

Algorithm for computing μ -bases of univariate polynomials.

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Abstract

We present a new algorithm for computing a μ -basis of the syzygy module of n polynomials in one variable over an arbitrary field \mathbb{K} . The algorithm is conceptually different from the previously-developed algorithms by Cox, Sederberg, Chen, Zheng, and Wang for $n = 3$, and by Song and Goldman for an arbitrary n . The algorithm involves computing a “partial” reduced row-echelon form of a $(2d + 1) \times n(d + 1)$ matrix over \mathbb{K} , where d is the maximum degree of the input polynomials. The proof of the algorithm is based on standard linear algebra and is completely self-contained. The proof includes a proof of the existence of the μ -basis and as a consequence provides an alternative proof of the freeness of the syzygy module. The theoretical (worst case asymptotic) computational complexity of the algorithm is $O(d^2n + d^3 + n^2)$. We have implemented this algorithm (HHK) and the one developed by Song and Goldman (SG). Experiments on random inputs indicate that SG is faster than HHK when d is sufficiently large for a fixed n , and that HHK is faster than SG when n is sufficiently large for a fixed d .

Keywords: μ -basis; syzygy module; polynomial vectors; rational curves.

1. Introduction

Let $a[s] = [a_1(s), \dots, a_n(s)]$ be a vector of univariate polynomials over a field \mathbb{K} . It is well-known that the syzygy module of a , consisting of linear relations over $\mathbb{K}[s]$ among $a_1(s), \dots, a_n(s)$:

$$\text{syz}(a) = \{h \in \mathbb{K}[s]^n \mid a_1 h_1 + \dots + a_n h_n = 0\}$$

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is free.¹ This means that the syzygy module has a basis, and, in fact, infinitely many bases. A μ -basis is a basis with particularly nice properties, which we describe in more detail in the next section.

The concept of a μ -basis first appeared in Cox et al. (1998b), motivated by the search for new, more efficient methods for solving implicitization problems for rational curves, and as a further development of the method of moving lines (and, more generally, moving curves) proposed in Sederberg and Chen (1995). Since then, a large body of literature on the applications of μ -bases to various problems involving vectors of univariate polynomials has appeared, such as Chen et al. (2005); Song and Goldman (2009); Jia and Goldman (2009); Tesemma and Wang (2014).² The variety of possible applications motivates the development of algorithms for computing μ -bases. Although a proof of the existence of a μ -basis for arbitrary n appeared already in Cox et al. (1998b), the algorithms were first developed for the $n = 3$ case only Cox et al. (1998b); Zheng and Sederberg (2001); Chen and Wang (2002). The first algorithm for arbitrary n appeared in Song and Goldman (2009), as a generalization of Chen and Wang (2002).

This paper presents an alternative algorithm for an arbitrary n . The proof of the algorithm does not rely on previously established theorems about the freeness of the syzygy module or the existence of a μ -basis, and, therefore, as a by-product, provides an alternative, self-contained, constructive proof of these facts. In the rest of the introduction, we informally sketch the main idea underlying this new algorithm, compare it with previous algorithms, and briefly describe its performance.

Main idea: It is well-known that the syzygy module of a , $\text{syz}(a)$, is generated by the set $\text{syz}_d(a)$ of syzygies of degree at most $d = \deg(a)$. The set $\text{syz}_d(a)$ is obviously a \mathbb{K} -subspace of $\mathbb{K}[s]^n$. Using the standard monomial basis, it is easy to see that this subspace is isomorphic to the kernel of a certain linear map $A: \mathbb{K}^{n(d+1)} \rightarrow \mathbb{K}^{2d+1}$ (explicitly given by (7) below). Now we come to the *key idea*: one can *systematically* choose a suitable finite subset of the kernel of A so that the corresponding subset of $\text{syz}_d(a)$ forms a μ -basis. We elaborate on how this is done. Recall that a column of a matrix is called *non-pivotal* if it is either the first column and zero, or it is a linear combination of the previous columns. Now we observe and prove a remarkable fact: the set of indices of non-pivotal columns of A splits into exactly $n - 1$ sets of modulo- n -equivalent integers. By taking the smallest representative in each set, we obtain $n - 1$ integers, which we call *basic non-pivotal* indices. The set of non-pivotal indices of A is equal to the set of non-pivotal indices of its reduced row-echelon form E . From each non-pivotal column of E , an element of $\ker(A)$ can easily be read off, that, in turn, gives rise to an element of $\text{syz}(a)$, which we call a *row-echelon syzygy*. We prove that the row-echelon syzygies corresponding to the $n - 1$ *basic non-pivotal* indices comprise a μ -basis. Thus, a μ -basis can be found by computing the reduced row-echelon form of a single $(2d + 1) \times n(d + 1)$ matrix A over \mathbb{K} . Actually, it is sufficient to compute only a “partial” reduced row-echelon form containing only the basic non-pivotal columns and the preceding pivotal columns.

Relation to the previous algorithms: Cox, Sederberg and Chen Cox et al. (1998b) implicitly suggested an algorithm for the $n = 3$ case. Later, it was explicitly described in the Introduction

¹ Freeness of the syzygy module in the one-variable can be deduced from the Hilbert Syzygy Theorem Hilbert (1890). In the multivariable case, the syzygy module of a polynomial vector is not always free (see, for instance, Cox et al. (1998a))

²A notion of a μ -basis for vectors of polynomials in two variables also has been developed and applied to the study of rational surfaces in three-dimensional projective space (see, for instance, Chen et al. (2005); Shi et al. (2012)). This paper is devoted solely to the one-variable case.

of Zheng and Sederberg (2001). The algorithm relies on the fact that, in the $n = 3$ case, there are only two elements in a μ -basis, and their degrees (denoted as μ_1 and μ_2) can be determined *prior* to computing the basis (see Corollary 2 on p. 811 of Cox et al. (1998b) and p. 621 of Zheng and Sederberg (2001)). Once the degrees are determined, two syzygies are constructed from null vectors of two linear maps $A_1: \mathbb{K}^{3(\mu_1+1)} \rightarrow \mathbb{K}^{\mu_1+d+1}$ and $A_2: \mathbb{K}^{3(\mu_2+1)} \rightarrow \mathbb{K}^{\mu_2+d+1}$ (similar to the one described above). Special care is taken to ensure that these syzygies are linearly independent over $\mathbb{K}[s]$. These two syzygies comprise a μ -basis. It is not clear, however, how this method can be generalized to arbitrary n . First, as far as we are aware, there is not yet an efficient way to determine the degrees of μ -basis members *a priori*. Second, there is not yet an efficient way for choosing appropriate null vectors so that the resulting syzygies are linearly independent.

Zheng and Sederberg (2001) gave a different algorithm for the $n = 3$ case, based on Buchberger-type reduction. A more efficient modification was proposed by Chen and Wang (2002), and was subsequently generalized to arbitrary n by Song and Goldman (2009). The general algorithm starts by observing that the set of the obvious syzygies $\{[-a_j \ a_i] \mid 1 \leq i < j \leq n\}$ generates $\text{syz}(a)$, provided $\text{gcd}(a) = 1$. Then Buchberger-type reduction is used to reduce the degree of one of the syzygies at a time. It is proved that when such reduction becomes impossible, one is left with exactly $n - 1$ non-zero syzygies that comprise a μ -basis. If $\text{gcd}(a)$ is non-trivial, then the output is a μ -basis multiplied by $\text{gcd}(a)$. We note that, in contrast, the algorithm developed in this paper outputs a μ -basis even in the case when $\text{gcd}(a)$ is non-trivial. See Section 8 for more details.

Performance: We show that the algorithm in this paper has theoretical complexity $O(d^2n + d^3 + n^2)$, assuming that the arithmetic takes constant time (which is the case when the field \mathbb{K} is finite). We have implemented our algorithm (HHK), as well as Song and Goldman's (2009) algorithm (SG) in Maple. Experiments on random inputs indicate that SG is faster than HHK when d is sufficiently large for a fixed n and that HHK is faster than SG when n is sufficiently large for a fixed d .

Structure of the paper: In Section 2, we give a rigorous definition of a μ -basis, describe its characteristic properties, and formulate the problem we are considering. In Section 3, we prove several lemmas about the vector space of syzygies of degree at most d , and the role they play in generating the syzygy module. In Section 4, we define the notion of *row-echelon syzygies* and explain how they can be computed. This section contains our main theoretical result, Theorem 27, which explicitly identifies a subset of *row-echelon syzygies* that comprise a μ -basis. In Section 5, we present an algorithm for computing a μ -basis. In Section 6, we analyze the theoretical (worst case asymptotic) computational complexity of this algorithm. In Section 7, we discuss implementation and experiments, and compare the performance of the algorithm presented here with the one described in Song and Goldman (2009). We conclude the paper with a more in-depth discussion and comparison with previous works on μ -bases and related problems in Section 8.

2. μ -basis of the syzygy module.

Throughout this paper, \mathbb{K} denotes a field and $\mathbb{K}[s]$ denotes a ring of polynomials in one indeterminate s . The symbol n will be reserved for the length of the polynomial vector a , whose syzygy module we are considering, and from now on we assume $n > 1$, because for the $n = 1$ case the problem is trivial. The symbol d is reserved for the degree of a . We also will assume that a is a *non-zero vector*. All vectors are implicitly assumed to be *column vectors*, unless specifically stated otherwise. Superscript T denotes transposition.

Definition 1 (Syzzygy). *Let $a = [a_1, \dots, a_n] \in \mathbb{K}[s]^n$ be a row n -vector of polynomials. The syzygy set of a is*

$$\text{syz}(a) = \{h \in \mathbb{K}[s]^n \mid a h = 0\}.$$

We emphasize that h is by default a column vector and a is explicitly defined to be a row vector, so that the product $a h$ is well-defined. It is easy to check that $\text{syz}(a)$ is a $\mathbb{K}[s]$ -module. To define a μ -basis, we need the following terminology:

Definition 2 (Leading vector). *For $h \in \mathbb{K}[s]^n$ we define the degree and the leading vector of h as follows:*

- $\deg(h) = \max_{i=1, \dots, n} \deg(h_i)$.
- $LV(h) = [\text{coeff}(h_1, t), \dots, \text{coeff}(h_n, t)]^T \in \mathbb{K}^n$, where $t = \deg(h)$ and $\text{coeff}(h_i, t)$ denotes the coefficient of s^t in h_i .

Example 3. Let $h = \begin{bmatrix} 1 - 2s - 2s^2 - s^3 \\ 2 + 2s + s^2 + s^3 \\ -3 \end{bmatrix}$. Then $\deg(h) = 3$ and $LV(h) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Before giving the definition of a μ -basis, we state a proposition that asserts the equivalence of several statements, each of which can be taken as a definition of a μ -basis.

Proposition 4. *For a subset $u = \{u_1, \dots, u_{n-1}\} \subset \text{syz}(a)$, ordered so that $\deg(u_1) \leq \dots \leq \deg(u_{n-1})$, the following properties are equivalent:*

1. [independence of the leading vectors] *The set u generates $\text{syz}(a)$, and the leading vectors $LV(u_1), \dots, LV(u_{n-1})$ are independent over \mathbb{K} .*
2. [minimality of the degrees] *The set u generates $\text{syz}(a)$, and if h_1, \dots, h_{n-1} is any generating set of $\text{syz}(a)$, such that $\deg(h_1) \leq \dots \leq \deg(h_{n-1})$, then $\deg(u_i) \leq \deg(h_i)$ for $i = 1, \dots, n-1$.*
3. [sum of the degrees] *The set u generates $\text{syz}(a)$, and $\deg(u_1) + \dots + \deg(u_{n-1}) = \deg(a) - \deg(\text{gcd}(a))$.*
4. [reduced representation] *For every $h \in \text{syz}(a)$, there exist $g_1, \dots, g_{n-1} \in \mathbb{K}[s]$ such that $\deg(g_i) \leq \deg(h) - \deg(u_i)$ and*

$$h = \sum_{i=1}^{n-1} g_i u_i. \tag{1}$$

5. [outer product] *There exists a non-zero constant $\alpha \in \mathbb{K}$ such that the outer product of u_1, \dots, u_{n-1} is equal to $\alpha a / \text{gcd}(a)$.*

Here and below $\text{gcd}(a)$ denotes the greatest common monic divisor of the polynomials a_1, \dots, a_n . The above proposition is a slight rephrasing of Theorem 2 in Song and Goldman (2009). The only notable difference is that we drop the assumption that $\text{gcd}(a) = 1$ and modify Statements 3 and 5 accordingly. After making an observation that $\text{syz}(a) = \text{syz}(a/\text{gcd}(a))$, one can easily check that a proof of Proposition 4 can follow the same lines as the proof of Theorem 2 in Song and Goldman (2009). We do not use Proposition 4 to derive and justify our algorithm

for computing a μ -basis, and therefore we are not including its proof. We include this proposition to underscore several important properties of a μ -basis and to facilitate comparison with the previous work on the subject.

Following Song and Goldman (2009), we base our definition of a μ -basis on Statement 1 of Proposition 4. We are making this choice, because in the process of proving the existence of a μ -basis, we explicitly construct a set of $n - 1$ syzygies for which Statement 1 can be easily verified, while verification of the other statements of Proposition 4 is not immediate. The original definition of a μ -basis (p. 824 of Cox et al. (1998b)) is based on the sum of the degrees property (Statement 2 of Proposition 4). In Section 8, we discuss the advantages of the original definition.

Definition 5 (μ -basis). *For a non-zero row vector $a \in \mathbb{K}[s]^n$, a subset $u \subset \mathbb{K}[s]^n$ of polynomial vectors is called a μ -basis of a , or, equivalently, a μ -basis of $\text{syz}(a)$, if the following three properties hold:*

1. *u has exactly $n - 1$ elements;*
2. *$LV(u_1), \dots, LV(u_{n-1})$ are independent over \mathbb{K} ;*
3. *u is a basis of $\text{syz}(a)$, the syzygy module of a.*

As we show in Lemma 26 below, the \mathbb{K} -linear independence of leading vectors of any set of polynomial vectors immediately implies the $\mathbb{K}[s]$ -linear independence of the polynomial vectors themselves. Therefore, a set u satisfying Statement 1 of Proposition 4 is a basis of $\text{syz}(a)$. Thus, the apparently stronger Definition 5 is, in fact, equivalent to Statement 1 of Proposition 4.

In the next two sections, through a series of lemmas culminating in Theorem 27, we give a self-contained constructive proof of the existence of a μ -basis. This, in turn, leads to an algorithm, presented in Section 5, for solving the following problem:

Problem:

Input: $a \neq 0 \in \mathbb{K}[s]^n$, row vector, where $n > 1$ and \mathbb{K} is a computable field.³

Output: $M \in \mathbb{K}[s]^{n \times (n-1)}$, such that the columns of M form a μ -basis of a .

Example 6 (Running example). *We will be using the following simple example throughout the paper to illustrate the theoretical ideas/findings and the resulting algorithm.*

Input: $a = \begin{bmatrix} 1 + s^2 + s^4 & 1 + s^3 + s^4 & 1 + s^4 \end{bmatrix} \in \mathbb{Q}[s]^3$

Output: $M = \begin{bmatrix} -s & 1 - 2s - 2s^2 - s^3 \\ 1 & 2 + 2s + s^2 + s^3 \\ -1 + s & -3 \end{bmatrix}$

In contrast to the algorithm developed by Song and Goldman in Song and Goldman (2009), the algorithm presented in this paper produces a μ -basis even when the input vector a has a non-trivial greatest common divisor (see Section 8 for more details).

It is worthwhile emphasizing that not every basis of $\text{syz}(a)$ is a μ -basis. Indeed, let u_1 and u_2 be the columns of matrix M in Example 6. Then $u_1 + u_2$ and u_2 is a basis of $\text{syz}(a)$, but

³ A field is *computable* if there are algorithms for carrying out the arithmetic $(+, -, \times, /)$ operations among the field elements.

not a μ -basis, because $LV(u_1 + u_2) = LV(u_2)$. A μ -basis is not canonical: for instance, u_1 and $u_1 + u_2$ will provide another μ -basis for $\text{syz}(a)$ in Example 6. However, Statement 2 of Proposition 4 implies that the degrees of the members of a μ -basis are canonical. In Cox et al. (1998b), these degrees were denoted by μ_1, \dots, μ_{n-1} and the term “ μ -basis” was coined. A more in-depth comparison with previous works on μ -bases and discussion of some related problems can be found in Section 8.

3. Syzygies of bounded degree.

From now on, let $\langle \square \rangle_{\mathbb{K}[s]}$ stand for the $\mathbb{K}[s]$ -module generated by \square . It is known that $\text{syz}(a)$ is generated by polynomial vectors of degree at most $d = \deg(a)$. To keep our presentation self-contained, we provide a proof of this fact (adapted from Lemma 2 of Song and Goldman (2009)).

Lemma 7. *Let $a \in \mathbb{K}[s]^n$ be of degree d . Then $\text{syz}(a)$ is generated by polynomial vectors of degree at most d .*

Proof. Let $\tilde{a} = a/\gcd(a) = [\tilde{a}_1, \dots, \tilde{a}_n]$. For all $i < j$, let

$$u_{ij} = [-\tilde{a}_j \ \tilde{a}_i]^T,$$

with $-\tilde{a}_j$ in i -th position, \tilde{a}_i in j -th position, and all the other elements equal to zero. We claim that the u_{ij} 's are the desired polynomial vectors. First note that

$$\max_{1 \leq i < j \leq n} \deg(u_{ij}) = \max_{1 \leq i \leq n} \tilde{a}_i \leq \deg a = d.$$

It remains to show that $\text{syz}(a) = \langle u_{ij} \mid 1 \leq i < j \leq n \rangle_{\mathbb{K}[s]}$. Obviously we have

$$\text{syz}(a) = \text{syz}(\tilde{a}) \tag{2}$$

Since u_{ij} belongs to $\text{syz}(\tilde{a})$, we have

$$\text{syz}(\tilde{a}) \supset \langle u_{ij} \mid 1 \leq i < j \leq n \rangle_{\mathbb{K}[s]}. \tag{3}$$

Since $\gcd(\tilde{a}) = 1$, there exists a polynomial vector $f = [f_1, \dots, f_n]^T$ such that

$$\tilde{a}_1 f_1 + \dots + \tilde{a}_n f_n = 1.$$

For any $h = [h_1, \dots, h_n]^T \in \text{syz}(\tilde{a})$, by definition

$$\tilde{a}_1 h_1 + \dots + \tilde{a}_n h_n = 0.$$

Therefore, for each h_i ,

$$\begin{aligned} h_i &= (\tilde{a}_1 f_1 + \dots + \tilde{a}_n f_n) h_i \\ &= \tilde{a}_1 f_1 h_i + \dots + \tilde{a}_{i-1} f_{i-1} h_i + \tilde{a}_i f_i h_i + \tilde{a}_{i+1} f_{i+1} h_i + \dots + \tilde{a}_n f_n h_i \\ &= \tilde{a}_1 f_1 h_i + \dots + \tilde{a}_{i-1} f_{i-1} h_i - f_i \sum_{k \neq i, k=1}^n \tilde{a}_k h_k + \tilde{a}_{i+1} f_{i+1} h_i + \dots + \tilde{a}_n f_n h_i \end{aligned}$$

$$= \tilde{a}_1(f_1h_i - f_ih_1) + \cdots + \tilde{a}_n(f_nh_i - f_ih_n) = \sum_{k \neq i, k=1}^n [k, i] \tilde{a}_k,$$

where we denote $f_k h_i - f_i h_k$ by $[k, i]$. Since $[k, i] = -[i, k]$, it follows that

$$h = [h_1, \dots, h_n]^T = \sum_{1 \leq i < j \leq n} [i, j] \begin{bmatrix} -\tilde{a}_j & \tilde{a}_i \end{bmatrix}^T.$$

That is,

$$h = \sum_{1 \leq i < j \leq n} (f_i h_j - f_j h_i) u_{ij}.$$

Therefore

$$\text{syz}(\tilde{a}) \subset \langle u_{ij} \mid 1 \leq i < j \leq n \rangle_{\mathbb{K}[s]}. \quad (4)$$

Putting (2), (3) and (4) together, we have

$$\text{syz}(a) = \langle u_{ij} \mid 1 \leq i < j \leq n \rangle_{\mathbb{K}[s]}.$$

□

Let $\mathbb{K}[s]_d$ denote the set of polynomials of degree at most d , let $\mathbb{K}[s]_d^n$ denote the set of polynomial vectors of degree at most d , and let

$$\text{syz}_d(a) = \{h \in \mathbb{K}[s]_d^n \mid a h = 0\}$$

be the set of all syzygies of degree at most d .

It is obvious that $\mathbb{K}[s]_d$ is a $(d+1)$ -dimensional vector space over \mathbb{K} . Therefore, the set $\mathbb{K}[s]_d^n$ is an $n(d+1)$ -dimensional vector space over \mathbb{K} . It is straightforward to check that $\text{syz}_d(a)$ is a vector subspace of $\mathbb{K}[s]_d^n$ over \mathbb{K} and, therefore, is finite-dimensional. The following lemma states that a \mathbb{K} -basis of the vector space $\text{syz}_d(a)$ generates the $\mathbb{K}[s]$ -module $\text{syz}(a)$. The proof of this lemma follows from Lemma 7 in a few trivial steps and is included for the sake of completeness.

Lemma 8. *Let $a \in \mathbb{K}[s]^n$ be of degree d and h_1, \dots, h_l be a basis of the \mathbb{K} -vector space $\text{syz}_d(a)$. Then $\text{syz}(a) = \langle h_1, \dots, h_l \rangle_{\mathbb{K}[s]}$.*

Proof. From Lemma 7, it follows that there exist $u_1, \dots, u_r \in \text{syz}_d(a)$ that generate the $\mathbb{K}[s]$ -module $\text{syz}(a)$. Therefore, for any $f \in \text{syz}(a)$, there exist $g_1, \dots, g_r \in \mathbb{K}[s]$, such that

$$f = \sum_{i=1}^r g_i u_i. \quad (5)$$

Since h_1, \dots, h_l is a basis of the \mathbb{K} -vector space $\text{syz}_d(a)$, there exist $\alpha_{ij} \in \mathbb{K}$ such that

$$u_i = \sum_{j=1}^l \alpha_{ij} h_j. \quad (6)$$

Combining (5) and (6) we get:

$$f = \sum_{i=1}^r g_i \sum_{j=1}^l \alpha_{ij} h_j = \sum_{j=1}^l \left(\sum_{i=1}^r \alpha_{ij} g_i \right) h_j.$$

□

The next step is to show that the vector space $\text{syz}_d(a)$ is isomorphic to the kernel of a linear map $A: \mathbb{K}^{n(d+1)} \rightarrow \mathbb{K}^{2d+1}$ defined as follows: for $a = \sum_{0 \leq j \leq d} c_j s^j \in \mathbb{K}_d^n[s]$, where $c_j = [c_{1j}, \dots, c_{nj}] \in \mathbb{K}^n$ are *row* vectors, define

$$A = \begin{bmatrix} c_0 & & & & \\ \vdots & \ddots & & & \\ c_d & \vdots & c_0 & & \\ & \ddots & & \ddots & \\ & & & & c_d \end{bmatrix} \in \mathbb{K}^{(2d+1) \times n(d+1)}, \quad (7)$$

with the blank spaces filled by zeros.

For this purpose, we define an explicit isomorphism between vector spaces $\mathbb{K}[s]_t^m$ and $\mathbb{K}^{m(t+1)}$, where t and m are arbitrary natural numbers. Any polynomial m -vector h of degree at most t can be written as $h = w_0 + sw_1 + \dots + s^t w_t$ where $w_i = [w_{1i}, \dots, w_{mi}]^T \in \mathbb{K}^m$. It is clear that the map

$$h \rightarrow h^{\sharp_t} = \begin{bmatrix} w_0 \\ \vdots \\ w_t \end{bmatrix} \quad (8)$$

is linear. It is easy to check that the inverse of this map

$$h^{\flat_t}: \mathbb{K}^{m(t+1)} \rightarrow \mathbb{K}[s]_t^m$$

is given by a linear map:

$$v \rightarrow v^{\flat_t} = S_t^m v \quad (9)$$

where

$$S_t^m = \begin{bmatrix} I_m & sI_m & \dots & s^t I_m \end{bmatrix} \in \mathbb{K}[s]^{m \times m(t+1)}.$$

Here I_m denotes the $m \times m$ identity matrix. For the sake of notational simplicity, we will often write \sharp , \flat and S instead of \sharp_t^m , \flat_t^m and S_t^m when the values of m and t are clear from the context.

Example 9. For

$$h = \begin{bmatrix} 1 - 2s - 2s^2 - s^3 \\ 2 + 2s + s^2 + s^3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + s \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + s^2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s^3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

we have

$$h^\sharp = [1, 2, -3, -2, 2, 0, -2, 1, 0, -1, 1, 0]^T.$$

Note that

$$h = (h^\sharp)^\flat = S h^\sharp = \begin{bmatrix} I_3 & sI_3 & s^2 I_3 & s^3 I_3 \end{bmatrix} h^\sharp.$$

With respect to the isomorphisms \sharp and \flat , the \mathbb{K} -linear map $a: \mathbb{K}[s]_d^n \rightarrow \mathbb{K}[s]_{2d}$ corresponds to the \mathbb{K} linear map $A: \mathbb{K}^{n(d+1)} \rightarrow \mathbb{K}^{2d+1}$ in the following sense:

Lemma 10. Let $a = \sum_{0 \leq j \leq d} c_j s^j \in \mathbb{K}_d^n[s]$ and $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ defined as in (7). Then for any $v \in \mathbb{K}^{n(d+1)}$:

$$av^\flat = (Av)^\flat. \quad (10)$$

Proof. A vector $v \in \mathbb{K}^{n(d+1)}$ can be split into $(d+1)$ blocks

$$\begin{bmatrix} w_0 \\ \vdots \\ w_d \end{bmatrix},$$

where $w_i \in \mathbb{K}^n$ are column vectors. For $j < 0$ and $j > d$, let us define $c_j = 0 \in \mathbb{K}^n$. Then Av is a $(2d+1)$ -vector with $(k+1)$ -th entry

$$(Av)_{k+1} = c_k w_0 + c_{k-1} w_1 + \cdots + c_{k-d} w_d = \sum_{0 \leq i \leq d} c_{k-i} w_i,$$

where $k = 0, \dots, 2d$. Then

$$\begin{aligned} av^\flat &= a S_d^n v = \left(\sum_{0 \leq j \leq d} c_j s^j \right) \left(\sum_{0 \leq i \leq d} w_i s^i \right) = \sum_{0 \leq i, j \leq d} c_j w_i s^{i+j} \\ &= \sum_{0 \leq k \leq 2d} s^k \left(\sum_{0 \leq i \leq d} c_{k-i} w_i \right) = \sum_{0 \leq k \leq 2d} s^k (Av)_{k+1} = S_{2d}^1 (Av) = (Av)^\flat. \end{aligned}$$

□

Example 11. For the row vector a in the running example (Example 6), we have $n = 3, d = 4$,

$$c_0 = [1, 1, 1], c_1 = [0, 0, 0], c_2 = [1, 0, 0], c_3 = [0, 1, 0], c_4 = [1, 1, 1]$$

and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ & & & & & & & 1 & 1 & 1 & 0 & 1 & 0 \\ & & & & & & & & 1 & 1 & 1 \end{bmatrix}.$$

Let $v = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]^T$. Then

$$Av = [6, 15, 25, 39, 60, 33, 48, 47, 42]^T$$

and so

$$(Av)^\flat = S_{2d}^1 (Av) = S_8^1 (Av) = 6 + 15s + 25s^2 + 39s^3 + 60s^4 + 33s^5 + 48s^6 + 47s^7 + 42s^8.$$

On the other hand, since

$$v^\flat = S_d^n v = S_4^3 v = \begin{bmatrix} 1 + 4s + 7s^2 + 10s^3 + 13s^4 \\ 2 + 5s + 8s^2 + 11s^3 + 14s^4 \\ 3 + 6s + 9s^2 + 12s^3 + 15s^4 \end{bmatrix},$$

we have

$$\begin{aligned} av^\flat &= \begin{bmatrix} 1 + s^2 + s^4 & 1 + s^3 + s^4 & 1 + s^4 \end{bmatrix} \begin{bmatrix} 1 + 4s + 7s^2 + 10s^3 + 13s^4 \\ 2 + 5s + 8s^2 + 11s^3 + 14s^4 \\ 3 + 6s + 9s^2 + 12s^3 + 15s^4 \end{bmatrix} \\ &= 42s^8 + 47s^7 + 48s^6 + 33s^5 + 60s^4 + 39s^3 + 25s^2 + 15s + 6. \end{aligned}$$

We observe that

$$av^\flat = (Av)^\flat.$$

Lemma 12. $v \in \ker(A)$ if and only if $v^\flat \in \text{syz}_d(a)$.

Proof. Follows immediately from (10). \square

We conclude this section by describing an explicit generating set for the syzygy module.

Lemma 13. Let b_1, \dots, b_l comprise a basis of $\ker(A)$, then

$$\text{syz}(a) = \langle b_1^\flat, \dots, b_l^\flat \rangle_{\mathbb{K}[s]}.$$

Proof. Lemma 12 shows that the isomorphism (9) between vector spaces $\mathbb{K}^{n(d+1)}$ and $\mathbb{K}(s)_d^n$ induces an isomorphism between their respective subspaces $\ker(A)$ and $\text{syz}_d(a)$. Therefore, $b_1^\flat, \dots, b_l^\flat$ is a basis of $\text{syz}_d(a)$. The conclusion then follows from Lemma 8. \square

4. “Row-echelon” generators and μ -bases.

In the previous section, we proved that any basis of $\ker(A)$ gives rise to a generating set of $\text{syz}(a)$. In this section, we show that a particular basis of $\ker(A)$, which can be “read off” the reduced row-echelon form of A , contains $n - 1$ vectors that give rise to a μ -basis of $\text{syz}(a)$. In this and the following sections, $\text{quo}(i, j)$ denotes the quotient and $\text{rem}(i, j)$ denotes the remainder generated by dividing of an integer i by an integer j .

We start with the following important definition:

Definition 14. A column of any matrix N is called *pivotal* if it is either the first column and is non-zero or it is linearly independent of all previous columns. The rest of the columns of N are called *non-pivotal*. The index of a pivotal (non-pivotal) column is called a *pivotal* (non-pivotal) index.

From this definition, using induction, it follows that every non-pivotal column can be written as a linear combination of the preceding *pivotal columns*.

We denote the set of pivotal indices of A as p and the set of its non-pivotal indices as q . In the following two lemmas, we show how the specific structure of the matrix A is reflected in the structure of the set of non-pivotal indices q .

Lemma 15 (periodicity). *If $j \in q$ then $j + kn \in q$ for $0 \leq k \leq \left\lfloor \frac{n(d+1)-j}{n} \right\rfloor$. Moreover,*

$$A_{*j} = \sum_{r < j} \alpha_r A_{*r} \implies A_{*j+kn} = \sum_{r < j} \alpha_r A_{*r+kn}, \quad (11)$$

where A_{*j} denotes the j -th column of A .

Proof. To prove the statement, we need to examine the structure of the $(2d+1) \times n(d+1)$ matrix A :

$$\left[\begin{array}{cccccc} c_{01} & \cdots & c_{0n} & & & & \\ \vdots & \cdots & \vdots & c_{01} & \cdots & c_{0n} & \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \ddots \\ c_{d1} & \cdots & c_{dn} & \vdots & \cdots & \vdots & \ddots & c_{01} & \cdots & c_{0n} \\ & & & c_{d1} & \cdots & c_{dn} & \ddots & \vdots & \cdots & \vdots \\ & & & & & & \ddots & \vdots & \cdots & \vdots \\ & & & & & & & c_{d1} & \cdots & c_{dn} \end{array} \right]. \quad (12)$$

The j -th column of A has the first $\text{quo}(j-1, n)$ and the last $(d - \text{quo}(j-1, n))$ entries zero. For $1 \leq j \leq nd$ the $(n+j)$ -th column is obtained by shifting all entries of the j -th column down by 1 and then putting an extra zero on the top. We consider two cases:

1. Integer $j = 1$ is in q if and only if the first column of A is zero. From the structure of A it follows that any column indexed by $1 + kn$ is zero and therefore, $(1 + kn) \in q$ for $\left\lfloor \frac{n(d+1)-1}{n} \right\rfloor = d \geq k \geq 0$.
2. Let us embed A in an infinite matrix indexed by integers. By inspection of the structure of A given by (12), we see immediately

$$A_{i,r+kn} = A_{i-k,r}. \quad (13)$$

Then, for a non pivotal index $j > 1$ and $0 \leq k \leq \left\lfloor \frac{n(d+1)-j}{n} \right\rfloor$ we have:

$$\begin{aligned} A_{*j} &= \sum_{r < j} \alpha_r A_{*r} \\ \iff \bigvee_{i \in \mathbb{Z}} A_{i,j} &= \sum_{r=1}^{j-1} \alpha_r A_{i,r} \\ \iff \bigvee_{i \in \mathbb{Z}} A_{i-k,j} &= \sum_{r=1}^{j-1} \alpha_r A_{i-k,r} \quad (\text{by reindexing the row}) \\ \iff \bigvee_{i \in \mathbb{Z}} A_{i,j+kn} &= \sum_{r=1}^{j-1} \alpha_r A_{i,r+kn} \quad (\text{from (13)}) \\ \implies A_{*j+kn} &= \sum_{r < j} \alpha_r A_{*r+kn}, \end{aligned}$$

Therefore $(j + kn) \in q$ for $\left\lfloor \frac{n(d+1)-j}{n} \right\rfloor \geq k \geq 0$ and equation (11) holds.

□

Definition 16. Let q be the set of non-pivotal indices. Let $q/(n)$ denote the set of equivalence classes of q modulo n . Then the set $\tilde{q} = \{\min \varrho \mid \varrho \in q/(n)\}$ will be called the set of *basic non-pivotal indices*.

Example 17. For the matrix A in Example 11, we have $n = 3$ and $q = \{6, 9, 11, 12, 14, 15\}$. Then $q/(n) = \{\{6, 9, 12, 15\}, \{11, 14\}\}$ and $\tilde{q} = \{6, 11\}$.

Lemma 18. *There are exactly $n - 1$ basic non-pivotal indices: $|\tilde{q}| = n - 1$.*

Proof. We prove this lemma by showing that $|\tilde{q}| < n$ and $|\tilde{q}| > n - 2$.

1. Since there are at most n equivalence classes in q modulo n , it follows from the definition of \tilde{q} that $|\tilde{q}| \leq n$. Moreover, the $(2d + 1)$ -th row of the last block of n -columns of A is given by the row vector $c_d = (c_{1d}, \dots, c_{nd}) = LV(a)$, which is non-zero. Thus, there exists $r \in \{1, \dots, n\}$, such that $c_{rd} \neq 0$. Suppose r is minimal such that $c_{rd} \neq 0$. Then the $(nd + r)$ -th column of A is independent from the first $nd + r - 1$ columns (since each of these columns has a zero in the $(2d + 1)$ -th position). Hence, there exists $r \in \{1, \dots, n\}$ such that $nd + r$ is a pivotal index. From the periodicity Lemma 15, it follows that for every $k = 0, \dots, d$, index $r + kn$ is pivotal and therefore no integer from the class r modulo n belongs to \tilde{q} . Thus $|\tilde{q}| < n$.
2. Assume $|\tilde{q}| \leq n - 2$. From the periodicity Lemma 15, it follows that the set of non-pivotal indices is the union of the sets $\{j + kn \mid j \in \tilde{q}, 0 \leq k \leq l_j\}$, where $l_j \leq d$ is some integer. Therefore

$$|q| \leq |\tilde{q}|(d + 1) \leq (n - 2)(d + 1) = nd + n - 2d - 2.$$

On the other hand, $|q| = n(d + 1) - |p|$. It is well-known (and easy to check) that $|p| = \text{rank}(A)$. Since $\text{rank}(A)$ cannot exceed the number of rows of A , $|p| \leq 2d + 1$. Therefore

$$|q| \geq n(d + 1) - (2d + 1) = nd + n - 2d - 1.$$

Contradiction. Hence $|\tilde{q}| > n - 2$.

□

From the matrix A we will now construct a square $n(d + 1) \times n(d + 1)$ matrix V in the following way. For $i \in p$, the i -th column of V has 1 in the i -th row and 0's in all other rows. For $i \in q$ we define the i -th column from the linear relationship

$$A_{*i} = \sum_{\{j \in p \mid j < i\}} \alpha_j A_{*j} \quad (14)$$

as follows: for $j \in p$ such that $j < i$ we set $V_{ji} = \alpha_j$. All the remaining elements of the column V_{*i} are zero. For simplicity we will denote the i -th column of V as v_i . We note two important properties of V :

1. Matrix V has the same linear relationships among its columns as A .

2. Vectors $\{b_i = e_i - v_i \mid i \in q\}$, where by e_i we denote a column vector that has 1 in the i -th position and 0's in all others, comprise a basis of $\ker(A)$.

The corresponding syzygies $\{b_i^b \mid i \in q\}$ will be called *row-echelon syzygies* because the α 's appearing in (14) can be read off the reduced row-echelon form E of A . (We remind the reader that the $(2d+1) \times n(d+1)$ matrix E has the following property: for all $i \in q$, the non-zero entries of the i -th column consist of $\{\alpha_j \mid j \in p, j < i\}$ and α_j is located in the row that corresponds to the place of j in the *increasingly ordered* list p .) The reduced row-echelon form can be computed using Gauss-Jordan elimination or some other methods.

Example 19. For the matrix A in Example 11, we have $n = 3, d = 4$, and

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The non-pivotal indices are $q = \{6, 9, 11, 12, 14, 15\}$. We have

$$\begin{aligned} b_6 &= e_6 - v_6 = [0, 1, -1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \\ b_9 &= e_9 - v_9 = [0, 1, -1, -1, 1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0]^T \\ b_{11} &= e_{11} - v_{11} = [1, 2, -3, -2, 2, 0, -2, 1, 0, -1, 1, 0, 0, 0, 0, 0]^T \\ b_{12} &= e_{12} - v_{12} = [0, 1, -1, -1, 1, 0, -1, 1, 0, -1, 0, 1, 0, 0, 0, 0]^T \\ b_{14} &= e_{14} - v_{14} = [-1, 1, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, -1, 1, 0, 0]^T \\ b_{15} &= e_{15} - v_{15} = [-1, -1, 2, 1, -1, 0, 1, 0, 0, 0, 0, -1, 0, 1]^T \end{aligned}$$

and the corresponding row-echelon syzygies are

$$\begin{aligned} b_6^b &= \begin{bmatrix} -s \\ 1 \\ -1 + s \end{bmatrix} & b_9^b &= \begin{bmatrix} -s - s^2 \\ 1 + s \\ -1 + s^2 \end{bmatrix} \\ b_{11}^b &= \begin{bmatrix} 1 - 2s - 2s^2 - s^3 \\ 2 + 2s + s^2 + s^3 \\ -3 \end{bmatrix} & b_{12}^b &= \begin{bmatrix} -s - s^2 - s^3 \\ 1 + s + s^2 \\ -1 + s^3 \end{bmatrix} \\ b_{14}^b &= \begin{bmatrix} -1 - s^3 - s^4 \\ 1 + s^2 + s^4 \\ 0 \end{bmatrix} & b_{15}^b &= \begin{bmatrix} -1 + s + s^2 - s^4 \\ -1 - s \\ 2 + s^4 \end{bmatrix}. \end{aligned}$$

The following lemma shows a crucial relationship between the row-echelon syzygies. Note that, in this lemma, we use i to denote a non-pivotal index and ι to denote a *basic* non-pivotal index.

Lemma 20. *Let v_r , $r \in \{1, \dots, n(d+1)\}$ denote columns of the matrix V . For $i \in q$, let*

$$b_i = e_i - v_i \quad (15)$$

Then for any $\iota \in \tilde{q}$ and any integer k such that $0 \leq k \leq \left\lfloor \frac{n(d+1)-\iota}{n} \right\rfloor$

$$b_{\iota+kn}^b = s^k b_\iota^b + \sum_{\{j \in p \mid j < \iota, j+kn \in q\}} \alpha_j b_{j+kn}^b, \quad (16)$$

where constants α_j appear in the expression of the ι -th column of A as a linear combination of the previous pivotal columns:

$$A_{*\iota} = \sum_{\{j \in p \mid j < \iota\}} \alpha_j A_{*j}.$$

Proof. We start by stating identities, which we use in the proof. By definition of V , we have for any $j \in p$:

$$v_j = e_j \quad (17)$$

and for any $\iota \in \tilde{q}$:

$$v_\iota = \sum_{\{j \in p \mid j < \iota\}} \alpha_j v_j = \sum_{\{j \in p \mid j < \iota\}} \alpha_j e_j. \quad (18)$$

Since V has the same linear relationships among its columns as A , it inherits periodicity property (11). Therefore, for any $\iota \in \tilde{q}$ and any integer k such that $0 \leq k \leq \left\lfloor \frac{n(d+1)-\iota}{n} \right\rfloor$:

$$v_{\iota+kn} = \sum_{\{j \in p \mid j < \iota\}} \alpha_j v_{j+kn}. \quad (19)$$

We also will use an obvious relationship for any $r \in \{1, \dots, n(d+1)\}$ and $0 \leq k \leq \left\lfloor \frac{n(d+1)-r}{n} \right\rfloor$:

$$e_{r+kn}^b = s^k e_r^b \quad (20)$$

and the fact that the set $\{1, \dots, n(d+1)\}$ is a disjoint union of the sets p and q . Then

$$\begin{aligned} b_{\iota+kn}^b &= (e_{\iota+kn} - v_{\iota+kn})^b = s^k e_\iota^b - \sum_{\{j \in p \mid j < \iota\}} \alpha_j v_{j+kn}^b && \text{by (15), (20) and (19)} \\ &= s^k e_\iota^b - \sum_{\{j \in p \mid j < \iota, j+kn \in p\}} \alpha_j v_{j+kn}^b - \sum_{\{j \in p \mid j < \iota, j+kn \in q\}} \alpha_j v_{j+kn}^b && \text{(disjoint union)} \\ &= s^k e_\iota^b - \sum_{\{j \in p \mid j < \iota, j+kn \in p\}} \alpha_j e_{j+kn}^b - \sum_{\{j \in p \mid j < \iota, j+kn \in q\}} \alpha_j v_{j+kn}^b && \text{by (17)} \\ &= s^k e_\iota^b - \sum_{\{j \in p \mid j < \iota\}} \alpha_j e_{j+kn}^b + \sum_{\{j \in p \mid j < \iota, j+kn \in q\}} \alpha_j (e_{j+kn}^b - v_{j+kn}^b) && \text{(disjoint union)} \\ &= s^k e_\iota^b - \sum_{\{j \in p \mid j < \iota\}} s^k \alpha_j e_j^b + \sum_{\{j \in p \mid j < \iota, j+kn \in q\}} \alpha_j b_{j+kn}^b && \text{by (20) and (15)} \end{aligned}$$

$$\begin{aligned}
&= s^k \left(e_\iota - \sum_{\{j \in p \mid j < \iota\}} \alpha_j e_j \right) + \sum_{\{j \in p \mid j < \iota, j + kn \in q\}} \alpha_j b_{j+kn}^\flat \\
&= s^k b_\iota^\flat + \sum_{\{j \in p \mid j < \iota, j + kn \in q\}} \alpha_j b_{j+kn}^\flat.
\end{aligned} \tag{18) and (15)$$

□

Example 21. Continuing with Example 19, where $q = \{6, 9, 11, 12, 14, 15\}$ and $\tilde{q} = \{6, 11\}$ and $p = \{1, 2, 3, 4, 5, 7, 8, 10, 13\}$, we have:

$$\begin{aligned}
b_9^\flat &= s b_6^\flat + 1 b_6^\flat, \\
b_{12}^\flat &= s^2 b_6^\flat + 1 b_9^\flat + 0 b_{11}^\flat, \\
b_{14}^\flat &= s b_{11}^\flat + 3 b_6^\flat + (-1) b_{11}^\flat \\
b_{15}^\flat &= s^3 b_6^\flat + (-1) b_{11}^\flat + 1 b_{12}^\flat + 0 b_{14}^\flat.
\end{aligned} \tag{21}$$

In the next lemma, we show that the subset of row-echelon syzygies indexed by the $n - 1$ basic non-pivotal indices is sufficient to generate $\text{syz}(a)$.

Lemma 22. *Let \tilde{q} denote the set of basic non-pivotal indices of A . Then*

$$\text{syz}(a) = \langle b_r^\flat \mid r \in \tilde{q} \rangle_{\mathbb{K}[s]}.$$

Proof. Since $\{b_i \mid i \in q\}$ comprise a basis of $\ker(A)$, we know from Lemma 13 that $\text{syz}(a) = \langle b_i^\flat \mid i \in q \rangle_{\mathbb{K}[s]}$. Equation (16) implies that for all $i \in q$, there exist constant β 's such that

$$b_i^\flat = s^k b_\iota^\flat + \sum_{\{r \in q \mid r < i\}} \beta_r b_r^\flat, \tag{22}$$

where $\iota \in \tilde{q}$ is equal to i modulo n . It follows that inductively we can express b_i^\flat as a $\mathbb{K}[s]$ -linear combination of $\{b_r \mid r \in \tilde{q}\}$ and the conclusion of the lemma follows. □

Example 23. Continuing with Example 19, we have from (21):

$$\begin{aligned}
b_9^\flat &= (s + 1) b_6^\flat, \\
b_{12}^\flat &= (s^2 + s + 1) b_6^\flat + 0 b_{11}^\flat, \\
b_{14}^\flat &= 3 b_6^\flat + (s - 1) b_{11}^\flat, \\
b_{15}^\flat &= (s^3 + s^2 + s + 1) b_6^\flat + (-1) b_{11}^\flat.
\end{aligned}$$

We next establish linear independence of the corresponding leading vectors:

Lemma 24. *The leading vectors $LV(b_r^\flat)$, $r \in \tilde{q}$ are linearly independent over \mathbb{K} .*

Proof. The leading vector $LV(b_r^\flat)$ is equal to the last non-zero n -block in the $n(d + 1)$ -vector b_r . By construction, the last non-zero element of b_r is equal to 1 and occurs in the r -th position. Then $LV(b_r^\flat)$ has 1 in $\bar{r} = (r \bmod n)$ (the remainder of division of r by n) position. All elements of $LV(b_r^\flat)$ positioned after \bar{r} are zero. Since all integers in \tilde{q} are distinct (modulo n), $LV(b_r^\flat)$, $r \in \tilde{q}$ are linearly independent over \mathbb{K} . □

Example 25. The basic non-pivotal columns of the matrix V in Example (19) are columns 6 and 11. We previously computed

$$\begin{aligned} b_6 &= e_6 - v_6 = [0, 1, -1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \\ b_{11} &= e_{11} - v_{11} = [1, 2, -3, -2, 2, 0, -2, 1, 0, -1, 1, 0, 0, 0, 0, 0]^T. \end{aligned}$$

The last non-zero n -blocks of b_6 and b_{11} are $[-1, 0, 1]$ and $[-1, 1, 0]$, respectively. These blocks coincide with $LV(b_6^b)$ and $LV(b_{11}^b)$ computed in Example 19. We observe that these vectors are linearly independent, as expected.

Lemma 26. *Let polynomial vectors $h_1, \dots, h_l \in \mathbb{K}[s]^n$ be such that $LV(h_1), \dots, LV(h_l)$ are independent over \mathbb{K} . Then h_1, \dots, h_l are independent over $\mathbb{K}[s]$.*

Proof. Assume that h_1, \dots, h_l are linearly dependent over $\mathbb{K}[s]$, i.e. there exist polynomials $g_1, \dots, g_l \in \mathbb{K}[s]$, not all zero, such that

$$\sum_{i=1}^l g_i h_i = 0. \quad (23)$$

Let $m = \max_{i=1, \dots, l} (\deg(g_i) + \deg(h_i))$ and let \mathcal{I} be the set of indices on which this maximum is achieved. Then (23) implies

$$\sum_{i \in \mathcal{I}} LC(g_i) LV(h_i) = 0,$$

where $LC(g_i)$ is the leading coefficient of g_i and is non-zero for $i \in \mathcal{I}$. This identity contradicts our assumption that $LV(h_1), \dots, LV(h_l)$ are linearly independent over \mathbb{K} . \square

Theorem 27 (Main). *The set $u = \{b_r^b \mid r \in \tilde{q}\}$ is a μ -basis of a .*

Proof. We will check that u satisfies the three conditions of a μ -basis in Definition 5.

1. From Lemma 18, there are exactly $n - 1$ elements in \tilde{q} . Since $b_{r_1} \neq b_{r_2}$ for $r_1 \neq r_2 \in \tilde{q}$ and since b is an isomorphism, the set u contains exactly $n - 1$ elements.
2. From Lemma 24, we know that the leading vectors $LV(b_r^b)$, $r \in \tilde{q}$ are linearly independent over \mathbb{K} .
3. Lemma 22 asserts that the set u generates $\text{syz}(a)$. By combining Lemmas 24 and 26, we see that the elements of this set are independent over $\mathbb{K}[s]$. Therefore u is a basis of $\text{syz}(a)$.

\square

Remark 28. *We note that by construction the last non-zero entry of vector b_r is in the r -th position, and therefore*

$$\deg(b_r^b) = \lceil r/n \rceil - 1.$$

Thus we can determine the degrees of the μ -basis elements prior to computing the μ -basis from the set of basic non-pivotal indices.

Example 29. For the row vector a given in the running example (Example 6), we determined that $\tilde{q} = \{6, 11\}$. Therefore, prior to computing a μ -basis, we can determine the degrees of its members: $\mu_1 = \lceil 6/3 \rceil - 1 = 1$ and $\mu_2 = \lceil 11/3 \rceil - 1 = 3$. We now can apply Theorem 27 and the computation we performed in Example 19 to write down a μ -basis:

$$b_6^b = \begin{bmatrix} -s \\ 1 \\ -1 + s \end{bmatrix} \text{ and } b_{11}^b = \begin{bmatrix} 1 - 2s - 2s^2 - s^3 \\ 2 + 2s + s^2 + s^3 \\ -3 \end{bmatrix}.$$

We observe that our degree prediction is correct.

5. Algorithm

In this section, we describe an algorithm for computing μ -bases of univariate polynomials. We assume that the reader is familiar with Gauss-Jordan elimination (for computing reduced row-echelon forms and in turn null vectors), which can be found in any standard linear algebra textbook. The theory developed in the previous sections can be recast into the following computational steps:

1. Construct a matrix $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ whose null space corresponds to $\text{syz}_d(a)$.
2. Compute the reduced row-echelon form E of A .
3. Construct a matrix $M \in \mathbb{K}[s]^{n \times (n-1)}$ whose columns form a μ -basis of a , as follows:
 - (a) Construct the matrix $B \in \mathbb{K}^{n(d+1) \times (n-1)}$ whose columns are the null vectors of E corresponding to its basic non-pivot columns:
 - $B_{\tilde{q}_j, j} = 1$
 - $B_{p_r, j} = -E_{r, \tilde{q}_j}$ for all r
 - All other entries are zero

where p is the list of the pivotal indices and \tilde{q} is the list of the basic non-pivotal indices of E .
 - (b) Translate the columns of B into polynomials.

However, steps 2 and 3 do some wasteful operations and they can be improved, as follows:

- Note that step 2 constructs the entire reduced row-echelon form of A , even though we only need $n - 1$ null vectors corresponding to its basic non-pivot columns. Hence, it is natural to optimize this step so that only the $n - 1$ null vectors are constructed: instead of using Gauss-Jordan elimination to compute the entire reduced row-echelon form, one performs operations column by column only on the pivot columns and basic non-pivot columns. One aborts the elimination process as soon as $n - 1$ basic non-pivot columns are found, resulting in a partial reduced row-echelon form of A .
- Note that step 3 constructs the entire matrix B even though many entries are zero. Hence, it is natural to optimize this step so that we bypass constructing the matrix B , but instead construct the matrix M directly from the matrix E . This is possible because the matrix E contains all the information about the matrix B .

Below, we describe the resulting algorithm in more detail and illustrate its operation on our running example (Example 6).

μ -Basis Algorithm

Input: $a \neq 0 \in \mathbb{K}[s]^n$, row vector, where $n > 1$ and \mathbb{K} is a computable field

Output: $M \in \mathbb{K}[s]^{n \times (n-1)}$ such that its columns form a μ -basis of a

1. *Construct a matrix $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ whose null space corresponds to $\text{syz}_d(a)$.*

(a) $d \leftarrow \deg(a)$

(b) Identify the row vectors $c_0, \dots, c_d \in \mathbb{K}^n$ such that $a = c_0 + c_1 s + \dots + c_d s^d$.

$$(c) A \leftarrow \begin{bmatrix} c_0 & & & \\ \vdots & \ddots & & \\ c_d & \vdots & c_0 & \\ & \ddots & \vdots & \\ & & & c_d \end{bmatrix}$$

2. *Construct the “partial” reduced row-echelon form E of A .*

This can be done by using Gauss-Jordan elimination (forward elimination, backward elimination, and normalization), with the following optimizations:

- Stop the forward elimination as soon as $n - 1$ basic non-pivot columns are detected.
- Skip over periodic non-pivot columns.
- Carry out the row operations only on the required columns.

3. *Construct a matrix $M \in \mathbb{K}[s]^{n \times (n-1)}$ whose columns form a μ -basis of a .*

Let p be the list of the pivotal indices and let \tilde{q} be the list of the basic non-pivotal indices of E .

(a) Initialize an $n \times n - 1$ matrix M with 0 in every entry.

(b) For $j = 1, \dots, n - 1$

$$r \leftarrow \text{rem}(\tilde{q}_j - 1, n) + 1$$

$$k \leftarrow \text{quo}(\tilde{q}_j - 1, n)$$

$$M_{r,j} \leftarrow M_{r,j} + s^k$$

(c) For $i = 1, \dots, |p|$

$$r \leftarrow \text{rem}(p_i - 1, n) + 1$$

$$k \leftarrow \text{quo}(p_i - 1, n)$$

For $j = 1, \dots, n - 1$

$$M_{r,j} \leftarrow M_{r,j} - E_{i,\tilde{q}_j} s^k$$

Theorem 30. *Let M be the output of the μ -Basis Algorithm on the input $a \in \mathbb{K}[s]^n$. Then the columns of M form a μ -basis for a .*

Proof. In step 1, we construct the matrix A whose null space corresponds to $\text{syz}_d(a)$ as has been shown in Lemma 12. In step 2, we perform partial Gauss-Jordan operations on A to identify the $n - 1$ basic non-pivot columns of its reduced row-echelon form E . In Lemma 18, we showed that there are exactly $n - 1$ such columns. In step 3, we convert the basic non-pivot columns of E into polynomial vectors, using the \mathbb{b} -isomorphism described in Section 3, and return these polynomial vectors as columns of the matrix M . From Theorem 27 it follows that the columns of M indeed form a μ -basis of a , because they satisfy the generating, leading vector, and linear independence conditions of Definition 5 of a μ -basis. \square

Example 31. We trace the algorithm (with partial Gauss-Jordan) on the input vector from Example 6:

$$a = \begin{bmatrix} 1 + s^2 + s^4 & 1 + s^3 + s^4 & 1 + s^4 \end{bmatrix} \in \mathbb{Q}[s]^3$$

1. Construct a matrix $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ whose null space corresponds to $\text{syz}_d(a)$:

(a) $d \leftarrow$

(b) $c_0, c_1, c_2, c_3, c_4 \leftarrow [1\ 1\ 1], [0\ 0\ 0], [1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 1]$

$$(c) A \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & & 1 & 1 & 1 & 0 & 1 & 0 \\ & & & & & & & & 1 & 1 & 1 & 0 & 1 \\ & & & & & & & & & 1 & 1 & 1 & 0 \\ & & & & & & & & & & 1 & 1 & 1 \end{array} \right]$$

A blank indicates that the entry is zero due to structural reasons.

2. Construct the “partial” reduced row-echelon form E of A :

For this step, we will maintain/update the following data structures.

- E : the matrix initialized with A and updated by the Gauss-Jordan process.
- p : the set of the pivotal indices found.
- \tilde{q} : the set of the basic non-pivotal indices found.
- O : the list of the row operations, represented as follows.

(i, i') : swap $E_{i,j}$ with $E_{i',j}$

$$(i, w, i') : E_{i,i} \leftarrow E_{i,i} + w \cdot E_{i',i}$$

where j is the current column index

We will also indicate the update status of the columns of E using the following color codings.

gray	:	not yet updated
blue	:	pivot
red	:	basic non-pivot
brown	:	periodic non-pivot

Now we show the trace.

(a) Initialize.

$$p \leftarrow \{\}$$

$$\tilde{q} \leftarrow \{\}$$

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

$$O \leftarrow []$$

(b) $j \leftarrow 1$

Carry out the row operations in O on column 1. (Nothing to do.)

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} \textcolor{blue}{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \textcolor{blue}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \textcolor{blue}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \textcolor{blue}{0} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \textcolor{blue}{1} & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

Identify column 1 as a pivot.

$$p \leftarrow \{1\}$$

$$\tilde{q} \leftarrow \{\}$$

Carry out the row operations $(3, -1, 1), (5, -1, 1)$ on column 1.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} \textcolor{blue}{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

Append $(3, -1, 1), (5, -1, 1)$ to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1)]$$

(c) $j \leftarrow 2$

Carry out the row operations in O on column 2.

$$E \leftarrow \left[\begin{array}{cc|cc|cc|cc|cc|cc} \textcolor{blue}{1} & \textcolor{blue}{1} & 1 & & & & & & & & & \\ & \textcolor{blue}{0} & 0 & 1 & 1 & 1 & & & & & & \\ & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & \\ & \textcolor{blue}{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ & \textcolor{blue}{0} & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \\ & & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & \\ & & & & & & & & 1 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array} \right]$$

Identify column 2 as a pivot.

$$p \leftarrow \{1, 2\}$$

$$\tilde{q} \leftarrow \{\}$$

Carry out the row operations $(3, 2), (4, 1, 2)$ on column 2.

$$E \leftarrow \left[\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 1 & 1 & & & & & & & & & \\ & -1 & 0 & 1 & 1 & 1 & & & & & & \\ & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & \\ & & & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 1 & 0 & 0 \\ & & & & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & & & & & & & 1 & 1 & 0 \\ & & & & & & & & & & & & & & & & 1 & 1 & 1 \end{array} \right]$$

Append $(3, 2), (4, 1, 2)$ to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2)]$$

(d) $j \leftarrow 3$

Carry out the row operations in O on column 3.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & & & & & & & & & & \\ -1 & -1 & & 1 & 1 & 1 & & & & & & & \\ 0 & 0 & 0 & 0 & & & 1 & 1 & 1 & & & & \\ -1 & & & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \end{array} \right]$$

Identify column 3 as a pivot.

$$p \leftarrow \{1, 2, 3\}$$

$$\tilde{q} \leftarrow \{ \}$$

Carry out the row operation (4, 3) on column 3.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

Append (4, 3) to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3)]$$

(e) $j \leftarrow 4$

Carry out the row operations in O on column 4.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

Identify column 4 as a pivot.

$$p \leftarrow \{1, 2, 3, 4\}$$

$$\tilde{q} \leftarrow \{\}$$

Carry out the row operation (6, -1, 4) on column 4.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

Append (6, -1, 4) to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4)]$$

(f) $j \leftarrow 5$

Carry out the row operations in O on column 5.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ \end{array} \right]$$

Identify column 5 as a pivot.

$p \leftarrow \{1, 2, 3, 4, 5\}$

$\tilde{q} \leftarrow \{\}$

No row operations needed on column 5.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ \end{array} \right]$$

Nothing to append to O .

$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4)]$

(g) $j \leftarrow 6$

Carry out the row operations in O on column 6.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \end{array} \right]$$

Identify column 6 as a basic non-pivot: column 6 is non-pivotal because it does not have non-zero entries below the 5-th row and therefore it is a linear combination of the five previous pivotal columns: $E_{*6} = -E_{*2} + E_{*3} + E_{*4}$. Column 6 is basic because its index is minimal in its equivalence class $q/(3)$.

$p \leftarrow \{1, 2, 3, 4, 5\}$

$\tilde{q} \leftarrow \{6\}$

No row operations needed on column 6.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

Nothing to append to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4)]$$

(h) $j \leftarrow 7$

Carry out the row operations in O on column 7.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{array} \right]$$

Identify column 7 as a pivot.

$$p \leftarrow \{1, 2, 3, 4, 5, 7\}$$

$$\tilde{q} \leftarrow \{6\}$$

Carry out the row operations (7, 6) on column 7.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{array} \right]$$

Append (7, 6) to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4), (7, 6)]$$

(i) $j \leftarrow 8$

Carry out the row operations in O on column 8.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

Identify column 8 as a pivot.

$$p \leftarrow \{1, 2, 3, 4, 5, 7, 8\}$$

$$\tilde{q} \leftarrow \{6\}$$

No row operations needed on column 8.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

Nothing to append to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4), (7, 6)]$$

$$(j) \ j \leftarrow 9$$

Identify column 9 as periodic non-pivot.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$(k) \ j \leftarrow 10$$

Carry out the row operations in O on column 10.

$$E \leftarrow \left[\begin{array}{cccc|cc|c|c|c|c|c} 1 & 1 & 1 & & & & & & & & & \\ & -1 & -1 & 0 & 0 & 0 & 1 & 1 & & 0 & & \\ & & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & & \\ & & & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & \\ & & & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & 1 & 1 & 1 & 1 & 0 & 0 & \\ & & & & & & & 1 & 1 & 1 & 0 & 0 & \\ & & & & & & & & 1 & 1 & 0 & 0 & \\ & & & & & & & & & 0 & 1 & 0 & \\ & & & & & & & & & & 0 & 1 & \\ & & & & & & & & & & & 1 & 1 & \\ \end{array} \right]$$

Identify column 10 as a pivot.

$$p \leftarrow \{1, 2, 3, 4, 5, 7, 8, 10\}$$

$$\tilde{q} \leftarrow \{6\}$$

No row operations needed on column 10.

$$E \leftarrow \left[\begin{array}{ccc|ccc|c|ccc|ccc} 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & 1 & 1 & 1 & & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ & & & & & 1 & 0 & & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ & & & & & & & & & & 1 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & & & & & & 1 & 0 & 0 & 0 & 1 \\ & & & & & & & & & & & & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

Nothing to append to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4), (7, 6)]$$

(1) $j \leftarrow 11$

Carry out the row operations in O on column 11.

Identify column 11 as a basic non-pivot.

$$p \leftarrow \{1, 2, 3, 4, 5, 7, 8, 10\}$$

$$\tilde{q} \leftarrow \{6, 11\}$$

No row operations needed on column 11.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

Nothing to append to O .

$$O \leftarrow [(3, -1, 1), (5, -1, 1), (3, 2), (4, 1, 2), (4, 3), (6, -1, 4), (7, 6)]$$

We have identified $n - 1$ basic non-pivot columns, so we abort forward elimination.

(m) Perform backward elimination on the pivot columns and basic non-pivot columns.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & & & 0 & & & -1 & & & & & & & & & \\ -1 & & & 1 & & & 2 & & & & & & & & & \\ -1 & & & -1 & & & 1 & & -3 & & & & & & & \\ 1 & & & 1 & & & 0 & & 2 & & 1 & & & & & \\ 1 & & & 0 & & & 0 & & -2 & & 0 & & 1 & & 1 & 1 \\ 1 & & & 0 & & & 0 & & 2 & & 0 & & 0 & & 0 & 0 \\ 1 & & & 1 & & & 0 & & -1 & & 0 & & 1 & & 0 & 0 \\ 1 & & & 1 & & & 1 & & 1 & & 1 & & 0 & & 1 & 0 \\ 1 & & & 1 & & & 1 & & 1 & & 1 & & 1 & & 1 & 1 \end{array} \right]$$

(n) Perform normalization on the pivot columns and basic non-pivot columns.

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & & & 0 & & & -1 & & & & & & & & & \\ 1 & & & -1 & & & -2 & & & & & & & & & \\ 1 & & & 1 & & & 3 & & & & & & & & & \\ 1 & & & 1 & & & 1 & & 2 & & 1 & & & & & \\ 1 & & & 0 & & & 0 & & -2 & & 0 & & 1 & & 1 & 1 \\ 1 & & & 0 & & & 0 & & 2 & & 0 & & 0 & & 0 & 0 \\ 1 & & & 1 & & & 0 & & -1 & & 0 & & 1 & & 0 & 0 \\ 1 & & & 1 & & & 1 & & 1 & & 1 & & 0 & & 1 & 0 \\ 1 & & & 1 & & & 1 & & 1 & & 1 & & 1 & & 1 & 1 \end{array} \right]$$

3. Construct a matrix $M \in \mathbb{K}[s]^{n \times (n-1)}$ whose columns form a μ -basis of a :

$$(a) M \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(b) M \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & s^3 \\ s & 0 \end{bmatrix}$$

$$(c) M \leftarrow \begin{bmatrix} -s & 1 - 2s - 2s^2 - s^3 \\ 1 & 2 + 2s + s^2 + s^3 \\ -1 + s & -3 \end{bmatrix}$$

6. Theoretical Complexity Analysis

In this subsection, we analyze the theoretical (asymptotic worst case) complexity of the μ -basis algorithm given in the previous section. We will do so under the assumption that the time for any arithmetic operation is constant.

Theorem 32. *The complexity of the μ -basis algorithm given in the previous section is*

$$O(d^2n + d^3 + n^2).$$

Proof. We will trace the theoretical complexity for each step of the algorithm.

1. (a) To determine d , we scan through each of the n polynomials in a to identify the highest degree term, which is always $\leq d$. Thus, the complexity for this step is $O(dn)$.
(b) We identify $n(d + 1)$ values to make up c_0, \dots, c_d . Thus, the complexity for this step is $O(dn)$.
(c) We construct a matrix with $(2d + 1)n(d + 1)$ entries. Thus, the complexity for this step is $O(d^2n)$.
2. With the partial Gauss-Jordan elimination, we perform row operations only on the (at most) $2d + 1$ pivot columns of A and the $n - 1$ basic non-pivot columns of A . Thus, we perform Gauss-Jordan elimination on a $(2d + 1) \times (2d + n)$ matrix. In general, for a $k \times l$ matrix, Gauss-Jordan elimination has complexity $O(k^2l)$. Thus, the complexity for this step is $O(d^2(d + n))$.
3. (a) We fill 0 into the entries of an $n \times (n - 1)$ matrix M . Thus, the complexity of this step is $O(n^2)$.
(b) We update entries of the matrix $n - 1$ times. Thus, the complexity of this step is $O(n)$.
(c) We update entries of the matrix $|p| \times (n - 1)$ times. Note that $|p| = \text{rank}(A) \leq 2d + 1$. Thus the complexity of this step is $O(dn)$.

By summing up, we have

$$O(dn + dn + d^2n + d^2(d + n) + n^2 + n + dn) = O(d^2n + d^3 + n^2)$$

□

Remark 33. Note that the n^2 term in the above complexity is solely due to step 3(a), where the matrix M is initialized with zeros. If one uses a *sparse* representation of the matrix (storing only non-zero elements), then one can skip the initialization of the matrix M . As a result, the complexity can be improved to $O(d^2n + d^3)$.

Remark 34. [Comparison with Song-Goldman Algorithm] As far as we are aware, the theoretical complexity of the algorithm by Song and Goldman (2009) has not yet been published. Here we roughly estimate the complexity of this algorithm to be $O(dn^5 + d^2n^4)$. It will require a more rigorous analysis to prove/refute this apparent complexity, which is beyond the scope of this paper. For the readers' convenience, we reproduce the algorithm published in Song and Goldman (2009) on pp. 220 – 221 in our notation, before analyzing its complexity.

Input: $a \in \mathbb{K}[s]^n$ with $\text{gcd}(a) = 1$

Output: A μ -basis of a

1. Create the $r = C_2^n$ “obvious” syzygies as described in Lemma 7 and label them u_1, \dots, u_r .
2. Set $m_i = LV(u_i)$ and $d_i = \deg(u_i)$ for $i = 1, \dots, r$.
3. Sort d_i so that $d_1 \geq d_2 \geq \dots \geq d_r$ and re-index u_i, m_i .
4. Find real numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that $\alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_r m_r = 0$.
5. Choose the lowest integer j such that $\alpha_j \neq 0$, and update u_j by setting

$$u_j = \alpha_j u_j + \alpha_{j+1} s^{d_j - d_{j+1}} u_{j+1} + \dots + \alpha_r s^{d_j - d_r} u_r.$$

If $u_j \equiv 0$, discard u_j and set $r = r - 1$; otherwise set $m_j = LV(u_j)$ and $d_j = \deg(u_j)$.

6. If $r = n - 1$, then output the $n - 1$ non-zero vector polynomials u_1, \dots, u_{n-1} and stop; otherwise, go to Step 3.

Finding a null vector in step 4 by partial Gauss-Jordan elimination requires performing row operations on (at most) $n + 1$ columns. Since each column contains n entries, we conclude that this step has complexity $O(n^3)$. Performing the “update” operation in step 5 of the algorithm has complexity $O(dn^2)$. Step 6 implies that, in the worst case, the algorithm repeats steps 4 and 5 at most $(d \frac{n(n-1)}{2} - d)$ times. The reason is as follows. Since the algorithm starts with the $C_2^n = \frac{n(n-1)}{2}$ obvious syzygies and each has degree $\leq d$, the (worst case) total degree of the syzygies at the beginning of the algorithm is $d \frac{n(n-1)}{2}$. The algorithm ends only when the total degree is d . If each repetition of steps 4 and 5 reduces the total degree by 1, then the steps are repeated $(d \frac{n(n-1)}{2} - d)$ times. Thus, the total computational complexity appears to be $O(dn^5 + d^2n^4)$.

7. Implementation, experiments, comparison

We implemented the μ -basis algorithm presented in this paper and the one described in Song-Goldman Song and Goldman (2009). For the sake of simplicity, from now on, we will call these two algorithms HHK and SG. In Section 7.1, we discuss our implementation. In Section 7.2, we describe the experimental performance of both algorithms. An experimental timing corresponds to a point (d, n, t) , where d is the degree, n is the length of the input polynomial vector, and t is the time in seconds it took for our codes to produce the output. For each algorithm, we fit a surface through the experimental data points. Our fitting models are based on the theoretical complexities obtained in Section 6. In Section 7.3, we compare the performance of the two algorithms.

7.1. Implementation

We implemented both algorithms (HHK and SG) in the computer algebra system Maple Bernardin et al. (2015). The codes and examples are available on the web:

<http://www.math.ncsu.edu/~zchough/mubasis.html>

We post two versions of the code:

`program_rf` : over rational number field \mathbb{Q} .

`program_ff` : over finite field \mathbb{F}_p where p is an arbitrary prime number.

Now we explain how the two algorithms (HHK and SG) have been implemented.

- Although both algorithms could be written in a non-interpreted language such as the C-language, making the running time significantly shorter, we implemented both algorithms in Maple [Bernardin et al. \(2015\)](#) for the following reasons.
 1. Maple allows fast prototyping of the algorithms, making it easier to write and read the programs written in Maple.
 2. It is expected that potential applications of μ -bases will often be written in computer algebra systems such as Maple.
- Both algorithms contain a step in which null vectors are computed (step 2 of HHK and step 4 of SG). Although Maple has a built-in routine for computing a basis of the null space for the input matrix, we did not use this built-in routine because we do not need the entire null basis, but only a certain subset of basis vectors with desired properties, consisting of $n - 1$ vectors for HHK and a single vector for SG. For this reason, we implemented partial Gauss-Jordan elimination.
- For the rational field implementation of the SG algorithm, we produced the null vector in step 4 with integer entries in order to avoid rational number arithmetic (which is expensive due to integer gcd computations) in the subsequent steps of the algorithm.
- Dense representations of matrices were used for both algorithms. As shown in Remark 33, it is easy to exploit sparse representations for HHK, but it was not clear how one could exploit sparse representations for SG. Thus, in order to ensure fair comparison, we used dense representations for both algorithms.

7.2. Timing and fitting

We explain the setup for our experiments so that the timings reported here can be reproduced independently.

- The programs were executed using Maple 2015 version running on Apple iMac (Intel i7-2600, 3.4 GHz, 16GB).
- The inputs were randomly-generated: for various values of d and n , the coefficients were taken randomly from \mathbb{F}_5 , with a uniform distribution.
- In order to get reliable timings, especially when the computing time is small relative to the clock resolution, we ran each program several times on the same input and computed the average of the computing times.
- The execution of a program on an input was cut off if its computing time exceeded 120 seconds.

Figure 1 shows the experimental timing for the HHK algorithm, while Figure 2 shows the experimental timing for the SG algorithm. The algorithms were run on randomly-generated examples with specified d and n , and they ran in time t . For each figure, the axes represent the range of values $d = 3, \dots, 200$, $n = 3, \dots, 200$, and $t = 0, \dots, 120$, where t is the timing in seconds. Each dot (d, n, t) represents an experimental timing.

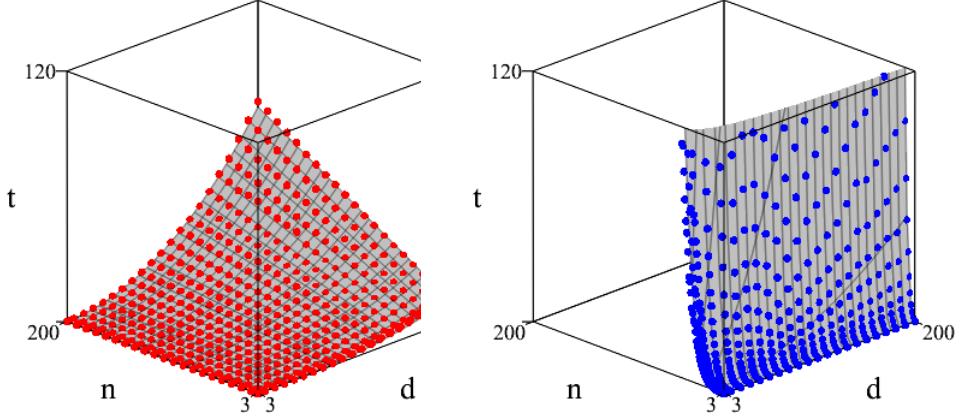


Figure 1: HHK algorithm timing

Figure 2: SG algorithm timing

The background gray surfaces are fitted to the experimental data. The fitting model is based on the theoretical complexities from Section 6. The fitting was computed using least squares. For HHK, based on Theorem 32, we chose a model for the timing, $t = \alpha_1 d^2 n + \alpha_2 d^3 + \alpha_3 n^2$, where α 's are unknown constants to be determined. After substituting the experimental values (d, n, t) , we obtain an over-determined system of linear equations in the α 's. We find α 's that minimize the sum of squares of errors. For SG, we used the same procedure with the timing model $t = \beta_1 d n^5 + \beta_2 d^2 n^4$ based on Remark 34.

We generated the following functions:

$$t_{HHK} \approx 10^{-6} \cdot (7.4 d^2 n + 1.2 d^3 + 1.2 n^2) \quad (24)$$

$$t_{SG} \approx 10^{-7} \cdot (2.6 d n^5 + 0.6 d^2 n^4) \quad (25)$$

For our experimental data, the residual standard deviation for the HHK-timing model (24) is 0.686 seconds, while the residual standard deviation for the SG-timing model (25) is 11.886 seconds.

We observe from Figures 1 and 2 that for a fixed d , the HHK algorithm's dependence on n is almost linear, while the SG algorithm's dependence on n is highly nonlinear. In fact, for the latter, the dependence is so steep that the algorithm was unable to terminate in under 120 seconds for most values of n , thus explaining the large amount of blank space in Figure 2. For a fixed n , the HHK algorithm's dependence on d is nonlinear, while the SG algorithm's dependence on d is almost linear.

7.3. Comparison

Two pictures below represent performance comparisons.

- Figure 3 shows the fitted surfaces from Figures 1 and 2 on the same graph. The axes represent the range of values $n = 3, \dots, 200$, $d = 3, \dots, 200$, and $t = 0, \dots, 120$, where t is the timing of the algorithms in seconds.

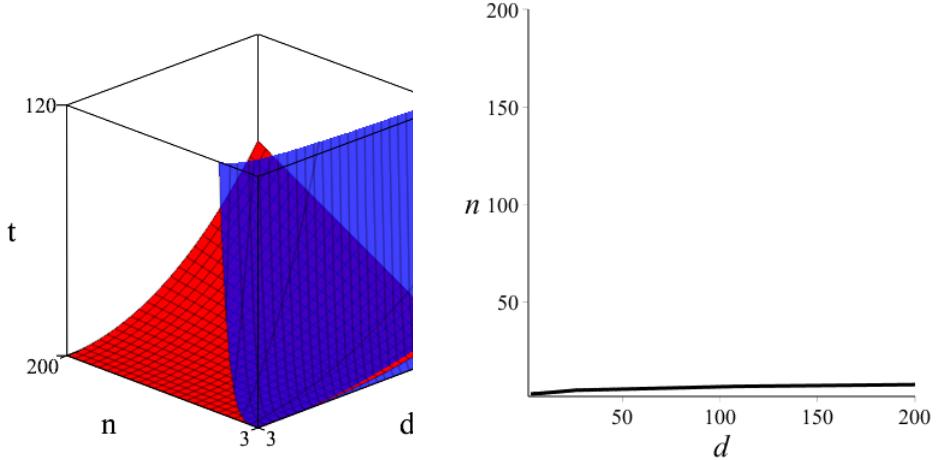


Figure 3: HHK (red) and SG (blue).

Figure 4: Tradeoff graph

- Figure 4 shows a tradeoff graph for the two algorithms. The curve in the figure represents values of d and n for which the two algorithms run with the same timing. Below the curve, the SG algorithm runs faster, while above the curve, the HHK algorithm runs faster. The ratio of the dominant terms in the fitted formulae is $d : n^4$. This ratio manifests itself in the shape of the tradeoff curve presented in Figure 4.

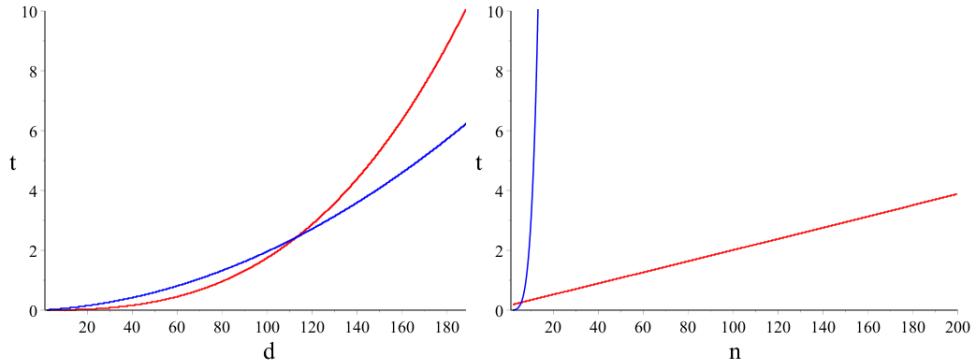


Figure 5: $n = 7$

Figure 6: $d = 50$

From Figure 3, we observe that for a fixed d , as n increases the HHK algorithm vastly outperforms the SG algorithm. In contrast, for a fixed value of n , as d increases the SG algorithm outperforms the HHK algorithm. The order by which SG runs faster is less than the order by which HHK runs faster for fixed d and increasing n . We underscore this observation by displaying two-dimensional slices of Figure 3. Figure 5 represents the slice in the d -direction with $n = 7$, while Figure 6 represents the slice in the n -direction with $d = 50$. As before, HHK is represented by red and SG by blue.

8. Discussion

In this section, we elaborate on some topics that were briefly discussed in the Introduction and Section 2 and discuss a natural generalization of the μ -basis computation problem – a problem of computing minimal bases of the kernels of $m \times n$ polynomial matrices.

The original definition and proof of existence: The original definition of a μ -basis appeared on p 824 of a paper by Cox, Sederberg, and Chen Cox et al. (1998b) and is based on the “sum of the degrees” property (Statement 2 of Proposition 4). The definition also mentions an equivalent “reduced representation” (Statement 4 of Proposition 4). The proof of the existence theorem (Theorem 1 on p. 824 of Cox et al. (1998b)) appeals to the celebrated Hilbert Syzygy Theorem Hilbert (1890) and utilizes Hilbert polynomials, which first appeared in the same paper Hilbert (1890) under the name of characteristic functions. The definition of μ -basis in terms of the degrees, given in Cox et al. (1998b), is compatible with the tools that have been chosen to show its existence.

The homogeneous version of the problem: It is instructive to compare the inhomogeneous and homogenous versions of the problem. In fact, in order to invoke the Hilbert Syzygy Theorem in the proof of the existence of a μ -basis, Cox, Sederberg, and Chen restated the problem in the homogeneous setting (see pp. 824-825 of Cox et al. (1998b)).

Let $\hat{a} = [\hat{a}_1(x, y), \dots, \hat{a}_n(x, y)]$ be a row vector of n homogeneous polynomials over a field \mathbb{K} , each of which has the same degree. As before, a syzygy of \hat{a} is a column vector $h = [h_1(x, y), \dots, h_n(x, y)]^T$ of polynomials (not necessarily homogeneous), such that $\hat{a}h = 0$. The set $\text{syz}(\hat{a})$ is a module over $\mathbb{K}[x, y]$, and the Hilbert Syzygy Theorem implies that it is a free module of rank $n - 1$ possessing a homogeneous basis. Let $n - 1$ homogeneous polynomial vectors $\hat{u}_1(x, y), \dots, \hat{u}_{n-1}(x, y)$ comprise an *arbitrary* homogeneous basis of $\text{syz}(\hat{a})$. Define dehomogenizations: $a(s) = [a_1(s), \dots, a_n(s)]$, where $a_i(s) = \hat{a}_i(s, 1) \in \mathbb{K}[s]$, $i = 1, \dots, n$ and $u_j(s) = \hat{u}_j(s, 1) \in \mathbb{K}[s]^n$, $j = 1, \dots, n - 1$. An argument, involving Hilbert polynomials on p. 825 of Cox et al. (1998b), shows that u_1, \dots, u_{n-1} is a μ -basis of $\text{syz}(a)$.

Let us now start with a polynomial vector $a(s) = [a_1(s), \dots, a_n(s)] \in \mathbb{K}[s]^n$ of degree d in the sense of Definition 2, and consider its homogenization $\hat{a} = [\hat{a}_1(x, y), \dots, \hat{a}_n(x, y)]$, where $\hat{a}_i(x, y) = y^d a_i\left(\frac{x}{y}\right)$, $i = 1, \dots, n$. It is not true that homogenization of an arbitrary basis of $\text{syz}(a)$ produces a basis of $\text{syz}(\hat{a})$. Indeed, let u_1 and u_2 be the columns of matrix M in Example 6. Then $\widehat{u_1 + u_2}$ and $\widehat{u_2}$ is a basis of $\text{syz}(a)$, with each vector having degree 3. Their homogenizations $\widehat{u_1 + u_2}$ and $\widehat{u_2}$ are homogeneous polynomial vectors of degree 3, and, therefore, they can not possibly generate a homogeneous vector $\hat{u}_1(x, y) = y u_1\left(\frac{x}{y}\right) = [-x, y, x - y]^T$ of degree 1, which clearly belongs to $\text{syz}(\hat{a})$. A rather simple argument that utilizes the “reduced representation” property (Statement 4 of Proposition 4) can be used to show that for an arbitrary non-zero vector $a \in \mathbb{K}[s]^n$, homogenization of any μ -basis of $\text{syz}(a)$ produces a homogeneous basis of $\text{syz}(\hat{a})$.

The above discussion can be summarized in the following statement: the set of homogeneous bases of $\text{syz}(\hat{a})$ is in one-to-one correspondence with the set of μ -bases of $\text{syz}(a)$, where $a \in \mathbb{K}[s]$ is the dehomogenization of $\hat{a} \in \mathbb{K}[x, y]$. Therefore, the algorithm developed in this paper can be used to compute homogeneous bases of $\text{syz}(\hat{a})$.

μ -basis algorithms and gcd computation: In contrast to the algorithm developed by Song and Goldman in Song and Goldman (2009), the algorithm presented in this paper produces a μ -basis even when the input vector a has a non-trivial greatest common divisor. Moreover, once a μ -basis is computed, one can *immediately* find $\text{gcd}(a)$ using Statement 5 of Proposition 4. Indeed, let h

denote the outer product of a μ -basis u_1, \dots, u_{n-1} . If M is the matrix generated by the algorithm, then $h_i = (-1)^i |M_i|$, where M_i is an $(n-1) \times (n-1)$ submatrix of M constructed by removing the i -th row. By Statement 5 of Proposition 4, there exists a non-zero $\alpha \in \mathbb{K}$ such that

$$a = \alpha \gcd(a) h.$$

Let $i \in \{1, \dots, n\}$ be such that a_i is a non-zero polynomial. Then $\gcd(a)$ is computed by long division of a_i by h_i and then dividing the quotient by its leading coefficient to make it monic. In comparison, the algorithm developed in Song and Goldman (2009) produces a μ -basis of a multiplied by $\gcd(a)$. From the output of this algorithm and Statement 5 of Proposition 4, one finds $\gcd(a)^{n-2}$. Song and Goldman discuss how to recover $\gcd(a)$ itself by *repeatedly* running their algorithm. They also run computational experiments to compare the efficiency of computing \gcd by iterating the SG μ -basis algorithm versus the standard Euclidean algorithm. Investigation of the efficiency of computing \gcd by using the HHK μ -basis algorithm and a long division can be a subject of a future work.

Kernels of $m \times n$ polynomial matrices: A natural generalization of the μ -basis problem is obtained by considering kernels, or nullspaces, of $m \times n$ polynomial matrices of rank m . A basis of the nullspace is called minimal if the “minimal degree” Statement 2 of Proposition 4 is satisfied (with $n-1$ replaced by $n-m$). One can easily adapt the argument in the proof of Theorem 2 in Song and Goldman (2009) to show that, in this more general setting, Statement 2 is equivalent to the “independence of the leading vectors” Statement 1 and to the “reduced representation” Statement 4 of Proposition 4. One can also show with an example that the “sum of the degrees” Statement 3 (with the degree of a polynomial matrix defined to be the maximum of the degrees of its entries) is no longer equivalent to Statements 1 and 4. There is a large body of work on computing minimal bases (see for instance Beelen (1987), Antoniou et al. (2005), Zhou et al. (2012) and references therein). This research direction seems to be developing independently of the body of work devoted to μ -bases. The algorithm presented in this paper can be generalized to compute minimal bases of the kernels of $m \times n$ polynomial matrices. The details and comparison with existing algorithms will be the subject of a forthcoming work.

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