Automorphic equivalence within gapped phases in the bulk

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Abstract

We develop a new adiabatic theorem for unique gapped ground states which does not require the gap for local Hamiltonians. We instead require a gap in the bulk and a smoothness of expectation values of sub-exponentially localized observables in the unique gapped ground state $\varphi_s(A)$. This requirement is weaker than the requirement of the gap of the local Hamiltonians, since a uniform spectral gap for finite dimensional ground states implies a gap in the bulk for unique gapped ground states, as well as the smoothness.

1 Introduction

Hastings's [H] [HW] adiabatic method is a powerful tool in the analysis of gapped Hamiltonians in quantum many-body systems. Seminal mathematical developments from [BMNS], [NSY], [Y] and onwards have established a strong mathematical framework of adiabatic theory for quantum many-body systems. The adiabatic theorems from these works state that for a smooth path of gapped Hamiltonians, there is an automorphic equivalence between ground state spaces along the path. Furthermore, these automorphisms are quasi-local.

This framework has proven to be broadly applicable to many situations. In [HM], the long standing problem of explaining the quantization of the Hall conductance was finally solved with this method. The Kubo formula was derived in [BDF] using the method.

Another use of the adiabatic theorem is the analysis of symmetry protected topological (SPT) phase, in [O2] and [O3]. In [O2] and [O3], indices for SPT phases which extend the indices by Pollmann et.al. [PTBO1],[PTBO2] were introduced. The adiabatic theorem was used to show the stability of these indices. See [Mo] for the extension of [O2] to interactions with unbounded interaction range with fast decay.

All of the adiabatic theorems developed so far require a uniform spectral gap for local Hamiltonians. Therefore, even if what we are interested in is the bulk, the use of known adiabatic theorems requires conditions on the gap in finite boxes. This is conceptually unsatisfactory because bulk-classification of gapped Hamiltonians can be coarser than the classification in finite

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volume [O1]. For this reason, many works have been carried on torus. In this paper, we develop a new adiabatic theorem for unique gapped ground states which does not require the gap for local Hamiltonians. We instead require a gap in the bulk and a smoothness of expectation values of sub-exponentially localized observables in the unique gapped ground state $\varphi_s(A)$. This requirement is weaker than the requirement of the gap of the local Hamiltonians, since a uniform spectral gap for finite dimensional ground states implies a gap in the bulk for unique gapped ground states, as well as the smoothness. (See Remark 4.15.) Under such conditions, we show that there is a smooth path of quasi-local automorphisms α_s, such that $\omega_s = \omega_0 \circ \alpha_s$. This α_s is the same as the one given in the literatures [BMNS], [NSY].

Although the result is analogous to those of finite systems, there is a crucial difference for the proof. For the finite system $\mathcal{A}_\Lambda$, there is a Hamiltonian $H_s(\Lambda)$ in the C^*-algebra $\mathcal{A}_\Lambda$. By considering a differential equation satisfied by the spectral projection $P_s(\Lambda)$ of the Hamiltonian $H_s(\Lambda)$ corresponding to the lowest eigenvalue, we may explicitly define in this case the automorphisms connecting the ground state spaces. In contrast, for infinite systems, we do not have a Hamiltonian H_s in the C^*-algebra of quantum spin systems. Of course we can consider the bulk Hamiltonian H_s, but H_s depends on the GNS representation, and the meaning of $\frac{d}{ds}H_s$ is ambiguous. Therefore, we have to find an alternative way to prove our adiabatic theorem.

In particular, for finite systems, the parallel transport condition $P_s(\Lambda)\dot{P}_s(\Lambda)P_s(\Lambda) = 0$ plays a crucial role. In infinite systems, this condition is replaced by Proposition 2.2.

Let us now give a more precise description of our result. We start by summarizing the standard setup of quantum spin systems [BR1, BR2]. Let $\nu \in \mathbb{N}$ and $d \in \mathbb{N}$. Throughout this article, we fix these numbers. We denote the algebra of $d \times d$ matrices by M_d.

We denote the set of all finite subsets in $\mathbb{Z}^\nu$ by $\mathfrak{S}_{\mathbb{Z}^\nu}$. For each $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $\mathrm{diam}(X)$ denotes the diameter of X. For $X, Y \subset \mathbb{Z}^\nu$, we denote by $d(X,Y)$ the distance between them. The number of elements in a finite set $\Lambda \subset \mathbb{Z}^\nu$ is denoted by $|\Lambda|$. For each $n \in \mathbb{N}$, we denote $[-n,n]^\nu \cap \mathbb{Z}^\nu$ by Λ_n. The complement of $\Lambda \subset \mathbb{Z}^\nu$ in $\mathbb{Z}^\nu$ is denoted by Λ^c.

For each $z \in \mathbb{Z}^\nu$, let $\mathcal{A}_{\{z\}}$ be an isomorphic copy of M_d, and for any finite subset $\Lambda \subset \mathbb{Z}^\nu$, let $\mathcal{A}_\Lambda = \otimes_{z\in\Lambda}\mathcal{A}_{\{z\}}$, which is the local algebra of observables in Λ. For finite Λ, the algebra $\mathcal{A}_\Lambda$ can be regarded as the set of all bounded operators acting on the Hilbert space $\otimes_{z\in\Lambda}\mathbb{C}^d$. We use this identification freely. If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}_{\Lambda_1}$ is naturally embedded in $\mathcal{A}_{\Lambda_2}$ by tensoring its elements with the identity. The algebra $\mathcal{A}$, representing the quantum spin system on $\mathbb{Z}^\nu$ is given as the inductive limit of the algebras $\mathcal{A}_\Lambda$ with $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$. Note that $\mathcal{A}_\Lambda$ for $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$ can be regarded naturally as a subalgebra of $\mathcal{A}$. We denote the set of local observables by $\mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda\in\mathfrak{S}_{\mathbb{Z}^\nu}} \mathcal{A}_\Lambda$.

A uniformly bounded interaction on $\mathcal{A}$ is a map $\Psi : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ such that

$$\Psi(X) = \Psi(X)^* \in \mathcal{A}_X, \quad X \in \mathfrak{S}_{\mathbb{Z}^\nu}, \tag{1.1}$$

and

$$\sup_{X\in\mathfrak{S}_{\mathbb{Z}^\nu}} \|\Psi(X)\| < \infty. \tag{1.2}$$

It is of finite range with interaction length less than or equal to $R \in \mathbb{N}$ if $\Psi(X) = 0$ for any $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ whose diameter is larger than R. We denote by Ψ_n for each $n \in \mathbb{N}$ the interaction given by

$$\Psi_n(X) := \begin{cases} \Psi(X), & \text{if} \quad X \subset \Lambda_n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

For a uniformly bounded and finite range interaction Ψ and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$ define the local Hamiltonian

$$(H_\Psi)_\Lambda := \sum_{X \subset \Lambda} \Psi(X), \tag{1.4}$$

and denote the dynamics

$$\tau^t_{\Psi,\Lambda}(A) := e^{it(H_\Psi)_\Lambda} A e^{-it(H_\Psi)_\Lambda}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}. \tag{1.5}$$

By the uniform boundedness and finite rangeness of Ψ, for each $A \in \mathcal{A}$, the following limit exists:

$$\lim_{\Lambda \to \mathbb{Z}^\nu} \tau^t_{\Psi,\Lambda}(A) =: \tau^t_\Psi(A), \quad t \in \mathbb{R}, \tag{1.6}$$

and defines the dynamics τ_Ψ on $\mathcal{A}$. Note that $\tau_{\Psi_n} = \tau_{\Psi,\Lambda_n}$. We denote by δ_Ψ the generator of τ_Ψ.

For a uniformly bounded and finite range interaction Ψ, a state φ on $\mathcal{A}$ is called a τ_Ψ-ground state if the inequality $-i\,\varphi(A^*\delta_\Psi(A)) \geq 0$ holds for any element A in the domain $\mathcal{D}(\delta_\Psi)$ of δ_Ψ. Let φ be a τ_Ψ-ground state, with the GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$. Then there exists a unique positive operator $H_{\varphi,\Psi}$ on $\mathcal{H}_\varphi$ such that $e^{itH_{\varphi,\Psi}}\pi_\varphi(A)\Omega_\varphi = \pi_\varphi(\tau^t_\Psi(A))\Omega_\varphi$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi,\Psi}$ the bulk Hamiltonian associated with φ. Note that Ω_φ is an eigenvector of $H_{\varphi,\Psi}$ with eigenvalue 0. See [BR2] for the general theory.

Let $\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N}$ be the conditional expectation with respect to the trace state. Let us consider the following subset of $\mathcal{A}$. (See [BDN] and [Ma] for analogous definitions.)

Definition 1.1. Let $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. For each $A \in \mathcal{A}$, let

$$\|A\|_f := \|A\| + \sup_{N\in\mathbb{N}} \left(\frac{\|A - \mathbb{E}_N(A)\|}{f(N)} \right). \tag{1.7}$$

We denote by $\mathcal{D}_f$ the set of all $A \in \mathcal{A}$ such that $\|A\|_f < \infty$.

Properties of $\mathcal{D}_f$ are collected in Appendix B. The set $\mathcal{D}_f$ is a $*$-algebra which is a Banach space with respect to the norm $\|\cdot\|_f$ (see Lemma B.1).

Assumption 1.2. Let $\Phi(\cdot\,; s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ be a family of uniformly bounded, finite range interactions parameterized by $s \in [0,1]$. We assume the following:

(i) For each $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$, the map $[0,1] \ni s \to \Phi(X;s) \in \mathcal{A}_X$ is continuous and piecewise C^1. We denote by $\dot{\Phi}(X;s)$ the corresponding derivatives. The interaction obtained by differentiation is denoted by $\dot{\Phi}(s)$, for each $s \in [0,1]$.

(ii) There is a number $R \in \mathbb{N}$ such that $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ and $\mathrm{diam}(X) \geq R$ imply $\Phi(X;s) = 0$, for all $s \in [0,1]$.

(iii) Interactions are bounded as follows

$$\sup_{s\in[0,1]} \sup_{X\in\mathfrak{S}_{\mathbb{Z}^\nu}} \left(\|\Phi(X;s)\| + |X| \left\| \dot{\Phi}(X;s) \right\| \right) < \infty. \tag{1.8}$$

(iv) Setting

$$b(\varepsilon) := \sup_{Z\in\mathfrak{S}_{\mathbb{Z}^\nu}} \sup_{s,s_0\in[0,1],0<|s-s_0|<\varepsilon} \left\| \frac{\Phi(Z;s)-\Phi(Z;s_0)}{s-s_0} - \dot{\Phi}(Z;s_0) \right\| \tag{1.9}$$

for each $\varepsilon > 0$, we have $\lim_{\varepsilon\to 0} b(\varepsilon) = 0$.

(v) For each $s \in [0,1]$, there exists a unique $\tau_{\Phi(s)}$-ground state φ_s.

(vi) There exists a $\gamma > 0$ such that $\sigma(H_{\varphi_s,\Phi(s)}) \setminus \{0\} \subset [2\gamma, \infty)$ for all $s \in [0,1]$, where $\sigma(H_{\varphi_s,\Phi(s)})$ is the spectrum of $H_{\varphi_s,\Phi(s)}$.

(vii) There exists $0 < \beta < 1$ satisfying the following: Set $\zeta(t) := e^{-t^\beta}$. Then for each $A \in D_\zeta$, $\varphi_s(A)$ is differentiable with respect to s, and there is a constant C_ζ such that:

$$|\dot{\varphi}_s(A)| \le C_\zeta \|A\|_\zeta, \tag{1.10}$$

for any $A \in D_\zeta$.

The main theorem of this paper is that under the Assumption 1.2, there is a strongly continuous path of automorphisms $[0,1] \ni s \mapsto \alpha_s$ such that $\varphi_s = \varphi_0 \circ \alpha_s, \quad s \in [0,1]$.

In fact, this α_s is the same one as in [BMNS] and [NSY], which is given through some differential equation. Let us recall it.

We use the function ω_1 introduced in [NSY]. Set

$$a_n := \frac{a_1}{n\ln(n)^2}, \quad n \ge 2, \tag{1.11}$$

and choose a_1 so that $\sum_{n=1}^\infty a_n = \frac{1}{2}$. Let $\omega_1(t) \in L^1(\mathbb{R})$ be the function on $\mathbb{R}$ defined by

$$\omega_1(t) := \begin{cases} c, & t = 0, \\ c\prod_{n=1}^\infty \left(\frac{\sin(a_n t)}{a_n t}\right)^2, & t \ne 0 \end{cases} \tag{1.12}$$

with normalization factor $c > 0$ such that

$$\int dt\omega_1(t) = 1. \tag{1.13}$$

As shown in [BMNS] and [NSY], ω_1 is indeed an even nonnegative L^1-function and

$$\omega_1(t) \le c_1 \frac{t}{\ln(t)^2} e^{-\frac{\eta t}{\ln(t)^2}}, \quad t > e, \tag{1.14}$$

$$W_1(x) := \int_x^\infty dt\omega_1(t) \le \begin{cases} c_1 \left(\frac{x}{\ln(x)^2}\right)^2 e^{-\frac{\eta x}{\ln(x)^2}}, & x > e^9, \\ 1, & x \le e^9 \end{cases} \tag{1.15}$$

for constants $\eta = 2a_1 \in (\frac{2}{7}, 1)$ and $c_1 = (27/14)ce^4$. We set $\omega_\gamma(t) := \gamma\omega_1(\gamma t)$, where $\gamma > 0$ is from Assumption 1.2, and $W_\gamma(x) := W_1(\gamma x)$, for $x \in \mathbb{R}_+$. The function ω_γ is an even nonnegative L^1-function with

$$\int dt\omega_\gamma(t) = 1. \tag{1.16}$$

We also have

$$W_\gamma(x) = \int_x^\infty dt \omega_\gamma(t), \quad x \in \mathbb{R}_+. \tag{1.17}$$

Furthermore, the Fourier transform of ω_γ is supported in the interval $[-\gamma, \gamma]$.(See [NSY].)

For each $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$, let U_Λ be the solution of the differential equation

$$-i \frac{d}{ds} U_\Lambda(s) = D_\Lambda(s) U_\Lambda(s), \quad U_\Lambda(0) = \mathbb{I}. \tag{1.18}$$

Here, $D_\Lambda(s)$ is defined by

$$D_\Lambda(s) := \int_{-\infty}^\infty dt\, \omega_\gamma(t) \quad \int_0^t du \tau^u_{\Phi(s),\Lambda} \left(\frac{d}{ds} \left(H_{\Phi(s)} \right)_\Lambda \right), \quad s \in [0,1]. \tag{1.19}$$

We set

$$\alpha_{s,\Lambda}(A) := U_\Lambda(s)^* A U_\Lambda(s), \quad A \in \mathcal{A}, \quad s \in [0,1]. \tag{1.20}$$

By the results of [BMNS] and [NSY], conditions (i), (ii) and (iii) of Assumption 1.2 imply that the thermodynamic limit.

$$\alpha_s(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} \alpha_{s,\Lambda}(A), \quad A \in \mathcal{A}, \quad s \in [0,1], \tag{1.21}$$

exists and defines a strongly continuous path of automorphisms $[0,1] \ni s \mapsto \alpha_s$. We also have the limit of the inverse

$$\alpha_s^{-1}(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} \alpha_{s,\Lambda}^{-1}(A), \quad A \in \mathcal{A}, \quad s \in [0,1]. \tag{1.22}$$

See [NSY]. Our main theorem is as follows.

Theorem 1.3. *Under the Assumption 1.2, we have*

$$\varphi_s = \varphi_0 \circ \alpha_s, \quad s \in [0,1], \tag{1.23}$$

for α_s given in (1.21).

Remark 1.4. In fact the conditions (v), (vi), (vii) in Assumption 1.2 can be relaxed as follows. Suppose that there is a path of pure states $[0,1] \ni s \mapsto \varphi_s$ such that

(v) for each $s \in [0,1]$, φ_s is a $\tau_{\Phi(s)}$-ground state,

(vi) There exists a $\gamma > 0$ such that $\sigma(H_{\varphi_s,\Phi(s)}) \setminus \{0\} \subset [2\gamma, \infty)$ for all $s \in [0,1]$, where $\sigma(H_{\varphi_s,\Phi(s)})$ is the spectrum of $H_{\varphi_s,\Phi(s)}$. The eigenvalue 0 of $H_{\varphi_s,\Phi(s)}$ is non-degenerate.

(vii) The condition (vii) of Assumption 1.2 holds for the path.

Then we have (1.23) for α_s given in (1.21).

Our motivation to develop this bulk version of automorphic equivalence was the index theorems for SPT-phases [O2] and [O3]. In [O2] and [O3], the path of interactions was required to have a uniform spectral gap for corresponding local Hamiltonians. It is a bit unpleasant that we have to ask for the existence of the gap for local Hamiltonians while what we really would like to investigate is the bulk. From our Theorem 1.3, combined with Theorem 2.6, and the proof of Proposition 3.5 of [O2], we obtain the following version of the index theorem for the time reversal symmetry.

Theorem 1.5. *Let $\Phi(\cdot\ ;s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\rm loc}$ be a path of time-reversal interactions satisfying Assumption 1.2. Then $\mathbb{Z}_2$-index defined in Definition 3.3 of [O2] is constant along the path.*

From our Theorem 1.3, combined with Theorem 2.9 of [O3], and the proof of Proposition 3.5 of [O2], we obtain the following version of the index theorem for the reflection symmetry.

Theorem 1.6. *Let $\Phi(\cdot\ ;s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\rm loc}$ be a path of reflection invariant interactions satisfying Assumption 1.2. Then $\mathbb{Z}_2$-index defined in Definition 3.3 of [O3] is constant along the path.*

The rest of the paper is devoted to the proof of Theorem 1.3.

2 Proof of the Theorem 1.3

Throughout this Section, we will always assume Assumption 1.2. For $s \in [0,1]$ and $A \in \mathcal{A}$, we set

$$I_s(A) := \int dt\, \omega_\gamma(t) \tau^t_{\Phi(s)}(A). \tag{2.1}$$

The integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$.

We need the following Lemma for the proof.

Lemma 2.1. *Fix $0 < \beta = \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f(t) := t^{-1}\exp(-t^{\beta_1})$, $f_0(t) := \exp(-t^{\beta_1})$, $f_1(t) := \exp(-t^{\beta_2})$, $f_2(t) := t^{-2(\nu+2)}\exp(-t^{\beta_3})$, $g(t) := \exp(-t^{\beta_4})$, $\zeta(t) := \exp(-t^{\beta_5})$. (Here β is the one in (vii) of Assumption 1.2.) Then we have the following.*

1. *For any $s \in [0,1]$, we have*

$$\alpha_s^{-1}(\mathcal{A}_{\rm loc}) \subset \mathcal{D}_f \subset \mathcal{D}_{f_0} \subset \mathcal{D}_{f_1} \subset \mathcal{D}_{f_2} \subset \mathcal{D}_g \subset \mathcal{D}_\zeta.$$

2. *We have $\tau^t_{\Phi(s)}(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$ and there is a non-negative non-decreasing function on $\mathbb{R}_{\ge 0}$, $b_{f,f_1}(t)$ such that*

$$\int dt\, \omega_\gamma(t)\ |t| \cdot b_{f,f_1}(|t|) < \infty, \tag{2.2}$$

$$\sup_{s\in[0,1]} \left\| \tau^t_{\Phi(s)}(A) \right\|_{f_1} \le b_{f,f_1}(|t|) \left\|A\right\|_f, \quad A \in \mathcal{D}_f. \tag{2.3}$$

3. *We have $\mathcal{D}_\zeta \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)})$ for any $s \in [0,1]$.*

4. *There is a constant $C^{(1)}_{f_2,\zeta} > 0$ such that*

$$\sup_{s\in[0,1]} \left\|\delta_{\Phi(s)}(A)\right\|_\zeta,\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\Phi_N(s)}(A)\right\|_\zeta \le C^{(1)}_{f_2,\zeta} \left\|A\right\|_{f_2} \tag{2.4}$$

$$\sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}(s)}(A)\right\|_\zeta,\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}_N(s)}(A)\right\|_\zeta \le C^{(1)}_{f_2,\zeta} \left\|A\right\|_{f_2} \tag{2.5}$$

$$\sup_{s,s_0\in[0,1], 0<|s-s_0|\le\varepsilon} \left\|\delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0} - \dot{\Phi}(s_0)}(A)\right\|_\zeta,\ \sup_{N\in\mathbb{N}} \sup_{s,s_0\in[0,1], 0<|s-s_0|\le\varepsilon} \left\|\delta_{\frac{\Phi_N(s)-\Phi_N(s_0)}{s-s_0} - \dot{\Phi}_N(s_0)}(A)\right\|_\zeta$$
$$\le b(\varepsilon) C^{(1)}_{f_2,\zeta} \left\|A\right\|_{f_2} \tag{2.6}$$

for all $A \in \mathcal{D}_{f_2}$. (Here the meaning of the inequality is that each term on the left hand side is bounded by the right hand side. We use this notation throughout this article.) In particular, $\delta_{\Phi(s)}(\mathcal{D}_{f_2}) \subset \mathcal{D}_\zeta$, for any $s \in [0,1]$. (Recall $b(\varepsilon)$ in Assumption 1.2 (iv).)

5. *For any* $A \in \mathcal{D}_f$, *and* $(s', u', s'', s''') \in [0,1] \times \mathbb{R} \times [0,1] \times [0,1]$, *we have* $\tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_{f_2} \subset \mathcal{D}_\zeta \subset D(\delta_{\Phi(s')}) \cap D(\delta_{\dot{\Phi}(s')})$ *and* $\delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A), \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta$. *For any compact intervals* $[a,b]$, $[c,d]$ *of* $\mathbb{R}$ *and* $A \in \mathcal{D}_f$, *the maps:*

$$[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \mapsto \tau^u_{\Phi(s)} \circ \delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A}, \tag{2.7}$$

and

$$[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \mapsto \tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A} \tag{2.8}$$

are uniformly continuous with respect to $\|\cdot\|$, *and maps*

$$[0,1] \times [c,d] \times [0,1] \times [0,1] \ni (s', u', s'', s''') \mapsto \delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta \tag{2.9}$$

$$[0,1] \times [c,d] \times [0,1] \times [0,1] \ni (s', u', s'', s''') \mapsto \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta \tag{2.10}$$

are uniformly continuous with respect to $\|\cdot\|_\zeta$.

6. *For any* $A \in \mathcal{D}_f$, $\alpha_s^{-1}(A)$ *is differentiable with respect to* $\|\cdot\|$ *and*

$$\frac{d}{ds}\alpha_s^{-1}(A) = \int dt \omega_\gamma(t) \int_0^t du \tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)} \left(\alpha_s^{-1}(A) \right) \right) \tag{2.11}$$

The right hand side can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$.

7. *For any* $A \in \mathcal{D}_f$, *the integral*

$$\int dt \omega_\gamma(t) \int_0^t du \tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)}(A) \right) \tag{2.12}$$

$$\int dt \ \omega_\gamma(t) \int_0^t du \ \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\dot{\Phi}(s)} \right) \circ \tau^u_{\Phi(s)}(A) \tag{2.13}$$

are well-defined as a Bochner integral with respect to $(\mathcal{A}, \|\cdot\|)$.

8. *For any* $A \in \mathcal{D}_f$ *and* $s \in [0,1]$, *we have* $I_s(A) \in \mathcal{D}_{f_1}$.

9. *For each* $A \in \mathcal{A}$, $\mathbb{R} \times [0,1] \ni (u,s) \to \tau^u_{\Phi(s)}(A) \in \mathcal{A}$ *is continuous with respect to the norm* $\|\cdot\|$.

10. *For any* $A \in \mathcal{D}_f$, *the integrals*

$$\int dt \omega_\gamma(t) \int_0^t du \delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A), \quad \int_0^t du \delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A), \tag{2.14}$$

are well-defined as Bochner integrals with respect to $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$.

The proof of Lemma 2.1 is given in Section 4. Throughout Section 2 and Section 3 (but not in Section 4), we fix $0 < \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f, f_0, f_1, f_2, g, \zeta$, given in Lemma 2.1, and apply Lemma 2.1.

In Section 3, we prove the following:

Proposition 2.2. *For any $A \in \mathcal{D}_f$, we have*

$$\dot{\varphi}_s\left(I_s(A)\right) = 0, \quad s \in [0,1]. \tag{2.15}$$

Note that by *8.* of Lemma 2.1, $I_s(A)$ belongs to $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta$, and that $\dot{\varphi}_s\left(I_s(A)\right)$ in Proposition 2.2 is well-defined by (vii) of Assumption 1.2. Note that from its definition, $I_s(A)$ does not have "off-diagonal parts," which holds for finite systems as well by the equation

$$\forall A \in \mathcal{A}_\Lambda, \ \left[\int dt\ \omega_\gamma(t)\tau^t_{\Phi(s),\Lambda}(A), P_s(\Lambda)\right] = 0."$$

We now prove Theorem 1.3 using this proposition. In order to prove the Theorem, it suffices to show

$$\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = 0, \tag{2.16}$$

for any $X \in \mathcal{A}_{\rm loc}$. Note that from Assumption 1.2 (vii), and *1.* of Lemma 2.1, the function $[0,1] \ni s \to \varphi_s \circ \alpha_{s_0}^{-1}(X)$ is differentiable for any $X \in \mathcal{A}_{\rm loc}$ and $s_0 \in [0,1]$. Furthermore, from *6.* of Lemma 2.1, $[0,1] \ni s \mapsto \alpha_s^{-1}(X) \in \mathcal{A}$ is differentiable with respect to the norm for any $X \in \mathcal{A}_{\rm loc} \subset \mathcal{D}_f$. Therefore, for any $X \in \mathcal{A}_{\rm loc}$, $[0,1] \ni s \to \varphi_s \circ \alpha_s^{-1}(X)$ is differentiable, the left hand side of (2.16) makes sense, and we have

$$\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = \dot{\varphi}_s \circ \alpha_s^{-1}(X) + \varphi_s \circ \frac{d}{ds}\alpha_s^{-1}(X), \quad X \in \mathcal{A}_{\rm loc}. \tag{2.17}$$

For the proof of (2.16), we use the following Lemma.

Lemma 2.3. *For any $A \in \mathcal{D}_f$,*

$$A - I_s(A) = -\int dt\omega_\gamma(t)\int_0^t du\delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A). \tag{2.18}$$

The integrand of the right hand side is continuous with respect to $\|\cdot\|_\zeta$ and the integral can be understood as the Bochner integral of $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$.

Proof. The latter part is *5., 10.* of lemma 2.1. To show (2.18), recall the Duhamel formula

$$A - \tau^t_{\Phi(s)}(A) = \int_0^t du\ \left(-\delta_{\Phi(s)}\right) \circ \tau^u_{\Phi(s)}(A), \quad A \in \mathcal{D}_f. \tag{2.19}$$

Here we used the fact that $\tau^u_{\Phi(s_0)}\left(\mathcal{D}_f\right) \subset \mathcal{D}_{f_1} \subset \mathcal{D}_\zeta \subset D\left(\delta_{\Phi(s)}\right)$, which follows from *2.,1,.3.* of Lemma 2.1.

We multiply (2.19) by $\omega_\gamma(t)$ and integrate over $t \in \mathbb{R}$. Then recalling (1.16), we obtain

$$\begin{aligned} A - I_s(A) &= \int dt\ \omega_\gamma(t)A - \int dt\ \omega_\gamma(t)\tau^t_{\Phi(s)}(A) \\ &= \int dt\ \omega_\gamma(t)\int_0^t du\ \left(-\delta_{\Phi(s)}\right) \circ \tau^u_{\Phi(s)}(A), \quad A \in \mathcal{D}_f. \end{aligned} \tag{2.20}$$

□

In order to show (2.16), we need to know $\dot{\varphi_s}$ on $\mathcal{D}_f$. From Proposition 2.2 and Lemma 2.3, for any $A \in \mathcal{D}_f$, we have

$$\left(\dot{\varphi_s}\right)(A) = \left(\dot{\varphi_s}\right)(A) - \left(\dot{\varphi_s}\right)(I_s(A)) = -\int dt\omega_\gamma(t)\int_0^t du\dot{\varphi_s}\left(\delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A)\right). \tag{2.21}$$

Here we used the Bochner integrability of the right hand side of (2.18) with respect to $\|\cdot\|_\zeta$, and the continuity of $\dot{\varphi_s}$ (1.10) with respect to $\|\cdot\|_\zeta$.

As φ_s is the $\tau_{\Phi(s)}$-ground state, we have

$$\varphi_s \circ \delta_{\Phi(s)}(B) = 0, \quad B \in \mathcal{D}_{f_1}, \quad s \in [0,1]. \tag{2.22}$$

(Recall that $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta \subset D(\delta_{\Phi(s)})$, from *1., 3.* of Lemma 2.1.) Differentiating this by s, we obtain

$$\dot{\varphi_s} \circ \delta_{\Phi(s)}(B) + \varphi_s \circ \delta_{\dot{\Phi}(s)}(B) = 0, \quad B \in \mathcal{D}_{f_1}, \quad s \in [0,1]. \tag{2.23}$$

More precisely, note that

$$\delta_{\Phi(s)}\left(\mathcal{D}_{f_1}\right) \subset \delta_{\Phi(s)}\left(\mathcal{D}_{f_2}\right) \subset \mathcal{D}_\zeta, \quad s \in [0,1], \tag{2.24}$$

by Lemma 2.1, *1., 4.*. Therefore, for $B \in \mathcal{D}_{f_1}$, we have $\delta_{\Phi(s)}(B) \in \mathcal{D}_\zeta$, $s \in [0,1]$, and for any $s, s_0 \in [0,1]$ with $s \neq s_0$, we have

$$\begin{aligned}
&\left|-\left(\dot{\varphi}_{s_0} \circ \delta_{\Phi(s_0)}(B) + \varphi_{s_0} \circ \delta_{\dot{\Phi}(s_0)}(B)\right)\right| \\
&= \left|\frac{\varphi_s \circ \delta_{\Phi(s)}(B) - \varphi_{s_0} \circ \delta_{\Phi(s_0)}(B)}{s - s_0} - \left(\dot{\varphi}_{s_0} \circ \delta_{\Phi(s_0)}(B) + \varphi_{s_0} \circ \delta_{\dot{\Phi}(s_0)}(B)\right)\right| \\
&\leq \left|\varphi_s\left(\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B)\right)\right| + \left|\frac{\varphi_s \circ \delta_{\Phi(s_0)}(B) - \varphi_{s_0} \circ \delta_{\Phi(s_0)}(B)}{s - s_0} - \left(\dot{\varphi}_{s_0} \circ \delta_{\Phi(s_0)}(B)\right)\right| \\
&+ \left|\left(\varphi_s - \varphi_{s_0}\right)\left(\delta_{\dot{\Phi}(s_0)}(B)\right)\right|.
\end{aligned} \tag{2.25}$$

As $\delta_{\Phi(s_0)}(B) \in \mathcal{D}_\zeta$, the second and the third terms of the last line converge to 0 as $s \to s_0$. The first term of the last line can be bounded as

$$\begin{aligned}
&\left|\varphi_s\left(\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B)\right)\right| \leq \left\|\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B)\right\| \\
&\leq b(|s - s_0|)C^{(1)}_{f_2,\zeta}\|B\|_{f_2} \to 0, \quad s \to s_0,
\end{aligned} \tag{2.26}$$

and goes to 0 as $s \to s_0$. Here, in the last line, we used *4.* of Lemma 2.1 and recalled $\mathcal{D}_{f_1} \subset \mathcal{D}_{f_2}$, from *1.* of Lemma 2.1, and (iv) of Assumption 1.2. Hence we obtain (2.23).

From this and (2.21), for $A \in \mathcal{D}_f$, recalling $\tau^u_{\Phi(s)}(A) \in \mathcal{D}_{f_1}$ by *2.* of Lemma 2.1, we have

$$\left(\dot{\varphi_s}\right)(A) = \int dt\omega_\gamma(t)\int_0^t du\varphi_s \circ \delta_{\dot{\Phi}(s)}\left(\tau^u_{\Phi(s)}(A)\right). \tag{2.27}$$

For any $X \in \mathcal{A}_{\mathrm{loc}}$, recall that $\alpha_s^{-1}(X) \in \alpha_s^{-1}\left(\mathcal{A}_{\mathrm{loc}}\right) \subset \mathcal{D}_f \subset \mathcal{D}_\zeta$ by *1.* of Lemma 2.1. From (2.17), (2.27) and *6.* of Lemma 2.1, we have

$$\begin{aligned}
&\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = \dot{\varphi}_s \circ \alpha_s^{-1}(X) + \varphi_s \circ \frac{d}{ds}\alpha_s^{-1}(X) \\
&= \int dt\omega_\gamma(t)\int_0^t du\varphi_s \circ \delta_{\dot{\Phi}(s)}\left(\tau^u_{\Phi(s)} \circ \alpha_s^{-1}(X)\right) + \int dt\omega_\gamma(t)\int_0^t du\varphi_s\left(\tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)}\left(\tau^{-u}_{\Phi(s)}\left(\alpha_s^{-1}(X)\right)\right)\right) = 0
\end{aligned} \tag{2.28}$$

Here we used the fact that ω_γ is an even function, and that φ_s is $\tau_{\Phi(s)}$-invariant because it is the $\tau_{\Phi(s)}$-ground state.

Hence we have proven the Theorem 1.3.

3 Proof of Proposition 2.2

Throughout this Section, we keep Assumption 1.2. We also continue to use the same $0 < \beta = \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f, f_0, f_1, f_2, g, \zeta$, as given in Lemma 2.1.

Let $(\mathcal{H}_s, \pi_s, \Omega_s)$ be the GNS triple of φ_s. Let $H_s := H_{\varphi_s,\Phi(s)}$ be the associated bulk Hamiltonian. The key property of I_s we use is the following.

Lemma 3.1. *For any $A \in \mathcal{A}$, we have*

$$\pi_s\left(I_s\left(A\right)\right)\Omega_s = \varphi_s(A)\Omega_s. \tag{3.1}$$

Proof. As the Fourier transform $\hat{\omega}_\gamma$ of ω_γ has support in $[-\gamma, \gamma]$, (v) and (vi) of Assumption 1.2 and (1.16) implies:

$$\hat{\omega}_\gamma\left(H_s\right) = \frac{1}{\sqrt{2\pi}}\left|\Omega_s\right\rangle\left\langle\Omega_s\right|. \tag{3.2}$$

From the definition of I_s, substituting (3.2), we have

$$\begin{aligned}\pi_s\left(I_s\left(A\right)\right)\Omega_s &= \int dt\ \omega_\gamma(t)\pi_s\left(\tau^t_{\Phi(s)}(A)\right)\Omega_s\\ &= \int dt\ \omega_\gamma(t)e^{itH_s}\pi_s(A)\Omega_s = \sqrt{2\pi}\hat{\omega}_\gamma(H_s)\pi_s(A)\Omega_s = \varphi_s(A)\Omega_s.\end{aligned} \tag{3.3}$$

□

From this, we immediately obtain the following decoupling.

Lemma 3.2. *For any $A, B \in \mathcal{A}$ and $s \in [0, 1]$, we have*

$$\varphi_s\left(B^* I_s(A)\right) = \varphi_s(B^*)\varphi_s(A). \tag{3.4}$$

Lemma 3.3. *For each $s \in [0, 1]$ and $A \in \mathcal{D}_f$, the integrand of*

$$V_s(A) := \int dt\ \omega_\gamma(t)\int_0^t du\ \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\dot{\Phi}(s)}\right) \circ \tau^u_{\Phi(s)}(A), \tag{3.5}$$

is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$. For any $A \in D_f$, $[0, 1] \ni s \to I_s(A) \in \mathcal{A}$ is differentiable with respect to $\|\cdot\|$ and

$$\frac{d}{ds}I_s(A) = V_s(A). \tag{3.6}$$

Proof. Let $A \in \mathcal{D}_f$. That the integrand of (3.5) is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$, follow from *5.* and *7.*, of Lemma 2.1, respectively.

Next, recall the Duhamel formula

$$\tau^t_{\Phi(s)}(A) - \tau^t_{\Phi(s_0)}(A) = \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\Phi(s)} - \delta_{\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A), \quad A \in \mathcal{D}_f. \tag{3.7}$$

Here we used the fact that $\tau^u_{\Phi(s_0)}(\mathcal{D}_f) \subset \mathcal{D}_\zeta \subset D\left(\delta_{\Phi(s)}\right)$, which follows from *2.*, *1.*, *3.*, of Lemma 2.1. By *5.* of Lemma 2.1, the integrand on the right hand side is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$.

We multiply (3.7) by $\omega_\gamma(t)$ and integrate over $t \in \mathbb{R}$. Then we obtain

$$\begin{aligned} I_s(A) - I_{s_0}(A) &= \int dt\, \omega_\gamma(t) \tau^t_{\Phi(s)}(A) - \int dt\, \omega_\gamma(t) \tau^t_{\Phi(s_0)}(A) \\ &= \int dt\, \omega_\gamma(t) \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\Phi(s)} - \delta_{\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A), \quad A \in \mathcal{D}_f. \end{aligned} \tag{3.8}$$

By *5.* of Lemma 2.1, all the integrands are continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$. For any $A \in \mathcal{D}_f$,

$$\begin{aligned} &\left\| \frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A) \right\| \\ &\le \int dt\, \omega_\gamma(t) \int_{[0,t]} du \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} \right) \circ \tau^u_{\Phi(s_0)}(A) - \tau^{t-u}_{\Phi(s_0)} \circ \left(\delta_{\dot\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \\ &\le \int dt\, \omega_\gamma(t) \int_{[0,t]} du \left(\begin{aligned} &\left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)} \right) \circ \left(\delta_{\dot\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \\ &+ \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot\Phi(s_0)}\right) \right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \end{aligned} \right). \end{aligned} \tag{3.9}$$

Here and after, $\int_{[0,t]} du$ always indicates Lebesgue integral (i.e. without sign) over the measurable set $[0, t]$. From *9.* of Lemma 2.1, for each t, u, we have

$$\lim_{s \to s_0} \left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)} \right) \circ \left(\delta_{\dot\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| = 0, \quad A \in \mathcal{D}_f. \tag{3.10}$$

By *4.* of Lemma 2.1, for each t, u, we have

$$\lim_{s \to s_0} \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot\Phi(s_0)}\right) \right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \le \limsup_{s \to s_0} b(|s - s_0|) C^{(1)}_{f_2,\zeta} \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_2} = 0, \quad A \in \mathcal{D}_f. \tag{3.11}$$

Here we used $\tau^u_{\Phi(s_0)}(A) \in \mathcal{D}_{f_1} \subset \mathcal{D}_{f_2}$ which follows from Lemma 2.1, *1.*,*2.* Furthermore, from *2.*, *4.* of Lemma 2.1, for $A \in \mathcal{D}_f$,

$$\begin{aligned} &\left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)} \right) \circ \left(\delta_{\dot\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \le 2C^{(1)}_{f_2,\zeta} \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_2} \\ &\le 2C^{(1)}_{f_2,\zeta} \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_1} \le 2C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \|A\|_f. \end{aligned} \tag{3.12}$$

Note that from $0 < \beta_3 < \beta_2 < 1$, we have $\sup_N \frac{f_1(N)}{f_2(N)} < \infty$. Similarly, from *2.*, *4.* of Lemma 2.1,

$$\left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot{\Phi}(s_0)} \right) \right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \le b(1) C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \|A\|_f . \tag{3.13}$$

Combining this (2.2) in *2.* of Lemma 2.1, from Lebesgue's convergence theorem, we obtain

$$\lim_{s \to s_0} \left\| \frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A) \right\| = 0, \quad A \in \mathcal{D}_f. \tag{3.14}$$

□

Lemma 3.4. *For any $A, B \in \mathcal{D}_f$ and $s \in [0,1]$, $A, B^*, B^* I_s(A)$ belong to $\mathcal{D}_\zeta$ and we have*

$$\begin{aligned} &\dot{\varphi}_s \left(B^* I_s(A) \right) + \int dt\, \omega_\gamma(t) \int_0^t du \varphi_s \left(B^* \tau^{t-u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \circ \tau^u_{\Phi(s)}(A) \right) \\ &= \dot{\varphi}_s(B^*) \varphi_s(A) + \varphi_s(B^*) \dot{\varphi}_s(A). \end{aligned} \tag{3.15}$$

Proof. For any $A, B \in \mathcal{D}_f \subset \mathcal{D}_\zeta$ and $s_0 \in [0,1]$, $B^* I_{s_0}(A)$ belongs to $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta$ (the inclusion *1.* of Lemma 2.1) because of *8.*, of Lemma 2.1 and Lemma B.1. Therefore, by (vii) of Assumption 1.2, $[0,1] \ni s \mapsto \varphi_s \left(B^* I_{s_0}(A) \right) \in \mathbb{C}$ is differentiable. For any $s, s_0 \in [0,1]$ with $s \neq s_0$, we have

$$\begin{aligned} &\frac{1}{s - s_0} \left(\varphi_s \left(B^* I_s(A) \right) - \varphi_{s_0} \left(B^* I_{s_0}(A) \right) \right) - \varphi_{s_0} \left(B^* V_{s_0}(A) \right) - \dot{\varphi}_{s_0} \left(B^* I_{s_0}(A) \right) \\ &= \varphi_s \left(B^* \left(\frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A) \right) \right) - \dot{\varphi_{s_0}} \left(B^* I_{s_0}(A) \right) + \frac{1}{s - s_0} \left(\varphi_s - \varphi_{s_0} \right) \left(B^* I_{s_0}(A) \right) \\ &+ \left(\varphi_s - \varphi_{s_0} \right) \left(B^* V_{s_0}(A) \right). \end{aligned} \tag{3.16}$$

The right hand side goes to 0 as $s \to s_0$, because of Lemma 3.3 and the differentiability of $[0,1] \ni s \mapsto \varphi_s \left(B^* I_{s_0}(A) \right) \in \mathbb{C}$. On the other hand, the first part of the left hand side of (3.16) is

$$\frac{1}{s - s_0} \left(\varphi_s \left(B^* I_s(A) \right) - \varphi_{s_0} \left(B^* I_{s_0}(A) \right) \right) = \frac{1}{s - s_0} \left(\varphi_s \left(B^* \right) \varphi_s \left(A \right) - \varphi_{s_0} \left(B^* \right) \varphi_{s_0} \left(A \right) \right), \tag{3.17}$$

because of Lemma 3.2 and converges to

$$\dot{\varphi_{s_0}}(B^*) \varphi_{s_0}(A) + \varphi_{s_0}(B^*) \dot{\varphi_{s_0}}(A), \tag{3.18}$$

as $s \to s_0$. Hence we obtain (3.15).

□

For each $s \in [0,1]$, we introduce the left ideal $\mathcal{L}_s$ of $\mathcal{A}$ by

$$\mathcal{L}_s := \left\{ A \in \mathcal{A} \mid \varphi_s(A^* A) = 0 \right\}. \tag{3.19}$$

Lemma 3.5. *For any $A \in \mathcal{D}_f$ and $s \in [0,1]$, $I_s(A) - \varphi_s(A)\mathbb{I}$ belongs to $\mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$.*

Proof. Let $A \in \mathcal{D}_f$. Let $(\mathcal{H}_s, \pi_s, \Omega_s)$ be the GNS triple of φ_s. That $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{D}_{f_1}$ is Lemma 2.1 *8.*. To show $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^*$, recall Lemma 3.1. From the latter Lemma, we obtain

$$\pi_s\left(I_s(A) - \varphi_s(A)\right)\Omega_s = \pi_s\left(I_s(A^*) - \varphi_s(A^*)\right)\Omega_s = 0, \tag{3.20}$$

which means $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^*$, because $I_s(A)^* = I_s(A^*)$. □

Lemma 3.6. *For any $A \in \mathcal{L}_s \cap \mathcal{D}_{f_1}$, there is a positive sequence $u_{N,A} \in \mathcal{A}_{\Lambda_N}$, $N \in \mathbb{N}$ with $\|u_{N,A}\| \le 1$ such that*

$$\|A(1 - u_{N,A})\|_g \to 0, \tag{3.21}$$

and

$$\lim_{N\to\infty} \varphi_s(u_{N,A}) = 0, \tag{3.22}$$

and

$$\operatorname{dist}\left(u_{N,A}, \mathcal{L}_s\right) := \inf_{x \in \mathcal{L}_s} \|x - u_{N,A}\| \to 0, \quad N \to \infty. \tag{3.23}$$

Proof. Choose $\beta_4 < \beta' < \beta_2$ and set $h(t) := e^{t^{\beta'}}$. Then we have

$$\lim_{N\to\infty} \frac{1}{g(N)\sqrt{h(N)}} = 0, \quad \lim_{N\to\infty} h(N) f_1(N) = 0. \tag{3.24}$$

Let $A \in \mathcal{L}_s \cap \mathcal{D}_{f_1}$. Set

$$u_{N,A} := \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} h(N)\mathbb{E}_N(A^*A). \tag{3.25}$$

Clearly, $\|u_{N,A}\| \le 1$, and $0 \le u_{N,A} \le 1$. Then we have

$$\begin{aligned}
&\left\| u_{N,A} - \left(1 + h(N)(A^*A)\right)^{-1} h(N)(A^*A) \right\| \\
&= \left\| \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} h(N)\mathbb{E}_N(A^*A) - \left(1 + h(N)(A^*A)\right)^{-1} h(N)(A^*A) \right\| \\
&= \left\| \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} - \left(1 + h(N)(A^*A)\right)^{-1} \right\| \\
&= \left\| \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} \left(h(N)\left(A^*A - \mathbb{E}_N(A^*A)\right)\right) \left(1 + h(N)(A^*A)\right)^{-1} \right\| \\
&\le h(N) f_1(N) \left\|A^*A\right\|_{f_1} \to 0, \quad N \to \infty,
\end{aligned} \tag{3.26}$$

from (3.24). As $\left(1 + h(N)(A^*A)\right)^{-1} h(N)(A^*A) \in \mathcal{L}_s$, we obtain (3.22), (3.23). We also have

$$\begin{aligned}
&\|A(1 - u_{N,A})\|^2 \le \|(1 - u_{N,A})(A^*A - \mathbb{E}_N(A^*A))(1 - u_{N,A})\| + \|(1 - u_{N,A})(\mathbb{E}_N(A^*A))(1 - u_{N,A})\| \\
&\le \left\|A^*A\right\|_{f_1} f_1(N) + \left\|(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\mathbb{E}_N(A^*A)(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\right\| \\
&= \left\|A^*A\right\|_{f_1} f_1(N) + \frac{1}{h(N)} \left\|(1 + h(N)\mathbb{E}_N(A^*A))^{-1}h(N)\mathbb{E}_N(A^*A)(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\right\| \\
&\le \left\|A^*A\right\|_{f_1} f_1(N) + \frac{1}{h(N)} =: \varepsilon_N^2.
\end{aligned} \tag{3.27}$$

For $M > N$, we have

$$\begin{aligned}\frac{\|A(1-u_{N,A}) - \mathbb{E}_M\left(A(1-u_{N,A})\right)\|}{g(M)} &= \frac{\|\left(A - \mathbb{E}_M\left(A\right)\right)(1-u_{N,A})\|}{g(M)} \\ \le \|A\|_{f_1} \sup_{M>N}\left(\frac{f_1(M)}{g(M)}\right) &=: \varepsilon_N^{'} \to 0, \quad N \to \infty.\end{aligned} \tag{3.28}$$

For $M \le N$, we have

$$\frac{\|A(1-u_{N,A}) - \mathbb{E}_M\left(A\left(1-u_{N,A}\right)\right)\|}{g(M)} \le \frac{2\,\|A(1-u_{N,A})\|}{g(N)}\frac{g(N)}{g(M)} \le \frac{2\,\|A(1-u_{N,A})\|}{g(N)} \le \frac{2\varepsilon_N}{g(N)} \to 0, \quad N\to\infty, \tag{3.29}$$

from (3.24) and $0 < \beta_4 < \beta_2 < 1$. Hence we obtain,

$$\|A(1-u_{N,A})\|_g \to 0, \tag{3.30}$$

proving the Lemma. □

Now we can prove Proposition 2.2.

Proof of Proposition 2.2. Fix $A \in \mathcal{D}_f$, and $s \in [0,1]$. By Lemma 3.5, $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$. Applying Lemma 3.6 to $(I_s(A) - \varphi_s(A)\mathbb{I})^* \in \mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$ we obtain a sequence $u_N \in \mathcal{A}_{\Lambda_N}$, $N \in \mathbb{N}$ such that $\|u_N\| \le 1$

$$\|(1-u_N)^*\left(I_s(A) - \varphi_s(A)\mathbb{I}\right)\|_g = \|\left(I_s(A) - \varphi_s(A)\mathbb{I}\right)^*(1-u_N)\|_g \to 0, \tag{3.31}$$

$$\mathrm{dist}(u_N, \mathcal{L}_s) \to 0, \tag{3.32}$$

as $N \to 0$. Applying Lemma 3.4 to $u_N \in \mathcal{D}_f$ and $A \in \mathcal{D}_f$, we have

$$\begin{aligned}&\dot{\varphi}_s\left(u_N^*\left(I_s(A) - \varphi_s(A)\mathbb{I}\right)\right) \\ &= -\int dt\, \omega_\gamma(t) \int_0^t du \varphi_s\left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) + \varphi_s(u_N^*)\dot{\varphi}_s(A).\end{aligned} \tag{3.33}$$

By (3.32), we have $\lim_{N\to\infty} \varphi_s\left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) = 0$. On the other hand, from *2.*, and *4.*, of Lemma 2.1, since $\|u_N\| \le 1$, we have, as in (3.12), the bound

$$\left|\varphi_s\left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right)\right| \le C_{f_2,\zeta}^{(1)} b_{f,f_1}(|u|)\left(1 + \sup_N \frac{f_1(N)}{f_2(N)}\right)\|A\|_f < \infty. \tag{3.34}$$

From *2.* of Lemma 2.1,

$$\int dt\, \omega_\gamma(t) \int_{[0,t]} du b_{f,f_1}(|u|) < \infty. \tag{3.35}$$

Therefore, by Lebesgue's convergence theorem, we have

$$\lim_{N\to\infty} \int dt\, \omega_\gamma(t) \int_0^t du \varphi_s\left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) = 0. \tag{3.36}$$

We also have $\lim_{N\to\infty} \varphi_s(u_N^*)\dot{\varphi}_s(A) = 0$, from (3.32). Therefore, the right hand side of (3.33) goes to 0 as $N \to \infty$. The left hand side of (3.33) goes to $\dot{\varphi}_s\left((I_s(A) - \varphi_s(A)\mathbb{I})\right)$ as $N \to \infty$, because of the continuity (1.10) of $\dot{\varphi}_s$ and (3.31). Clearly, $\dot{\varphi}_s(\mathbb{I}) = 0$. Therefore, we obtain $\dot{\varphi}_s\left(I_s(A)\right) = 0$.

□

4 Technical Lemmas

In this Section, we prove various lemmas used in this paper. We assume (i), (ii), (iii) of Assumption 1.2 throughout this section. For $t \in \mathbb{R}$, $[t]$ indicates the largest integer less than or equal to t.

4.1 Properties of $\tau_{\Phi(s)}$

First we recall several facts from [BMNS] and [NSY]. Define positive functions $F(r)$ and $F_1(r)$ on $\mathbb{R}_{\geq 0}$ by $F(r) := (1+r)^{-(\nu+1)}$, $F_1(r) := (1+r)^{-(\nu+1)} e^{-r}$. For a path of interactions satisfying Assumption 1.2, there exist positive constants C_1', v satisfying the following Lieb-Robinson bound: For any $X, Y \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $s \in [0,1]$ and $t \in \mathbb{R}$, we have

$$\left\| \left[\tau^t_{\Phi(s)}(A), B \right] \right\|, \quad \left\| \left[\tau^t_{\Phi(s),\Lambda}(A), B \right] \right\| \leq C_1' e^{v|t|} \sum_{x \in X, y \in Y} F_1(d(x,y)) \|A\| \|B\| . \tag{4.1}$$

We fix the constant v and call it the Lieb-Robinson velocity. From this and Corollary 4.4. of [NSY] (Proposition A.1) we obtain the following.

Lemma 4.1. *There is a positive constant $C_1 > 0$ such that*

$$\left\| \tau^t_{\Phi(s),\Lambda}(A) - \mathbb{E}_N \left(\tau^t_{\Phi(s),\Lambda}(A) \right) \right\|, \ \left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N \left(\tau^t_{\Phi(s)}(A) \right) \right\| \leq C_1 |\Lambda_M| e^{v|t| - (N-M)} \|A\| , \tag{4.2}$$

for any $M, N \in \mathbb{N}$ with $M \leq N$, $A \in \mathcal{A}_{\Lambda_M}$ and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$.

We also have the following (see Corollary 3.6 (3.80) of [NSY].)

Lemma 4.2. *There is a constant $C_4 > 0$ such that*

$$\sup_{s \in [0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(B) - \tau^{-u}_{\Phi(s)}(B) \right\| \leq C_4 |\Lambda_M| |u| e^{|u|v - (n-M)} \|B\| , \quad n \geq M, \quad u \in \mathbb{R}, \quad B \in \mathcal{A}_{\Lambda_M}. \tag{4.3}$$

It is standard to derive the following from Lemma 4.2 (cf. [BR1]).

Lemma 4.3. *For any $A \in \mathcal{A}$,*

$$\sup_{s \in [0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right\| \to 0, \tag{4.4}$$

uniformly in compact $u \in \mathbb{R}$. In particular, for each $A \in \mathcal{A}$, $\mathbb{R} \times [0,1] \ni (u,s) \to \tau^{-u}_{\Phi(s)}(A) \in \mathcal{A}$ is continuous with respect to the norm $\|\cdot\|$.

Lemma 4.4. *Suppose $f_1, f_2 : (0,\infty) \to (0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_i(t) = 0$, for $i = 1, 2$. Suppose that we have*

$$\lim_{N \to \infty} \left(\frac{|\Lambda_{[\frac{N}{2}]}| e^{-(N - [\frac{N}{2}])}}{f_2(N)} \right) = 0. \tag{4.5}$$

and

$$\lim_{N \to \infty} \frac{f_1\left([\frac{N}{2}] \right)}{f_2(N)} = 0. \tag{4.6}$$

Then

$$\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}\to 0,\quad A\in\mathcal{D}_{f_1}, \tag{4.7}$$

uniformly in compact $u\in\mathbb{R}$. *In particular, for each* $A\in\mathcal{D}_{f_1}$, $\mathbb{R}\times[0,1]\ni(u,s)\to\tau^{-u}_{\Phi(s)}(A)\in\mathcal{D}_{f_2}$ *is continuous with respect to the norm* $\|\cdot\|_{f_2}$.

Proof. Let $A\in\mathcal{D}_{f_1}$. From Lemma 4.3, we have

$$\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|\to 0. \tag{4.8}$$

Applying Lemma 4.2, for $N\le[\frac{n}{2}]$, we have

$$\begin{aligned}&\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right)\right\|\\&\le\left\|\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{n}{2}]}(A)\right)-\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{n}{2}]}(A)\right)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{n}{2}]}(A)\right)-\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{n}{2}]}(A)\right)\right)\right\|\\&+4\left\|\mathbb{E}_{[\frac{n}{2}]}(A)-A\right\|\\&\le 2C_4\left|\Lambda_{[\frac{n}{2}]}\right||u|e^{|u|v-(n-[\frac{n}{2}])}\|A\|+4f_1\left(\left[\frac{n}{2}\right]\right)\|A\|_{f_1}\end{aligned}\tag{4.9}$$

On the other hand, from Lemma 4.1 $N\ge\left[\frac{n}{2}\right]$,

$$\begin{aligned}&\left\|\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)\right)\right\|\le C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])},\\&\left\|\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)\right)\right\|\le C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}.\end{aligned}\tag{4.10}$$

Therefore, for $N\ge\left[\frac{n}{2}\right]$, we have

$$\begin{aligned}&\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right)\right\|\\&\le\left\|\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)-\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)-\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)-\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A)\right)\right)\right\|\\&+4\left\|\mathbb{E}_{[\frac{N}{2}]}(A)-A\right\|\\&\le 2C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}+4f_1\left(\left[\frac{N}{2}\right]\right)\|A\|_{f_1}\end{aligned}\tag{4.11}$$

Hence we obtain

$$\begin{aligned}\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}\le\max&\left\{\begin{array}{c}2C_4|u|e^{|u|v}\|A\|\dfrac{\left|\Lambda_{[\frac{n}{2}]}\right|e^{-(n-[\frac{n}{2}])}}{f_2\left([\frac{n}{2}]\right)}+4\dfrac{f_1\left([\frac{n}{2}]\right)}{f_2\left([\frac{n}{2}]\right)}\|A\|_{f_1},\\2C_1\|A\|\sup\limits_{N\ge[\frac{n}{2}]}\left(\dfrac{|\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}}{f_2(N)}\right)+\sup\limits_{N\ge[\frac{n}{2}]}\left(\dfrac{4f_1\left([\frac{N}{2}]\right)\|A\|_{f_1}}{f_2(N)}\right)\end{array}\right\}\\&+\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|,\end{aligned}\tag{4.12}$$

and $\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}$ converges to 0 as $n\to\infty$, uniformly in compact u.

□

Lemma 4.5. *Let $f, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = 0$. Suppose that*

$$\begin{aligned} &\int_{4v|t|\geq 1} dt\omega_\gamma(t) \frac{2|t|}{f_1(4v|t|)} < \infty, \\ &\sup_{N\in\mathbb{N}} \left(\frac{f(N - \left[\frac{N}{2}\right])}{f_1(N)} \right) < \infty, \\ &\sup_{N\in\mathbb{N}} \left(\frac{|\Lambda_N| e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_1(N)} \right) < \infty. \end{aligned} \tag{4.13}$$

Then $\tau^t_{\Phi(s)}(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$ and there is a non-negative non-decreasing function on $\mathbb{R}_{\geq 0}$, $b_{f,f_1}(t)$ such that

$$\int dt\ \omega_\gamma(t)\ |t| \cdot b_{f,f_1}(|t|) < \infty. \tag{4.14}$$

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \tau^t_{\Phi_n(s)}(A) \right\|_{f_1},\ \sup_{s\in[0,1]} \left\| \tau^t_{\Phi(s)}(A) \right\|_{f_1} \leq b_{f,f_1}(|t|) \left\|A\right\|_f, \quad A \in \mathcal{D}_f. \tag{4.15}$$

Proof. Let $A \in \mathcal{D}_f$. We have to estimate

$$\frac{\left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A) \right) \right\|}{f_1(N)}, \quad N \in \mathbb{N}. \tag{4.16}$$

From Lemma 4.1 for $A \in \mathcal{D}_f$, $N, k \in \mathbb{N}$ with $k < N$, we obtain

$$\begin{aligned} &\left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A) \right) \right\| \leq \left\| \tau^t_{\Phi(s)}(\mathbb{E}_k(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_k(A)) \right) \right\| + 2\left\| A - (\mathbb{E}_k(A)) \right\| \\ &\leq 2\left\|A\right\|_f f(k) + C_1 \|A\| |\Lambda_k| e^{v|t| - (N-k)}. \end{aligned} \tag{4.17}$$

For $N \in \mathbb{N}$ with $4v\,|t| \leq N$, we use this bound with $k := N - \left[\frac{N}{2}\right]$ to estimate (4.16). Then we have

$$\begin{aligned} \left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A) \right) \right\| &\leq 2\left\|A\right\|_f \left(f(N - \left[\frac{N}{2}\right]) \right) + C_1 \|A\| |\Lambda_N| e^{v|t| - \left[\frac{N}{2}\right]} \\ &\leq 2\left\|A\right\|_f \left(f(N - \left[\frac{N}{2}\right]) \right) + C_1 \|A\| |\Lambda_N| e^{-\frac{\left[\frac{N}{2}\right]}{2} + \frac{1}{2}}. \end{aligned} \tag{4.18}$$

On the other hand, for $N \in \mathbb{N}$ with $4v\,|t| > N$, we simply have

$$\left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A) \right) \right\| \leq 2\left\|A\right\|. \tag{4.19}$$

Hence we obtain

$$\left\| \tau^t_{\Phi(s)}(A) \right\|_{f_1} \leq \left(1 + \max\left\{ \begin{array}{c} 2\sup_{N\in\mathbb{N}} \left(\frac{\left(f(N - \left[\frac{N}{2}\right]) \right)}{f_1(N)} \right) + C_1 \sup_{N\in\mathbb{N}} \left(\frac{|\Lambda_N| e^{-\frac{\left[\frac{N}{2}\right]}{2} + \frac{1}{2}}}{f_1(N)} \right), \\ \frac{2}{f_1\left((4v\,|t|)\right)} \mathbb{I}_{4v|t|\geq 1} \end{array} \right\} \right) \left\|A\right\|_f =: b_{f,f_1}(t) \left\|A\right\|_f, \tag{4.20}$$

for $A \in \mathcal{D}_f$ and $t \in \mathbb{R}$, $s \in [0,1]$. Here $\mathbb{I}_{4v|t|\geq 1}$ is the characteristic function for $\{t \in \mathbb{R} \mid 4v\,|t| \geq 1\}$. From the assumptions and (1.16), $b_{f,f_1}(t)$ satisfies the required condition. The inequality for $\tau^t_{\Phi_n(s)}(A)$ can be proven in the same way. □

Lemma 4.6. *Let $f, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_1(t) = 0$. Suppose that*

$$\sup_{N\in\mathbb{N}} \frac{f(N - \left[\frac{N}{2}\right])}{f_1(N)} < \infty, \qquad \sup_{N\in\mathbb{N}} \frac{|\Lambda_N|\, e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_1(N)} < \infty, \qquad \sup_{N\in\mathbb{N}} \frac{W_\gamma\left(\frac{\left[\frac{N}{2}\right]}{2v}\right)}{f_1(N)} < \infty. \tag{4.21}$$

(Recall (1.17).) For $s \in [0,1]$ and $A \in \mathcal{A}$, we set

$$I_s(A) := \int dt\; \omega_\gamma(t) \tau^t_{\Phi(s)}(A). \tag{4.22}$$

The integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$. Then for any $A \in \mathcal{D}_f$ and $s \in [0,1]$, we have $I_s(A) \in \mathcal{D}_{f_1}$.

Proof. That the integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$ is from the continuity of $\mathbb{R} \ni t \to \tau^t_{\Phi(s)}(A) \in \mathcal{A}$, Lemma 4.3 and $\omega_\gamma \in L^1(\mathbb{R})$.

From (4.2), we obtain

$$\|\tau^t_{\Phi(s)}(\mathbb{E}_k(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_k(A))\right)\| \leq C_1\, |\Lambda_k|\, e^{v|t|-(N-k)}\, \|A\|\,, \tag{4.23}$$

for any $A \in \mathcal{D}_f$, $s \in [0,1]$, $t \in \mathbb{R}$, $N, k \in \mathbb{N}$, with $k \leq N$.

For any $A \in \mathcal{D}_f$, $s \in [0,1]$, $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\|I_s(A) - \mathbb{E}_N\left(I_s(A)\right)\| \\
&\leq \left\| I_s\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right) - \mathbb{E}_N\left(I_s\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right)\right)\right\| + 2\left\|A - \mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right\| \\
&\leq \int_{|t|\leq \frac{\left[\frac{N}{2}\right]}{2v}} dt\omega_\gamma(t) \|\tau^t_{\Phi(s)}(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A))\right)\| \\
&+ \int_{|t|\geq \frac{\left[\frac{N}{2}\right]}{2v}} dt\omega_\gamma(t) \|\tau^t_{\Phi(s)}(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A))\right)\| + 2\,\|A\|_f\, f(N - \left[\frac{N}{2}\right]) \\
&\leq \int_{|t|\leq \frac{\left[\frac{N}{2}\right]}{2v}} dt\omega_\gamma(t) C_1\, |\Lambda_N|\, e^{v|t|-\left[\frac{N}{2}\right]}\, \|A\| + \int_{|t|\geq \frac{\left[\frac{N}{2}\right]}{2v}} dt\omega_\gamma(t) 2\,\|A\| + 2\,\|A\|_f\, f(N - \left[\frac{N}{2}\right]) \\
&\leq C_1\, |\Lambda_N|\, e^{-\frac{\left[\frac{N}{2}\right]}{2}}\, \|A\| + 4\,\|A\|\, W_\gamma\left(\frac{\left[\frac{N}{2}\right]}{2v}\right) + 2\,\|A\|_f\, f(N - \left[\frac{N}{2}\right]).
\end{aligned} \tag{4.24}$$

For the first and the fourth inequality, we used (1.16). We used (4.23), with $k = N - [\frac{N}{2}]$, for the third inequality.

Hence we obtain

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \frac{\|I_s(A) - \mathbb{E}_N\left(I_s(A)\right)\|}{f_1(N)} \\
&\le C_1\|A\| \sup_{N\in\mathbb{N}} \frac{|\Lambda_N| e^{-\frac{[\frac{N}{2}]}{2}}}{f_1(N)} + 4\|A\| \sup_{N\in\mathbb{N}} \frac{W_\gamma\left(\frac{[\frac{N}{2}]}{2v}\right)}{f_1(N)} + 2\|A\|_f \sup_{N\in\mathbb{N}} \frac{f(N - [\frac{N}{2}])}{f_1(N)} < \infty,
\end{aligned} \tag{4.25}$$

for any $A \in \mathcal{D}_f$ and $s \in [0,1]$. Hence we obtain $I_s(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$, for any $s \in [0,1]$. □

4.2 Estimates on α_s

In the following, we prove estimates on quasi-locality of the automorphisms α_s and $\alpha_{s,\Lambda}$. To do this, we first recall a theorem from [BMNS] on Lieb-Robinson bounds.

Define $\tilde{h}(x) = \frac{x}{\ln^2(x)}$ for $x > 1$. Define the weight function as:

$$h(x) = \begin{cases} \tilde{h}(e^2) & \text{if } 0 \le x \le e^2 \\ \tilde{h}(x) & \text{otherwise} \end{cases}.$$

The Lieb-Robinson bound for the automorphisms α_s is given as follows: there exists a constant $C_2 > 0$, $\eta_1 > 0$, $\tilde{a} > 0$ satisfying the following: setting $\hat{h}(x) := \eta_1 h(\tilde{a}x)$, we have

$$\|[\alpha_s(B), A]\|, \|[\alpha_{s,\Lambda_n}(B), A]\| \le \frac{C_2}{2}\|A\|\|B\||X|e^{-\hat{h}(d(X,Y))} \tag{4.26}$$

for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ with $X, Y \in \mathfrak{S}_{\mathbb{Z}^\nu}$, and $s \in [0,1]$. See Theorem 4.5 of [BMNS] and Corollary 6.14 of [NSY]. (Note that in [BMNS], Assumption 4.3 about a spectral gap is assumed but for the proof of (4.26), this assumption is not used.) From Corollary 3.6 (3.80) of [NSY], there is a constant $C_3 > 0$ such that

$$\sup_{s\in[0,1]} \left\|\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A)\right\| \le C_3\,|\Lambda_M|\, e^{-\hat{h}(n-M)}\,\|A\|\,, \quad n \ge M, \quad M \in \mathbb{N}, \quad \text{and } A \in \mathcal{A}_{\Lambda_M}. \tag{4.27}$$

From (4.26), we obtain the following.

Lemma 4.7. *For any $M, N \in \mathbb{N}$ with $M < N$, we have*

$$\|\alpha_s^{-1}(A) - \mathbb{E}_N(\alpha_s^{-1}(A))\| \le C_2\,|\Lambda_M|\,\|A\|\,e^{-\hat{h}(N-M)}, \quad A \in \mathcal{A}_{\Lambda_M}. \tag{4.28}$$

Proof. If $A \in \mathcal{A}_{\Lambda_M}$ and $B \in \mathcal{A}_{\Lambda_N^c}$, then $B = \lim_{n\to\infty} B_n$ in norm for a sequence of local observables $B_n \in \mathcal{A}_{\Lambda_N^c} \cap \mathcal{A}_{\rm loc}$ and:

$$\begin{aligned}
\|[B, \alpha_s^{-1}(A)]\| = \|[\alpha_s(B), A]\| &\le \limsup_n \left(2\|A\|\|B - B_n\| + \frac{C_2}{2}\|A\||\Lambda_M|\|B_n\|e^{-\hat{h}(N-M)}\right) \\
&= \frac{C_2}{2}|\Lambda_M|\|A\|\|B\|e^{-\hat{h}(N-M)}.
\end{aligned} \tag{4.29}$$

And so by Corollary 4.4. of [NSY] (Proposition A.1) we conclude (4.28). □

From this Lemma we immediately obtain the following:

Lemma 4.8. *Suppose $f : (0,\infty) \to (0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. Suppose that for all $M \in \mathbb{N}$, we have*

$$\sup_n \frac{e^{-\hat{h}(n)}}{f(M+n)} < \infty \tag{4.30}$$

then $\alpha_s^{-1}(\mathcal{A}_{loc}) \subset D_f$.

Proof. Let $M \in \mathbb{N}$ and $A \in \mathcal{A}_{\Lambda_M}$. From (4.28), we have

$$\sup_{R\in\mathbb{N}} \left(\frac{\|\alpha_s^{-1}(A) - \mathbb{E}_{M+R}(\alpha_s^{-1}(A))\|}{f(M+R)} \right) \le \sup_{R\in\mathbb{N}} \left(C_2 |\Lambda_M| \frac{e^{-\hat{h}(R)}}{f(M+R)} \right) \|A\| < \infty. \tag{4.31}$$

Hence we obtain $\alpha_s^{-1}(A) \in \mathcal{D}_f$. □

Lemma 4.9. *Let $f_1, f_2 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f_i(t) = 0$, $i = 1,2$. Suppose that*

$$\begin{aligned} \sup_{N\in\mathbb{N}} \left(\frac{f_1\left(N - \left[\frac{N}{2}\right]\right)}{f_2(N)} \right) &< \infty, \\ \sup_{N\in\mathbb{N}} \left(\frac{e^{-\hat{h}\left(\left[\frac{N}{2}\right]\right)} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|}{f_2(N)} \right) &< \infty. \end{aligned} \tag{4.32}$$

Then we have $\alpha_s^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$, $\alpha_{s,\Lambda}^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$ for any $s \in [0,1]$, and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$. Furthermore we have the following inequalities:

$$\sup_{s\in[0,1]} \left\|\alpha_s^{-1}(A)\right\|_{f_2}, \sup_{s\in[0,1]} \left\|\alpha_{s,\Lambda}^{-1}(A)\right\|_{f_2} \le \|A\|_{f_1} \left(1 + \sup_{N\in\mathbb{N}} \left(\frac{2f_1\left(N - \left[\frac{N}{2}\right]\right) + C_2 e^{-\hat{h}\left(\left[\frac{N}{2}\right]\right)} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|}{f_2(N)} \right) \right), \tag{4.33}$$

for any $A \in \mathcal{D}_{f_1}$.

Proof. This follows from the following inequality: for each $N \in \mathbb{N}$ and $A \in \mathcal{D}_{f_1}$,

$$\begin{aligned} &\left\|\alpha_s^{-1}(A) - \mathbb{E}_N\left(\alpha_s^{-1}(A)\right)\right\| \\ &\le \left\|\alpha_s^{-1}\left(A - \mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right) - \mathbb{E}_N\left(\alpha_s^{-1}\left(A - \mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right)\right)\right\| \\ &+ \left\|\alpha_s^{-1}\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right) - \mathbb{E}_N\left(\alpha_s^{-1}\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right)\right)\right\| \\ &\le \|A\|_{f_1}\left(2f_1\left(N - \left[\frac{N}{2}\right]\right) + C_2 e^{-\hat{h}\left(\left[\frac{N}{2}\right]\right)}\left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|\right). \end{aligned} \tag{4.34}$$

□

Lemma 4.10. *Suppose $f:(0,\infty)\to(0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. Suppose that for all $M\in\mathbb{N}$, we have*

$$\lim_{n\to\infty}\sup_{N\geq n}\left(\frac{e^{-\hat{h}(N-M)}}{f(N)}\right)=0. \tag{4.35}$$

Then we have

$$\sup_{s\in[0,1]}\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right\|_f\to 0,\quad A\in\mathcal{A}_{\mathrm{loc}}. \tag{4.36}$$

In particular, for each $A\in\mathcal{A}_{\mathrm{loc}}$, $\mathbb{R}\ni s\to\alpha_s^{-1}(A)\in\mathcal{D}_f$ is continuous with respect to the norm $\|\cdot\|_f$.

Proof. Let $A\in\mathcal{A}_{\Lambda_M}$. From (4.27), for $n\geq N\geq M$, we have

$$\sup_{s\in[0,1]}\frac{\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)-\mathbb{E}_N\left(\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right)\right\|}{f(N)}\leq 2C_3\left|\Lambda_M\right|\frac{e^{-\hat{h}(n-M)}}{f(n)}\|A\|. \tag{4.37}$$

On the other hand, for $M\leq n\leq N$, from (4.28)

$$\begin{aligned}&\sup_{s\in[0,1]}\frac{\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)-\mathbb{E}_N\left(\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right)\right\|}{f(N)}=\sup_{s\in[0,1]}\frac{\left\|\alpha_s^{-1}(A)-\mathbb{E}_N\left(\alpha_s^{-1}(A)\right)\right\|}{f(N)}\\&\leq C_2\left|\Lambda_M\right|\|A\|\frac{e^{-\hat{h}(N-M)}}{f(N)}\leq C_2\left|\Lambda_M\right|\|A\|\sup_{N\geq n}\left(\frac{e^{-\hat{h}(N-M)}}{f(N)}\right).\end{aligned} \tag{4.38}$$

Furthermore, for $n\geq M>N$, we have

$$\begin{aligned}&\sup_{s\in[0,1]}\frac{\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)-\mathbb{E}_N\left(\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right)\right\|}{f(N)}\\&\leq 2C_3\left|\Lambda_M\right|\frac{e^{-\hat{h}(n-M)}}{f(M)}\|A\|.\end{aligned} \tag{4.39}$$

Hence we obtain

$$\begin{aligned}&\sup_{s\in[0,1]}\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right\|_f\\&\leq\|A\|\left(1+\max\left\{2C_3\left|\Lambda_M\right|\frac{e^{-\hat{h}(n-M)}}{f(n)},\ C_2\left|\Lambda_M\right|\sup_{N\geq n}\left(\frac{e^{-\hat{h}(N-M)}}{f(N)}\right),\ 2C_3\left|\Lambda_M\right|\frac{e^{-\hat{h}(n-M)}}{f(M)}\right\}\right)\to 0,\quad n\to\infty.\end{aligned} \tag{4.40}$$

□

Lemma 4.11. *Let $f, f_0, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_0(t) = \lim_{t\to\infty} f_1(t) = 0$. Suppose that for all $M \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \sup_{N\geq n} \left(\frac{e^{-\hat{h}(N-M)}}{f(N)} \right) = 0. \tag{4.41}$$

Suppose that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \frac{f_1(N - [\frac{N}{2}])}{f(N)} < \infty, \\ &\sup_{N\in\mathbb{N}} \frac{e^{-\hat{h}([\frac{N}{2}])}}{f(N)} \left|\Lambda_{N-[\frac{N}{2}]}\right| < \infty. \end{aligned} \tag{4.42}$$

Suppose that

$$\lim_{N\to\infty} \frac{f_0(N)}{f_1(N)} = 0. \tag{4.43}$$

Then we have $\alpha_s^{-1}(\mathcal{D}_{f_0}) \subset \mathcal{D}_f$ and

$$\sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \to 0, \quad A \in \mathcal{D}_{f_0}. \tag{4.44}$$

In particular, for each $A \in \mathcal{D}_{f_0}$, $[0,1] \ni s \to \alpha_s^{-1}(A) \in \mathcal{D}_f$ is continuous with respect to the norm $\|\cdot\|_f$.

Proof. As

$$\sup_{N\in\mathbb{N}} \frac{f_0(N)}{f_1(N)} < \infty, \tag{4.45}$$

we have $\mathcal{D}_{f_0} \subset \mathcal{D}_{f_1}$. By Lemma 4.9 with (f_1, f_2) replaced by (f_1, f), we get $\alpha_s^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_f$. Hence we have $\alpha_s^{-1}(\mathcal{D}_{f_0}) \subset \mathcal{D}_f$. For any $A \in \mathcal{D}_{f_0}$,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \\ &= \limsup_{n\to\infty} \sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A - \mathbb{E}_M(A)) - \alpha_s^{-1}(A - \mathbb{E}_M(A)) + \alpha_{s,\Lambda_n}^{-1}(\mathbb{E}_M(A)) - \alpha_s^{-1}(\mathbb{E}_M(A)) \right\|_f \\ &\leq \limsup_{n\to\infty} \sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(\mathbb{E}_M(A)) - \alpha_s^{-1}(\mathbb{E}_M(A)) \right\|_f \\ &+ 2\|A - \mathbb{E}_M(A)\|_{f_1} \left(\sup_{N\in\mathbb{N}} \frac{\left(2f_1\left(N - [\frac{N}{2}]\right) + C_2 e^{-\hat{h}([\frac{N}{2}])} \left|\Lambda_{N-[\frac{N}{2}]}\right|\right)}{f(N)} + 1 \right) \\ &= 2\|A - \mathbb{E}_M(A)\|_{f_1} \left(\sup_{N\in\mathbb{N}} \frac{\left(2f_1\left(N - [\frac{N}{2}]\right) + C_2 e^{-\hat{h}([\frac{N}{2}])} \left|\Lambda_{N-[\frac{N}{2}]}\right|\right)}{f(N)} + 1 \right) \to 0, \quad M \to \infty. \end{aligned} \tag{4.46}$$

For the inequality, we used Lemma 4.9. For the last line we used Lemma 4.10. As we have $\lim_{M\to\infty} \|A - \mathbb{E}_M(A)\|_{f_1} = 0$ by Lemma B.3 with (f, f_1) replaced by (f_0, f_1), we have proven the claim. □

4.3 Properties of $\delta_{\Phi(s)}$, $\delta_{\dot{\Phi}(s)}$

Lemma 4.12. *Let $f_2 : (0,\infty) \to (0,\infty)$ be a continuous decreasing function such that*

$$\sum_{k=2}^{\infty} k^{\nu} f_2(k-1) < \infty. \tag{4.47}$$

Let $f_3 : (0,\infty) \to (0,\infty)$ be continuous decreasing function with $\lim_{t\to\infty} f_3(t) = 0$ such that

$$\lim_{N\to\infty} \frac{\sum_{k=N-R}^{\infty} k^{\nu} f_2(k-1)}{f_3(N)} = 0. \tag{4.48}$$

Then $\mathcal{D}_{f_2} \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)})$, and there is a constant $C^{(1)}_{f_2,f_3} > 0$ such that

$$\sup_{s\in[0,1]} \left\| \delta_{\Phi(s)}(A) \right\|_{f_3},\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \delta_{\Phi_N(s)}(A) \right\|_{f_3} \le C^{(1)}_{f_2,f_3} \left\| A \right\|_{f_2} \tag{4.49}$$

$$\sup_{s\in[0,1]} \left\| \delta_{\dot{\Phi}(s)}(A) \right\|_{f_3},\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \delta_{\dot{\Phi}_N(s)}(A) \right\|_{f_3} \le C^{(1)}_{f_2,f_3} \left\| A \right\|_{f_2} \tag{4.50}$$

for all $A \in \mathcal{D}_{f_2}$, and $\varepsilon > 0$. If we assume Assumption 1.2 (iv) in addition, then we may also take $C^{(1)}_{f_2,f_3} > 0$ so that

$$\sup_{s,s_0\in[0,1],0<|s-s_0|\le\varepsilon} \left\| \delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0} - \dot{\Phi}(s_0)}(A) \right\|_{f_3},\ \sup_{N\in\mathbb{N}} \sup_{s,s_0\in[0,1],0<|s-s_0|\le\varepsilon} \left\| \delta_{\frac{\Phi_N(s)-\Phi_N(s_0)}{s-s_0} - \dot{\Phi}_N(s_0)}(A) \right\|_{f_3}$$
$$\le b(\varepsilon) C^{(1)}_{f_2,f_3} \left\| A \right\|_{f_2}. \tag{4.51}$$

Proof. We prove (4.49). The proof of (4.50) and (4.51) are same. Note that there exists a constant $C_5 > 0$ such that

$$\left\| (H_{\Phi(s)})_{\Lambda_{N+R}} \right\| \le C_5 \left| \Lambda_{N+R} \right|, \quad s \in [0,1], \quad N \in \mathbb{N}. \tag{4.52}$$

Therefore ,we have

$$\left\| \delta_{\Phi(s)}(A_N) \right\| = \left\| \left[(H_{\Phi(s)})_{\Lambda_{N+R}}, A_N \right] \right\| \le 2C_5 \left| \Lambda_{N+R} \right| \left\| A_N \right\|, \quad A_N \in \mathcal{A}_{\Lambda_N},\ s \in [0,1]. \tag{4.53}$$

From this, for any $A \in \mathcal{D}_{f_2}$ and $N, M \in \mathbb{N}$ with $M > N$, we have

$$\left\| \delta_{\Phi(s)}\left(\mathbb{E}_N(A) - \mathbb{E}_M(A) \right) \right\| = \left\| \sum_{k=N+1}^{M} \delta_{\Phi(s)}\left(\mathbb{E}_k(A) - \mathbb{E}_{k-1}(A) \right) \right\| \le 2C_5 \sum_{k=N+1}^{M} \left| \Lambda_{k+R} \right| \left\| \mathbb{E}_k(A) - \mathbb{E}_{k-1}(A) \right\|$$
$$\le 4C_5 \left\| A \right\|_{f_2} \sum_{k=N+1}^{M} \left| \Lambda_{k+R} \right| f_2(k-1). \tag{4.54}$$

Hence $\{\delta_{\Phi(s)}(\mathbb{E}_N(A))\}_N$ with $A \in \mathcal{D}_{f_2}$ is a Cauchy sequence in $\mathcal{A}$, hence there exists a limit $\lim_{N\to\infty} \delta_{\Phi(s)}(\mathbb{E}_N(A))$. On the other hand, $\mathbb{E}_N(A)$ converges to A in $\left\|\cdot\right\|$. By the closedness of $\delta_{\Phi(s)}$, $A \in \mathcal{D}_{f_2}$ belongs to the domain $D(\delta_{\Phi(s)})$ of $\delta_{\Phi(s)}$, and

$$\delta_{\Phi(s)}(A) = \lim_{N\to\infty} \delta_{\Phi(s)}\left(\mathbb{E}_N(A) \right). \tag{4.55}$$

Hence we get $\mathcal{D}_{f_2} \subset D(\delta_{\Phi(s)})$. From (4.54), we have

$$\begin{aligned}
&\left\|\delta_{\Phi(s)}\left(A\right)\right\| = \lim_{N\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_N(A)\right)\right\| = \lim_{N\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_N(A)-\mathbb{E}_1(A)+\mathbb{E}_1(A)\right)\right\| \\
&\le 4C_5\left\|A\right\|_{f_2}\sum_{k=2}^{\infty}\left|\Lambda_{k+R}\right| f_2(k-1) + 2C_5\left|\Lambda_{1+R}\right|\left\|A\right\| \\
&\le \left(4C_5\sum_{k=2}^{\infty}\left|\Lambda_{k+R}\right| f_2(k-1) + 2C_5\left|\Lambda_{1+R}\right|\right)\left\|A\right\|_{f_2},
\end{aligned} \tag{4.56}$$

for any $A \in \mathcal{D}_{f_2}$.

Next note that

$$\begin{aligned}
&\left\|\delta_{\Phi(s)}(A) - \mathbb{E}_N\left(\delta_{\Phi(s)}(A)\right)\right\| = \lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)\right) - \mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)\right)\right)\right\| \\
&= \lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)+\mathbb{E}_{N-R}(A)\right) - \mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)+\mathbb{E}_{N-R}(A)\right)\right)\right\| \\
&= \lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)\right) - \mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)\right)\right)\right\| \\
&\le 8C_5\left\|A\right\|_{f_2}\sum_{k=N-R+1}^{\infty}\left|\Lambda_{k+R}\right| f_2(k-1),
\end{aligned} \tag{4.57}$$

for any $A \in \mathcal{D}_{f_2}$. Here, in the third line we used the fact that $\delta_{\Phi(s)}\left(\mathbb{E}_{N-R}(A)\right) \in \mathcal{A}_{\Lambda_N}$. In the fourth line, we used (4.54). Therefore, we obtain

$$\left\|\delta_{\Phi(s)}(A)\right\|_{f_3} \le \left(8C_5\sup_{N\in\mathbb{N}}\frac{\sum_{k=N-R+1}^{\infty}\left|\Lambda_{k+R}\right| f_2(k-1)}{f_3(N)} + 4C_5\sum_{k=2}^{\infty}\left|\Lambda_{k+R}\right| f_2(k-1) + 2C_5\left|\Lambda_{1+R}\right|\right)\left\|A\right\|_{f_2}. \tag{4.58}$$

The right hand side is finite from the assumptions. Hence we have shown (4.49). □

Lemma 4.13. *Let $f, f_3 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_3(t) = 0$ such that*

$$\sum_{k=1}^{\infty} k^{\nu}\sqrt{f(k-1)} < \infty, \tag{4.59}$$

$$\lim_{N\to\infty}\frac{\sum_{k=N-R}^{\infty} k^{\nu}\sqrt{f(k-1)}}{\left(f_3(N)\right)^2} = 0. \tag{4.60}$$

Then we have $\mathcal{D}_f \subset \mathcal{D}\left(\delta_{\dot{\Phi}(s)}\right)$, $\delta_{\dot{\Phi}(s)}\left(\mathcal{D}_f\right) \subset \mathcal{D}_{f_3}$ and

$$\lim_{N\to\infty}\sup_{s\in[0,1]}\left\|\left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)}\right)(A)\right\|_{f_3} = 0, \quad A \in \mathcal{D}_f. \tag{4.61}$$

In particular, for each $A \in \mathcal{D}_f$, $[0,1] \ni s \to \delta_{\dot{\Phi}(s)}(A) \in \mathcal{D}_{f_3}$ is continuous with respect to the norm $\left\|\cdot\right\|_{f_3}$. The same statement, with $\delta_{\dot{\Phi}(s)}$ replaced by $\delta_{\Phi(s)}$ also holds.

Proof. We prove the claim for $\delta_{\dot{\Phi}(s)}$. The proof for $\delta_{\Phi(s)}$ is the same. Set $f_2(t) := \sqrt{f(t)}$ and $f_4(t) := (f_3(t))^2$.As we have $\sup_N \frac{f(N)}{f_2(N)} < \infty$, $\sup_N \frac{f_4(N)}{f_3(N)} < \infty$ we have $\mathcal{D}_f \subset \mathcal{D}_{f_2}$ and $\mathcal{D}_{f_4} \subset \mathcal{D}_{f_3}$. From Lemma 4.12 with (f_2, f_3) replaced by $(f_2 = \sqrt{f}, f_4 = f_3^2)$, we have $\mathcal{D}_f \subset \mathcal{D}_{f_2} \subset D(\delta_{\dot{\Phi}(s)})$, and $\delta_{\dot{\Phi}(s)}(\mathcal{D}_f) \subset \delta_{\dot{\Phi}(s)}(\mathcal{D}_{f_2}) \subset \mathcal{D}_{f_4} \subset \mathcal{D}_{f_3}$. From Lemma 4.12, with (f_2, f_3) replaced by (f_2, f_4) for $N > R$, we have

$$\begin{aligned}
&\left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) \right\| \le \left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A - \mathbb{E}_{N-R}(A)) \right\| + \left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right) \mathbb{E}_{N-R}(A) \right\| \\
&= \left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A - \mathbb{E}_{N-R}(A)) \right\| \le 2C^{(1)}_{f_2 f_4} \left\| A - \mathbb{E}_{N-R}(A) \right\|_{f_2} \\
&= 2C^{(1)}_{f_2 f_4} \begin{pmatrix} \left\| A - \mathbb{E}_{N-R}(A) \right\| \\ + \sup_{M \in \mathbb{N}} \dfrac{\left\| A - \mathbb{E}_{N-R}(A) - \mathbb{E}_M \left(A - \mathbb{E}_{N-R}(A) \right) \right\|}{f_2(M)} \end{pmatrix} \\
&\le 2C^{(1)}_{f_2 f_4} \begin{pmatrix} \left\| A - \mathbb{E}_{N-R}(A) \right\| \\ + \max \left\{ \begin{matrix} \sup_{N-R \le M \in \mathbb{N}} \dfrac{\left\| A - \mathbb{E}_M(A) \right\|}{f_2(M)}, \\ \sup_{N-R > M \in \mathbb{N}} \dfrac{\left\| A - \mathbb{E}_{N-R}(A) \right\|}{f_2(N-R)} \end{matrix} \right\} \end{pmatrix} \\
&\le 2C^{(1)}_{f_2 f_4} \begin{pmatrix} f(N-R) \left\| A \right\|_f \\ + \max \left\{ \begin{matrix} \sup_{N-R \le M \in \mathbb{N}} \left\| A \right\|_f \dfrac{f(M)}{f_2(M)}, \\ \sup_{N-R > M \in \mathbb{N}} \left\| A \right\|_f \dfrac{f(N-R)}{f_2(N-R)} \end{matrix} \right\} \end{pmatrix} \\
&= 2C^{(1)}_{f_2 f_4} \left(f(N-R) + \sup_{N-R \le L} \left(\frac{f(L)}{f_2(L)} \right) \right) \left\| A \right\|_f = 2C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right) \left\| A \right\|_f .
\end{aligned} \tag{4.62}$$

Here $C^{(1)}_{f_2 f_4}$ is a constant independent of N, s. Therefore, we have

$$\lim_{N \to \infty} \sup_{s \in [0,1]} \left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) \right\| = 0, \quad A \in \mathcal{D}_f. \tag{4.63}$$

Furthermore, for $A \in \mathcal{D}_f$, we have

$$\begin{aligned}
&\frac{\left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) - \mathbb{E}_M \left(\left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) \right) \right\|}{f_3(M)} \\
&\le \begin{cases} \dfrac{f_4(M)}{f_3(M)} \left(\left\| \delta_{\dot{\Phi}(s),N}(A) \right\|_{f_4} + \left\| \delta_{\dot{\Phi}(s)}(A) \right\|_{f_4} \right) \le 2C^{(1)}_{f_2 f_4} \dfrac{f_4(M)}{f_3(M)} \left\| A \right\|_{f_2} \le 2 f_3(N-R) C^{(1)}_{f_2 f_4} \left\| A \right\|_{f_2}, \\ \qquad \text{for } M > N-R, \\ \dfrac{4C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right)}{f_3(M)} \left\| A \right\|_f \le \dfrac{4C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right)}{f_3(N-R)} \left\| A \right\|_f, \\ \qquad \text{for } M \le N-R. \end{cases}
\end{aligned} \tag{4.64}$$

For $M > N - R$, we used Lemma 4.12, with (f_2, f_3) replaced by (f_2, f_4). For $M \le N - R$, we used (4.62). As

$$\lim_{N\to\infty} 2f_3(N-R)C^{(1)}_{f_2f_4}\|A\|_{f_2} = \lim_{N\to\infty} \frac{4C^{(1)}_{f_2f_4}\left(f(N-R)+\sqrt{f(N-R)}\right)}{f_3(N-R)}\|A\|_f = 0, \tag{4.65}$$

we get

$$\lim_{N\to\infty}\sup_{M\in\mathbb{N}}\left(\frac{\left\|\left(\delta_{\dot\Phi(s),N}-\delta_{\dot\Phi(s)}\right)(A)-\mathbb{E}_M\left(\left(\delta_{\dot\Phi(s),N}-\delta_{\dot\Phi(s)}\right)(A)\right)\right\|}{f_3(M)}\right) = 0, \quad A\in\mathcal{D}_f. \tag{4.66}$$

From this and (4.62), we have shown the claim of the Lemma. □

4.4 Proof of Lemma 2.1

Below, we use the following facts repeatedly: for any $0<\beta<\beta'\le 1$, $0<c,c'$, $0<a,a'$, $s\in\mathbb{R}$, $l\in\mathbb{N}$, $r=0,1$, and $k\in\mathbb{Z}$, we have

$$\lim_{t\to\infty}\frac{t^k e^{-\hat h(t-s)}}{e^{-t^\beta}} = \lim_{t\to\infty}\frac{t^k e^{-\hat h([\frac{t}{2}])}}{e^{-t^\beta}} = \lim_{t\to\infty}\frac{t^k e^{-\hat h(t-[\frac{t}{2}])}}{e^{-t^\beta}} = 0, \tag{4.67}$$

$$\lim_{t\to\infty}\frac{e^{-t^\beta}}{e^{-(\frac{t}{2})^\beta}} = 0, \tag{4.68}$$

$$\lim_{t\to\infty}\frac{t^k e^{-t^{\beta'}}}{e^{-(t)^\beta}} = \lim_{t\to\infty}\frac{t^k e^{-c([\frac{t}{2}])^{\beta'}}}{e^{-t^\beta}} = \lim_{t\to\infty}\frac{t^k e^{-(t-[\frac{t}{2}])^{\beta'}}}{e^{-t^\beta}} = 0, \tag{4.69}$$

$$\sum_{m=1}^{\infty} m^k e^{-c(m-r)^\beta} < \infty, \tag{4.70}$$

$$\lim_{N\to\infty}\frac{\sum_{m=N-l}^{\infty} m^k e^{-c(m-r)^{\beta'}}}{e^{-c'N^\beta}} \le \sum_{m=1}^{\infty} m^k e^{-\frac{c}{2}(m-r)^{\beta'}} \lim_{N\to\infty}\frac{e^{-\frac{c}{2}(N-l-r)^{\beta'}}}{e^{-c'N^\beta}} = 0. \tag{4.71}$$

We also note that for $0<\beta<1$, $0<c,c'$, and $l\in\mathbb{N}$, $|t|^l e^{-\hat h(ct)}/e^{-(c't)^\beta}$ is integrable with respect to $t>0$. From this and (1.14), for any $0<\beta<1$, $0<c$, and $l\in\mathbb{N}$, we have

$$\int_{-\infty}^{\infty} dt\omega_\gamma(t)\,|t|^l e^{(c|t|)^\beta} < \infty. \tag{4.72}$$

We also have for any $0<\beta<1$ and $c>0$

$$\sup_{t\ge 1}\frac{W_\gamma(c\left[\frac{t}{2}\right])}{e^{-t^\beta}} < \infty, \tag{4.73}$$

from (1.15).

Lemma 4.14. *Fix $0<\beta_5<\beta_1<1$ and set $f(t):=\frac{\exp(-t^{\beta_1})}{t}$, and $\zeta(t):=\exp(-t^{\beta_5})$. Then for any $A\in\mathcal{D}_f$, and $(s',u',s'',s''')\in[0,1]\times\mathbb{R}\times[0,1]\times[0,1]$, we have $\tau^{-u'}_{\Phi(s'')}\circ\alpha^{-1}_{s'''}(A)\in\mathcal{D}_{f_2}\subset\mathcal{D}_\zeta\subset$*

$D(\delta_{\Phi(s')}) \cap D(\delta_{\dot{\Phi}(s')})$ and $\delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A), \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta$. For any $A \in \mathcal{D}_f$ and any compact intervals $[a,b]$, $[c,d]$ of $\mathbb{R}$, the maps

$$[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \mapsto \tau^{u}_{\Phi(s)} \circ \delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A} \tag{4.74}$$

and

$$[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \mapsto \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A} \tag{4.75}$$

are uniformly continuous with respect to $\|\cdot\|$, and the maps

$$[0,1] \times [c,d] \times [0,1] \times [0,1] \ni (s', u', s'', s''') \mapsto \delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta \tag{4.76}$$

and

$$[0,1] \times [c,d] \times [0,1] \times [0,1] \ni (s', u', s'', s''') \mapsto \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta \tag{4.77}$$

are uniformly continuous with respect to $\|\cdot\|_\zeta$. For any $A \in \mathcal{D}_f$, the integral

$$\int dt \omega_\gamma(t) \int_0^t du \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)}(A) \right), \tag{4.78}$$

and

$$\int dt\, \omega_\gamma(t) \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\dot{\Phi}(s)} \right) \circ \tau^{u}_{\Phi(s)}(A), \tag{4.79}$$

are well-defined as Bochner integrals of $(\mathcal{A}, \|\cdot\|)$. Furthermore, for any $A \in \mathcal{D}_f$, $\alpha_s^{-1}(A)$ and $\alpha_s(A)$ are differentiable with respect to $\|\cdot\|$ and

$$\frac{d}{ds} \alpha_s^{-1}(A) = \int dt \omega_\gamma(t) \int_0^t du \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)} \left(\alpha_s^{-1}(A) \right) \right). \tag{4.80}$$

The right hand side can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$ and there is a constant $C_{9,f} > 0$ such that

$$\left\| \frac{d}{ds} \alpha_s^{-1}(A) \right\|, \left\| \frac{d}{ds} \alpha_s(A) \right\| \le C_{9,f} \|A\|_f, \quad A \in \mathcal{D}_f. \tag{4.81}$$

Remark 4.15. As mentioned in the introduction, α_s is the same automorphism given in [BMNS] and [NSY]. In particular, if a C^1-path of interactions satisfy *Condition B* in [O2] except for the time reversal condition (iii) *6*, for each $s \in [0,1]$, the unique ground state φ_s is given by $\varphi_s = \varphi_0 \circ \alpha_s$, with the α_s. Lemma 4.14 implies for any $A \in \mathcal{D}_f$, $\varphi_s(A) = \varphi_0 \circ \alpha_s(A)$ is differentiable and the derivative is bounded by $C_{9,f} \|A\|_f$, corresponding to Assumption 1.2 (vii). It is well known that the local gap implies the existence of the gap in the bulk in the sense of Assumption 1.2 (vi), [O1].

Proof. We prove the continuity for (4.75) and (4.77). The proof for (4.74) and (4.76) are the same. We also prove only (4.78). The proof for (4.79) is the same. We prove (4.81) only for α_s^{-1}. The proof for α_s is analogous.

Choose real numbers $\beta_4, \beta_3, \beta_2$ so that $0 < \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and fix. Define $f_0(t) := \exp(-t^{\beta_1})$, $f_1(t) := \exp(-t^{\beta_2})$, $f_2(t) := t^{-2(\nu+2)} \exp(-t^{\beta_3})$, $g(t) := \exp(-t^{\beta_4})$.

Note that $f_1, f, f_0 : (0, \infty) \to (0, \infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_0(t) = 0$. From (4.67), we have

$$\lim_{N\to\infty} \left(\frac{e^{-\hat{h}(N-M)}}{f_1(N)} \right) = 0, \text{ for all } M \in \mathbb{N}, \tag{4.82}$$

$$\sup_{N\in\mathbb{N}} \frac{e^{-\hat{h}([\frac{N}{2}])}}{f_1(N)} \left| \Lambda_{N-[\frac{N}{2}]} \right| < \infty. \tag{4.83}$$

Furthermore, from (4.69) and $0 < \beta_2 < \beta_1 < 1$, we have

$$\sup_{N\in\mathbb{N}} \frac{f_0(N - [\frac{N}{2}])}{f_1(N)} < \infty. \tag{4.84}$$

We also have

$$\lim_{M\to\infty} \frac{f(M)}{f_0(M)} = \lim_{M\to\infty} \frac{1}{M} = 0. \tag{4.85}$$

Therefore, from Lemma 4.11 with (f, f_0, f_1) replaced by (f_1, f, f_0), we have $\alpha_s^{-1}(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$ and

$$\sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_{f_1} \to 0, \quad A \in \mathcal{D}_f. \tag{4.86}$$

Therefore, for each $A \in \mathcal{D}_f$, $[0,1] \ni s \to \alpha_s^{-1}(A) \in \mathcal{D}_{f_1}$ is continuous with respect to the norm $\|\cdot\|_{f_1}$.

Note that $f, f_1 : (0, \infty) \to (0, \infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} f(t) = 0$. From (4.69), and $0 < \beta_2 < \beta_1 < 1$, we have

$$\sup_{N\in\mathbb{N}} \left(\frac{f\left(N - [\frac{N}{2}]\right)}{f_1(N)} \right) < \infty. \tag{4.87}$$

From this and (4.83), Lemma 4.9 with (f_1, f_2) replaced by (f, f_1) implies the existence of a constant $C_{8,f,f_1} > 0$ such that

$$\sup_{s\in[0,1]} \left\| \alpha_s^{-1}(A) \right\|_{f_1} \le C_{8,f,f_1} \|A\|_f, \quad A \in \mathcal{D}_f. \tag{4.88}$$

The functions $f_1, f_2 : (0, \infty) \to (0, \infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_i(t) = 0$, $i = 1, 2$. From (4.69), we have

$$\lim_{N\to\infty} \left(\frac{|\Lambda_{[\frac{N}{2}]}| e^{-(N-[\frac{N}{2}])}}{f_2(N)} \right) = 0. \tag{4.89}$$

From (4.69) and $0 < \beta_3 < \beta_2 < 1$, we have

$$\lim_{N\to\infty} \frac{f_1\left([\frac{N}{2}]\right)}{f_2(N)} = 0. \tag{4.90}$$

Therefore, from Lemma 4.4, we have

$$\sup_{s\in[0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right\|_{f_2} \to 0, \quad A \in \mathcal{D}_{f_1}, \tag{4.91}$$

uniformly in compact $u \in \mathbb{R}$. Therefore, for each $A \in \mathcal{D}_{f_1}$, $\mathbb{R} \times [0,1] \ni (u,s) \to \tau^{-u}_{\Phi(s)}(A) \in \mathcal{D}_{f_2}$ is continuous with respect to the norm $\|\cdot\|_{f_2}$.

Note that $f_2, \zeta : (0,\infty) \to (0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \zeta(t) = 0$. From (4.70) and (4.71), and $0 < \beta_5 < \beta_3 < 1$, we have

$$\sum_{k=1}^{\infty} k^{\nu} \sqrt{f_2(k)} < \infty, \tag{4.92}$$

$$\lim_{N\to\infty} \frac{\sum_{k=N-R}^{\infty} k^{\nu} \sqrt{f_2(k)}}{\zeta(N)^2} = 0. \tag{4.93}$$

Therefore applying Lemma 4.13 with (f, f_3) replaced by (f_2, ζ), we have $\delta_{\dot{\Phi}(s)}(\mathcal{D}_{f_2}) \subset \mathcal{D}_{\zeta}$ and

$$\lim_{N\to\infty} \sup_{s\in[0,1]} \left\| \left(\delta_{\dot{\Phi}_N(s)} - \delta_{\dot{\Phi}(s)} \right)(A) \right\|_{\zeta} = 0, \quad A \in \mathcal{D}_{f_2}. \tag{4.94}$$

Therefore, for each $A \in \mathcal{D}_{f_2}$, $[0,1] \ni s \to \delta_{\dot{\Phi}(s)}(A) \in \mathcal{D}_{\zeta}$ is continuous with respect to the norm $\|\cdot\|_{\zeta}$.

Note that $f_2 : (0,\infty) \to (0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f_2(t) = 0$. From (4.72), we have

$$\int_{(4v|t|)\geq 1} dt\ \omega_{\gamma}(t) \frac{|t|}{f_2((4v\,|t|))} < \infty. \tag{4.95}$$

We also have

$$\sup_{N\in\mathbb{N}} \frac{f_1(N - \left[\frac{N}{2}\right])}{f_2(N)} < \infty, \tag{4.96}$$

$$\sup_{N\in\mathbb{N}} \frac{|\Lambda_N|\, e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_2(N)} < \infty, \tag{4.97}$$

from (4.69) with $0 < \beta_3 < \beta_2 < 1$ and (4.67). Therefore, from Lemma 4.5, with (f, f_1) replaced by (f_1, f_2) we have $\tau^{t}_{\Phi(s)}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$ and there is a non-negative non-decreasing function on $\mathbb{R}_+$, $b_{1,f_1,f_2}(t)$ such that

$$\int dt\ \omega_{\gamma}(t)\ |t| \cdot b_{1,f_1,f_2}(|t|) < \infty \tag{4.98}$$

and

$$\sup_{s\in[0,1]} \left\| \tau^{t}_{\Phi(s)}(A) \right\|_{f_2}, \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \tau^{t}_{\Phi_N(s)}(A) \right\|_{f_2} \leq b_{1,f_1,f_2}(|t|)\, \|A\|_{f_1}, \quad A \in \mathcal{D}_{f_1}. \tag{4.99}$$

Note that $f_2, \zeta : (0,\infty) \to (0,\infty)$ are continuous decreasing functions such that $\lim_{t\to\infty} f_2(t) = \lim_{t\to\infty} \zeta(t) = 0$. By (4.70) and (4.71) with $0 < \beta_5 < \beta_3 < 1$, we have

$$\sum_{k=2}^{\infty} k^{\nu} f_2(k-1) < \infty, \tag{4.100}$$

$$\limsup_{N} \frac{\sum_{k=N-R}^{\infty} k^{\nu} f_2(k)}{\zeta(N)} = 0. \tag{4.101}$$

Therefore, from Lemma 4.12 with (f_2, f_3) replaced by (f_2, ζ), we have $\mathcal{D}_{f_2} \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)}) \cap D(\delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0} - \dot{\Phi}(s_0)})$, and there exists a constant $C^{(1)}_{7,f_2,\zeta} > 0$ such that

$$\sup_{s\in[0,1]} \left\|\delta_{\Phi(s)}(A)\right\|_\zeta, \quad \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\Phi_N(s)}(A)\right\|_\zeta \le C^{(1)}_{7,f_2,\zeta} \|A\|_{f_2} \tag{4.102}$$

$$\sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}(s)}(A)\right\|_\zeta, \quad \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}_N(s)}(A)\right\|_\zeta \le C^{(1)}_{7,f_2,\zeta} \|A\|_{f_2} \tag{4.103}$$

for all $A \in \mathcal{D}_{f_2}$ and $\varepsilon > 0$.

We claim that for any compact intervals $[a,b]$, $[c,d]$ of $\mathbb{R}$ and $A \in \mathcal{D}_f$,

$$[a,b]\times[0,1]\times[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (u,s,s',u',s'',s''') \mapsto \tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A} \tag{4.104}$$

is continuous with respect to $\|\cdot\|$. We also claim that

$$[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (s',u',s'',s''') \mapsto \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta \tag{4.105}$$

is continuous with respect to $\|\cdot\|_\zeta$.

To see this, let $A \in \mathcal{D}_f$ and fix any $\varepsilon > 0$. Note that from the continuity of $[0,1] \ni s''' \mapsto \alpha^{-1}_{s'''}(A) \in \mathcal{D}_{f_1}$ in $\|\cdot\|_{f_1}$, there exists a finite sequence $s_0 = 0 < s_1 < \cdots < s_{N_\varepsilon} = 1$ such that

$$\left\|\alpha^{-1}_{s'''}(A) - \alpha^{-1}_{s_i}(A)\right\|_{f_1} < \varepsilon, \text{for all } s''' \in [s_{i-1}, s_{i+1}], \text{and } i = 1,\ldots,N_\varepsilon - 1. \tag{4.106}$$

For $\alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_1}$, $i = 0,\ldots,N_\varepsilon$, from the continuity of $(u', s'') \mapsto \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_2}$, in $\|\cdot\|_{f_2}$ we get $\tilde{s}_0 = 0 < \tilde{s}_1 < \cdots < \tilde{s}_{\tilde{N}_\varepsilon} = 1$ and $u_0 = c < u_1 < \cdots < u_{M_\varepsilon} = d$ such that

$$\left\|\left(\tau^{-u'}_{\Phi(s'')} - \tau^{-u_k}_{\Phi(\tilde{s}_j)}\right) \circ \alpha^{-1}_{s_i}(A)\right\|_{f_2} < \varepsilon, \tag{4.107}$$

$$\begin{aligned} &\text{for all } s'' \in [\tilde{s}_{j-1}, \tilde{s}_{j+1}], \text{and } j = 1,\ldots,\tilde{N}_\varepsilon - 1,\\ &\text{for all } u' \in [u_{k-1}, u_{k+1}], \text{and } k = 1,\ldots,M_\varepsilon - 1,\\ &\text{and } i = 1,\ldots,N_\varepsilon - 1. \end{aligned} \tag{4.108}$$

From the continuity of $[0,1] \ni s' \to \delta_{\dot{\Phi}(s')} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_\zeta$ for $\tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_2}$ in $\|\cdot\|_\zeta$, there exists a finite sequence $\hat{s}_0 = 0 < \hat{s}_1 < \cdots < \hat{s}_{\hat{N}_\varepsilon} = 1$ such that

$$\begin{aligned} &\left\|\left(\delta_{\dot{\Phi}(s')} - \delta_{\dot{\Phi}(\hat{s}_l)}\right) \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A)\right\|_\zeta < \varepsilon.\\ &\text{for all } s' \in [\hat{s}_{l-1}, \hat{s}_{l+1}], \text{and } l = 1,\ldots,\hat{N}_\varepsilon - 1,\\ &\text{and } j = 1,\ldots,\tilde{N}_\varepsilon - 1, \text{and } k = 1,\ldots,M_\varepsilon - 1,\\ &\text{and } i = 1,\ldots,N_\varepsilon - 1. \end{aligned} \tag{4.109}$$

Finally, from the continuity of $\mathbb{R}\times[0,1] \ni (u,s) \to \tau^u_{\Phi(s)}\left(\delta_{\dot{\Phi}(\hat{s}_l)} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A)\right) \in \mathcal{A}$ in the norm $\|\cdot\|$, (Lemma 4.3,) we have finite sequences $\check{s}_0 = 0 < \check{s}_1 < \cdots < \check{s}_{\check{N}_\varepsilon} = 1$ and $\hat{u}_0 = a <$

$\hat{u}_1 < \cdots < \hat{u}_{\hat{M}_\varepsilon} = b$ such that

$$\begin{aligned}\left\| \left(\tau^{u}_{\Phi(s)} - \tau^{\hat{u}_x}_{\Phi(\check{s}_y)} \right) \circ \delta_{\dot{\Phi}(\hat{s}_l)} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \right\| < \varepsilon, \\ \text{for all } s \in [\check{s}_{y-1}, \check{s}_{y+1}], \text{and } y = 1, \ldots, \check{N}_\varepsilon - 1, \\ \text{and } u \in [\hat{u}_{x-1}, \hat{u}_{x+1}], \text{and } x = 1, \ldots, \hat{M}_\varepsilon - 1, \\ \text{and } l = 1, \ldots, \hat{N}_\varepsilon - 1, \\ \text{and } j = 1, \ldots, \tilde{N}_\varepsilon - 1, \text{and } k = 1, \ldots, M_\varepsilon - 1, \\ \text{and } i = 1, \ldots, N_\varepsilon - 1.\end{aligned} \tag{4.110}$$

Now for any $(u, s, s', u', s'', s''') \in [a, b] \times [0, 1] \times [0, 1] \times [c, d] \times [0, 1] \times [0, 1]$, there is (x, y, l, k, j, i) such that

$$u \in [\hat{u}_{x-1}, \hat{u}_{x+1}], s \in [\check{s}_{y-1}, \check{s}_{y+1}], s' \in [\hat{s}_{l-1}, \hat{s}_{l+1}], u' \in [u_{k-1}, u_{k+1}], s'' \in [\tilde{s}_{j-1}, \tilde{s}_{j+1}], s''' \in [s_{i-1}, s_{i+1}]. \tag{4.111}$$

For any such (x, y, l, k, j, i), we have

$$\begin{aligned}&\left\| -\tau^{\hat{u}_x}_{\Phi(\check{s}_y)} \circ \delta_{\dot{\Phi}(\hat{s}_l)} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) + \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \right\| \\ &\leq \left\| \left(\tau^{u}_{\Phi(s)} - \tau^{\hat{u}_x}_{\Phi(\check{s}_y)} \right) \circ \delta_{\dot{\Phi}(\hat{s}_l)} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \right\| + \left\| \tau^{u}_{\Phi(s)} \circ \left(-\delta_{\dot{\Phi}(\hat{s}_l)} + \delta_{\dot{\Phi}(s')} \right) \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \right\| \\ &+ \left\| \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \left(-\tau^{-u_k}_{\Phi(\tilde{s}_j)} + \tau^{-u'}_{\Phi(s'')} \right) \circ \alpha^{-1}_{s_i}(A) \right\| + \left\| \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \left(-\alpha^{-1}_{s_i}(A) + \alpha^{-1}_{s'''}(A) \right) \right\| \\ &\leq 2\varepsilon + C^{(1)}_{7,f_2,\zeta}\varepsilon + C^{(1)}_{7,f_2,\zeta} \sup_{u \in [c,d]} b_{1,f_1,f_2}(|u|)\varepsilon.\end{aligned} \tag{4.112}$$

We also have

$$\begin{aligned}&\left\| -\delta_{\dot{\Phi}(\hat{s}_l)} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) + \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \right\|_\zeta \\ &\leq \left\| \left(-\delta_{\dot{\Phi}(\hat{s}_l)} + \delta_{\dot{\Phi}(s')} \right) \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \right\|_\zeta \\ &+ \left\| \delta_{\dot{\Phi}(s')} \circ \left(-\tau^{-u_k}_{\Phi(\tilde{s}_j)} + \tau^{-u'}_{\Phi(s'')} \right) \circ \alpha^{-1}_{s_i}(A) \right\|_\zeta + \left\| \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \left(-\alpha^{-1}_{s_i}(A) + \alpha^{-1}_{s'''}(A) \right) \right\|_\zeta \\ &\leq \varepsilon + C^{(1)}_{7,f_2,\zeta}\varepsilon + C^{(1)}_{7,f_2,\zeta} \sup_{u \in [c,d]} b_{1,f_1,f_2}(|u|)\varepsilon.\end{aligned} \tag{4.113}$$

As b_{1,f_1,f_2} is an $\mathbb{R}$-valued nondecreasing function, $\sup_{u \in [c,d]} b_{1,f_1,f_2}(|u|)$ is finite. Hence we have proven the continuity of (4.75) and (4.77).

Furthermore, for any $A \in \mathcal{D}_f$, we have

$$\begin{aligned}&\sup_{s \in [0,1]} \int dt\ \omega_\gamma(t) \int_{[0,t]} du \left\| \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)} \left(\alpha^{-1}_{s}(A) \right) \right) \right\| \\ &\leq \sup_{s \in [0,1]} \int dt\ \omega_\gamma(t) \int_{[0,t]} du C^{(1)}_{7,f_2,\zeta} b_{1,f_1,f_2}(|u|) C_{8,f,f_1} \|A\|_f \\ &\leq C^{(1)}_{7,f_2,\zeta} C_{8,f,f_1} \|A\|_f \int dt \omega_\gamma(t) b_{1,f_1,f_2}(|t|)|t| < \infty.\end{aligned} \tag{4.114}$$

In the last line we used the fact that b_{1,f_1,f_2} is nondecreasing and (4.98). Therefore, the right hand side of (4.80) is a well-defined Bochner integral of $(\mathcal{A}, \|\cdot\|)$ for any $A \in \mathcal{D}_f$. By the same argument, (4.78) is a well-defined Bochner integral of $(\mathcal{A}, \|\cdot\|)$ for any $A \in \mathcal{D}_f$. By the definition of α_{s,Λ_n}, we have

$$\begin{aligned}\frac{d}{ds}\alpha_{s,\Lambda_n}^{-1}(A) &= i\left[D_{\Lambda_n}(s), \alpha_{s,\Lambda_n}^{-1}(A)\right] = i\int dt\omega_\gamma(t)\int_0^t du\left[\tau^u_{\Phi(s),\Lambda_n}\left(H_{\dot\Phi(s),\Lambda_n}\right), \alpha_{s,\Lambda_n}^{-1}(A)\right]\\ &= \int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(s),\Lambda_n}\circ\delta_{\dot\Phi_n(s)}\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\alpha_{s,\Lambda_n}^{-1}(A)\right)\right), \quad A\in\mathcal{D}_f.\end{aligned} \tag{4.115}$$

Hence we obtain

$$\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_{s_0,\Lambda_n}^{-1}(A) = \int_{s_0}^s dv\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot\Phi_n(v)}\left(\tau^{-u}_{\Phi(v),\Lambda_n}\left(\alpha_{v,\Lambda_n}^{-1}(A)\right)\right), \quad A\in\mathcal{D}_f. \tag{4.116}$$

For each (u, v), for any $A \in \mathcal{D}_f$, we have

$$\begin{aligned}&\left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot\Phi_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\circ\alpha^{-1}_{v,\Lambda_n}(A) - \tau^u_{\Phi(v)}\circ\delta_{\dot\Phi(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right\|\\
&\le \left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot\Phi_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\left(\alpha^{-1}_{v,\Lambda_n}(A) - \alpha_v^{-1}(A)\right)\right\| + \left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot\Phi_n(v)}\circ\left(\tau^{-u}_{\Phi(v),\Lambda_n} - \tau^{-u}_{\Phi(v)}\right)\alpha_v^{-1}(A)\right\|\\
&+ \left\|\tau^u_{\Phi(v),\Lambda_n}\circ\left(\delta_{\dot\Phi_n(v)} - \delta_{\dot\Phi(v)}\right)\left(\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\| + \left\|\left(\tau^u_{\Phi(v),\Lambda_n} - \tau^u_{\Phi(v)}\right)\circ\left(\delta_{\dot\Phi(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\|\\
&\le C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)\left\|\alpha^{-1}_{v,\Lambda_n}(A) - \alpha_v^{-1}(A)\right\|_{f_1} + C^{(1)}_{7,f_2,\zeta}\left\|\left(\tau^{-u}_{\Phi(v),\Lambda_n} - \tau^{-u}_{\Phi(v)}\right)\alpha_v^{-1}(A)\right\|_{f_2}\\
&+ \left\|\left(\delta_{\dot\Phi_n(v)} - \delta_{\dot\Phi(v)}\right)\left(\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\| + \left\|\left(\tau^u_{\Phi(v),\Lambda_n} - \tau^u_{\Phi(v)}\right)\circ\left(\delta_{\dot\Phi(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\|.\end{aligned} \tag{4.117}$$

From (4.86), (4.91), (4.94) and Lemma 4.3, the last part converges to 0 as $n \to \infty$. Furthermore, we have

$$\sup_{n\in\mathbb{N}}\left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot\Phi_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\circ\alpha^{-1}_{v,\Lambda_n}(A) - \tau^u_{\Phi(v)}\circ\delta_{\dot\Phi(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right\| \le 2C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\|A\|_f, \tag{4.118}$$

with

$$\int_0^1 ds\int dt\ \omega_\gamma(t)\int_{[0,t]} du2C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\|A\|_f < \infty. \tag{4.119}$$

Therefore, applying Lebesgue's convergence theorem for (4.116), we obtain

$$\alpha_s^{-1}(A) - \alpha_{s_0}^{-1}(A) = \int_{s_0}^s dv\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(v)}\circ\delta_{\dot\Phi(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right), \quad A\in\mathcal{D}_f. \tag{4.120}$$

From this, for $A \in \mathcal{D}_f$, we get

$$\begin{aligned}&\left\|\frac{\alpha_s^{-1}(A) - \alpha_{s_0}^{-1}(A)}{s - s_0} - \int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(s_0)}\circ\delta_{\dot\Phi(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right\|\\
&\le \int dt\omega_\gamma(t)\int_{[0,t]} du\left\|\frac{1}{s-s_0}\int_{s_0}^s dv\left(\tau^u_{\Phi(v)}\circ\delta_{\dot\Phi(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right) - \tau^u_{\Phi(s_0)}\circ\delta_{\dot\Phi(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right)\right\|.\end{aligned} \tag{4.121}$$

By the continuity of $(s,u) \to \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \left(\tau^{-u}_{\Phi(s)} \left(\alpha_s^{-1}(A) \right) \right) \in \mathcal{A}$ with respect to $\|\cdot\|$ for $A \in \mathcal{D}_f$, we have

$$\lim_{s \to s_0} \frac{1}{s-s_0} \int_{s_0}^{s} dv \left(\tau^{u}_{\Phi(v)} \circ \delta_{\dot{\Phi}(v)} \left(\tau^{-u}_{\Phi(v)} \left(\alpha_v^{-1}(A) \right) \right) - \tau^{u}_{\Phi(s_0)} \circ \delta_{\dot{\Phi}(s_0)} \left(\tau^{-u}_{\Phi(s_0)} \left(\alpha_{s_0}^{-1}(A) \right) \right) \right) = 0, \tag{4.122}$$

for each u. On the other hand, we have

$$\begin{aligned} &\left\| \frac{1}{s-s_0} \int_{s_0}^{s} dv \left(\tau^{u}_{\Phi(v)} \circ \delta_{\dot{\Phi}(v)} \left(\tau^{-u}_{\Phi(v)} \left(\alpha_v^{-1}(A) \right) \right) - \tau^{u}_{\Phi(s_0)} \circ \delta_{\dot{\Phi}(s_0)} \left(\tau^{-u}_{\Phi(s_0)} \left(\alpha_{s_0}^{-1}(A) \right) \right) \right) \right\| \\ &\leq 2 C^{(1)}_{7,f_2,\zeta} b_{1,f_1,f_2}(|u|) C_{8,f,f_1} \|A\|_f , \end{aligned} \tag{4.123}$$

with (4.119). From Lebesgue's convergence theorem, we obtain

$$\lim_{s \to s_0} \left\| \frac{\alpha_s^{-1}(A) - \alpha_{s_0}^{-1}(A)}{s-s_0} - \int dt \omega_\gamma(t) \int_0^t du \tau^{u}_{\Phi(s_0)} \circ \delta_{\dot{\Phi}(s_0)} \left(\tau^{-u}_{\Phi(s_0)} \left(\alpha_{s_0}^{-1}(A) \right) \right) \right\| = 0, \quad \text{for } A \in \mathcal{D}_f. \tag{4.124}$$

Hence for $A \in \mathcal{D}_f$, $[0,1] \ni s \mapsto \alpha_s^{-1}(A)$ is differentiable with respect to $\|\cdot\|$, and we have

$$\frac{d}{ds} \alpha_s^{-1}(A) = \int dt \omega_\gamma(t) \int_0^t du \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \circ \tau^{-u}_{\Phi(s)} \left(\alpha_s^{-1}(A) \right). \tag{4.125}$$

From this formula, we obtain

$$\left\| \frac{d}{ds} \alpha_s^{-1}(A) \right\| \leq \left(\int dt \omega_\gamma(t) \int_{[0,t]} C^{(1)}_{7,f_2,\zeta} b_{1,f_1,f_2}(|u|) C_{8,f,f_1} \right) \|A\|_f =: C_{9,f} \|A\|_f . \tag{4.126}$$

□

Now we prove Lemma 2.1.

Proof of Lemma 2.1.

1. The inclusions $\mathcal{D}_f \subset \mathcal{D}_{f_0} \subset \mathcal{D}_{f_1} \subset \mathcal{D}_{f_2} \subset \mathcal{D}_g \subset \mathcal{D}_\zeta$ follow by the monotone choice of the β_i, $i = 1, \ldots, 5$. From (4.67), we can see that f satisfies the condition required in Lemma 4.8. Therefore, from Lemma 4.8, we have $\alpha_s^{-1}(\mathcal{A}_{loc}) \subset \mathcal{D}_f$ for all $s \in [0,1]$.

2. This is from Lemma 4.5. From (4.72), (4.69) (f, f_1) satisfies the conditions required in Lemma 4.5.

3. Fix $0 < \beta_6 < \beta_5$ and set $\zeta_0(t) := e^{-t^{\beta_6}}$ for $t > 0$. We apply Lemma 4.12, replacing (f_2, f_3) in it by (ζ, ζ_0). To see that (ζ, ζ_0) satisfy the required conditions in Lemma 4.12, we recall (4.70) and (4.71). Hence from Lemma 4.12, we obtain $\mathcal{D}_\zeta \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)})$.

4. This also follows by Lemma 4.12 with (f_2, f_3) replaced by (f_2, ζ). The required conditions in Lemma 4.12 can be checked by (4.70) and (4.71).

5., 6., and 7. are proven in Lemma 4.14.

8. This follows from Lemma 4.6 for (f, f_1). The conditions for (f, f_1) can be checked from (4.69) and (4.73).

9. This is Lemma 4.3.

10. For any $A \in \mathcal{D}_f$, from *5.* above, $(u,s) \mapsto \delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A) \in \mathcal{D}_\zeta$ is continuous with respect to $\|\cdot\|_\zeta$. Furthermore, from *4., 2.,* above, as in (3.12), we have

$$\begin{aligned}&\left\|\delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A)\right\|_\zeta \le C^{(1)}_{f_2,\zeta} \left\|\tau^u_{\Phi(s)}(A)\right\|_{f_2} \le C^{(1)}_{f_2,\zeta} \left(1 + \sup_{N\in\mathbb{N}} \frac{f_1(N)}{f_2(N)}\right) \left\|\tau^u_{\Phi(s)}(A)\right\|_{f_1} \\ &\le C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)}\right) \|A\|_f .\end{aligned} \tag{4.127}$$

From *2.* above, the inequality (2.2) holds and (2.14) is well-defined as the Bochner integral with respect to $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$.

□

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A Conditional expectation $\mathbb{E}_N$

We now briefly describe a family of conditional expectations $\{\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N} \mid N \in \mathbb{N}\}$ are used extensively in this paper. Let $N \in \mathbb{N}$ be fixed and let Λ denote any finite set containing Λ_N. Define:

$$\mathbb{E}^\Lambda_N = \mathrm{id}_{\Lambda_N} \otimes \rho_{\Lambda\setminus\Lambda_N} \tag{A.1}$$

where ρ_X is the product state whose factors are normalized trace:

$$\rho_X = \frac{1}{d^{|X|}} \bigotimes_{x\in X} \mathrm{tr}_x. \tag{A.2}$$

Each $\mathbb{E}^\Lambda_N$ is bounded and linear, and as $\Lambda \subset \Sigma$ implies $\mathbb{E}^\Sigma_N|_{\mathcal{A}_\Lambda} = \mathbb{E}^\Lambda_N$, there exists a unique bounded map and conditional expectation $\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N}$ such that for all Λ containing Λ_N:

$$\mathbb{E}_N|_{\mathcal{A}_\Lambda} = \mathbb{E}^\Lambda_N \tag{A.3}$$

Furthermore, by the definition (A.1) of the finite-volume maps, $\mathbb{E}_N(A^*) = \mathbb{E}_N(A)^*$ for all $A \in \mathcal{A}$ and if $M \in \mathbb{N}$ and $M \ge N$,

$$\mathbb{E}_M\mathbb{E}_N = \mathbb{E}_N\mathbb{E}_M = \mathbb{E}_N. \tag{A.4}$$

The family $\{\mathbb{E}_N\}$ provides local approximations of quasi-local observables. For completeness, we record this as the following proposition and refer to [NSY] for the proof.

Proposition A.1. *Let* $\varepsilon \geq 0$. *Suppose* $A \in \mathcal{A}$ *is such that for all* $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap \Lambda_N = \emptyset}} \mathcal{A}_X$*:*

$$\|[A, B]\| \leq \varepsilon \|B\|. \tag{A.5}$$

Then $\|A - \mathbb{E}_N(A)\| \leq 2\varepsilon$.

Proof. See Corollary 4.4 of [NSY]. □

B Properties of $\mathcal{D}_f$

The map $\|\cdot\|_f : \mathcal{D}_f \to \mathbb{R}_{\geq 0}$ is a norm on $\mathcal{D}_f$. Note that $\|A^*\|_f = \|A\|_f$, and $\|\mathbb{E}_N(A)\|_f \leq \|A\|_f$. Furthermore, if $\sup_{N \in \mathbb{N}} \frac{f(N)}{g(N)} < \infty$, then $\mathcal{D}_f \subset \mathcal{D}_g$.

Lemma B.1. *Let* $f : (0, \infty) \to (0, \infty)$ *be a continuous decreasing function with* $\lim_{t\to\infty} f(t) = 0$. *The set* $\mathcal{D}_f$ *is a* $*$*-algebra which is a Banach space with respect to the norm* $\|\cdot\|_f$.

Proof. That $\mathcal{D}_f$ is $*$-closed is trivial from $\|A^*\|_f = \|A\|_f$. To see that $\mathcal{D}_f$ is closed under multiplication, let $A, B \in \mathcal{D}_f$. For each $N \in \mathbb{N}$, we have

$$\begin{aligned} &\|AB - \mathbb{E}_N(AB)\| \leq \|(A - \mathbb{E}_N(A)) \cdot B\| + \|-\mathbb{E}_N((A - \mathbb{E}_N(A)) \cdot B)\| + \|\mathbb{E}_N(A) \cdot (B - \mathbb{E}_N(B))\| \\ &\leq \left(2\|A\|_f \|B\| + \|A\| \|B\|_f\right) f(N) \leq 3\|A\|_f \|B\|_f f(N). \end{aligned} \tag{B.1}$$

Hence we obtain $AB \in \mathcal{D}_f$, and $\mathcal{D}_f$ is closed under the multiplication.

To prove that $\mathcal{D}_f$ is complete with respect to $\|\cdot\|_f$, let $\{A_n\}_n$ be a Cauchy sequence in $\mathcal{D}_f$ with respect to $\|\cdot\|_f$. As $\{A_n\}_n$ is Cauchy with respect to $\|\cdot\|$ as well, there is an $A \in \mathcal{A}$ such that $\lim_{n\to\infty} \|A - A_n\| = 0$. This A belongs to $\mathcal{D}_f$ because

$$\sup_{N \in \mathbb{N}} \frac{\|A - \mathbb{E}_N(A)\|}{f(N)} = \sup_{N \in \mathbb{N}} \left(\lim_{M\to\infty} \frac{\|A_M - \mathbb{E}_N(A_M)\|}{f(N)} \right) \leq \sup_M \|A_M\|_f < \infty. \tag{B.2}$$

Furthermore, we have

$$\sup_N \frac{\|A - A_m - \mathbb{E}_N(A - A_m)\|}{f(N)} = \sup_N \lim_{n\to\infty} \left(\frac{\|A_n - A_m - \mathbb{E}_N(A_n - A_m)\|}{f(N)} \right) \leq \limsup_{n\to\infty} \|A_n - A_m\|_f. \tag{B.3}$$

Therefore, A_m converges to $A \in \mathcal{D}_f$ in $\|\cdot\|_f$-norm. □

Lemma B.2. *Let* $f : (0, \infty) \to (0, \infty)$ *be a continuous decreasing function with* $\lim_{t\to\infty} f(t) = 0$ *with* $M \in \mathbb{N}$. *For any* $A \in \mathcal{D}_f$ *and* $B \in \mathcal{A}_{\Lambda_M}$ *and* $M \in \mathbb{N}$ *we have*

$$\|BA\|_f \leq \left(1 + \max\left\{\frac{2}{f(M)}, 1\right\}\right) \|B\| \|A\|_f. \tag{B.4}$$

Proof. This follows from the following inequality:

$$\begin{aligned}
&\|BA - \mathbb{E}_N(BA)\| \\
&\leq \begin{cases} 2\|B\|\,\|A\|, & N \leq M, \\ \|B\,(A - \mathbb{E}_N(A))\|, & N > M. \end{cases} \\
&\leq \begin{cases} 2\|B\|\,\|A\|, & N \leq M, \\ \|B\|\,\|A\|_f\, f(N) & N > M. \end{cases}
\end{aligned} \tag{B.5}$$

□

Lemma B.3. *Let $f, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions. Suppose that and*

$$\lim_{N\to\infty} \frac{f(N)}{f_1(N)} = 0. \tag{B.6}$$

Then we have

$$\lim_{M\to\infty} \|A - \mathbb{E}_M(A)\|_{f_1} = 0, \quad A \in \mathcal{D}_f. \tag{B.7}$$

Proof. Let $A \in \mathcal{D}_f$. By the definition of $\mathcal{A}$, we have $\lim_{M\to\infty} \|A - \mathbb{E}_M(A)\| = 0$. We note that for $N \in \mathbb{N}$,

$$\begin{aligned}
\frac{\|A - \mathbb{E}_M(A) - \mathbb{E}_N\,(A - \mathbb{E}_M(A))\|}{f_1(N)} &= \begin{cases} \dfrac{\|A - \mathbb{E}_N\,(A)\|}{f_1(N)}, & M \leq N, \\ \dfrac{\|A - \mathbb{E}_M(A)\|}{f_1(N)}, & M > N, \end{cases} \\
&= \begin{cases} \dfrac{\|A - \mathbb{E}_N\,(A)\|}{f(N)}\dfrac{f(N)}{f_1(N)}, & M \leq N, \\ \dfrac{\|A - \mathbb{E}_M(A)\|}{f(M)}\dfrac{f(M)}{f_1(N)}, & M > N \end{cases} \\
&\leq \|A\|_f \begin{cases} \dfrac{f(N)}{f_1(N)}, & M \leq N, \\ \dfrac{f(M)}{f_1(M)}, & M > N \end{cases} \\
&\leq \|A\|_f \sup_{M\leq L\in\mathbb{N}} \left(\frac{f(L)}{f_1(L)}\right) \to 0,\ M \to \infty.
\end{aligned} \tag{B.8}$$

Hence we obtain (B.7). □

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