NONLOCAL MODELS WITH HETEROGENEOUS LOCALIZATION AND THEIR APPLICATION TO SEAMLESS LOCAL-NONLOCAL COUPLING*

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Abstract. Motivated by recent development on nonlocal mechanical models like peridynamics, we consider nonlocal integral models with a spatially varying horizon that allows the finite range of nonlocal interactions to be position-dependent. In particular, we focus on linear variational problems of such nonlocal models with heterogeneous localization on co-dimension one interfaces. The well-posedness is established for several classes of variational problems involving homogeneous and inhomogeneous Neumann and Dirichlet type constraints. We also study their seamless coupling with local models. In addition, we present numerical studies of the nonlocal models and local-nonlocal coupled models based on the asymptotically compatible Galerkin finite element discretizations.

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1. Introduction. This work is concerned with nonlocal models that have a finite range of nonlocal interactions. An example is given by the peridynamic models of continuum mechanics [21], which are parametrized by a finite horizon parameter measuring the range of nonlocal interactions. One can find similar models for nonlocal diffusion and transport processes [8, 6]. The horizon parameter is often chosen as a positive constant over the spatial domain in most of the earlier studies. In [23], models with a variable horizon having a positive lower bound over the domain have also been examined. In [28], the variable horizon adopted there is allowed to vanish as the material points approach a co-dimension one hyper-surface. With a vanishing horizon, nonlocal models get localized heterogeneously. The corresponding models are thus called nonlocal models with heterogeneous localization [28]. The latter is the topic that we like to further explore here, given its potential impact to nonlocal modeling. For example, unlike nonlocal models with a constant horizon that are generically accompanied by volumetric constraints (nonlocal analog of boundary conditions, see [8]), models with heterogeneous localization on co-dimension one hyper-surfaces allow conventional boundary conditions to be imposed [28].

In this paper, we aim to deliver the following messages. First, by introducing the spatially varying horizon, one can allow the nonlocal models to get localized, especially on the boundary or interface. In those cases, in contrast to the general theory on nonlocal volumetric constraints for nonlocal problems discussed in [8] and other related works [15, 16, 17, 25], we end up with well-posed nonlocal models with local boundary conditions. Furthermore, these local boundary value problems of nonlo-

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cal models recovers the local limit as the nonlocal interaction vanishes everywhere in the domain. However, for some horizon functions that have unbounded second order derivative, *e.g.*, a piecewise linear horizon parameter, directly imposing the local boundary conditions might produce nonlocal solutions that fail to accurately capture all the interesting solution properties in the local limit such as the boundary flux associated with local Dirichlet data, which can be seen from the numerical experiments presented in Section 5. To address such issues, we discuss two remedies. One is to introduce an auxiliary function to handle the boundary effects, as shown in Section 3, for several cases that cover both Dirichlet and Neumann type conditions. Well-posedness results of linear variational problems associated with these nonlocal models are also established. An added advantage, in the case where the constructed auxiliary function is linear, is that one can pass the so called patch test straightforwardly. However, auxiliary functions are not always readily available, especially for the complex geometry in high dimensional spaces. We thus discuss the alternative using a smoother horizon function, *e.g.*, of C^2 class. Although such approach cannot pass the traditional patch test, we demonstrate in the numerical experiments that the effects of ghost forces can be controlled. In addition, as illustrated in Section 4, we demonstrate that it is possible to seamlessly couple classical local models with nonlocal models with heterogeneous localization through a common interface instead of an overlapping domain of nonzero measure. Detailed computational studies are presented in Section 5 on the numeral discretization of both nonlocal models with the heterogeneous localization and seamlessly coupled local-nonlocal models. The simulations are done using standard Galerkin continuous piecewise linear finite element methods that are known to be asymptotically compatible, a concept developed in [26, 27] for the consistent discretization of nonlocal models and their local limits, and are thus ideal as a robust discretization of nonlocal models involving the heterogeneous localization.

2. Background and existing works. There are various challenges in nonlocal modeling that motivate our study of variable horizon such as the multiscale nature of materials, the boundary or surface effects in nonlocal modeling, and the coupling of local and nonlocal models, etc [5, 14, 10, 11, 18, 20]. Among these challenges, the treatment of interface and boundary conditions for nonlocal models has received much attention, given their differences from local PDEs. Discussions on a variety of nonlocal constraints have been given in [8] and a number of other works [1, 15, 25, 29]. In particular, [25] presented formulations of Neumann conditions and demonstrated how to get the second order convergence of the nonlocal models to their local limit in the horizon parameter δ as $\delta \to 0$. Our study first focuses on models with Neumann type volumetric constraints and a piecewise linear horizon parameter. To handle the extra complexity associated with the Dirichlet type constraints, we then consider two approaches using either a smoother horizon parameter or some auxiliary functions.

In terms of local-nonlocal coupling that offers great potential in multiscale coupling and effective simulations, an overlapping domain of local and nonlocal regions is often used in earlier works [5, 10, 20]. In this paper, we present a seamless coupling strategy that allows a local model to connect through a co-dimension one interface with a nonlocal model having heterogeneous localization on such an interface. Hence, the local region may be seen naturally as an extension of the nonlocal model with a zero (localized) horizon. Such a seamless coupling is possible due to the new trace theorems established in [28] and it removes the overlapping domain. Our results here demonstrate that such coupled models can be formulated as well-posed variational problems and the so-called ghost force effect can be controlled properly.

Concerning the numerical solution of nonlocal models with heterogeneous localization, constructing a robust and convergent discretization scheme is important for practical applications. For nonlocal models characterized by a constant horizon parameter δ, it is known that as $\delta \to 0$, one might encounter consistency issues at discrete levels between the nonlocal models and the local PDEs, when the latter remain valid. Answering such questions on the discrete level is an important task of code validation and verification. In [27], a theory of asymptotically compatible (AC) schemes was developed. It was successfully applied to nonlocal models with constant horizon for various boundary constraints [25, 26] and nonlocal gradient recovery technique [12]. Given the extra complications involved in the variable horizon, we demonstrate that asymptotically compatible schemes are ideally suited for the nonlocal models that allows heterogeneous localization and seamlessly coupled local-nonlocal problems. In this paper, we use standard Galerkin piecewise linear finite element methods that are known to be AC [27]. We offer computational studies on the convergence rates for different types of volume constraints and horizons as the nonlocality vanishes over the whole computation domain.

3. Nonlocal variational problems with heterogeneous localization.

We now present a nonlocal variational problem with heterogeneous localization [13, 28]. Given a spatial domain Ω of interest, we let Ω_I denote the corresponding interaction domain, a concept introduced in [7, 8] that will be explained later. We introduce the following nonlocal energy functional

$$E_\Omega(u) = \frac{1}{2}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \rho_\delta(x',x)\left(u(x')-u(x)\right)^2 dx'dx\,, \tag{3.1}$$

where $\rho_\delta(x',x) = \rho_\delta(x,x')$ is a symmetric, nonnegative interaction kernel with more details specified later.

The energy accounting for the contribution due to the work done by a given external force $f = f(x)$ on Ω is given by

$$E_f(u) = E_\Omega(u) - \int_\Omega f(x)u(x)dx\,. \tag{3.2}$$

We consider the constrained minimization problem

$$\min\, E_f(u) \quad \text{subject to} \quad E_c(u) = 0\,, \tag{3.3}$$

where $E_c(u)$ denotes a constraint functional, see [8] for more details. For example,

$$E_c(u) := \int_{\Omega_I} u^2\,dx \tag{3.4}$$

leads to a homogeneous nonlocal Dirichlet constraint on Ω_I. On the other hand, if $\Omega_I = \emptyset$ and

$$E_c(u) := (c_N - \int_\Omega u\,dx)^2 \tag{3.5}$$

for a given constant c_N, then $E_c(u) = 0$ implies that the integral of $u = u(x)$ over Ω is c_N and we end up with a nonlocal Neumann type problem. For homogeneous pure Neumann type problems discussed later, we always assume the compatibility condition

$$\int_\Omega f(x)\,dx = 0\,. \tag{3.6}$$

The weak form of Euler-Lagrange equation for (3.3) and (3.5) with $\Omega_I = \emptyset$ is

$$B_\delta(u,v) := \int_\Omega\int_\Omega \rho_\delta(x,x')\,(u(x')-u(x))\,(v(x')-v(x))\,dx' = (f,v)\,, \tag{3.7}$$

where

$$(f,v) = \int_\Omega f(x)v(x)dx\,,$$

and $B_\delta(u,v)$ defines a symmetric bilinear form for the solution u and any test function v in suitable function spaces.

3.1. Nonlocal kernels, variable horizon and function spaces. Recently, nonlocal problems with heterogeneous localization have been introduced in [28]. We recall some basic definitions here.

Let Ω be an open and bounded domain in $\mathbb{R}^n$. Following [28], we define, for all $x, x' \in \Omega$,

$$\rho_\delta(x,x') = c_\delta(x)\gamma_{\delta(x)}(|x'-x|) + c_\delta(x')\gamma_{\delta(x')}(|x'-x|) \tag{3.8}$$

such that

$$c_\delta(x)\int_\Omega \gamma_{\delta(x)}(|x-x'|)(x'-x)^2dx' = \frac{n}{2}\,, \tag{3.9}$$

where $\delta(x)$ represents a variable horizon such that $\gamma_{\delta(x)}(|x'-x|) \geq 0$ for $|x-x'| \leq \delta(x)$ and $\gamma_{\delta(x)}(|x'-x|) = 0$ for $|x-x'| > \delta(x)$. For example, for a nonnegative function $\hat{\gamma}$ with compact support in the interval $(0,1)$, we may define [28]

$$\gamma_{\delta(x)}(r) = \hat{\gamma}\left(\frac{r}{\delta(x)}\right)\,. \tag{3.10}$$

For much of this work, we restrict to the case that $\hat{\gamma}$ is a constant function but the theory can be readily extended to more singular kernels, following the discussions given in [28] concerning the nonlocal energy spaces corresponding to both the constant and more general kernels.

By heterogeneous localization [28], we are interested in the case where $\delta(x) = 0$ at some isolated points. Here, throughout this section, we consider the case that $\delta(x)$ vanishes as x goes to the boundary $\partial\Omega$. For example, one choice of $\delta(x)$ is simply given by:

$$\delta(x) = \min(\mathrm{dist}(x,\partial\Omega),\delta)\,. \tag{3.11}$$

Here we have $\max\delta(x) = \delta$. Besides (3.11), other forms of the function $\delta(x)$, particularly ones with much more smoothness, will be considered later. Without loss of generality, we take as a notation convention to use δ representing both the heterogeneously defined horizon function and its maximum value. In this way, we may use $\delta \to 0$ to represent the localization of the model throughout the domain.

Let us now define some function spaces of interest to us. The space $\mathcal{S}_\delta(\Omega) \subset L^2(\Omega)$ is given by

$$\mathcal{S}_\delta(\Omega) = \left\{u \in L^2(\Omega) : E_\Omega(u) < \infty\right\}\,.$$

Now the solution space for the nonlocal homogeneous Dirichlet type problem is defined by the closure of smooth functions in the space of $\mathcal{S}_\delta(\Omega)$. Namely, we define

$$\mathcal{S}_\delta^D(\Omega) = \{u \in \mathcal{S}_\delta(\Omega) : \exists u_n \in C_c^\infty(\Omega) \text{ such that } u_n \to u \text{ in } \mathcal{S}_\delta(\Omega)\}\,.$$

It is worth noting that by the trace theorem in [28], any function in $\mathcal{S}_\delta(\Omega)$ has a well-defined trace in the space $H^{1/2}(\partial\Omega)$. So it is reasonable to impose local boundary constraints in the space $\mathcal{S}_\delta(\Omega)$. Naturally, the nonlocal Neumann-type constrained energy space can be similarly defined as a subspace of $\mathcal{S}_\delta(\Omega)$ with a normalization condition:

$$\mathcal{S}_\delta^N(\Omega) = \left\{ u \in \mathcal{S}_\delta(\Omega) : \int_\Omega u dx = c_N \right\} ,$$

for some constant c_N. Again, without loss of generality, we assume $c_N = 0$ unless noted otherwise. It is not hard to see that $\mathcal{S}_\delta(\Omega)$ and the constrained energy spaces are real Hilbert spaces with the inner product $(\cdot,\cdot)_s$ defined as $(u,w)_s = B_\delta(u,w)+(u,w)$, see, for example, similar results given in [16]. We use $\|u\|_{L^2}$ to denote the L^2 norm of u, $|u|_e$ to denote the energy seminorm $\sqrt{(B_\delta(u,u))}$ of u in $\mathcal{S}_\delta(\Omega)$ and $\|\cdot\|_s$ to denote the norm on $\mathcal{S}_\delta(\Omega)$ defined by $\|u\|_s^2 = \|u\|_{L^2}^2 + |u|_e^2$.

Similar to the constant horizon case, it is not hard to see that the nonlocal energy space $\mathcal{S}(\Omega)$ is a Hilbert space. Moreover, we may observe that, if $|u|_e = 0$ for some $u \in \mathcal{S}(\Omega)$, then $u = u(x)$ is a constant function for a.e. $x \in \Omega$. Indeed, let K be any subset compactly contained in Ω, we have $\sigma := \inf_{x\in K} \delta(x) > 0$. We can restrict the choice of the kernel $\rho_\delta(x,x')$ so that it is strictly above zero for any $x, x' \in K$ with $|x'-x| < \sigma/2$. This implies that

$$0 = |u|_e^2 \geq \int_K \int_{K\cap B_{\sigma/2}(x)} \rho_\delta(x,x')(u(x)-u(x'))^2 dx'dx ,$$

which forces u be a constant on K. Thus, u is a constant almost everywhere over Ω.

The well-posedness of the nonlocal boundary value problems proposed respectively in sections 3.2, 3.3 and 3.4 can be derived using the conventional Lax-Milgram theorem with the help of a nonlocal Poincaré-type inequality. The latter, shown below, is applicable to any subspace of $\mathcal{S}_\delta(\Omega)$ that intersects $\mathbb{R}$ trivially, including particularly $\mathcal{S}_\delta^D(\Omega)$ and $\mathcal{S}_\delta^N(\Omega)$ as cases of interests here.

PROPOSITION 3.1. *Suppose V is a closed subspace of $L^2(\Omega)$ that intersects $\mathbb{R}$ trivially. Then there exists a constant $C = C(\rho_\delta, V, \Omega)$ such that*

$$\|u\|_{L^2} \leq C|u|_e , \quad \forall u \in V \cap \mathcal{S}_\delta(\Omega). \tag{3.12}$$

Proof. We prove the inequality by contradiction, which is a standard technique for establishing Poincaré inequality. Suppose the inequality (3.12) is false. Then there exist $\{u_n\} \in V$ such that for all n, $\|u_n\|_{L^2} = 1$, and as $n \to \infty$, $|u_n|_e \to 0$. We claim that in such case $\|u_n\|_{L^2} \to 0$ as $n \to \infty$, resulting in a contradiction. To prove the claim, suppose that u is the weak limit in L^2 of the bounded sequence u_n. Since V is a closed subspace of $L^2(\Omega)$, we have $u \in V$.

Step 1. We show that *the weak limit u is in fact* 0. We claim that $|\cdot|_e$ is L^2-weakly lower semicontinuous, namely

$$|u|_e \leq \liminf_n |u_n|_e . \tag{3.13}$$

In fact, since $|\cdot|_e$ is a convex functional, then the weak lower semicontinuity is equivalent to strong lower semicontinuity. So we only need to show (3.13) under the assumption that u_n converges to u strongly in L^2. Indeed, under such assumption, we can extract a subsequence of $\{u_{n_k}\}$ such that it converges to u pointwisely up to a set of measure zero. Then we arrive at (3.13) by applying Fatou's lemma. Now from (3.13)

we have $|u|_e = 0$ so that u equals to a constant in Ω. Therefore, u must be identical to zero since the only constant function in V is the zero function by assumption.
Step 2. We show next that, *as* $n \to \infty$, $u_n \to 0$ *strongly in* $L^2(\Omega)$. First we observe the following fact for $\gamma_{\delta(x)}$ to be of the rescaled type (3.10) and $\delta(x)$ only vanishes on the boundary $\partial\Omega$ continuously. Fix a constant $c > 0$, then for any $\epsilon > 0$, we could choose $M > 0$, such that by defining $\rho_\delta^M = \min(M, \rho_\delta)$, the integral defined by

$$d(x) := \int_\Omega \rho_\delta^M(x, x')dx'$$

has a lower bound c for all x in the interior of Ω characterized by the distance ϵ to the boundary $\partial\Omega$ (denoted as Ω_ϵ). Then we have

$$\begin{aligned}
|u_k|_e^2 &\geq \int_\Omega\int_\Omega \rho_\delta^M(x,x')(u_k(x) - u_k(x'))^2 dxdx' \\
&\geq 2\int_\Omega \left(\int_\Omega \rho_\delta^M(x,x')dx'\right) u_k^2(x)dx - 2\int_\Omega \left(\int_\Omega \rho_\delta^M(x,x')u_k(x')dx'\right) u(x)dx \\
&\geq 2\int_{\Omega_\epsilon} d(x)u_k^2(x)dx - 2\int_\Omega \left(\int_\Omega \rho_\delta^M(x,x')u_k(x')dx'\right) u_k(x)dx \\
&\geq 2c\|u_k\|^2_{L^2(\Omega_\epsilon)} - \int_\Omega \mathcal{K}u_k(x)u_k(x)dx\,,
\end{aligned}$$

where $\mathcal{K}$ is a Hilbert-Schmidt operator defined by

$$\mathcal{K}u_k(x) = \int_\Omega \rho_\delta^M(x,x')u_k(x')dx'\,,$$

since $\rho_\delta^M \in L^2(\Omega \times \Omega)$. Now since $u_k \rightharpoonup 0$ from the first step, we have $\mathcal{K}u_k \to 0$ strongly in L^2. Thus we have

$$0 = \limsup_k |u_k|_e \geq \sqrt{2c} \cdot \limsup_k \|u_k\|_{L^2(\Omega_\epsilon)}\,.$$

By letting $\epsilon \to 0$ we have $\|u_k\|_{L^2(\Omega)} \to 0$ as $k \to \infty$, which contradicts to $\|u_k\|_{L^2(\Omega)} = 1$. This proves the claim above and completes the proof of the proposition. ∎

Let $\mathcal{V}_s$ denote a generic subspace $V \cap \mathcal{S}_\delta$ with its dual denoted by $\mathcal{V}_s'$. A standard application of the Lax-Milgram theorem (which is based on the Riesz representation theorem) yields the well-posedness of the variational problems in $\mathcal{V}_s$. The important examples of $\mathcal{V}_s$ considered in this work are the spaces $\mathcal{S}_\delta^D(\Omega)$ and $\mathcal{S}_\delta^N(\Omega)$.

LEMMA 3.2. *For a given $f \in \mathcal{V}_s'$, there exists a unique $u \in \mathcal{V}_s$ such that*

$$B_\delta(u, v) = (f, v)\,,$$

for all $v \in \mathcal{V}_s$. Moreover, $|u|_e = |f|_{\mathcal{V}_s'}$.

We will study different types of problems in the subsequent sections with different choices of $\mathcal{V}_s$ and f.

3.2. Homogeneous Neumann-type problems. Working with $\mathcal{S}_\delta^N(\Omega)$, one can formulate the homogeneous Neumann-type problem in a variational way. For a given $f \in L^2(\Omega)$, find $u_\delta \in \mathcal{S}_\delta^N(\Omega)$, such that

$$B_\delta(u_\delta, v) = (f, v) \quad \forall v \in \mathcal{S}_\delta^N(\Omega)\,. \tag{3.14}$$

Notice that the problem has a solution u_δ for any f in the dual space of $\mathcal{S}_\delta^N(\Omega)$ based on the Lax-Milgram/Riesz-representation theory. But in order for the solution to have properly defined homogeneous Neumann data on the boundary and to be valid pointwise in the interior of Ω, we let f in $L^2(\Omega)$, a subspace of the dual of $\mathcal{S}_\delta^N(\Omega)$. Then the nonlocal Neumann problem (3.14) converges as $\delta \to 0$ to the classical local homogeneous Neumann value problem: for $f \in L^2(\Omega)$, find $u_0 \in H^1(\Omega)\backslash\mathbb{R}$ such that

$$(\nabla u_0, \nabla v) = (f, v) \quad \forall v \in H^1(\Omega)\,.$$

We may formally examine the limiting weak form of (3.14). Indeed, for smooth functions u and v, we have

$$\begin{aligned}
B_\delta(u,v) &= \int_\Omega \int_\Omega \rho_\delta(x,x')(u(x')-u(x))(v(x')-v(x))dx'dx \\
&= 2\int_\Omega \int_{\mathbb{R}^n} c_\delta(x)\gamma_{\delta(x)}(|s|)(u(x+s)-u(x))(v(x+s)-v(x))dsdx \\
&= \int_\Omega \nabla u(x)\cdot\nabla v(x)\left(\int_{\mathbb{R}^n} \frac{2}{n}c_\delta(x)\gamma_{\delta(x)}(|s|)|s|^2ds + O(\delta^2)\right)dx \\
&\to \int_\Omega \nabla u(x)\cdot\nabla v(x)dx\,,
\end{aligned}$$

as $\delta \to 0$. So the limiting weak form of (3.14) corresponds to the weak form of the local homogeneous Neumann problem. The rigorous proof of the convergence of solutions u_δ to u_0 requires a new compactness result, which is out of the scope of this paper. Detailed derivations will be conduced in separate works.

The strong form of (3.14) is given as

$$\begin{cases} \mathcal{L}_\delta u(x) = f(x) & \text{in } \Omega\,, \\ \displaystyle\int_\Omega u\,dx = 0\,. \end{cases}$$

where $\mathcal{L}_\delta$ is found to be

$$\mathcal{L}_\delta u(x) := -2\int_\Omega \rho_\delta(x,x')(u(x')-u(x))dx'\,,$$

thanks to the Fubini theorem, which can be applied in the case that u has vanishing normal derivatives on the boundary of Ω.

Finally, the strong form of the local limit is given by

$$\begin{cases} \mathcal{L}_0 u := -\Delta u = f & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\,, \\ \displaystyle\int_\Omega u\,dx = 0\,. \end{cases} \tag{3.15}$$

3.3. Inhomogeneous Neumann-type problems. We now extend the study earlier on the homogeneous Neumann-type problems to those involving inhomogeneous Neumann data. Instead of (3.15), we want to get a nonlocal solution which is an approximation of the solution to classical inhomogeneous Neumann problems

$$\begin{cases} \mathcal{L}_0 u = f \quad \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = g \quad \text{on } \partial\Omega\,, \\ \displaystyle\int_\Omega u\,dx = 0\,, \end{cases} \tag{3.16}$$

for some inhomogeneous boundary data g. In general, (3.16) is well-defined for $f \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$ that satisfy the compatibility condition

$$\int_\Omega f = -\int_{\partial\Omega} g.$$

It has the following weak formulation: find $u_0 \in H^1(\Omega)\backslash\mathbb{R}$, such that

$$(\nabla u_0, \nabla v) = (f, v) + (g, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)\,. \tag{3.17}$$

A nonlocal version of (3.17) naturally becomes: find $u_\delta \in \mathcal{S}_\delta^N(\Omega)$, such that

$$B_\delta(u_\delta, v) = (f, v) + (g, v)_{\partial\Omega} \quad \forall v \in \mathcal{S}_\delta^N(\Omega)\,. \tag{3.18}$$

Note that (3.18) is well defined for $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$, since by the trace theorem in [28], the space $\mathcal{S}_\delta^N(\Omega)$ has $H^{1/2}(\partial\Omega)$ as its trace space, so $(g, \cdot)_{\partial\Omega}$ is a continuous functional on $\mathcal{S}_\delta^N(\Omega)$. To avoid technicality, we consider more regular data with $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Otherwise, the less regular data may be first mollified. Experiments on the numerical solution of (3.18) can be found in Section 5.

While a valid weak formulation, (3.18) also leads to some inconveniences. The first is that we lack a proper definition of its strong form since the Fubini theorem cannot be applied when the normal derivative of u_δ does not vanish on the boundary. The second is that the horizon function $\delta(x)$ generally has to be smoothly varying in order to get a high order approximation to the corresponding local problem, as shown in Section 5.2. Indeed, as illustrated in the experiments presented in Section 5, the solution to nonlocal problem (3.18) with a piecewise linear horizon converges to its local limit only in first order in δ. This can be seen based on the results reported in Table 5.4. A possible remedy is to consider a relaxed version of (3.18). If the goal is to find a proper nonlocal problem that approximates the local Neumann boundary value problem accurately, then instead of using $(g, v)_{\partial\Omega}$ exactly in (3.18), one may replace but some functional (g_δ, v) that approximates $(g, v)_{\partial\Omega}$ in the asymptotic limit.

3.3.1. Auxiliary function approach. The auxiliary function approach has been used to formulate nonlocal problem before. For nonlocal problems with a constant horizon, [25] discussed the use of auxiliary functions to transfer a nonlocal inhomogeneous Neumann problem to a new homogeneous one while maintaining second order consistency with the local limit in terms of δ.

Let us now demonstrate how to modify (3.18) via an auxiliary function: let u_a be an auxiliary function to be specified later that satisfies $u_\mathrm{a} \in H^2(\Omega)$ and $\frac{\partial u_\mathrm{a}}{\partial \mathbf{n}} = g$ on $\partial\Omega$. Instead of (3.18), we consider the solution u_δ to the following variational problem

$$B_\delta(u_\delta, v) = (f, v) + (\Delta u_\mathrm{a}, v) + B_\delta(u_\mathrm{a}, v) \quad \forall v \in \mathcal{S}_\delta^N(\Omega)\,. \tag{3.19}$$

The derivation of (3.19) comes as follows. If we define (g_δ, v) as

$$(g_\delta, v) = (g, v)_{\partial\Omega} + B_\delta(u_\mathrm{a}, v) - (\nabla u_\mathrm{a}, \nabla v)\,,$$

then it is obvious that (g_δ, v) approximates $(g, v)_{\partial\Omega}$ as $\delta \to 0$, by the consistency of the nonlocal bilinear form to the local bilinear form. Then using integration by parts and the fact that the normal derivative of u_a corresponds to g on the boundary, we arrive at

$$(g_\delta, v) = B_\delta(u_{\rm a}, v) + (\Delta u_{\rm a}, v)\,,$$

which leads to the conclusion of (3.19). This modification of (3.18) preserves the limit of u_δ, as shown in the numerical experiments in Section 5.

The well-posedness in this case can be given in the following proposition.

Proposition 3.3. *Given $f \in L^2$ and $u_a \in H^2(\Omega)$ with $\frac{\partial u_a}{\partial \mathbf{n}} = g$ on $\partial\Omega$, there exists a unique u_δ such that $u_\delta - u_a \in \mathcal{S}_\delta^N(\Omega)$ and u_δ solves (3.19) for all $v \in \mathcal{S}_\delta^N(\Omega)$. Moreover, with $\mathcal{L}_\delta$ given in Section 3.2, (3.19) can be written in a strong from:*

$$\mathcal{L}_\delta(u_\delta - u_a)(x) = f(x) + \Delta u_a(x) \quad \text{in } \Omega\,.$$

Now let us discuss how to select the auxiliary function $u_{\rm a}$. A possible way to determine $u_{\rm a}$ is to solve the following local equation

$$\begin{cases} -\Delta u_{\rm a} = \displaystyle\fint_\Omega f dx \quad \text{in } \Omega\,, \\ \dfrac{\partial u_{\rm a}}{\partial \mathbf{n}} = g \quad \text{on } \partial\Omega\,, \\ \displaystyle\int_\Omega u_{\rm a}\, dx = 0\,. \end{cases} \tag{3.20}$$

Notice that this equation is solvable because the forcing term is compatible with the Neumann data. By the particular choice of the the auxiliary function, the strong form of (3.19) is then reduced to

$$\begin{cases} \mathcal{L}_\delta(u_\delta - u_{\rm a})(x) = f(x) - \displaystyle\fint_\Omega f dx \quad \text{in } \Omega\,, \\ \displaystyle\int_\Omega u_\delta\, dx = 0\,. \end{cases} \tag{3.21}$$

In 1d, working on $\Omega = (0,1)$, the auxiliary function $u_{\rm a}$ can be simply given by

$$u_{\rm a}(x) = -\frac{x^2}{2}\int_\Omega f dx + g(0)x\,,$$

where f and g satisfy the compatibility condition

$$\int_\Omega f dx = g(0) - g(1)\,.$$

One can easily check that $u_{\rm a}'(0) = g(0)$ and $u_{\rm a}'(1) = g(1)$.

3.3.2. Modified forcing for high order truncation. As also proposed in [25], we can cancel low order terms in the nonlocal modeling/truncation error through a modification of the right hand side for problems with a constant horizon. Let us elaborate next this approach in the more general context of heterogeneous nonlocal interactions discussed here.

Consider for now $\Omega = (0,1)$ and $u \in C^3(\Omega)$ is sufficiently regular. For x close to the left boundary $x = 0$, we can get from a careful Taylor expansion that

$$\begin{aligned}
\mathcal{L}_0 u - \mathcal{L}_\delta u &= 2\int_\Omega \rho_\delta(x,x')(u(x') - u(x))dx' - u''(x) \\
&= 2\int_\Omega \rho_\delta(x,x')(x'-x)u'(x)dx' + \int_\Omega \rho_\delta(x,x')(x'-x)^2 u''(x)dx' - u''(x) + \mathcal{O}(\delta) \\
&= 2\int_\Omega \rho_\delta(x,x')(x'-x)(u'(0) + xu''(x))dx' \\
&\quad + \int_\Omega \rho_\delta(x,x')(x'-x)^2 u''(x)dx' - u''(x) + \mathcal{O}(\delta) \\
&= 2u'(0)\int_\Omega \rho_\delta(x,x')(x'-x)dx' + u''(x)\left(\int_\Omega \rho_\delta(x,x')(x'^2 - x^2) - 1\right)dx' + \mathcal{O}(\delta)\,.
\end{aligned}$$

Similarly, when x is close to the right boundary or in the middle of Ω, we can derive the truncation error accordingly. Consequently, similar to the illustration provided in [25] for the constant horizon case, one may introduce some modified right hand side to the nonlocal model near the boundary to get higher order consistency. For example, when x is close to $x = 0$, we have

$$\mathcal{L}_\delta u(x) = f_\delta(x) := -2u'(0)\int_\Omega \rho_\delta(x,x')(x'-x)dx' + f(x)\int_\Omega \rho_\delta(x,x')(x'^2 - x^2)dx'\,.$$

Indeed, if we consider the constant horizon case, then we can get from the above that

$$\begin{aligned}
f_\delta(x) &= -2u'(0)\int_0^{x+\delta} \rho_\delta(x,x')(x'-x)dx' + f(x)\int_0^{x+\delta} \rho_\delta(x,x')(x'^2 - x^2)dx' \\
&= 2u'(0)\int_{x-\delta}^0 \rho_\delta(x,x')(x'-x)dx' + f(x)\left(1 - \int_{x-\delta}^0 \rho_\delta(x,x')(x'^2 - x^2)dx'\right) \\
&= 2u'(0)\int_{-\delta}^0 \rho_\delta(x,x')(x'-x)dx' + f(x)\left(1 - \int_\Omega \rho_\delta(x,-x')(x'^2 - x^2)dx'\right)\,,
\end{aligned}$$

which is similar to the formula given in [25] but has a simpler form.

3.4. Dirichlet-type problems. For nonlocal diffusion models with a constant horizon, problems subject to Dirichlet volumetric constraints have been studied in various earlier works, for example [7, 26]. The local limiting problem corresponds to

$$\begin{cases} \mathcal{L}_0 u = f & \text{in } \Omega\,, \\ \quad u = g & \text{on } \partial\Omega\,. \end{cases} \tag{3.22}$$

The corresponding weak from is given by: find the solution $u \in \{w \in H^1(\Omega) : w|_{\partial\Omega} = g\}$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)\,. \tag{3.23}$$

While intuitively we may use the trace theorem given in [28] to specify a Dirichlet condition on the boundary where the nonlocal interaction gets localized, the derivatives of the resulting solution could develop undesirable oscillations if the horizon function is not sufficiently smooth, which is similar to the inhomogeneous Neumann case that will be presented in Section 5.2. It turns out that we can again use the idea of modifying the bilinear form by the use of auxiliary functions discussed below.

3.4.1. Auxiliary function formulation. We again want to seek auxiliary function $u_{\rm a} \in C^2$ with $\frac{\partial u_{\rm a}}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}}$. However, the Neumann data is not specified in this case, we thus cannot solve for $u_{\rm a}$ a priori without solving the corresponding local problem as in the case of (3.20) and (3.21). Instead, in order to use the auxiliary function, we have to solve the coupled system

$$\begin{cases} \mathcal{L}_\delta(u_\delta - u_{\rm a})(x) = f(x) - \int_\Omega f dx \quad \text{in } \Omega\,, \\ -\Delta u_{\rm a} = \int_\Omega f dx \quad \text{in } \Omega\,, \\ u_\delta = g \quad \text{on } \partial\Omega\,, \\ \dfrac{\partial u_\delta}{\partial \mathbf{n}} - \dfrac{\partial u_{\rm a}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{3.24}$$

The coupled system (3.24) can be decoupled in the following way. By denoting $\tilde{u} = u_\delta - u_{\rm a}$, one can find that solving (3.24) is equivalent to solve the following two systems separately:

$$\begin{cases} \mathcal{L}_\delta \tilde{u}(x) = f(x) - \int_\Omega f dx \quad \text{in } \Omega\,, \\ \int_\Omega \tilde{u}\, dx = 0\,, \end{cases} \tag{3.25}$$

and

$$\begin{cases} -\Delta u_{\rm a} = \int_\Omega f dx \quad \text{in } \Omega\,, \\ u_{\rm a} = g - \tilde{u} \quad \text{on } \partial\Omega\,. \end{cases} \tag{3.26}$$

Then u_δ is obtained by letting $u_\delta = \tilde{u} + u_{\rm a}$.

We note that (3.26) can be seen as a post-processing step. Its validity depends crucially on the superposition principle of linear problems.

3.4.2. An equivalent formulation in one dimension. In 1d, it is easy to solve the coupled system (3.24) directly. Assuming that $\Omega = (0, 1)$. Using the fact that, in 1d, $u_{\rm a}$ in (3.26) takes on the form of

$$u_{\rm a}(x) = -\frac{x^2}{2}\int_\Omega f dx + ax\,, \tag{3.27}$$

with an undetermined parameter $a = u'(0)$. Using the boundary condition $\frac{\partial u_{\rm a}}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}}$, we can test the first equation in (3.24) with a test function v, and use Fubini's theorem to arrive at

$$B_\delta(u_\delta - u_{\rm a}, v) = (f - \int_\Omega f dx, v)\,.$$

Now if we let $v(x) = x$, then the left side of the above is equal to

$$B_\delta(u_\delta - u_{\rm a}, x) = B_\delta(u_\delta, x) - B_\delta(u_{\rm a}, x) = B_\delta(u_\delta, x) + \frac{1}{2}\int_\Omega f dx - a\,,$$

and the right hand side is equal to

$$\int_0^1 \left(f(x) - \int_\Omega f dx\right) x dx = g(1) - g(0) - u'(1) - \frac{1}{2}\int_\Omega f dx$$

Using the compatibility condition for f:

$$\int_\Omega f dx = u'(0) - u'(1) = a - u'(1)$$

one arrives at $B_\delta(u_\delta, x) = g(1) - g(0)$ which can be viewed as a nonlocal compatibility condition to determine a. In the end, in 1d, the coupled system (3.24) can be written as: find $u = u(x)$ and a parameter a such that

$$\left\{ \begin{aligned} \mathcal{L}_\delta(u_\delta - u_{\mathrm{a}})(x) &= f(x) - \int_0^1 f dx \quad \text{ on } \Omega = (0,1)\ , \\ u_{\mathrm{a}}(x) &= -\frac{x^2}{2}\int_\Omega f dx + ax\ , \\ u_\delta &= g \quad \text{ on } \partial\Omega\ , \\ B_\delta(u_\delta, x) &= g(1) - g(0)\,. \end{aligned} \right. \tag{3.28}$$

3.5. Nonlocal models with mixed boundary conditions. The treatment for a nonlocal problem with mixed boundary conditions is a combination of the discussions in Sections 3.3 and 3.4. Briefly, if we want to take $u_\delta = g$ on $\Gamma_1 \subset \partial\Omega$, and $\frac{\partial u_\delta}{\partial \mathbf{n}} = h$ on $\Gamma_2 \subset \partial\Omega$, then we solve the following two systems separately:

$$\left\{ \begin{aligned} &\mathcal{L}_\delta \tilde{u}(x) = f(x) - \int_\Omega f dx \quad \text{in } \Omega\ , \\ &\int_\Omega \tilde{u}\, dx = 0\ , \end{aligned} \right. \tag{3.29}$$

and

$$\left\{ \begin{aligned} -\Delta u_{\mathrm{a}} &= \int_\Omega f dx \quad \text{in } \Omega\ , \\ u_{\mathrm{a}} &= g - \tilde{u} \quad \text{on } \Gamma_1\ . \\ \frac{\partial u_{\mathrm{a}}}{\partial \mathbf{n}} &= h \quad \text{on } \Gamma_2\ . \end{aligned} \right. \tag{3.30}$$

Then the solution u_δ for the nonlocal mixed problem is obtained by $u_\delta = \tilde{u} + u_{\mathrm{a}}$.

REMARK 1. *To end this section, we note that the introduction of auxiliary functions serves as a remedy to have better recovery of the local limit from the nonlocal models with a non-smooth horizon function such as the one given by* (3.11)*, and it is not necessary when the horizon function is smoothly defined such as the one that is given in Fig 4.1.*

4. Local-nonlocal coupled problems. While nonlocal modeling has its advantage on complex physical processes, nonlocal model based numerical simulations often incur higher computational cost than those based on traditional local models. Local-nonlocal coupling is a natural approach in practice and various strategies have been proposed [5, 10, 11, 18, 20]. In this section, we propose an energy-based seamless coupling approach to define local-nonlocal coupled problems. We first adopt the auxiliary function approach and use a general horizon parameter (one representative could be the one given by (3.11)), which is a direct application from the discussion in Section 3.4. For this approach, we only discuss its application in 1d. We then use a smoother variable horizon to solve the coupled problems without the auxiliary functions, which is directly applicable to high dimensional problems. The goal of this section is to derive a coupled model which approximates the traditional PDEs (3.22).

We consider an open bounded and connected domain Ω in $\mathbb{R}^n$. Ω is decomposed into two parts $\Omega = \Omega_- \cup \Omega_+$, where we consider nonlocal model on Ω_+ and local model on Ω_-. We denote $\Gamma = \partial\Omega_- \cap \partial\Omega_+$. The nonlocal interaction domain Ω_I is defined as

$$\Omega_I = \{y \in \mathbb{R}^n \setminus \Omega_+ \text{ such that } \rho_\delta(x,y) \neq 0 \text{ for some } x \in \Omega_+\}\,.$$

We define the coupled energy functional as

$$\begin{aligned} E_f(u) =& \frac{1}{2}\int_{\Omega_+\cup\Omega_I}\int_{\Omega_+\cup\Omega_I} \rho_\delta(x,x')(u(x')-u(x))^2 dx'dx \\ &+ \frac{1}{2}\int_{\Omega_-} |\nabla u(x)|^2\, dx - \int_\Omega f(x)u(x)dx\,. \end{aligned} \tag{4.1}$$

Let us also denote the space

$$\mathcal{W}_\delta(\Omega) = \left\{u \in H^1(\Omega_-) \cap \mathcal{S}_\delta(\Omega_+) \text{ such that } u_- = u_+ \text{ at } \Gamma\right\}\,,$$

which is well-defined by the trace theorem given in [28]. For simplicity, we will only consider the minimization problem of the energy functional subject to Dirichlet boundary conditions. So the solution space is given by

$$\widetilde{\mathcal{W}_\delta}(\Omega) = \{u \in \mathcal{W}_\delta(\Omega), u = g \text{ on } \Omega_I \cup (\partial\Omega_-\backslash\Gamma)\}\,.$$

4.1. Coupled problems with auxiliary functions for general horizons. In this subsection, we use auxiliary function to solve the coupled nonlocal and local model, following the discussion in Section 3.4. Since in 1d the auxiliary function is easily obtained, we will only consider the 1d case here. We consider $\Omega_- = (-1,0]$, $\Omega_+ = [0,1)$ and $\Omega_I = \emptyset$. We use the piecewise linear horizon function on Ω_+ that is localized at $\partial\Omega_+$.

As an application of the auxiliary function approach for solving nonlocal Dirichlet problems with heterogeneous localization, we let u_{a} be the auxiliary function be given by (3.27), where Ω should be replaced by Ω_+. We now consider the minimizer of the following modified energy functional

$$\begin{aligned} E_f(u;a) =& E_{\Omega_+}(u-u_{\text{a}}) - \int_{\Omega_+} u_{\text{a}}''(x)u(x)dx \\ &+ \frac{1}{2}\int_{\Omega_-} u'(x)^2\, dx - \int_\Omega f(x)u(x)dx\,. \end{aligned} \tag{4.2}$$

The solution space now is defined by

$$\{u \in \mathcal{W}_\delta(\Omega), u = g \text{ on } \{-1,1\}\}\,.$$

The weak form of Euler-Lagrange equation for the minimization problem becomes

$$B_0(u,v)_{\Omega_-} + B_\delta(u-u_{\text{a}},v)_{\Omega_+} - (u_{\text{a}}'',v)_{\Omega_+} = (f,v)\,, \tag{4.3}$$

where the subscripts Ω_- and Ω_+ represent the integral domain of the bilinear forms, and the test function v are in $\mathcal{W}_\delta(\Omega)$ but with homogeneous Dirichlet conditions on $\{x = \pm 1\}$. By Fubini's theorem, (4.3) leads to

$$\mathcal{L}_\delta(u-u_{\text{a}}) = f - \int_{\Omega_+} f\, dx \quad \text{in } \Omega_+\,.$$

In order to find a nonlocal compatibility condition for the coupled problems, we again multiply the linear function $h(x) = x$ on both sides and integrate with respect to x from 0 to 1, which leads to

$$\int_{\Omega_+}\int_{\Omega_+} \rho_\delta(x,x')(x'-x)(u(x')-u(x))\, dx'dx - a = \int_{\Omega_+} f(x)(x-1)\, dx\,. \tag{4.4}$$

4.2. Coupled problems with C^2 horizon functions. As demonstrated earlier, constructing suitable auxiliary functions can help retain good consistency of the nonlocal model with the local limit for general kernels and horizons. However, such constructions in general geometry in higher dimensional spaces are challenging. Instead, we now consider the use of a smoother horizon function in the kernel over the domain Ω_+. Discussion on smooth (and strictly positive) horizon has been made also in [23] along with numerical experiments. Here we discuss the case that allows heterogeneous localization with the horizon becoming zero at Γ.

We consider the minimization problem of the original energy functional (4.1). The weak form of the coupled problem is simply given by

$$B_0(u,v)_{\Omega_-} + B_\delta(u,v)_{\Omega_+} = (f,v)\,,$$

for $u \in \widetilde{\mathcal{W}}_\delta(\Omega)$ and the test function v is from $\mathcal{W}_\delta(\Omega)$ with homogeneous Dirichlet condition at $\Omega_I \cup (\partial\Omega_-\backslash\Gamma)$.

In Fig. 4.1, we give a 1d example of a second order differentiable horizon function $\delta(x)$ on the domain $\Omega_+ = (0,1)$, which will be used in the numerical experiments in Section 5. We plot $\delta(x)$ up to its second order derivative. $\delta''(x)$ is bounded by some power of the maximum value of $\delta(x)$ on $(0,1)$.

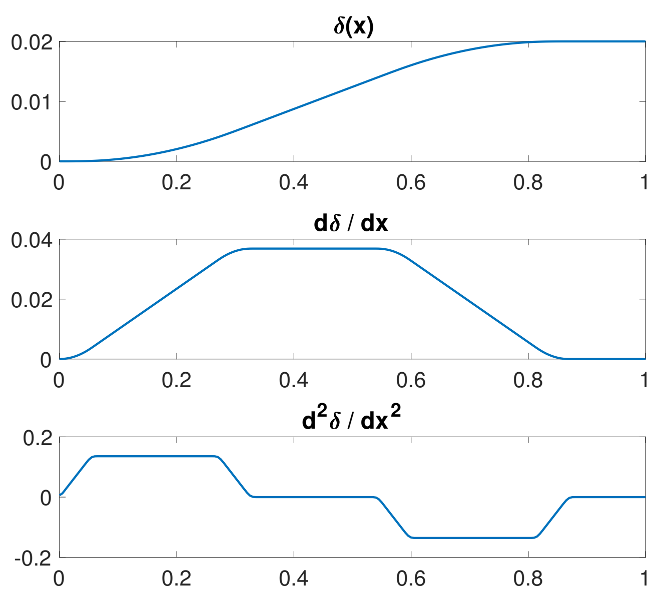

FIG. 4.1. *Horizon as a C^2 function of x.*

In the numerical experiments presented in Section 5, we can see that the coupled solutions converge to the limiting local solution in an optimal order. Moreover, although the ghost forces do not disappear totally, we demonstrate that they vanish as $\delta \to 0$. We also present numerical experiments on using such a horizon to solve nonlocal problems subject to various constraints. Convergence can again be observed in an optimal order without auxiliary functions.

5. Numerical simulations. We now report numerical experiments in 1d that on one hand substantiate the analysis given earlier and on the other hand, offer quantitative pictures to the solution behavior (particularly in the local limit) of nonlocal models and coupled local-nonlocal models. We adopt Galerkin piecewise linear finite element methods to do discretization.

5.1. Nonlocal homogeneous Neumann problems. We work on the 1d domain $\Omega = (0, 1)$. After numerical discretization, we get the stiffness matrix $\mathbb{A}$, where $\mathbb{A}$ is assembled by conforming piecewise linear finite element Galerkin approximations, see [25] for details. For Neumann type problems, however, the matrix is singular without the uniqueness constraint. In principle, we may have several ways to deal with such issue. One possible approach is to modify the stiffness matrix. Let $\mathbf{e}$ be the column vector in the null space of the stiffness matrix $\mathbb{A}$ and define $\mathbb{B} = \mathbb{A} + \mathbf{e}\mathbf{e}^T$, we solve the linear system with coefficient matrix $\mathbb{B}$ to get a solution $\mathbf{U}'$. Then the vector $\mathbf{U}'$ has the property $\mathbf{e}^T\mathbf{U}' = \mathbf{0}$. To get the solutions with different mean values (say to match with suitable benchmark solutions), we need to set $\mathbf{U} = \mathbf{U}' - C_h\mathbf{e}$ for suitably chosen constant C_h.

For numerical experiments, we first fix the horizon δ and calculate the right hand side of the nonlocal equation based on an exact solution $u(x) = x^2(1 - x)^2 - 1/30$. This naturally leads to a δ-dependent right hand side $f = f_\delta(x) = \mathcal{L}_\delta u$. Meanwhile, since the integral of u over Ω is zero, which leads to our target weak form:

$$B_\delta(u_\delta, v) = (f_\delta, v) \quad \forall v \in \mathcal{S}_\delta^N(\Omega)\ , \tag{5.1}$$

where $u_\delta \in \mathcal{S}_\delta^N(\Omega)$. We take the kernel

$$\rho_\delta(x', x) = c_\delta(x)\chi_{\delta(x)}(|x' - x|) + c_\delta(x')\chi_{\delta(x')}(|x' - x|),$$

where

$$c_\delta(x) = \frac{3}{4\delta(x)^3}, \quad \delta(x) = \min(\delta, x, 1 - x).$$

We solve the nonlocal problem on a uniform mesh with mesh size h and take δ to be constant and reduce h to check the convergence properties. As an illustration we choose $\delta = \frac{1}{4}$ and refine the mesh with decreasing h. For each h, the constant C_h is chosen as the trapezoidal rule of the numerical solution on the grid points for each h which is an approximation of the integral of u.

Table 5.1 shows errors and error orders of the piecewise linear finite element approximations to the solution $x^2(1-x)^2-1/30$ with a fixed $\delta = \frac{1}{4}$ while refining mesh with a decreasing h, where $\mathcal{I}_h$ denotes the piecewise linear interpolation operator. From the data in the table, we see that the convergence rate for fixed δ is $\mathcal{O}(h^2)$ for finite element approximations.

We now establish the numerical experiments to show the order of convergence as δ and h both go to 0. In this example, we discretize and solve the following equation:

$$B_\delta(u_\delta, v) = (f, v) \quad \forall v \in \mathcal{S}_\delta^N(\Omega)\,. \tag{5.2}$$

We choose the same kernel and the same local limit of the nonlocal solution $u(x) = x^2(1-x)^2-1/30$, hence the right hand side would be $f(x) = -12x^2+12x-2$. Let $\mathbf{U}_\delta^h$

h	$\|\mathbf{U}^h - \mathcal{I}_h u\|_\infty$	Order
2^{-5}	6.14×10^{-4}	–
2^{-6}	1.86×10^{-4}	1.73
2^{-7}	3.64×10^{-5}	2.35
2^{-8}	1.10×10^{-5}	1.73
2^{-9}	2.74×10^{-6}	2.00
2^{-10}	6.25×10^{-7}	2.13
2^{-11}	1.63×10^{-7}	1.94
2^{-12}	3.63×10^{-8}	2.16

Table 5.1

L^∞ errors and error orders of piecewise linear finite element approximations for fixed $\delta = \frac{1}{4}$ to the solution $x^4 - 2x^3 + x^2 - 1/30$ as $h \to 0$.

denote the numerical solution of (5.2) with mesh size h and horizon δ. Then from the above example, with fixed δ, $\mathbf{U}_\delta^h$ will converge to the interpolant of nonlocal solution u_δ with decreasing h. Therefore, when we keep reducing δ while picking a relative small enough h, the resulting solutions are expected to approximate the local limit. On the other side, we can also fix the ratio between horizon and mesh size: $r = \delta/h$ and refine the mesh, which is also a popular limiting process as it roughly preserves the number of nonzero entries in each row of the stiffness matrix.

	Fixed small h		fFxed $r = 2$	
δ	$\|\mathbf{U}_\delta^h - \mathcal{I}_h u\|_\infty$	Order	$\|\mathbf{U}_\delta^h - \mathcal{I}_h u\|_\infty$	Order
2^{-2}	7.81×10^{-3}	–	7.88×10^{-3}	–
2^{-3}	3.12×10^{-3}	1.35	3.47×10^{-3}	1.18
2^{-4}	9.46×10^{-4}	1.71	1.14×10^{-3}	1.60
2^{-5}	2.60×10^{-4}	1.86	3.26×10^{-4}	1.81
2^{-6}	6.81×10^{-5}	1.93	8.70×10^{-5}	1.91
2^{-7}	1.74×10^{-5}	1.97	2.25×10^{-5}	1.95
2^{-8}	4.41×10^{-6}	1.99	5.71×10^{-6}	1.98

Table 5.2

L^∞ errors and error orders of piecewise linear finite element method as $\delta \to 0$ to solution $x^4 - 2x^3 + x^2 - 1/30$.

Table 5.2 shows errors and error orders to the local limit of the piecewise linear finite element approximations as δ goes to 0 for a fixed small enough h (Column 2 and 3) and a fixed ratio of horizon size to mesh size (Column 4 and 5). From the table, we can see the convergence rate to the local limit is $\mathcal{O}(\delta^2)$. This example shows the asymptotic compatibility. More thorough discussion can be found in other references, for example [12, 25, 26, 27].

For further illustration, we can also estimate the derivative of nonlocal solutions to see if it will recover the local Neumann boundary conditions as $\delta \to 0$. Table 5.3 shows that the numerical derivative of the nonlocal solution converges to the derivative of the local limit in the first order, where D_h is the numerical difference

δ	$\lVert D_h \mathbf{U}^h_\delta - \mathcal{I}_h u' \rVert_\infty$	Order
2^{-3}	3.83×10^{-2}	–
2^{-4}	2.11×10^{-2}	0.86
2^{-5}	1.11×10^{-2}	0.92
2^{-6}	5.74×10^{-3}	0.97
2^{-7}	2.93×10^{-3}	0.98
2^{-8}	1.47×10^{-3}	0.99

TABLE 5.3
L^∞ errors and error orders of the numerical derivative of u_δ to derivative of local solution: $4x^3 - 6x^2 + 2x$ as $\delta \to 0$.

quotient operator (forward, backward or central). Moreover, when we only focus on the derivative at the boundary, we can find that for any δ, $u'_\delta(0) = u'_\delta(1) = 0$, which means the nonlocal solutions also satisfy the homogeneous Neumann conditions. For example, we can see this phenomenon in Fig. 5.1, where we fix $\delta = \frac{1}{8}$ and get the corresponding u_δ.

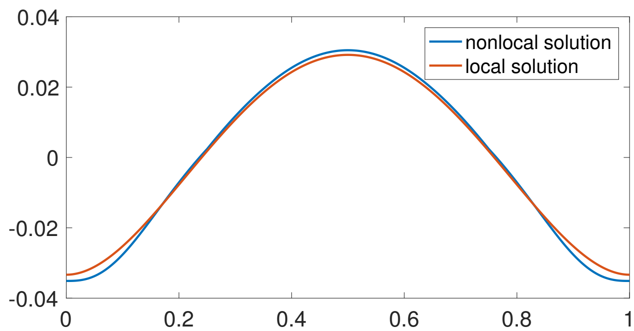

FIG. 5.1. *The nonlocal solution with $\delta = \frac{1}{8}$ and the local solution.*

We remark that we can get the same convergence results when using the smoother horizon shown in Fig. 4.1. The results are omitted here but more interesting examples of solving nonlocal problems by such horizon will be given shortly.

5.2. Nonlocal inhomogeneous Neumann problems. In this example, we present the numerical study on inhomogeneous nonlocal Neumann problems by both the smoother horizon shown in Fig. 4.1 with local boundary conditions and the piecewise linear horizon (3.11) with auxiliary functions, which has been discussed in Section 3.3. Before that, we first give an example on the issue when imposing local inhomogeneous Neumann boundary conditions on the nonlocal model with the piecewise linear horizon (3.11). We discretize and solve the following equation:

$$B_\delta(u_\delta, v) = (f, v) + (g, v)_{\partial\Omega}\,, \tag{5.3}$$

where $u, v \in \mathcal{S}^N_\delta(\Omega)$, g is the Neumann boundary condition and $(\cdot,\cdot)_{\partial\Omega}$ denotes the boundary integration. In 1D domain $\Omega = (0, 1)$, it is defined by

$$(g, v)_{\partial\Omega} = u'(1)v(1) - u'(0)v(0)\,. \tag{5.4}$$

For illustration, we take an exact local solution $u = x^4 - 2x^3 + x^2 - 2x + 29/30$. Table 5.4 shows that the nonlocal solution only has first order convergence as $\delta \to 0$. Moreover, there are oscillations of derivatives with length 2δ at the boundary layers. See Fig. 5.2 for example where $\delta = 1/8$.

	Numerical solution	
δ	L^∞ Error	Order
2^{-3}	3.69×10^{-2}	–
2^{-4}	1.80×10^{-2}	1.03
2^{-5}	8.80×10^{-3}	1.03
2^{-6}	4.34×10^{-3}	1.02
2^{-7}	2.15×10^{-3}	1.01
2^{-8}	1.07×10^{-3}	1.00

TABLE 5.4

L^∞ errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with the piecewise linear horizon function and inhomogeneous Neumann conditions, but without the use of auxiliary function.

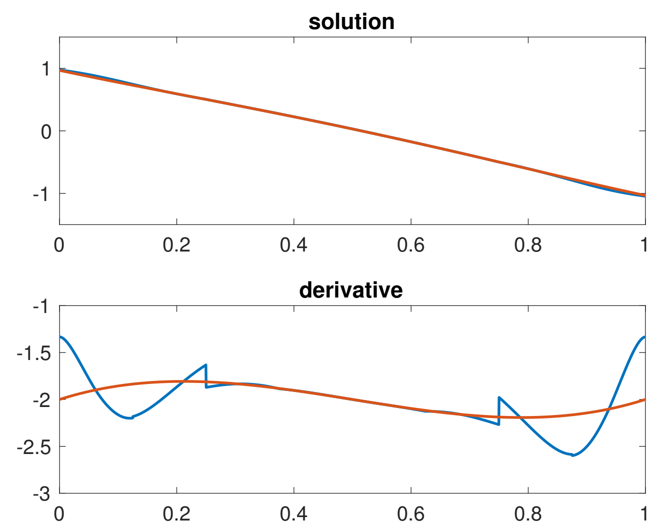

FIG. 5.2. *Derivatives oscillate around the boundary when imposing local Neumann boundary conditions on nonlocal models with piecewise linear variable horizon.*

Instead, we can consider to use the auxiliary functions. Since $g = -2$ on the boundary, we can simply take $u_{\mathrm{a}}(x) = -2x$. Table 5.5 shows that the numerical solution and numerical derivative of the nonlocal case converges to its local limit, which means that this approach works and the boundary condition can still be achieved when the horizon vanishes.

Moreover, we can use the horizon shown in Fig. 4.1 without auxiliary functions and impose the local boundary conditions. As noted before, we use the constant δ to also represent the maximum value of $\delta(x)$. In order to check the asymptotic

	Numerical solution		Numerical derivative	
δ	L^∞ Error	Order	L^∞ Error	Order
2^{-3}	2.81×10^{-3}	–	3.82×10^{-2}	–
2^{-4}	9.47×10^{-4}	1.55	2.12×10^{-2}	0.85
2^{-5}	2.64×10^{-4}	1.84	1.13×10^{-2}	0.91
2^{-6}	5.99×10^{-5}	2.14	6.02×10^{-3}	0.91
2^{-7}	1.21×10^{-5}	2.28	3.00×10^{-3}	0.98
2^{-8}	3.12×10^{-6}	2.02	1.50×10^{-3}	0.99

Table 5.5

L^∞ errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with the piecewise linear horizon function and inhomogeneous Neumann conditions.

compatibility of this case, we instead fix $r = \delta/h$ and let $\delta \to 0$. Table 5.6 shows the convergence of numerical solution and numerical derivative to the local limit with a fixed $r = 2$ in L^∞ sense. The order of convergence for numerical derivatives can be confirmed as the optimal order, while the convergence of numerical solutions is close, but not exactly second order. More convincing data on second order convergence for the solutions are presented in Table 5.7, where one can see that the numerical solutions converge in an optimal order in L^2 norm.

	Numerical solution		Numerical derivative	
$r = 2$	L^∞ Error	Order	L^∞ Error	Order
$\delta = 0.02$	3.13×10^{-4}	–	1.96×10^{-2}	–
$\delta/2$	9.06×10^{-5}	1.79	9.90×10^{-3}	0.99
$\delta/4$	2.58×10^{-5}	1.81	4.97×10^{-3}	0.99
$\delta/8$	7.45×10^{-6}	1.79	2.49×10^{-3}	1.00
$\delta/16$	2.20×10^{-6}	1.76	1.25×10^{-3}	1.00

Table 5.6

L^∞ errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ with fixed r to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with C^2 horizon function and inhomogeneous Neumann conditions.

5.3. Nonlocal Dirichlet problems. In this example, we still calculate $f = f(x)$ based on the choice of a local limiting solution $u_0(x) = x^4 - 2x^3 + x^2 - 2x + 29/30$. Similar to the example given in Section 5.2, we can observe undesirable oscillations of the solution derivative around the boundary when directly imposing local Dirichlet boundary conditions on the nonlocal model with the piecewise linear horizon function. Therefore, we turn to use auxiliary functions and solve the modified problem (3.28). where a and b are assumed to be the boundary derivatives of the local limiting solution at $x = 0$ and $x = 1$ respectively. In this example, we expect that the derivatives of numerical solution at 0 and 1 matches those of the local limiting solution, which is given by $u_0'(0) = u_0'(1) = -2$. Table 5.8 shows that the numerical solution and numerical derivative of the nonlocal case again converge in an optimal order. The auxiliary function approach with extra unknowns still work in nonlocal Dirichlet

$r=2$	$\|\mathbf{U}_\delta^h - \mathcal{I}_h u\|_2$	Order
δ	1.93×10^{-4}	–
$\delta/2$	4.98×10^{-5}	1.95
$\delta/4$	1.26×10^{-5}	1.99
$\delta/8$	3.19×10^{-6}	1.98
$\delta/16$	8.28×10^{-7}	1.95

Table 5.7

L^2 errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ with fixed r to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with C^2 horizon function and inhomogeneous Neumann conditions.

problems. Moreover, from Column 4 of the table we can see that the assumption that $u_\delta - u_{\mathrm{a}}$ has homogeneous Neumann boundary conditions is satisfied for each δ.

	Numerical solution			Numerical derivative	
δ	L^∞ Error	Order	$u'_\delta(0)$	L^∞ Error	Order
2^{-3}	3.09×10^{-3}	–	-2.00	3.80×10^{-2}	–
2^{-4}	9.46×10^{-4}	1.71	-2.00	2.09×10^{-2}	0.86
2^{-5}	2.60×10^{-4}	1.86	-2.00	1.09×10^{-2}	0.94
2^{-6}	6.81×10^{-5}	1.93	-2.00	5.57×10^{-3}	0.97
2^{-7}	1.74×10^{-5}	1.97	-2.00	2.81×10^{-3}	0.99
2^{-8}	4.41×10^{-6}	1.98	-2.00	1.41×10^{-3}	0.99

Table 5.8

L^∞ errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ as well as the solution derivatives at $x = 0$ (value of a) with the piecewise linear horizon function and Dirichlet conditions.

Besides, if we use the horizon shown in Fig. 4.1 and impose the local boundary conditions, we can also get convergence of solutions and derivatives both in optimal orders. The results are listed in Table 5.9.

	Numerical solution				Numerical derivative	
$r=2$	L^∞ Error	Order	L^2 Error	Order	L^∞ Error	Order
$\delta = 0.02$	2.76×10^{-4}	–	1.54×10^{-4}	–	1.95×10^{-2}	–
$\delta/2$	7.92×10^{-5}	1.80	3.96×10^{-5}	1.96	9.87×10^{-3}	0.98
$\delta/4$	2.29×10^{-5}	1.79	1.03×10^{-5}	1.95	4.97×10^{-3}	0.99
$\delta/8$	6.78×10^{-6}	1.75	2.73×10^{-6}	1.91	2.49×10^{-3}	1.00
$\delta/16$	2.06×10^{-6}	1.72	7.42×10^{-7}	1.88	1.25×10^{-3}	1.00

Table 5.9

L^∞ and L^2 errors and error orders of piecewise linear finite element approximations as $\delta \to 0$ with fixed r to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with C^2 horizon function and Dirichlet conditions.

5.4. Local-nonlocal coupled problems. The examples so far have introduced the auxiliary function approach on nonlocal diffusion models with various constraints. Furthermore, the numerical scheme is verified as the asymptotically compatible scheme. We now present how this method can be combined with local PDEs to stimulate coupled local and nonlocal models in a seamless fashion without overlapping domains. In Section 4, the coupled problems are presented with two different horizon functions, both cases are explored more in this example.

As a direct application of the last example, we first consider the same variable horizon used in previous examples with an auxiliary function $u_{\mathrm{a}}(x) = -\frac{1}{2}(\int_{\Omega_+} f)x^2 + ax$. We discretize and solve the variational problem (4.3) and the compatibility condition (4.4). Table 5.10 shows the convergence of numerical solution and numerical derivative of the coupled case to the local limits. Moreover, we observe that for each δ, we can solve $u'_\delta(0)$ correctly as the mesh size goes to zero.

	Numerical solution			Numerical derivative	
δ	L^∞ Error	Order	$u'_\delta(0)$	L^∞ Error	Order
2^{-3}	3.09×10^{-3}	–	-2.00	3.80×10^{-2}	–
2^{-4}	9.46×10^{-4}	1.71	-2.00	2.09×10^{-2}	0.86
2^{-5}	2.60×10^{-4}	1.86	-2.00	1.09×10^{-2}	0.94
2^{-6}	6.81×10^{-5}	1.93	-2.00	5.57×10^{-3}	0.97
2^{-7}	1.74×10^{-5}	1.97	-2.00	2.81×10^{-3}	0.99

TABLE 5.10

L^∞ errors and error orders of piecewise linear finite element approximations of coupled problem as $\delta \to 0$ to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ as well as the solution derivatives at $x = 0$ (value of a) with the piecewise linear horizon function.

	Numerical solution				Numerical derivative	
$r = 2$	L^∞ Error	Order	L^2 Error	Order	L^∞ Error	Order
$\delta = 0.02$	2.95×10^{-4}	–	1.74×10^{-4}	–	1.29×10^{-1}	–
$\delta/2$	8.49×10^{-5}	1.79	4.49×10^{-5}	1.95	6.49×10^{-2}	1.00
$\delta/4$	2.43×10^{-5}	1.80	1.15×10^{-5}	1.97	3.25×10^{-2}	1.00
$\delta/8$	7.12×10^{-6}	1.77	2.97×10^{-6}	1.95	1.62×10^{-2}	1.00
$\delta/16$	2.13×10^{-6}	1.74	7.86×10^{-7}	1.92	8.12×10^{-3}	1.00

TABLE 5.11

L^∞ and L^2 errors and error orders of piecewise linear finite element approximations of coupled problem as $\delta \to 0$ with fixed r to solution $x^4 - 2x^3 + x^2 - 2x + 29/30$ with C^2 horizon function.

Another energy-based method is to consider the original energy functional (4.1) but with a smoother horizon function, which is shown in Fig. 4.1. Again we fix $r = \delta/h$ and let $\delta \to 0$ since the model will get localized as the maximum value of horizon vanishes. Table 5.11 shows that the numerical solution and numerical derivative converge to the local limits with a fixed $r = 2$ both in optimal orders. For the convergence of numerical solutions, one can see this more clearly with respect to the L^2 norm.

Although the solution derivatives can converge as $\delta \to 0$, this model does not satisfy the patch test, which could lead to the existence of ghost forces. However, the ghost forces vanishes as $\delta \to 0$. They are also much smaller than those produced in the case with $\delta(x) = \min(x, \delta, 1-x)$ but without an auxiliary function. In Fig. 5.3, we choose a linear profile $u(x) = -2x + 1$ which implies a zero source term. We can see that the ghost strains vanish as δ decreases. Although we cannot pass the patch test for a fixed $\delta > 0$, the ghost forces do not affect the convergence of solution derivatives and they get reduced with smaller δ.

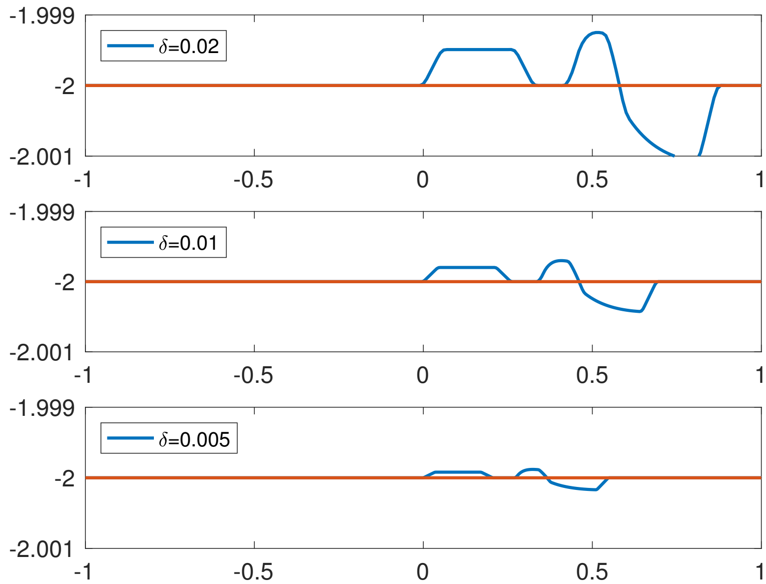

Fig. 5.3. *In patch test, ghost forces vanish as $\delta \to 0$.*

5.5. Discussion. The numerical experiments presented so far with the spatially varying horizon parameters have mimicked those earlier experiments give in [26], where the nonlocal diffusion model with Dirichlet type volume constraints and the constant horizon is studied, and in [25] where the nonlocal diffusion model with Neumann type volume constraints and the constant horizon are studied. In addition, we have applied the variable horizon approach to local-nonlocal coupled problems. We carried out experiments using either the auxiliary function method for inhomogeneous boundary value problems with a piecewise linear horizon function or a C^2 horizon function without the auxiliary function. Both approaches can provide optimal order convergence to the local limit.

Nevertheless, the simulations are mostly confined to smooth solutions. We leave experiments involving singular solutions to future works. Simulations of our coupling model to higher dimensional spaces and real physical processes will be explored in future works.

6. Conclusion. In this paper, we have developed a linear nonlocal model with a spatially varying horizon that captures a spatial change of scales in nonlocal interactions. Our work has extended existing studies on the nonlocal diffusion and nonlocal peridynamic models, and their finite element discrete approximations.

On the modeling side, we are able to significantly expand the nonlocal modeling technique by allowing heterogeneous localization. The latter in turn offers, in particular, a way to pose local boundary value problems for nonlocal models and provides a seamless coupling of local and nonlocal models. This contribution should also be of interests to situations where nonlocal models are utilized as relaxations to study or approximate local PDE models (such as in the development of SPH [6]). Our studies here, though limited to one-dimensional space, are quite extensive as they have covered a number of cases involving piecewise linear or smooth horizons, homogeneous and inhomogeneous, local or nonlocal, and Neumann or Dirichlet constraints.

On the computational side, we have demonstrated that the asymptotically compatible schemes are naturally designed for nonlocal problems with a heterogeneously defined horizon that can be positive and zero in different part of the computational domain. They provide the necessary robustness with respect to the change of length scales.

We note that the current work is restricted to the linear diffusion models and in the numerical experiments, the computational mesh is taken to be uniform since a smooth solution is assumed. While these serve the purpose of illustration well, additional complications may arise in practice. It will be interesting to study further extensions to more general models in higher dimensions and more general discretization, as well as more applications in real physical processes.

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