

A GENERALIZATION OF AN INTEGRABILITY THEOREM OF DARBOUX

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ABSTRACT. In his monograph “Systèmes Orthogonaux” [4] Darboux stated three theorems providing local existence and uniqueness of solutions to first order systems of the type

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha(x, u(x)), \quad i \in I_\alpha \subseteq \{1, \dots, n\}.$$

For a given point $\bar{x} \in \mathbb{R}^n$ it is assumed that the values of the unknown u_α are given locally near $\bar{x}$ along $\{x \mid x_i = \bar{x}_i \text{ for each } i \in I_\alpha\}$. The more general of the theorems, Théorème III, was proved by Darboux only for the cases $n = 2$ and 3 .

In this work we formulate and prove a generalization of Darboux’s Théorème III which applies to systems of the form

$$\mathbf{r}_i(u_\alpha)|_x = f_i^\alpha(x, u(x)), \quad i \in I_\alpha \subseteq \{1, \dots, n\}$$

where $\mathcal{R} = \{\mathbf{r}_i\}_{i=1}^n$ is a fixed local frame of vector fields near $\bar{x}$. The data for u_α are prescribed along a manifold Ξ_α containing $\bar{x}$ and transverse to the vector fields $\{\mathbf{r}_i \mid i \in I_\alpha\}$. We identify a certain Stable Configuration Condition (SCC). This is a geometric condition that depends on both the frame $\mathcal{R}$ and on the manifolds Ξ_α ; it is automatically met in the case considered by Darboux [4]. Assuming the SCC and the relevant integrability conditions are satisfied, we establish local existence and uniqueness of a C^1 -solution via Picard iteration for any number of independent variables n .

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1. INTRODUCTION

Darboux, in Chapitre I, Livre III in his monograph “Systèmes Orthogonaux” [4], stated three integrability theorems (“Théorèmes I-III”) for certain types of first order systems of PDEs. The theorems apply to systems of the form

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha(x, u(x)) \tag{1}$$

where $u = (u_1, \dots, u_m)$ denotes the vector of unknown functions, the independent variables $x = (x_1, \dots, x_n)$ range over an open set about a fixed point $\bar{x} \in \mathbb{R}^n$, and the f_i^α are given C^1 -functions from an appropriate open subset of $\mathbb{R}^{n+m}$ to $\mathbb{R}$. The data for these systems consist of C^1 -functions g_α prescribing the unknowns u_α along certain affine subspaces through the point $\bar{x} \in \mathbb{R}^n$.

For each $\alpha = 1, \dots, m$, we let I_α be the set of all indices $i \in \{1, \dots, n\}$ for which the system contains the equation $\partial_{x_i} u_\alpha = f_i^\alpha(x, u(x))$. The cases covered by Darboux's three theorems can then be described as follows:

Théorème I applies to determined systems, which are characterized by the requirement that $|I_\alpha| = 1$ for all $\alpha = 1, \dots, m$. For each α , letting $I_\alpha = \{i_\alpha\}$, the data for u_α are prescribed near $\bar{x}$ along the hyperplane $\{x \mid x_{i_\alpha} = \bar{x}_{i_\alpha}\}$. In the special case that i_α is the same index for all α , Darboux's Théorème I reduces to the standard local existence and uniqueness result for ODEs with parameters (see [5]).

Théorème II is the PDE version of Frobenius' theorem for completely overdetermined systems. This situation is characterized by $|I_\alpha| = n$ for all $\alpha = 1, \dots, m$, i.e. the derivatives of all unknowns are prescribed in all coordinate directions. In this case, the data prescribes the value of each u_α at the point $\bar{x}$. The integrability conditions require the partial derivatives given by the system to be consistent with equality of 2nd order mixed partial derivatives. Under these conditions Darboux's Théorème II guarantees a unique local solution of the PDE system with the assigned data. As noted by Darboux ([4], p. 326) this result is well-known; the novelty is in his proof which is by induction on the number of independent variables and exploits Théorème I.

Théorème III applies to the general case where the elements as well as the cardinality of I_α may vary with α . The data are assigned as follows: if $I_\alpha = \{i_1, \dots, i_{p_\alpha}\}$, we prescribe a function g_α along the affine subspace $\Xi_\alpha := \{x \mid x_{i_j} = \bar{x}_{i_j}, 1 \leq j \leq p_\alpha\}$, and require that $u_\alpha|_{\Xi_\alpha} = g_\alpha$. Under the appropriate integrability conditions, detailed below in Section 2.1, Darboux's Théorème III guarantees a unique local solution of the PDE system with the assigned data.

We note that Théorèmes I and II are particular cases of Théorème III. Another special case of systems, for which the index sets I_α are the same for all α , was addressed separately in [1].

Darboux stated his Théorème III for any number of independent variables. He provided a proof only in the cases with two and three independent variables ($n = 2$ or 3), which sufficed for his investigation of triply orthogonal systems in [4]. For $n = 2$ his proof of Théorème III used Théorème I; for $n = 3$ he used both the result for $n = 2$ as well as his Théorème I. Given Darboux's partial proof, it is natural to try and establish his Théorème III via induction on the number n of independent variables. While this is possible, we have been able to do so only through an involved, combinatorial argument (see our unpublished note [2]). Furthermore, this inductive approach does not apply to the more general situation we consider in the present paper. Instead, we shall provide a direct proof that applies to more general systems with any number of independent variables.

Our results generalize Darboux's Théorème III in two ways:

- (i) The unknowns may be differentiated along vector fields in a fixed frame $\mathcal{R} = \{\mathbf{r}_i\}_{i=1}^n$ defined near $\bar{x}$. That is, for each $\alpha = 1, \dots, m$, there is an index set $I_\alpha \subseteq \{1, \dots, n\}$ such that the system contains the equations

$$\mathbf{r}_i(u_\alpha)|_x = f_i^\alpha(x, u(x)) \quad \text{for each } i \in I_\alpha. \quad (2)$$

As in Darboux's Théorème III, the elements and cardinality of the index sets I_α may vary with α .

- (ii) The prescribed data g_α for the unknown u_α may be given along a manifold Ξ_α through the point $\bar{x}$ which is transverse to the vector fields $\mathbf{r}_i$ with $i \in I_\alpha$.

The claim is that, under the appropriate integrability conditions (generalizing those of Darboux's Théorème III), the PDE system (2) has a unique local solution which takes on the assigned data. A precise formulation is provided in our Theorem 1 below.

However, our proof requires what we refer to as a *Stable Configuration Condition* (SCC) to be satisfied. The formulation of the SCC is somewhat technical (see Section 2.2 and Definitions 3.1-3.3 below). Roughly speaking, this condition is required to guarantee that the natural Picard iteration scheme is well defined. We note that the validity of the SCC depends on both the frame $\mathcal{R}$ and on the relative location of the manifolds Ξ_α that carry the data; see Section 2.2 below for a concrete example. Also, it is immediate to verify that the SCC is met in the setting of Darboux's original treatment where $\mathbf{r}_i \equiv \partial_{x_i}$, $i = 1, \dots, n$, and $\Xi_\alpha = \{x \mid x_i = \bar{x}_i, i \in I_\alpha\}$, see Remark 3.4.

Concerning regularity, we assume that the frame $\{\mathbf{r}_i\}_{i=1}^n$, the functions f_i^α , the manifolds Ξ_α , and the data g_α are all C^1 . A solution refers to a C^1 function $u = (u_1, \dots, u_m)$ which satisfies the PDEs and the data in a classic, pointwise manner on a neighborhood of the given point $\bar{x}$.

We note that, even in an analytic setting, the form of the system and the data assignment in Darboux's theorems precludes any direct application of the Cauchy-Kowalvskaya or the Cartan-Kähler theorems. The Cauchy-Kowalvskaya theorem would only apply to determined systems of the form (2) where I_α is the same singleton for all α , i.e., the case of an analytic ODE system, possibly with parameters.

The Cartan-Kähler theorem applies to overdetermined system of analytic¹ PDEs (and more generally to exterior differential systems). In the analytic setting, under conditions (viii)-(x) of Theorem 1 below, the Cartan-Kähler theorem asserts that an analytic solution of (2) exists locally, and that the general solution depends on a certain number of arbitrary functions of certain numbers of variables. Moreover, these arbitrary functions are

¹In [11] it has been shown, by involving a rather technical machinery, that the Cartan-Kähler theorem can be applied to systems of a certain special type (involutive, hyperbolic) in the C^∞ setting.

uniquely determined by prescribing some combination of the unknowns of the system and their derivatives along certain submanifolds (see [6] [Theorem 7.3.3] and discussion on p. 87 in [3]). However, these data are different from the data considered in the present paper: in the setting of the Cartan-Kähler theorem, the data manifolds of equal dimension coincide, and lower dimensional initial manifolds are contained in the higher dimensional ones. These data manifolds are determined post-factum, in the process of building up the graph of a solution dimension-by-dimension. Since the solutions are built in the jet space, the values of some combination of the unknown functions and their derivatives are prescribed along these data manifolds. For example, in the simple case of the determined system $u_x = v, v_y = u$ for two unknown functions of two variables, an application of the Cartan-Kähler theorem will assert the existence of the unique solution near the origin with prescribed values of u and v at the origin, and prescribed values of the derivatives u_y and v_x along a common non-characteristic curve. In contrast, in the setting of Darboux's Théorème I, the data manifolds are distinct and prescribed in advance.

The rest of the present paper is organized as follows. In Section 2.1 we review Darboux's original Théorème III and discuss the partial proof provided by Darboux. We also indicate how our approach in this paper differs from that of Darboux. Section 2.2 considers a simple system of equations to highlight the role of the Stable Configuration Condition (SCC): for a determined system of two equations for two unknowns in the plane, we show how the relative location of the two data manifolds Ξ_1 and Ξ_2 can yield radically different behavior in terms of the domains of definition of the natural Picard iterates. Finally, in Section 3 we formulate and prove our Theorem. A key part of the proof is a technical lemma about the "restricted" system obtained by considering the same set of equations as in the original system, but restricted to certain sub-manifolds defined in terms of the frame vector fields $\mathbf{r}_i$; see Lemma 3.6 below.

2. REVIEW OF DARBOUX'S WORK AND THE STABLE CONFIGURATION CONDITION (SCC)

2.1. Darboux's setup and result. We first consider the situation addressed by Darboux in his Théorème III: for each unknown u_α , the system consists of the equations

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha(x, u(x)) \quad \text{for } i \in I_\alpha \subseteq \{1, \dots, n\}, \quad (3)$$

where f_α^i are C^1 functions on $\mathbb{R}^{n+m}$. Setting

$$\Xi_\alpha := \{x \mid x_i = \bar{x}_i \text{ for } i \in I_\alpha\}, \quad (4)$$

we prescribe the data

$$u_\alpha|_{\Xi_\alpha} = g_\alpha, \quad (5)$$

for a given C^1 function $g_\alpha : \Xi_\alpha \rightarrow \mathbb{R}$.

Next consider the integrability conditions which need to be imposed. Let u_α be an unknown for which the system prescribes two distinct partial derivatives, say

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha(x, u(x)) \quad \text{and} \quad \partial_{x_j} u_\alpha(x) = f_j^\alpha(x, u(x))$$

where $i \neq j$ and $i, j \in I_\alpha$. The derivatives prescribed by the system need to be consistent with equality of mixed partial derivatives. That is, the expressions

$$\partial_{x_j x_i}^2 u_\alpha(x) = \partial_{x_j} f_i^\alpha(x, u(x)) + \sum_{\beta=1}^m \partial_{u_\beta} f_i^\alpha(x, u(x)) \partial_{x_j} u_\beta(x) \quad (6)$$

and

$$\partial_{x_i x_j}^2 u_\alpha(x) = \partial_{x_i} f_j^\alpha(x, u(x)) + \sum_{\beta=1}^m \partial_{u_\beta} f_j^\alpha(x, u(x)) \partial_{x_i} u_\beta(x) \quad (7)$$

should agree. Since the system (1) may not prescribe all the partials $\partial_{x_j} u_\beta$ and $\partial_{x_i} u_\beta$ appearing on the right-hand sides of (6) and (7), this puts constraints on which dependent variables u_β the functions f_i^α and f_j^α may depend on. This is brought out in the following example which is the simplest case of an overdetermined system where Darboux's Théorème III applies.

Example 2.1. *Consider a system of 3 equations for 2 unknowns in 2 independent variables. Let the unknowns be u and v , the independent variables be x and y , and assume that the equations are*

$$u_x = f(x, y, u, v) \quad (8)$$

$$v_x = \phi(x, y, u, v) \quad (9)$$

$$v_y = \psi(x, y, u, v). \quad (10)$$

The data in this case take the form

$$u(\bar{x}, y) = g_1(y) \quad (11)$$

$$v(\bar{x}, \bar{y}) = g_2, \quad (12)$$

where g_1 is a given function and g_2 is a given constant. The integrability condition is imposed to ensure that the prescription of the two partial derivatives of the unknown v is consistent with the equality of the partial derivatives $(v_x)_y = (v_y)_x$. To derive these conditions we expand $\partial_y[\phi(x, y, u, v)] = \partial_x[\psi(x, y, u, v)]$ applying the chain rule, to obtain

$$\phi_y + \phi_u u_y + \phi_v v_y = \psi_x + \psi_u u_x + \psi_v v_x. \quad (13)$$

We next substitute the derivatives given by the system (8)-(10) into (13). However, the system does not provide an expression for u_y , and we must therefore impose the condition

$$\phi_u = 0. \quad (14)$$

All other partial derivatives of u and v appearing in (13) are prescribed by (8)-(10), and we obtain the condition:

$$\phi_y + \phi_v \psi = \psi_x + \psi_u f + \psi_v \phi. \quad (15)$$

Conditions (14) and (15) comprise the integrability conditions for the system (8)-(10). If these conditions hold as identities in an (x, y, u, v) -neighborhood of $(\bar{x}, \bar{y}, g_1(\bar{y}), g_2)$, then Darboux's Théorème III guarantees the existence of a unique local C^1 solution $(u(x, y), v(x, y))$ to (8)-(10) near $(\bar{x}, \bar{y})$ taking on the data (11)-(12).

In the general setting of the system (3), the integrability conditions require that, whenever $\alpha \in \{1, \dots, m\}$ and $i, j \in I_\alpha$ with $i \neq j$, then the following should hold: for all $\beta \in \{1, \dots, m\}$ with $i \notin I_\beta$ we have

$$\partial_{u_\beta} f_j^\alpha = 0$$

and

$$\partial_{x_i} f_j^\alpha + \sum_{\beta: i \in I_\beta} (\partial_{u_\beta} f_j^\alpha) f_i^\beta \equiv \partial_{x_j} f_i^\alpha + \sum_{\beta: j \in I_\beta} (\partial_{u_\beta} f_i^\alpha) f_j^\beta.$$

If these conditions hold as identities in a neighborhood of $(\bar{x}, g(\bar{x}))$, then Darboux's Théorème III guarantees the existence of a unique local C^1 solution $u(x)$ to (3) in a neighborhood of $\bar{x}$ that takes on the data (5).

Due to the particular structure of the systems under consideration (viz., each equation contains a single derivative for which it is solved), it is natural to base a proof of existence on Picard iteration. Indeed, it is immediate to write down a functional map for which any solution of (3)-(5) must be a fixed point (see (31) below).

Now, in the particular situation addressed by Darboux's Théorème I (for determined systems), one can verify that a fixed point exists and provides a solution of the original system. This is how Darboux established his Théorème I. However, in the more general situation of overdetermined systems addressed by his Théorème III, such an approach appears to be more challenging. This circumstance might explain why Darboux [4] did not provide a general proof based directly on Picard iteration for his Théorème III. Instead, for $n = 2$ case he exploited Théorème I. For $n = 3$ case he identified sub-systems that can be treated by Théorème I or by $n = 2$ case. These sub-systems are solved in a "right" order so that the solution of one sub-system provides initial data to the next. Darboux states that the general proof will be too technical and, therefore, he restricts himself to the cases of $n = 2$ and $n = 3$ as they are sufficient for the applications he considers.² Darboux's treatment of $n = 3$ case suggests an inductive proof for an arbitrary n , which we accomplish in [2]. The inductive proof turns out, indeed, to be quite technical. Moreover, the same approach can not be applied to

²"Pour établir cette importante proposition, sans employer un trop grand luxe de notations, nous nous bornerons au cas de deux et de trois variables indépendantes, qui suffira d'ailleurs pour les applications que nous avons en vue." [4] p. 336.

a more general problem considered here, because even if the initial system satisfies the hypothesis of our Theorem 1, the sub-systems appearing in the inductive proof may not satisfy these hypothesis.

In fact, as we shall show below, it is possible to provide a direct argument based on Picard iteration also for overdetermined systems of the more general type (2) with any number of independent variables. The key observation is that it suffices to consider a certain “restricted” system which consists of the same equations as the original system, but now required to hold only along certain submanifolds containing the given point $\bar{x}$. For this, seemingly weaker, restricted system, we establish existence of a solution $\tilde{u}$ via Picard iteration. This result (Lemma 3.6 below) is the first main new ingredient in our approach. (We note that this part of the argument does not involve any use of the integrability conditions.)

In the more general setting described in (i) and (ii) above, the argument of Lemma 3.6 becomes complicated by the need to work with different coordinate systems for different components u_α of the solution u . This is where we have found it necessary to introduce the Stable Configuration Condition (SCC) illustrated in the next section. We stress that, in the more general setting, the SCC is relevant already for generalizing Darboux’s Théorème I (i.e., the it is not about determinacy or over-determinacy of the system). However, in the original setting described by Darboux [4], the SCC condition is trivially satisfied (Remark 3.4). Therefore, our paper contains a direct proof of Darboux’s Théorème III, for an arbitrary number of variables.

Before considering an example explaining the relevance of the SCC, we outline the last step of the proof: showing that the solution of the restricted system $\tilde{u}$ is a solution of the original system on a *full* $\mathbb{R}^n$ -neighborhood of $\bar{x}$. This is accomplished by showing that the quantities

$$A_i^\alpha(x) = \mathbf{r}_i(\tilde{u}_\alpha)|_x - f_i^\alpha(x, \tilde{u}(x)), \quad 1 \leq \alpha \leq m, i \in I_\alpha,$$

satisfy certain linear, homogeneous equations which form a restricted system of the type covered by our Lemma. Only at this point are the integrability conditions used. As this latter system admits the trivial solutions $A_i^\alpha \equiv 0$ on a full neighborhood of $\bar{x}$, it follows from the uniqueness part of the Lemma that $\mathbf{r}_i(\tilde{u}_\alpha)|_x = f_i^\alpha(x, \tilde{u}(x))$ for all x near $\bar{x}$, thereby completing the proof.

2.2. The Stable Configuration Condition (SCC). To illustrate the relevance of this condition we analyze two simple examples that illustrate the existence and uniqueness parts of our main result.

2.2.1. Example 1: Iteration scheme. Consider the following system for the two unknown scalar functions $u(x, y)$ and $v(x, y)$ of two independent variables:

$$u_x = v \tag{16}$$

$$v_y = u. \tag{17}$$

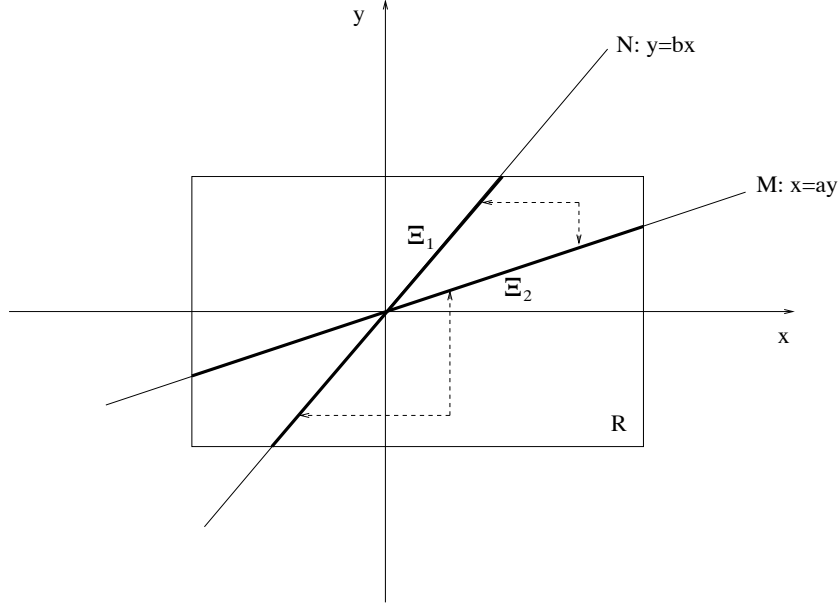


FIGURE 1. Stable configuration.

The system (16)-(17) is a determined system of the form (2) with $m = n = 2$, $u_1 = u$, $u_2 = v$, $\mathbf{r}_1 = \partial_x$, and $\mathbf{r}_2 = \partial_y$. We consider the system near the origin in the (x, y) -plane and let M and N be the two straight lines

$$M := \{(x, y) \mid x = ay\} \quad N := \{(x, y) \mid y = bx\},$$

where $0 < \frac{1}{a} < b$. We shall study separately the two cases:

- (a) The data for u are prescribed along a line segment $\Xi_1 \subset N$, and the data for v are prescribed along a line segment $\Xi_2 \subset M$.
- (b) Vice versa: u is prescribed along $\Xi_1 \subset M$ and v is prescribed along $\Xi_2 \subset N$.

Here and below we assume that the origin is an interior point for both Ξ_1 and Ξ_2 . Clearly, in both cases the transversality condition in (ii) above is met. Letting g and h be given scalar functions of a single argument and defined near zero, we consider two scenarios:

- (a) With $u(x, bx) = g(x)$ and $v(ay, y) = h(y)$, say, the natural iteration scheme is to set:

$$u^{(0)}(x, y) := g(x), \quad v^{(0)}(x, y) := h(y),$$

and then define

$$u^{(n+1)}(x, y) := g\left(\frac{y}{b}\right) + \int_{\frac{y}{b}}^x v^{(n)}(\xi, y) d\xi,$$

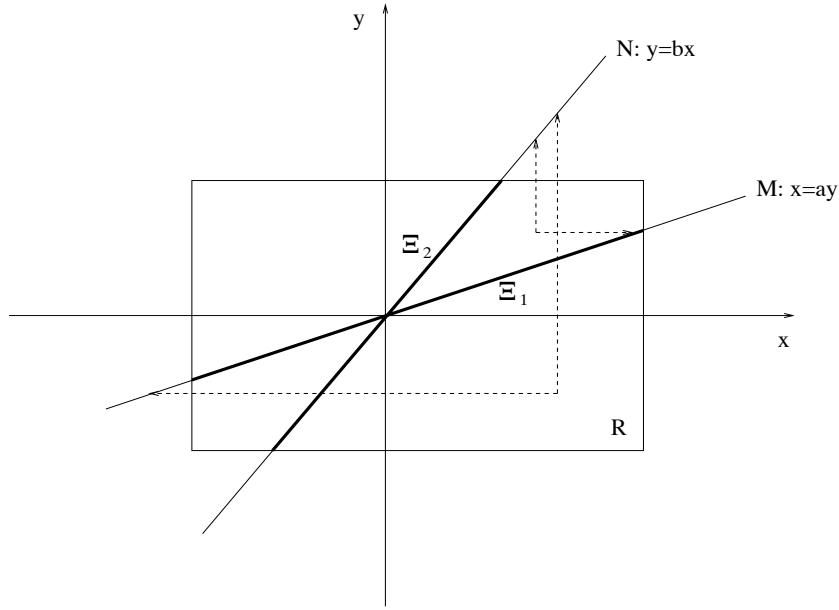


FIGURE 2. Not a stable configuration.

and

$$v^{(n+1)}(x, y) := h\left(\frac{x}{a}\right) + \int_{\frac{x}{a}}^y u^{(n)}(x, \eta) d\eta,$$

for $n \geq 0$. The first step of an existence proof based on this scheme is to specify Ξ_1 and Ξ_2 so that all iterates are defined on a fixed set. For this we may introduce the following domains of influence:

$$\Omega_1 := \{(x, y) \mid (x, y) \text{ is reachable from } \Xi_1 \text{ by flowing along } \mathbf{r}_1\},$$

and

$$\Omega_2 := \{(x, y) \mid (x, y) \text{ is reachable from } \Xi_2 \text{ by flowing along } \mathbf{r}_2\}.$$

It is evident (cf. Figure 1) that it is possible to choose Ξ_1 and Ξ_2 so that Ω_1 contains Ξ_2 and Ω_2 contains Ξ_1 , and, in this example, this property ensures that all iterates $(u^{(n)}, v^{(n)})$ are defined on the rectangle $R := \Omega_1 \cap \Omega_2$. Conversely, if R is any rectangular neighborhood of the origin with the property that its upper-right corner and its lower-left corner both lie between the lines M and N in the first and third quadrants, respectively, then $\Xi_1 := N \cap R$ and $\Xi_2 := M \cap R$ satisfies the above property and all iterates are defined on R . (Of course, we need to choose R small enough that the given functions g and h are defined on Ξ_1 and Ξ_2 , respectively.) The crucial property of the neighborhood R , is that whenever we start at a point in R and move toward N or M along the integral curves of $\mathbf{r}_1 = \partial_x$ or $\mathbf{r}_2 = \partial_y$, respectively, we remain within R until we meet N and M ,

respectively. We express this by saying that the *Stable Configuration Condition* (SCC) is met in this case. This is a key property that must be met in order that the iteration scheme is well-defined.

- (b) The situation changes if we instead prescribe u along a line segment $\Xi_1 \subset M$ and v along a line segment $\Xi_2 \subset N$, say, $u(ay, y) = g(y)$ and $v(x, bx) = h(x)$. The natural iteration scheme is then:

$$u^{(0)}(x, y) := g(y), \quad v^{(0)}(x, y) := h(x),$$

and then define

$$u^{(n+1)}(x, y) := g(ay) - \int_x^{ay} v^{(n)}(\xi, y) d\xi,$$

and

$$v^{(n+1)}(x, y) := h(bx) - \int_y^{bx} u^{(n)}(x, \eta) d\eta,$$

for $n \geq 0$. The trouble with this is that there is no choice of bounded segments Ξ_1 and Ξ_2 with the property that the domain of influence of each segment contains the other segment. Indeed, it is easily verified that given any bounded neighborhood U of the origin, there is always a point $(x_0, y_0) \in U$, such that, either in moving horizontally till intersecting M , or in moving vertically till intersecting N , one meets M or N , respectively, in a point outside U . The upshot is that there is no fixed, bounded neighborhood on which all iterates are defined, and the SCC is not met.

While g and h may only be defined locally near zero, it is natural, as a possible remedy, to consider extending them to all of $\mathbb{R}$, and then running the iteration above on all of $\mathbb{R}^2$. However, even assuming convergence to a solution of the original system, this raises another issue which is illustrated in a simpler setting by our next example: the limit might depend on the choice of extensions. As is clear from Example 2 below, a change in g or h away from the origin may influence the solution in a region closer to the origin than where the change was made. We have not been able to determine, for the system (16)-(17), if this influence is sufficiently “strong” that a change in data away from the origin will necessarily propagate (through the iterates of the scheme) all the way to the origin. This would signal a somewhat surprising manifestation of non-uniqueness. It appears to be a subtle issue that we plan to pursue further elsewhere.

The example above highlights the need for the SCC in order to obtain a well-defined iteration scheme. Before turning to the issue of uniqueness, we note that things are more involved in the general case covered by our main result. Specifically, for cases where an unknown u_α appears differentiated with respect to several variables (i.e., $|I_\alpha| > 1$), it requires more work to formulate the SCC (see Section 3).

2.2.2. *Example 2: Uniqueness.* This example shows that the uniqueness part of our Theorem might fail when the SCC is not met. It also provides a simple case to see how changes in data affect the solution. To describe this accurately, we introduce the following distinction: in the setting of Section 1 we say that

- *strong uniqueness* holds if there is a fixed neighborhood U of the point $\bar{x}$ with the property that any two solutions of the system, both of which are defined on U , agree on U ;
- *weak uniqueness* holds if, given any two solutions of the system, each of which is defined on an open set about the point $\bar{x}$, there is a neighborhood V (possibly depending on the given solutions) on which the two solutions agree.

The uniqueness claim in the Theorem below is that strong uniqueness holds under the SCC. In contrast, the following simple example shows that strong uniqueness might fail when the SCC is not met. For this, consider the trivial system

$$u_x = 0 \tag{18}$$

$$v_y = 0. \tag{19}$$

For a bounded open neighborhood U of the origin, let $\Xi_1 = M \cap U$ and $\Xi_2 = N \cap U$. Without loss of generality we may assume that U is chosen so that Ξ_1 and Ξ_2 are connected. We prescribe vanishing data $u \equiv 0$ along Ξ_1 and vanishing data $v \equiv 0$ along Ξ_2 . Clearly, the trivial solution $(u(x, y), v(x, y)) \equiv (0, 0)$ is a solution on U , satisfying this data.

Now, as in case (b) of Example 1 above, the SCC is not met. Thus, there is a point such that, either the horizontal or vertical line through it will meet M or N , respectively, outside U . As U is open it follows that there is either:

- an interval (y_0, y_1) such that every horizontal line $y = \tilde{y}$ with $\tilde{y} \in (y_0, y_1)$ intersects U but only meets M outside of U , or
- an interval (x_0, x_1) such that every vertical line $x = \tilde{x}$ with $\tilde{x} \in (x_0, x_1)$ intersects U but only meets N outside of U .

Suppose the former. Let F be a smooth function, say, with compact support contained in the interval (y_0, y_1) , and define $\hat{u}(x, y) := F(y)$ and $\hat{v}(x, y) \equiv 0$ for $(x, y) \in U$. If instead the latter situation occurs, we let G be a smooth function with compact support contained in the interval (x_0, x_1) and define $\hat{u}(x, y) \equiv 0$ and $\hat{v}(x, y) := G(x)$. In either case, the pair $(\hat{u}, \hat{v})$ provides a non-trivial solution to the system (18)-(19) on U also with the vanishing data $u \equiv 0$ prescribed along Ξ_1 and vanishing data $v \equiv 0$ prescribed along Ξ_2 . This shows that strong uniqueness fails.

This example also shows how a change in data along M or N can change a solution at points closer to the origin than the points at which the change is made.

On the other hand, we note that weak uniqueness does hold for the system (18)-(19), even though the SCC is not met.

Remark 2.2. *It would be of interest to know if weak uniqueness holds for more general systems in the absence of the SCC. As noted above, already for the system (16)-(17) we do not know the answer.*

Remark 2.3. *Returning to the system (16)-(17) above, we note that applying ∂_y to the first equation and using the second equation, yield the second order hyperbolic equation $u_{xy} = u$. Conversely, a solution of the latter equation yields, upon setting $v := u_x$, a solution to the system (16)-(17). Thus, at least at the level of C^2 -solutions, these are equivalent problems.*

Equations of the form $u_{xy} = F(x, y, u, u_x, u_y)$ have been studied extensively, starting with classical treatments by Riemann, Darboux, and Goursat. In particular, various types of boundary value problems have been considered; see [7, 8] and references therein. However, we are not aware of results that cover the particular situation above, i.e. with general data for u and u_x prescribed along two different non-characteristic curves through the origin, and a solution is sought on a full neighborhood of the origin.

3. STATEMENT AND PROOF

We start by stating and proving a Lemma about certain restricted systems; and then state and prove our main theorem.

We are given open sets $\Omega \subseteq \mathbb{R}^n$ and $\Upsilon \subseteq \mathbb{R}^m$ together with a C^1 frame $\{\mathbf{r}_i\}_{i=1}^n$ on Ω , and we fix a point $\bar{x} \in \Omega$. For all positive integers p , on all open subsets of $\mathbb{R}^p$, we will use the max norm, denoted by $\|\cdot\|_\infty$. The corresponding open ball of radius $\epsilon > 0$ about a point $y \in \mathbb{R}^p$ is denoted $B_\epsilon(y)$.

Throughout this section we use the following conventions: the integers i, j , and k satisfy $1 \leq i, j, k \leq n$ and index the vector fields in the frame (derivations). The integers α and β satisfy $1 \leq \alpha, \beta \leq m$ and index the unknown functions. For each index α , I_α will denote the set the indices of vector fields with respect to which the unknown function u_α is differentiated in the original system (2). Both the elements and the cardinality of I_α may vary with α . However, in order to avoid an extra index, whenever α is fixed, we simply write $I_\alpha = \{i_1, i_2, \dots, i_{p_\alpha}\}$, where it is assumed that $1 \leq i_1 < i_2 < \dots < i_{p_\alpha} \leq n$.

Finally, for each α we fix an $(n - p_\alpha)$ -dimensional C^1 submanifold Ξ_α of Ω that is transverse to the span of vector fields $\{\mathbf{r}_i \mid i \in I_\alpha\}$. We assume that the point $\bar{x}$ belongs to $\cap_{\alpha=1}^m \Xi_\alpha$ and that each Ξ_α is small enough to be covered by a single coordinate chart centered at $\bar{x}$. In other words, there exist C^1 -diffeomorphisms $\xi_\alpha: \mathcal{U}_\alpha \rightarrow \Xi_\alpha$, where $\mathcal{U}_\alpha \subset \mathbb{R}^{n-p_\alpha}$ is an open neighborhood of the origin and $\xi_\alpha(0) = \bar{x}$. The submanifold Ξ_α will be where the unknown function u_α is prescribed. We shall refer to the Ξ_α as *data manifolds*.

The goal is to establish existence of a solution to the system (2) when each unknown u_α is prescribed along its data manifold Ξ_α . Given the form of the equations, the most natural approach is to employ an iteration scheme, much

like what is done in standard ODE-theory: an iterate $u^{(N)}$ is substituted into the right hand side of each equation, followed by integrations with respect to the appropriate independent variables, to produce the the next iterate $u^{(N+1)}$.

However, to implement this approach in the present situation requires some care because different unknowns are integrated along different directions (i.e., along the integral curves of different vector fields), and also because several integrations are required to generate the iterate of the component u_α (whenever $p_\alpha > 1$).

Specifically, to have a well-defined scheme it is necessary that all components of the next iterate $u^{(N+1)}$ remain defined on *the same neighborhood* of $\bar{x}$ as where the components of $u^{(N)}$ were defined. In general, as the discussion in Section 2.2 shows, this may or may not be the case. We shall therefore introduce a *Stable Configuration Condition* (SCC), which amounts to the existence of such “stable neighborhoods.” As the data manifolds Ξ_α may need to be shrunk we have found it necessary to require that there are arbitrarily small such stable neighborhoods about $\bar{x}$.

To formulate the iteration scheme and the SCC we start by introducing the flows W_i^τ of the $\mathbf{r}_i$:

$$\frac{d}{d\tau} W_i^\tau(x) = \mathbf{r}_i|_{W_i^\tau(x)}, \quad W_i^0(x) = x \quad (i = 1, \dots, n).$$

As the vector fields $\mathbf{r}_i$ are C^1 , their flows are defined and C^1 on an $\mathbb{R} \times \mathbb{R}^n$ -neighborhood of $(0, \bar{x})$ (Theorem 2.6, p. 81 in [10]).

We need to ensure that the set of points reachable by starting from a data manifold Ξ_α and flowing along its corresponding vector fields $\mathfrak{R}_\alpha := \{\mathbf{r}_i \mid i \in I_\alpha\}$, i.e., roughly the “domain of influence of Ξ_α ,” is *the same set for each* α , $1 \leq \alpha \leq m$. In arranging this we are free to change the Ξ_α further (as long as they all contain the point $\bar{x}$), and also the amount by which to flow along each vector field. Note that, already for a fixed α , the latter amounts should be allowed to be different for different starting points in Ξ_α . (E.g., this is the case in Example 1 in Section 2.2.) To ensure this freedom we proceed as follows.

First, for each α , with $I_\alpha = \{i_1, i_2, \dots, i_{p_\alpha}\}$, there exists an open neighborhood $\Theta_\alpha \subset \mathbb{R}^n$ of the origin which is small enough that for each $t = (t_1, \dots, t_n) \in \Theta_\alpha$ the map $\psi_\alpha : \Theta_\alpha \rightarrow \mathbb{R}^n$ given by

$$\psi_\alpha(t) := W_{i_{p_\alpha}}^{t_{p_\alpha}} \cdots W_{i_2}^{t_2} W_{i_1}^{t_1} \xi_\alpha(t_{p_\alpha+1}, \dots, t_n). \quad (20)$$

is well-defined and C^1 . Note that the $n - p_\alpha$ last components of $t \in \Theta_\alpha$ parametrizes the data manifold Ξ_α via the map ξ_α introduced above.

Next, by assumption the manifold Ξ_α is everywhere transverse to the span of vector fields $\{\mathbf{r}_i \mid i \in I_\alpha\}$, and it follows from the C^1 -version of the inverse mapping theorem (Theorem 5.2, p. 13 in [10]) that, possibly after shrinking Θ_α , we may assume that ψ_α is a C^1 -diffeomorphism between Θ_α

and its image $\Omega_\alpha := \psi_\alpha(\Theta_\alpha) \subset \mathbb{R}^n$. Thus we now have m coordinate charts $(\Omega_\alpha, \psi_\alpha^{-1})$ near $\bar{x}$. From now on we shrink Ξ_α to $\Xi_\alpha \cap \Omega_\alpha$.

The ψ_α will be used to define the iteration map $u^{(N)} \mapsto \Phi[u^{(N)}] =: u^{(N+1)}$ (see (31) below) which, for each α , involves integrations along the vector fields $\{\mathbf{r}_i \mid i \in I_\alpha\}$ in the order prescribed by I_α . To be of relevance for the scheme the neighborhoods Θ_α therefore need to be “accessible” in the following sense.

Definition 3.1. *Let $p \in \{1, \dots, n\}$. An open neighborhood $\Theta \subset \mathbb{R}^n$ of the origin is called p -accessible if for each $t = (t_1, t_2, \dots, t_n) \in \Theta$, the piecewise linear path from $(0, \dots, 0, t_{p+1}, \dots, t_n)$ to t , with vertices*

$$\begin{aligned} & (0, \dots, 0, t_{p+1}, \dots, t_n), \\ & (t_1, 0, \dots, 0, t_{p+1}, \dots, t_n), \\ & (t_1, t_2, 0, \dots, 0, t_{p+1}, \dots, t_n), \\ & \vdots \\ & (t_1, t_2, \dots, t_n), \end{aligned}$$

belongs to Θ .

We can now introduce the following notion:

Definition 3.2 (Domain of influence). *Let Ξ_α be a submanifold of codimension p_α , transversal to the set of vector-fields $\mathfrak{R}_\alpha := \{\mathbf{r}_i \mid i \in I_\alpha\}$. We say that an open neighborhood Ω_α is a domain of influence of $(\Xi_\alpha, \mathfrak{R}_\alpha)$ if it is the image of a p_α -accessible neighborhood Θ_α under the C^1 -diffeomorphism ψ_α defined in (20).*

Thus, whenever Ω_α is a domain of influence of $(\Xi_\alpha, \mathfrak{R}_\alpha)$, then from every point $x \in \Omega_\alpha$ we can flow back to Ξ_α along integral curves of the vector fields $\{\mathbf{r}_i \mid i \in I_\alpha\}$ (in decreasing order of indexes) in a unique way, without leaving Ω_α . Equivalently, for all $x \in \Omega_\alpha$ there exists a unique point $x_\alpha \in \Xi_\alpha$ and a unique ordered p_α -tuple of real numbers $(\hat{t}_1, \dots, \hat{t}_{p_\alpha})$, such that

$$x = W_{i_{p_\alpha}}^{\hat{t}_{p_\alpha}} \dots W_{i_2}^{\hat{t}_2} W_{i_1}^{\hat{t}_1} x_\alpha \quad (21)$$

and such that for all t_i between 0 and $\hat{t}_i$, $i = 1, \dots, p_\alpha$, we also have

$$W_{i_{p_\alpha}}^{t_{p_\alpha}} \dots W_{i_2}^{t_2} W_{i_1}^{t_1} x_\alpha \in \Omega_\alpha \quad (22)$$

Condition (21) follows from the requirement that ψ_α is a diffeomorphism, while (22) follows from the requirement that Θ_α is p_α -accessible.

We finally use the notion of domain of influence to formulate the SCC:

Definition 3.3 (Stable Configuration Condition). *In the context described above we say that the frame $\mathfrak{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_n\}$ and the set of m submanifolds $\Xi := \{\Xi_1, \dots, \Xi_m\}$ is a stable configuration provided the following holds: for every $\epsilon > 0$ there exists an open neighborhood $\Omega_\epsilon \subset B_\epsilon(\bar{x})$ of $\bar{x}$ which*

is a domain of influence for each of the pairs $(\Xi_\alpha, \mathfrak{R}_\alpha)$, $\alpha = 1, \dots, m$. The neighborhoods Ω_ϵ are called stable neighborhoods.

The significant part of this definition is that the same neighborhood Ω_ϵ works for all $\alpha = 1, \dots, m$.

Remark 3.4. In Darboux's original setting the initial data submanifolds are the coordinate affine subspaces $\Xi_\alpha := \{x \mid x_{i_j} = \bar{x}_{i_j}, 1 \leq j \leq p_\alpha\}$, and the vector fields $r_{i_j} = \frac{\partial}{\partial x_{i_j}}$, $1 \leq j \leq p_\alpha$, are transversal coordinate vectors. It follows that the open balls $B_\epsilon(\bar{x})$ (in the sup metric) are stable and so the SCC is satisfied.

Remark 3.5. In the general setting, when an unknown function is differentiated with respect to several non-commuting vector-fields, we need to fix an order in which we move along an integral curves of various vector-fields. In this setting, the definitions of the domain of influence and the SCC depends on the choice of this order.

Lemma 3.6 below provides a key technical step and will be applied twice in the proof of the main theorem. It can be regarded as a “restricted” form of our main theorem, in which integrability conditions are not assumed and, consequently, we cannot conclude that the equations in the original system (2) have solutions defined on a full $\mathbb{R}^n$ -neighborhood of $\bar{x}$. Instead, we will only conclude that each equation is satisfied on a certain submanifold of Ω (which varies from equation to equation). To define these submanifolds, for each α , we introduce a sequence of manifolds $\Xi_\alpha^0, \dots, \Xi_\alpha^{p_\alpha}$ by:

$$\Xi_\alpha^j := \{\psi_\alpha(t) \mid t = (t_1, \dots, t_j, 0, \dots, 0, t_{p_\alpha+1}, \dots, t_n) \in \Theta_\alpha\}. \quad (23)$$

We observe that $\Xi_\alpha^0 \equiv \Xi_\alpha$, while Ξ_α^j is the set of points obtained by starting from a point in Ξ_α and then applying the flows $W_{i_1}^{t_1}, \dots, W_{i_j}^{t_j}$, in that order. In particular, $\Xi_\alpha^{p_\alpha} \equiv \Omega_\alpha$.

Lemma 3.6. Let $\Omega \subseteq \mathbb{R}^n$ and $\Upsilon \subseteq \mathbb{R}^m$ be open subsets. For $\alpha = 1, \dots, m$, let $I_\alpha = \{i_1, i_2, \dots, i_{p_\alpha}\}$ be an ordered set of indices: $1 \leq i_1 < i_2 < \dots < i_{p_\alpha} \leq n$. Assume the following:

- (i) $\mathfrak{R} = \{\mathbf{r}_i\}_{i=1}^n$ is a C^1 frame on Ω ;
- (ii) for each α , Ξ_α is an embedded C^1 manifold in Ω , and $\bar{x} \in \cap_\alpha \Xi_\alpha$;
- (iii) the manifold Ξ_α is of codimension p_α and is everywhere transverse to the span of $\{\mathbf{r}_i\}_{i \in I_\alpha}$;
- (iv) the frame $\mathfrak{R}$ and the set of manifolds $\Xi := \{\Xi_1, \dots, \Xi_m\}$ is a stable configuration near $\bar{x} \in \Omega$ according to Definition 3.3;
- (v) for each α , $g_\alpha : \Xi_\alpha \rightarrow \mathbb{R}$ is C^1 and bounded, and

$$\bar{g} := (g_1(\bar{x}), g_2(\bar{x}), \dots, g_m(\bar{x})) \in \Upsilon;$$

- (vi) for each α and each $i \in I_\alpha$, $f_i^\alpha : \Omega \times \Upsilon \rightarrow \mathbb{R}$ is uniformly bounded and continuous on $\Omega \times \Upsilon$ and also uniformly Lipschitz in the second argument.

Then there is a neighborhood $\tilde{\Omega} \ni \bar{x}$ on which there is a unique solution $u : \tilde{\Omega} \rightarrow \Upsilon$ to the system

$$\mathbf{r}_{i_j}(u_\alpha)|_x = f_{i_j}^\alpha(x, u(x)) \quad \text{for } 1 \leq \alpha \leq m, i_j \in I_\alpha, \text{ and } x \in \Xi_\alpha^j \cap \tilde{\Omega} \quad (24)$$

satisfying the data

$$u_\alpha(x) = g_\alpha(x) \quad \text{for } x \in \Xi_\alpha \cap \tilde{\Omega}. \quad (25)$$

Remark 3.7. Note that, in (24), we need to employ double indices i_j for the elements of the ordered set I_α since the position (the j th, say) of an index in I_α relates to which submanifold Ξ_α^j is considered.

Proof. Let L be a common Lipschitz constant for the functions f_i^α :

$$|f_i^\alpha(x, u) - f_i^\alpha(x, v)| \leq L\|u - v\|_\infty \quad \text{whenever } x \in \Omega, u, v \in \Upsilon. \quad (26)$$

We let M be a common bound for the functions f_i^α :

$$|f_i^\alpha(x, u)| \leq M \quad \text{for } (x, u) \in \Omega \times \Upsilon. \quad (27)$$

Choose $r > 0$ such that $\bar{B}_r(\bar{g})$ (the closed ball under the sup norm) is contained in Υ and put

$$\tilde{\Upsilon} := \bar{B}_r(\bar{g}). \quad (28)$$

Shrink Ω to $\Omega' \ni \bar{x}$ such that

$$|g_\alpha(x) - g_\alpha(\bar{x})| \leq \frac{1}{2}r \quad \text{for each } \alpha \text{ and for } x \in \Xi_\alpha \cap \Omega'. \quad (29)$$

Finally, using assumption (iv), we choose a bounded and stable neighborhood $\tilde{\Omega} \subset \Omega'$, with accessible neighborhoods Θ_α and diffeomorphisms $\psi_\alpha : \Theta_\alpha \rightarrow \tilde{\Omega}$, according to Definitions 3.1, 3.3. If necessary, we may shrink $\tilde{\Omega}$, and hence each Θ_α , so as to have

$$nL\|t\|_\infty \leq \frac{1}{2} \quad \text{and} \quad nM\|t\|_\infty \leq \frac{1}{2}r \quad \text{for all } t \in \Theta_\alpha, \alpha = 1, \dots, m. \quad (30)$$

Next, let $\mathcal{C}$ denote the set of continuous functions $\tilde{\Omega} \rightarrow \tilde{\Upsilon}$. Define a functional $\Phi : \mathcal{C} \rightarrow \mathcal{C}$ by defining its components according to

$$\begin{aligned} \Phi[u]_\alpha(x) &:= g_\alpha(\xi_\alpha) + \int_0^{t_1} f_{i_1}^\alpha(\psi_\alpha(s, 0, \dots, 0, t_{p_\alpha+}), u(\psi_\alpha(s, 0, \dots, 0, t_{p_\alpha+}))) ds \\ &+ \int_0^{t_2} f_{i_2}^\alpha(\psi_\alpha(t_1, s, 0, \dots, 0, t_{p_\alpha+}), u(\psi_\alpha(t_1, s, 0, \dots, 0, t_{p_\alpha+}))) ds \\ &\vdots \\ &+ \int_0^{t_{p_\alpha}} f_{i_{p_\alpha}}^\alpha(\psi_\alpha(t_1, \dots, t_{p_\alpha-1}, s, t_{p_\alpha+}), u(\psi_\alpha(t_1, \dots, t_{p_\alpha-1}, s, t_{p_\alpha+}))) ds. \end{aligned} \quad (31)$$

In the above equation, the values $(t_1, t_2, \dots, t_n)$ are chosen so that $x = \psi_\alpha(t_1, \dots, t_n)$. Since ψ_α , given by (20) is a diffeomorphism, this choice is unique. We have also used the abbreviations: $t_{p_\alpha+} := (t_{p_\alpha+1}, \dots, t_n)$, $\xi_\alpha := \xi_\alpha(t_{p_\alpha+})$. Note that the function $\Phi[u]$ is well-defined since $u \in \mathcal{C}$ and since the neighborhoods Θ_α are accessible; this is the technical reason for imposing the Stable Configuration Condition.

To verify that Φ in fact maps $\mathcal{C}$ into itself, we assume $u \in \mathcal{C}$ and $x \in \tilde{\Omega}$ and show that the right hand side of (31) belongs to $\tilde{\Upsilon} = \bar{B}_r(\bar{g})$. To this end, observe that

$$\begin{aligned} \|\Phi[u](x) - \bar{g}\|_\infty &= \max_\alpha |\Phi[u]_\alpha(x) - g_\alpha(\bar{x})| \\ &\leq \max_\alpha \left\{ |g_\alpha(\xi) - g_\alpha(\bar{x})| + \left| \sum_{j=1}^{p_\alpha} \int_0^{t_j} f_{i_\alpha}^\alpha(\psi_\alpha, u(\psi_\alpha)) ds \right| \right\} \end{aligned} \quad (32)$$

$$\leq \max_\alpha \left(\frac{1}{2}r + nM \cdot \left(\max_{1 \leq j \leq p_\alpha} |t_j| \right) \right) \leq r, \quad (33)$$

where we have omitted the arguments of ψ_α . Here, to obtain line (32) we used the definition (31) of the functional Φ and the triangle inequality. To obtain (33) we used statement (29) for the first term and the triangle inequality together with (27) and the fact that $p_\alpha \leq n$ for the second term. Finally, we have used (30).

Equip $\mathcal{C}$ with the metric $d(u, v) = \sup_{x \in \tilde{\Omega}} \|u(x) - v(x)\|_\infty$. With this metric, $\mathcal{C}$ is a complete metric space. We now show that Φ is a contraction mapping. Let $u, v \in \mathcal{C}$, and estimate

$$d(\Phi[u], \Phi[v]) = \sup_{x \in \tilde{\Omega}} \|\Phi[u](x) - \Phi[v](x)\|_\infty = \sup_{x \in \tilde{\Omega}} \max_\alpha |\Phi[u]_\alpha(x) - \Phi[v]_\alpha(x)|$$

$$\leq \sup_{x \in \tilde{\Omega}} \max_\alpha \sum_{j=1}^{p_\alpha} \int_0^{t_j^\alpha} \left| f_{i_j}^\alpha(\psi_\alpha, u(\psi_\alpha)) - f_{i_j}^\alpha(\psi_\alpha, v(\psi_\alpha)) \right| ds \quad (34)$$

$$\leq n \max_{\substack{1 \leq \alpha \leq m \\ 1 \leq j \leq p_\alpha}} |t_j^\alpha| L \cdot \sup_{y \in \tilde{\Omega}} \|u(y) - v(y)\|_\infty \quad (35)$$

$$\leq \frac{1}{2}d(u, v), \quad (36)$$

where, again, we have omitted the arguments of ψ_α . To obtain line (34), we used the definition (31) of Φ and the triangle inequality. To obtain line (35), we used the fact that each f_i^α has Lipschitz constant L and that $p_\alpha \leq n$. To obtain line (36), we used the first inequality in (30) and the definition of $d(u, v)$. Thus Φ is a (uniformly) strict contraction. It follows that Φ has a unique fixed point in $\mathcal{C}$, which we denote $\tilde{u}$. Here we used notation t_j^α , to emphasize that the pre-image of x under ψ_α depends on α . Thus

$$\begin{aligned} \tilde{u}_\alpha(x) &= g_\alpha(\xi_\alpha) + \int_0^{t_1} f_{i_1}^\alpha(\psi_\alpha(s, 0, \dots, 0, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(s, 0, \dots, 0, t_{p_\alpha+}))) ds \\ &+ \int_0^{t_2} f_{i_2}^\alpha(\psi_\alpha(t_1, s, 0, \dots, 0, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(t_1, s, 0, \dots, 0, t_{p_\alpha+}))) ds \\ &\vdots \\ &+ \int_0^{t_{p_\alpha}} f_{i_{p_\alpha}}^\alpha(\psi_\alpha(t_1, t_2, \dots, t_{p_\alpha-1}, s, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(t_1, t_2, \dots, t_{p_\alpha-1}, s, t_{p_\alpha+}))) ds. \end{aligned} \quad (37)$$

Since $t_1 = \dots = t_{p_\alpha} = 0$ and $\xi_\alpha \equiv \xi_\alpha(t_{p_\alpha+}) = x$, whenever $x \in \Xi_\alpha$, the function $\tilde{u}$ satisfies the data (25). Note that on the manifold Ξ_α^j , we have $t_k = 0$ for $j < k \leq p_\alpha$, and also $\mathbf{r}_{i_j} = \partial_{t_j}$. The latter follows since, for any smooth function $h : \tilde{\Omega} \rightarrow \mathbb{R}$, whenever $x \in \Xi_\alpha^j$, we have

$$\partial_{t_j} h(x) = \partial_{t_j} h(W_j^{t_j} \dots W_1^{t_1} \xi_\alpha) = \nabla_x h \cdot \mathbf{r}_{i_j} \big|_{W_j^{t_j} \dots W_1^{t_1} \xi_\alpha} = \mathbf{r}_{i_j} h(x). \quad (38)$$

Thus, with equation (37) restricted to Ξ_α^j , we apply the fundamental theorem of calculus and obtain that, for $x \in \Xi_\alpha^j$,

$$\begin{aligned} \mathbf{r}_{i_j} \tilde{u}_\alpha(x) &= \partial_{t_j} \tilde{u}_\alpha(x) \\ &= f_{i_j}^\alpha(\psi_\alpha(t_1, \dots, t_j, 0, \dots, 0, t_{p_\alpha+1}), \tilde{u}(\psi_\alpha(t_1, \dots, t_j, 0, \dots, 0, t_{p_\alpha+1}))) \\ &= f_{i_j}^\alpha(x, \tilde{u}(x)), \end{aligned}$$

showing that $\tilde{u}$ is indeed a solution of (24)-(25).

It remains to show that $\tilde{u}$ is the unique solution of (24)-(25) on $\tilde{\Omega}$. Assuming $v : \tilde{\Omega} \rightarrow \Upsilon$ is also a solution to (24)-(25) in $\tilde{\Omega}$, we have

$$\begin{aligned} d(\tilde{u}, v) &= \sup_{x \in \tilde{\Omega}} \|\tilde{u}(x) - v(x)\|_\infty = \sup_{x \in \tilde{\Omega}} \max_\alpha |\tilde{u}_\alpha(x) - v_\alpha(x)| \\ &= \sup_{x \in \tilde{\Omega}} \max_\alpha \left| \sum_{j=1}^{p_\alpha} \int_0^{t_{i_j}} f_{i_j}^\alpha(\psi_\alpha, \tilde{u}(\psi_\alpha)) - f_{i_j}^\alpha(\psi_\alpha, v(\psi_\alpha)) dx \right| \end{aligned} \quad (39)$$

$$\leq \sup_{y \in \tilde{\Omega}} \max_\alpha \sum_{j=1}^{p_\alpha} |t_{i_j}| L \|\tilde{u}(y) - v(y)\|_\infty \quad (40)$$

$$\leq \frac{1}{2} \sup_{y \in \tilde{\Omega}} \|\tilde{u}(y) - v(y)\|_\infty = \frac{1}{2} d(\tilde{u}, v), \quad (41)$$

where, as above, $x = \psi_\alpha(t_1, \dots, t_n)$ and we have omitted the arguments of ψ_α . Line (39) follows from the fact that both $\tilde{u}$ and v , being solutions to (24)-(25), satisfy (37). Line (40) follows from the triangle inequality, the Lipschitz property (26), and the fact that the neighborhoods Θ_α are accessible (so that the ψ_α take values in $\tilde{\Omega}$). The inequality in (41) follows from the first inequality in (30). Now, $\tilde{\Omega}$ is, by choice, a bounded set in which each point can be reached by starting on any data manifold Ξ_α and then moving along a finite number of integral curves of the vector fields $\mathbf{r}_i$. Since we assume that the initial data g_α are bounded, and also that the functions $f_i^\alpha : \Omega \times \Upsilon \rightarrow \mathbb{R}$ are uniformly bounded, it follows that the solutions $\tilde{u}$ and v are both bounded on $\tilde{\Omega}$. Therefore, $\sup_{x \in \tilde{\Omega}} \|\tilde{u}(x) - v(x)\|_\infty < \infty$ and it follows from the inequalities above that $d(\tilde{u}, v) = 0$, i.e. $\tilde{u} \equiv v$ on $\tilde{\Omega}$. $\square$

Theorem 1. *Suppose that, in addition to hypotheses (i)–(vi) of Lemma 3.6, we also have that:*

- (vii) *the functions f_i^α belong to $C^1(\Omega \times \Upsilon)$;*

- (viii) for each α , for each $i, j \in I_\alpha$ with $i \neq j$, and for each β : if $i \notin I_\beta$, then $\partial_{u_\beta} f_j^\alpha = 0$;
- (ix) for each α , the vector fields $\{\mathbf{r}_i\}_{i \in I_\alpha}$ are in involution, i.e. $[r_j, r_k] \in \text{span}\{\mathbf{r}_i\}_{i \in I_\alpha}$ whenever $j, k \in I_\alpha$;
- (x) for each α , for each $i, j \in I_\alpha$ with $i \neq j$, and for all $(x, u) \in \Omega \times \Upsilon$:

$$\begin{aligned} & (\nabla_x f_j^\alpha)|_{(x,u)} \cdot \mathbf{r}_i|_x + \sum_{\beta: i \in I_\beta} \partial_{u_\beta} f_j^\alpha(x, u) f_i^\beta(x, u) \\ & - \nabla_x (f_i^\alpha)|_{(x,u)} \cdot \mathbf{r}_j|_x - \sum_{\beta: j \in I_\beta} \partial_{u_\beta} f_i^\alpha(x, u) f_j^\beta(x, u) = \sum_{k \in I_\alpha} c_{ij}^k(x) f_k^\alpha(x, u), \end{aligned} \quad (42)$$

where here and below $\nabla_x(f_i^\alpha)$ denotes the gradient with respect to the variables x and c_{ij}^k denote the structure coefficients of the frame:

$$[\mathbf{r}_i, \mathbf{r}_j]|_x = \sum_{k=1}^n c_{ij}^k(x) \mathbf{r}_k|_x \quad \text{for } x \in \Omega \text{ and } 1 \leq i, j \leq n.$$

- (xi) for each α and each $i, j, k \in I_\alpha$, the structure coefficient $c_{ij}^k(x)$ is uniformly bounded on Ω .

Then there is a neighborhood $\tilde{\Omega}$ of $\bar{x}$ on which there is a unique solution to the system

$$\mathbf{r}_i(u_\alpha)|_x = f_i^\alpha(x, u(x)) \quad \text{for } 1 \leq \alpha \leq m, i \in I_\alpha, \text{ and } x \in \tilde{\Omega}, \quad (43)$$

satisfying the data

$$u_\alpha(x) = g_\alpha(x) \quad \text{for } x \in \Xi_\alpha \cap \tilde{\Omega}. \quad (44)$$

Before giving the proof we make a few remarks. First, as remarked above, the difference between the conclusion of Theorem 1 and the conclusion of Lemma 3.6 is that in the Theorem, the equations of system (43) are satisfied everywhere in $\tilde{\Omega}$, while in the Lemma, each equation of system (24) is only guaranteed to be satisfied only for $x \in \Xi_\alpha^j \cap \tilde{\Omega}$. The data (25) and (44) are identical.

The integrability conditions appearing in equations (42) are the generalization of the condition of mixed partial derivatives being equal, to the case of non-commutative derivations. They correspond to the integrability conditions in the PDE version of the classic Frobenius Theorem (see the equations marked (**)) in Theorem 1 of Chapter 6 in [9]). The conditions appear from expanding the equation

$$\mathbf{r}_i(\mathbf{r}_j(u_\alpha)) - \mathbf{r}_j(\mathbf{r}_i(u_\alpha)) = \sum_{k=1}^n c_{ij}^k \mathbf{r}_k(u_\alpha), \quad (45)$$

which should hold for any function u_α , and then, once fully expanded, making substitutions of the form $\mathbf{r}_i(u_\alpha) = f_i^\alpha(x, u)$, which should hold for any solution $u = (u_1, \dots, u_m)$ of (43).

The restricted summations of the form $\{\beta : i \in I_\beta\}$, $\{\beta : j \in I_\beta\}$, and $\{k \in I_\alpha\}$ in (42) ensure that (42) only contains functions f_i^α which actually

are defined by the system (43). For instance, examining the first summation in (42), if we included an index β with $i \notin I_\beta$, then the factor $f_i^\beta(x, u)$ would be a function that is not given by the system (43). Similar remarks apply to the other two summations.

Hypotheses (viii) and (ix) are necessary so that in making the restrictions on summation indices just described, we have not actually omitted any terms that should appear in the expansion of (45). For instance, for the omitted indices β with $i \notin I_\beta$ from the first summation of (42), hypothesis (viii) guarantees that $\partial_{u_\beta} f_j^\alpha(x, u) \equiv 0$, and so we have not actually missed any terms. Similar remarks apply to hypothesis (ix) and the last summation.

Proof. Apply Lemma 3.6 to obtain a neighborhood $\tilde{\Omega}$ of $\bar{x}$ on which there is a unique solution $\tilde{u}$ to the system (24) satisfying (44). It remains only to show that $\tilde{u}$ is a solution to the full system (43); uniqueness is already established since any solution of (43) is also a solution of (24).

Fix (for now) an index α . As in the formulation of Lemma 3.6, we will use double indices i_j for the elements of the ordered set $I_\alpha = \{i_1, \dots, i_{p_\alpha}\}$. For each $i_j \in I_\alpha$ (i.e., $1 \leq j \leq p_\alpha$) we define the function $A_{i_j}^\alpha : \tilde{\Omega} \rightarrow \mathbb{R}$ by

$$A_{i_j}^\alpha(x) := \mathbf{r}_{i_j}(\tilde{u}_\alpha)|_x - f_{i_j}^\alpha(x, \tilde{u}(x)). \quad (46)$$

Then, for each $i_k \in I_\alpha$ with $j < k \leq p_\alpha$, we apply $\mathbf{r}_{i_k}$ to (46) to obtain

$$\begin{aligned} \mathbf{r}_{i_k}(A_{i_j}^\alpha)|_x &= \mathbf{r}_{i_k}(\mathbf{r}_{i_j}(\tilde{u}_\alpha))|_x - (\nabla_x f_{i_j}^\alpha)|_{(x, \tilde{u}(x))} \cdot \mathbf{r}_{i_k}|_x \\ &\quad - \sum_{\beta: i_k \in I_\beta} \partial_{u_\beta} f_{i_j}^\alpha(x, \tilde{u}(x)) \mathbf{r}_{i_k}(\tilde{u}_\beta)|_x \end{aligned} \quad (47)$$

$$\begin{aligned} &= \mathbf{r}_{i_j}(\mathbf{r}_{i_k}(\tilde{u}_\alpha))|_x + \sum_{l \in I_\alpha} c_{i_k i_j}^l(x) \mathbf{r}_l(\tilde{u}_\alpha)|_x \\ &\quad - (\nabla_x f_{i_j}^\alpha)|_{(x, \tilde{u}(x))} \cdot \mathbf{r}_{i_k}|_x - \sum_{\beta: i_k \in I_\beta} \partial_{u_\beta} f_{i_j}^\alpha(x, \tilde{u}(x)) \mathbf{r}_{i_k}(\tilde{u}_\beta)|_x. \end{aligned} \quad (48)$$

We note that the summation in line (47) is restricted to $\{\beta : i_k \in I_\beta\}$ by hypothesis (viii), and that the summation in line (48) is restricted to $\{l \in I_\alpha\}$ by hypothesis (ix). We now restrict the last equation above to $x \in \Xi_\alpha^k$, where, according to the conclusion of Lemma 3.6, $\mathbf{r}_{i_k}(\tilde{u}_\alpha) = f_{i_k}^\alpha(x, \tilde{u})$. Thus,

for $x \in \Xi_\alpha^k$ we have

$$\begin{aligned}
\mathbf{r}_{i_k}(A_{i_j}^\alpha)|_x &= \mathbf{r}_{i_j}(f_{i_k}^\alpha(x, \tilde{u}))|_x + \sum_{l \in I_\alpha} c_{i_k i_j}^l(x) \mathbf{r}_l(\tilde{u}_\alpha)|_x \\
&\quad - (\nabla_x f_{i_j}^\alpha)|_{(x, \tilde{u})} \cdot \mathbf{r}_{i_k}|_x - \sum_{\beta: i_k \in I_\beta} \partial_\beta f_{i_j}^\alpha(x, \tilde{u}) \mathbf{r}_{i_k}(\tilde{u}_\beta)|_x \\
&= (\nabla_x f_{i_k}^\alpha)|_{(x, \tilde{u})} \cdot \mathbf{r}_{i_j}|_x + \sum_{\beta: i_k \in I_\beta} \partial_{u_\beta} f_{i_k}^\alpha(x, \tilde{u}) \mathbf{r}_{i_j}(\tilde{u}_\beta)|_x \\
&\quad + \sum_{l \in I_\alpha} c_{i_k i_j}^l(x) \mathbf{r}_l(\tilde{u}_\alpha)|_x \\
&\quad - (\nabla_x f_{i_j}^\alpha)|_{(x, \tilde{u})} \cdot \mathbf{r}_{i_k}|_x - \sum_{\beta: i_k \in I_\beta} \partial_{u_\beta} f_{i_j}^\alpha(x, \tilde{u}) \mathbf{r}_{i_k}(\tilde{u}_\beta)|_x.
\end{aligned} \tag{49}$$

(Here we begin omitting the argument x of $\tilde{u}$.) We now apply the integrability condition (42) with $u = \tilde{u} \equiv \tilde{u}(x)$, $i = i_j$, $j = i_k$, and rearrange to obtain

$$\begin{aligned}
&(\nabla_x f_{i_k}^\alpha)|_{(x, \tilde{u})} \cdot \mathbf{r}_{i_j}|_x + \sum_{\beta: i_j \in I_\beta} \partial_{u_\beta} f_{i_k}^\alpha(x, \tilde{u}) f_{i_j}^\beta(x, \tilde{u}) + \sum_{l \in I_\alpha} c_{i_k i_j}^l(x) f_l^\alpha(x, \tilde{u}) \\
&\quad - (\nabla_x f_{i_j}^\alpha)|_{(x, \tilde{u})} \cdot \mathbf{r}_{i_k}|_x - \sum_{\beta: j \in I_\beta} \partial_{u_\beta} f_{i_j}^\alpha(x, \tilde{u}) f_{i_k}^\beta(x, \tilde{u}) = 0.
\end{aligned} \tag{51}$$

Subtracting (51) from (50) and recalling the definition (46) of the $A_{i_j}^\alpha$, we obtain: whenever $i_j, i_k \in I_\alpha$ with $k > j$, and $x \in \Xi_\alpha^k$, then

$$\begin{aligned}
\mathbf{r}_{i_k}(A_{i_j}^\alpha)|_x &= \sum_{\beta: i_j \in I_\beta} \partial_{u_\beta} f_{i_k}^\alpha(x, \tilde{u}) A_{i_j}^\beta(x) - \sum_{\beta: i_k \in I_\beta} \partial_{u_\beta} f_{i_j}^\alpha(x, \tilde{u}) A_{i_k}^\beta(x) \\
&\quad + \sum_{l \in I_\alpha} c_{i_k i_j}^l(x) A_l^\alpha(x).
\end{aligned} \tag{52}$$

We now consider (52) as a system of differential equations for the unknowns $A_{i_j}^\alpha$, with vanishing data prescribed along appropriate submanifolds. The trivial functions $A_{i_j}^\alpha \equiv 0$ clearly provide a solution. The goal is to apply the uniqueness part of Lemma 3.6 to the new system (52), and conclude that the trivial solution is the only one. That is, $A_{i_j}^\alpha$ defined by (46) must vanish identically near $\bar{x}$. We proceed with verifying that the assumptions of Lemma 3.6 are satisfied for the system (52).

To do so we introduce the following notations. Let $J = \{(j, \alpha) \mid 1 \leq \alpha \leq m; 1 \leq j \leq p_\alpha\}$, and for each double index $(j, \alpha) \in J$ define $\Lambda_{j, \alpha} := \Xi_\alpha^j$, where the right-hand side is given by (23). Thus, the C^1 submanifolds $\Lambda_{j, \alpha}$ can be parametrized by the functions

$$\begin{aligned}
\lambda_{j, \alpha}(s_1, \dots, s_{n-p_\alpha+j}) &:= W_{i_j}^{s_j} \cdots W_{i_1}^{s_1} \xi_\alpha(s_{j+1}, \dots, s_{n-p_\alpha+j}) \\
&= \psi_\alpha(s_1, \dots, s_j, 0, \dots, 0, s_{j+1}, \dots, s_{n-p_\alpha+j}),
\end{aligned}$$

which are defined for all $(s_1, \dots, s_{n-p_\alpha+j})$ with the property that the point $(s_1, \dots, s_j, 0, \dots, 0, s_{j+1}, \dots, s_{n-p_\alpha+j})$ belongs to Θ_α . The domain of $\lambda_{j,\alpha}$ is, therefore, the intersection of Θ_α with the coordinate subset of $\mathbb{R}^n$ where $t_{j+1} = \dots = t_{p_\alpha} = 0$.

We let the data $h_{j,\alpha}$ for the unknown $A_{i_j}^\alpha$ in the system (52) vanish identically on the manifolds $\Lambda_{j,\alpha}$: $h_{j,\alpha}(x) \equiv 0$ for $x \in \Lambda_{j,\alpha}$. Finally, setting $J_{j,\alpha} := \{k \mid j < k \leq p_\alpha\}$, we have that (52) yields an equation for $\mathbf{r}_{i_k}(A_{i_j}^\alpha)$ whenever $(j, \alpha) \in J$ and $k \in J_{j,\alpha}$.

Now consider the hypotheses of Lemma 3.6 in the context of system (52):

- (i)' $\mathfrak{R} = \{\mathbf{r}_i\}_{i=1}^n$ is a C^1 frame on $\tilde{\Omega}$;
- (ii)' $\Lambda_{j,\alpha}$, $(j, \alpha) \in J$, are embedded C^1 submanifolds of $\tilde{\Omega}$, and they share a common point $\bar{x} \in \bigcap_{(j,\alpha) \in J} \Lambda_{j,\alpha}$;
- (iii)' each manifold $\Lambda_{j,\alpha}$ is of codimension $p_\alpha - j$ and is everywhere transverse to the span of $\{\mathbf{r}_{i_k}\}_{k \in J_{j,\alpha}}$;
- (iv)' the frame $\mathfrak{R}$ and the set of manifolds $\Lambda = \{\Lambda_{j,\alpha} \mid (j, \alpha) \in J\}$ is a stable configuration near a point $\bar{x} \in \tilde{\Omega}$;
- (v)' for each double index $(j, \alpha) \in J$, $h_{j,\alpha} : \Lambda_{j,\alpha} \rightarrow \mathbb{R}$ is bounded and C^1 , with $h(\bar{x}) \in \mathbb{R}^{|J|}$;
- (vi)' for each double index $(j, \alpha) \in J$ and each $k \in J_{j,\alpha}$, the right-hand side of (52) is a function $\tilde{\Omega} \times \mathbb{R}^{|J|} \rightarrow \mathbb{R}$ which is bounded, continuous, and also uniformly Lipschitz in its second argument.

The statements (i)', (ii)', (iii)', and (v)' are self-evident, while statement (vi)' follows from assumption (vi) of Lemma 3.6 together with assumptions (vii) and (xi) of the present theorem.

Finally, to verify (iv)' we show that stability of the configuration $(\mathfrak{R}, \Lambda)$ follows from that of the configuration $(\mathfrak{R}, \Xi)$ (which holds according to assumption (vi) of Lemma 3.6). By stability of the configuration $(\mathfrak{R}, \Xi)$, let $\hat{\Omega}$ be a neighborhood of $\bar{x}$ for which there is, for each $\alpha = 1, \dots, m$, a p_α -accessible neighborhood $\Theta_\alpha \subset \mathbb{R}^n$ such that the map $\psi_\alpha : \Theta_\alpha \rightarrow \hat{\Omega}$ given by (20) is a C^1 -diffeomorphism (see Definitions 3.1-3.3). Then, for $(j, \alpha) \in J$, the map

$$\begin{aligned} \hat{\psi}_{j,\alpha}(t_1, \dots, t_n) &:= W_{i_{p_\alpha-j}}^{t_{p_\alpha-j}} \dots W_{i_1}^{t_1} \lambda_{j,\alpha}(t_{p_\alpha-j+1}, \dots, t_n) \\ &= \psi_\alpha(t_{p_\alpha-j+1}, \dots, t_{p_\alpha}, t_1, \dots, t_{p_\alpha-j}, t_{p_\alpha+1}, \dots, t_n) \end{aligned}$$

is defined for all $(t_1, \dots, t_n) \in \mathbb{R}^n$ with the property that

$$(t_{p_\alpha-j+1}, \dots, t_{p_\alpha}, t_1, \dots, t_{p_\alpha-j}, t_{p_\alpha+1}, \dots, t_n) \in \Theta_\alpha.$$

Thus, the domain $\hat{\Theta}_{j,\alpha}$ of $\hat{\psi}_{j,\alpha}$ is the image of Θ_α under a permutation-of-coordinates map Π that interchanges the block of 1st-through- j th coordinates with the block of $(j+1)$ th-through- $(p_\alpha-1)$ th coordinates. The map $\hat{\psi}_{j,\alpha} = \psi_\alpha \circ \Pi^{-1}$ is therefore a C^1 -diffeomorphism $\hat{\Theta}_{j,\alpha} \rightarrow \hat{\Omega}$. It is straightforward to verify that $(p_\alpha - j)$ -accessibility of $\hat{\Theta}_{j,\alpha}$ follows from p_α -accessibility of Θ_α (see Definition 3.1.) This shows that assumption (iv)' is satisfied.

Thus, according to Lemma 3.6, there is a neighborhood $\tilde{\Omega}'$ of $\bar{x}$ in $\mathbb{R}^n$ on which there exists a unique set of functions $\{A_{ij}^\alpha(x) \mid (j, \alpha) \in J\}$ solving (52) (with vanishing data on the $\Lambda_{j,\alpha}$), for all $x \in \Lambda_{j,\alpha}^{k-j} \cap \tilde{\Omega}'$, where $\Lambda_{j,\alpha}^{k-j}$ are defined similarly to (23) by

$$\Lambda_{j,\alpha}^{k-j} := \{\hat{\psi}_{j,\alpha}(t) \mid t = (t_1, \dots, t_{k-j}, 0, \dots, 0, t_{p_\alpha-j+1}, \dots, t_n) \in \hat{\Theta}_{j,\alpha}\} \quad (53)$$

Unwinding the definition of $\hat{\psi}_{j,\alpha}(t)$ we see that $\Lambda_{j,\alpha}^{k-j}$ is obtained by starting at a point $\Lambda_{j,\alpha}$, and then flowing in turn along the vectors fields whose indices appear no later than the $(k-j)$ th member of $J_{j,\alpha}$. Recalling that $\Lambda_{j,\alpha} = \Xi_\alpha^j$ and observing that the $(k-j)$ th member of $J_{j,\alpha}$ is the k -th member of I_α , we conclude that $\Lambda_{j,\alpha}^{k-j}$ equals to Ξ_α^k , on which (52) is to hold.

Finally, since the identically vanishing functions $A_{ij}^\alpha(x) \equiv 0$ satisfy (52) as well as the data, it follows from the uniqueness part of Lemma 3.6 that the functions defined by (46) are identically zero on $\tilde{\Omega}'$. That is,

$$\mathbf{r}_{ij}(\tilde{u}_\alpha)|_x = f_{ij}^\alpha(x, \tilde{u}(x)) \quad \text{for all } x \in \tilde{\Omega}',$$

as was to be shown. $\square$

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