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Preservice teachers learning to teach proof through classroom implementation: Successes and challenges

Orly Buchbinder*, Sharon McCrone

Department of Mathematics and Statistics, University of New Hampshire, United States

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ABSTRACT

Proof and reasoning are central to learning mathematics with understanding. Yet proof is seen as challenging to teach and to learn. In a capstone course for preservice teachers, we developed instructional modules that guided prospective secondary mathematics teachers (PSTs) through a cycle of learning about the logical aspects of proof, then planning and implementing lessons in secondary classrooms that integrate these aspects with traditional mathematics curriculum in the United States. In this paper we highlight our framework on mathematical knowledge for teaching proof and focus on some of the logical aspects of proof that are seen as particularly challenging (four proof themes). We analyze 60 lesson plans, video recordings of a subset of 13 enacted lessons, and the PSTs' seif- reported data to shed light on how the PSTs planned and enacted lessons that integrate these proof themes. The results provide insights into successes and challenges the PSTs encountered in this process and illustrate potential pathways for preparing PSTs to enact reasoning and proof in secondary classrooms. We also highlight the design principles for supporting the development of PSTs' mathematical knowledge for teaching proof.

1. Introduction

Educators and researchers agree that reasoning and proof are essential to learning mathematics in secondary school. In support of this claim, researchers have portrayed reasoning and proof as key mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996; Hanna & deVillers, 2012; Schoenfeld, 2009). Similarly, the importance of proof is reflected in policy documents such as the *Common Core State Standards* (CCSS) (National Governors' Association Center for Best Practices and Council of Chief State School Officers, NGA & CCSSO, 2010) and National Council of Teachers of Mathematics (2000, 2009). Nevertheless, in reality, the place of proof in mathematics classrooms is far from that vision. Studies have shown that proof is a subject that is "hard to learn and hard to teach" (Stylianides, Stylianides, & Weber, 2017). Thus, teachers tend to focus on procedural skills that are easier to teach and assess (Harel & Rabin, 2010; Kotelawala, 2016). Even when proof and proving are the focus in a mathematics lesson, students often attempt to follow what was shown in dass or in a textbook, and struggle with understanding the logical aspects of proof (Martin, McCrone, Bower, & Dindyal, 2005).

Teachers are instrumental to changing the culture of mathematics classrooms (Association of Mathematics Teacher Educators, 2017; Nardi & Knuth, 2017), however, teachers themselves often Jack the mathematical knowledge for teaching proof and hence are not disposed to support students to learn proving (Ko, 2010). In addition, teachers might not have a strong sense of what proof can look like in secondary mathematics outside of the traditional geometry curriculum.

To address the need to better prepare teachers for the challenges related to teaching proof and reasoning, we developed and

* Corresponding author at: University of New Hampshire, 33 Academic Way, Kingsbury Building, Durham, NH 03824, United States.
E-mail address: orly.buchbinder@unh.edu (0. Buchbinder).

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systematically studied a capstone course *Mathematical Reasoning and Proving for Secondary Teachers,* for seniors in the mathematics education program. In line with a practice-based teacher education paradigm (e.g., Grossman, Hammerness, & McDonald, 2009; Lampert, 2010) and recommendations for developing content-specific methods courses (Steele, Hillen, & Smith, 2013), our course contains a blend of content, methods and clinical practice. The course design and activities were aimed at enhancing preservice secondary teachers' (PSTs') knowledge and dispositions for integrating reasoning and proving into their classroom practices, at any grade level and in any mathematical topic. Specifically, we designed the course activities to advance PSTs' awareness of the importance of teaching students the logical aspects of proof and to equip PSTs with tools and skills for integrating proof-related tasks within regular secondary school curricula. The logical aspects of proof chosen as the foci of the course were: (1) quantification and the role of examples in proving, (2) conditional statements, (3) direct proof and argument evaluation and (4) indirect reasoning (hereafter, we refer to these areas as *proof themes* and describe them in more depth below). In the practical component of the course, the PSTs developed and taught in local schools lessons that intended to integrate one of four proof themes with the ongoing mathematical topic.

Thus, the goals of the course were to advance PSTs' Mathematical Knowledge for Teaching Proof (MKT-P), as we describe in the theoretical framework below. The overarching goals of our research project were to examine how PSTs' knowledge developed as a result of PSTs' participation in this specially designed course. In this paper, we focus on one particular manifestation of MKT-P - an ability to design and enact lessons that integrate the four proof themes with topics from traditional secondary curricula.

We begin this paper by explaining what we mean by proof in the context of secondary school and justify our choice of the four proof themes. We present our theoretical framework on Mathematical Knowledge for Teaching Proof (MKT-P) and explain how it guided the design of the course, specifically the lesson planning component. In the results section we present the analysis of the lesson plans at !arge, and then more focused analysis of enactment of lessons on one proof theme, Conditional Statements (CS). In this, we aim to demonstrate that our course was successful in supporting PSTs in developing and teaching proof-oriented lessons.

Teaching proof is a complex practice since most PSTs did not experience this type of teaching as school students themselves. Hence, we also examined the PSTs' challenges, associated with both planning and implementing proof-oriented lessons.

2. Theoretical framework

2.1. Proof and proving at the secondary Level: the chaUenge

While multiple definitions of proof exist (Balacheff, 2002; Reid & Knipping, 2010), we adopt the definition of proof by Stylianides and Stylianides (2017) as "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (p. 212). We chose this definition because it reflects disciplinary practices and is appropriate for the secondary school context. lt is not tied to any particular form, such as two column proofs in geometry or a formal algebraic proof, rather the emphasis is put on mathematical soundness of an argument and its accessibility to the community of learners. A related phrase, *reasoning and proving* refers to a wide range of cognitive and social processes, such as exploring, generalizing, conjecturing, making and evaluating arguments on the basis of mathematical deductions (Ellis, Bieda, & Knuth, 2012; Stylianides, 2008). We make reference to these processes as weil, as they arise in our work with the PSTs and our analysis of their lessons.

The key words in these definitions that distinguish proof from other types of arguments, are "mathematically sound" and "mathematical deductions." But what does it mean to produce a mathematically sound argument or a "viable argument" as the *Common Core State Standards* (NGA & CCSO, 2010, p.6) refers to it, especially ifwe do not want to limit students to formal arguments and two-column proofs? On the one hand, deductive arguments are often contrasted in the literature with arguments based on authority of a teacher or a textbook and with arguments based on empirical evidence or on perception (Harel & Sowder, 2007; Healy & Hoyles, 2000). While this clarifies what types of arguments do not constitute proof, it remains unclear what kinds of arguments do constitute proof. Harel and Sowder help to shed light on this subject in their definition of *deductive proof scheme:* a mode of reasoning characterized by intentional application of logical inference to produce a general argument, an argument that establishes truth of a statement for all cases. The authors further suggest that the goal of mathematics education is to help students advance from argumentation based on empirical evidence and authority towards deductive argumentation. Stylianides et al. (2017) explain that the types of arguments that constitute proof in the context of school mathematics are characterized by the use of "statements, valid forms of reasoning, and appropriate forms of expression in the sense of commonly accepted mathematical knowledge" (p. 239). This again brings up a dilemma of balancing the teacher's desire for students to produce arguments based on logic and deductive reasoning with the need to stay within the conceptual reach of secondary students. This dilemma is exacerbated by the lack of research-based suggestions for how to teach logical reasoning at the secondary level (for an exception, see Yopp, 2017).

In addition, studies have identified many challenges that students and even prospective teachers experience with logical aspects of proof. In the next section we describe four specific areas of difficulty with proof which we chose to address in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers.* We also describe the instructional approach for engaging secondary students in proof and proving, which we promoted in the course by having the PSTs develop lesson plans that integrate logical aspects of proof within regular mathematical curricula.

2.2. The Jour proof themes

Research on students' conceptions of proof conducted in recent decades around the world identified several areas of deductive reasoning that are crucial for proof production and evaluation (Stylianides et al., 2017). However, these areas constitute persistent difficulties for students at all levels, including college students and prospective teachers (Stylianides et al., 2017). Of these areas, which we term the proof themes, we chose four as the foci of our course: (1) quantification and the role of examples in proving, (2) conditional statements, (3) direct proof and argument evaluation and (4) indirect reasoning. We acknowledge that the four proof themes we chose are not the only areas of difficulty identified by research. For example, multiple challenges with understanding logical connectors, such as conjunction and disjunction have been described in the literature (e.g., Dawkins & Cook, 2017). Our choice of these four proof themes stems from the literature and our own experience as instructors observing PSTs' challenges in university coursework. The four proof themes are obviously interrelated, yet they are sufficiently independent to serve as a basis for the four instructional modules of the capstone course.

With respect to *quantified statements,* studies have shown that students have difficulty recognizing universally and existentially quantified statements, especially when the quantifier is implicit (e.g., the sum of an even and odd integer is odd, or a triangle can be inscribed in a circle) ortend to interpret existential quantifiers as universal (Buchbinder & Zaslavsky, 2019; Epp, 2003). Students also struggle to understand that a universally quantified statement must be proved for all elements in the domain, and fail to recognize the limitation of relying on supportive examples for proving universal statements. This phenomenon was recognized by the Education Committee of the European Mathematical Society (2011) as one of the "solid finding[s]" produced by the international research community through accumulated research evidence. A related concern is students' difficulty to recognize the different roles that examples and counterexamples play in proving or disproving universal and existential statements (Buchbinder & Zaslavsky, 2019; McCrone & Martin, 2004; Zazkis, Liljedahl, & Chernoff, 2008).

The challenges associated with *conditional statements* include difficulty recognizing their logical structure, understanding their generality and determining their truth-value (e.g., Durand-Guerrier, 2003; Epp, 2003; Seiden & Seiden, 1995). Several studies have shown that students tend to wrongly assume that a conditional statement is equivalent to its converse or inverse, while at the same time having difficulty accepting that it is equivalent to a contrapositive (Hoyles & Küchemann, 2002). When producing *directproofs and/or evaluating proofs,* one of the prevalent student mistakes is assuming the conclusion and deducing the antecedent, which essentially proves the converse of the given statement (Inglis & Alcock, 2012). Yet students are usually unaware of these logical fallacies and therefore struggle to distinguish between valid and invalid arguments (Alcock & Weber, 2005; Seiden & Seiden, 2003). These challenges are also associated with students' difficulties in forming negations of conditional statements, which is critical when using *indirect reasoning,* such as proof by contradiction and by contrapositive (Antonini & Mariotti, 2008; Lin, Lee, & Wu Yu, 2003).

Our research stems from the assertion that these proof themes can be integrated into secondary school mathematics in intellectually honest ways that preserve the integrity of mathematics as a discipline while honoring students as learners (Bruner, 1960; Stylianides, 2008). The instructional approach that we envision has a teacher choosing some subset of formal logical aspects of deductive reasoning (e.g., a contrapositive, an existential statement), and developing an instructional task for students in which these logical elements are integrated within the ongoing mathematical topic of a regular school curriculum. Such integration requires adjustment of language and imagery to the conceptual level of the students.

Engaging students in such tasks conveys an important message that proof is not a unit or a topic and it is not confined to high school geometry; rather, proof and logical reasoning are ways of thinking and understanding mathematics applicable to all mathematical subjects. Moreover, following Harel's (2008) notion of repeated reasoning, we maintain that repeated exposure to logical aspects of the four proof themes in multiple mathematical topics is likely to have cumulative effects on students' understanding of deductive reasoning. Over time these ideas could crystalize into knowledge that can support students' production of formal proofs, although we are not aware of empirical studies that support this assertion.

2.3. Mathematical knowledge for teaching proof

Examining connections between mathematical knowledge for teaching (MKT) and teachers' knowledge of reasoning and proof led many researchers to conjecture that there is a special type of teacher knowledge: Mathematical Knowledge for Teaching Proof (MKT-P) (Corleis, Schwarz, Kaiser, & Leung, 2008; Lesseig, 2016; Steele & Rogers, 2012; Stylianides, 2011). Our analysis of the various frameworks revealed that there are certain similarities and differences in terms of how MKT-P is conceptualized and measured. Our own conceptualization of MKT-P draws on the existing literature in term of identifying aspects of teacher knowledge specific to proof. We also draw on Steele and Rogers' notion that MKT-P is reflected in both the declarative knowledge and teacher's classroom practices, although we found it difficult to use their MKT-P categories in our study. Our resulting MKT-P framework is presented in Table 1.

Similar to most scholars (e.g. Lesseig, 2016; Stylianides, 2011) we consider MKT-P as comprised of subject matter knowledge and pedagogical content knowledge. Our categories of pedagogical content knowledge align closely with that of others. Similar to Stylianides and Steele and Rogers (2012), but contrary to Lesseig, we consider subject matter knowledge broadly, without necessarily distinguishing between common and specialized content knowledge. We focus on aspects of subject matter knowledge directly related to proof and term them Knowledge of the Logical Aspects of Proof (KLAP). It includes knowledge of various types of proof (e.g., proof by cases, direct proof, proof by contradiction or by contrapositive, disprove by a counterexample), valid and invalid modes of reasoning, logical relations (e.g., negation, conjunction, biconditional), knowledge of various definitions (e.g., range of definitions of a trapezoid) and knowledge of multiple methods of proof (e.g., various proofs of the Pythagorean theorem). This type of knowledge is manifested in teachers' own ability to construct viable proofs, evaluate validity and generality of arguments, identify invalid arguments and the sources of flaws in these arguments. In classrooms, KLAP can be visible in a teacher's ability to use precise mathematical language, evaluate logical correctness of students' arguments and facilitate discussion that make logical aspects of

Table 1
Mathematical Knowledge for Teaching Proof.

Type of MKT-P	KLAP: Knowledge of the LogicaJ Aspects of Proof	KCS-P: Knowledge of Content and Students	KCT-P: Knowledge of Content and Teaching
Description	Knowledge of different types of proofs, valid and invalid modes of reasoning, the roles of examples in proving, logical relations and range of definitions and theorems	Knowledge of students' proof-related conceptions, rnisconceptions and common mistakes	Knowledge of pedagogical practices for supporting students engagement with proof
Related classroom practices	Use precise mathematical language and notation within students' conceptual reach	ldentify / anticipate common misconceptions about R&P[a] in students' utterances or written work	ldentify curriculum opportunities for R&P in diverse mathematical contents
	Identify and correct students' logical mistakes or inaccurate language, and support students' use of correct logical reasoning and language	Facilitate discussions to address common misconceptions about R&P	Design tasks that embed rich opportunities for R&P and make logical aspects of proof explicit
		Make proof concepts explicit and accessible to students' conceptual level	Use productive instructional moves to utilize learning potential of R&P tasks

Note (a): R&P stands for reasoning and proof.

proof explicit.

The pedagogical content knowledge for teaching proof (PCK-P) comprise knowledge of content and students (KCS-P) and knowledge of content and teaching (KCT-P). The KCS-P component includes knowledge of students' common conceptions and misconceptions of proof, such as a tendency to rely on inductive reasoning or on unstated assumptions. The KCT-P component includes knowledge of pedagogical strategies for supporting students' engagement with proof, and designing and implementing tasks that help to advance students' understanding of proof. Table 1 shows the three types of MKT-P and the related classroom practices.

lt is important to note that the three types of MKT-P are interrelated, especially in their classroom manifestations. For instance, designing tasks that address common student misconceptions of proof requires both knowledge of these misconceptions (KCS-P) and the types of tasks for addressing these misconceptions (KCT-P). Similarly, enacting proof-related tasks requires knowledge of productive instructional strategies (KCT-P), while facilitating discussions that make logical aspects of proof explicit requires knowledge of students' conceptions of proof (KCS-P) and being able to interpret and correct students' mathematical reasoning (KLAP). In the context of expert teaching, the three types of MKT-P would be seamlessly integrated, making it difficult, and artificial, to separate the knowledge domains. However, the usefulness of these distinctions becomes more apparent in the context of supporting novices, whether PSTs or practicing teachers, and in the context of assessing classroom enactrnent of proof-related instruction.

In addition to having strong MKT-P, planning and enacting mathematics instruction that emphasizes reasoning and proving requires teachers to hold productive dispositions towards proof. This includes recognizing the value of proof for teaching and learning mathematics, viewing students as capable of doing proofs and of teachers themselves as capable of teaching it.

The design of the course *Mathemati.cal Reasoning and Proving for Secondary Teachers* was grounded in the MKT-P framework described above to provide the PSTs with opportunities to enhance all aspects of their MKT-P and to promote productive dispositions towards teaching and learning proof.

3. The setting

3.1. Applying the theoreti.cal frameworks to the course design

The capstone course comprised four modules corresponding to the four proof themes: quantified statements, conditional statements, direct proof and indirect reasoning. Each module lasted three weeks and contained activities aimed to enhance various aspects of PSTs' MKT-P. Given the extensive mathematical coursework required for mathematics education majors at our university, which includes specialized coursework in secondary content, we oriented our main efforts towards enhancing PSTs' knowledge of the logical aspects of proof related to the four proof themes, and the two types of PCK-P. Fig. 1 shows the structure of the course (top row) and

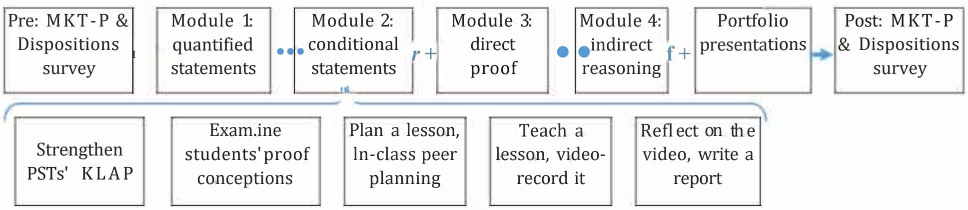


Fig. 1. Design of the course *Mathematical Reasoning and Proving for Secondary Teachers* (adapted from Buchbinder & McCrone, 2018).

the general structure of one course module (bottom row).

As shown in Fig. 1, each module began with activities aimed to refresh and strengthen PSTs' own knowledge of the logical aspects related to a particular proof theme. For instance, in the CS module, we had PSTs sort eleven logical forms of conditional statements into true / false, as weil as equivalent / not equivalent to a given conditional statement, and then reflect on the concepts of truth-value versus logical equivalence (Buchbinder & McCrone, 2018).

Next, PSTs learned about students' conceptions of proof, specific to that proof theme, such as student tendency to rely on empirical evidence, misunderstanding the role of counterexamples, or difficulties with indirect reasoning. We used research literature and our prior studies to create scenarios, cases and sample student work to illustrate these conceptions and difficulties for each of the four proof themes (e.g., Buchbinder et al., 2017; Buchbinder, 2018; Thompson, 1996). Learning experiences induded analyzing and interpreting sample student mathematical arguments and envisioning responding to student thinking. In addition, the activities induded sharing strategies for supporting student thinking, and addressing student difficulties in the ways that would help students develop conceptions of proof that are in line with conventional mathematics.

An important aspect of MKT-P is designing or modifying mathematical tasks to create opportunities to engage students in reasoning and proving; specifically about the four proof themes. While teachers are expected to engage students with constructing and evaluating arguments and proofs (NGA & CCSO, 2010), textbooks offer limited proof-oriented instructional activities (Otten, Gilbertson, Males, & Clark, 2014; Sears & Chavez, 2014; Thompson, Senk, & Johnson, 2012). Thus, much of the work to design and implement tasks that promote students' understanding of proof is left to teachers, who need special pedagogical content knowledge to carry out this work in dassrooms. Thus, one of the core activities for PSTs in the course was to design and implement, in local schools, lessons that integrate a certain proof theme with the ongoing mathematical topic from the secondary school curriculum.

To support PSTs in this task, we built into the course activities that (a) engaged PSTs in identifying opportunities to integrate proof-related tasks within traditional US school curricula, (b) provided dass time for lesson planning that induded peer and instructor feedback on early drafts of the lesson plans, and (c) modeled types of proof tasks that can be enacted in secondary dassrooms. We describe these supports below.

3.2. Instructi.onal support for lesson planning

Prior to planning the lessons, the PSTs communicated with the cooperating teachers[1] to whose dassroom they were assigned to inquire about students' prior knowledge and the mathematical topic they were supposed to teach. Same teachers had additional requests for the PSTs, such as devoting some portion of the dass time to review for an upcoming exam. Based on this information the PSTs developed lesson plans integrating a particular proof theme. We also encouraged (but did not require) the PSTs to visit their assigned dassrooms to get a better sense of the physical layout of the dassroom and student participation norms.

When designing their lesson plans, the PSTs were encouraged to rely on multiple resources, induding textbooks and web resources. However, traditional textbooks and web resources were of limited help in terms of proof theme integration. For example, although certain tasks on conditional statements can be found on the web, there is no guarantee that they would also match the mathematical topic requested by the cooperating teacher. Thus, adapting these resources required ingenuity and creativity on behalf of the PSTs.

A range of proof tasks was modeled in the course. We encouraged the PSTs to modify some of these tasks for use in their lessons, such as *True-or-False, Always-Someti.mes-Never, Who is right?* and *Is it a coincidence?* (Buchbinder & Zaslavsky, 2013). These types of tasks have the potential to elicit rich student engagement with the logical aspects of proof, and can be adapted for various mathematical topics, while maintaining their original structure and goals. These features of the tasks were made explicit to PSTs and discussed during the course.

In addition, to support the PSTs' lesson planning process, prior to each lesson, we devoted a full two-hour dassroom session for PSTs to share their draft lesson plans with each other, give and receive feedback from their peers and the course instructors (the authors of this paper). The course instructors circulated the room spending about 10 minutes with each PST individually, discussing their plan, answering questions and suggesting resources. The instructors did not provide templates or strict guidelines for the PSTs to follow. The decisions on how to integrate a proof theme in the lesson, what pedagogical features to use and how to engage students was left to the PSTs. We intentionally avoided influencing the PSTs' decisions both to preserve the integrity of the data and to promote the PSTs' pedagogical autonomy. At the same time, we encouraged the PSTs to flesh out, as much as possible, the details of their lesson plans.

Our goal was that, by the end of the semester, the PSTs would build a repertoire of pedagogical tools for engaging students in proving and gain practical experience in enacting proof-oriented activities in dass.

4. Objectives and research questions

Teaching mathematics by integrating reasoning and proof is a complex skill, which evolves over time and is affected by multiple experiences of the PSTs. Understanding how MKT-P evolves requires combining evidence from multiple sources, induding but not

[1] The cooperating teachers volunteered to work with the PSTs in their classrooms. We did not collect any information about proof-related classroom practices of the cooperating teachers prior to or during the study, except that college-preparatory geometry curriculum includes a chapter on "proof".

limited to MKT-P and Dispositions for Proof questionnaires (Fig. 1), course participation data, lesson plans and enactments. This is the ultimate goal of our project but it is beyond the scope of this paper.

In this paper, we focus on the manifestations of MKT-P reflected in classroom practices (Table 1). Specifically, we focus on the PST's knowledge of *designing* and *enacting* lessons that integrate the four proof themes in traditional mathematical curricula. Since the goal of our course was to support PSTs' development of this type of knowledge, we also seek to examine what challenges PSTs encountered in the process. We examine the following research questions:

1 How did PSTs integrate the four proof themes with ongoing mathematical topics from the secondary curriculum in their lesson plans?
2 How did PSTs implement lessons on conditional statements in secondary classrooms?
3 What challenges did PSTs encounter when designing lesson plans on the four proof themes, and when implementing lessons on conditional statements in secondary classrooms?

We turn now to describing the study methods and results. We also address the mechanisms built into the course structure to advance the PSTs' MKT-P.

5. Methods

5.1. Participants

Fifteen PSTs took part in the capstone course (six males and nine females); of them 11 PSTs were in the high school track and four were in the middle school track. All PSTs were in their senior year of the program, meaning that they had completed the majority of their mathematical coursework, which included courses that introduced them to mathematical proving. In addition, the PSTs had completed two educational courses, one on general issues in mathematics education, common across grade levels, and one course specific to methods for teaching secondary mathematics. In these educational courses, the PSTs had opportunities to observe mathematics classrooms and to practice lesson planning. However, none of these courses included a practicum component; nor were the PSTs involved in any additional practicum setting concurrently with our course, due to the structure of our teacher preparation program. In fact, this was the first structured opportunity for the PSTs to teach students in a classroom setting.

5.2. Data sources and analytic methods

The data for this paper came from two primary sources: the PSTs' lesson plans and the video recordings of their enacted lessons. As PSTs taught their lessons, they videotaped themse]ves using 360° video cameras to capture both teacher actions as weil as student work and speech. PSTs then watched the video and wrote a report that invited them to reflect on their own practice, an activity that is known tobe beneficial to professional growth of teachers (Santagata & Guarino, 2011). These reflections, as weil as the overall course reflection constitute supplementary data sources for the study.

5.2.1. Analysis of the lesson plans

A total of 60 lesson plans (15 plans on each of the four proof themes) were analyzed to answer research question 1: *How* did *PSTs integrate the Jour proof themes with ongoing mathematical topics from the secondary curriculum in their lesson plans?*

PSTs were required to include in their lesson plans general information about the lesson (e.g., grade level, subject area, lesson topic), lesson objectives, and student prior knowledge, estimated based on their communication with the cooperating teachers. This information was not analyzed separately, but was used to gain a background on the lesson plan and the PSTs' goals. The main portion of the lesson plan contained the outline of the lesson activities, tasks or worksheets for the students with complete solutions, description of how the students would be engaged, any anticipated difficulties and potential ways to assist students. Due to practical arrangement with the cooperating schools, all lessons were planned for a 50-minute timeframe but intended for small groups of four to eight students. Tue latter detail affected the PSTs' planning process, so, for example, the PSTs planned for close interaction with the students.

Analysis of the PSTs' lesson plans was inspired by the Silver, Mesa, Morris, Star, and Benken (2009) framework, and proceeded in several stages. First, in each lesson plan, we mapped out the grade level, mathematical content and *pedagogical features,* which are pedagogical strategies intended to support students' developing mathematical understanding. Silver and colleagues identified five types of pedagogical features: context outside mathematics, hands-on materials, multi-person collaboration, technology, and tasks that necessitate student-generated explanations. However, this !ist of categories could not be applied to our analysis, since some of them, such as multi-person collaboration, were present in all lesson plans in our data set, while others, e.g., technology, were not present at all. On the other hand, some features, such as concept maps and logic riddles seemed tobe unique to our data set. Thus, we Jet the categories of pedagogical features emerge naturally from the data, as features of the types of tasks that the PSTs used in their lessons.

Since pedagogical features describe the types of tasks designed by the PSTs to promote students' understanding of the four proof themes, worksheets with exercises that merely required students to recall vocabulary or practice standard procedures were not considered among pedagogical features. Also, in some cases, a particular pedagogical feature was a major component of a lesson, e.g., a lesson built around President Garfield's proof of the Pythagorean theorem, while in other cases a pedagogical feature was a relatively minor component of a lesson, e.g., introducing conditional statements using non-mathematical content of commercial ads. We did not assign weight to these pedagogical features, but rather sought to document the variety and frequency of the use of these features. If a certain feature appeared multiple times in the lesson, it was only counted once, as representing one feature type (e.g., three logical riddles in Rebecca's lesson were counted as one pedagogical feature "logical riddles"). Thus, some lesson plans contained no pedagogical features, while others had up to three distinct features. In total, we identified 70 pedagogical features across all lesson plans.

The second step of the analysis examined the extent to which a lesson plan was focused on the particular proof theme for which it was developed. The unit of analysis was the whole lesson plan, since a proof theme could be addressed in multiple parts of the lesson e.g., exposition, warm-up, some or all student tasks. Although all lesson plans intended to integrate some proof theme, the plans varied in the extent to which they addressed the proof theme. To capture this variation, each lesson, as a whole, was assigned a rating: high, medium or low, according to the richness of proof-related aspects specific to the proof-theme that were addressed in the lesson.

Lesson plans coded as having *low* focus on the proof theme only minimally addressed a particular proof theme, if at all, or the activity did not require students to attend to the proof theme. For example, Ellen's (all names are pseudonyms) lesson in geometry contained two two-column proofs about vertical angles and parallel lines, but nothing related to the intended proof theme: quantification and the role of examples in proving. Such a lesson was coded as having low focus on the proof theme. Another PST, Chuck, designed an eighth grade lesson on conditional statements and exponents. The lesson plan included three true-or-false questions such as: "If a negative number is raised to an even power, the result will be a positive number." This question is embedded with great potential to unpack the structure of a conditional statement, discuss how one can prove or disprove it, and use the rules of exponents to create a generic proof which is within the conceptual reach of eighth grade students. However, Chuck's answer sheet suggested that he expected students to produce "a proof by example": $(-2)^2 = (-2)(-2) = 4$, missing the opportunity to address limitations of using empirical evidence as proof and even potentially enforcing this misconception.

Lesson plans that were coded as having *high* focus on a proof theme, contained mathematical tasks that exhibited a clear integration of a proof theme, an explanation of how the proof theme would be introduced and explained to students; also, the time allocated to the proof theme comprised a substantial portion of the lesson. An example of such a lesson plan was Rebecca's lesson on quadrilaterals with the proof theme: quantification and the role of examples in proving. The lesson began with an exposition on universal statements conveyed through a non-mathematical example: "A man who is wearing a suit and tie is attending a funeral." Rebecca explained that universal statements require a general proof but can be disproved by a single counterexample. As a main lesson activity, Rebecca had students explore and develop a conjecture about midpoint quadrilaterals. Although students did not have to prove their conjectures in full, they were asked to explain what would be needed to prove or disprove them.

Lessons coded as having *medium* focus on the proof theme, could be described as lying somewhere in the middle of a low-high continuum. Such lessons may have contained rich mathematical activities requiring students to reason and construct arguments, but the specific proof theme was underspecified. An example of one such lesson plan is Nate's lesson plan that purported to integrate quantification and the role of examples in proving with an Algebra 1 topic on unit conversion. Nate designed a real-world problem about two investors buying land in the US and Europe and four faux student arguments claiming that one or the other investor got a better deal. The task for students was to solve the problem and to evaluate the four arguments. Nate wrote that he expected students to use their own solutions as counterexamples to the claims made by the imaginary students. Although it might be possible to use Nate's problem in this way, we did not find sufficient evidence in the lesson plan that the problem would in fact be used to explicate ideas related to quantification and the role of examples in proving. Hence, we coded it as having medium focus on the proof theme.

When coding the lesson plans, the two researchers, the authors of this paper, coded each lesson separately, and then met to discuss and resolve any discrepancies.

5.2.2. Analysis of enacted lessons

To explore research question 2: *How did PSTs implement lessons on conditional statements in secondary classrooms?* we analyzed the 360'-video recordings of these lessons. In this paper, we chose to report on the CS proof theme enacted lessons, because it had the largest number of lesson plans with high proof theme focus (see Table 3 in the results section). Since these were the lesson plans with the highest potential to engage students with proving, we were curious to know how such lessons were implemented in the classroom.

The coding of the video proceeded as following. Each video was divided into thematic episodes, no langer than 5 minutes, resulting in a total of 168 episodes, averaging 13 episodes per lesson. Then, we scored each episode using the Lesson Enactrnent rubric (see Appendix A), which we developed for the purpose of this study (Buchbinder & McCrone, 2019). The Lesson Enactrnent rubric contained four dimensions, three related to teacher actions: Accuracy of Language, Explicating the Proof Theme, and Actions to Promote Student Engagement (with proof); and one dimension related to Student Engagement. The Accuracy ofLanguage category was further partitioned into language specifically related to the CS proof theme, and general mathematical language. Similarly, the category of Student Engagement was partitioned to capture student engagement with the proof theme versus student engagement with non-proof related aspects of the lesson.

The categories of the Lesson Enactrnent rubric correspond to the MKT-P framework (Table 1) in the following ways. The category of Accuracy of Language allows for capturing PSTs' KLAP, as expressed in their use of precise mathematical language and notation. The category of Explicating the Proof Theme describes the extent to which the key ideas about proof are made explicit and conceptually accessible to students at a particular grade level. Thus, this category corresponds to KCS-P in the framework. Actions to Promote Student Engagement with proof describe general pedagogical moves, such as re-voicing and pressing for explanations. When enacted in the context of proof-oriented tasks with the specific aim of engaging students in proving, we consider these actions as manifestations of KCT-P. As opposed to PSTs' actions related to general classroom management, not specific to proof, which were not included in this category.

Each video episode was assigned a score of 3 (high), 2 (medium), 1 (low) or not applicable, an each of the dimensions of the rubric (see Appendix A); then we averaged the scores across all episodes for a particular lesson, along a particular dimension of the rubric. In the results section we illustrate these dimensions as weil as the scoring system.

Ta calibrate the coding scheme, the research team consisting of the two authors and two graduate students collaboratively coded the video of two full lessons. Next, the two graduate students coded the rest of the videos, and weekly group meetings were held to discuss and reconcile episodes that were difficult to code. Then, 25 % of the episodes were assigned to a different coder for validation. The discrepancies between coders were minimal, and those instances in which discrepancies occurred were discussed as a group and resolved.

Based an the coded and scored episodes, we calculated the number of minutes in each lesson devoted to the CS proof theme by considering the episodes in which the PST explicitly discussed conditional statements, and the episodes in which students were working an tasks specific to the proof theme. These data were compared to the coding of the lesson plans to compare the enacted lesson to the planned one.

5.2.3. Analysis of PSTs' challenges

The data for answering research question 3: *What challenges did PSTs encounter when designing lesson plans on the Jour proof themes, and when implementing lessons on conditional statements in secondary classrooms?* came from three sources: the PSTs' written comments in the reflection reports following each lesson, the PSTs' summative course reflection, and the analysis of video recordings of the CS lessons. The PSTs' self-reported data allowed for examining the lesson design and implementation challenges from the PSTs' perspectives. Using open coding (Wiersma & Jurs, 2005) and the constant comparative method (Strauss & Corbin, 1994) we identified and organized by themes those challenges that PSTs self-identified in their written reflections, especially the summative reflection of the course. The identified categories include challenges associated with planning lessons that integrate the four proof themes.

In addition, the analysis of video data described above provided insights into specific PSTs' challenges related to lesson enactment. These challenges were identified by examining the episodes that scored low an some of the dimensions in the Lesson Enactment rubric. For example, although PSTs did not specifically state accuracy of language as a challenge in proof-lesson enactment, the analysis of the video data revealed that PSTs indeed often struggled to maintain accurate mathematical language specific to proof or to use proof specific mathematical vocabulary, as we will show in the results section. We turn now to describing the results of our study.

6. Results

6.1. Lesson plans integrating reasoning and proving

6.1.1. Mathematical topics and pedagogicalfeatures of the lesson plans

Throughout the course each PST developed four lesson plans that integrated one of the four proof themes in a variety of subject areas across multiple topics. Mathematical content of these lesson plans is summarized in Table 2.

Table 2 shows that the PSTs integrated the four proof themes in a wide variety of topics, requested by classroom teachers including topics outside geometry and at the eighth-grade level. In addition, the analysis of pedagogical features sheds light an the types of pedagogical strategies the PSTs used in their lesson plans to engage students in reasoning (i.e., types of proof tasks).

The analysis of pedagogical features of the lesson plans generated a !ist of 70 features across 60 lessons: Nine lesson plans contained no pedagogical features, 34 lessons had a single feature, 15 lesson plans had two features, and two lesson plans contained as many as three pedagogical features. Overall, the PSTs implemented a wide variety of pedagogical features. Fig. 2 shows frequencies of categories of pedagogical features which appeared in four or more lesson plans. The features that appeared in three lesson plans were use of manipulatives, such as algebra tiles, as weil as exploring and conjecturing. Two lesson plans asked students to work with concept maps and one lesson plan contained logic riddles.

Fig. 2 shows the three most prevalent categories of pedagogical features in the lesson plans were: analyzing work or arguments of others, problems embedded in contexts outside mathematics, and games. The most common pedagogical feature was analyzing work

Table 2
Mathematical content and number of lesson plans by subject matter areas.

Eighth Grade Mathematics Pre-Algebra (high-school) • 22 lessons	Algebra 1 College-Preparatory Algebra 1 18 lessons	Geometry College-Preparatory Geometry 20 lessons
• Rules of exponents • Scientific notation • Order of operations • Problem solving • Variable expressions • Distributive property	• Proportions • Order of operations • Combining "like" terms • Solving equations • Linear functions/graphs	• Quadrilaterals • Parallel lines • Vertical angles • Line and angle proofs • Pythagorean theorem • Simplifying square roots

Note: (a) High-school pre-algebra is considered a remedial course since its content is traditionally taught in middle school.

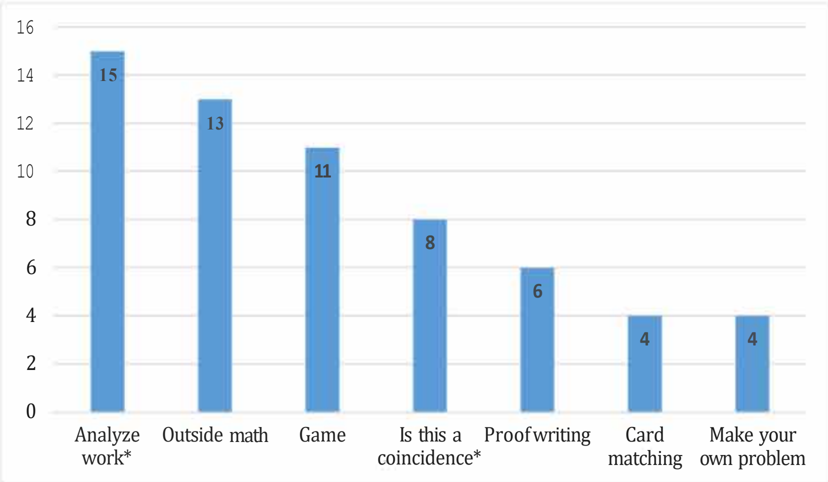


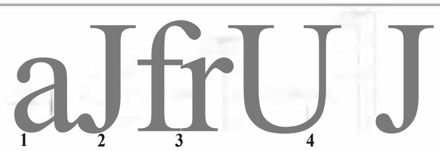
Fig. 2. Number of lesson plans containing each pedagogical feature, when used in four or more lesson plans. Features marked with (*) are the ones modeled in the course.

or arguments of others; that is, the PSTs designed mathematical problems and a set of faux solutions to it, and asked school students to analyze and critique these solutions. This type of task was modeled in the course as "Who is right?" (Buchbinder, 2018). The advantage of this type of task is that it allows teachers to strategically plan the types of arguments and common mistakes for students to analyze. It also helps to reduce social pressure potentially associated with examining erroneous solutions of actual students in dass. Thus, it is not surprising that PSTs often used this sort of task in their lessons. An example of one such task, created by Laura for a ninth grade Algebra 1 dass to discuss ideas of direct proof and argument evaluation, is shown in Fig. 3. The task consists of a visual pattern and three fictitious students' arguments for how to mathematically describe the pattern. The task calls for determining which argument is most correct and providing a justification for the choice.

The second most frequent pedagogical feature was the use of a context outside mathematics. In this category we induded the original problems designed by the PSTs, or modified from existing resources, such as Nate's problem on unit conversion described in the methods section, or Logan's problem on Pythagorean theorem (Fig. 5). PSTs also frequently used games, such as "Jeopardy!" or "Mathematical Baseball" (a game with problems written on cards and a scoring system explained in the lesson plan) to engage students in argumentation and reasoning.

Another commonly used type of task was *Is this a coincidence?* which was modeled and discussed in one of the in-dass sessions of the course. This type of task contains a description of a mathematical exploration by an imaginary student, accompanied by one or two examples and a pattern the students observed in these examples, which implies some general rule, either correct or not. The task for students is to formulate the rule as a conjecture and prove or disprove it. This type of task allows certain important aspects of proof to surface such as the need to formulate conjectures dearly, the limitation of reliance on examples, the need for a general proof, and using counterexamples to disprove or refine a conjecture (Buchbinder & Zaslavsky, 2013). Fig. 4 shows an example of one such task designed by Angela for a high-school geometry dass.

The distribution of pedagogical features among different proof themes was relatively homogeneous across the three proof themes: quantification, conditional statements and direct proof/argument evaluation contained 18, 19 and 21 pedagogical features (respectively); the lessons on indirect reasoning had 12 pedagogical features in them. Although indirect reasoning is known in



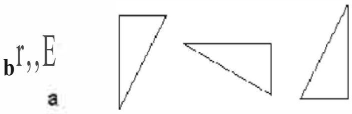
Kelly thinks that you can describe the pattem with the expression: $n(n-1)$: "Eicicn pcittern. t,cis "'blocR.S dlM)(w 1 blocR.S o'" top wnlc"1 ls wnl:J I t,cive wrltte"' t'1e expresslo"' ,,.,(,,.,-1):

Brandon thinks that the pattern must be described by the expression: $n-1+n$. "I disagree with Kelly because each figure is made of n blocks minus one block plus n more blocks, so the expression must be n-l+n blocks."

Tom believes that the pattem can only be described by the expression: $2n-1$: '13randon and KellY are both wrang. Each figure consiStS Oftwice the amount Of n blOCKS minus one blOCK. Thus, the expression is 2n-1."

Fig. 3. Laura's task on direct proof and argurnent evaluation.

A student said: I took four congruent triangles, with side lengths 3in, 4in, and 5in, and found that I could rearrange them in a square. I tried to do the same thing with four triangles of side lengths 6in, 7in, and 8in and I couldn't make a square.
Is this a coincidence?



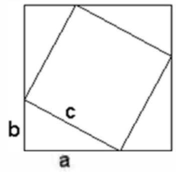


Fig. 4. Angela's task using the model *Is this a coincidence?*

You're working as an independent contractor and your latest client needs a ramp built at one oftheir properties. The client knows that the ramp must come to an elevation ofthree feet and that they only have enough room for the ramp to come out six feet from the wall. The client mentions that the length ofthe ramp's surface will be 9 feet. *Explain to the client why the length of the ramp cannot be ninefeet.* Also include what the correct third measurement is for the rarno.

Fig. 5. Logan's task on indirect reasoning. Emphasis added.

mathematics education literature to pose particular difficulties to both students and teachers (e.g., Antonini & Mariotti, 2008), we cannot draw direct connections with the lower number of pedagogical features observed in indirect reasoning lessons.

6. 1.2. *Focus on the proof themes*

The PSTs' lesson plans varied substantially with respect to the richness of proof-related aspects and the depth in which the proof themes were addressed in the plan. As described in the methods section above, each lesson plan was coded as having high, medium or low focus on the proof themes. Table 3 summarizes the results across the four proof themes.

Table 3 shows that 28 out of 60 lesson plans were coded as having high focus on the proof theme. Within those lessons, two proof themes stood out as having a !arge number of high-focused plans: conditional statements (11 out of the 28 marked as high) and direct proof/argument evaluation (10 out of 28). Nineteen out of 60 lessons were coded as having low focus on the proof themes. Of those lessons, the two proof themes which had the largest number of low-focused lesson plans are quantification and the rote of examples in proving (8 out of the 19 marked low) and indirect reasoning (8 out of 19). The majority of these lesson plans were only tangentially related to the proof themes. However, some PSTs found creative ways to incorporate these proof themes in their lessons, such as Rebecca's lesson on quantification and midpoint quadrilaterals described in the methods section. Another strong example is Logan's lesson on indirect reasoning. Logan created six problems on applications of the Pythagorean theorem, each asking students to explain why certain measures of triangle sides cannot be true (see Fig. 5 for one problem). Indirect reasoning would come into play by assuming that the ramp is 9 feet long, and using the Pythagorean theorem to calculate the length of the ramp to arrive at a contradiction.

Another way to look at the data in Table 3 is by the proof theme, each of which had 15 lesson plans. For conditional statements and direct proof/argument evaluation proof themes, 11 and 10 (out of 15) lesson plans respectively were coded as having high-focus on the proof theme. For quantification and indirect reasoning proof themes the majority of lesson plans (8 out of 15, or slightly more than half, in each theme) were coded as having low proof theme focus.

Both ways of looking at the data suggest a similar trend that some proof themes were potentially easier for the PSTs to integrale with the ongoing mathematical topics then others. We treat this observation cautiously, given that the PSTs had to balance the school mathematics schedule with the course structure which followed a particular order of proof themes (Fig. 1). That is, the lesson plans had to integrale specific proof themes at given times, while the mathematical topic of a lesson taught at that week might have been more conducive to another proof theme. For example, Ellen's lesson contained an exploration and proof of the exterior angle theorem for triangles; but almost nothing on conditional statements. Thus, although this lesson had high proof content, it was rated as having a low focus on the conditional statements proof theme. The data in Table 3 above should be interpreted in light of this complexity.

Table 3
Distribution of high-medium-low focus on proof themes in the lesson plans.

Proof eherne	Focus of the lesson of a proof eherne		
	High	Medium	Low
Quantification and ehe role of examples in proving	3	4	8
Conditional statements	11	1	3
Direct proof, argument evaluation	10	5	0
Indirect reasoning	4	3	8
Total:	28	13	19

Table 4
Data on classroom enactment of lessons on the CS proof theme.

Name ofthe PST	Focus on CS proof theme in the plan	% of time devoted to CS	Explicating CS proof theme	Language accuracy		Student engagement		PSTs' actions to promote engagement
				General	Specific to CS	General	Specific to CS	
Rebecca	**M**	0%	n/a	3.00	n/a	2.67	n/a	2.75
Cindy	L	6%	1.00	2.78	n/a	3.00	3.00	2.73
Ellen	L	21%	2.00	2.75	2.00	2.67	2.50	2.75
Audrey	**H**	40%	3.00	2.2	2.43	2.75	2.57	2.60
Ethan	**H**	46%	2.17	3.00	3.00	2.64	3.00	2.58
Laura	**H**	46%	2.67	2.75	2.50	2.50	3.00	2.67
Logan	**H**	60%	2.14	2.36	1.75	1.82	2.00	2.55
Grace	**H**	67%	2.80	3.00	3.00	3.00	3.00	2.57
Dylan	**H**	71%	2.43	2.33	1.73	2.53	3.00	2.53
Angela	**H**	83%	3.00	3.00	2.43	2.00	2.43	2.22
Sam	**H**	82%	1.50	2.86	2.00	2.88	3.00	2.73
Bill	**H**	97%	2.36	2.75	2.00	2.25	2.60	2.92
Nate	**H**	100%	2.92	3.00	2.73	2.4	2.44	2.23
Mean		**51.44%**	**2.33**	**2.75**	**2.32**	**2.55**	**2.71**	**2.60**
Min		0%	1.00	2.20	1.73	1.82	2.00	2.22
Max		100%	3.00	3.00	3.00	3.00	3.00	2.92
Mode		--	3.00	3.00	2.00	2.67	3.00	2.75
Median		60.16%	2.40	2.78	2.43	2.64	2.80	2.60

Note: The score n/a indicates that the tasks did not !end themselves to reasoning and proof, making the coding irrelevant (see Lesson Enactment Rubric, Appendix A).

Consistent with our MKT-P framework, we interpret PSTs' ability to identify curricular opportunities to integrate the reasoning and proof, specifically the four proof themes, as manifestation of their evolving KCT-P. The analysis of enacted lessons illuminates additional aspects of KCT-P and other areas of MKT-P.

6.2. Classroom implementation of lessons

In order to present an in-depth analysis of the lessons as they were implemented in classrooms, we focus this section on lessons on the *conditional statements* (CS) proof theme. We chose the CS proof theme because this theme had the greatest number of lesson plans coded as having high proof theme focus (Table 3), making it interesting to analyze in terms of classroom implementation. The high score for the proof theme focus reflected the *potential* of the lessons to address the proof theme, but the enactment of the lesson is what determines the actual learning opportunities for students to engage with the proof theme (Stein, Remillard, & Smith, 2007). Table 4 summarizes the data on PSTs' implementation of lessons on the CS proof theme. The numbers in the table represent mean values for a given lesson averaged across individual episodes (as described in the analysis of enacted lessons in the Methods section). Tue sections that follow describe each of the categories in Table 4 and our analysis of the results. Due to technical problems with two of the video recordings, hereafter our analysis only includes 13 out of 15 lessons.

6.2.1. Time devoted to conditional statements in the lesson

To compare the planned lessons to the enacted ones, we calculated the percent of the time, in minutes, devoted to conditional statements during the lessons. This included video episodes in which the proof theme was the explicit focus of the instructional activity, either through PSTs' explanations or student work (individual or in groups) on tasks specifically related to CS. Based on the data for percent of time devoted to the CS proof theme, the PSTs feil roughly evenly into the groups as indicated in shading in Table 4.

Not surprisingly, the lessons whose original plans had low or medium focus on the proof theme, when enacted, indeed provided limited opportunities for students to engage with the proof theme, devoting 0 %-21 % of dass time to conditional statements. For example, Rebecca, did not mention conditional statements at all, although her lesson dealt with reasoning about logic riddles. Cindy had students decide whether some conditional statements, such as "If x is a negative number, then x^5 is a positive number" are true or false. However, she never explicitly discussed with the students the meaning, the structure or how to prove or disprove conditional statements, in general.

The other 10 lessons, whose lesson plans were originally coded as having high focus on the CS proof theme, when enacted, devoted varying amounts of time to the CS proof theme, from 40 % to 100 %. We emphasize that when requesting that PSTs create a lesson plan that integrates a certain proof theme with the ongoing mathematical topic, we did not require the PSTs to devote the whole lesson to the proof theme. Thus, it was expected that the majority of the lessons would spend less than 100 % of time on the proof theme.

To provide more context to the data in Table 4, we note that the PSTs whose lesson plan had high focus on the proof theme, but who, in reality, spent between 40 % to 46 % of class time on it, did so due to factors that were beyond their control. One PST had an unexpected fire drill during her lesson, which significantly affected the enactment of her initial plan. The other two PSTs had to accommodate a cooperating teacher's request of devote class time to reviewing material for an upcoming test. For that, the PSTs designed a "Jeopardy!" game on the topic of exponents, which was not directly related to CS, although it asked students to justify their reasoning. Thus, the differences between the high score for the lesson plan and the time spent on CS described in Table 4 were sometimes due to factors external to the PSTs teaching performance. To better understand the *quality* of the PSTs' teaching of the proof themes, we looked beyond the time devoted to it, and analyzed the PSTs' pedagogical moves according to the dimensions of the Lesson Enactment rubric; the results of this analysis are described below.

6.2.2. Explicating the conditional statements proof theme

By explicating the CS proof theme, we mean the extent to which conditional statements were made the focus of a particular lesson episode, were sufficiently discussed and the relevant mathematical-logical ideas were made explicit to students. Explicating a proof theme requires intertwining the knowledge of mathematical-logical content with knowledge of pedagogical practices for supporting student learning, thus illuminating PSTs' KCS-P.

The mathematical content specific to CS could include a subset of the following: the structure of a conditional statement: *If P then Q* ($P \Rightarrow Q$), where P is a hypothesis and Q is a conclusion, the fact that a conditional statement describes a general claim and therefore requires a general proof to establish its truth for all cases, the limitation of empirical evidence as proof, disproof by a counterexample, logical forms that are equivalent to a conditional statement such as contrapositive ($\sim Q \Rightarrow \sim P$) and non-equivalent forms, such as a converse ($Q \Rightarrow P$) or inverse ($\sim P \Rightarrow \sim Q$). The choice of which knowledge components of conditional statements to make explicit during the lesson, to what extent, and in what ways, was left to the PSTs; the course instructors did not intervene with that.

As shown in Table 4, the level of explication of the CS proof theme ranged from 1 to 3 (out of 3), with the average of 2.33 across all lessons. It is interesting to note two PSTs' lessons in this regard. Audrey only devoted 40 % of her lesson to conditional statements, but she explicitly addressed key ideas about conditional statements with her students. In Sam's lesson, although students engaged with conditional statements 82 % of time, she missed many opportunities to make the mathematical-logical content explicit to students (Table 4). Below we illustrate some of the opportunities and missed opportunities for explicating conditional statements in the PSTs' enacted lessons.

One example of a missed opportunity was Cindy's lesson on exponents which contained two statements about raising a negative number to an odd power, whose truth-value students had to determine. Even these minimal opportunities to explicate ideas about conditional statements were not utilized in the enacted lesson, resulting in a total lesson score of 1 (low) for explicating the CS proof theme. Another PST, Ellen, wrote on the board the Exterior Angle Theorem in the conditional form, and asked students to identify what is given and what needs to be proved. When students hesitated on how to respond to this question, Ellen explained that "given always comes first and the prove statement second." This episode was coded as 2 (medium) on explicating the CS proof theme, since although conditional statements seemed to be the focus of the episode, the PST did little to support students' understanding of the relevant mathematical ideas pertaining to conditional statements.

In contrast to these examples, seven PSTs scored above the average on explicating the proof theme with scores ranging between 2.36 to 3 out of 3 (see Table 4). These PSTs explicitly and creatively addressed the CS proof theme in their lessons. Some of these PSTs opted for a rather informal approach, which they felt was appropriate to their students' grade level. For example, Ethan, who taught an eighth-grade lesson on exponents, gave the following definition of conditional statements (which was written on students' worksheets):

> An if/then statement that has an implication and a conclusion. It can be either true or false. Example 1: If I do my homework, then I get my allowance. Example 2: If a polygon has three sides, then it is a triangle.

Although the use of the word "implication" instead of hypothesis is questionable, Ethan's definition provides important information about the structure of a conditional statement, its possible truth-values, and illustrates these ideas with two examples: one mathematical and one non-mathematical using the context from students' personal experiences. In comparison, Audrey, who taught the same grade level, provided students with the following written definition:

> An if/then statement which can be symbolized by *p* and *q*, where *p* is the proposition and *q* is the conclusion. In the examples below the *p* is shown in red and the *q* is shown in blue. Example 1: If it is raining, then it is cloudy. (True) Example 2: If it is cloudy, then it is raining. (False). Notice how if we switch the *p* and *q* of the statement, it may change the truth value of the statement.[2]

During Audrey's lesson, she introduced students to P and Q notation as a part of the definition of a conditional statement and its

[2] Here we use straight and waived underline for illustration. Audrey's worksheet and board work used color. The text in parenthesis appeared in the original.

structure, and used color to highlight P and Q in two real-life examples. The examples served to show the relationship between a statement and its converse *(P* Q vs. Q *P)* without formally introducing the definition of the converse. Audrey was the only person who introduced formal notation to eighth graders; but the PSTs who taught in high school were more likely to use this notation when introducing conditional statements.

Nate, whose CS Jesson dealt with parallel Jines, in the context of tenth grade geometry, came up with a unique approach. He went as far as including a converse, inverse and contrapositive in his Jesson, but instead of using formal notation, Nate used the Janguage of hypothesis and conclusion to compare and contrast different types of statements about parallel lines. First, he introduced a statement "Two parallel Jines cut by a transversal will have congruent alternate interior angles" and asked students to identify hypothesis and conclusion. Then, Nate asked students to formulate a converse in the form "If Conclusion, then Hypothesis," the inverse statement as "If not Hypothesis, then not Conclusion," and a contrapositive as "If not Conclusion, then not Hypothesis." Finally, Nate asked students to determine the truth-values of all statements, compare, and contrast them.

Another important aspect of conditional statements, which many PSTs addressed in their Jessons, was the concept of counter-example and its role in disproving a conditional statement. The Jevel of explicitness varied among different PSTs' approaches. Some PSTs had students disprove conditional statements, without clearly explaining that the object used for that purpose is called a counterexample. For example, Angela Jed students to recognize that the statement "All quadrilaterals with four right angles are squares" is false by drawing a rectangle on the board and explaining that "we cannot say that the statement is true, because it is not true 100 % of the time." Then she invited students to check whether other statements in the worksheet have similar "loop-holes", instead of using the word "counterexample."

Other PSTs did explicitly introduce the concept of a counterexample, although often the definitions they used Jacked clarity. This happened when the PSTs tried to avoid using formal Janguage, for instance, Dylan defined a counterexample as "something that does not fulfil the statement." This explanation was accompanied by a few false statements, which students disproved as a group using correct counterexamples. Yet, since Dylan's explanation was so vague, as the Jesson progressed students occasionally provided ir-relevant examples instead of counterexamples, that is, examples that do not satisfy the hypothesis but do satisfy the conclusion of a given conditional statement (Buchbinder & Zaslavsky, 2019). In general, the PSTs' efforts to explicate the CS proof theme were Jargely affected by their use of appropriate mathematical Janguage, as we show in the next section.

6.2.3. Accuracy of language

The analysis of the PSTs' use of mathematical language aims to illuminate how PSTs' KLAP is manifested in their use of precise mathematical Janguage and their use of proper mathematical vocabulary, while adjusting it to the Jevel of students. We also noted whether PSTs noticed and corrected themselves if they misspoke or whether they corrected or re-voiced students' contributions, which were not mathematically precise. In our analysis, we distinguished between instances when the PSTs spoke about mathe-matical content unrelated to proof, such as solving equations, exponents, word problems, geometrical properties, etc., versus content that pertained to the CS proof theme. The two categories of language accuracy were evaluated on the scale of 1 (low) to 3 (high) (see Appendix A for details).

Table 4 shows that the PSTs' scores for accuracy of mathematical Janguage were generally Jower for proof-related language than for content specific Janguage, with average score of 2.32 (out of 3) for CS-specific language and average score of 2.75 (out of 3) for general mathematical Janguage. Although, it can be claimed that spending more time on conditional statements contributes to accumulation of inaccuracies, our analysis does not support such interpretation. Regardless of time spent on conditional statements, all PSTs, except Audrey, had lower or same scores for proof specific language than for content specific mathematical accuracy.

The accuracy of proof-related language was primarily determined by whether the PST introduced and consistently used proof-related vocabulary to communicate with students about conditional statements. Some PSTs used the Janguage of hypothesis and conclusion while others preferred softer Janguage of "given" and "what needs tobe proved" to convey the same idea. Either of these options could be considered appropriate depending on students' grade Jevel. As teacher educators, during the course, we frequently discussed the need to adjust one's mathematical Janguage to student audience and the importance of not overwhelming students with formal notation and new vocabulary. The data suggests that finding the right balance was challenging for the PSTs. Some tried to oversimplify their Janguage or avoid introducing new vocabulary altogether, relying solely on informal language. One example of this phenomenon is particularly revealing.

Bill designed a tenth-grade geometry lesson creatively incorporating the topic of triangles with the CS proof theme. Students were given a set of color-coded index cards: yellow cards containing hypotheses, such as "a triangle is equilateral" while green cards contained conclusions, such as "a triangle is isosceles." Bill had students create conditional statements by matching cards from the two colored sets, determine their truth-values and then switch the order of the cards to examine the truth-value of the converse. Both the Jesson plan and the enacted Jesson scored high on the focus on the proof theme, the time devoted to it in the Jesson, and explicating the CS proof theme. However, when talking about conditional statements, Bill tried as much as possible to use only informal Janguage: throughout the Jesson, he referred to hypothesis and conclusion as "if-part" and "then-part" of a statement; defined a counterexample as "an example that does not fit the statement and disproves it"; and a converse as "a statement in which the 'then' becomes the 'if.'' Although, one can clearly see a kerne! of correct mathematical idea in each of these "definitions", they are cumbersome. In reality, Bill's use of informal language complicated, rather than simplified classroom communication, since the students struggled to express their thoughts when responding to Bill's prompts, saying "this", "that" and pointing, instead of using proper vocabulary.

On the other hand, the PSTs who did introduce the mathematical vocabulary of hypothesis and conclusion (or some version of it, such as given and conclusion) or who used the P and Q notation, were able to generate more mathematically accurate classroom discourse.

6.2.4. Student engagement and PSTs' actions to promote engagement

Two other important dimensions of PSTs' lesson implementation that we analyzed were student engagement and the PSTs' actions for holding students accountable and promoting student engagement. The PSTs' actions were coded on a 3-point scale, where 3 (high) represents ensuring that students are on task, following up on students' input, being attentive to student work, providing autonomy during group activities, effectively using pedagogical moves such as re-voicing, inviting contributions, exercising a proper wait time, and pressing for explanation (see Appendix A for details). Some of the PSTs' moves or questions were pre-planned, but many actions occurred in response to students' interaction with the lessons and were motivated by students' verbal or non-verbal feedback.

It is important to note that working with small groups did not automatically result in productive student participation. For example, Logan had a very active group of nineth graders, who were easily distracted, so he had to work hard to keep them on track. On the contrary, Laura's students were very quiet, and although they individually completed the tasks she gave them, the students remained quiet when Laura asked them to share their thoughts with the group. Thus, Laura had to extend extra effort to promote student engagement with each other and with the group, not just with the worksheet. The scores for PSTs' actions to promote student engagement ranged from 2.22 to 2.92, with the average of 2.6 (see Table 4).

When analyzing student engagement in response to the PSTs' lessons, we broke it down into engagement with proof- and non-proof-related components of the lesson to gain a more detailed picture of students' interaction with the lesson. The scores for student engagement were also assigned using a 3-point scale, where 3 indicated that most students were attentive towards the lesson or activity; participated in meaningful ways, such as actively listening, asking questions or sharing ideas (Appendix A). Across all lessons, the aggregated scores for student engagement ranged from 1.82 to 3, with the average of 2.55 (see Table 4). Table 4 shows that overall the PSTs did a good job promoting student engagement with the CS tasks, which can be interpreted as an indication of emerging KCT-P. We find it encouraging considering that most PSTs had almost no prior classroom experience. Students, on their part, collaborated weil with the CS lessons and, as Table 4 shows, were actively engaged with the proof-related components of the lesson, which we interpret as a positive outcome.

It is important to keep in mind, that due to the nature of the teaching setting it is not possible to infer direct relationships between PSTs' actions to promote student engagement and actual levels of student engagement. On the one hand, teaching small groups of students is conducive to close involvement of the PST with the students and among them. The novelty of the content and of the instructional activities, the mere change in learning routine, and having the lesson being taught by a PST might have contributed to high student engagement. On the other hand, these very same factors can have a negative effect on students' participation. In the absence of base line data on student participation in regular mathematics classrooms, we are treating these data with caution. Nevertheless, we feel that these data provide important evidence for feasibility of secondary students to actively and meaningfully participate in lessons that integrate proof themes, especially in the lesson components dedicated to conditional statements.

6.3. PSTs' challenges with proof integration and implementation, and dealing with challenges

As mentioned in the methods section, the data on the PSTs' challenges with designing and implementing their lessons came from three sources: classroom video recordings, the PSTs' written reflection reports on each lesson, and reflections on the course as a whole. With respect to lesson implementation, the main challenges emerging from analysis of the video data were PSTs' difficulties to adjust their mathematical language to the level of the students, to use proof-related vocabulary accurately and consistently, and to make ideas about conditional statements explicit for the students (see Table 4). The PSTs self-reported data did not explicitly mention language as a particular source of difficulty, however, all PSTs reported that developing and teaching lessons that integrated the four proof themes was the most challenging aspect of the course for them. Tue PSTs wrote about being "outside of their comfort zone," and being "pushed to do things they have never done before."

The types of challenges encountered by the PSTs fall under two interrelated themes: (a) the PSTs' own doubts about utility and feasibility of teaching proof themes at the secondary level, and (b) challenges specific to planning and enacting lessons on the four proof themes. We elaborate on these challenges below.

The instructional activities of the course aimed to enhance PSTs' perceptions of the usefulness of proof for students' learning at all grade levels. The PSTs learned about the role of proof in promoting students' mathematical understanding and about different forms a proof can take in school mathematics. Nevertheless, as the PSTs designed and taught their own lessons, they voiced concerns about teaching the four proof themes outside Geometry. For example, Grace wrote that she "lucked out" because two out of four lessons she taught were in geometry and thus lent themselves more easily to proof theme integration.

A related concern was teaching proof to low achieving students, as Rebecca's comment from the summative reflection illustrates:

> For many of the lower level classes we taught in, this proofs material was very unrelated and unnecessary for those students. It was quite a different story in the CP [college preparatory] courses though. These students were much more receptive to the material and capable of understanding the connections between proofs and the topic in the curriculum that they were at.

The majority of the PSTs expressed more positive views of students' abilities to cope with proof-related content. Moreover, when responding to a reflection prompt about what they found interesting or surprising in student thinking, most PSTs noted positive student responses to the proof content of the lessons. For example, Laura wrote that she found it "surprising and interesting that students at such a low mathematical skill level were able to understand basic concepts of conditional statements."

The second type of challenge was more specific to the details of planning lessons that integrated the proof themes. Namely, the

PSTs were unsure of the students' prior knowledge of proof; and more importantly, the PSTs were uncertain what leaming objectives, specific to proof themes, they could expect to achieve with secondary students in a single lesson. Same PSTs shared that they found some proof themes to be more challenging to integrate than others. For example, Nate wrote that indirect reasoning and conditional statements were his "favorite topics to teach and expand upon" because he feit they "could be taken in many directions." However, Ethan, who taught eighth grade mathematics, feit exactly the opposite. Ethan found conditional statements and proof by contradiction to be challenging to relate to topics such as exponents and linear equations, and difficult to teach to middle school students "even at the most basic level." Overall, no explicit pattern emerged from the PSTs' reflections on the relative difficulty of integrating various proof themes in lesson planning; although the data from Table 3 provide some clues that two of the proof themes: quantification and the role of examples in proving, and indirect reasoning, might have been more challenging to integrate than others.

For each proof theme (i.e., each course module), the PSTs needed to carefully contemplate which proof related concepts to include in their lessons and how to make them accessible and meaningful for students. The following comment shows how Sam approached this dilemma:

> As the semester progressed, I realized that you can make these ideas accessible to students of all levels. You don't necessarily have to make a lesson that screams direct reasoning but incorporating some of those ideas to get students thinking about logical reasoning and the role of examples in proving can be easier than you think. Ultimately, I enjoyed the challenge of creating lesson plans that forced me to think about what students should get out of each lesson, not just the procedural knowledge.

This quote is informative as it describes the kind of strategies we were hoping our PSTs would adopt. Throughout the course we engaged in multiple conversations, cautioning the PSTs that the university level "Introduction to Proof' course content cannot (and should not) be directly taken into secondary classrooms. Integrating proof themes within school mathematics required the PSTs to distill the main logical ideas and apply them to school content. We were encouraged to see that PSTs, like Sam and others, embraced that idea. Moreover, Sam's comment indicates that she viewed the integration of proof in her lessons as a way to rise above procedural knowledge towards more meaningful leaming for students.

Sam's comment also indicates shifts in her views of feasibility of teaching proof to students, suggesting that she dealt productively with this type of challenge. In the similar fashion, Angela wrote:

> …incorporating the four proof themes into lessons designed for lower level mathematics curriculum [was] the most challenging aspect of the course, but I think that's what made it so worthwhile by really forcing us to show ingenuity and originality. After it was all said and done, it was not beyond our means as college students to incorporate these higher level ideas into our lessons and most of us found creative and natural methods of doing so in games or real-world applications.

These examples are representative of the PSTs' comments in the summative course reflection. The majority of the PSTs indicated that the course experience challenged but also changed their perceptions on feasibility of teaching proof to secondary students, as weil as their perceptions of their own ability to integrate proof themes in their teaching.

7. Discussion

Our research agenda stemmed from the need to prepare prospective secondary teachers to teach reasoning and proving as an integral part of doing mathematics. We addressed this need in the specially designed capstone course *Mathematical Reasoning and Provingfor Secondary Teachers* through a variety of activities aimed to foster PSTs' MKT-P. We paid particular attention to developing practical aspects of this type of knowledge by having PSTs design and teach lessons that integrate mathematical topics from secondary classrooms with one of the four proof themes: (a) quantification and the role of examples in proving, (b) conditional statements, (c) direct proof and argument evaluation and (d) indirect reasoning. In this paper, we presented the analysis of all lesson plans and of classroom implementations of the lessons on conditional statements. Our work was guided by the research questions: (1) How did PSTs integrate the four proof themes with ongoing mathematical topics from the secondary curriculum in their lesson plans?, (2) How did PSTs implement lessons on conditional statements in secondary classrooms?, and (3) What challenges did PSTs encounter in this process? We view PSTs' ability to design and enact lessons that integrate the four proof themes as manifestation of their emerging MKT-P. We do not claim that our PSTs have fully developed MTK-P, nor do we intend to compare performance across PSTs. Our goal is to capture how PSTs performed in a particular moment in time and to provide a proof of existence for the possibility of supporting PSTs' development of MKT-P. In addition, the challenges encountered by the PSTs provide feedback on how to improve the course to better support future PSTs' MKT-P development. In the following, we summarize our outcomes.

The ability to identify curricular opportunities for proof integration and designing lessons that make proof-related content explicit and accessible to students across grade levels corresponds to KCT-P (see Table 1, MKT-P framework). With respect to integration of the proof themes, our analysis showed that PSTs were able to develop lesson plans that successfully integrated the four proof themes with a variety of mathematical topics and grade-levels. In particular, we are encouraged by the number and the variety of mathematical topics, including subject areas outside geometry, such as Algebra, Pre-Algebra and eighth grade mathematics. This is impressive, since despite the NCTM (2000, 2009) and *Common Core State Standards* (NGA & CCSO, 2010) emphasis on proof and argumentation in all subject areas, proof still has limited presence outside high school geometry (Stylianides et al., 2017). During the course, we tried to help PSTs develop a view of proof as a natural part of students' mathematical leaming. This message was conveyed throughout all course activities, but it became most tangible when the PSTs had to implement these ideas in real classrooms.

We acknowledge that the variety of mathematical topics for which PSTs developed proof-oriented lessons is a function of the course setting, in which the PSTs were placed in particular classrooms and were required to integrate the ongoing mathematical topic with a proof theme. Nevertheless, we point to the impressive number and variety of pedagogical features utilized by the PSTs in their Jesson plans (Fig. 2). The richness of pedagogical features suggests that PSTs were able to make use of existing resources, induding proof-tasks modeled in the course, and their own creativity to develop activities intended to promote students' understanding of the four proof themes. We also note that 41 out of 60 Jesson plans had high or medium focus on the proof themes (Table 3) indicating the depth to which the proof themes were addressed in the Jessons[3].

Moreover, the PSTs achieved these goals of incorporating the proof themes in their lessons despite the complexity of the need to match the given mathematical topic with the predetermined proof theme, while accommodating occasional requests of cooperating teachers to devote dass time to non-proof related activities. With these complexities in mind, we are encouraged by the PSTs' Jesson planning performance and view it as a possible indication of evolving KCT-P.

In terms of dassroom implementation, we presented data from Jessons on the CS proof theme, which had the largest number of Jesson plans with a high focus on the proof theme. Not surprisingly, the enactment of the Jessons differed from what was planned (Steinetal., 2007). Specifically, our data show variation among PSTs in both the actual percent of time devoted to CS proof theme in dass and the quality of PSTs' teaching of proof-specific aspects of the Jessons. The Jatter was assessed by quantifying PSTs' performance on explicating the proof-theme (a reflection of KCS-P), the accuracy of proof-specific Janguage (associated with KLAP), and actions to promote student engagement with proof (KCT-P). Language accuracy related to proofs and proving seemed to present a challenge to the PSTs: the scores for proof-related mathematical accuracy were Jower than for non-proof related aspects (Table 4). Our analysis suggests that explication of the proof-themes is related to accuracy of mathematical Janguage, especially with the proof-related vocabulary, since one of the challenges the PSTs encountered was deciding how to adjust concepts and notation specific to conditional statements to the conceptual Jevel of students. We observed !arge variation in the ways the PSTs' approached this challenge: from Bill's decision to rely only on informal Janguage without introducing any proof-specific vocabulary, which ultimately complicated the dassroom communication, to Nate's decision to indude converse, inverse and contrapositive in one Jesson, although such abundance of concepts might not be conducive to students' understanding or memory retention.

Our data suggests that the PSTs can benefit from even doser attention to decisions pertaining to proof-related vocabulary, notation and explicit use of proof-related Janguage during Jesson planning. In the next iterations of the course, following the one described herein, we added a section to the Jesson plan template requiring PSTs to !ist all proof-related concepts pertaining to their Jesson (e.g., a counterexample, an existential statement). We also added a Jesson plan section in which we asked the PSTs to describe, using exact words and imagery, how they plan to introduce students to these concepts. We hope that this careful planning will aid PSTs' explication of the proof themes.

Another notable aspect of Jesson implementation was the high Jevel of PSTs' actions to promote student engagement with proof and high Jevels of student engagement with proof-related components of the Jessons (Table 4). Student participation can be affected by multiple factors, induding dassroom norms. For example, middle school students were more accustomed to working in groups, while the high school dassrooms had rather traditional settings, and students were Jess used to Jeaming in small group settings. As mentioned above, our research setting does not allow drawing direct connections between PSTs' actions and student participation, since the relative novelty of setting and content taught can have either positive or negative effects on student participation. Also, it is important to keep in mind the limited teaching experience of our PSTs, for most of whom these were their first enacted Jessons. Nevertheless, our data show high Jevels of student engagement with Jessons on conditional statements, especially with their proof-specific components (Table 4). We assert that it is a noteworthy and non-trivial outcome that the PSTs were able to generate dassroom situations in which secondary students were actively and meaningfully engaged with lessons on conditional statements[4]•

Our analysis also identified a number of challenges the PSTs encountered in designing and implementing their lessons. One type of challenge, mentioned above, was deciding what aspects of the proof themes to indude in the lesson plan and in what ways to do it. Another type of challenge was related to PSTs' personal doubts about the feasibility of teaching proof themes to secondary students. Our data suggest that many of the PSTs encountered this challenge, at least initially in the course. At the same time, comments, such as those of Angela and Sam presented in the results section, indicate positive shifts in PSTs' perceptions of the feasibility of teaching proof at the secondary Jevel and their increased confidence in doing so. Establishing and quantifying these shifts is one of our further research questions, which is beyond the scope of this paper[5].

In their recent review of the state of research in the area of proof, Stylianides, Stylianides and Weber (2017) note the scarcity of research studies on dassroom implementation of proof. The research base on PSTs' dassroom experiences with proof is even more limited. One study examined how three elementary PSTs in the US tried to enact proof in their intemship sites (Stylianides, Stylianides, & Shilling-Traina, 2013). Despite having strong knowledge and productive dispositions towards proof, these PSTs encountered multiple challenges such as: maintaining cognitive demand of proof-oriented tasks, managing time, responding to students in the moment and managing students' pre-existing habits of mind. Although a direct comparison with our study is not possible due to differences in methodology and the settings, our study adds to this literature by describing additional challenges that are specific to integrating the four proof themes in secondary mathematics curricula.

[3] Our data show variation between the four proof themes in distribution of pedagogical features and the number of high, medium or low-focus on proof themes. However, no direct connections can be drawn based on these data.

[4] Our study design does not allow evaluating student learning of the proof themes. First, most PSTs taught multiple different groups of students throughout the semester which resulted in a lack of continuity of instruction for individual students. Second, studying Student learning was not one of our research goals.

[5] For some preliminary results, see Buchbinder and McCrone (2018).

We conclude this summary by retuming to the primary goal of our study: supporting the development of PSTs' MKT-P, in particular, its practice-related aspects such as lesson planning and enactment. Our analysis suggests that with appropriate pedagogical support throughout the course and PSTs' own creative efforts it is possible for PSTs to plan and enact lessons that integrate reasoning and proof across a multitude of mathematical topics, even within traditional US mathematics curricula. In light of the limited proof-related tasks in mathematical textbooks in the United States (e.g., Otten et al., 2014; Sears & Chavez, 2014) developing teachers' knowledge for creating instructional materials that emphasize reasoning and proof is especially critical. Our data provide empirical support for the feasibility of supporting the development of these aspects of PSTs' MKT-P. We base this assertion on the data that the PSTs' lesson plans included many of the mathematical notation, language, and pedagogical features that were discussed and emphasized in the course. Although our study design does not include a control group, we doubt that it is possible to obtain the same richness of lesson plans without the instructional supports of the course.

Another course feature, which contributed to PSTs' MKT-P development, was inclusion of the school-based practicum. Enacting the proof-related lessons, even with small groups of students, allowed the PSTs to engage in the classroom practices associated with aspects of MKT-P. It also helped the PSTs to experience first-hand that it is possible to engage secondary students with proof. Thus, our study adds to the literature on the importance of practice-based teacher education (Grossman et al., 2009; Lampert, 2010; Zeichner, 2010).

We conclude by noting some interesting research questions that spur from our data, reaching beyond the scope of our study. For example, how would the PSTs' lesson plans play out in full-size classrooms? What do school students learn about the four proof themes? And what are the long-term effects of our course on the PSTs' teaching practices specific to proof and proving? Future studies should examine these important questions.

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Appendix A. Lesson Enactment Video Coding Rubric

	Accuracy of Language		Explicating Specific Reasoning and Proof Themes	Actions to promote student engagement with proof	Student Engagement	
	General	Proof specific			General	Proof specific
3	Tue PST's mathematical language is precise and the PST's mathematical language/vocab is properly adjusted to the students' level. Tue PST attends to their own mathematical precision **(correct their own mistakes).**	Replace "mathematical" with "reasoning and proof'	Reasoning and proof themes are made the focus of the lesson, and are properly integrated, discussed, and explicated. Visual, symbolic, or verbal methods are used by the PST to support argumentation/proving activity.	The PST holds students accountable for learning during the activity or lesson by: (a) ensuring students are on task and (b) following up on students' input. The PST uses pedagogical **moves such as re-voicing,** inviting contributions, exercising a proper wait time, and pressing for explanation. If group work: the PST is attentive to the work of students, allows for autonomy.	Most students seem attentive towards the PST's lesson or activity. Students participate in meaningful ways, such as asking questions, sharing ideas, or actively listening.	Students' engagement is specifically related to the proof theme or proving in general. May include exploring, generalizing, conjecturing, and/or justifying.
2	**No major mistakes or one to two inaccuracies / minor mistakes, that are** left uncorrected. Tue PST slightly struggles with adjusting language to the level of the students.	Replace "mathematical" with **"reasoning and** proof'	Reasoning and proof themes should be the focus of the task at hand, but these themes are only mentioned briefly or are not explicitly integrated into the episode. Tue PST does little to support students' argumentation/ proving activity.	The PST attempts to hold students accountable for learning, but struggles to implement pedagogical **moves, such as re-voi**eing, inviting contribu**tions, exercising a proper** wait time, and pressing for explanation. Tue PST rarely inquires for students to share their ideas, **or may not encourage** students to collaborate (depending on the task at hand). Tue PST does not attend to uneven participation.	Some students appear engaged, but participate **in a limited manner, e.g.,** provide single-word responses to PST's prompts. Or: Some students ap**pear inanentive.**	Students' activity may include some ex**ploring, generalizing,** conjecturing, and/or j ustifying, but it is not the focus of their participation / engage**rnent.**

1	**One or rnore major** mathematical mistakes **or inaccurate vocabu-** lary/ representations. Or: No explicit connections made between concepts or to prior knowledge.	Example of major mistake: using **contrapositive** instead of converse and not **correcting oneself**	Reasoning and proof should be the focus of the task at hand, but these themes are not mentioned nor integrated into the episode. The PST does not support students' ar**gumentation/proving ac**tivity.	The PST does little, if anything, to hold students accountable for learning; stands at a distance, does not seem focused on student activity, does not engage in one-**on-one interactions** (based on the task at hand). Uncooperative or idle student behavior is not addressed by the PST. The PST does most of the talking, without making attempts to engage students.	Most of the students appear inattentive towards the lesson or activity.	Students' engagement is not related to the proof theme or proving in general.
n/a	The task at hand does not !end itself to teacher explaining.		The task at hand does not !end itself to reasoning and proof	The task at hand does not !end itself to PST intervention	The task at hand does not !end itself to general student engagement (or proof-related engagement).	

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