ASYMPTOTICALLY COMPATIBLE SCHEMES FOR ROBUST DISCRETIZATION OF PARAMETRIZED PROBLEMS WITH APPLICATIONS TO NONLOCAL MODELS*

XIAOCHUAN TIAN† AND QIANG DU‡

Abstract. Many problems in nature, being characterized by a parameter, are of interest both with a fixed parameter value and with the parameter approaching an asymptotic limit. Numerical schemes that are convergent in both regimes offer robust discretizations, which can be highly desirable in practice. The asymptotically compatible schemes studied in an earlier published version of this paper meet such objectives for a class of parametrized problems. An extended version of the abstract mathematical framework is established rigorously here, together with applications to the numerical solution of both nonlocal models and their local limits. In particular, the framework can be applied to nonlocal models of diffusion and a general state-based peridynamic system parametrized by the horizon radius. Recent findings have exposed the risks associated with some discretizations of nonlocal models when the horizon radius is proportional to the discretization parameter. Thus, it is desirable to develop asymptotically compatible schemes for such models so as to offer robust numerical discretizations to problems involving nonlocal interactions on multiple scales. This work provides new insight in this regard through a careful analysis of related conforming finite element discretizations and the finding is valid under minimal regularity assumptions on exact solutions. It reveals that for the nonlocal models under consideration and their local limit, as long as the finite element space contains continuous piecewise linear functions, the Galerkin finite element approximation is always asymptotically compatible. For piecewise constant finite element, whenever applicable, it is shown that a correct local limit solution can also be obtained as long as the discretization (mesh) parameter decreases faster than the modeling (horizon) parameter does. These results can be used to guide future computational studies of nonlocal problems. Some other applications, such as the fractional PDE limit of nonlocal models, and open questions are also presented.

Key words. nonlocal diffusion, peridynamics, fractional PDEs, local limit, convergence analysis, asymptotically compatible schemes, finite element

AMS subject classifications. 65J10, 49M25, 65N30, 65R20, 82C21, 46N40, 45A05

1. Introduction. Asymptotically compatible schemes, as first defined in [57], are aimed at providing approximations to parameterized problems that are robust with respect to changes in parameters. It can be formulated as a general mathematical framework, even though the original theory presented in [57] was largely inspired by the study of robust numerical methods for nonlocal models and their local limits [56]. Nonlocal phenomena are ubiquitous in nature and nonlocal models have appeared in many subjects, from physics and biology to materials and social sciences [16]. For example, there has been a great deal of interest recently in the nonlocal peridynamic (PD) continuum theory introduced first by Silling in [49]. PD models avoid the explicit use of spatial derivatives and provide alternatives to the classical partial differential equation (PDE) based continuum mechanics, especially when dealing with cracks and materials failure [5, 34, 43, 53, 54]. Mathematical analysis of PD models and other related nonlocal models, such as nonlocal diffusion, can be found in [1, 3, 10, 20, 28,

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†Department of Mathematics, University of Texas, Austin, TX 78712 (xtian@math.utexas.edu).

‡Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027 (qd2125@columbia.edu).

30, 41, 42]. Numerical approximations have also been studied in [7, 12, 23, 34, 37, 48, 51, 56, 62, 63] (see [16] for a more recent update on the references).

A common feature of PD models is the introduction of the horizon parameter δ that characterizes the range (radius) of nonlocal interactions [20, 49]. As $\delta \to 0$, nonlocal effect diminishes and the zero-horizon limit of nonlocal PD models becomes a classical local differential equation model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models, and has immense practical significance especially for multiscale modeling and simulations. A natural question leading to the work here is how such limiting behaviors can be preserved in various discrete approximations. This is a critical issue in the applications of PD like models to problems involving possibly different scales, given the popularity and practicality to perform PD simulations with a coupled horizon δ and mesh spacing h. Recently, we have showed that some standard numerical methods for nonlocal diffusion (ND) and PD models may approximate the wrong local limit when the ratio of δ and h is kept constant, while convergence to the desired local limit can also be established for some other discretizations [56]. To keep the discussion relatively simple, the results presented in [56] have mostly been confined to one-dimensional models. Still, they have clearly exposed the risks involved in some popular practices for dealing with nonlocal models and exemplified the need for more comprehensive numerical analysis of the relevant issues.

This paper recapitulates a large part of the work of [57] in introducing asymptotically compatible schemes and the corresponding abstract mathematical framework for their rigorous numerical analysis with respect to certain classes of parametrized problems and their asymptotic limits. We note that the original framework given in [57] was stated for self-adjoint operators and associated variational problems. In the present version, the framework is generalized to cover for non self-adjoint and indefinite operators as well as parameter-dependent data. As pointed out in [57], the abstraction allows us to go beyond the discussion on approximations of nonlocal models and their local limits to establish a more general mathematical theory with a much broader perspective. Indeed, numerical analysis of parametrized problems has been a classical subject [9, 45, 46, 47]. The vanishing nonlocality of the ND/PD models as δ tends to zero reminds us of many classical problems with a changing parameter. For instance, the vanishing viscosity limit for nonlinear conservation laws [11], the convergence of phase field models to their sharp interface limits [17], as well as the linear elasticity problem as the Lamé constant tends to infinity [4], etc. All these problems share a common feature that properties of the underlying equations change significantly in the limit process, so that it is not at all obvious what numerical methods may be effective for a vast range of parameter values and in some limiting cases. It is interesting and challenging to develop numerical methods that behave as desired while taking limits of the problems, and we consider such methods here and name them as asymptotically compatible schemes. While it is perhaps impossible to develop a theory that would encompass problems of many different types, the work in [57] and the discussion in this extended version serve as an attempt to develop an abstract framework that can not only be applied to linear ND and PD models and their local limits but also to other applications. This may offer new insights into the study of problems involving both a modeling parameter (or a relaxation parameter as in the development of numerical techniques such as the smoothed particle hydrodynamics [26]) and a discretization parameter. Moreover, the abstraction also illustrates the key features that distinguish the nonlocal models, which are the main applications considered here, from other parametrized problems.

An immediate consequence of the abstract framework is the identification of asymptotically compatible finite element methods for the robust discretization of linear ND/PD models, as summarized later by a commutative diagram in Figure 2.1. The results are for models with very general nonlocal interaction kernels and solutions with minimal regularity assumptions, as well as general geometric meshes with no restrictions on the space dimension. They significantly generalize earlier exploratory findings [56] and offer another major contribution of this paper that will be of particular interest to people working on numerical simulations of ND/PD models. Furthermore, the concept of asymptotically compatible schemes is applicable to not only Galerkin approximation but also other discretizations such as finite difference and collocations methods using quadratures [25, 56], as well as DG, particle and spectral approximations [22, 27, 36, 61].

The paper is organized as follows. In section 2, we introduce the asymptotically compatible scheme and an abstract framework for the convergence analysis. In section 3, we consider the application to linear ND problems and characterize asymptotically compatible finite element methods. We also show that the discontinuous piecewise constant finite element, which is not reliable if the ratio δ/h is fixed [56], may be conditionally asymptotically compatible. Section 4 contains an application to state-based PD models. Results of numerical experiments are reported in section 5 to complement the theoretical analysis and to illustrate the order of accuracy of numerical schemes. Section 6 provides a few concluding remarks and some discussions on subsequent works inspired by [57].

2. An abstract framework.

In this section, we introduce the notion of asymptotically compatible schemes and propose an abstract framework for their numerical analysis when they are applied to a special class of parametrized problems.

2.1. Notation and assumptions.

Before stating the main results, let us introduce notation and state the main assumptions. The assumptions are given in the order of (infinite-dimensional) function spaces, then bilinear forms, followed by induced linear operators, and finally the approximations. While the original version in [57] was presented for self-adjoint problems, we provide an extended version that works also for non self-adjoint problems.

We begin by considering two families of reflexive Banach spaces $\{\mathcal{T}_\sigma, \sigma \in [0,\infty]\}$ and $\{\mathcal{X}_\sigma, \sigma \in (0,\infty]\}$ over $\mathbb{R}$ with corresponding norms $\{\|\cdot\|_{\mathcal{T}_\sigma}, \sigma \in [0,\infty]\}$ and $\{\|\cdot\|_{\mathcal{X}_\sigma}, \sigma \in (0,\infty]\}$. Denote the dual space of $\mathcal{T}_\sigma$ by $\mathcal{T}_{-\sigma} = \mathcal{T}_\sigma^*$, and the dual space of $\mathcal{X}_\sigma$ by $\mathcal{X}_{-\sigma} = \mathcal{X}_\sigma^*$. For simplicity, we use the same notation $\langle\cdot,\cdot\rangle$ to stand for the duality pairing between $\mathcal{T}_\sigma$ and $\mathcal{T}_{-\sigma}$ or between $\mathcal{X}_\sigma$ and $\mathcal{X}_{-\sigma}$, without specifying any subscript related to the spaces.

We note that both spaces $\mathcal{T}_0$ and $\mathcal{T}_\infty$ are of particular interest to our discussions here. Indeed, we let $\mathcal{T}_0$ be a Hilbert space and identify the dual space of $\mathcal{T}_0$ with itself, that is, $\mathcal{T}_0^* = \mathcal{T}_0$. A typical example of $\mathcal{T}_0$ is given by the standard L^2 function space in applications that we consider later. Moreover, we assume that $\mathcal{T}_0$ serves as the pivot space between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ so that a realization of the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ is given as the extension of the inner product on $\mathcal{T}_0$.

Assumptions given above on the spaces $\{\mathcal{T}_\sigma\}$ and $\{\mathcal{X}_\sigma\}$ are very generic so far. To discuss the special class of problems defined on spaces $\{\mathcal{T}_\sigma\}$ and $\{\mathcal{X}_\sigma\}$, we state the following assumptions, which are crucial to the problems under consideration.

Assumption 2.1. The spaces $\{\mathcal{T}_\sigma\}$ and $\{\mathcal{X}_\sigma\}$ are assumed to satisfy the properties below.

(i) *Uniform embedding* property: there are positive constants M_1 and M_2, independent of $\sigma \in (0,\infty)$, such that

$$M_1\|u\|_{\mathcal{T}_0} \le \|u\|_{\mathcal{T}_\sigma} \quad \forall u \in \mathcal{T}_\sigma \quad \text{and} \quad \|v\|_{\mathcal{X}_\sigma} \le M_2\|v\|_{\mathcal{X}_\infty} \quad \forall v \in \mathcal{X}_\infty.$$

(ii) *Asymptotically compact embedding* property for $\{\mathcal{T}_\sigma\}$: for any sequence $(u_n \in \mathcal{T}_n)$, if there is a constant $C > 0$ independent of n such that

$$\|u_n\|_{\mathcal{T}_n} \le C \quad \forall n,$$

then the sequence (u_n) is relatively compact in $\mathcal{T}_0$ and each limit point is in $\mathcal{T}_\infty$.

With spaces $\{\mathcal{T}_\sigma\}$ and $\{\mathcal{X}_\sigma\}$ given, we now consider some parametrized bilinear forms.

Assumption 2.2. Let $a_\sigma : \mathcal{T}_\sigma \times \mathcal{X}_\sigma \to \mathbb{R}$ be a symmetric bilinear form, $\sigma \in (0,\infty]$.

(i) a_σ is *bounded*: there exists a constant $C_2 > 0$ such that

$$a_\sigma(u,v) \le C_2\|u\|_{\mathcal{T}_\sigma}\|v\|_{\mathcal{X}_\sigma} \quad \forall u \in \mathcal{T}_\sigma,\ v \in \mathcal{X}_\sigma.$$

(ii) Inf-sup condition: there exists a constant $\alpha > 0$ independent of σ such that

$$\inf_{u\in\mathcal{T}_\sigma}\sup_{v\in\mathcal{X}_\sigma}\frac{a_\sigma(u,v)}{\|u\|_{\mathcal{T}_\sigma}\|v\|_{\mathcal{X}_\sigma}} \ge \alpha > 0.$$

(iii) If $a_\sigma(u,v) = 0$ for all $u \in \mathcal{T}_\sigma$, then $v = 0$.

Given the above assumption on the bilinear forms, for any $\sigma \in (0,\infty]$, we see that $a_\sigma(\cdot, v)$ defines a bounded linear functional for any $v \in \mathcal{X}_\sigma$. Moreover, by the Lax-Milgram/Banach-Necas-Babuska theorem [31], it induces naturally a bounded linear operator, denoted by $\mathcal{A}^*_\sigma$, from $\mathcal{X}_\sigma$ to $\mathcal{T}_{-\sigma}$, with a bounded inverse $(\mathcal{A}^*_\sigma)^{-1} : \mathcal{T}_{-\sigma} \to \mathcal{X}_\sigma$. In other words, using the notation given above, we have

$$\langle u, \mathcal{A}^*_\sigma v\rangle = a_\sigma(u,v) \quad \forall u \in \mathcal{T}_\sigma, v \in \mathcal{X}_\sigma. \tag{2.1}$$

Moreover, the adjoint of $\mathcal{A}^*_\sigma$, denoted by $\mathcal{A}_\sigma$, is also a bounded linear operator from $\mathcal{T}_\sigma$ to $\mathcal{X}_{-\sigma}$ with a bounded inverse, and we have

$$\langle \mathcal{A}_\sigma u, v\rangle = \langle u, \mathcal{A}^*_\sigma v\rangle \quad \forall u \in \mathcal{T}_\sigma, v \in \mathcal{X}_\sigma\,. \tag{2.2}$$

We next give some assumptions on $\mathcal{A}_\sigma$.

Assumption 2.3. For $\mathcal{A}^*_\sigma$ defined by (2.1), we assume the following:

(i) A subspace $\mathcal{X}_*$ is dense in $\mathcal{X}_\infty$, and also dense in any $\mathcal{X}_\sigma$ with $\sigma \ge 0$, such that

$$\mathcal{A}^*_\sigma v \in \mathcal{T}_0 \quad \forall v \in \mathcal{X}_*.$$

(ii) $\mathcal{A}^*_\infty$ is the limit of $\mathcal{A}^*_\sigma$ in $\mathcal{X}_*$ in the sense that

$$\lim_{\sigma\to\infty}\|\mathcal{A}^*_\sigma v - \mathcal{A}^*_\infty v\|_{\mathcal{T}_{-\sigma}} = 0 \quad \forall v \in \mathcal{X}_*. \tag{2.3}$$

Since we are concerned with numerical approximations of problems associated with the operators $\{\mathcal{A}_\sigma\}$ for $\sigma \in (0,\infty]$, we consider families of closed subspaces $\{W_{\sigma,h} \subset \mathcal{T}_\sigma\}$ and $\{X_{\sigma,h} \subset \mathcal{X}_\sigma\}$ parametrized by an additional real parameter $h \in$

$(0, h_0]$. The fact that we take $W_{\sigma,h}$ as a subspace of $\mathcal{T}_\sigma$ and $X_{\sigma,h}$ as a subspace of $\mathcal{X}_\sigma$ implies that we are effectively adopting a standard, internal, or equivalently conforming type Galerkin approximation approach.

We assume that $W_{\sigma,h}$ and $X_{\sigma,h}$ are finite dimensional spaces, and $\dim W_{\sigma,h} = \dim X_{\sigma,h}$. Moreover, we need some basic assumptions on the approximation properties as stated below. The first assumption guarantees the discrete problems to be solvable. The second assumption ensures the convergence of $W_{\sigma,h}$ to $\mathcal{T}_\sigma$ as $h \to 0$ for each σ, and the third assumption on $\{X_{\sigma,h}\}$ is concerned with the limiting behavior as both $h \to 0$ and $\sigma \to \infty$ at the same time.

Assumption 2.4. Assume that the family of subspaces $\{X_{\sigma,h} \subset \mathcal{X}_\sigma\}$ parametrized by $\sigma \in (0, \infty]$ and $h \in (0, h_0]$ satisfies the following properties:

(i) Discrete inf-sup: there exists a constant $\tilde{\alpha} > 0$ independent of σ such that

$$\inf_{u \in W_{\sigma,h}} \sup_{v \in X_{\sigma,h}} \frac{a_\sigma(u, v)}{\|u\|_{\mathcal{T}_\sigma} \|v\|_{\mathcal{X}_\sigma}} \geq \tilde{\alpha} > 0.$$

(ii) For each $\sigma \in (0, \infty]$, the family $\{W_{\sigma,h}, h \in (0, h_0]\}$ is dense in $\mathcal{T}_\sigma$ in the sense that $\forall u \in \mathcal{T}_\sigma$, there exists a sequence $\{u_n \in W_{\sigma,h_n}\}$ with $h_n \to 0$ as $n \to \infty$ such that

$$\|u - u_n\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

(iii) $\{X_{\sigma,h}, \sigma \in (0, \infty), h \in (0, h_0]\}$ is *asymptotically dense* in $\mathcal{X}_\infty$, i.e., $\forall v \in \mathcal{X}_\infty$, there exists a sequence $\{v_n \in X_{\sigma_n,h_n}\}_{h_n \to 0, \sigma_n \to \infty}$ as $n \to \infty$ such that

$$\|v - v_n\|_{\mathcal{X}_\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

2.2. The parametrized problems and their approximations. Consider a family of parametrized problems defined by the following: given $f_\sigma \in \mathcal{X}_{-\sigma}$,

$$\text{find } u_\sigma \in \mathcal{T}_\sigma \text{ such that } \ a_\sigma(u_\sigma, v) = \langle f_\sigma, v \rangle \quad \forall v \in \mathcal{X}_\sigma \tag{2.6}$$

for any parameter $\sigma \in (0, \infty]$. The approximation to u_σ in a subspace $W_{\sigma,h}$ is defined by the following:

$$\text{find } u_{\sigma,h} \in W_{\sigma,h} \text{ such that } \ a_\sigma(u_{\sigma,h}, v) = \langle f_\sigma, v \rangle \quad \forall v \in X_{\sigma,h}. \tag{2.7}$$

Note that f_σ was taken to be independent of σ in [57]. Here, we allow for a generalization that is subject to the following assumption on the data f_σ.

Assumption 2.5. $\|f_\sigma\|_{\mathcal{X}_{-\sigma}} \leq C$ for a constant C independent of σ and $\|f_\sigma - f_\infty\|_{\mathcal{X}_{-\infty}} \to 0$ as $\sigma \to \infty$.

The existence and uniqueness of u_σ and $u_{\sigma,h}$ follow from assumptions made earlier. We may also express (2.6) and (2.7) in strong forms as

$$\mathcal{A}_\sigma u_\sigma = f_\sigma, \tag{2.8}$$
$$\mathcal{A}_\sigma^h u_{\sigma,h} = \pi_\sigma^h f_\sigma, \tag{2.9}$$

where π_σ^h is the projection operator onto the subspace $X_{\sigma,h}^*$, which can be identified with $X_{\sigma,h}$ due to the finite dimensionality, and $\mathcal{A}_\sigma^h : X_{\sigma,h} \to X_{\sigma,h}^*$ is the operator induced by the bilinear form a_σ on $W_{\sigma,h} \times X_{\sigma,h}$ (or the solution operator of (2.7) in the specified subspace).

We are interested in establishing an abstract framework to analyze the various limits of $\{u_{\sigma,h}\}$ as we take limits in the parameters. We first state a convergence result for the solutions of the parametrized problems as $\sigma \to \infty$.

THEOREM 2.6 (convergence of solutions as $\sigma \to \infty$). *Given the assumptions on the family of spaces and the bilinear forms and operators, we have*

$$\|u_\sigma - u_\infty\|_{\mathcal{T}_0} \to 0 \quad \text{as } \sigma \to \infty.$$

Proof. By (2.6) and the assumptions, we have

$$\alpha \|u_\sigma\|_{\mathcal{T}_\sigma} \le \sup_{v \in \mathcal{X}_\sigma} \frac{|a_\sigma(u_\sigma, v)|}{\|v\|_{\mathcal{X}_\sigma}} = \sup_{v \in \mathcal{X}_\sigma} \frac{|\langle f_\sigma, v\rangle|}{\|v\|_{\mathcal{X}_\sigma}} = \|f_\sigma\|_{\mathcal{X}_{-\sigma}} \le C,$$

which leads to the uniform boundedness of $\{u_\sigma \in \mathcal{T}_\sigma\}$, and thus by the asymptotically compact embedding property, we get the convergence of a subsequence of $\{u_\sigma\}$ in $\mathcal{T}_0$ to a limit point $u_* \in \mathcal{T}_\infty$. For notational convenience, we use the same $\{u_\sigma\}$ to denote the subsequence. Now, taking $v \in \mathcal{X}_* \subset \mathcal{X}_\infty$, we have

$$\begin{aligned} \langle f_\infty - \mathcal{A}_\infty u_*, v\rangle &= \langle f_\infty - f_\sigma, v\rangle + \langle \mathcal{A}_\sigma u_\sigma - \mathcal{A}_\infty u_*, v\rangle \\ &= \langle f_\infty - f_\sigma, v\rangle + \langle u_\sigma, \mathcal{A}_\sigma^* v - \mathcal{A}_\infty^* v\rangle + \langle u_\sigma - u_*, \mathcal{A}_\infty^* v\rangle. \end{aligned}$$

The first term in the above equation goes to zero as $\sigma \to \infty$ by the assumption that $\|f_\infty - f_\sigma\|_{\mathcal{X}_{-\infty}} \to 0$. The second term goes to zero as a result of the uniform boundedness of u_σ in $\mathcal{T}_\sigma$ and equation (2.3). The third term goes to zero since $\mathcal{A}_\infty^* v \in \mathcal{T}_0$ and $u_\sigma - u_*$ goes to zero in $\mathcal{T}_0$. Together we arrive at

$$\langle f_\infty - \mathcal{A}_\infty u_*, v\rangle \to 0 \quad \text{as } \sigma \to \infty.$$

Moreover, since $\mathcal{X}_*$ is dense in $\mathcal{X}_\infty$, we see that u_* is the unique solution u_∞ of $\mathcal{A}_\infty u_\infty = f$ and the convergence of the whole sequence also follows. ∎

Next, we consider the convergence of approximations as $h \to 0$ for a given σ.

THEOREM 2.7 (convergence with a fixed $\sigma \in [0,\infty]$ as $h \to 0$). *For any given $\sigma \in [0,\infty]$, let u_σ and $u_{\sigma,h}$ be defined by* (2.6) *and* (2.7). *Given the assumptions on the approximate spaces and the approximate bilinear forms, there exists a constant $C > 0$, independent of h, such that*

$$\|u_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \le C \inf_{v_{\sigma,h} \in W_{\sigma,h}} \|v_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0.$$

Proof. The proof is similar to the standard best approximation property of the Galerkin approximation. Given $\sigma \in [0,\infty]$, for any $v_{\sigma,h} \in W_{\sigma,h}$,

$$\tilde{\alpha}\|u_{\sigma,h} - w\|_{\mathcal{T}_\sigma} \le \sup_{v \in X_{\sigma,h}} \frac{|a_\sigma(u_{\sigma,h} - w, v)|}{\|v\|_{\mathcal{X}_\sigma}} = \sup_{v \in X_{\sigma,h}} \frac{|a_\sigma(u_\sigma - w, v)|}{\|v\|_{\mathcal{X}_\sigma}} \le C\|u_\sigma - w\|_{\mathcal{T}_\sigma}.$$

So we have

$$\begin{aligned} \|u_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} &\le \inf_{w \in W_{\sigma,h}} (\|u_{\sigma,h} - w\|_{\mathcal{T}_\sigma} + \|w - u_\sigma\|_{\mathcal{T}_\sigma}) \\ &\le (1 + C/\tilde{\alpha}) \inf_{w \in W_{\sigma,h}} \|u_\sigma - w\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0. \end{aligned}$$

This proves the theorem. ∎

We now move on to an analogue of Theorem 2.6 for approximate problems, that is, we consider the convergence as $\sigma \to \infty$ but for a fixed $h > 0$. For this, we need a few additional assumptions on the approximation spaces.

THEOREM 2.8 (convergence of approximate solutions with $h > 0$ as $\sigma \to \infty$). *Given the assumptions on the family of spaces, bilinear forms, operators, and approximate spaces, and assume in addition that for a given $h > 0$, we have the following:*

(i) *Limit of approximate spaces:*

$$W_{\infty,h} = \mathcal{T}_\infty \cap \left(\bigcap_{\sigma \geq 0} W_{\sigma,h} \right), \text{ and } X_{\infty,h} = \mathcal{X}_\infty \cap \left(\bigcap_{\sigma \geq 0} X_{\sigma,h} \right). \tag{2.10}$$

(ii) *Approximation property of bilinear forms:*

$$\lim_{\sigma \to \infty} a_\sigma(u_h, v_h) = a_\infty(u_h, v_h) \quad \forall u_h \in W_{\infty,h}, v \in X_{\infty,h}. \tag{2.11}$$

(iii) *A strengthened continuity property: for any sequence ($w_{\sigma,h} \in W_{\sigma,h}$) with uniformly bounded ($\|w_{\sigma,h}\|_{\mathcal{T}_\sigma}$) and satisfying $w_{\sigma,h} \to 0$ in $\mathcal{T}_0$ as $\sigma \to \infty$, we have*

$$\lim_{\sigma \to \infty} a_\sigma(w_{\sigma,h}, v_h) = 0 \quad \forall v_h \in X_{\infty,h}. \tag{2.12}$$

Then, for the approximate solutions $u_{\sigma,h}$ of (2.7) with $\sigma \in (0, \infty)$, we have

$$\|u_{\sigma,h} - u_{\infty,h}\|_{\mathcal{T}_0} \to 0 \quad \text{as } \sigma \to \infty. \tag{2.13}$$

Proof. Similar to the proof of Theorem 2.6, we have

$$\tilde{\alpha}\|u_{\sigma,h}\|_{\mathcal{T}_\sigma} \leq \sup_{v \in X_{\sigma,h}} \frac{|a_\sigma(u_{\sigma,h}, v)|}{\|v\|_{\mathcal{X}_\sigma}} = \sup_{v \in X_{\sigma,h}} \frac{|\langle f_\sigma, v\rangle|}{\|v\|_{\mathcal{X}_\sigma}} \leq \|f_\sigma\|_{\mathcal{X}_{-\sigma}} \leq C\,.$$

This leads to the uniform boundedness of $\{u_{\sigma,h} \in \mathcal{T}_\sigma\}$, and thus by the asymptotically compact embedding property, we get the convergence of a subsequence in $\mathcal{T}_0$ to a limit point $u_{*,h} \in \mathcal{T}_\infty$. By assumption (2.10), we have necessarily that $u_{*,h} \in W_{\infty,h}$. Using again the same $\{u_{\sigma,h}\}$ to denote the subsequence and taking $v_h \in W_{\infty,h} \subset W_{\sigma,h}$,

$$\begin{aligned}
\langle f_\infty - \mathcal{A}_\infty u_{*,h}, v_h\rangle &= \langle f_\infty - f_\sigma, v_h\rangle + \langle \mathcal{A}_\sigma u_{\sigma,h}, v_h\rangle - \langle \mathcal{A}_\infty u_{*,h}, v_h\rangle \\
&= \langle f_\infty - f_\sigma, v_h\rangle + \langle \mathcal{A}_\sigma (u_{\sigma,h} - u_{*,h}), v_h\rangle + \langle (\mathcal{A}_\sigma - \mathcal{A}_\infty) u_{*,h}, v_h\rangle \\
&= \langle f_\infty - f_\sigma, v_h\rangle + a_\sigma(u_{\sigma,h} - u_{*,h}, v_h) + \big[a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h)\big] \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

The fist term I_1 in the above equation goes to zero since $\|f_\infty - f_\sigma\|_{\mathcal{X}_{-\infty}} \to 0$ as $\sigma \to \infty$. Now, to estimate the second term, we let $w_{\sigma,h} = u_{\sigma,h} - u_{*,h} \in W_{\sigma,h}$ and apply the strengthened continuity property of a_σ to get $|\mathrm{I}_2| \to 0$. Assumption (2.11) implies that $\mathrm{I}_3 \to 0$. Thus, $u_{*,h}$ is the unique solution of (2.7) with $\sigma = \infty$ and the unique limit point of the whole sequence $\{u_{\sigma,h}\}$. The theorem thus follows. ∎

2.3. Asymptotically compatible schemes. While we have the convergence of $\{u_{\sigma,h}\}$ for a given σ as $h \to 0$, as well as the convergence of $\{u_\sigma\}$ to u_∞ and $\{u_{\sigma,h}\}$ to $u_{\infty,h}$ as $\sigma \to \infty$, we are also interested in the behavior as both $\sigma \to \infty$ and $h \to 0$. We summarize this in Figure 2.1 and introduce the concept of asymptotically compatible schemes.

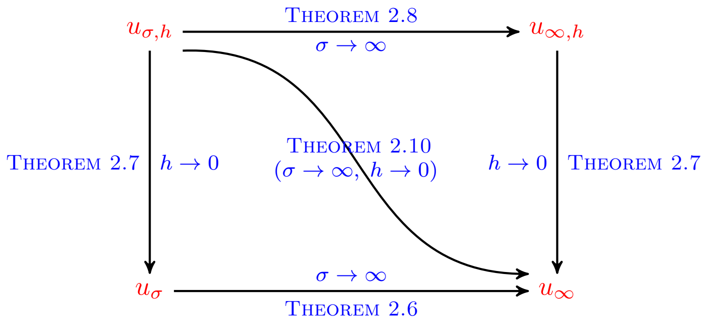

FIG. 2.1. *A diagram for asymptomatically compatible schemes and convergence results.*

DEFINITION 2.9. *The family of convergent approximations* $\{u_{\sigma,h}\}$ *defined by* (2.7) *is said to be asymptotically compatible to the solution* u_∞ *defined by* (2.6) *with* $\sigma = \infty$ *if for any sequence* $\sigma_n \to \infty$ *and* $h_n \to 0$*, we have* $\|u_{\sigma_n,h_n} - u_\infty\|_{\mathcal{T}_0} \to 0$.

Note that since u_{σ_n,h_n} and u_∞ may live in different spaces, the space $\mathcal{T}_0$ is the most natural space that contains all the elements involved. In cases where u_{σ_n,h_n} represent discrete solutions, one may even use different meshes and basis functions. Nevertheless, additional compatibilities on the spaces are needed for the convergence of $u_{\sigma,h}$ and to $u_{\infty,h}$ as $\sigma \to \infty$, as suggested by (2.10), for example.

THEOREM 2.10 (asymptotically compatibility). *Under Assumptions* 2.1–2.5*, the family of approximations is asymptotically compatible.*

Proof. The first step is again similar to that in the proof of Theorems 2.6 and 2.8, that is, we can get $\|u_{\sigma,h}\|_{\mathcal{T}_\sigma}$ being uniformly bounded by some constant,

$$\|u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le C. \tag{2.14}$$

Then for any sequences $\{\sigma_n\}$ and $\{h_n\}$, where $\sigma_n \to \infty$, $h_n \to 0$, the sequence $(u_{\sigma_n,h_n})_n$ is relatively compact in $\mathcal{T}_0$, and any limit point u_* of the convergent subsequence in $\mathcal{T}_0$, still denoted by ($u_n = u_{\sigma_n,h_n}$) without loss of generality, is in $\mathcal{T}_\infty$. Let us show that u_* solves (2.6) with $\sigma = \infty$ and therefore is unique so that the entire sequence actually converges to the unique solution $u_* = u_\infty$. That is, for $\|u_* - u_n\|_{\mathcal{T}_0} \to 0$ as $n \to \infty$, we need to prove for every $v \in \mathcal{X}_*$, u_* satisfies (2.6).

By the asymptotically dense property (2.5) of $X_{\sigma,h}$ in $\mathcal{X}_\infty$, we can choose $v_n \in X_{\sigma_n,h_n}$ such that $\|v - v_n\|_{\mathcal{X}_\infty} \to 0$. Then we have the following equation:

$$\begin{aligned} a_\infty(u_*, v) - \langle f_\infty, v\rangle &= \langle \mathcal{A}_\infty(u_* - u_n), v\rangle + \langle(\mathcal{A}_\infty - \mathcal{A}_{\sigma_n})u_n, v\rangle \\ &\quad + \langle \mathcal{A}_{\sigma_n} u_n, v - v_n\rangle + [\langle f_{\sigma_n}, v_n\rangle - \langle f_\infty, v\rangle] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{2.15}$$

We now show that as $n \to \infty$, all four terms vanish. Now for the first part, since the adjoint operator of $\mathcal{A}_\infty$ satisfies $\mathcal{A}_\infty^a stv \in \mathcal{T}_0$, we can rewrite I as

$$|\mathrm{I}| = |\langle \mathcal{A}_\infty(u_* - u_n), v\rangle| = |\langle u_* - u_n, \mathcal{A}_\infty^* v\rangle| \le \|u_* - u_n\|_{\mathcal{T}_0} \|\mathcal{A}_\infty^* v\|_{\mathcal{T}_0} \to 0.$$

Similarly we can rewrite the second part and use Assumption 2.3 to obtain

$$\begin{aligned} |\mathrm{II}| &= |\langle u_n, \mathcal{A}_\infty^* v - \mathcal{A}_{\sigma_n}^* v\rangle| \le \|u_n\|_{\mathcal{T}_{\sigma_n}} \|\mathcal{A}_\infty^* v - \mathcal{A}_{\sigma_n}^* v\|_{\mathcal{T}_{-\sigma_n}} \\ &\le C\|\mathcal{A}_\infty^* v - \mathcal{A}_{\sigma_n}^* v\|_{\mathcal{T}_{-\sigma_n}} \to 0. \end{aligned}$$

We then use the bound on the bilinear form a_σ and the uniform embedding to get

$$|\mathrm{III}| = |a_{\sigma_n}(u_n, v - v_n)| \le C\|u_n\|_{\mathcal{T}_{\sigma_n}}\|v - v_n\|_{\mathcal{X}_{\sigma_n}} \\ \le \tilde{C}\|v - v_n\|_{\mathcal{X}_{\sigma_n}} \le \tilde{C}M_2\|v - v_n\|_{\mathcal{X}_\infty} \to 0.$$

Finally, for the last term,

$$|\mathrm{IV}| \le |\langle f_{\sigma_n}, v_n\rangle - \langle f_\infty, v\rangle| \le |\langle f_{\sigma_n} - f_\infty, v_n\rangle| + |\langle f_\infty, v_n - v\rangle| \\ \le \|f_{\sigma_n} - f_\infty\|_{\mathcal{X}_{-\infty}}\|v_n\|_{\mathcal{X}_\infty} + \|f_\infty\|_{\mathcal{X}_{-\infty}}\|v_n - v\|_{\mathcal{X}_\infty} \to 0.$$

This shows that u_* solves (2.6). This completes the proof of the theorem. ∎

3. Applications to ND problem. The first example we consider is a homogeneous Dirichlet volume constrained value problem associated with a linear ND model. We refer [16, 18, 38, 41] for more discussions on related mathematical theory. These problems are nonlocal in nature and they can be cast into the parametrized form described in the above section since a parameter δ is often used in these models to denote the range of nonlocal interactions.

3.1. Model equation. Let $\Omega \subset \mathbb{R}^d$ denote a bounded, open domain with a piecewise planar boundary. The corresponding *interaction domain* is defined as

$$\Omega_{\mathcal{I}} = \{\mathbf{y} \in \mathbb{R}^d \backslash \Omega \text{ such that } \mathrm{dist}(\mathbf{y}, \partial\Omega) \le 1\}. \tag{3.1}$$

Also we let $\Omega_w = \Omega \cup \Omega_{\mathcal{I}}$ be the domain containing both Ω and $\Omega_{\mathcal{I}}$. A nonlocal operator $\mathcal{L}$ is defined as, for any function $u(\mathbf{x}) : \Omega_w \to \mathbb{R}$,

$$\mathcal{L}u(\mathbf{x}) = -2\int_\Omega (u(\mathbf{y}) - u(\mathbf{x}))\gamma(|\mathbf{x} - \mathbf{y}|)d\mathbf{y}, \tag{3.2}$$

where the kernel $\gamma = \gamma(|\mathbf{x} - \mathbf{y}|)$ is assumed to be radial with $\mathrm{supp}(\gamma(|\cdot|)) \subset B_1(\mathbf{0})$ (the unit ball centered at the origin), and there exists a constant $\lambda > 0$ such that $B_\lambda(\mathbf{0}) \subset \mathrm{supp}(\gamma(|\cdot|))$. Moreover, γ is a nonnegative and nonincreasing function with a bounded second order moment. That is,

$$\hat{\gamma}(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^2\gamma(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d) \quad \text{and} \quad \int_{B_1(\mathbf{0})} \hat{\gamma}(|\boldsymbol{\xi}|)d\boldsymbol{\xi} = d. \tag{3.3}$$

And we denote the rescaled kernels

$$\hat{\gamma}_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^d}\hat{\gamma}\left(\frac{|\boldsymbol{\xi}|}{\delta}\right), \quad \gamma_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^{d+2}}\gamma\left(\frac{|\boldsymbol{\xi}|}{\delta}\right) \tag{3.4}$$

for $\delta \in (0, 1]$ and let $\mathcal{L}_\delta$ denote the nonlocal operator corresponding to the kernel γ_δ. Note that more general forms of the kernels can be considered as well [16, 18]; the essential features are that they in some sense are approximations of the distributional Laplacian of the Dirac-delta measure near the origin. The assumptions of the above form are made to simplify the presentation.

The model equation to be studied is the *nonlocal volume-constrained problem* [18]:

$$\begin{cases} \mathcal{L}_\delta u = f & \text{on } \Omega, \\ \quad u = 0 & \text{on } \Omega_{\mathcal{I}_\delta} \end{cases} \tag{3.5}$$

where $\Omega_{\mathcal{I}_\delta} = \{\mathbf{y} \in \mathbb{R}^d \backslash \Omega \ \operatorname{dist}(\mathbf{y}, \partial\Omega) \leq \delta\}$ and $\Omega_\delta = \Omega \cup \Omega_{\mathcal{I}_\delta}$. The second equation in (3.5) is a constraint imposed on a domain $\Omega_{\mathcal{I}_\delta}$ with a nonzero measure. It is a natural extension of *Dirichelet* boundary condition for classical PDEs [18]. With $\mathcal{L}_0 = -\Delta$, the nonlocal equation (3.5) is an analogue of the classical problem

$$\begin{cases} \mathcal{L}_0 u = f & \text{on } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Let $\mathcal{S}_0 = H_0^1(\Omega)$ be the standard Sobolev space of real-valued functions that are square integrable and have square-integrable derivatives on Ω with trace zero on the boundary $\partial\Omega$. It serves as the natural energy space of (3.6) equipped with an inner product and norm

$$(u, v)_{\mathcal{S}_0} = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}, \quad \|u\|_{\mathcal{S}_0} = \left(\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2}.$$

The natural energy space associated with (3.5) is [18, 38]

$$\mathcal{S}_\delta = \left\{ u \in L^2(\Omega_\delta) : \int_{\Omega_\delta} \int_{\Omega_\delta} \gamma_\delta(|\mathbf{x} - \mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} < \infty, \ u|_{\Omega_{\mathcal{I}_\delta}} = 0 \right\}$$

for $\delta \in (0, 1]$. It is clear that $\mathcal{S}_\delta$ is a subspace of $L^2(\Omega_\delta)$ (the space of all real-valued square-integrable functions on Ω_δ) with an inner product $(\cdot, \cdot)_{\mathcal{S}_\delta}$ defined as

$$(u, v)_{\mathcal{S}_\delta} = \int_{\Omega_\delta} \int_{\Omega_\delta} \gamma_\delta(|\mathbf{x} - \mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y})) d\mathbf{x} d\mathbf{y}$$

and $\| \cdot \|_{\mathcal{S}_\delta}$ the associated norm. We note that $\{\| \cdot \|_{\mathcal{S}_\delta}\}$ are usually seminorms, but for $\{\mathcal{S}_\delta\}$ they are equivalent to full norms, as demonstrated by the Poincaré inequality given later (see also [38]), just as on $H_0^1(\Omega)$, the norms $| \cdot |_{H^1(\Omega)}$ and $\| \cdot \|_{H^1(\Omega)}$ are equivalent. It can be shown that $\mathcal{S}_\delta$ is also the completion of $C_0^\infty(\Omega)$ in $L^2(\Omega_\delta)$ under the norm $\| \cdot \|_{\mathcal{S}_\delta}$ [38].

In order to apply the framework given earlier, it is convenient to have functions in the different spaces $\{\mathcal{S}_\delta, \delta \in [0, 1]\}$ be specified in a common spatial domain, say, $\Omega_w = \Omega \cup \Omega_{\mathcal{I}}$; we thus make all functions in $\mathcal{S}_\delta$ equivalent to themselves with zero extension outside Ω and norms defined by

$$\left\{ \int_{\Omega_\delta} \int_{\Omega_\delta} \gamma_\delta(|\mathbf{x} - \mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \right\}^{1/2}$$

and

$$\left\{ \int_{\Omega_w} \int_{\Omega_w} \gamma_\delta(|\mathbf{x} - \mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \right\}^{1/2}$$

are also equivalent for such functions, independently of δ. These equivalence properties will be implicitly used throughout the manuscript unless otherwise noted.

Now we present weak formulations for the nonlocal (and local) diffusion models. Define a family of bilinear forms $\{b_\delta\}$ by

$$b_\delta(u, v) = \begin{cases} \displaystyle\int_{\Omega_\delta} \int_{\Omega_\delta} \gamma_\delta(|\mathbf{y} - \mathbf{x}|)(u(\mathbf{y}) - u(\mathbf{x}))(v(\mathbf{y}) - v(\mathbf{x})) d\mathbf{y} d\mathbf{x} & (\delta > 0), \\ \displaystyle\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x}) & (\delta = 0) \end{cases} \tag{3.7}$$

for $u, v \in \mathcal{S}_\delta$. Then the weak formulations of (3.5) and (3.6) are as follows:

$$\text{Find } \ u_\delta \in \mathcal{S}_\delta \text{ such that } b_\delta(u_\delta, v) = (f, v)_{L^2} \quad \forall v \in \mathcal{S}_\delta. \tag{3.8}$$

Note that, for simplicity, f is taken to be independent of δ as in [57], so that the Assumption 2.5 is automatically satisfied, though the results remain valid if f depends on δ and satisfies the Assumption 2.5.

Now for each δ, we introduce the finite element spaces $\{V_{\delta,h}\} \subset \mathcal{S}_\delta$ associated with the triangulation $\tau_h = \{K\}$ of the domain Ω_δ (or Ω_w). We set

$$V_{\delta,h} := \{v \in \mathcal{S}_\delta : v|_K \in P(K) \quad \forall K \in \tau_h\},$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less than or equal to p. Again, for different δ, in order to have the finite element functions defined on a common spatial domain, we also assume, as in the case for $\mathcal{S}_\delta$, that any element in $V_{\delta,h}$ automatically vanishes outside Ω. As $h \to 0$, $\{V_{\delta,h}\}$ is dense in $\mathcal{S}_\delta$, i.e., for any $v \in \mathcal{S}_\delta$, there exists a sequence $(v_h \in V_{\delta,h})$ such that

$$\|v_h - v\|_{\mathcal{S}_\delta} \to 0 \quad \text{as} \quad h \to 0. \tag{3.9}$$

These properties are easily satisfied by standard finite element spaces.

The Galerkin approximation is to replace $\mathcal{S}_\delta$ by $V_{\delta,h}$ in (3.8):

$$\text{Find } \ u_{\delta,h} \in V_{\delta,h} \text{ such that } b_\delta(u_{\delta,h}, v) = (f, v)_{L^2} \quad \forall v \in V_{\delta,h}. \tag{3.10}$$

3.2. Asymptotically compatible schemes. To apply the abstract framework to the ND model, we define $\mathcal{T}_\sigma$ and $\mathcal{X}_\sigma$ in the context of section 2 as

$$\mathcal{T}_\sigma = \mathcal{X}_\sigma = \begin{cases} \mathcal{S}_{1/\sigma} & \text{for } \sigma \in [1, \infty], \\ L^2_0(\Omega) & \text{for } \sigma = 0, \\ \mathcal{S}_1 & \text{for } \sigma \in (0, 1), \end{cases} \tag{3.11}$$

where $L^2_0(\Omega)$ contains all elements in $L^2(\Omega)$ that vanish outside Ω. Note that for this example, the spaces $\mathcal{T}_\sigma$ and $\mathcal{X}_\sigma$ are the same and a_σ is a symmetric bilinear form on $\mathcal{T}_\sigma \times \mathcal{X}_\sigma$. We define $\mathcal{T}_\sigma$ and $\mathcal{X}_\sigma$ for $\sigma \in (0, 1)$ to be the same as $\mathcal{S}_1$, since this would not affect the limiting behavior as $\sigma \to \infty$, or equivalently, $\delta \to 0$. Indeed, we are interested in approximations of both nonlocal problems with a finite horizon parameter and their local limits.

For the family of spaces, we need to verify the assumptions made in the earlier section. First, let us state a simple lemma below where fractional Sobolev spaces are used. We use $H^{\alpha/2}_0(\Omega)$ to denote the closure of $C^\infty_c(\Omega)$ in $H^{\alpha/2}(\Omega)$ for $\alpha \in (0, 2]$. More discussions on these spaces can be found in [8, 15].

LEMMA 3.1. *For $\alpha \in (0, 2]$ and a kernel γ_δ satisfying $|\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) \in L^1(\mathbb{R}^d)$, we have a constant C depending only on Ω such that*

$$\|u\|^2_{\mathcal{S}_\delta} \le C \left(\int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right) \|u\|^2_{H^{\alpha/2}(\Omega)} \quad \forall\, u \in H^{\alpha/2}_0(\Omega) \cap L^2_0(\Omega). \tag{3.12}$$

Proof. We consider the zero extension of functions to $\mathbb{R}^d$, so that there exists a constant $C = C(\Omega)$, independent of α and δ, such that $\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} \le C\|u\|_{H^{\alpha/2}(\Omega)}$

$\forall u \in H_0^{\alpha/2}(\Omega) \cap L_0^2(\Omega)$. Here we denote the extension of u by the same notation. The lemma is then a consequence of the following:

$$\int_{\mathbb{R}^d} |u(\mathbf{x}+\boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} \leq C|\boldsymbol{\xi}|^\alpha \|u\|^2_{H^{\alpha/2}(\mathbb{R}^d)}.$$

To see the above, we have by the Fourier transform that

$$|u|^2_{H^{\alpha/2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathbf{k}|^\alpha \hat{u}^2(\mathbf{k}) d\mathbf{k}, \quad \int_{\mathbb{R}^d} |u(\mathbf{x}+\boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \hat{u}^2(\mathbf{k}) d\mathbf{k}.$$

So the desired inequality follows from an elementary inequality $|e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \leq 2|\boldsymbol{\xi}\cdot\mathbf{k}|^\alpha$ for $\alpha \in (0, 2]$. Hence, we get

$$\begin{aligned}\int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{y}-\mathbf{x}|)(u(\mathbf{y}) - u(\mathbf{x}))^2 d\mathbf{y} d\mathbf{x} &\leq \int_{\mathbb{R}^d} \gamma_\delta(|\boldsymbol{\xi}|) \int_{\mathbb{R}^d} (u(\mathbf{x}+\boldsymbol{\xi}) - u(\mathbf{x}))^2 d\mathbf{x} d\boldsymbol{\xi} \\ &\leq C\|u\|_{H^{\alpha/2}(\Omega)} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi},\end{aligned}$$

which leads to the lemma. □

By applying the above to functions in $\mathcal{S}_0 = \mathcal{X}_\infty$ for the case of $\alpha = 2$, we have the uniform embedding of $\mathcal{X}_\infty$ in $\mathcal{X}_\sigma$ since for the kernel γ_δ, we have

$$\int |\boldsymbol{\xi}|^2 \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = 1.$$

To verify Assumption 2.1(i) for $\{\mathcal{T}_\sigma\}$ and $\{\mathcal{X}_\sigma\}$, it remains to apply a uniform Poincaré type inequality for the uniform embedding of $\mathcal{T}_\sigma$ in $\mathcal{T}_0$.

LEMMA 3.2 (uniform Poincaré inequality). *There exists $C > 0$ independent of δ such that $\forall \delta \in (0, 1]$,*

$$\|u\|^2_{L^2(\Omega_\delta)} \leq C\|u\|^2_{\mathcal{S}_\delta} \quad \forall u \in \mathcal{S}_\delta. \tag{3.13}$$

The above is a special case of [39, Proposition 5.3] for scalar valued functions (see [39] for the proof). Also from [38], we know that $\mathcal{S}_\delta$ is complete, thus a Hilbert space.

To check Assumption 2.1(ii), we need a compactness lemma that can be found in [8, Theorem 4] and [44, Theorems 1.2, 1.3].

LEMMA 3.3. *Suppose $u_n \in \mathcal{S}_{\delta_n}$ with $\delta_n \to 0$. If*

$$\sup_n \int_{\Omega_{\delta_n}} \int_{\Omega_{\delta_n}} \gamma_{\delta_n}(|\mathbf{x}-\mathbf{y}|)(u_n(\mathbf{x}) - u_n(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \leq \infty,$$

then u_n is precompact in $L_0^2(\Omega)$. Moreover, any limit point $u \in \mathcal{S}_0$.

We note that in establishing Lemmas 3.2 and 3.3, an argument of [8] often can be used, which requires that $\hat{\gamma}$ is nonincreasing. It has been noted that by techniques introduced in [2], the results remain true under a less restrictive condition where γ is assumed to be nonincreasing. Moreover, [44] provided an even more general argument that works for $d \geq 2$ without the assumption on γ being nonincreasing. Related discussions on these issues can be found in [38, 39].

We next move to the bilinear forms. Note that b_δ is exactly the inner product defined on $\mathcal{S}_\delta$, so Assumption 2.2(i) is naturally satisfied with $C = 1$. Assumption 2.2(ii) and (iii) are also satisfied, since b_δ is a coercive bilinear form.

Since $\mathcal{T}_\sigma = \mathcal{X}_\sigma$ and a_σ is a symmetric bilinear form, the associated linear operator $\mathcal{A}_\sigma$ is then self-adjoint. Assumption 2.3 is about the convergence of the operator $\mathcal{L}_\delta$, a result that has been shown in many works, such as [20, 39]. We state it here as a proposition without proof. It is a pointwise convergence property of a smooth function under the action of the nonlocal integral operator $\mathcal{L}_\delta$ (generally interpreted in the principal value sense [39]) to that under the negative Laplace operator.

PROPOSITION 3.4. *For all $v \in C_c^\infty(\Omega)$, and all $\mathbf{x} \in \Omega$, we have*

$$\mathcal{L}_\delta v(\mathbf{x}) \longrightarrow -\Delta v(\mathbf{x}) \quad \text{as } \delta \to 0. \tag{3.14}$$

Moreover, there exists a constant $C = C(d, v)$ such that

$$\sup_{\delta\in(0,1)} \sup_{\mathbf{x}\in\Omega} |\mathcal{L}_\delta v(\mathbf{x})| \le C. \tag{3.15}$$

With pointwise convergence and uniform boundedness estimate of $\mathcal{L}_\delta v$, Assumption 2.3 is obviously true by the bounded convergence theorem. This is stated in the following lemma, which is a stronger result than what Assumption 2.3(ii) requires.

LEMMA 3.5. *For $\mathcal{L}_\delta$ and $\mathcal{L}_0$ defined in section* 3.1, *$\forall v \in C_c^\infty(\Omega)$,*

$$\|\mathcal{L}_\delta v - \mathcal{L}_0 v\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } \delta \to 0.$$

For the approximate spaces, we have $W_{\sigma,h} = X_{\sigma,h} = V_{1/\sigma,h}$ for $\sigma \ge 1$. So Assumption 2.4(i) is satisfied as a result of coercivity and $V_{\delta,h} \subset \mathcal{S}_\delta$. The property (3.9) ensures that $V_{\delta,h}$ satisfies Assumption 2.4(ii). To check Assumption 2.4(iii), for convenience, we define a special family of spaces $\hat{V}_{\delta,h}$.

DEFINITION 3.6. *Let $\hat{V}_{\delta,h} \subset V_{0,h} \subset \mathcal{S}_0$ be the continuous piecewise linear finite element space that corresponds to the same mesh τ_h with $V_{\delta,h}$.*

The following lemma is simply a restatement of a simple fact that the continuous piecewise linear subspace of H_0^1 approximates the whole space as mesh size goes to zero.

LEMMA 3.7. *The family $\{\hat{V}_{\delta,h}\}$ is asymptotically dense in $\mathcal{S}_0$, that is, it satisfies Assumption* 2.4(ii).

Now we see that if $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $V_{\delta,h}$ also satisfies Assumption 2.4(iii).

With all assumptions 2.1–2.5 verified, the following theorem offers a remedy for developing asymptotically compatible schemes when one wants to solve ND equations.

THEOREM 3.8. *Let u_δ and $u_{\delta,h}$ be solutions of* (3.8) *and* (3.10)*, respectively, and $\hat{V}_{\delta,h}$ is defined in Definition* 3.6*. If $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $\|u_{\delta,h} - u_0\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, $h \to 0$.*

Proof. Taking $\mathcal{T}_\sigma = \mathcal{X}_\sigma := \mathcal{S}_{1/\sigma}$, $a_\sigma := b_{1/\sigma}$, $\mathcal{A}_\sigma = \mathcal{A}_\sigma^* := \mathcal{L}_{1/\sigma}$, and $W_{\sigma,h} = X_{\sigma,h} := V_{1/\sigma,h}$, we see that the above theorem follows from Theorem 2.10, since in the above discussions we have verified all assumptions 2.1–2.5 for this case. ∎

In short, we see that if the finite element spaces contain a continuous finite element subspace that have desired approximation properties in $\mathcal{S}_0$, then the corresponding discretization is asymptotically compatible. This is particularly true for any continuous or discontinuous finite element spaces containing at least the subspace of continuous piecewise linear elements.

We now examine the local limit of discrete solutions on a fixed mesh, following the discussions in Theorem 2.8. By condition (2.10), we see that some compatibility of the discrete spaces is needed. We choose to work with a fixed mesh as a simplification, although there is still ample freedom to choose different finite element spaces for nonlocal and local problems. We assume that $V_{0,h} = \mathcal{S}_0 \cap (\bigcap_{\delta>0} V_{\delta,h})$ so that (2.10) is satisfied. To verify all the additional assumptions required for Theorem 2.8, we state a couple of technical results.

For a given triangulation τ_h, we define a space of continuous and piecewise smooth functions given by

$$\mathcal{V}_h := \{v \in C(\overline{\Omega_\delta}) : \ v|_K \in C^\infty(\overline{K}), \, K \in \tau_h, \ v|_{\Omega_{\mathcal{I}_\delta}} = 0\}.$$

Note that functions in $\mathcal{V}_h$ are again set to vanish outside Ω. Then, we have the convergence of the bilinear forms on the subspace $\mathcal{V}_h$.

LEMMA 3.9. *For any u, $v \in \mathcal{V}_h$, as $\delta \to 0$, we have*

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} \to 0.$$

Consequently, for any $u_h, v_h \in V_{0,h}$,

$$\lim_{\delta \to 0} b_\delta(u_h, v_h) = b_0(u_h, v_h).$$

Proof. First, we note that

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K \in \tau_h} \int_K u \mathcal{L}_\delta v - \sum_{K \in \tau_h} \int_K \nabla u \cdot \nabla v.$$

Now, for any mesh element $K \in \tau_h$, integration by parts on each K gives

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K \in \tau_h} \int_K u(\mathcal{L}_\delta v + \Delta v) - \sum_{e \in \mathcal{E}_h^0} \int_e u [\![\nabla v]\!]_e,$$

where $\mathcal{E}_h^0$ is the set of internal edges of τ_h and $[\![\nabla v]\!]_e$ is the jump of the vector on the edge e. For the first term, by [23, Theorem 3.7], which remains valid for the kernel under consideration here, we have

$$\int_K u(\Delta v + \mathcal{L}_\delta v) \to \frac{1}{2} \sum_{e \in edge(K)} \int_e u [\![\nabla v]\!]_e \quad \text{as } \delta \to 0.$$

Summing over $K \in \tau_h$, we get

$$\sum_{K \in \tau_h} \int_K u(\Delta v + \mathcal{L}_\delta v) \to \sum_{e \in \mathcal{E}_h^0} \int_e u [\![\nabla v]\!]_e.$$

Thus we have $(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} \to (\nabla u, \nabla v)_{L^2(\Omega)}$ and the lemma follows. ∎

Next, to show condition (2.12) in Theorem 2.8, we present the following lemma.

LEMMA 3.10. *Assume that $w_{\delta,h} \in V_{\delta,h} \subset \mathcal{S}_\delta$, $v_h \in V_{0,h} \subset \mathcal{S}_0$ and $\|w_{\delta,h}\|_{L^2} \to 0$ as $\delta \to 0$, then $b_\delta(w_{\delta,h}, v_h) \to 0$ as $\delta \to 0$.*

Proof. Since $w_{\delta,h}$ and v_h are smooth on each element $K \subset \tau_h$, we will prove the result on each $K \subset \tau_h$. Also, we define Γ_K for each $K \subset \tau_h$ by

$$\Gamma_K := \{\mathbf{x} \notin K \mid \operatorname{dist}(\mathbf{x}, K) \le \delta\}.$$

Then

$$b_\delta(w_{\delta,h}, v_h) = \sum_{K \in \tau_h} \int_K \int_{K \cup \Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x})) d\mathbf{x}' d\mathbf{x}.$$

By [8, Theorem 1], for smooth $w_{\delta,h}$ and v_h restricted on K,

$$\int_K \int_K \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}')-w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}')-v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \\ \leq \|w_{\delta,h}\|_{\mathcal{S}_\delta(K)}\|v_h\|_{\mathcal{S}_\delta(K)} \leq C\|w_{\delta,h}\|_{H^1(K)}\|v_h\|_{H^1(K)}.$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{H^1(K)} \leq C\|w_{\delta,h}\|_{L^2(K)} \to 0 \quad \text{as } \delta \to 0,$$

so

$$\int_K \int_K \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}')-w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}')-v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \to 0.$$

For the second term,

$$\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}')-w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}')-v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \\ \leq 2\|w_{\delta,h}\|_{L^\infty} \int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}')-v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}.$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{L^\infty} \leq C\|w_{\delta,h}\|_{L^2} \to 0,$$

it remains to prove that for any $v_h \in V_{0,h}$,

$$\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}')-v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}$$

is bounded uniformly in δ.

Since v_h is piecewise smooth for $\mathbf{x} \in K$ and $\mathbf{x}' \in \Gamma_K$, respectively, we use $\mathbf{s}$ to denote the intersection of ∂K and the line between $\mathbf{x}'$ and $\mathbf{x}$. By Taylor expansion, we have

$$v_h(\mathbf{x}') = v_h(\mathbf{x}) + \nabla v_h(\mathbf{x})\cdot(\mathbf{s}-\mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s})\cdot(\mathbf{x}'-\mathbf{s}) + o(\delta).$$

Denote $K_{out} := K \cap B_\delta(\partial K)$ (the latter being a δ neighborhood of ∂K); then

$$\begin{aligned}
&\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}')-v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x} \\
&= \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|\nabla v_h(\mathbf{x})\cdot(\mathbf{s}-\mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s})\cdot(\mathbf{x}'-\mathbf{s})|d\mathbf{x}'d\mathbf{x} \\
&\quad + \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta \cdot o(\delta)d\mathbf{x}'d\mathbf{x} \\
&\leq 2\|\nabla v\|_{L^\infty} \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|d\mathbf{x}'d\mathbf{x} + \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta \cdot o(\delta)d\mathbf{x}'d\mathbf{x}.
\end{aligned}$$

Now it is easy to see that the second term on the above right-hand side tends to zero as $\delta \to 0$. For the first term, following the proof of [23, Theorem 3.7] and in particular [23, (3.35)], we have

$$2\int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|d\mathbf{x}'d\mathbf{x} \leq \int_{B_\delta(0)} \gamma_\delta(\mathbf{z})|\mathbf{z}|^2 d\mathbf{z} \left(\sum_{e\in K}|e|\right),$$

which is bounded uniformly in δ under the assumption on the kernel γ_δ.

In summary, we now have proved that for each $K \in \tau_h$,

$$\int_K \int_{K \cup \Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \to 0,$$

which implies $b_\delta(w_{\delta,h}, v_h) \to 0$. Hence, the Lemma is proved. ∎

THEOREM 3.11. *Let $u_{\delta,h}$ and $u_{0,h}$ be discrete solutions as defined in* (3.10) *with $\delta > 0$ and $\delta = 0$, respectively. Assume further that $V_{\delta,h} \subset \mathcal{S}_\delta$ is a finite element space that contains all continuous piecewise linear functions. Moreover, $V_{0,h} = \mathcal{S}_0 \cap (\bigcap_{\delta>0} V_{\delta,h})$. Then, for fixed h and τ_h, we have $\|u_{\delta,h} - u_{0,h}\|_{L^2} \to 0$ as $\delta \to 0$.*

Proof. The theorem is a direct application of Theorem 2.8, where conditions (2.11) and (2.12) are verified by Lemma 3.9 and Lemma 3.10 respectively. ∎

We note that the above theorem implies that as long as all piecewise continuous linear elements are included, the finite element spaces for nonlocal problems may not be conforming subspaces of the local limit problem but can still have solutions that converge to the conforming local finite element solution.

3.3. A case of conditional asymptotic stability. It is known that some finite element approximations to the ND models are not asymptotically compatible [56]. In particular, the counterexamples given in [56] demonstrate that if δ is taken to be proportional to h, then as $h \to 0$, the discrete solutions may be convergent, but to the wrong limit. It is interesting from a practical point of view to provide some constructive remedies to avoid such undesirable effects. This is the purpose of the discussion here. We show that as long as the condition $h = o(\delta)$ is met as $\delta \to 0$, then we are able to obtain the correct local limit even for discontinuous piecewise constant finite element approximations when they are of conforming type.

THEOREM 3.12. *Let u_δ, $u_{\delta,h}$ be solutions of* (3.8) *and* (3.10)*. If $V_{\delta,h}$ is the piecewise constant space, then $\|u_{\delta,h} - u_0\|_{L^2} \to 0$ if $h = o(\delta)$ as $\delta \to 0$.*

Proof. We revisit the proof of Theorem 2.10. Recall that $a_\infty(u,v) - \langle f_\infty, v\rangle$ is split into four parts. Without Assumption 2.4(iii), the estimates for three of the four terms are not affected. We only need to estimate the term III and prove that III $\to 0$ if $\sigma_n h_n \to 0$ as $n \to \infty$. In fact, by Lemma 3.1,

$$\text{III} \le C\|v - v_n\|_{\mathcal{S}_{\sigma_n}} \le C\|v - v_n\|_{H^{\alpha/2}(\Omega)} \left(\int |\boldsymbol{\xi}|^\alpha \gamma_{\delta_n}(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right)^{1/2},$$

where $v_n \in V_{\delta_n,h_n} = W_{\sigma_n,h_n}$. A direct calculation shows

$$\int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = \delta^{\alpha-2} \int_{B(0,1)} |\boldsymbol{\xi}|^\alpha \gamma(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = C\delta^{\alpha-2}.$$

So,

$$\text{III} \le C\sigma_n^{1-\alpha/2} \|v - v_n\|_{H^{\alpha/2}(\Omega)}, \quad \text{for} \quad \alpha \in [0,1].$$

Now, by taking v_n as the piecewise constant L^2-orthogonal projection of $v \in \mathcal{S}_0$ onto V_{δ_n,h_n}, we have [6, (1.3)]

$$\|v - v_n\|_{H^{\alpha/2}(\Omega)} \le Ch_n^{1-\alpha/2} \|v\|_{H^1(\Omega)}.$$

Thus, III $\le C(\sigma_n \cdot h_n)^{1-\alpha/2} \|v\|_{H^1(\Omega)} \to 0$ as $n \to \infty$, which completes the proof. ∎

4. Applications to the state-based PD system. State-based PD models were presented in [52, 50] as a generalization of bond-based PD models. We refer to [40] for the mathematical analysis. Given the similarity between linear state-based models and ND models in applying the abstract framework introduced in this work, we omit most of the technical details here but emphasize filling in the necessary ingredients (and references) for verifying all the needed assumptions.

4.1. Linear PD solids. Using the same notation as for the ND model, we present a PD model [50], using the terms similar to [19, 40], for a constitutively linear, isotropic solid undergoing deformation. For simplicity, we omit mechanical descriptions and define directly the corresponding bilinear form:

$$B_\delta(\mathbf{u},\mathbf{v}):=\int_\Omega\Bigg(\left(k(\mathbf{x})-\frac{\alpha(\mathbf{x})\,m(\mathbf{x})}{d^2}\right)\mathrm{Tr}(\mathscr{D}^*\mathbf{u})(\mathbf{x})\mathrm{Tr}(\mathscr{D}^*\mathbf{v})(\mathbf{x}) \\ +\alpha(\mathbf{x})\int_\Omega\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x})d\mathbf{x}'\Bigg)\,d\mathbf{x}, \quad (4.1)$$

where $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are scalar functions that are closely related to the bulk and shear modulus of the material, respectively, and γ_δ is a kernel as defined for the ND model given earlier. The function $m(\mathbf{x})$ is defined as

$$m(\mathbf{x})=\int_\Omega\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}'.$$

$\mathrm{Tr}(\mathcal{D}^*)$ is the trace of the nonlocal operator $\mathcal{D}^*$ defined by

$$\mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y}):=(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\otimes\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|},$$

and $\mathrm{Tr}(\mathscr{D}^*)$ is the trace of the nonlocal gradient $\mathscr{D}^*$ defined by:

$$\mathscr{D}^*(\mathbf{u})(\mathbf{x}):=\int_\Omega\mathcal{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y})\omega(\mathbf{y},\mathbf{x})d\mathbf{y},\quad\text{where}\quad\omega(\mathbf{y},\mathbf{x})=\frac{d}{m(\mathbf{x})}\gamma_\delta(|\mathbf{y}-\mathbf{x}|)|\mathbf{y}-\mathbf{x}|.$$

We refer to [16, 20, 42] for more detailed discussions on these basic nonlocal operators. Let $\mathcal{S}^*=L^2(\Omega_\delta;\mathbb{R}^d)$. Using the same notation for vector-valued function spaces as for the scalar ND model, the energy spaces are given by

$$\mathcal{S}_\delta=\left\{\mathbf{u}\in\mathcal{S}^*:\int_{\Omega_\delta}\int_{\Omega_\delta}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\left(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\right)^2d\mathbf{x}'d\mathbf{x}<\infty,\mathbf{u}=0\text{ on }\Omega_{\mathcal{I}_\delta}\right\}$$

with an inner product

$$(\mathbf{u},\mathbf{v})_{\mathcal{S}_\delta}=\int_{\Omega_\delta}\int_{\Omega_\delta}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x})\,d\mathbf{x}'d\mathbf{x}$$

and an induced norm $\|\cdot\|_{\mathcal{S}_\delta}$, where the δ-dependence is due to the kernel γ_δ. Zero extensions to functions in $\mathcal{S}_\delta$ are again assumed as in the scalar case.

By the following uniform Poincaré-type inequality proved in [40, Proposition 3], we know that $(\cdot,\cdot)_{\mathcal{S}_\delta}$ is indeed a well-defined inner product that also induces a well-defined norm $\|\cdot\|_{\mathcal{S}_\delta}$.

LEMMA 4.1 (uniform Poincaré inequality). *There exists a constant $C>0$ independent of δ such that $\forall\delta\in(0,1]$,*

$$\|\mathbf{u}\|^2_{L^2(\Omega_\delta;\mathbb{R}^d)}\le C\|\mathbf{u}\|^2_{\mathcal{S}_\delta}\quad\forall\mathbf{u}\in\mathcal{S}_\delta.$$

Furthermore, by [40, Lemma 3], B_δ is a bounded and coercive bilinear operator on $\mathcal{S}_\delta$, i.e., there exist positive constants C_1 and C_2 independent of δ such that, $\forall \mathbf{u},\mathbf{v}\in\mathcal{S}_\delta$,

$$B_\delta(\mathbf{u},\mathbf{v}) \le C_2\|\mathbf{u}\|_{\mathcal{S}_\delta}\|\mathbf{v}\|_{\mathcal{S}_\delta} \quad \text{and} \quad B_\delta(\mathbf{u},\mathbf{u}) \ge C_1\|\mathbf{u}\|^2_{\mathcal{S}_\delta}.$$

Thus B_δ induces the nonlocal PD Navier operator $\mathcal{L}_\delta : \mathcal{S}_\delta \to \mathcal{S}_\delta^*$, which is a linear operator bounded uniformly in δ and is defined by

$$B_\delta(\mathbf{u},\mathbf{v}) = \langle \mathcal{L}_\delta(\mathbf{u}),\mathbf{v}\rangle \;\; \forall\, \mathbf{u},\mathbf{v}\in\mathcal{S}_\delta. \tag{4.2}$$

We also denote the space $\mathcal{S}_0$ to be

$$\mathcal{S}_0 = \left\{ \mathbf{u}\in L^2(\Omega;\mathbb{R}^d) : \int_\Omega |(\nabla\mathbf{u}+\nabla\mathbf{u}^T)(\mathbf{x})|^2 d\mathbf{x} < \infty, \mathbf{u}|_{\partial\Omega} = 0 \right\} \tag{4.3}$$

equipped with a norm equivalent to $\|\cdot\|_{H_0^1(\Omega)}$.

Concerning the local limit of $\mathcal{L}_\delta$, we quote the following result [40, Thoerem 3].

Lemma 4.2. *Assume that $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are smooth functions (say, of the class C^1). Then for $\mathbf{w}\in C_c^\infty(\Omega;\mathbb{R}^d)$, $\mathcal{L}_\delta\mathbf{w}$ is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^d)$, and*

$$\mathcal{L}_\delta\mathbf{w}(\mathbf{x}) \longrightarrow \mathcal{L}_0\mathbf{w}(\mathbf{x}) \quad \text{as} \;\; \delta\to 0 \quad \forall\, \mathbf{x}\in\Omega,$$

where $\mathcal{L}_0$ is defined by $\mathcal{L}_0\mathbf{w}(\mathbf{x}) = -\mathrm{div}(\mu(\mathbf{x})\nabla\mathbf{w}(\mathbf{x})) - \nabla((\mu(\mathbf{x})+\lambda(\mathbf{x}))\,\mathrm{div}\,\mathbf{w}(\mathbf{x}))$ with $\mu(\mathbf{x}) = \alpha(\mathbf{x})/[d(d+2)]$ and $\lambda(\mathbf{x}) = k(\mathbf{x}) - 2\alpha(\mathbf{x})/[d^2(d+2)]$.

For the given $\mathbf{w}$, combining the above pointwise convergence of $\mathcal{L}_\delta\mathbf{w}$ to $\mathcal{L}_0\mathbf{w}$ with the uniform boundedness of $\mathcal{L}_\delta\mathbf{w}$, we get $\|\mathcal{L}_\delta\mathbf{w}-\mathcal{L}_0\mathbf{w}\|_{L^2(\Omega)}\to 0$ as $\delta\to 0$, a result stronger than what is needed later.

4.2. Asymptotically compatible scheme. As before, we define the spaces $\mathcal{T}_\sigma$ and $\mathcal{X}_\sigma$ in Assumption 2.1 the same way as in (3.11) except with $\{\mathcal{S}_\delta\}$ denoting vector-valued function spaces associated with the state-based PD model. We then define $\mathbf{u}_\delta$ as follows.

$$\text{Find} \;\; \mathbf{u}_\delta\in\mathcal{S}_\delta \text{ such that } B_\delta(\mathbf{u}_\delta,\mathbf{v}) = (\mathbf{f},\mathbf{v})_{L^2} \quad \forall v\in\mathcal{S}_\delta. \tag{4.4}$$

with $\mathbf{f}\in L^2(\Omega_w;\mathbb{R}^d)$ independent of δ so that the Assumption 2.5 is satisfied. Similarly, we define the limiting bilinear form on $\mathcal{S}_0$:

$$B_0(\mathbf{u},\mathbf{v}) := \langle\mathcal{L}_0\mathbf{u},\mathbf{v}\rangle \quad \forall\mathbf{u},\mathbf{v}\in\mathcal{S}_0.$$

It is well known that B_0 is bounded and coercive on $\mathcal{S}_0$ [13, 29]. We set $a_\sigma := B_{1/\sigma}$ and $\mathcal{A}_\sigma = \mathcal{A}_\sigma^* := \mathcal{L}_{1/\sigma}$ for $\sigma\in[1,\infty]$. Then one part of Assumption 2.1(i) is given by Lemma 4.1, while the other is precisely [39, Lemma 2.2] restated as the lemma below.

Lemma 4.3. *There exists a constant $C>0$ only depending on Ω such that*

$$\|\mathbf{u}\|^2_{\mathcal{S}_\delta} \le C\left(\int_{\mathbb{R}^d}|\boldsymbol{\xi}|^2\gamma_\delta(|\boldsymbol{\xi}|)d\boldsymbol{\xi}\right)\|\mathbf{u}\|^2_{H^1(\Omega)} \quad \forall\mathbf{u}\in H^1(\Omega)\cap\mathcal{S}_\delta.$$

Assumption 2.1(ii) is just [40, Lemma 7] that is restated below without proof.

Lemma 4.4. *Let $\mathbf{u}_\delta\in\mathcal{S}_\delta$ for $\delta>0$. If $\sup_{\delta>0}\|\mathbf{u}_\delta\|_{\mathcal{S}_\delta}<\infty$, then the sequence $(\mathbf{u}_\delta)$ is precompact in $L^2(\Omega;\mathbb{R}^d)$. Moreover, any limit point $\mathbf{u}\in\mathcal{S}_0$.*

Meanwhile, the discussions in the previous subsection easily lead to Assumptions 2.2 and 2.3 in the present context.

For discrete approximations, as in the ND case, let $\{V_{\delta,h}\} \subset \mathcal{S}_\delta$ denote a family of finite element subspaces where h characterizes the mesh size and for any $\mathbf{v} \in \mathcal{S}_\delta$, we have a family of elements $\{\mathbf{v}_h \in V_{\delta,h}\}$ such that $\|\mathbf{v}_h - \mathbf{v}\|_{\mathcal{S}_\delta} \to 0$ as $h \to 0$. Then, the Galerkin approximation is to replace $\mathcal{S}_\delta$ by $V_{\delta,h}$ in (4.4):

$$\text{Find } \mathbf{u}_{\delta,h} \in V_{\delta,h} \text{ such that } B_\delta(\mathbf{u}_{\delta,h}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in V_{\delta,h}. \tag{4.5}$$

Clearly, $W_{\sigma,h} = X_{\sigma,h} := V_{1/\sigma,h}$ satisfies Assumption 2.4(i) and (ii). The Assumption 2.4(iii) is also satisfied with $\hat{V}_{\delta,h} \subset V_{\delta,h}$ and $\hat{V}_{\delta,h}$ being a vector-valued version of the continuous piecewise linear element subspace that approximates $\mathcal{S}_0 = \mathcal{T}_\infty = \mathcal{X}_\infty$ as $h \to 0$.

Now we are ready to state the convergence theorem on the finite element approximations of the linear state-based PD model, as a direct consequence of Theorem 2.10. We skip the detailed proof.

THEOREM 4.5. *Let $\mathbf{u}_\delta$, $\mathbf{u}_{\delta,h}$ be the solutions of* (4.4) *and* (4.5)*, and $\hat{V}_{\delta,h} \subset \mathcal{S}_\delta$ is described as above. If $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $\|\mathbf{u}_{\delta,h} - \mathbf{u}_0\|_{L^2} \to 0$ as $\delta \to 0$, $h \to 0$.*

Consequently, we see also that for the state-based PD models, the asymptotic compatibility is preserved for conforming finite element approximations that contain continuous piecewise linear finite element subspaces.

By extending the convergence of the discrete linear forms from the ND models to the state-based PD models, we can also get similar results on the convergence of the discrete solutions between the PD models and the local Navier equations as $\delta \to 0$ on a fixed mesh.

Again, the conclusions remain valid for the data $\mathbf{f}$ in (4.4) that depends on δ, if the Assumption 2.5 is satisfied.

5. Numerical experiments. Here, we report numerical results that validate our analysis and provide results on the order of convergence that cannot be seen from our convergence theorems. We use a discontinuous piecewise linear finite element to solve a one-dimensional nonlocal problem $-\mathcal{L}_\delta u = f$ on $(0, 1)$ with the nonlocal constraint $u = 0$ outside $(0, 1)$ and the nonlocal operator given by

$$\mathcal{L}_\delta u = 2 \int_{-\delta}^{\delta} \gamma_\delta(s)(u(x+s) - u(x))ds.$$

A special kernel is chosen to be $\gamma_\delta(s) = \delta^{-2}|s|^{-1}$ in our numerical examples.

We choose a relatively smooth function as u_0 given by a fourth order B-spline:

$$u_0(x) = \frac{1}{h^5} \begin{cases} 0, & x < 0, \\ \dfrac{x^4}{120}, & 0 \le x < 0.2, \\ -\dfrac{x^4}{30} + \dfrac{x^3}{30} - \dfrac{x^2}{100} + \dfrac{x}{750} - \dfrac{1}{15000}, & 0.2 \le x < 0.4, \\ \dfrac{x^4}{20} - \dfrac{x^3}{10} + \dfrac{7x^2}{100} - \dfrac{x}{50} + \dfrac{31}{15000}, & 0.4 \le x < 0.5, \end{cases}$$

with u_0 symmetric (even) with respect to $x = 0.5$. Its graph shown in Figure 5.1.

We then calculate analytically $f := -u_0''$ and solve nonlocal problems on a uniform mesh using a discontinuous piecewise linear finite element space. The corresponding

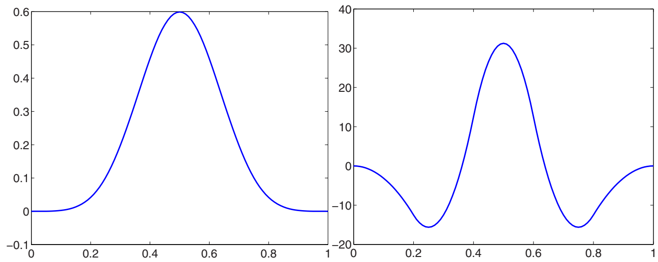

FIG. 5.1. *Graph of $u_0(x)$ and its second order derivative.*

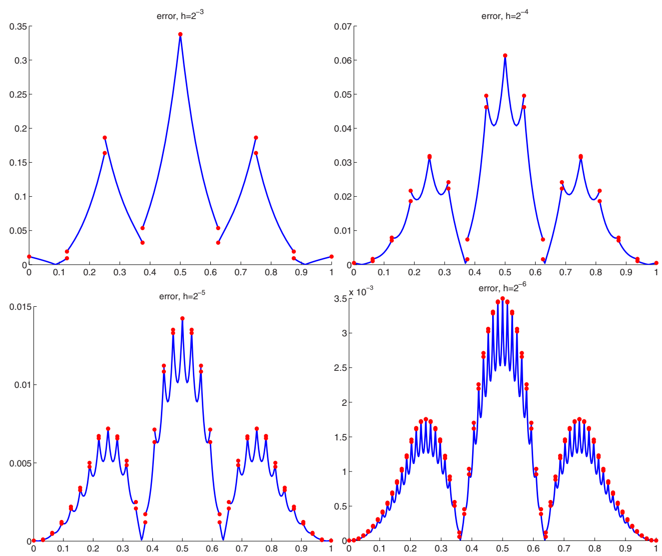

FIG. 5.2. *Pointwise error $u_{\delta,h}(x) - u_0(x)$ with $r = \frac{\delta}{h} = 3$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.*

pointwise errors $e(x) = u_{\delta,h}(x) - u_0(x)$ are plotted in Figures 5.2–5.4 for the three cases, respectively. Note that the red dots are highlighted to show errors at nodal points. Qualitatively, one may observe some common features in these plots: first, while the errors are generally discontinuous at the nodal points given the use of discontinuous finite element functions, the magnitude of discontinuity diminishes as $\delta \to 0$, leading to a continuous (and conforming) approximation to the local limit solution as predicted by the theory; second, the error profiles, in particular, the maximum and minimum envelopes of the errors, are all nicely correlated with the second derivatives of u_0 shown in Figure 5.1. While this does not follow from our analytical framework here, this is consistent with the errors of typical piecewise linear interpolations and may not also tie this with the more detailed truncation error analysis given in [56]. Meanwhile, the error plots also show different oscillation patterns of the errors inside the mesh intervals in comparison with those at nodal points for the three cases. A possible explanation is that oscillations are related to discretization errors that become more pronounced with smaller δ due to the reduction of modeling errors (between nonlocal and local equations).

To be more quantitative, errors of the numerical solutions in various norms and with different relations between δ and h were measured and reported in [57]. These

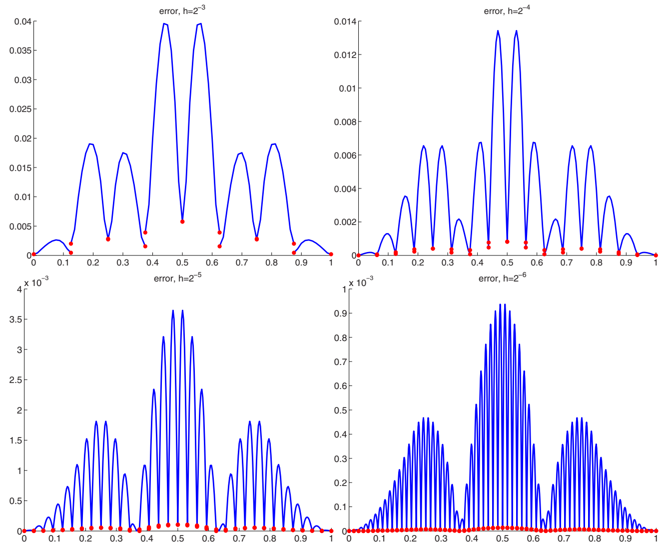

FIG. 5.3. *Pointwise error* $u_{\delta,h}(x) - u_0(x)$ *with* $\delta = h^2$ *and* $h = 2^{-k}$*,* $k = 3, 4, 5, 6$*.*

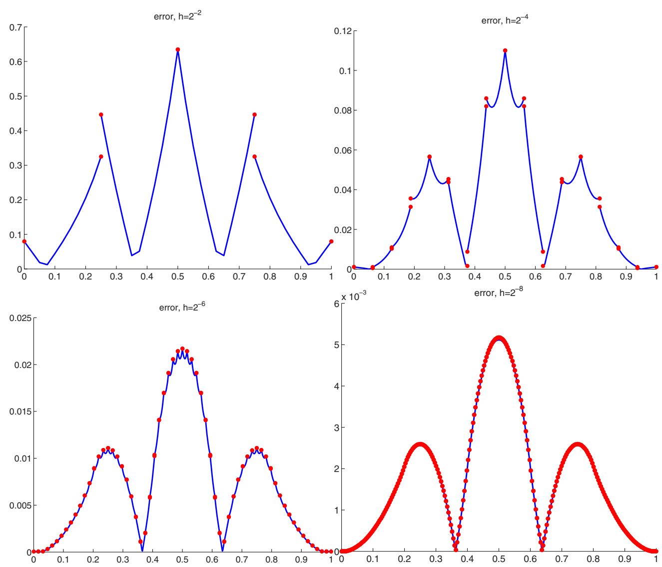

FIG. 5.4. *Pointwise error* $u_{\delta,h}(x) - u_0(x)$ *with* $\delta = \sqrt{h}$ *and* $h = 2^{-k}$*,* $k = 3, 4, 6, 8$*.*

data complemented the theoretical framework that addresses the convergence of numerical solutions but without precise estimates on the orders of approximation errors. While the detailed data on numerical errors are removed in this work (but can be found in tables presented in [57]), we summarize the main observations here for the difference cases. In the first case, when δ/h is taken as a fixed constant as the mesh is refined with a decreasing h, it was observed that the L^2 convergence rate for function values is of second order and that for piecewise first order derivatives is of first order, which is consistent with the optimal orders predicted by standard approximation theory. Meanwhile, with $\delta = h^2$ when refining the mesh, the L^2 convergence orders stay the same as in the previous case. However, in the case of $\delta = \sqrt{h}$, the L^2 convergence order for function values drops to first order. A possible explanation is that the modeling error dominates and it is of the order $O(\delta^2) = O(h)$.

Some superconvergence order can also be observed from the data reported in [57]. We refer to some related findings in [56]. In addition, it was noted in [57] that with the same mesh spacing, say, $h = 2^{-6}$, the errors decrease as δ changes from $O(\sqrt{h})$ to $O(h)$ and $O(h^2)$, a reasonable and desirable behavior showing the efficiency of localization (small horizon) if the objective is to capture the local limit when the latter is well defined.

6. Conclusion. In this work, we established in [57] and subsequent works on the analysis of a class of asymptotically compatible schemes for the approximations of parametrized problems. The original motivation was to develop a robust discretization of nonlocal models for multiscale problems where the nonlocal models can be seen as parametrized by the horizon that measures the range of nonlocal interactions. Yet, the abstract framework allows us to put the discussion in an even broader context. Not only does it reveal the true essence of the nonlocal problems, but it may also be applicable to other parametrized problems. The analytical set-up underlying the abstract framework is valid with minimal assumptions on the underlying problems, the solution regularity, and the approximation spaces. Among various studies of numerical methods and their asymptotic behavior with a parameter approaching to a limit (ranging from uniformly convergent schemes for singularly perturbed problems [47], numerical viscosity solutions of conservation laws [11], lock-free approximations for shells [4], to asymptotically preserving schemes for kinetic equations [33]), perhaps the analysis in [32] offers the closest resemblance to the work here in spirit. In [32], the approximations to the zero mean free path $\epsilon \to 0$ limit or diffusive limit of radiative transport models have been studied. The models studied there share similar features as the nonlocal models considered here in that the parametrized problems may have singular solutions but they approach a more regular solution of the diffusive limit. It has been concluded in [32] that piecewise constant approximations would only lead to a uniform constant solution in such a limit but that finite elements containing enough continuous elements can recover the correct limit as both mesh size and mean free path go to zero, a phenomenon that is reminiscent to our finding here for the local limit of nonlocal problems. This provides additional motivation to present the more abstract framework developed here so it may be applied to problems that arise from different applications.

Meanwhile, the illustrative applications of the framework discussed here offered some new results on the numerical analysis of nonlocal problems as well. For a homogeneous Dirichlet type nonlocal constrained value problems associated with a scalar ND equation, we showed that any finite element discretization that contains piecewise linear functions provides an asymptotically compatible scheme and thus is a robust discretization to both the nonlocal problems and the local limit. The convergence of approximations to the correct solutions and models is ensured independent of the relations between the horizon parameter δ and the discretization parameter h as shown in the diagram 2.1. Moreover, we showed that such discrete schemes of the nonlocal problem converge to the conforming finite element scheme of the local differential problem as the horizon goes to zero for fixed h. We further extended similar results to a nonlocal state-based PD system.

The main results presented so far, following what were given in [57], focused on the regime with nonlocal horizon parameters going to zero that resulted in local limits given by PDEs. To show the generality of the abstract framework, let us mention some subsequent works inspired by [57]. First, [58] considered a type of nonconforming approximations to nonlocal problems involving nonlocal interaction kernels that are sufficiently singular near the origin. Through the removal of singularity of the nonlocal interaction near a sufficiently small neighborhood of the origin (parametrized by the cut-off radius ϵ) conventional discontinuous finite element space becomes conforming for the parametrized problems with $\epsilon \to 0$. Then, the convergence theory can be established by generalizing the relevant compactness results given in [8] as $\epsilon \to 0$ and applying the framework of asymptotically compatible discretization with respect to the mesh size and the truncation radius ϵ. Another example is the successful application of the framework to the discretization of fractional PDE (fPDE) associated with the fractional Laplacian that can be viewed as the global (with an infinite nonlocal horizon parameter) limit of nonlocal models with properly scaled fractional type nonlocal interaction kernels [60]. We recall the fPDE and the homogeneous nonlocal Dirichlet boundary condition given by

$$\begin{cases} (-\Delta)^{\alpha} u = f \quad \text{on } \Omega \subset \mathbb{R}^d \\ \qquad\quad u = 0 \quad \text{on } \mathbb{R}^d \backslash \Omega \end{cases} . \tag{6.1}$$

Here, $(-\Delta)^{\alpha}$ represents an integral form of the fractional Laplacian operator and it is defined by

$$(-\Delta)^{\alpha} u(\mathbf{x}) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{u(\mathbf{x}) - u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{d+2\alpha}} d\mathbf{x}' \,, \tag{6.2}$$

where $C_{d,\alpha}$ is a constant related to the dimension d and the fractional order α. By truncations of both the spatial domain and the range of nonlocal interactions, we may end up with a class of parametrized problem

$$\begin{cases} -\mathcal{L}_\delta u(\mathbf{x}) = -\displaystyle\int_{\mathbb{R}^d} (u(\mathbf{x}') - u(\mathbf{x})) \gamma_\delta(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' = f \quad \text{on } \Omega \\ \qquad\qquad u = 0 \quad \text{on } \Omega_\delta \,. \end{cases} \tag{6.3}$$

where

$$\gamma_\delta(|\mathbf{x}' - \mathbf{x}|) = \begin{cases} \dfrac{C_{d,\alpha}}{|\mathbf{x} - \mathbf{x}'|^{d+2\alpha}} & \mathbf{x}' \in B_\delta(\mathbf{x}) \\ 0 & \mathbf{x}' \in \mathbb{R}^d \backslash B_\delta(\mathbf{x}) \,. \end{cases} \tag{6.4}$$

The framework of asymptotically compatible discretization can then be adopted to get the convergence of the Galerkin approximations of (6.3) to the solution of the fractional equation (6.1) as the numerical resolution level gets refined while $\delta \to \infty$. Moreover, the study of $\delta \to \infty$ fractional equation limit, in combination with the discussions on the $\delta \to 0$ local PDE limit, demonstrates that nonlocal models with δ being a measure of the finite range of nonlocal interactions, with different scalings of the kernels and limits, serve to effective bridge local PDE models and fractional equation models, as shown in Fig. 6.1 [16]. The same picture remains valid on the discrete level for asymptotically compatible schemes.

We note also that while some of our earlier works exposed possible risks in using piecewise constant finite element for nonlocal problem when the horizon parameter

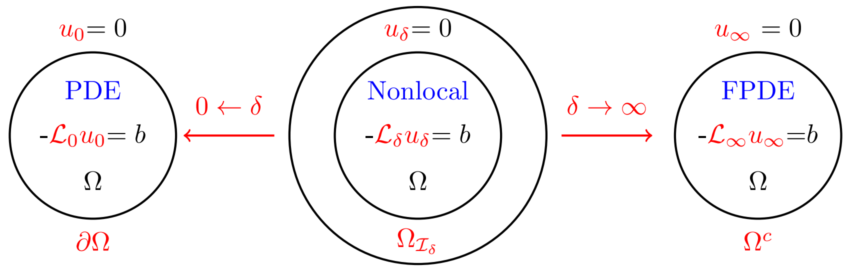

FIG. 6.1. *Bridging local and fractional diffusion equations: PDE as the local limit ($\delta \to 0$) and fPDE model as the global limit ($\delta \to \infty$).*

is proportional to the mesh size [56], the present study provided new remedy to deal with the issue by showing that piecewise constant finite element for the ND problem, when conforming, would be a conditionally asymptotically compatible discretization, under the natural condition that $h = o(\delta)$, which has been pointed out in some simulation-based studies [7, 12].[1]

In addition, to compensate for the lack of analysis on the order of convergence, we carried out numerical experiments of a one-dimensional ND equation discretized with conforming discontinuous piecewise linear finite elements. The discontinuous linear finite element solutions of the nonlocal problem converge to the solution of the correct local differential problem as predicted no matter how δ varies with h, but the convergence rates show dependence on the choices of δ and h. The convergence and superconvergence orders observed lead to interesting theoretical issues to be studied further along with the development of possible postprocessing techniques [14] to improve the order of convergence especially for derivatives and stress variables when singular behaviors are likely to be present in practice.

Finally, we note that although the original study [57] is restricted to Galerkin conforming approximations and linear problems, there have been subsequent studies to extend the notion of asymptotically compatible schemes to other varieties of approximation methods including particle-based (or meshfree) methods, quadrature-based difference methods, Discontinuous Galerkin methods, spectral type methods and also to nonlinear nonlocal models of hyperbolic conservation laws and phase field equations (see for example [21, 22, 25, 27, 36, 55, 61] and a review and additional references in [16]). With the extension here to non-self-adjoint and indefinite cases, it opens up possibilities to treat nonlocal convection-diffusion problems and problems involving both repulsive and attractive nonlocal interactions (that may be associated with sign-changing kernels [38]), as well as saddle-point problems [26]. Naturally, there are also many interesting and relevant theoretical and practical issues remain to be investigated. For example, error estimates that are consistent with the asymptotic compatibility can be further studied. Concerning practical applications, additional studies are needed to explore the constructive roles of asymptotically compatible schemes in dealing with multiscale problems involving coupled local and nonlocal models, particularly those models adopting a spatially heterogeneous horizon parameter [16, 24, 59]. Similarly, more studies on asymptotically compatible schemes may also shed new light on improving the robustness of numerical methods based on various nonlocal smoothing of the local PDE models such as the Smoothed Particle Hydrodynamics [26, 35].

[1]This has also been discussed in personal communications with M. Parks and R. Lehoucq.

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