A MIXED BOUNDARY VALUE PROBLEM FOR $u_{xy} = f(x, y, u, u_x, u_y)$

HELGE KRISTIAN JENSSEN AND IRINA A. KOGAN

Abstract. Consider a single hyperbolic PDE $u_{xy} = f(x, y, u, u_x, u_y)$, with locally prescribed data: u along a non-characteristic curve M and u_x along a non-characteristic curve N. We assume that M and N are graphs of one-to-one functions, intersecting only at the origin, and located in the first quadrant of the (x, y)-plane.

It is known that if M is located above N, then there is a unique local solution, obtainable by successive approximation. We show that in the opposite case, when M lies below N, the uniqueness can fail in the following strong sense: for the same boundary data, there are two solutions that differ at points arbitrarily close to the origin.

In the latter case, we also establish existence of a local solution (under a Lipschitz condition on the function f). The construction, via Picard iteration, makes use of a careful choice of additional u-data which are updated in each iteration step.

Keywords: Second order hyperbolic partial differential equations, mixed problems, non-uniqueness.
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1. Introduction

We consider existence and uniqueness for a certain type of boundary value problem for the second order wave equation

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{1}$$

(All quantities and variables are real valued.) Our main goal is to draw attention to an issue related to uniqueness of solutions when the data prescribe the unknown itself along one non-characteristic curve, together with one of its partial derivatives, u_x say, along a different non-characteristic curve. The curves are assumed to be located in the first quadrant, both passing through the origin, but otherwise disjoint. We shall see that, depending on the relative position of the data curves, uniqueness may fail. We also show how a non-standard Picard iteration scheme yields existence of a local solution. For this we join the two given data curves by a third one, along which we iteratively prescribe the values of the solution itself.

Our motivation stems from our earlier work [2] which provided a generalization of an integrability theorem due to Darboux. Darboux's original results concerned (possibly overdetermined) systems of first order PDEs where each equation contains a single partial derivative, for which it is solved, and

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where the data for the unknowns are prescribed locally along certain affine subspaces. In [2] we generalized Darboux's results to cases where the data are given along more general manifolds and where partial derivatives are replaced by differentiation along vector fields. However, to establish existence of a unique local solution, we found it necessary to assume a certain *Stable Configuration Condition* (SCC). This is a geometric condition on the relative location of the data manifolds for the unknowns. As such it has nothing to do with over-determinacy or nonlinearity of the PDE system under consideration.

Therefore, to investigate the necessity of the SCC it is reasonable to consider a transparent case which highlights its relevance. For this the simplest setting appears to be a linear system of two, fully coupled, equations, such as $u_x = v$ and $v_y = u$, which yields the equation $u_{xy} = u$. In this case, the SCC puts a restriction on the relative location of the data manifolds M and N for u and $v = u_x$, respectively.

Given the extensive literature on second order hyperbolic PDEs (see Section 2 for a selective review), our expectation was that, either the SCC was not actually necessary, or it was "well-known" that a condition like the SCC is indeed required to establish well-posedness in this situation.

However, we have not been able to find a treatment of this issue in the literature. In the present work we show that the SCC is indeed required for the uniqueness of a local solution by providing an example of non-uniqueness for the equation $u_{xy} = u$ when the SCC is not met. We then establish an existence result for more general (possibly nonlinear) second order hyperbolic equations.

In subsection 1.1 we explain in more detail the issue at hand. Section 2 reviews related works. In Section 3 we establish existence via successive approximations for the model equation $u_{xy} = u$ in the case when the SCC fails. For the same case, we show in Section 4 that uniqueness can fail. As the proof of existence in Section 3 involves a choice of additional data assignment along a certain curve (located away from the origin), the non-uniqueness result in Section 4 may appear unsurprising. However, we show that the additional data assignment influences the solution all the way to the origin. Finally, in Section 5 we apply the same technique (additional data assignment) to show existence of a solution to (1) in the case when the SCC fails. However, the additional data assignment must now be done adaptively at each step in the iteration scheme.

1.1. **The issue.** To highlight the issue, consider the following simple situation where both data curves are straight lines through the origin:

$$M := \{x = ay\} \qquad N := \{y = bx\},$$

where $a > 0$ and $b > 0$ are constants. We assume that u and u_x are given along parts of M and N, respectively, in the first quadrant:

$$u(ay, y) = \varphi(y) \qquad 0 \le y \le y_A, \tag{2}$$

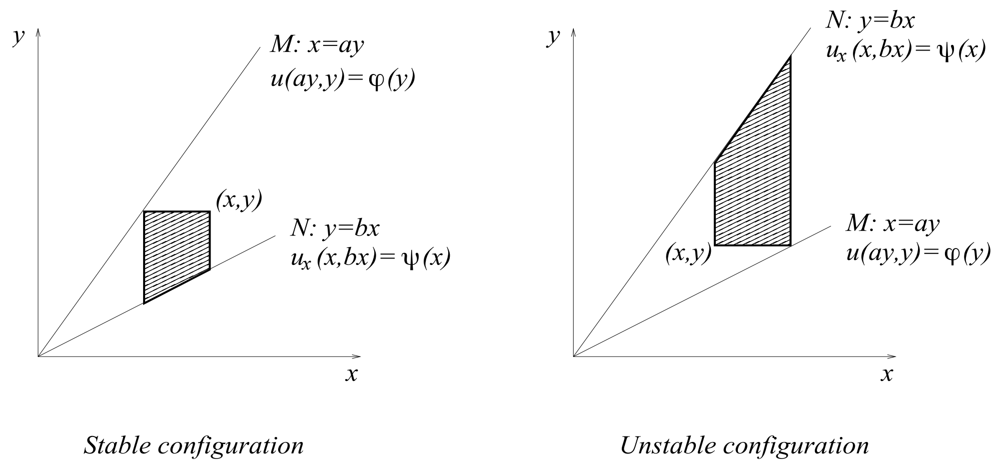

FIGURE 1. Stable and unstable configurations.

and

$$u_x(x, bx) = \psi(x) \qquad 0 \le x \le x_B. \tag{3}$$

Note that the data are only prescribed locally (the natural setting for non-linear equations). Now consider the two cases,

(I) $ab < 1$: the data curve N for u_x lies below the data curve M for u;
(II) $ab > 1$: vice versa;

see Figure 1. Already for the elementary case where the right-hand side of (1) is independent of (u, u_x, u_y), i.e., $f = f(x, y)$, there is a notable difference in how the value of u is determined at a point (x, y) located between M and N. In either case we first integrate with respect to y, exploiting the equation and the given u_x-data, followed by an x-integration to exploit the u-data, obtaining:

$$\begin{aligned} u(x, y) &= \varphi(y) + \int_{ay}^{x} \left(\psi(\xi) + \int_{b\xi}^{y} f(\xi, \eta)\, d\eta \right) d\xi \\ &= \varphi(y) + \int_{ay}^{x} \psi(\xi)\, d\xi + \int_{ay}^{x} \int_{b\xi}^{y} f(\xi, \eta)\, d\eta d\xi. \end{aligned} \tag{4}$$

However, in Case (I), the upper limits of integrations, given by the coordinates of the point (x, y), are larger than the lower limits, and so all points in the integration region are closer to the origin than the point (x, y) (shaded region in left part of Figure 1). In contrast, for Case (II), the upper limits of integrations, given by the coordinates of the point (x, y), are smaller than the lower limits. Thus we need to know f, ϕ and ψ at points located *further away from the origin than* (x, y) in order to determine $u(x, y)$ (right part of Figure 1).

This difference between the two cases is harmless when the right-hand side of (1) is independent of (u, u_x, u_y): we obtain a (local) solution in either case.

However, if we instead have a nontrivial equation such as

$$u_{xy} = u, \tag{5}$$

a standard approach would apply Picard iteration where the nth iterate $u^{(n)}$ defines the next iterate $u^{(n+1)}$ from the equation

$$u_{xy}^{(n+1)} = u^{(n)}(x, y),$$

together with the original data requirements along M and N. That is, $u^{(n)}(x, y)$ now plays the role of $f(x, y)$ above. It is clear that Case (II) will, in a finite number of iterations, require data values that have not been assigned if the data are only prescribed locally.

In contrast, it can be shown (see [2]) that in Case (I) the iteration scheme outlined above will converge to a unique, local solution to (5). We thus have a situation where the natural iteration scheme is well-defined and converges provided the data curves are located in a certain manner, while the same iteration scheme is undefined when this is not the case. We express this by saying that the *Stable Configuration Condition* (SCC) is satisfied in Case (I), while it is violated in Case (II).

As a remedy for Case (II) we shall fix a bounded set whose boundary consists of one part of each of M and N, together with a new, chosen curve joining these, along which we prescribe values of u. (These new values are subject to some mild compatibility conditions.) For concreteness we choose the new curve to be a vertical line segment. In Section 3 we use this approach to establish local existence in Case (II) for the model equation $u_{xy} = u$ with general boundary data (2)-(3).

However, we also establish non-uniqueness. More precisely, fix a triangle $\overline{OAB}$ as in Figure 2 below, and let u and u_x be prescribed to vanish along $\overline{OA}$ and $\overline{OB}$, respectively. Clearly, $u \equiv 0$ is a solution in this case. On the other hand, by prescribing a non-trivial function along the vertical segment $\overline{AB}$, we show that there exist a non-zero solution to the same problem which takes nonzero values at points arbitrarily close to the origin. I.e., the values that we "artificially" assign to u along $\overline{AB}$, propagate inward and influence the solution arbitrarily close to the origin.

In Section 5, we use a similar iteration scheme to prove the existence of the solution to a general, nonlinear equation of the form (1) with a boundary data prescribed in the non-stable configuration case. The crucial difference with the linear case treated in Section 3, is that additional u-data along $\overline{AB}$ has to be chosen adaptively as a part of the iteration scheme. The local convergence is of the scheme is proven under Lipschitz conditions on f.

Before starting the analysis, we include a selective review of the substantial literature on boundary value problems for second order hyperbolic equations of the type (1).

2. Review of related results

It is an understatement to say that the literature on hyperbolic second order PDEs of the form (1) is extensive. Yet, to the best of our knowledge, the particular boundary problem described above has not been resolved when the relative location of the data curves is as in Case (II).

For linear equations, *viz.*

$$u_{xy} = \alpha u_x + \beta u_y + \gamma u + \delta, \tag{6}$$

where the coefficients α-δ are functions of (x, y), the consideration of various types of boundary value problems dates, at least, back to Riemann's seminal work [14] where he introduced the Riemann function for a specific case.

In [5] Darboux generalized and systemized Riemann's work, and applied Picard iteration to solve the characteristic boundary value problem for (6), i.e., the case when u is prescribed along one vertical and one horizontal line. Part IV [6] of the same work contains a note by Picard outlining the method of successive approximations and its use for 2nd order hyperbolic equations. In this note, Picard treats the characteristic boundary value problem for both linear and nonlinear equations (1), and for the former he also considers the problem where u-data are prescribed along one characteristic curve as well along one non-characteristic curve.

In Chapter XXVI of his Cours d'Analyse, Goursat [9] provided a detailed exposition of these results and also treated the Cauchy problem. In addition, Goursat considered the issue of integrating the elementary equation $u_{xy}(x, y) = F(x, y)$, where F is a given function, with prescribed u-values along two non-characteristic curves in the first quadrant.

The notes [15] by Picard again treat a series of boundary value problems for hyperbolic second order equations, both linear and nonlinear:

- (i) the characteristic boundary value problem (u prescribed along two characteristics, one of each family);
- (ii) the Cauchy problem (u together with one of its first partial derivatives prescribed along a single, non-characteristic curve);
- (iii) u prescribed along one characteristic and one non-characteristic curve (both passing through a given point);
- (iv) u prescribed along two non-characteristic curves (with both passing through a given point; two cases are considered: both curves lie in the first quadrant, or one lies in the first and one lies in the fourth).

For linear equations the existence of a unique solution is obtained in each of these "classical" cases via successive approximations, [15].

While everybody agrees that problem (ii) above be named the *Cauchy problem*, the terminology for the other problems is not uniform in the literature, [13]. E.g., Pogorzelski [16] refers to problems (i) and (iii) above as the *Darboux problem* and the *Picard problem*, respectively. Walter [19] and Kharibegashvili [10] refer to (i) as the *Goursat problem*, while Lieberstein

[12] uses this for problem (iv). Other authors [4, 7, 8] call (i) the *characteristic initial value (or characteristic Goursat) problem.* Kharibegashvili [10] also refers to problems (iii) and (iv) as the *1st and 2nd Darboux problems*, respectively.

In any case, none of the works mentioned thus far addresses the boundary value problem (1)-(2)-(3) where u is prescribed along one non-characteristic curve and u_x is prescribed along another non-characteristic curve (with both curves located in the first quadrant).

The 1940s and 50s saw renewed interest in hyperbolic equations. In particular, various new types of boundary value problems were considered for equations of the form (1). Among the many works in this area we shall only comment on those that are most closely related to the problem described in Section 1. (For an extensive bibliography, see Walter [19].) We shall see that, while the results in these works do apply to Case (I) above, none of them covers Case (II). The reason is essentially the same in each case: the various setups put a restriction on the relative location of the data curves for u and u_x, excluding Case (II).

Szmydt [17, 18] considered two generalized boundary value problems for equations of the form (1). For both types of problems, the data are prescribed along two curves, $\Gamma = \{y = \gamma(x)\}$ defined for $-\alpha \le x \le \alpha$, and $\Lambda := \{x = \lambda(y)\}$ defined for $-\beta \le y \le \beta$. It is explicitly assumed that α, β are finite, and that Γ and Λ are situated within the rectangle $D = [-\alpha, \alpha] \times [-\beta, \beta]$. For the first type of problem Szmydt prescribes $u(x_0, y_0) = u_0$ at an arbitrary point $(x_0, y_0) \in D$,

$$u_x(x, y) = G(x, u(x, y), u_y(x, y)) \qquad \text{along } y = \gamma(x), \tag{7}$$

and similarly u_y as a function $H(y, u, u_x)$ along $x = \lambda(y)$. For the second type of problem, u_x is again prescribed according to (7), while it is required that

$$u(\lambda(y), y) = u_0 + \int_{y_0}^{y} B(t, u(\lambda(t), t), u_x(\lambda(t), t))\, dt \tag{8}$$

holds for $-\beta \le y \le \beta$. Here, u_0 is a constant, $-\beta \le y_0 \le \beta$, and B is a continuous function. As Szmydt points out in Remark 1 of [17], if λ is of the class C^1, then the two problems are equivalent. Under various conditions, existence of a local solution is obtained for each type of problem. As pointed out in [18], the results cover the classical problems (i)-(iii) listed above.

Now consider the linear equation (1) with data prescribed along two straight lines M and N as in (2)-(3), where we assume $ab > 1$ (Case (II)). To formulate this in the setup of Szmydt [17, 18] we must let $\Gamma = M$ and $\Lambda = N$, since the curves are to be contained within the rectangle D. However, in either problem considered by Szmydt, u_x is prescribed along $\Gamma = M$, while in our assignment (2), u is assigned along M. Thus, Szmydt's setup does not apply to Case (II).

Next, the concise work [3] by Ciliberto considers the following boundary value problem for (1) ("Problem (A)" in [3]). We are given two curves

$$C_1 = \{y = \alpha(x) : a \leq x \leq b\} \qquad \text{and} \qquad C_2 = \{x = \beta(y) : c \leq y \leq d\},$$

where

$$c \leq \alpha(x) \leq d \quad \text{for } a \leq x \leq b, \text{ and} \quad a \leq \beta(y) \leq b \quad \text{for } c \leq y \leq d, \tag{9}$$

and $\alpha(0) = \beta(0) = 0$. Then: determine a solution of (1) with assigned values of u along C_2 and with assigned values of u_x along C_1. Under suitable conditions, Ciliberto establishes the existence of a (global) solution to this problem. (There is considerable overlap between Ciliberto's and Szmydt's results; see footnote (^{3}) on p. 384 in [3].)

If we try to apply this setup to (1) with data prescribed along two straight lines M and N as in (2)-(3), we encounter the same issue as with Szmydt's setup. Namely, since, according to (9), the curves should be situated within the rectangle $[a, b] \times [c, d]$, C_1 must be chosen as M and C_2 must be chosen as N. However, this is precisely Case (I) in Section 1, showing that the analysis in [3] does not apply to Case (II).

Ciliberto remarks (footnote (^{2}) on p. 384 of [3]) that an entirely equivalent problem is obtained if u is prescribed along C_1 and u_y is prescribed along C_2. This being the case, it is surprising to us that no remark is made about the (non-equivalent) problem where u is prescribed along C_1 and u_x is prescribed along C_2 (i.e., Case (II) above).

Next we consider the detailed work [1] by Aziz & Diaz on linear equations of the form

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y) = d(x, y), \tag{10}$$

and with boundary conditions of the form

$$\alpha_0(x)u + \alpha_1(x)u_x + \alpha_2(x)u_y = \sigma(x) \qquad \text{on } y = f_1(x),\ x \in [0, x_0], \tag{11}$$

and

$$\beta_0(y)u + \beta_1(y)u_x + \beta_2(y)u_y = \tau(y) \qquad \text{on } x = f_2(y),\ y \in [0, y_0], \tag{12}$$

and

$$u(0, 0) = \gamma, \tag{13}$$

for given functions α_i, β_i, f_i, and a given constant γ. The problem (10)-(11)-(12)-(13) is posed on a characteristic rectangle $R := [0, x_0] \times [0, y_0]$, and it is explicitly assumed in [1] that

$$0 \leq f_1(x) \leq y_0 \qquad \text{with } f_1(x) \text{ defined for all } x \in [0, x_0], \tag{14}$$

and that

$$0 \leq f_2(y) \leq y_0 \qquad \text{with } f_2(y) \text{ defined for all } y \in [0, y_0]. \tag{15}$$

Three cases are treated by Aziz & Diaz: (here, e.g. "α_1, $\beta_2 \neq 0$" means that $\alpha_1(x) \neq 0$ for all $x \in [0, x_0]$ and $\beta_2(y) \neq 0$ for all $y \in [0, y_0]$)

(*) α_1, $\beta_2 \neq 0$ and $\alpha_2 = \beta_1 \equiv 0$.

(**) α_1, $\beta_2 \neq 0$, but with additional conditions imposed on f_i, σ and τ.

(***) α_0, $\beta_0 \neq 0$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 \equiv 0$.

Under suitable assumptions, the authors establish existence and uniqueness of a solution to (10)-(11)-(12)-(13) in each case.

To apply this setup in Case (II) of the problem (1)-(2)-(3) in Section 1 we would need to differentiate (2) with respect to y, and then see if (**) can be applied. This, however, fails for the following reason. The differentiated condition is

$$au_x(ay, y) + u_y(ay, y) = \varphi'(y), \qquad \text{(together with } u(0,0) = \varphi(0)\text{)},$$

and this must play the role of (12) above (because the other boundary condition is $u_x(x, bx) = \psi(x)$, which does not contain u_y). It follows that the curves Γ_1 and Γ_2 in [1] must correspond to the curves N and M, respectively, in our Case (II). Thus, Γ_1 lies above Γ_2 in the first quadrant. But then it is not possible to find x_0 and y_0 so that both (14) and (15) are satisfied. (We note that the work [1] provides a very detailed review and criticism of earlier works.)

We finally mention the more recent work [11] which treats nonlinear equations of the type (1) in angular domains. Again, a careful reading reveals that the analysis in [11] applies to Case (I), but not to Case (II).

3. Existence for $u_{xy} = u$ in Case (II)

In this section, we establish existence of a local solution to the following mixed boundary value problem:

$$u_{xy} = u \tag{16}$$

with data

$$u(ay, y) = \varphi(y) \qquad 0 \le y \le y_A, \tag{17}$$

and

$$u_x(x, bx) = \psi(x) \qquad 0 \le x \le x_B, \tag{18}$$

where φ is a C^1 function on $[0, y_A]$ and ψ is a C^0 function on $[0, x_A]$. We assume $ab > 1$, so that we are in Case (II) (see Figure 2), and we proceed to write down an iteration scheme which will provide existence of a local solution.

As outlined above, we circumvent the problem in Case (II) (i.e., requiring data progressively farther away from the origin for the iterates), by artificially prescribing u-data along the vertical segment $\overline{AB}$; see Figure 2. This results in a "split" scheme with different expressions in the triangles $\overline{OAC}$ and $\overline{ABC}$, where $C = (\frac{y_A}{b}, y_A)$. The u-data along $\overline{AB}$ are given by

$$u(x_A, y) = \theta(y) \qquad y_A \le y \le y_B, \tag{19}$$

where θ remains to be specified. It must be chosen so that the iterates remain continuous on the full closed triangle $\overline{OAB}$.

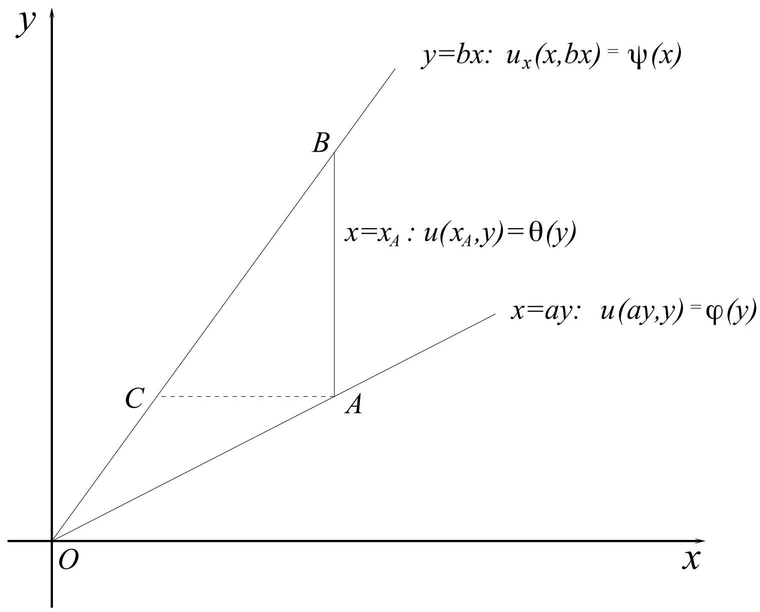

FIGURE 2. $u_{xy} = u$: in addition to the original data φ and ψ, we also prescribe u-data $u(x_A, y) = \theta(y)$ along the vertical segment $\overline{AB}$.

3.1. **Iteration scheme.** The iteration scheme is obtained by having $u^{(n+1)}$ be the solution of $u_{xy}(x, y) = u^{(n)}(x, y)$ in $\overline{OAB}$, with the originally assigned boundary data along $\overline{OA}$ and $\overline{OB}$, together with u-data assigned according to (19) along $\overline{AB}$. However, continuity of the solution throughout the triangle $\overline{OAB}$ (in particular, across the horizontal segment $\overline{AC}$) puts restrictions on θ. To determine these we first consider the elementary case of $u_{xy} = f(x, y)$. The solution in this case is given by the split expression

$$u(x, y) = \varphi(y) - \int_x^{ay} \psi(\xi)\, d\xi + \int_x^{ay} \int_y^{b\xi} f(\xi, \eta)\, d\eta d\xi \qquad \text{on } \overline{OAC}, \quad (20)$$

and

$$u(x, y) = \theta(y) - \int_x^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A} \int_y^{b\xi} f(\xi, \eta)\, d\eta d\xi \qquad \text{on } \overline{ABC}. \quad (21)$$

A first issue is what restrictions need to be imposed on θ to guarantee that the function u given by (20)-(21) is consistently defined on the segment $\overline{AC}$ and is of the class C^1. This is answered by:

Lemma 3.1. *Consider the mixed boundary value problem for the equation $u_{xy} = f(x, y)$ with data (17)-(18)-(19), where $ab > 1$ and*

(A1) $\varphi \in C^1[0, y_A]$
(A2) $\psi \in C^0[0, x_A]$
(A3) $f \in C^0(\overline{OAB})$.

Let $\theta : [y_A, y_B] \to \mathbb{R}$ be any function satisfying

(A4) $\theta \in C^1[y_A, y_B]$
(A5) $\theta(y_A) = \varphi(y_A)$
(A6) $\theta'(y_A) = \varphi'(y_A) - a\psi(x_A) + a \int_{y_A}^{y_B} f(x_A, \eta)\, d\eta.$

Then the piecewise defined function $u(x, y)$ given by (20)-(21) provides a solution in the following sense: u, u_x, u_y, and $u_{xy} = u_{yx}$ are continuous on $\overline{OAB}$, $u_{xy}(x, y) = f(x, y)$ on $\overline{OAB}$, and u takes the values (17)-(18)-(19) on the boundary of $\overline{OAB}$.

Proof. Continuity of u in the triangles $\overline{OAC}$ and $\overline{ABC}$ is clear from (20)-(21), combined with (A1)-(A4). To show that (20)-(21) consistently define u along the common side $\overline{AC}$, we substitute

$$y = y_A = \frac{x_A}{a} \tag{22}$$

in (20)-(21), respectively. We get for $(x, y_A) \in \overline{AC}$:

$$\begin{aligned} u(x, y_A) &= \varphi(y_A) - \int_x^{ay_A} \psi(\xi)\, d\xi + \int_x^{ay_A} \int_{y_A}^{b\xi} f(\xi, \eta)\, d\eta d\xi, &(23)\\ &= \varphi(y_A) - \int_x^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A} \int_{y_A}^{b\xi} f(\xi, \eta)\, d\eta d\xi,\\ u(x, y_A) &= \theta(y_A) - \int_x^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A} \int_{y_A}^{b\xi} f(\xi, \eta)\, d\eta d\xi. &(24) \end{aligned}$$

We see that (A5) is the only condition that we need in order to guarantee that u is consistently defined along $\overline{AC}$ and is continuous.

To check that u takes the assigned values on $\overline{OA}$, we substitute $x = ay$ into (20):

$$u(ay, y) = \varphi(y) - \int_{ay}^{ay} \psi(\xi)\, d\xi + \int_{ay}^{ay} \int_y^{b\xi} f(\xi, \eta)\, d\eta d\xi = \varphi(y). \tag{25}$$

Differentiation of (20) and (21) with respect to x produces the same formula:

$$u_x = \psi(x) - \int_y^{bx} f(x, \eta) d\eta, \tag{26}$$

which is clearly continuous on the entire region $\overline{OAB}$ and takes the assigned values $\psi(x)$ along $\overline{OB}$, where $y = bx$. Differentiation of (20) and (21) with respect to y produces two different formulas:

$$u_y(x, y) = \varphi'(y) - a\psi(ay) + a \int_y^{aby} f(ay, \eta)\, d\eta - \int_x^{ay} f(\xi, y)\, d\xi \qquad \text{on } \overline{OAC}, \tag{27}$$

$$u_y(x, y) = \theta'(y) - \int_x^{x_A} f(\xi, y) d\xi \qquad \text{on } \overline{ABC}. \tag{28}$$

Continuity of u_y on $\overline{OAC}$ and $\overline{ABC}$ is clear from (27)-(28), combined with (A1)-(A4). To show that (27)-(28) consistently define u along the common side $\overline{AC}$, we substitute (22) into these formulas:

$$\begin{aligned} u_y(x, y_A) &= \varphi'(y_A) - a\psi(ay_A) + a \int_{y_A}^{aby_A} f(ay, \eta)\, d\eta - \int_x^{ay_A} f(\xi, y)\, d\xi \\ &= \varphi'(y_A) - a\psi(x_A) + a \int_{y_A}^{y_B} f(ay, \eta)\, d\eta - \int_x^{x_A} f(\xi, y)\, d\xi \qquad (29) \end{aligned}$$

$$u_y(x, y_A) = \theta'(y_A) - \int_x^{x_A} f(\xi, y) d\xi. \qquad (30)$$

We see that (A6) is the only condition needed in order to guarantee that u_y is is consistently defined along $\overline{AC}$ and is continuous.

Finally, by differentiating (26) with respect to y and differentiating (27) and (28) with respect to x we immediately see that

$$u_{xy} = u_{yx} = f(x, y).$$

□

We now return to the equation $u_{xy} = u$ with data (17)-(18)-(19). Let the nth iterate $u^{(n)}(x, y)$ play the role of f above. Substituting $\theta(y)$ for $u^{(n)}(x_A, y)$ in (A6), we now fix any function $\theta \in C^1[y_A, y_B]$ satisfying

$$\theta(y_A) = \varphi(y_A) \qquad \text{and} \qquad \theta'(y_A) = \varphi'(y_A) - a\psi(x_A) + a \int_{y_A}^{y_B} \theta(\eta)\, d\eta. \qquad (31)$$

It is clear that there are many possible choices for θ. E.g., we may choose $\theta(y)$ as an affine function or quadratic polynomial in y. A straightforward calculation shows that in either case, $|\theta(y)|$ may be bounded by a constant depending on $(y_B - y_A)$ and upper bounds on $|\psi|$, $|\varphi|$, and $|\varphi'|$. We can now specify a suitable iteration scheme. We start by setting

$$u^{(0)}(x, y) := \varphi(y) - \int_x^{ay} \psi(\xi)\, d\xi \qquad \text{on } \overline{OAC} \qquad (32)$$

and

$$u^{(0)}(x, y) := \theta(y) - \int_x^{x_A} \psi(\xi)\, d\xi \qquad \text{on } \overline{ABC}. \qquad (33)$$

Note that $u^{(0)}$ is well defined on the overlap $\overline{AC}$ of the two regions, thanks to $(31)_1$, and so is continuous throughout $\overline{OAB}$. In addition, $u^{(0)}$ equals to θ along $\overline{AB}$.

The first iterate $u^{(1)}(x, y)$ is then defined according to (20)-(21) with $u^{(0)}$ substituted for f. By definition, $u^{(1)}$ reduces to θ along $\overline{AB}$, and, due to the properties (31) of θ, $u^{(1)}$ is continuous throughout $\overline{OAB}$ according to Lemma 3.1.

Next, assuming the nth iterate $u^{(n)}$ has been determined, and is continuous, throughout $\overline{OAB}$, we define the next iterate $u^{(n+1)}$ according to (20)-(21) with $u^{(n)}$ substituted for f. Again, by definition $u^{(n+1)}$ reduces to θ along $\overline{AB}$, and, due to the properties (31) of θ, $u^{(n+1)}$ is continuous throughout $\overline{OAB}$ according to Lemma 3.1.

3.2. **Convergence.** According to (32)-(33) we have

$$|u^{(0)}(x,y)| \leq c \qquad \text{for } (x,y) \in \overline{OAB},$$

for a finite constant c depending on the size of the triangle $\overline{OAB}$ and upper bounds on $|\psi|$, $|\varphi|$, and $|\varphi'|$. (Recall that θ may be chosen so as to be similarly bounded.) It follows that

$$|u^{(1)}(x,y) - u^{(0)}(x,y)| \leq c(x_A - x)(y_B - y) \qquad \text{for } (x,y) \in \overline{OAB},$$

and that in general

$$|u^{(n)}(x,y) - u^{(n-1)}(x,y)| \leq \frac{c}{(n!)^2}(x_A - x)^n (y_B - y)^n \qquad \text{for } (x,y) \in \overline{OAB}.$$

It follows that the sequence $(u^{(n)})$ is uniformly Cauchy in $C^0(\overline{OAB})$, and thus converges uniformly to a limit function $u \in C^0(\overline{OAB})$. Recalling that

$$u^{(n+1)}(x,y) = \varphi(y) - \int_x^{ay} \psi(\xi)\, d\xi + \int_x^{ay}\int_y^{b\xi} u^{(n)}(\xi,\eta)\, d\eta d\xi \qquad \text{on } \overline{OAC},$$

and

$$u^{(n+1)}(x,y) = \theta(y) - \int_x^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A}\int_y^{b\xi} u^{(n)}(\xi,\eta)\, d\eta d\xi \qquad \text{on } \overline{ABC},$$

and passing to the limit $n \to \infty$, shows that u satisfies

$$u(x,y) = \varphi(y) - \int_x^{ay} \psi(\xi)\, d\xi + \int_x^{ay}\int_y^{b\xi} u(\xi,\eta)\, d\eta d\xi \qquad \text{on } \overline{OAC},$$

and

$$u(x,y) = \theta(y) - \int_x^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A}\int_y^{b\xi} u(\xi,\eta)\, d\eta d\xi \qquad \text{on } \overline{ABC}.$$

Direct calculations then show that u is a solution to the mixed boundary value problem (16)-(17)-(18).

Remark 3.2. *Unsurprisingly, for the linear problem under consideration we obtain a global solution defined on all of $\overline{OAB}$. Also, the occurrence in the calculations above of the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{(n!)^2} = J_0(2i\sqrt{z}),$$

where J_0 denotes the Bessel function of the 1st kind of order zero, is natural: it is known that the Riemann function for the linear equation $u_{xy} = cu$ (c constant) can be expressed in terms of J_0; e.g., see Section 4.4 in [8].

4. Non-uniqueness for $u_{xy} = u$ in Case (II)

We proceed to show that the solution to (16) found in the previous section is, in general, not uniquely determined by prescribing u locally along $\{x = ay\}$ and u_x locally along $\{y = bx\}$, when $ab > 1$.

Given the existence result in the previous section, and the freedom we have in assigning the function θ along $\overline{AB}$, it is immediate that different solutions can be obtained whose u-values agree along the segment $\overline{OA}$ of the line $\{x = ay\}$ and whose u_x-values agree along the segment $\overline{OB}$ of the line $\{y = bx\}$ (see Figure 2). Indeed, by continuity of the solutions on $\overline{OAB}$, it is clear that different choices of θ will give solutions that are distinct in some neighborhood of the segment $\overline{AB}$.

However, we shall establish a stronger version of non-uniqueness, showing that changes in θ-values along $\overline{AB}$ can "propagate" into the domain OAB and influence the resulting solution at points arbitrarily close to the origin.

For this the simplest thing appears to be to assign vanishing data for both u and u_x, such that $u_0(x, y) \equiv 0$ is one solution. The issue then is to show that there is another solution $\hat{u}(x, y)$ which is not identically zero. This will be done by carefully choosing a non-negative function θ, satisfying the constraints (31), i.e.,

$$\theta(y_A) = 0 \qquad \text{and} \qquad \theta'(y_A) = a \int_{y_A}^{y_B} \theta(\eta)\, d\eta, \tag{34}$$

and then run the iteration scheme described in the previous section. We shall show that this yields a solution $\hat{u}$ which takes on strictly positive values arbitrarily close to the origin.

Theorem 1. *Consider the boundary value problem (16)-(17)-(18) with vanishing boundary data $\varphi \equiv 0$ and $\psi \equiv 0$. For a given $\theta \in C^1[y_A, y_B]$ satisfying (34), let u_θ denote the solution generated by the scheme in Section 3. Then: It is possible to choose θ so that $u_\theta(x, y) > 0$ at all points (x, y) in the open triangle OAC.*

Proof. It is convenient to introduce the following notation (see Figure 3). For $(x, y) \in \overline{ABC}$, let $T(x, y)$ be the closed trapezoid with vertices (x, y), (x_A, y), B, and (x, bx). For $(x, y) \in \overline{OAC}$, let $\tau(x, y)$ be the closed trapezoid with vertices (x, y), (ay, y), (ay, bay), and (x, bx). Note that for $(x, y) \in \overline{AC}$, these trapezoids coincide. With θ as indicated, the iteration scheme in Section 3 for $u_{xy} = u$ with vanishing boundary data $\varphi \equiv 0$ and $\psi \equiv 0$, is then:

- for $(x, y) \in \overline{OAC}$:

$$u^{(0)}(x, y) = 0 \quad \text{and} \quad u^{(n+1)}(x, y) = \iint\limits_{\tau(x,y)} u^{(n)}(\xi, \eta)\, d\eta d\xi; \tag{35}$$

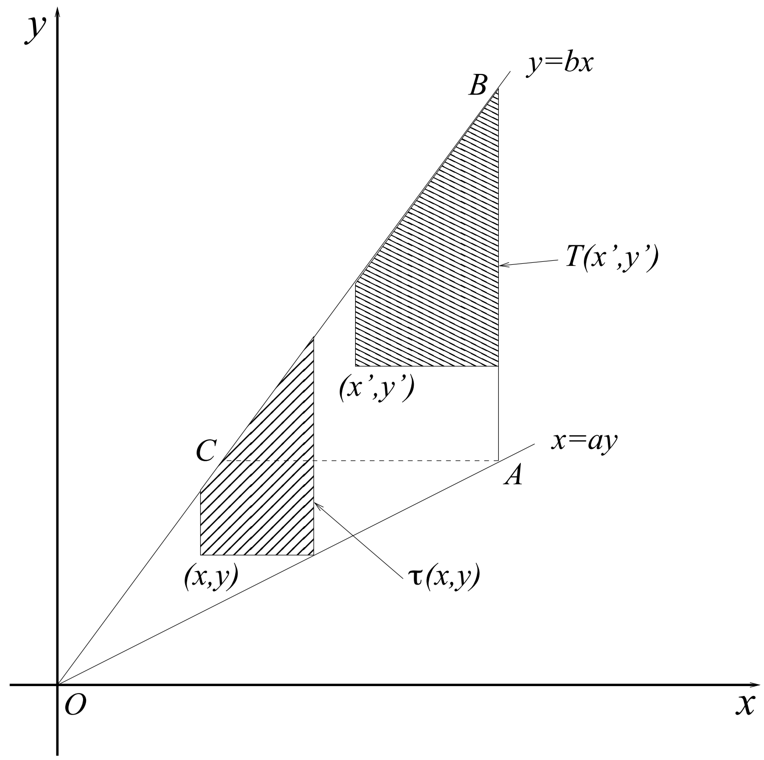

FIGURE 3. $u_{xy} = u$: the trapezoids $\tau(x, y)$ and $T(x', y')$.

- for $(x, y) \in \overline{ABC}$:

$$u^{(0)}(x, y) = \theta(y) \quad \text{and} \quad u^{(n+1)}(x, y) = \theta(y) + \iint\limits_{T(x,y)} u^{(n)}(\xi, \eta)\, d\eta d\xi. \tag{36}$$

We impose the additional requirement that θ satisfies $\theta(y) > 0$ on the open interval (y_A, y_B). It is not difficult to see that there are such functions which also satisfy (34).

The goal is to show that the strict positivity of θ "propagates" into the solution on all of OAC in the precise sense of Claim 4.1 below. First, we introduce the following notation. Set

$$A_0 := A, \quad C_0 := C,$$
$$A_i := \left(x_{C_{i-1}}, \tfrac{1}{a} x_{C_{i-1}}\right), \quad C_i := \left(\tfrac{1}{b} y_{A_i}, y_{A_i}\right) \quad \text{for } i = 1, 2, \ldots,$$

(see Figure 4), and define the trapezoidal regions:

$$\mathcal{T}_N := A_N A_{N-1} C_{N-1} C_N \cup C_{N-1} A_{N-1},$$

with the top side included and other three sides excluded, where $C_{N-1}A_{N-1}$ is an open segment.

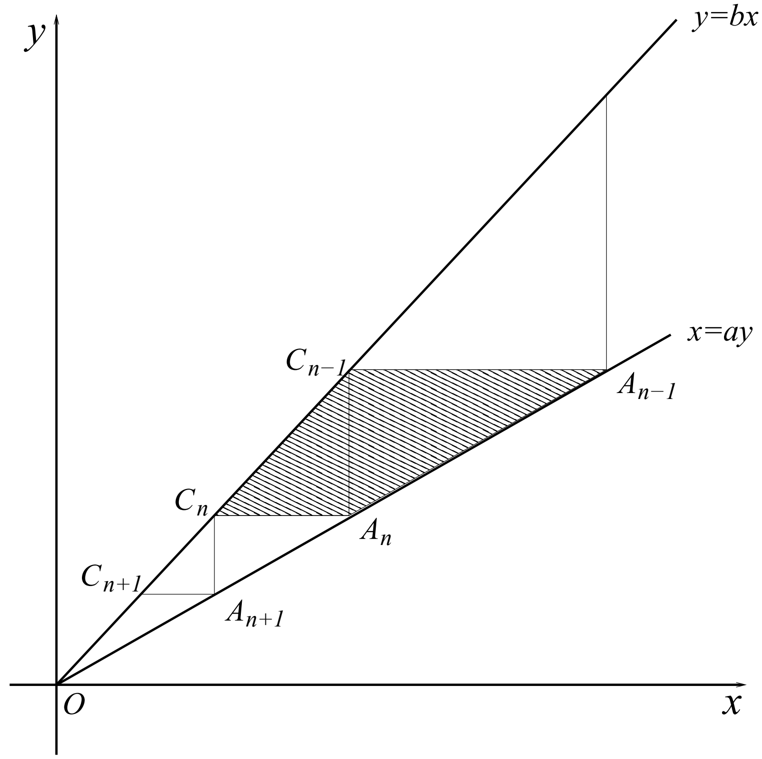

FIGURE 4. $u_{xy} = u$: the subdomains used in the proof of Theorem 1.

Claim 4.1. *For any $N \geq 1$, the following holds: for all $n \geq N$, $u^{(n)} > u^{(n-1)}$ on $\mathcal{T}_N$.*

Since the iterates $u^{(n)}$ converge uniformly on OAB to the limit solution u_θ, and since the trapezoids $\mathcal{T}_N$ in Claim 4.1 exhaust OAC as $N \to \infty$, Claim 4.1 implies that $u_\theta(x, y) > 0$ at all points $(x, y) \in OAC$, which is the conclusion of the theorem.

It thus remains to establish Claim 4.1. This is carried out via double induction on N and n. We first establish two auxiliary results.

Claim 4.2. *For all $n \geq 0$ we have,*

$$u^{(n)} \geq 0 \qquad \text{on the triangle } \overline{OAB}, \tag{37}$$

and

$$u^{(n)} \equiv 0 \qquad \text{on the triangle } \overline{OA_nC_n}. \tag{38}$$

Proof of Claim 4.2. Non-negativity of all iterates is immediate from the scheme in (35)-(36), while (38) follows easily by induction on n: the base step is provided by the first part of (35), and the induction step follows from the second part since $\tau(x, y) \subset \overline{OA_nC_n}$ whenever $(x, y) \in \overline{OA_{n+1}C_{n+1}}$. $\square$

For the base step of the main argument we shall also need the following.

Claim 4.3. *For all $n \geq 0$ we have*

$$u^{(n+1)} > u^{(n)} \text{ on the triangle } ABC. \tag{39}$$

Proof of Claim 4.3. We argue by induction on n. Fix $(x, y) \in ABC$. For $n = 0$ we have $u^{(0)}(x, y) = \theta(y)$, which is strictly positive on $T(x, y)$. According to (36) we thus have

$$\begin{aligned} u^{(1)}(x, y) &= \theta(y) + \iint\limits_{T(x,y)} u^{(0)}(\xi, \eta)\, d\eta d\xi \\ &= u^{(0)}(x, y) + \iint\limits_{T(x,y)} u^{(0)}(\xi, \eta) d\eta d\xi > u^{(0)}(x, y). \end{aligned}$$

Next, assuming that $u^{(n)} > u^{(n-1)}$ holds on ABC for index n, (36) gives

$$\begin{aligned} u^{(n+1)}(x, y) &= \theta(y) + \iint\limits_{T(x,y)} u^{(n)}(\xi, \eta)\, d\eta d\xi \\ &> \theta(y) + \iint\limits_{T(x,y)} u^{(n-1)}(\xi, \eta) d\eta d\xi = u^{(n)}(x, y). \end{aligned}$$

□

Proof of Claim 4.1. The proof is by double induction. We set

$$\begin{aligned} \tau_0(x, y) &:= \tau(x, y) \cap ABC, \\ \tau_N(x, y) &:= \tau(x, y) \cap \mathcal{T}_N \qquad \text{for } N \geq 1. \end{aligned} \tag{40}$$

Base step $N = 1$: Fix $(x, y) \in \mathcal{T}_1$ and $n \geq 1$; we want to show that $u^{(n)}(x, y) > u^{(n-1)}(x, y)$. This is done by induction on n.

Base step $n = 1$: We want show $u^{(1)}(x, y) > u^{(0)}(x, y) = 0$; we have

$$u^{(1)}(x, y) = \iint\limits_{\tau(x,y)} u^{(0)}(\xi, \eta) d\eta d\xi = \iint\limits_{\tau_0(x,y)} \theta(\eta) d\eta d\xi > 0. \tag{41}$$

Inductive step in n: Assuming that for some $n \geq 1$ we have

$$u^{(n)} > u^{(n-1)} \qquad \text{on } \mathcal{T}_1, \tag{42}$$

we want to show that $u^{(n+1)} > u^{(n)}$ on $\mathcal{T}_1$. We have

$$u^{(n+1)}(x, y) = \iint\limits_{\tau(x,y)} u^{(n)}(\xi, \eta) d\eta d\xi = \Big(\iint\limits_{\tau_0(x,y)} + \iint\limits_{\tau_1(x,y)} \Big) u^{(n)}(\xi, \eta) d\eta d\xi.$$

On τ_0, $u^{(n)}(\xi, \eta) > u^{(n-1)}(\xi, \eta)$ by Claim 4.3, while on τ_1, $u^{(n)}(\xi, \eta) > u^{(n-1)}(\xi, \eta)$, according to the induction hypothesis (42). Therefore,

$$\begin{aligned} u^{(n+1)}(x, y) &> \Big(\iint\limits_{\tau_0(x,y)} + \iint\limits_{\tau_1(x,y)} \Big) u^{(n-1)}(\xi, \eta) d\eta d\xi \\ &= \iint\limits_{\tau(x,y)} u^{(n-1)}(\xi, \eta) d\eta d\xi = u^{(n)}(x, y), \end{aligned}$$

completing the argument for the base step $N = 1$.

Inductive step in N: We now assume that for some $N \geq 1$ and for all $n \geq N$,

$$u^{(n)} > u^{(n-1)} \qquad \text{on } \mathcal{T}_N, \tag{43}$$

and we want to show that $u^{(n)} > u^{(n-1)}$ on $\mathcal{T}_{N+1}$ for all $n \geq N+1$. This is again done by induction on n. Fix $(x, y) \in \mathcal{T}_{N+1}$.

Base step $n = N + 1$: We want to show $u^{(N+1)}(x, y) > u^{(N)}(x, y)$; we have

$$\begin{aligned} u^{(N+1)}(x, y) &= \iint\limits_{\tau(x,y)} u^{(N)}(\xi, \eta)\, d\eta d\xi \\ &= \iint\limits_{\tau_{N+1}(x,y)} u^{(N)}(\xi, \eta) d\eta d\xi + \iint\limits_{\tau_N(x,y)} u^{(N)}(\xi, \eta) d\eta d\xi, \end{aligned}$$

where $\tau_N(x, y)$ is defined by (40). According to the first part of Claim 4.2, the first term in the sum is non-negative. For the second term, we apply the induction hypothesis (43) with $n = N$: since $\tau_N(x, y) \subset \mathcal{T}_N$, we have $u^{(N)} > u^{(N-1)}$ on $\tau_N(x, y)$. Therefore,

$$u^{(N+1)}(x, y) \geq \iint\limits_{\tau_N(x,y)} u^{(N)}(\xi, \eta) d\eta d\xi > \iint\limits_{\tau_N(x,y)} u^{(N-1)}(\xi, \eta) d\eta d\xi.$$

Finally, according to the second part of Claim 4.2, $u^{(N-1)}$ vanishes on $OA_{N-1}C_{N-1}$, which contains $\tau_{N+1}(x, y)$. It follows that

$$u^{(N+1)}(x, y) > \iint\limits_{\tau_N(x,y)} u^{(N-1)}(\xi, \eta) d\eta d\xi \equiv \iint\limits_{\tau(x,y)} u^{(N-1)}(\xi, \eta) d\eta d\xi = u^{(N)}(x, y),$$

which establishes the base step $n = N + 1$.

Inductive step in n: Assuming that for some $n \geq N + 1$ we have

$$u^{(n)} > u^{(n-1)} \qquad \text{on } \mathcal{T}_{N+1}, \tag{44}$$

we want to show that $u^{(n+1)}(x, y) > u^{(n)}(x, y)$. We have,

$$u^{(n+1)}(x, y) = \iint\limits_{\tau(x,y)} u^{(n)}(\xi, \eta)\, d\eta d\xi = \Big(\iint\limits_{\tau_{N+1}(x,y)} + \iint\limits_{\tau_N(x,y)} \Big) u^{(n)}(\xi, \eta) d\eta d\xi.$$

For the first integral we use that $\tau_{N+1}(x, y) \subset \mathcal{T}_{N+1}$ and apply the induction hypothesis (44) on n; for the second integral we use that $\tau_N(x, y) \subset \mathcal{T}_N$, and apply the induction hypothesis (43) on N. This yields

$$\begin{aligned} u^{(n+1)}(x, y) &> \iint\limits_{\tau_{N+1}(x,y)} u^{(n-1)}(\xi, \eta) d\eta d\xi + \iint\limits_{\tau_N(x,y)} u^{(n-1)}(\xi, \eta) d\eta d\xi \\ &= \iint\limits_{\tau(x,y)} u^{(n-1)}(\xi, \eta)\, d\eta d\xi = u^{(n)}(x, y), \end{aligned}$$

completing the induction on N. $\square$

As detailed above, with Claim 4.1 proved, Theorem 1 follows. $\square$

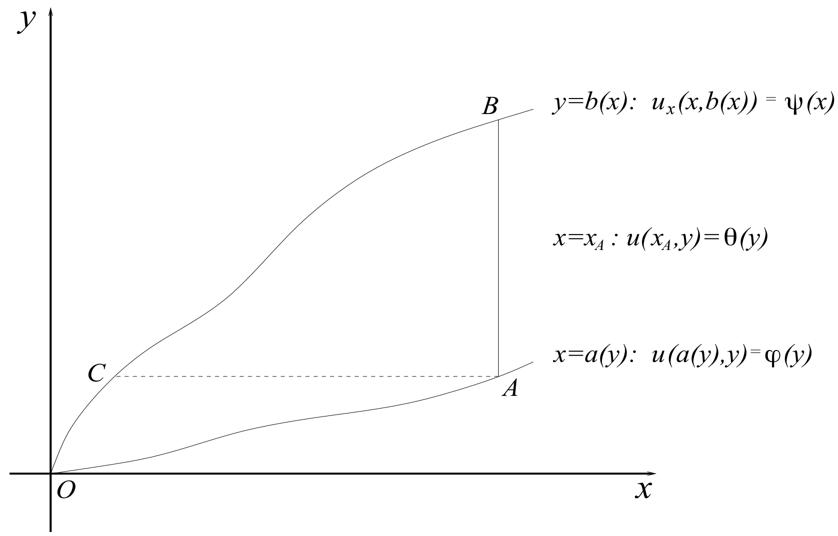

FIGURE 5. $u_{xy} = u$: the data for $u(x, y)$ is prescribed along the graph of $x = a(y)$, $y \in [0, y_A]$, while the data for $u_x(x, y)$ is prescribed along the graph of $y = b(x)$, $x \in [0, x_A]$.

5. Local existence for nonlinear equations

In this section, we consider nonlinear equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y), \tag{45}$$

with u-data prescribed along a curve $M = \{x = a(y)\}$, and with u_x-data prescribed along a curve $N = \{y = b(x)\}$. We assume that their graphs are located as in Figure 5. More precisely, we consider the following setup: $A = (x_A, y_A)$ and $B = (x_B, y_B)$ are two points in the first quadrant, with $x_A = x_B$ and $y_A < y_B$, and

(H1) $a \in C^1[0, y_A]$, a is strictly increasing, with $a(0) = 0$ and $a(y_A) = x_A$;
(H2) $b \in C^0[0, x_A]$, b is strictly increasing, with $b(0) = 0$ and $b(x_A) = y_B$;
(H3) $y < b(a(y))$ for all $y \in (0, y_A]$;
(H4) $\varphi \in C^1[0, x_A]$ and $\psi \in C^0[0, y_B]$.

For simplicity we make the following assumptions on the right-hand member f in (45):

(H5) $f : \overline{OAB} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, bounded, and satisfies a uniform Lipschitz condition with respect to $v = (u, p, q)$: there exists $L > 0$, such that

$$|f(x, y, v) - f(x, y, \bar{v})| \leq L|v - \bar{v}| \tag{46}$$

for all $(x, y) \in \overline{OAB}$ and all $v, \bar{v} \in \mathbb{R}^3$.

Here and below we use the notation

$$|v| = |u| + |p| + |q| \qquad \text{for } v = (u, p, q) \in \mathbb{R}^3.$$

With these assumptions we now pose the following boundary value problem:

$$u_{xy}(x,y) = f(x,y,u(x,y),u_x(x,y),u_y(x,y)) \qquad \text{for } (x,y) \in \overline{OAB} \tag{47}$$

$$u(a(y),y) = \varphi(y) \qquad \text{for } y \in [0, y_A] \tag{48}$$

$$u_x(x,b(x)) = \psi(x) \qquad \text{for } x \in [0, x_A]. \tag{49}$$

As for the particular case treated in Section 3, we shall obtain a solution of (47)-(48)-(49) by constructing a solution of the following modified problem:

$$u_{xy}(x,y) = f(x,y,u(x,y),u_x(x,y),u_y(x,y)) \qquad \text{for } (x,y) \in \overline{OAB} \tag{50}$$

$$u(a(y),y) = \varphi(y) \qquad \text{for } y \in [0, y_A] \tag{51}$$

$$u_x(x,b(x)) = \psi(x) \qquad \text{for } x \in [0, x_A] \tag{52}$$

$$u(x_A,y) = \theta(y) \qquad \text{for } y \in [y_A, y_B], \tag{53}$$

where $\theta(y)$ is a suitably chosen function defined on $[y_A, y_B]$. To obtain a solution of the latter problem, and hence of the original problem (47)-(48)-(49), we will employ Picard iteration. The convergence of the iteration scheme will be obtained on a sufficiently small subregion of $\overline{OAB}$.

The reformulation of the PDE (50) as an integral equation, and setting up an iteration scheme for the triple (u, u_x, u_y) is standard; see e.g., Section 21 in [19]. However, there is now an additional "twist" to this setup: differently from standard cases treated in the literature, the choice of θ is now part of the problem. Indeed, as we shall see, the presence of u_x on the right-hand side of (50) forces us to consider a scheme which includes iteration of the function θ as well.

Before formulating the iteration scheme we analyze the conditions that θ must satisfy in order to yield a classical C^1-solution to (50)-(51)-(52)-(53). By integrating (50), first with respect to x and then with respect to y, and making use of the boundary data (51)-(52)-(53), we obtain that

$$u(x,y) = \varphi(y) - \int_a^{a(y)} \psi(\xi)\, d\xi + \int_x^{a(y)} \int_y^{b(\xi)} f(\xi,\eta,v(\xi,\eta))\, d\eta d\xi \quad \text{on } \overline{OAC}, \tag{54}$$

and

$$u(x,y) = \theta(y) - \int_a^{x_A} \psi(\xi)\, d\xi + \int_x^{x_A} \int_y^{b(\xi)} f(\xi,\eta,v(\xi,\eta))\, d\eta d\xi \quad \text{on } \overline{ABC}. \tag{55}$$

As in the proof of Lemma 3.1 we obtain the following: in order that the expressions in (54) and (55) hold for a $C^1(\overline{OAB})$ function u, it is necessary that θ belongs to $C^1[y_A, y_B]$ and satisfies

$$\theta(y_A) = \varphi(y_A) \tag{56}$$

$$\theta'(y_A) = \varphi'(y_A) - a'(y_A)\Big[\psi(x_A) - \int_{y_A}^{y_B} f(x_A,\eta,v(x_A,\eta))\, d\eta\Big]. \tag{57}$$

We need to make sure that the iteration scheme incorporates these conditions. For simplicity we shall use affine functions as iterates for θ. We proceed as follows.

5.1. **Iteration scheme.**

5.1.1. *Base step.* We start by fixing the affine function $\theta^{(0)} = \varphi(y_A) + \sigma_0(y - y_A)$ characterized by the conditions

$$\theta^{(0)}(y_A) = \varphi(y_A) \qquad \text{and} \qquad \theta^{(0)\prime}(y_A) = \sigma_0 = \varphi'(y_A) - a'(y_A)\psi(x_A), \tag{58}$$

and then set

$$u^{(0)}(x,y) := \begin{cases} \varphi(y) - \int_x^{a(y)} \psi(\xi)d\xi & \text{for } (x,y) \in \overline{OAC} \\ \theta^{(0)}(y) - \int_x^{x_A} \psi(\xi)d\xi & \text{for } (x,y) \in \overline{ABC}, \end{cases} \tag{59}$$

$$p^{(0)}(x,y) := \psi(x) \qquad \text{for } (x,y) \in \overline{OAB}, \tag{60}$$

$$q^{(0)}(x,y) := \begin{cases} \varphi'(y) - a'(y)\psi(a(y)) & \text{for } (x,y) \in \overline{OAC} \\ \theta^{(0)\prime}(y) & \text{for } (x,y) \in \overline{ABC}, \end{cases} \tag{61}$$

The conditions in (58) ensure that $u^{(0)}$ and $q^{(0)}$ are defined consistently on $\overline{AC} = \overline{OAC} \cap \overline{ABC}$. It is immediate to verify that the following holds:

$$\begin{aligned}
&u_x^{(0)}(x,y) = p^{(0)}(x,y) && \text{for } (x,y) \in \overline{OAB},\\
&u_y^{(0)}(x,y) = q^{(0)}(x,y) && \text{for } (x,y) \in \overline{OAB},\\
&u^{(0)}(a(y),y) = \varphi(y) && \text{for } y \in [0,y_A],\\
&u^{(0)}(x_A,y) = \theta^{(0)}(y) && \text{for } y \in [y_A,y_B],\\
&p^{(0)}(x,b(x)) = \psi(x) && \text{for } x \in [0,x_A],\\
&q^{(0)}(x_A,y) = \theta^{(0)\prime}(y) && \text{for } y \in [0,y_A].
\end{aligned}$$

5.1.2. *Iteration step.* For $n \geq 0$, assume that $u^{(n)}$, $p^{(n)}$, $q^{(n)}$, and $\theta^{(n)}$ are continuous functions on $\overline{OAB}$ that satisfy

$$u_x^{(n)}(x,y) = p^{(n)}(x,y) \qquad \text{for } (x,y) \in \overline{OAB}, \tag{62}$$

$$u_y^{(n)}(x,y) = q^{(n)}(x,y) \qquad \text{for } (x,y) \in \overline{OAB}, \tag{63}$$

$$u^{(n)}(a(y),y) = \varphi(y) \qquad \text{for } y \in [0,y_A], \tag{64}$$

$$u^{(n)}(x_A,y) = \theta^{(n)}(y) \qquad \text{for } y \in [y_A,y_B], \tag{65}$$

$$p^{(n)}(x,b(x)) = \psi(x) \qquad \text{for } x \in [0,x_A], \tag{66}$$

$$q^{(n)}(x_A,y) = {\theta^{(n)}}'(y) \qquad \text{for } y \in [0,y_A]. \tag{67}$$

We first update the u-data along $\overline{AB}$ by letting $\theta^{(n+1)}$ be the affine function characterized by

$$\theta^{(n+1)}(y_A) = \varphi(y_A) \tag{68}$$

$$\theta^{(n+1)\prime}(y_A) = \varphi'(y_A) - a'(y_A)\Big[\psi(x_A) - \int_{y_A}^{y_B} f\big(x_A, \eta, \theta^{(n)}(\eta), p^{(n)}(x_A, \eta), \theta^{(n)\prime}(\eta)\big)d\eta\Big]. \tag{69}$$

In accordance with (56)-(57), by using this $\theta^{(n+1)}$ in the definitions of $u^{(n+1)}$ and $q^{(n+1)}$ below, we guarantee continuity of the next iterate $v^{(n+1)}$ across the horizontal line $\overline{AC}$. As remarked above, we note that the presence of $p^{(n)}(x_A, \eta)$ in the integrand on the right-hand side of (69) (i.e., the presence of u_x in the original PDE (45)), rules out the possibility of using a fixed function θ in all iteration steps.

We then update $v^{(n)} = (u^{(n)}, p^{(n)}, q^{(n)})$ by setting

$$u^{(n+1)}(x, y) := \varphi(y) - \int_x^{a(y)} \psi(\xi)d\xi + \int_x^{a(y)}\int_y^{b(\xi)} f\big(\xi, \eta, v^{(n)}(\xi, \eta)\big)\, d\eta d\xi \tag{70}$$

for $(x, y) \in \overline{OAC}$,

$$u^{(n+1)}(x, y) := \theta^{(n+1)}(y) - \int_x^{x_A} \psi(\xi)d\xi + \int_x^{x_A}\int_y^{b(\xi)} f\big(\xi, \eta, v^{(n)}(\xi, \eta)\big)\, d\eta d\xi \tag{71}$$

for $(x, y) \in \overline{ABC}$;

$$p^{(n+1)}(x, y) := \psi(x) - \int_y^{b(x)} f\big(x, \eta, v^{(n)}(x, \eta)\big)\, d\eta \tag{72}$$

for $(x, y) \in \overline{OAB}$;

$$q^{(n+1)}(x, y) := \varphi'(y) - a'(y)\Big[\psi(a(y)) - \int_y^{b(a(y))} f\big(a(y), \eta, v^{(n)}(a(y), \eta)\big)d\eta\Big] - \int_x^{a(y)} f\big(\xi, y, v^{(n)}(\xi, y)\big)d\xi \tag{73}$$

for $(x, y) \in \overline{OAC}$, and

$$q^{(n+1)}(x, y) := \theta^{(n+1)\prime}(y) - \int_x^{x_A} f\big(\xi, y, v^{(n)}(\xi, y)\big)\, d\xi \tag{74}$$

for $(x, y) \in \overline{ABC}$.

As noted above, the conditions (68) and (69) ensure continuity of $u^{(n+1)}$ and $q^{(n+1)}$, respectively, across $\overline{AC}$. It is immediate to verify that, with the definitions above, (62)-(67) are satisfied with n replaced by $n + 1$.

5.2. **Convergence.** We proceed to establish convergence of the sequence of iterates $v^{(n)} = (u^{(n)}, p^{(n)}, q^{(n)})$. The goal is to show that they form a Cauchy sequence in $C^0(\overline{OAB}; \mathbb{R}^3)$ equipped with the norm

$$\|v\| = \|u\| + \|p\| + \|q\| \equiv \sup |u(x, y)| + \sup |p(x, y)| + \sup |q(x, y)|,$$

where the supremums are taken over $(x, y) \in \overline{OAB}$. This will be established under the condition that the region $\overline{OAB}$ is sufficiently small. Set

$$\begin{aligned}
I &:= [y_A, y_B] \\
l &:= x_A \\
h &:= y_B \\
\alpha &:= \text{area}(OAB) \leq lh \\
\gamma &:= \max_{0 \leq y \leq y_A} |a'(y)|.
\end{aligned}$$

Fix $n \geq 0$. The first step is to estimate the difference between $\theta^{(n+1)}$ and $\theta^{(n)}$ on $[y_A, y_B]$. Denote the slope of the affine function $\theta^{(n)}$ by σ_n; this is given by $(58)_2$ and (69). Thus,

$$\theta^{(n)}(y) = \varphi(y_A) + \sigma_n(y - y_A) \qquad n \geq 0.$$

We thus have

$$|(\theta^{(n+1)} - \theta^{(n)})(y)| = |\sigma_{n+1} - \sigma_n||y - y_A| < h|\sigma_{n+1} - \sigma_n|. \tag{75}$$

For $n \geq 1$, (69) together with the Lipschitz property (46) give

$$\begin{aligned}
|\sigma_{n+1} - \sigma_n| &= |a'(y_A)| \cdot \int_{y_A}^{y_B} \big| f\big(x_A, \eta, \theta^{(n)}(\eta), p^{(n)}(x_A, \eta), \sigma_n\big) \\
&\qquad\qquad - f\big(x_A, \eta, \theta^{(n-1)}(\eta), p^{(n-1)}(x_A, \eta), \sigma_{n-1}\big)\big| d\eta \\
&\leq \gamma L \int_{y_A}^{y_B} |(\theta^{(n)} - \theta^{(n-1)})(\eta)| + |(p^{(n)} - p^{(n-1)})(x_A, \eta)| + |\sigma_n - \sigma_{n-1}|\, d\eta \\
&\leq \gamma Lh \Big[\sup_{\eta \in I} |(\theta^{(n)} - \theta^{(n-1)})(\eta)| + \sup_{\eta \in I} |(p^{(n)} - p^{(n-1)})(x_A, \eta)| + |\sigma_n - \sigma_{n-1}|\Big] \\
&\leq \gamma Lh \Big[\|u^{(n)} - u^{(n-1)}\| + \|p^{(n)} - p^{(n-1)}\| + \|q^{(n)} - q^{(n-1)}\|\Big] \\
&= \gamma Lh \|v^{(n)} - v^{(n-1)}\|,
\end{aligned} \tag{76}$$

where for the last inequality we have used (65) and (67).

For $n \geq 1$ we proceed with similar estimates for $u^{(n)}$, $p^{(n)}$, and $q^{(n)}$. From (70),

$$|(u^{(n+1)} - u^{(n)})(x, y)| \leq L\alpha \|v^{(n)} - v^{(n-1)}\| \tag{77}$$

for $(x, y) \in \overline{OAC}$, while (71) gives

$$\begin{aligned}
|(u^{(n+1)} - u^{(n)})(x, y)| &\leq |(\theta^{(n+1)} - \theta^{(n)})(y)| + L\alpha \|v^{(n)} - v^{(n-1)}\| \\
&\leq (\gamma Lh^2 + L\alpha) \|v^{(n)} - v^{(n-1)}\|
\end{aligned} \tag{78}$$

for $(x, y) \in \overline{ABC}$, where in the last inequality we have used (75) and (76). Combining (77) and (78) and using that $\alpha \le lh$, we get f

$$\|u^{(n+1)} - u^{(n)}\| \le Lh(\gamma h + l)\|v^{(n)} - v^{(n-1)}\|. \tag{79}$$

Next, (72) gives

$$|(p^{(n+1)} - p^{(n)})(x, y)| \le Lh\|v^{(n)} - v^{(n-1)}\| \tag{80}$$

for all $(x, y) \in \overline{OAB}$, (73) gives

$$|(q^{(n+1)} - q^{(n)})(x, y)| \le Lh\gamma||v^{(n)} - v^{(n-1)}|| + Ll\|v^{(n)} - v^{(n-1)}\| \tag{81}$$

for $(x, y) \in \overline{OAC}$, and (74) gives

$$\begin{aligned} |(q^{(n+1)} - q^{(n)})(x, y)| &\le |(\sigma_{n+1} - \sigma_n)(y)| + Ll\|v^{(n)} - v^{(n-1)}\| \\ &\le (\gamma Lh + Ll)\|v^{(n)} - v^{(n-1)}\| \end{aligned} \tag{82}$$

for $(x, y) \in \overline{ABC}$, where in the last inequality we have used (76). From (81) and (82), we obtain

$$|(q^{(n+1)} - q^{(n)})(x, y)| \le L(\gamma h + l)\|v^{(n)} - v^{(n-1)}\|. \tag{83}$$

for all $(x, y) \in \overline{OAB}$.

Finally, combining (79), (80), and (83), we obtain

$$\|v^{(n+1)} - v^{(n)}|| \le \mu(L, \gamma, h, l)||v^{(n)} - v^{(n-1)}\|,$$

where $\mu(L, \gamma, h, l) = L(2\gamma h + 2l + h)$. We then choose l, and hence h, sufficiently small, so that $\mu < 1$. It follows that the sequence $(v^{(n)}) = (u^{(n)}, p^{(n)}, q^{(n)})$ is Cauchy in $C^0(\overline{OAB}; \mathbb{R}^3)$, and thus converges uniformly to a continuous triplet of functions $v = (u, p, q) : \overline{OAB} \to \mathbb{R}^3$. As $(v^{(n)})$ is Cauchy it follows from (75) that (σ_n) is a Cauchy sequence of real numbers. Recalling that the σ_n are the slopes of the affine functions $\theta^{(n)} : [y_A, y_B] \to \mathbb{R}$, we obtain that $(\theta^{(n)})$ is Cauchy in $C^0[y_A, y_B]$; let its limit be θ. As a uniform limit of affine functions with value $\varphi(y_A)$ at $y = y_A$, θ is itself affine, $\theta(y) = \sigma(y - y_A) + \varphi(y_A)$, where σ is the limit of (σ_n).

Thanks to uniform convergences $v^{(n)} \to v$ and $\theta^{(n)} \to \theta$, sending $n \to \infty$ in (68)-(71), (72), (73)-(74) yields the corresponding equations with upper indices (n) and $(n + 1)$ removed. It is an immediate consequence of the resulting identities that $p = u_x$ and $q = u_y$ on $\overline{OAB}$. In particular, we obtain that

$$u(x, y) = \varphi(y) - \int_x^{a(y)} \psi(\xi) d\xi + \int_x^{a(y)} \int_y^{b(\xi)} f\left(\xi, \eta, (u, u_x, u_y)(\xi, \eta)\right) d\eta\, d\xi \tag{84}$$

for $(x, y) \in \overline{OAC}$, and

$$u(x, y) = \theta(y) - \int_x^{x_A} \psi(\xi) d\xi + \int_x^{x_A} \int_y^{b(\xi)} f\left(\xi, \eta, (u, u_x, u_y)(\xi, \eta)\right) d\eta\, d\xi \tag{85}$$

for $(x, y) \in \overline{ABC}$. Since θ satisfies (68) and (69) with upper indices (n) and $(n+1)$ removed, it follows from these identities that u, u_x, u_y, and u_{xy} exist and are continuous in $\overline{OAB}$, that $u_{xy}(x, y) = f(x, y, (u, u_x, u_y)(x, y))$ for $(x, y) \in OAB$, $u(y, (a(y)) = \varphi(y)$ for $y \in [0, y_A]$, and that $u_x(x, b(x)) = \psi(x)$ for $x \in [0, x_A]$. This concludes the proof of local existence for the mixed boundary value problem (47)-(48)-(49).

Combining our findings above with those from Section 4 we have the following theorem.

Theorem 2. *With assumptions (H1), (H2), (H3), (H4), and (H5) above, consider the mixed boundary value problem (47)-(48)-(49) on the region* $\overline{OAB}$ *as described above; see Figure 5. Then, for* $\overline{OAB}$ *sufficiently small, this problem has a solution* u *with the properties that* u, u_x, u_y, *and* u_{xy} *are continuous functions on* $\overline{OAB}$. *The solution is in general not unique.*

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H. K. Jenssen, Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA (jenssen@math.psu.edu).

I. A. Kogan, Department of Mathematics, North Carolina State University, Raleigh, NC, 27695-8205, USA (iakogan@ncsu.edu)