Prospective teachers enacting proof tasks in secondary mathematics classrooms

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We use a curriculum design framework to analyze how prospective secondary teachers (PSTs) designed and implemented in local schools, lessons that integrate ongoing mathematical topics with one of the four proof themes addressed in the capstone course Mathematical Reasoning and Proving for Secondary Teachers. In this paper we focus on lessons developed around the conditional statements proof theme. We examine the ways in which PSTs integrated conditional statements in their lesson plans, how these lessons were implemented in classrooms, and the challenges PSTs encountered in these processes. Our results suggest that even when PSTs designed rich lesson plans, they often struggled to adjust their language to the students' level and to maintain the cognitive demand of the tasks. We conclude by discussing possible supports for PSTs' learning in these areas.

Keywords: Reasoning and Proving, Preservice Secondary Teachers, Lesson Plans, Task Design

Integrating reasoning and proving in secondary schools has been an elusive goal of the mathematics education community. Despite agreement of the importance of reasoning and proving in school mathematics among scholars and policy makers, proof has been shown to be a "hard-to-teach and hard-to-learn" concept (Stylianides & Stylianides, 2017, p. 119). Areas that have been identified as being persistently difficult for students, but also critically important for proof production and comprehension, are understanding the role of examples in proving including recognition of limitation of supportive examples as proof and the role of a single counterexample as refuting evidence, conditional statements, argument evaluation, and indirect reasoning (e.g., Antonini & Mariotti, 2008; Durand-Guerrier, 2003). We term these areas "proof themes". Our choice of these four proof themes stems from the literature and our own experience as instructors observing PSTs' challenges in university coursework. Our study rests on the assumption that these proof themes can be integrated into secondary school mathematics in "intellectually honest" ways that are true to the discipline of mathematics and honor students as learners (Bruner, 1960; Stylianides, 2008). Towards this end we developed and systematically study a capstone course *Mathematical Reasoning and Proving for Secondary Teachers*; intended to support prospective secondary teachers (PSTs) in developing robust knowledge and pedagogical skills for integrating proof in their classroom practices.

The course aimed to increase PSTs' awareness of the logical aspects of proof and the place of proof in secondary curricula, expose PSTs to common student difficulties with proof, and provide PSTs with pedagogical tools to create or modify tasks that integrate reasoning and proving. In the practical component of the course, PSTs developed lessons on each of the four proof themes and taught them in local schools. In this paper, we focus on two PSTs' lessons plans that successfully integrated the conditional statements proof theme and analyze how these lessons were enacted in classrooms. Our analysis reveals the aspects of lesson enactment that were successful and those that posed challenges to PSTs. In the discussion, we contemplate potential reasons for these challenges and how we, as teacher educators, can further support PSTs in the process of integrating proof in their teaching practices.

Theoretical Perspectives

We adopt Stylianides and Stylianides' (2017) definition of proof as "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (p. 212). By "proving" we mean processes such as conjecturing, generalizing and making valid arguments grounded in mathematical deductions rather than authority or empirical evidence (Ellis, Bieda & Knuth, 2012). Implicit in this definition is that students must develop understanding of what a deductive argument is, and that teachers must provide opportunities for students to develop such understanding through instructional activities. Teachers, on their part, rely on curriculum materials to facilitate student learning of proof.

Stein, Remillard and Smith (2007) distinguish between *written curriculum*, which includes written artifacts that teachers and students use, *intended curriculum*, which is the teacher's lesson plan, and *enacted curriculum* that is the lesson as it unfolds in the classroom (Fig. 1).

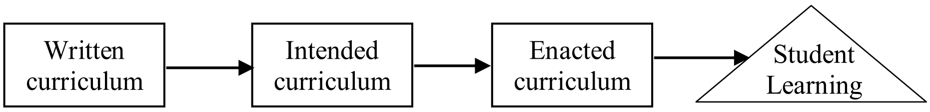


Figure 1: Phases of curriculum. Adapted from Stein, Remillard, & Smith, 2007, p.322

From the perspective of a secondary teacher aiming to integrate reasoning and proof into the mathematics curriculum, we find that each element of this model (Fig. 1) presents unique challenges. First, written curricula in the United States, such as textbooks, offer limited proof-related tasks outside high-school geometry (e.g., Thompson, Senk, & Johnson, 2012). Thus, it becomes the work of the teacher to design tasks and develop an intended curriculum. Next, as the teacher enacts the lesson in class, he/she must use appropriate language and conceptual tools that are within the reach of secondary students to highlight mathematical ideas. Here, again, curriculum materials offer little guidance to teachers on how to enact proof tasks in classrooms (Stylianides, 2008) in ways that support development of students' conceptions which are in line with conventional mathematics. Thus, much of the curriculum design and implementation around proof rests on teachers' own knowledge and beliefs about the importance of proof for their students' mathematical learning.

Supporting PSTs in developing such knowledge and productive dispositions towards proof were the goals of the capstone course. In Buchbinder and McCrone (2018) we describe the course structure and its theoretical underpinnings. Here, we illustrate the design features of the course that were intended to support PSTs in development and enactment of a lesson on conditional statements. We use conditional statements as an example, but our analyses apply to all four proof themes mentioned.

Setting

The course *Mathematical Reasoning and Proving for Secondary Teachers* contains four modules, each addressing one of the proof themes. In the Conditional Statements module, PSTs first engaged in activities designed to strengthen their knowledge of conditional statements. This knowledge includes understanding that a conditional statement has the form: $If\ P\ then\ Q$ ($P \Rightarrow Q$), where P is an hypothesis and Q is a conclusion; how to determine truth-value of such a statement; its equivalent forms such as a contrapositive ($\sim Q \Rightarrow \sim P$) and non-equivalent forms, such as a converse ($Q \Rightarrow P$).

PSTs then examined excerpts of hypothetical student work related to conditional statements, analyzed students' conceptions, and contemplated ways to address students' difficulties. Next, the PSTs reviewed a sample of mathematics textbooks to examine where conditional statements appear in the school curriculum. These activities aimed to equip PSTs with the background for creating their own lessons integrating conditional statements with mathematical topics taught in local schools.

During the lesson development stage, PSTs shared ideas and received feedback on their lesson plans from their peers and the course instructor. The lessons were 50-minutes in length, and were intended for small groups of 5-8 students rather than the whole class. All lessons were videotaped with 360° cameras to capture both the PSTs' actions and the students' participation. PSTs then watched their videos and wrote a reflective report. Sharing lesson plans with peers was intended to support PSTs' enactment of their lesson, while reflection reports and instructor's feedback on it aimed to serve as a mechanism for future improvement. Despite multiple means of support embedded in the course design, PSTs experienced challenges in developing and enacting lessons on conditional statements, as we will show in the results section.

Methods: Participants, Data Sources and Analysis

Fifteen PSTs in their last year of university studies took part in the research. Prior to the capstone course the PSTs completed most of the required courses in mathematics and pedagogy.

The data sources for the analysis reported in this paper comprise the PST-developed lesson plans on conditional statements, the video recordings of the enacted lessons, and PSTs' reflective reports. The lesson plans were analyzed in terms of their focus on the conditional statements proof theme, and assigned a rating of high, medium or low. The *low* focused lesson plans included no more than 3 conditional statements and the activity only required students to determine their truth value. For example, Chuck's (all names in the paper are pseudonyms) 8th grade lesson on exponents had three true or false questions, such as: "*If a negative number is raised to an even power, the result will be a positive number.*" This question offers opportunities to discuss what is needed to prove or disprove such a statement and use the rules of exponents to produce a generic proof accessible to 8th graders. Yet, Chuck's plan merely expected students to produce a "proof" by example, such as $(-2)^2 = (-2)(-2) = 4$, missing the opportunity to attend to a misconception about the limitations of empirical evidence as proof and even enforcing it. Lesson plans with *high* focus on the proof theme contained more than three conditional statements along with a clear plan on how they would be used to advance students' knowledge of conditional statements (as examples in the results section will show). A lesson plan with *medium* focus would be located between these two extremes, for example, Rebecca's lesson on logic riddles dealt with reasoning and justifying, but the place of conditional statements was unclear.

The classroom videos were analyzed using Schoenfeld's (2013) *Teaching for Robust Understanding* (TRU) rubric which was slightly modified to reflect aspects of teacher work and student interaction that are specific to proving. The revised rubric had five dimensions, four related to teacher actions: (a) Accuracy, language, and connections, (b) Explicating reasoning and proof theme, (c) Actions to promote student engagement, (d) Maintaining cognitive demand; and one dimension related to (e)

Student engagement. Each video was divided into thematic episodes, no more than 5 minutes long, and each episode was assigned a score of 3 (high), 2 (medium), 1 (low) on each dimension. In the results section we illustrate the different dimensions of the rubric and the scoring system.

Results

The intended curriculum: Lesson plans

The analysis of the lesson plans in terms of prevalence of the conditional statements revealed 3 plans with low focus on the proof theme, 1 medium and 11 high. Below are examples of two lesson plans with a high focus on the conditional statements theme, developed by Bill and Dylan for students in grade 10. We chose these lessons to illustrate creative integration of conditional statements with regular content in algebra and geometry; as well as the challenges that PSTs encountered while enacting the lessons in classrooms.

Bill's lesson plan integrated conditional statements with triangle geometry. Each pair of students had two sets of notecards: yellow cards had hypotheses written on them (e.g., a triangle is equilateral), and green cards had conclusions (e.g., a triangle is isosceles). Students had to create conditional statements by matching hypotheses to conclusions. Bill intended to use student-produced statements in a whole class discussion to introduce such concepts as *domain* of a statement and a *counterexample*. Bill also planned to have students physically switch between hypothesis and conclusion cards as a way to introduce a *converse*. The lesson plan did not contain any exposition about what a conditional statement is, how it is structured, and what is needed to prove or disprove it. Bill hoped that these ideas would come out naturally as the students engaged in and discussed the card-matching activity.

Dylan's lesson integrated conditional statements with evaluating expressions and solving simple algebraic equations. First, the concept of conditional statements and key vocabulary such as truth value, domain and proposition (in lieu of hypothesis and conclusion) was introduced through non-mathematical examples such as "If a motor vehicle has four wheels then it's a car." Next, students practiced identifying domain, proposition and determining the truth-value of four statements: (1) If a number is divisible by 10, then it is divisible by 5; (2) If a number is not divisible by 10, then the number is not divisible by 5; (3) If a number is not divisible by 5, then it is not divisible by 10; and (4) If a number is divisible by 5, then it is divisible by 10. Notating the first one as $P \Rightarrow Q$, the other statements have the forms: $\sim P \Rightarrow \sim Q$, $\sim Q \Rightarrow \sim P$, and $Q \Rightarrow P$, respectively, which allows making interesting connections. The third task had students identify domain and proposition in statements related to evaluating expressions, such as: "If we have the equation 11x – 12 = 1, then the solution of x is a whole number", or "If the side length of a cube is a whole number, then the volume is also a whole number." The students worked on these tasks in pairs and then discussed as a group.

Both lesson plans meaningfully integrated conditional statements with the mathematical topics Bill and Dylan planned to teach. The tasks in the lessons were of high cognitive demand (Silver et al., 2009) as they required formulating statements, exploring and justifying claims. In the next section we examine the transition from the intended to enacted curriculum in Bill's and Dylan's lessons.

The enacted curriculum: Classroom implementation

Both Bill's and Dylan's lessons were enacted in 10^{th} grade classrooms with a group of 4 students. The description of the enactment below follows the five dimensions of the modified TRU rubric.

Bill's enacted lesson. Within the dimension of *Accuracy, language and communication* we distinguish between accuracy related to geometry, in which Bill's performance was impeccable, versus accuracy related to the proof theme. Bill's lesson plan suggested that he intended to build on students' contributions to elicit ideas about conditional statements. Thus, throughout the lesson Bill tried to avoid unfamiliar vocabulary and only used informal language. For example, when introducing the card matching activity Bill instructed students to match "if-cards" with "then-cards" to create "if-then" statements. He never introduced the concept of conditional statement and referred to hypothesis and conclusion as the "if-part" and "then-part" of the statement throughout the entire lesson. The lack of proper mathematical language complicated the classroom communication. Towards the end of the lesson Bill wrote a statement and its converse next to each other on the board and asked the students: "what changed?" One student responded by saying "the 'then' became the 'if'." This is a correct observation on behalf of the student, which signals the lack of language to describe it. While intending to build on student knowledge, Bill missed the opportunity to introduce vocabulary that could help to streamline the communication around conditional statements.

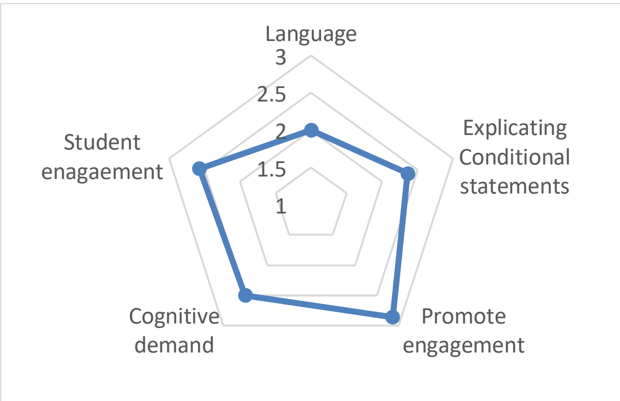
In terms of *explicating the conditional statements* proof theme, Bill seemed to follow a similar strategy of minimizing his input. The mathematical task lent itself naturally to discussing such important ideas as generality of a conditional statement and how to determine if it is true or false. Yet, the only concept Bill introduced in the lesson was a counterexample, which he informally defined as an example that "does not fit the statement and disproves it." Bill initiated a discussion on how to distinguish between examples that support, disprove or are irrelevant to the statement. For the latter point he used the statement: "If an angle in a triangle is 45°, then the measure of the third angle is 45°," and drew a triangle with no 45° angles. Students were initially confused whether this triangle constitutes a counterexample to the statement, so Bill explained that a counterexample must satisfy the "if-part" but not the "then-part". Overall, Bill explicated multiple aspects related to conditional statements, but his insistence on using only informal language kept the discussion at a basic level.

In terms of actions to *promote student engagement*, Bill's lesson was rated very high. He encouraged participation by asking multiple questions, pushing students to provide explanations and justify their thinking. He was attentive to student body language and when he sensed that some students did not follow the discussion, he asked a student to repeat what was said in their own words. Bill made sure that each student contributed something to the conversation, however the level of *student engagement* was rather moderate. Although all students were listening attentively, they appeared uncomfortable when pushed to speak in full sentences, only responding in a few words. As a result, Bill often had to break down his questions to a set of simpler ones, lowering the *cognitive demand* of the tasks. The aggregate scores for Bill along the five dimensions are shown in Figure 2a.

Dylan's enacted lesson. Similar to Bill, Dylan's *Accuracy, language and communication* was different when talking about solving equations versus aspects related to conditional statements. The latter was often imprecise or not properly adjusted to the students' level. For example, Dylan said that “when we prove the statement is false, we are providing a counterexample - something that does not fulfill the statement.” But since he did not provide a clear explanation of what it means to “not fulfill the statement” students occasionally confused an irrelevant example for a counterexample.

Dylan *explicated the conditional statement* proof theme much stronger than Bill. Dylan introduced some key concepts related to conditional statements, and even some logical notation, e.g., $P \Rightarrow Q$, which contributed to more seamless communication. However, Dylan missed multiple opportunities to draw connections among the concepts. For example, each of the four conditional statements on divisibility by 10 and by 5 was treated as a separate entity during the lesson. We are not claiming that Dylan should have delved deeper into logical notation or introduced a contrapositive, which could have been overwhelming for the students. However, Dylan missed the opportunity to draw students’ attention to the fact that that statements (2) and (4) are disproved by the same counterexample or that statements (1) and (3) require a general proof that uses the same key idea of 5 being a factor of 10. In his lesson reflection Dylan wrote how impressed he was with the students being able to identify the domain and proposition, and correctly justify the truth-value of statements that included negations, the converse, and the contrapositive. However, his lesson plan did not mention any of these connections, suggesting that the missed opportunity occurred at the planning stage.

Dylan did a good job in *promoting students’ engagement* by ensuring that students were on task, following up on students’ input, asking questions and pressing for explanations. For their part, *students participated* in the lesson in meaningful ways such as sharing ideas, responding to prompts, and justifying their work. As the lesson progressed, and its focus shifted to conditional statements involving equations, Dylan payed less attention to the logical aspects, focusing almost entirely on solving equations, possibly because students appeared to have difficulties in this area. While trying to support student thinking, Dylan often took over the explanation, thus lowering *the cognitive demand* of the tasks. The aggregate scores for Dylan along the five dimensions are shown in Figure 2b (four on teacher actions and the fifth on student engagement).



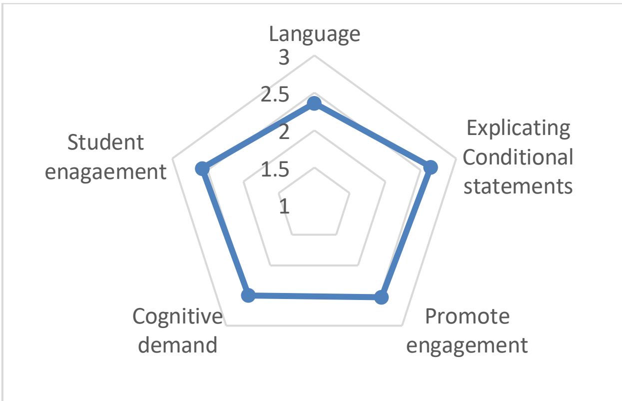


Figure 2a: Bill’s enacted lesson. **2b: Dylan’s enacted lesson**

Figure 2 allows us to compare the various dimensions of Bill’s and Dylan’s enacted lessons. Both written lesson plans were rated high on explicating the conditional statements theme, however, Bill’s insistence on using informal language resulted in lower scores in the areas of *Language* and *Explicating Conditional Statements*. Also, as described above, Bill put more effort into promoting student engagement, which is reflected in the high score for *Promote Engagement* (Fig. 2a). For the

Cognitive Demand dimension, both Bill and Dylan scored about 2.5 on the three-point scale, reflecting the fact that they both tried but did not completely succeeded in maintaining the cognitive demand of their intended tasks when enacting their lessons. The *Student Engagement* dimension was also about 2.5 for both Bill and Dylan. We emphasize that in our setting it is not possible to draw direct connections between teacher actions and student engagement. Students' active participation, or the lack of thereof, could be in response to the change in their learning routine, by having a standard mathematics lesson replaced by one taught by a PST. Nevertheless, we include the student engagement dimension in the analysis to show the feasibility of having secondary students participate meaningfully in lessons that integrate conditional statements with the ongoing mathematical topics.

Discussion

Our goal in this paper was to trace how PSTs who participate in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* developed and implemented, in real classrooms, lessons that integrate aspects of conditional statements with the regular mathematics curriculum. Our analysis was grounded in Stein et al. (2007) curriculum framework. Overall, we were impressed with the fact that 11 out of 15 PST-developed lesson plans rated high on the prevalence of the conditional statements proof theme in them. This is a non-trivial outcome, especially given the limited access to pre-existing proof-oriented tasks in traditional US mathematics textbooks (i.e., the written curriculum). The majority of our PSTs were able to overcome this limitation, with the appropriate instructional support, and use knowledge and skills acquired in the capstone course creatively to develop lesson plans (i.e., the intended curriculum) that integrate logical aspects of proof with a variety of standard mathematical topics.

Despite many of the lesson plans having high focus on the conditional statements proof theme, the actual enactment of the lesson was often challenging for PSTs, as Bill's and Dylan's lessons illustrate. The main difficulties observed were adjusting the language to the student audience and clearly explicating the proof theme. These difficulties can be due to the fact that the PSTs did not know the students prior to the lesson, which impeded their ability to anticipate how students would respond to the conditional statements content. We addressed this issue in the subsequent iteration of the course, by including a requirement that PSTs provide in their lesson plans a list of mathematical-logical concepts they plan to use during the lesson and write a verbatim description of how they intend to introduce these concepts to students. The intention is to have the PSTs play out these aspects of the lesson plans more explicitly, prior to their enactment, so that their lessons are more likely to match the intended curriculum.

Based on the structure of the course, most PSTs taught a different group of students each time; this lack of continuity impedes our ability to make claims about student learning across time. However, our analysis showed relatively high levels of student engagement with proving during the PSTs' enacted lessons. Although we cannot attribute this completely to the PSTs' pedagogical actions, we assume that if the content of the lessons was completely outside students' interest or conceptual reach, we would be seeing much lower levels of student participation.

The challenges encountered by the PSTs in our study can be partially explained by PSTs' lack of teaching experience. However, we assert that teaching conditional statements, or proof themes, in general, is inherently challenging. Identifying specific areas of challenge for PSTs can help us, as mathematics teacher educators, to develop support structures that promote PSTs' competence in enacting reasoning and proof in their future classrooms. Some of these support structures were tested in our course design. Through repeated cycles of planning lessons that integrate proof themes within regular school curriculum, enacting these lessons in classrooms, and reflecting on them, the PSTs gained valuable experiences and developed a sense of feasibility of engaging students in proving.

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