_2(\lambda) \qquad \text{and} \qquad a^2 := \quad \mathbf{r}_1(\lambda) \tag{143}$$

satisfy the following system (these are (92)–(97) in the case $\lambda^1 = \lambda^2$):

$$\mathbf{r}_1(\lambda) = \frac{1}{\Gamma_{21}^3}\big(\Gamma_{11}^3\,a^1 + 2\,\Gamma_{12}^3\,a^2\big), \tag{144}$$

$$\mathbf{r}_2(\lambda) = -\frac{1}{\Gamma_{12}^3}\big(2\,\Gamma_{21}^3\,a^1 + \Gamma_{22}^3\,a^2\big), \tag{145}$$

$$\mathbf{r}_2(a^1) = (c_{23}^3 - \Gamma_{21}^1)\,a^1 - \Gamma_{22}^1\,a^2, \tag{146}$$

$$\mathbf{r}_1(a^2) = -\Gamma_{11}^2\,a^1 + (c_{13}^3 - \Gamma_{12}^2)\,a^2, \tag{147}$$

$$\mathbf{r}_1(a^1) - \mathbf{s}(\lambda) = -(\Gamma_{11}^1 - c_{13}^3)\,a^1 - \Gamma_{12}^1\,a^2, \tag{148}$$

$$\mathbf{r}_2(a^2) - \mathbf{s}(\lambda) = -\Gamma_{21}^2\,a^1 + (c_{23}^3 - \Gamma_{22}^2)a^2, \tag{149}$$

where the functions Γ_{ij}^k are Christoffel symbols for $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s} := [\mathbf{r}_1, \mathbf{r}_2]\}$. Equation (143) immediately imply that

$$\mathbf{s}(\lambda) = \mathbf{r}_1(\mathbf{r}_2(\lambda)) - \mathbf{r}_2(\mathbf{r}_1(\lambda)) = -\mathbf{r}_1(a^1) - \mathbf{r}_2(a^2). \tag{150}$$

Then from (150), together with (148) and (149), we obtain:

$$\mathbf{s}(\lambda) = \frac{1}{3}(\Gamma_{11}^1 + \Gamma_{21}^2 - c_{13}^3)\, a^1 + \frac{1}{3}(\Gamma_{22}^2 + \Gamma_{12}^1 - c_{23}^3)\, a^2\,, \tag{151}$$

$$\mathbf{r}_1(a^1) = \frac{1}{3}(-2\,\Gamma_{11}^1 + \Gamma_{21}^2 + 2\,c_{13}^3)\, a^1 + \frac{1}{3}(\Gamma_{22}^2 - 2\,\Gamma_{12}^1 - c_{23}^3)\, a^2, \tag{152}$$

$$\mathbf{r}_2(a^2) = \frac{1}{3}(\Gamma_{11}^1 - 2\,\Gamma_{21}^2 - c_{13}^3)\, a^1 + \frac{1}{3}(-2\,\Gamma_{22}^2 + \Gamma_{12}^1 + 2\,c_{23}^3)\, a^2. \tag{153}$$

From Lemma 6.6 we know that λ is a non-constant function, and, therefore, at least one of its derivatives in the frame directions must be non-zero. Examining (143) and (151), we conclude that at least one of the functions a^1 or a^2 is non zero. Without loss of generality, we assume that $a^1 \neq 0$ (otherwise, relabel $\mathbf{r}_1$ and $\mathbf{r}_2$).

(3) Equations (143), (144), (145) imply

$$\begin{bmatrix} \Gamma_{11}^3 & \Gamma_{12}^3 + 1 \\ \Gamma_{21}^3 - 1 & \Gamma_{22}^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0. \tag{154}$$

Since $[a^1, a^2]^T$ is non-zero, the determinant of the matrix in (154) must vanish, i.e.

$$\Gamma_{11}^3\,\Gamma_{22}^3 - (\Gamma_{12}^3 + 1)(\Gamma_{21}^3 - 1) = 0. \tag{155}$$

Substituting $c_{12}^3 = 1$ in (155) and simplifying, we get the condition

$$\Gamma_{12}^3\,\Gamma_{21}^3 - (c_{12}^3)^2 = \Gamma_{11}^3\,\Gamma_{22}^3. \tag{156}$$

(4) At least one of the expressions $\Gamma_{12}^3 + 1$ or $\Gamma_{21}^3 - 1$ is non-zero: if both were zero, then $c_{21}^3 = -2$, which contradicts our assumption that the c_{ij}^k are the structure coefficients for the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2]\}$. Thus (154) has a one parameter family of solutions. In part (2) of the proof, we argued that we may assume $a^1 \neq 0$. Then, from (154), we can express

$$a^2 = \alpha\, a^1, \tag{157}$$

where $\alpha(u)$ is a known function expressible in terms of the Γ_{ij}^k. (Explicitly, if $\Gamma_{12}^3 \neq -1$, then $\alpha = \frac{\Gamma_{11}^3}{\Gamma_{12}^3 + 1}$, otherwise, we can show that $\Gamma_{22}^3 \neq 0$ and $\alpha = \frac{3}{\Gamma_{22}^3}$.) Substitution of (157) into (143), (146), (151), and (152), gives

$$\mathbf{r}_1(\lambda) = \alpha\, a^1 \tag{158}$$

$$\mathbf{r}_2(\lambda) = -a^1 \tag{159}$$

$$\mathbf{s}(\lambda) = \alpha_1\, a^1 \tag{160}$$

$$\mathbf{r}_1(a^1) = \alpha_2\, a^1 \tag{161}$$

$$\mathbf{r}_2(a^1) = \alpha_3\, a^1, \tag{162}$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are known functions, expressible in terms of the functions Γ_{ij}^k and their derivatives. Substituting (161) and (162) in the commutator relationship, we conclude that

$$\mathbf{s}(a^1) = \mathbf{r}_1(\mathbf{r}_2(a^1)) - \mathbf{r}_2(\mathbf{r}_1(a^1)) = \alpha_4\, a^1, \tag{163}$$

where α_4 is another known function, expressible in terms of the functions Γ_{ij}^k and their derivatives. The system (158)-(163) is a Frobenius system for the two unknowns λ and a^1, and so its solution depends on at most two arbitrary constants.

(5) Parts (1) and (2) of the proof show that there exist non-constant functions λ and $a^1 = -\mathbf{r}_2(\lambda)$, satisfying (158)-(163), giving a 2-parameter family of solutions $\lambda_{c,\bar{\lambda}} = c\,\lambda + \bar{\lambda}$, $a_c^1 = c\,a^1$, where $c, \bar{\lambda}$ are arbitrary constants. Part (4) shows that there are no other solution. Also, each $\lambda_{c,\bar{\lambda}}$, with $c \neq 0$, corresponds to a 3-parameter family of non-hyperbolic fluxes $c\,\mathbf{f} + \bar{\lambda}\bar{\mathbf{f}}$, where $\bar{\mathbf{f}} \in \mathcal{F}^{\mathrm{id}}$. We conclude that any non-hyperbolic flux in $\mathcal{F}(\mathfrak{R})$ is of the form $c\,\mathbf{f} +$ (a trivial flux). □

Remark 6.8. *From (143) it follows that if* $\mathbf{f}$ *is a non-hyperbolic flux for* $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$*, then* $\mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2]$ *is a generalized eigenvector field of* $\mathbf{f}$*. Indeed,*

$$\widetilde{\nabla}_{[\mathbf{r}_1,\mathbf{r}_2]}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_1}\widetilde{\nabla}_{\mathbf{r}_2}\mathbf{f} - \widetilde{\nabla}_{\mathbf{r}_2}\widetilde{\nabla}_{\mathbf{r}_1}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_1}(\lambda\,\mathbf{r}_2) - \widetilde{\nabla}_{\mathbf{r}_2}(\lambda\,\mathbf{r}_1) = a^1\,\mathbf{r}_1 + a^2\mathbf{r}_2 + \lambda\,[\mathbf{r}_1, \mathbf{r}_2]. \tag{164}$$

Proof of Theorem 6.1: 1. We want to show that a non-zero flux $\mathbf{f} \in \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\mathrm{triv}}$ is either strictly hyperbolic or non-hyperbolic. Assume that there exists a non-strictly hyperbolic flux $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$. Then $\mathbf{f}$ has the third eigenvector field $\mathbf{r}_3$ and at least two of the corresponding eigenvalue functions λ^1, λ^2 and λ^3 coincide in an neighborhood of a fixed point $\bar{u} \in \Omega$. Examining the $\mathbf{r}_3$ component of the expended flatness condition (100), we conclude that

$$\Gamma_{12}^3\,\lambda^2 - \Gamma_{21}^3\,\lambda^1 = c_{12}^3\,\lambda^3, \tag{165}$$

where here c_{ij}^k and Γ_{ij}^k denote structure coefficients and Christoffel symbols for $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$. (165) must hold as an identity near $\bar{u}$, and may be written as

$$\Gamma_{12}^3\,(\lambda^2 - \lambda^3) - \Gamma_{21}^3\,(\lambda^1 - \lambda^3) \equiv 0. \tag{166}$$

From the assumption of the theorem it follows that $\Gamma_{12}^3 \neq 0$, $\Gamma_{21}^3 \neq 0$, and $\Gamma_{12}^3 \neq \Gamma_{21}^3$. From (166) we conclude that if any two of the functions $\lambda^1, \lambda^2, \lambda^3$ are equal, then all three of them must be equal, to $\lambda(u)$, say. This implies that $\widetilde{\nabla}_{\mathbf{r}}\mathbf{f} = \lambda\mathbf{r}$ for any $r \in \mathcal{X}(\Omega)$. The flatness conditions

$$\widetilde{\nabla}_{[\mathbf{r}_1,\mathbf{r}_i]}\mathbf{f} = \widetilde{\nabla}_{\mathbf{r}_1}\widetilde{\nabla}_{\mathbf{r}_i}\mathbf{f} - \widetilde{\nabla}_{\mathbf{r}_i}\widetilde{\nabla}_{\mathbf{r}_1}\mathbf{f} \qquad \text{for } i = 2, 3,$$

then imply that

$$\lambda\,[\mathbf{r}_1, \mathbf{r}_i] = \mathbf{r}_1(\lambda\,\mathbf{r}_i) - \mathbf{r}_i(\lambda\,\mathbf{r}_1) \qquad \text{for } i = 2, 3.$$

As the right hand side of the above equation is $\lambda\,[\mathbf{r}_1, \mathbf{r}_i] + \mathbf{r}_1(\lambda)\,\mathbf{r}_i - \mathbf{r}_i(\lambda)\,\mathbf{r}_1$, and $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are independent, we conclude that $\mathbf{r}_i(\lambda) = 0$ for $i = 1, 2, 3$. Therefore, $\lambda \equiv \bar{\lambda} \in \mathbb{R}$ is a constant function. This implies that $\mathbf{f}$ is a trivial flux, and the statement is proven.

2. From Lemma 6.7, if $\mathcal{F}(R)$ contains strictly hyperbolic fluxes, then up to adding a trivial flux, it contains exactly a one parameter family of non-hyperbolic fluxes. Therefore, if $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\mathrm{triv}} > 1$, then $\mathcal{F}(\mathfrak{R})$ contains hyperbolic fluxes, and, from the first part of the theorem, we know that all non-trivial hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are strictly hyperbolic.

3. In the proof of Lemma 6.7 (see (156)), we showed that if $\mathcal{F}(\mathfrak{R})$ contains non-hyperbolic fluxes, then (88) holds with c_{ij}^k and Γ_{ij}^k being the structure coefficients and Christoffel symbols of the connection $\widetilde{\nabla}$ relative to the frame $\{\mathbf{r}_1, \mathbf{r}_2, [\mathbf{r}_1, \mathbf{r}_2]\}$. Then Lemma 6.3 asserts that (88) holds with c and Γ corresponding to any completion $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{s}\}$ of $\mathfrak{R}$ to a frame. □

Proof of Theorem 6.2: 1. We want to show that $0 \leq \dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\mathrm{triv}} \leq 4$. Lemma 6.5 asserts that under the assumptions of Theorem 6.2, the set of pairs of functions $\lambda(\mathfrak{R}) = \{(\lambda^1, \lambda^2)\}$ satisfying condition 2 of Lemma 6.4 is a real vector space of dimension at most 5. In addition, Lemma 6.4 implies that for every λ^1 and λ^2 satisfying condition 2, there exists a unique (up to a constant vector) flux $\mathbf{f}$ satisfying (86). Thus $\dim \mathcal{F}(\mathfrak{R}) \leq 8$. On the other hand, $\mathcal{F}(\mathfrak{R})$ contains a 4-dimensional subspace of trivial fluxes, giving the stated inequality.

2. For $k = 0, \ldots, 4$, Examples 7.4-7.8 exhibit partial frames, satisfying the assumptions of the theorem, and with $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\mathrm{triv}} = k$. □

7 Examples

The examples provided in this section illustrate the main results of the paper and also provide a proof for the existence statement in Theorem 6.2. The computations were performed with MAPLE by setting up systems of differential equations for $\mathbf{f}$ and λ and applying `pdsolove`.

7.1 Rich partial frames

For rich partial frames satisfying (51) Theorem 5.7 gives the degree of freedom for prescribing λ and $\mathbf{f}$ satisfying the $\mathcal{F}(\mathfrak{R})$-system (25). The theorem also asserts that $\mathcal{F}(\mathfrak{R})$ contains strictly hyperbolic fluxes. The following three examples demonstrate these results. They also illustrate the fact that a hyperbolic flux corresponding to a *rich partial frame* may have a *non-rich full frame*. In fact, there are three different scenarios: in Example 7.1 all strictly hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are rich, in Example 7.2 all hyperbolic (strictly and non-strictly) fluxes in $\mathcal{F}(\mathfrak{R})$ are non-rich, and in Example 7.3 $\mathcal{F}(\mathfrak{R})$ contains both rich and non-rich strictly hyperbolic fluxes.

In the following examples, $n = 3$ and $m = 2$. The standard affine coordinates in $\mathbb{R}^3$ for the connection $\widetilde{\nabla}$ are denoted by (u, v, w). We start with a simple example, a partial frame given by the first two standard vectors in $\mathbb{R}^3$.

Example 7.1. Let $\mathbf{r}_1 = [1, 0, 0]^T$ and $\mathbf{r}_2 = [0, 1, 0]^T$ comprise a partial frame $\mathfrak{R}$ on $\mathbb{R}^3$. It is clear that $\mathfrak{R}$ satisfies the assumptions of Theorem 5.7, and as predicted by this theorem λ^1 and λ^2 are parametrized by two functions of two variables:

$$\lambda^1 = \phi(u, w) \qquad \text{and} \qquad \lambda^2 = \psi(v, w). \tag{167}$$

For each such pair of λ^1 and λ^2 we get a family of fluxes in $\mathcal{F}(\mathfrak{R})$ parametrized by three arbitrary functions of one variable, g, h and k:

$$\mathbf{f} = \left[\int_*^u \phi(s, w)\, ds + g(w), \int_*^v \psi(s, w)\, ds + h(w), k(w)\right]^T. \tag{168}$$

On the other hand, we could start by parametrizing the set $\mathcal{F}(\mathfrak{R})$ by two arbitrary functions Φ and Ψ of two variables and an arbitrary function k of one variable:

$$\mathbf{f} = [\Phi(u, w), \Psi(v, w), k(w)]^T \qquad \text{with} \qquad \lambda^1 = \partial_u \Phi, \quad \lambda^2 = \partial_v \Psi. \tag{169}$$

Of course, (169) is equivalent to (167)-(168), but in (169) the functions g, h are absorbed into Φ and Ψ. While (169) is simpler, (167)-(168) more closely illustrates the argument in the proof of Theorem 5.7. Obviously, for most choices of Φ, Ψ, and k, the resulting flux is strictly hyperbolic.

We finally show that all strictly hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$ are rich. Let $\mathbf{r}_3$ be the third eigenvector field of a hyperbolic flux $\mathbf{f} \in \mathcal{F}(\mathfrak{R})$. Since $\mathbf{r}_3$ is linearly independent of $\mathbf{r}_1$ and $\mathbf{r}_2$, it can be, up to rescaling, written as $\mathbf{r}_3 = [a, b, 1]^T$, where a and b are functions on $\mathbb{R}^3$. As $\widetilde{\nabla}_{\mathbf{r}_3}\mathbf{r}_1 = \widetilde{\nabla}_{\mathbf{r}_3}\mathbf{r}_2 = 0$, we have in particular that

$$\Gamma_{31}^2 = \Gamma_{32}^1 = 0 \quad \text{and therefore} \qquad c_{13}^2 = \Gamma_{13}^2 \qquad \text{and} \qquad c_{23}^1 = \Gamma_{23}^2. \tag{170}$$

We also have

$$\Gamma_{12}^3 = \Gamma_{21}^3 = c_{12}^3 = 0. \tag{171}$$

Substituting (170) and (171) into (41) produces two equations:

$$\Gamma_{23}^1 \,(\lambda^3 - \lambda^1) = 0 \quad \text{and} \quad \Gamma_{13}^2 \,(\lambda^3 - \lambda^2) = 0. \tag{172}$$

If $\Gamma_{23}^1 \neq 0$ or $\Gamma_{13}^2 \neq 0$, then (172) implies that $\lambda^3 = \lambda^1$ or $\lambda^3 = \lambda^2$ and therefore $\mathbf{f}$ is not strictly hyperbolic. If $\Gamma_{23}^1 = 0$ and $\Gamma_{13}^2 = 0$, then (171) implies that $c_{23}^1 = 0$ and $c_{13}^2 = 0$, and therefore $\mathbf{f}$ is rich. Thus $\mathcal{F}(\mathfrak{R})$ does not contain non-rich strictly hyperbolic fluxes.

On the other hand, the following example presents a rich pair of vector fields satisfying (51) and which admits only non-rich hyperbolic fluxes.

Example 7.2. Consider the partial frame $\mathfrak{R}$ consisting of the vector fields $\mathbf{r}_1 = [1, 0, 0]^T$ and $\mathbf{r}_2 = [w, 1, 0]^T$ on a set $\Omega \subset \mathbb{R}^3$ where $w \neq 0$. As $[\mathbf{r}_1, \mathbf{r}_2] = 0$, $\widetilde{\nabla}_{\mathbf{r}_1} \mathbf{r}_2 = 0$, and $\widetilde{\nabla}_{\mathbf{r}_2} \mathbf{r}_1 = 0$, we are in the case considered in Theorem 5.7. As predicted by Theorem 5.7, the freedom for prescribing λ^1 and λ^2 consists of two arbitrary functions of two variables:

$$\lambda^1 = \phi\big(w, v - \tfrac{u}{w}\big) \qquad \text{and} \qquad \lambda^2 = \psi(v, w). \tag{173}$$

The corresponding family of fluxes is

$$\mathbf{f} = \left[w \int_*^v \psi(s, w)\, ds - w \int_*^{v - \frac{u}{w}} \phi(w, s)\, ds + g(w), \int_*^v \psi(s, w)\, ds + h(w), k(w) \right]^T,$$

where g, h and k are arbitrary functions of one variable.

Proposition 5.3 gives that for any $\bar{u} \in \Omega$ and any choices of ϕ and ψ such that the $\lambda^1(\bar{u}) \neq \lambda^2(\bar{u})$, one can find functions h, g and k so that the resulting flux $\mathbf{f}$ is strictly hyperbolic. For a concrete example, let $\phi\left(w, v - \frac{u}{w}\right) = -\frac{1}{w}$ and $\psi(v, w) = 0$, $g(w) = h(w) = 0$ and $k(w) = -\frac{1}{w} - \log w$. We observe that the flux

$$\mathbf{f} = \left[v - \frac{u}{w}, 0, -\frac{1}{w} - \log w \right]^T$$

is strictly hyperbolic with eigenvalues

$$\lambda^1 = -\frac{1}{w}, \qquad \lambda^2 = 0, \qquad \lambda^3 = \frac{1 - w}{w^2},$$

and with the third eigenvector given by $\mathbf{r}_3 = [u, 0, 1]^T$.

We now show that, although the partial frame $\mathfrak{R}$ is rich, the corresponding set of fluxes $\mathcal{F}(\mathfrak{R})$ does not contain any rich hyperbolic fluxes. Indeed, let $\mathbf{r}_3$ be the third eigenvector of a strictly hyperbolic flux in $\mathcal{F}(\mathfrak{R})$. Up to a scaling, any vector field which is linearly independent from $\mathbf{r}_1$ and $\mathbf{r}_2$, is of the form $\mathbf{r}_3 = [a, b, 1]^T$, where a and b are arbitrary functions on $\mathbb{R}^3$. Since $[\mathbf{r}_3, \mathbf{r}_2] = [1, 0, 0]^T$, we have $c^1_{32} = 1$, and, therefore, there is no rich hyperbolic fluxes in $\mathcal{F}(\mathfrak{R})$.

Finally, we present an example of a rich partial frame $\mathfrak{R}$, which admits both rich and non-rich strictly hyperbolic fluxes.

Example 7.3. Consider a partial frame $\mathfrak{R}$, consisting of the vector fields $\mathbf{r}_1 = [1, -\sqrt{u}, 0]^T$ and $\mathbf{r}_2 = [1, -\sqrt{u}, 0]^T$ on a set $\Omega \subset \mathbb{R}^3$ where $u \neq 0$. One can directly check that the assumption of Theorem 5.7 are satisfied.

Adjoining the third vector field $\mathbf{r}_3 = [0, 0, 1]^T$, we obtain a full rich frame, which also satisfies the hypothesis of Theorem 5.7, and thus admits strictly hyperbolic fluxes, all of which belong to $\mathcal{F}(\mathfrak{R})$ by construction. We do not include the general explicit expression for these fluxes, which is rather long and involves special functions.

On the other hand, if we adjoin vector field $\tilde{\mathbf{r}}_3 = [1, 0, -u]^T$, we obtain a non-rich full frame (with $c^2_{13} = -\frac{1}{4u}$), such that modulo $\mathcal{F}^{\text{triv}}$, it has a 1-parameter family of strictly hyperbolic fluxes:

$$\mathbf{f} = a\,[v, \frac{u^2}{2} + w, 0]^T, \text{ where } a \neq 0 \in \mathbb{R}, \tag{174}$$

with the eigenvalues

$$\lambda^1 = -\sqrt{u}; \quad \lambda^2 = \sqrt{u}; \quad \lambda^3 = 0.$$

By construction, $\mathcal{F}(\mathfrak{R})$ contains fluxes (174), and thus it contains both rich and non-rich strictly hyperbolic fluxes.

7.2 Non-involutive partial frames of two vectors fields in $\mathbb{R}^3$.

We next present examples of non-involutive partial frames $\mathfrak{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ on some open subsets of $\mathbb{R}^3$ which illustrate Theorems 6.1 and 6.2. We continue with examples satisfying the hypotheses of Theorem 6.2. These examples illustrate the second claim of this theorem, assuring that for each $k = 0, \dots, 4$, there exists $\mathfrak{R}$, meeting the assumptions of the theorem and satisfying $\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = k$.

Example 7.4 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 0$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [0, 1, u]^T$ and $\mathbf{r}_2 = [w, 0, 1]^T$ all fluxes are trivial.

Example 7.5 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 1$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [v, u, w]^T$ and $\mathbf{r}_2 = [u, w, v]^T$, on an open subset $\Omega \subset \mathbb{R}^3$ where these vectors are independent, the non-trivial fluxes form a 1-parameter family:

$$\mathbf{f} = \frac{c_1}{(u+v+w)^2}\left[-\frac{1}{2}u^2 - uv, -(u+v)(u+w) - \frac{1}{2}v^2, vw + \frac{1}{2}w^2\right]^T.$$

This frame does not satisfy condition (88) and, therefore, in agreement with Theorem 6.1, all non-trivial fluxes are strictly hyperbolic with eigenfunctions

$$\lambda^1 = c_1\frac{u-v}{(u+v+w)^2}, \quad \lambda^2 = c_1\frac{v-w}{(u+v+w)^2}, \quad \lambda^3 = 0.$$

The third eigenvector is equal to $\mathbf{r}_3 = [u, v, w]^T$.

Example 7.6 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 2$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [-1, 0, v+1]^T$ and $\mathbf{r}_2 = [\frac{w}{v^2-1}, -1, u]^T$ defined on an appropriate open subset of $\mathbb{R}^3$, the set of non-trivial fluxes forms a two-dimensional vector space[7]:

$$\mathbf{f} = c_1\begin{bmatrix} ((v-1)\,u + w)\,\mathrm{Ei}(v-1) - e^{1-v}u \\ \frac{1}{2}\left[(v-1)^2\,\mathrm{Ei}(v-1) - (3v+2)e^{1-v}\right] \\ (v+1)((1-v)u - w)\,\mathrm{Ei}(v-1) + (2(v+1)u + w)e^{1-v} \end{bmatrix} + c_2\begin{bmatrix} uv + w \\ \frac{v^2}{2} \\ u\,(1-v^2) - vw \end{bmatrix},$$

where $\mathrm{Ei}(x) = \int_1^\infty \frac{e^{-tx}}{t}\,dt$ is the exponential integral. This frame does not satisfy condition (88) and therefore, in agreement with Theorem 6.1, all non-trivial fluxes are strictly hyperbolic with eigenfunctions

$$\lambda^1 = -c_1(2\,\mathrm{Ei}(v-1) + e^{1-v}) - c_2, \quad \lambda^2 = c_1((v-1)\,\mathrm{Ei}(v-1) + v\,e^{1-v}) + c_2 v, \qquad \lambda^3 = c_1 e^{1-v} + c_2.$$

The third eigenvector of $[D\mathbf{f}]$ is:

$$\mathbf{r}_3 = \left[c_1\,\mathrm{Ei}(v-1) + c_2, 0, c_1\,(2\,e^{1-v} + (v-1)\,\mathrm{Ei}(v-1)) - c_2\,(v-1)\right]^T.$$

Example 7.7 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 3$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [1, \sqrt{w}, 0]^T$ and $\mathbf{r}_2 = [u, 0, -w]^T$ the set of non-trivial fluxes forms a three-dimensional vector space:

$$\mathbf{f} = c_1\begin{bmatrix} 3uv\sqrt{w} - v^2 - u^2 w \\ uvw \\ vw^{3/2} - uw^2 \end{bmatrix} + c_2\begin{bmatrix} v \\ uw \\ 0 \end{bmatrix} + c_3\begin{bmatrix} u\sqrt{w} - v \\ 0 \\ \frac{w^{3/2}}{3} \end{bmatrix}.$$

In this case, when $c_1 = 0$ and $c_2 = \frac{1}{2}c_3$, we obtain a 1-parameter family of non-hyperbolic fluxes

$$\mathbf{f}^{\text{nh}} = c\left[u\sqrt{w} - \frac{1}{2}\,v, \frac{1}{2}\,u\,w, \frac{w^{3/2}}{3}\right]^T,$$

[7]Technically, we should say "the set of non-trivial fluxes and the zero flux form a two-dimensional vector space."

with the eigenfunctions

$$\lambda^1 = \lambda^2 = \frac{1}{2} c\sqrt{w}.$$

We also confirm that $\mathfrak{R}$ satisfies the necessary condition (88) for admitting non-hyperbolic fluxes. To check this we complete $\mathfrak{R}$ to a frame, e.g., by adjoining the vector field $\mathbf{s} = [\mathbf{r}_1, \mathbf{r}_2] = [1, \frac{1}{2}\sqrt{w}, 0]^T$. We also confirm Remark 6.8: $\mathbf{s}$ is a generalized eigenvector, viz.

$$\widetilde{\nabla}_{\mathbf{s}} \mathbf{f} = \frac{1}{2} c\sqrt{w}\, \mathbf{s} + \frac{1}{4} c\sqrt{w}\, \mathbf{r}_1.$$

In agreement with Theorem 6.1, all other fluxes are strictly hyperbolic with the eigenfunctions:

$$\begin{aligned}
\lambda^1 &= c_1(v\sqrt{w} + uw) + c_2\sqrt{w};\\
\lambda^2 &= c_1\left(\frac{3}{2} v\sqrt{w} - uw\right) + \frac{c_3}{2}\sqrt{w};\\
\lambda^3 &= c_1(2v\sqrt{w} - 3uw) - c_2\sqrt{w} + c_3\sqrt{w}.
\end{aligned}$$

The third eigenvector of $[D\mathbf{f}]$ is:

$$\mathbf{r}_3 = \begin{bmatrix} c_1^2\,(2\,u^2\,w - 2\,v^2) + c_1\,c_2\,(3\,\sqrt{w}\,u - v) - c_1\,c_3\,(\sqrt{w}\,u + v) + c_2^2 - c_2\,c_3 \\ \frac{1}{2}\,(c_1\,v + c_2)\,\left(c_1\,(\sqrt{w}\,v + u\,w) + c_2\,\sqrt{w}\right) \\ 2\,c_1\,w\,\left(c_1\,(\sqrt{w}\,v + u\,w) + c_2\,\sqrt{w}\right) \end{bmatrix}.$$

We observe that, in this last case, the dependence of $\mathbf{r}_3$ on c_1, c_2 and c_3 is non-linear.

Example 7.8 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 4$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [1, 0, v]^T$ and $\mathbf{r}_2 = [0, 1, -u]^T$, the set of non-trivial fluxes forms a four-dimensional vector space:

$$\mathbf{f} = c_1 \begin{bmatrix} 2\,u\,(w + u\,v) \\ 2\,v\,(w - u\,v) \\ w^2 + 3u^2v^2 \end{bmatrix} + c_2 \begin{bmatrix} 2\,u^2 \\ w - u\,v \\ 2\,u^2\,v \end{bmatrix} + c_3 \begin{bmatrix} u\,v + w \\ -2\,v^2 \\ 2\,uv^2 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2\,v \\ w - uv \end{bmatrix}.$$

This frame does not satisfy condition (88) and, therefore, in the agreement with Theorem 6.1, all non-trivial fluxes are strictly hyperbolic with eigenfunctions

$$\begin{aligned}
\lambda^1 &= 2\,c_1\,(w + 3u\,v) + 4\,c_2 u - 2\,c_3 v;\\
\lambda^2 &= 2\,c_1\,(w - 3\,u\,v) - 2\,c_2 u - 4\,c_3\,v + 2\,c_4;\\
\lambda^3 &= 2\,c_1 w + c_2\,u - c_3 v + c_4.
\end{aligned}$$

The third eigenvector of $[D\mathbf{f}]$ is

$$\mathbf{r}_3 = \left[-2\,c_1\,u - c_3, 2\,c_1\,v + c_2, 2\,c_1\,u\,v + 2\,c_2\,u + 2\,c_3\,v - c_4\right]^T.$$

Finally, we give another maximal-dimensional case where $\mathcal{F}(\mathfrak{R})$ contains non-hyperbolic fluxes.

Example 7.9 ($\dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{\text{triv}} = 4$)**.** For a partial frame $\mathfrak{R}$ consisting of vector fields $\mathbf{r}_1 = [1, 0, 2\,v]^T$ and $\mathbf{r}_2 = [0, 1, u]^T$, the set of non-trivial fluxes forms a four-dimensional vector space:

$$\mathbf{f} = c_1 \begin{bmatrix} u\,(uv - w) \\ -2\,v\,(2\,uv - w) \\ -6\,u\,v(uv - w) - 2\,w^2 \end{bmatrix} + c_2 \begin{bmatrix} u^2 \\ 2\,(2uv - w) \\ 2\,u^2 v \end{bmatrix} + c_3 \begin{bmatrix} w - uv \\ 2\,v^2 \\ 2\,uv^2 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ v \\ 2uv - w \end{bmatrix}.$$

When $c_1 = c_3 = c_4 = 0$ and $c_2 = 1$, we obtain a 1-parameter family of non-hyperbolic fluxes

$$\mathbf{f}^{\text{nh}} = c\left[u^2, 2\,(2uv - w), 2\,u^2 v\right]^T$$

with eigenfunctions

$$\lambda^1 = \lambda^2 = 2\,c\,u.$$

In agreement with Theorem 6.1, all other fluxes are strictly hyperbolic with eigenfunctions

$$\begin{aligned}
\lambda^1 &= -c_1\,w + 2\,c_2\,u + c_3\,v\\
\lambda^2 &= 2\,c_1\,(w - 3\,u\,v) + 2\,c_2\,u + 4\,c_3\,v + c_4,\\
\lambda^3 &= 2\,c_1\,(3\,u\,v - 2\,w) + 2\,c_2\,u - 2\,c_3\,v - c_4,
\end{aligned}$$

and

$$\mathbf{r}_3 = \left[c_1\,u - c_3 c_1\,v - c_2(5\,u\,v - 3\,w) -\ c_2\,u -\ c_3\,v - c_4\right]^T.$$

Remark 7.10. *In Examples 7.6-7.9, where $2 \le \dim \mathcal{F}(\mathfrak{R})/\mathcal{F}^{triv} \le 4$, the following interesting phenomenon occurs. For the basis fluxes $\mathbf{f}_1, \dots, \mathbf{f}_k$ presented in these examples ($k = 2, ..., 4$, depending on an example), the corresponding Jacobian matrices $DF_1, \dots, DF_k$ have the additivity of eigenvalues property, called the property L by Motzkin and Taussky in [10, 11]. By construction, $\mathbf{r}_1$ and $\mathbf{r}_2$ are eigenvectors of $DF_1, \dots, DF_k$. It is therefore obvious that, if $\lambda^1_1, \dots, \lambda^1_k$ are the eigenvalues for $\mathbf{r}_1$ of $DF_1, \dots, DF_k$, respectively, and $\lambda^2_1, \dots, \lambda^2_k$ are the eigenvalues for $\mathbf{r}_2$ of $DF_1, \dots, DF_k$, respectively, then for $\mathbf{f} = c_1\mathbf{f}_1 + \cdots + c_k\mathbf{f}_k$ the Jacobian matrix DF has the eigenvalue $\lambda^1 = c_1\lambda^1_1 + \cdots + c_k\lambda^1_k$ for the eigenvector $\mathbf{r}_1$ and the eigenvalue $\lambda^2 = c_1\lambda^2_1 + \cdots + c_k\lambda^2_k$ for the eigenvectors $\mathbf{r}_2$. However, we note that also the third eigenvalues also "add up." Indeed, in all of the examples, $\lambda^3 = c_1\lambda^3_1 + \cdots + c_k\lambda^3_k$ is the third eigenvalue of Df, where $\lambda^3_1, \dots, \lambda^3_k$ are the third eigenvalues of $DF_1, \dots, DF_k$, despite the fact that these matrices have non-collinear third eigenvectors $\mathbf{r}_{3,1}, \dots, \mathbf{r}_{3,k}$.*

Our last example shows that even when the first assumption in Theorem 6.2, i.e., the necessary condition (87) for strict hyperbolicity, holds, the second assumption, i.e., (89), may not.

Example 7.11. Consider the partial frame $\mathfrak{R}$ defined consisting of the vector fields $\mathbf{r}_1 = [1, 0, w]^T$ and $\mathbf{r}_2 = [0, 1, -\frac{9}{8}\ln(w) + u]^T$, for which (87) holds. The vector field $\mathbf{s} = [0, 0, 1]^T$ completes $\mathfrak{R}$ to a frame, and relative to this frame we have

$$\Gamma^3_{22}(u)\,\Gamma^3_{11}(u) - 9\,\Gamma^3_{12}(u)\,\Gamma^3_{21}(u) \equiv 0. \tag{175}$$

(This example was obtained by setting up a differential equation on the components of vector fields $\mathbf{r}_1$ and $\mathbf{r}_2$, induced by the identity (175) and finding its particular solution.) For this partial frame the vector space $\mathcal{F}(\mathfrak{R})/\mathcal{F}^{\rm triv}$ is one dimensional:

$$\mathbf{f} = c\left[\tfrac{1}{8}e^{-u}, e^{-u}\,w, e^{-u}\,w\,\left(u - \tfrac{9}{8}\ln(w) + \tfrac{9}{8}\right)\right]^T.$$

This frame does not satisfy condition (88), and therefore, in agreement with Theorem 6.1, all non-trivial fluxes are strictly hyperbolic with eigenfunctions

$$\lambda^1 = -\tfrac{1}{8}c\,e^{-u}, \qquad \lambda^2 = c\,e^{-u}\,\left(u - \tfrac{9}{8}\ln(w)\right), \qquad \lambda^3 = 0,$$

and the third eigenvector field is $\mathbf{r}_3 = [0, 1, 0]^T$.

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