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ORIGINAL PAPER

A Discontinuous Galerkin Method with Penalty for One-Dimensional Nonlocal Diffusion Problems

Qiang Du[1] · Lili Ju[2] · Jianfang Lu[3] · Xiaochuan Tian[4]

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Abstract

There have been many theoretical studies and numerical investigations of nonlocal diffusion (ND) problems in recent years. In this paper, we propose and analyze a new discontinuous Galerkin method for solving one-dimensional steady-state and time-dependent ND problems, based on a formulation that directly penalizes the jumps across the element interfaces in the nonlocal sense. We show that the proposed discontinuous Galerkin scheme is stable and convergent. Moreover, the local limit of such DG scheme recovers classical DG scheme for the corresponding local diffusion problem, which is a distinct feature of the new formulation and assures the asymptotic compatibility of the discretization. Numerical tests are also presented to demonstrate the effectiveness and the robustness of the proposed method.

Keywords Nonlocal diffusion · Discontinuous Galerkin method · Interior penalty · Asymptotic compatibility · Strong stability preserving

Mathematics Subject Classification 65M60 · 65R20 · 45A05

✉ Jianfang Lu
jflu@m.scnu.edu.cn

Qiang Du
qd2125@columbia.edu

Lili Ju
ju@math.sc.edu

Xiaochuan Tian
xtian@math.utexas.edu

1 Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA

2 Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

3 South China Research Center for Applied Mathematics and Interdisciplinary Studies, South China Normal University, Guangzhou 510631, China

4 Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA

1 Introduction

Recent development of nonlocal modeling has attracted much attention in many application fields, ranging from solid mechanics and anomalous diffusion to imaging analysis and machine learning [12, 13, 20, 21, 28–32]. One of major differences between the nonlocal models and the local models is that the nonlocal models are integral-type equations, while the classical local models are often involved with differential operators. As an example, the peridynamic model was firstly introduced in [29] to study crack and fracture of materials, since the classical continuum models may not be effective when discontinuities occur. Nonlocal models can also be used to develop and study numerical schemes for local problems [17]. Indeed, nonlocal modeling can provide a new approach to describe both continuous and discontinuous phenomena in a unified mathematical model; it also offers a tool and bridge to understand and connect existing models.

As generalizations of classical PDE-based models, many nonlocal models like the peridynamics and nonlocal diffusion (ND) models are characterized by a horizon parameter δ, such that the nonlocal models would converge to the corresponding classical ones if the latter make sense as δ goes to zero. To introduce the nonlocal model under consideration in this paper, we recall that the ND operator $\mathcal{L}_\delta$ represented as follows:

$$\mathcal{L}_\delta u(\boldsymbol{x}) := -2\int_{\widetilde{\Omega}} (u(\boldsymbol{y}) - u(\boldsymbol{x}))\widehat{\gamma}_\delta(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \forall \boldsymbol{x} \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded, open domain (note that we focus on the case $n = 1$ in later sections), and $\widetilde{\Omega} = \Omega \cup \Omega_\delta$ with $\Omega_\delta \subseteq \mathbb{R}^n$ being of a nonzero volume that is not necessarily located near or at the boundary of Ω. The kernel function

$$\widehat{\gamma}_\delta(\boldsymbol{x},\boldsymbol{y}) : \widetilde{\Omega} \times \widetilde{\Omega} \to \mathbb{R}$$

is nonnegative and symmetric, i.e., $\widehat{\gamma}_\delta(\boldsymbol{x},\boldsymbol{y}) = \widehat{\gamma}_\delta(\boldsymbol{y},\boldsymbol{x}) \geq 0$. To connect with its local limit, we may make some extra assumptions (see [13, 26, 33]) that $\widetilde{\gamma}_\delta$ are radial (i.e., $\widehat{\gamma}_\delta(\boldsymbol{x},\boldsymbol{y}) = \gamma_\delta(|\boldsymbol{x}-\boldsymbol{y}|)$) and compactly supported in a ball $B_\delta(\boldsymbol{0})$ with bounded second-order moments defined by

$$(\boldsymbol{C}_\delta)_{ij} = \int_{B_\delta(\boldsymbol{0})} \gamma_\delta(\boldsymbol{\xi})\xi_i\xi_j \mathrm{d}\boldsymbol{\xi}, \quad i,j = 1,\dots,n.$$

Then, we have $\mathcal{L}_\delta \to \nabla\cdot(\boldsymbol{C}\cdot\nabla)$ as $\delta \to 0$, where $\boldsymbol{C} = \lim\limits_{\delta\to 0} \boldsymbol{C}_\delta$ is a second-order tensor. To preserve such mathematical property on the local limit in the discrete sense, a number of studies have been carried out to obtain the so-called *asymptotically compatible* schemes for solving nonlocal problems [33, 34]. In [33], Tian and Du pointed out that the solutions based on some numerical schemes would converge to the wrong local limits as the horizon goes to zero. They also showed numerical schemes that avoid such mishaps. Then, they further established in [34] an abstract mathematical framework to analyze a class of asymptotically compatible schemes for conforming Galerkin approximations of some parameterized linear nonlocal problems. Meanwhile, some numerical methods such as finite difference, finite element, Fourier spectral, and discontinuous Galerkin (DG) approaches have been designed and studied to satisfy the asymptotic compatibility (see, e.g., [14, 16, 18, 33, 35]). When the kernel function γ_δ is chosen such that the solution of the nonlocal diffusion problem contains spatial discontinuities, the DG method could be an advantageous choice for its discretization in space.

Since the major development in the 1990s [6–10], the DG methods have been widely used in many areas such as aero-acoustics, viscoelastic flows, electromagnetism, gas dynamics, and oceanography for their robustness and capability of handling discontinuities. Particularly, there exist various DG approximations for the classical elliptic problems (see, e.g., [1, 5, 24]). For the nonlocal diffusion and nonlocal mechanical models, different conforming and nonconforming Galerkin approximations using discontinuous elements have been considered in [4, 19, 25, 27, 35]. The DG scheme recently proposed in [14] for solving the ND equation is motivated by the local discontinuous Galerkin (LDG) method [11] and relies on the introduction of auxiliary variables. In this paper, we propose a DG method with penalty technique for solving one-dimensional ND problems without introducing auxiliary variables. The method is applied to both steady-state and time-dependent ND problems. We prove the Poincaré's inequality at the discrete level and derive the stability, boundedness, a priori error estimates, and asymptotic compatibility of the proposed scheme. In fact, the local limit of the proposed DG scheme (as the horizon goes to zero) is shown to be identical to the one proposed by Babuška and Zlámal in [3] for the classical diffusion problems. In numerical experiments for the time-dependent ND problem, we use the singly diagonal implicit Runge–Kutta method (SDIRK) for time stepping, which is of strong stability based on the analysis done in [15].

The paper is organized as follows. In Sect. 2, we briefly introduce the one-dimensional ND problem, including the steady-state and time-dependent ones. In Sect. 3, we present semi-discrete DG schemes for the ND problems, which directly penalize the jumps across element interfaces in the nonlocal sense. In Sect. 4, we first prove a Poincaré's inequality at the discrete level and then, we derive the stability, boundedness, and a priori error estimates of the DG scheme for steady-state ND problems. The error estimates imply that the DG scheme is asymptotically compatible. We also obtain the L^2-stability and a priori error estimates of the DG scheme for the time-dependent ND problems. In Sect. 5, numerical examples are given to demonstrate the effectiveness and the robustness of the proposed DG method. Some concluding remarks are finally given in Sect. 6.

2 The Model Problem

Let us consider a one-dimensional steady-state nonlocal diffusion problem with nonlocal volume constraints given as follows:

$$\begin{cases} \mathcal{L}_\delta u = f_\delta, & x \in \Omega = (a,b), \\ u = 0, & x \in \Omega_\delta = [a-\delta, a] \cup [b, b+\delta], \end{cases} \tag{1}$$

where $\delta > 0$ is the horizon and $f_\delta \in L^2(\Omega)$, $\widetilde{\Omega} = \Omega \cup \Omega_\delta$. The corresponding time-dependent problem of (1) is given by

$$\begin{cases} u_t + \mathcal{L}_\delta u = f_\delta, & (x,t) \in \Omega \times (0,T], \\ u = u_0, & x \in \Omega \times \{t=0\}, \\ u = 0, & (x,t) \in \Omega_\delta \times [0,T]. \end{cases} \tag{2}$$

The nonlocal diffusion operator $\mathcal{L}_\delta$ is defined as

$$\mathcal{L}_\delta u(x) := -2\int_{x-\delta}^{x+\delta}(u(y)-u(x))\widehat{\gamma}_\delta(x,y)\,\mathrm{d}y$$

for a kernel function $\widehat{\gamma}_\delta(x,y)$. For simplicity, we take the kernel function to be the form of $\hat{\gamma}_\delta(x,y)=\gamma_\delta(|x-y|)=\gamma_\delta(s)$ and assume it has a finite second moment, i.e.,

$$\gamma_\delta=\gamma_\delta(s)\text{ is nonnegative and symmetric, with } s^2\gamma_\delta(s)\in L^1_{\mathrm{loc}}(\mathbb{R}). \tag{3}$$

The solution space associated with (1) is

$$\mathcal{S}=\left\{u\in L^2(\widetilde{\Omega})\ :\ \|u\|_{\mathcal{S}}<\infty \text{ and } u|_{\Omega_\delta}=0\right\},$$

where the energy norm $\|u\|_{\mathcal{S}}$ is defined as

$$\|u\|^2_{\mathcal{S}}=2\int_0^\delta\gamma_\delta(s)\int_{\widetilde{\Omega}}(E_s^+u)^2\,\mathrm{d}x\mathrm{d}s$$

with $E_s^+u=u(x+s)-u(x)$. Note that the definition of this energy norm requires the values of u outside $\widetilde{\Omega}$; hence, we make the extension of u such that $u=0$ on $\Omega_{2\delta}$. The energy norm $\|\cdot\|_{\mathcal{S}}$ is in fact a norm on $\mathcal{S}$ (see, e.g., [26] for more details). With change of variables, the variational formulations of the steady-state problem (1) and the time-dependent problem (2) are, respectively, defined by

$$\text{find } u\in\mathcal{S}\text{ such that } B(u,v)=(f_\delta,v),\quad \forall v\in\mathcal{S}, \tag{4}$$

and

$$\text{find } u(\cdot,t)\in\mathcal{S}\text{ such that } (u_t,v)+B(u,v)=(f_\delta,v),\quad \forall v\in\mathcal{S}, \tag{5}$$

where $(\cdot,\cdot)$ is the L^2 inner product and

$$B(u,v)=2\int_0^\delta\int_{\widetilde{\Omega}}\gamma_\delta(s)E_s^+uE_s^+v\,\mathrm{d}x\mathrm{d}s,\quad \forall\, u,v\in\mathcal{S}. \tag{6}$$

Since $s^2\gamma_\delta(s)\in L^1_{\mathrm{loc}}(\mathbb{R})$, we have

$$C_\delta=2\int_0^\delta s^2\gamma_\delta(s)\mathrm{d}s<\infty.$$

To connect with the local limit, without loss of generality, it is assumed that

$$C_\delta=1,$$

which can always be achieved by a rescaling of $\gamma_\delta(s)$.

Thus, when $\delta\to 0$, the nonlocal diffusion operator becomes the classical (local) diffusion operator, which implies that (1) and (2) converge to the Poisson's equation and heat equation, respectively (see [13, 33] for more details).

3 Discontinuous Galerkin Approximations with Penalty

In this section, we propose a new DG method which directly imposes penalties on the jumps across element interfaces instead of introducing auxiliary variables as done in [14] for discretizing the problems (1) and (2).

First, we take the partition of the domain $\widetilde{\Omega}$ as $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=-m+1}^{N+m}$ with

$$x_{\frac{1}{2}} = a, \;\; x_{N+\frac{1}{2}} = b, \;\; x_{-m-\frac{1}{2}} \leq a - \delta < x_{-m+\frac{1}{2}}, \;\; x_{N+m-\frac{1}{2}} < b + \delta \leq x_{N+m+\frac{1}{2}}.$$

Let $h = \max_j h_j, \; \rho = \min_j h_j, \; h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. We assume the partition $\mathcal{T}_h$ is regular, i.e., there exists a constant $\nu > 0$, independent of h as $h \to 0$, such that

$$\nu h \leq \rho.$$

Now, we define a finite element space V_h as

$$V_h = V_h^k = \left\{ v \in L^2(\Omega) \;:\; v|_{I_j} \in \mathcal{P}_k(I_j), \, j = 1, \dots, N, v|_{\Omega_\delta} = 0 \right\}, \tag{7}$$

where $\mathcal{P}_k(I_j)$ is the space of polynomials on I_j whose degrees are at most k.

Let us rewrite the bilinear form $B(u, v)$ in (6) as follows:

$$B(u, v) = B_1(u, v) + B_2(u, v) + B_3(u, v), \quad \forall\, u, v \in \mathcal{S}, \tag{8}$$

where, for $\hat{h} = \min\{\rho, \delta\}$, we have that

$$\begin{cases} B_1(u, v) &= 2 \displaystyle\int_0^{\hat{h}} \gamma_\delta(s) \sum_j \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}} - s} E_s^+ u E_s^+ v \, \mathrm{d}x\mathrm{d}s, \\ B_2(u, v) &= 2 \displaystyle\int_0^{\hat{h}} \gamma_\delta(s) \sum_j \int_{x_{j+\frac{1}{2}} - s}^{x_{j+\frac{1}{2}}} E_s^+ u E_s^+ v \, \mathrm{d}x\mathrm{d}s, \\ B_3(u, v) &= 2 \displaystyle\int_{\hat{h}}^{\delta} \gamma_\delta(s) \sum_j \int_{I_j} E_s^+ u E_s^+ v \, \mathrm{d}x\mathrm{d}s. \end{cases} \tag{9}$$

We are interested in modifying $B(u, v)$ so that it may be defined in the discrete spaces. For u_h and v_h in V_h, one can see first that $B_1(u_h, v_h)$ and $B_3(u_h, v_h)$ in the RHS of (8) can be well defined but $B_2(u_h, v_h)$ could become problematic if u_h and v_h are discontinuous at the element interfaces. In fact, if we fix h_j, we have formally that

$$2 \int_0^{\hat{h}} \gamma_\delta(s) \sum_j \int_{x_{j+\frac{1}{2}} - s}^{x_{j+\frac{1}{2}}} E_s^+ u_h E_s^+ v_h \, \mathrm{d}x\mathrm{d}s \sim 2 \int_0^{\hat{h}} s\gamma_\delta(s)\mathrm{d}s \sum_j [\![u_h]\!]_{j+\frac{1}{2}} [\![v_h]\!]_{j+\frac{1}{2}}, \quad \text{as } \delta \to 0,$$

where $[\![w]\!]_{j+\frac{1}{2}} = w(x_{j+\frac{1}{2}}^+) - w(x_{j+\frac{1}{2}}^-)$ denotes the jump of w at $x_{j+\frac{1}{2}}$. For a general kernel $\gamma_\delta(s)$ with a bounded second moment (3) but an unbounded first moment, $s\gamma_\delta(s)$ may not be in $L^1_{\mathrm{loc}}(\mathbb{R})$, thus causing problems in the local limit. The remedy that we propose here is to introduce an extra penalty term and replace, in the problematic term, $E_s^+ u_h$ and $E_s^+ v_h$ by $E_s^+ u_h - [\![u_h]\!]_{j+\frac{1}{2}}$ and $E_s^+ v_h - [\![v_h]\!]_{j+\frac{1}{2}}$, respectively. Such a modification would make the previously identified problematic term well defined in the local limit. Hence, we obtain a new bilinear form in the following:

$$
\begin{aligned}
B_h(u_h, v_h) = {} & 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j-\frac12}}^{x_{j+\frac12}-s} E_s^+ u_h E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \\
& + 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j+\frac12}-s}^{x_{j+\frac12}} \Big(E_s^+ u_h - [\![u_h]\!]_{j+\frac12}\Big)\Big(E_s^+ v_h - [\![v_h]\!]_{j+\frac12}\Big) \mathrm{d}x\mathrm{d}s \\
& + 2\sum_j \int_{\hat h}^{\delta} \gamma_\delta(s) \int_{I_j} E_s^+ u_h E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \\
& + \int_0^{\hat h} s^2 \gamma_\delta(s)\mathrm{d}s \sum_j \mu_j [\![u_h]\!]_{j+\frac12} [\![v_h]\!]_{j+\frac12},
\end{aligned}
\tag{10}
$$

where the penalty parameters $\{\mu_j > 0\}$ are expected to be sufficiently large for deriving the error estimates later on. In particular, with $\mu_j = O(h^{-2k-1})$, we are able to recover the DG scheme designed by Babuška and Zlámal [3] in the local zero horizon limit.

To see the local limit more clearly, let us fix h_j. As $\delta \to 0$, we have

$$
\begin{aligned}
& 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j-\frac12}}^{x_{j+\frac12}-s} E_s^+ u_h E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \to \int_{I_j} (u_h)_x (v_h)_x \,\mathrm{d}x, \\
& 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j+\frac12}-s}^{x_{j+\frac12}} \Big(E_s^+ u_h - [\![u_h]\!]_{j+\frac12}\Big)\Big(E_s^+ v_h - [\![v_h]\!]_{j+\frac12}\Big) \mathrm{d}x\mathrm{d}s \to 0, \\
& 2\sum_j \int_{\hat h}^{\delta} \gamma_\delta(s) \int_{I_j} E_s^+ u_h E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \to 0, \\
& \int_0^{\hat h} s^2 \gamma_\delta(s)\mathrm{d}s \sum_j \mu_j [\![u_h]\!]_{j+\frac12} [\![v_h]\!]_{j+\frac12} \to \frac12 \sum_j \mu_j [\![u_h]\!]_{j+\frac12} [\![v_h]\!]_{j+\frac12},
\end{aligned}
$$

which correspond to the original DG scheme proposed in [3]. It is known in the literature that this superpenalty procedure makes the DG method behave like a standard conforming method and increases the condition number of the stiffness matrix significantly [1].

We now present the new DG scheme with penalty for the problem (1) as follows:

$$
\text{find } u_h \in V_h \text{ such that } \qquad B_h(u_h, v_h) = (f_\delta, v_h), \quad \forall\, v_h \in V_h. \tag{11}
$$

The corresponding semi-discrete DG scheme for solving the time-dependent problem (2) is given by:

$$
\text{find } u_h(\cdot, t) \in V_h \text{ such that } \quad ((u_h)_t, v_h) + B_h(u_h, v_h) = (f_\delta, v_h), \quad \forall\, v_h \in V_h, \tag{12}
$$

where the initial $u_h(x, 0) \in V_h$ is taken as the standard L^2 projection of u_0 onto V_h.

4 Stability, Boundedness, and a Priori Error Estimates

In this section, we first present the discrete Poincaré's inequality and then study the boundedness and stability results of (11), which enable us to derive a priori error estimates. Next, we prove the L^2-stability and a priori error estimates of the semi-discrete DG scheme (12). Throughout this section, we let $C > 0$ represent a generic positive constant independent of h and δ with possibly different values if not noted otherwise. Let us define the semi-norms for $v \in V_h$ as follows:

$$\begin{aligned} |v|^2_{\mathcal{S},h} =& 2\sum_j \int_0^{\hat{h}} \gamma_\delta(s) \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-s} (E_s^+ v)^2 \, \mathrm{d}x\mathrm{d}s + 2\sum_j \int_{\hat{h}}^{\delta} \gamma_\delta(s) \int_{I_j} (E_s^+ v)^2 \, \mathrm{d}x\mathrm{d}s \\ &+ 2\sum_j \int_0^{\hat{h}} \gamma_\delta(s) \int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} \left(E_s^+ v - [\![v]\!]_{j+\frac{1}{2}} \right)^2 \mathrm{d}x\mathrm{d}s, \qquad (13) \\ |v|^2_* =& \int_0^{\hat{h}} s^2 \gamma_\delta(s) \mathrm{d}s \sum_j \mu_j [\![v]\!]^2_{j+\frac{1}{2}}. \end{aligned}$$

By the definition of the bilinear form (10) and the semi-norms (13), we immediately have

$$B_h(v, v) = |v|^2_{\mathcal{S},h} + |v|^2_*.$$

Then, we define the semi-norm

$$|||v|||^2 := B_h(v, v), \quad \forall\, v \in V_h.$$

4.1 Discrete Poincaré's Inequality

To ensure that $|||\cdot|||$ is a norm on V_h, we need to derive some Poincaré's inequalities at the discrete level. First, let us present the result when the kernel γ_δ is bounded on $[0, \delta]$.

Lemma 4.1 *For a bounded γ_δ, it holds that*

$$\|v_h\|_{L^2} \le C\, |||v_h|||, \quad \forall\, v_h \in V_h, \qquad (14)$$

where the constant $C > 0$ is independent of δ and h.

Proof Following [2], we consider the problem $\mathcal{L}_\delta \phi = v_h$ for any $v_h \in V_h$ with a bounded kernel γ_δ. Then, the energy space $\mathcal{S}$ is indeed the L^2 space and it is implied by the nonlocal Poincaré's inequality ([26]) that the following energy inequality holds:

$$\|\phi\|_{\mathcal{S}} \le C\, \|v_h\|_{L^2},$$

where $C > 0$ is a constant independent of δ. For the DG scheme (11) and by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
(v_h, v_h) &= (\mathcal{L}_\delta \phi, v_h) \\
&= 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j-\frac12}}^{x_{j+\frac12}-s} E_s^+\phi E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \\
&\quad + 2\sum_j \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi\Big(E_s^+ v_h - [\![v_h]\!]_{j+\frac12}\Big)\,\mathrm{d}x\mathrm{d}s \\
&\quad + 2\sum_j \int_{\hat h}^{\delta} \gamma_\delta(s) \int_{I_j} E_s^+\phi E_s^+ v_h \,\mathrm{d}x\mathrm{d}s - \mathcal{C}(\phi, v_h) \\
&\leq \|\phi\|_{\mathcal{S}}\, |||v_h||| + |\mathcal{C}(\phi, v_h)| \\
&\leq C\, \|v_h\|_{L^2}\, |||v_h||| + |\mathcal{C}(\phi, v_h)|,
\end{aligned}
\tag{15}
$$

where $\mathcal{C}(\phi, v_h)$ is given as

$$
\mathcal{C}(\phi, v_h) = -2\sum_j [\![v_h]\!]_{j+\frac12} \int_0^{\hat h} \gamma_\delta(s) \int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi \,\mathrm{d}x\mathrm{d}s. \tag{16}
$$

We then apply the following inequality (to be shown in Lemma 4.2 later in the section):

$$
|\mathcal{C}(\phi, v_h)| \leq C\, |||v_h|||\, \|v_h\|_{L^2}, \tag{17}
$$

and plug (17) into (15) to get the discrete Poincaré's inequality (14).

Before proving the inequality (17) used in the proof above, we first explain the idea behind the proof. Let us think about the local limit, namely for δ small, the term

$$
\int_0^{\hat h} \gamma_\delta(s) \int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi \mathrm{d}x\mathrm{d}s
$$

is essentially bounded by $\|\phi\|_{C^{0,1}}$, since we have

$$
E_s^+\phi = s\frac{\phi(x+s)-\phi(x)}{s} \leq s\|\phi\|_{C^{0,1}}.
$$

By the Sobolev embedding theorem (or trace inequality), we have

$$
\|\phi\|_{C^{0,1}} \leq C\|\phi\|_{H^{3/2}}. \tag{18}
$$

The elliptic regularity can then be applied to conclude that $\|\phi\|_{H^{3/2}} \leq C\|v_h\|_{L^2}$ thus completing the proof, as argued in [2]. For the situation considered here, a difficulty is to obtain (17) with a uniform constant C. Indeed, for each finite $\delta > 0$, there is not enough elliptic regularity to make the argument in general. We thus have to avoid relying on the type of inequality as (18). Hence, we find another way to show the results in 1d that is analogous to the PDE counterpart in [2]. We use the fundamental theorem of calculus; we could rewrite $\phi'(x)$ as

$$\phi'(x) = \phi'(y) + \int_y^x \phi''(z)\mathrm{d}z, \tag{19}$$

which is true for all y. Also, notice that in the local limit ϕ satisfies the problem $-\phi''(z) = v_h(z)$. Now, we integrate y on some interval I and obtain

$$|I||\phi'(x)| \leq \int_I |\phi'(y)|\mathrm{d}y + \int_I \int_y^x |v_h(z)|\mathrm{d}z \leq C_1 |\phi|_{H^1} + C_2 \|v_h\|_{L^2}\,.$$

Here, we can bound $|\phi'(x)|$ by the H^1 norm of ϕ and the L^2 norm of $\|v_h\|_{L^2}$ and finally, we can use an energy inequality to bound $|\phi|_{H^1}$ by $\|v_h\|_{L^2}$. In the following lemma, equality (21) can be understood as an analog of (19).

Lemma 4.2 *Assume that $\phi \in \mathcal{S}$ solves the problem $\mathcal{L}_\delta \phi = v_h$, then we have, for some constant $C > 0$, independent of δ and h,*

$$|\mathcal{C}(\phi, v_h)| \leq C\, |||v_h|||\, \|v_h\|_{L^2}\,, \tag{20}$$

where $\mathcal{C}(\phi, v_h)$ is defined in (16).

Proof Firstly, with $E_s^- \phi(y) = \phi(y) - \phi(y - s)$, we can write

$$\begin{aligned}
\int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} E_s^+ \phi(y)\,\mathrm{d}y &= \int_x^{x_{j+\frac{1}{2}}} E_s^+ \phi(y)\,\mathrm{d}y + \int_{x_{j+\frac{1}{2}}-s}^{x} E_s^+ \phi(y)\,\mathrm{d}y \\
&= \int_x^{x_{j+\frac{1}{2}}} (\phi(y+s) - \phi(y))\,\mathrm{d}y - \int_{x+s}^{x_{j+\frac{1}{2}}} (\phi(y) - \phi(y-s))\,\mathrm{d}y \\
&= \int_x^{x+s} E_s^- \phi(y)\,\mathrm{d}y + \int_x^{x_{j+\frac{1}{2}}} (\phi(y+s) - 2\phi(y) + \phi(y-s))\,\mathrm{d}y \\
&= \int_{x-s}^{x} E_s^+ \phi(y)\,\mathrm{d}y + \int_x^{x_{j+\frac{1}{2}}} E_s^+ E_s^- \phi(y)\,\mathrm{d}y.
\end{aligned} \tag{21}$$

Integrating (21) over $(0, \hat{h}) \times I_j$ with the weight function $\gamma_\delta(s)$, we obtain

$$\begin{aligned}
h_j \int_0^{\hat{h}} \gamma_\delta(s) \int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} E_s^+ \phi(y)\,\mathrm{d}y\mathrm{d}s &= \int_0^{\hat{h}} \gamma_\delta(s) \int_{I_j} \int_{x-s}^{x} E_s^+ \phi(y)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&+ \int_0^{\hat{h}} \gamma_\delta(s) \int_{I_j} \int_x^{x_{j+\frac{1}{2}}} E_s^+ E_s^- \phi(y)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s = I + II.
\end{aligned} \tag{22}$$

By changing the order of the integration in I, we have

$$
\begin{aligned}
I &= \int_0^{\hat{h}} \gamma_\delta(s)\Big(\int_{x_{j-\frac{1}{2}}-s}^{x_{j-\frac{1}{2}}} \int_{x_{j-\frac{1}{2}}}^{y+s} + \int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} \int_{y}^{x_{j+\frac{1}{2}}} + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-s} \int_{y}^{y+s} \Big) E_s^+ \phi(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}s \\
&= \int_0^{\hat{h}} \gamma_\delta(s)\Big(\int_{x_{j-\frac{1}{2}}-s}^{x_{j-\frac{1}{2}}} \big(y+s-x_{j-\frac{1}{2}}\big) E_s^+\phi(y)\,\mathrm{d}y + \int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} \big(x_{j+\frac{1}{2}}-y\big)E_s^+\phi(y)\,\mathrm{d}y \\
&\quad + s\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-s} E_s^+\phi(y)\,\mathrm{d}y\Big)\mathrm{d}s \\
&\le \int_0^{\hat{h}} s\gamma_\delta(s) \int_{x_{j-\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} |E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
\tag{23}
$$

Similarly, by changing the order of the integration in II, we have

$$
\begin{aligned}
II &= \int_{I_j}\int_x^{x_{j+\frac{1}{2}}} \int_0^{\hat{h}} \gamma_\delta(s) E_s^+E_s^-\phi(y)\,\mathrm{d}s\mathrm{d}y\mathrm{d}x \\
&= \int_{I_j}\int_x^{x_{j+\frac{1}{2}}} \Big(v_h(y) - \int_{\hat{h}}^{\delta} \gamma_\delta(s)E_s^+E_s^-\phi(y)\mathrm{d}s\Big)\mathrm{d}y\mathrm{d}x \\
&= \int_{I_j}\int_x^{x_{j+\frac{1}{2}}} v_h(y)\,\mathrm{d}y - \int_{\hat{h}}^{\delta} \gamma_\delta(s) \int_{I_j}\Big(\int_x^{x_{j+\frac{1}{2}}} - \int_{x-s}^{x_{j+\frac{1}{2}}-s}\Big) E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\le h_j \int_{I_j} |v_h(y)|\,\mathrm{d}y + \frac{h_j}{\hat{h}} \int_{\hat{h}}^{\delta} s\gamma_\delta(s)\Big(\int_{I_j} |E_s^+\phi(y)|\,\mathrm{d}y + \int_{I_j-s} |E_s^+\phi(y)|\,\mathrm{d}y\Big)\mathrm{d}s,
\end{aligned}
\tag{24}
$$

where $I_j - s = \big(x_{j-\frac{1}{2}} - s, x_{j+\frac{1}{2}} - s\big)$. Plugging (23) and (24) into (22), we get

$$
\begin{aligned}
& h_j \int_0^{\hat{h}} \gamma_\delta(s) \int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}s \\
&\le \int_0^{\hat{h}} s\gamma_\delta(s) \int_{x_{j-\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}} |E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s + h_j \int_{I_j} |v_h(y)|\,\mathrm{d}y \\
&\quad + \frac{h_j}{\hat{h}} \int_{\hat{h}}^{\delta} s\gamma_\delta(s) \int_{I_j\cup(I_j-s)} |E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
\tag{25}
$$

Therefore, with (25) we have

$$
\begin{aligned}
&\sum_j h_j\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&=\sum_j \frac{1}{h_j}\Big(h_j\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&\leq \sum_j \frac{1}{h_j}\Big(\int_0^{\hat h}s\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}} |E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s + h_j\int_{I_j}|v_h(y)|\,\mathrm{d}y\\
&\quad+\frac{h_j}{\hat h}\int_{\hat h}^{\delta}s\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&\leq 3\sum_j\Bigg(\frac{1}{h_j}\Big(\int_0^{\hat h}s\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}|E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s\Big)^2 + h_j\Big(\int_{I_j}|v_h(y)|\,\mathrm{d}y\Big)^2\\
&\quad+\frac{h_j}{\hat h^2}\Big(\int_{\hat h}^{\delta}s\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s\Big)^2\Bigg).
\end{aligned}
\tag{26}
$$

By the Cauchy–Schwarz inequality, we then have

$$
\begin{aligned}
&\frac{1}{h_j}\Big(\int_0^{\hat h}s\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}|E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&\leq \frac{1}{h_j}\int_0^{\hat h}s^2\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}1\,\mathrm{d}y\mathrm{d}s\,\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s\Big)\\
&\leq \frac{h_{j-1}+h_j}{h_j}\int_0^{\hat h}s^2\gamma_\delta(s)\mathrm{d}s\,\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s\Big),
\end{aligned}
\tag{27}
$$

and

$$
h_j\Big(\int_{I_j}|v_h(y)|\,\mathrm{d}y\Big)^2\leq h_j^2\int_{I_j}v_h(y)^2\mathrm{d}y,
\tag{28}
$$

and

$$
\begin{aligned}
&\frac{h_j}{\hat h^2}\Big(\int_{\hat h}^{\delta}s\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&\leq \frac{h_j}{\hat h^2}\int_{\hat h}^{\delta}s^2\gamma_\delta(s)\int_{I_j\cup(I_j-s)}1\,\mathrm{d}y\mathrm{d}s\,\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s\Big)\\
&\leq \frac{2h_j^2}{\hat h^2}\int_{\hat h}^{\delta}s^2\gamma_\delta(s)\mathrm{d}s\,\Big(\int_{\hat h}^{\delta}\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s\Big).
\end{aligned}
\tag{29}
$$

Plugging (27)–(29) into (26), we then obtain

$$
\begin{aligned}
&\sum_j h_j\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j+\frac12}-s}^{x_{j+\frac12}} E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}s\Big)^2\\
&\le \frac{6h}{\rho}\int_0^{\hat h} s^2\gamma_\delta(s)\mathrm{d}s\sum_j\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j-\frac12}-s}^{x_{j+\frac12}}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s+3h^2\|v_h\|_{L^2}^2\\
&\quad+\frac{6h^2}{\rho^2}\int_{\hat h}^{\delta} s^2\gamma_\delta(s)\mathrm{d}s\sum_j\int_{\hat h}^{\delta}\gamma_\delta(s)\int_{I_j\cup(I_j-s)}|E_s^+\phi(y)|^2\,\mathrm{d}y\mathrm{d}s\\
&\le C\,\|\phi\|_{\mathcal S}^2+3h^2\|v_h\|_{L^2}^2\\
&\le C\,\|v_h\|_{L^2}^2,
\end{aligned}
\tag{30}
$$

where C depends only on ν, a positive constant given in the assumption on the regularity of the mesh partition. Therefore, by the Cauchy–Schwarz inequality again, we finally get

$$
\begin{aligned}
|C(\phi,v_h)| &\le 2\Big(\sum_j\frac{1}{h_j}[\![v_h]\!]_{j+\frac12}^2\Big)^{\frac12}\left(\sum_j h_j\Big(\int_0^{\hat h}\gamma_\delta(s)\int_{x_{j+\frac12}-s}^{x_{j+\frac12}}E_s^+\phi(y)\,\mathrm{d}y\mathrm{d}s\Big)^2\right)^{\frac12}\\
&\le C\,|||v_h|||\,\|v_h\|_{L^2},
\end{aligned}
$$

which completes the proof.

Proposition 4.1 *For the general kernels γ_δ satisfying (3), it holds that for some constant $C>0$ uniformly in δ and h,*

$$
\|v_h\|_{L^2}\le C\,|||v_h|||,\quad \forall\, v_h\in V_h. \tag{31}
$$

Proof Consider a cutoff of γ_δ as follows:

$$
\widetilde{\gamma}_\delta(s)=\begin{cases}\gamma_\delta(s), & \gamma_\delta(s)\le M,\\ M, & \gamma_\delta(s)>M.\end{cases}
$$

Take $M>0$ sufficiently large such that

$$
\widetilde{C_\delta}=2\int_0^\delta s^2\widetilde{\gamma}_\delta(s)\mathrm{d}s\ge\frac14.
$$

For the modified kernel $\widetilde{\gamma}_\delta/\widetilde{C_\delta}$, we assume the corresponding energy norm is $|||\cdot|||_{\widetilde{\mathcal S}}$; then by Lemma 4.1, we have

$$\|v_h\|_{L^2} \le C\,|||v_h|||_{\widetilde{S}}, \quad v_h \in V_h.$$

Since $\widetilde{\gamma}_\delta(s) \le \gamma_\delta(s)$ and $\widetilde{C_\delta} \ge \frac{1}{4}$, we have

$$|||v_h|||_{\widetilde{S}} \le \frac{1}{\sqrt{\widetilde{C_\delta}}}\,|||v_h||| \le 2\,|||v_h|||, \quad \forall\, v_h \in V_h.$$

Then, we obtain the desired inequality (31).

4.2 Boundedness, Stability, and a Priori Error Estimates for the DG Scheme (11)

From the definition of $B_h(\cdot,\cdot)$, it is straightforward to obtain the boundedness and stability results represented in the following:

$$B_h(v,w) \le |||v|||\,|||w|||, \quad \forall v,w \in V_h. \tag{32}$$

$$B_h(v_h,v_h) = |||v_h|||^2, \quad \forall v_h \in V_h. \tag{33}$$

Note that (32) also holds for $v,w \in \mathcal{S}$. Now, let us take the continuous interpolant $u_I \in V_h$ of the exact solution u such that $u - u_I \in \mathcal{S}$ and the jumps of $u - u_I$ are zero at the element interfaces. Note that this can be easily achieved when the exact solution u is smooth enough and $k \ge 1$ where k is the degree of the polynomials in V_h. Then, we have the following approximation property:

$$\|u-u_I\|_{L^2} \le Ch^{k+1}\,|u|_{H^{k+1}} \quad \text{and} \quad |||u-u_I||| \le Ch^k\,|u|_{H^{k+1}}, \tag{34}$$

where $C>0$ is independent of h and δ.

Corollary 4.1 *Let δ be fixed. Then, it holds that when the kernel $\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$ (i.e., integrable kernel),*

$$|||u-u_I||| \le C(\delta)\,h^{k+1},$$

and $s\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$ (i.e., finite first moment),

$$|||u-u_I||| \le C(\delta)\,h^{k+\frac{1}{2}}.$$

To derive the error estimates, we first need the following lemma.

Lemma 4.3 *For the solution u is smooth enough, we have that $B(u,v_h)$ in (8) is well defined. Moreover,*

$$B(u,v_h) = (f_\delta, v_h), \quad \forall\, v_h \in V_h.$$

Proof Recall the construction in (9) given by $B(u,v) = B_1(u,v) + B_2(u,v) + B_3(u,v)$. We consider the terms separately with $v = v_h$. By the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
B_1(u,v_h) &\le 2\int_0^{\hat{h}}\sum_j\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-s}|\gamma_\delta(s)E_s^+uE_s^+v_h|\,\mathrm{d}x\mathrm{d}s\\
&\le 2\,\|u\|_{C^{0,1}(\mathcal{T}_h)}\|v_h\|_{C^{0,1}(\mathcal{T}_h)}\int_0^{\hat{h}}\sum_j\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-s}\gamma_\delta(s)s^2\mathrm{d}x\mathrm{d}s\\
&\le C\,\|u\|_{C^{0,1}(\mathcal{T}_h)}\|v_h\|_{C^{0,1}(\mathcal{T}_h)}<\infty,
\end{aligned}
$$

where

$$
\|w\|_{C^{0,1}(\mathcal{T}_h)}=\max_j\|w\|_{C^{0,1}(I_j)}.
$$

Meanwhile, we have

$$
\begin{aligned}
B_2(u,v_h) &\le 2\int_0^{\hat{h}}\sum_j\int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}}|\gamma_\delta(s)E_s^+uE_s^+v_h|\,\mathrm{d}x\mathrm{d}s\\
&\le 2\int_0^{\hat{h}}\sum_j\int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}}|\gamma_\delta(s)E_s^+u\Big(E_s^+v_h-[\![v_h]\!]_{j+\frac{1}{2}}\Big)|\,\mathrm{d}x\mathrm{d}s\\
&\quad+2\int_0^{\hat{h}}\sum_j\Big|[\![v_h]\!]_{j+\frac{1}{2}}\Big|\int_{x_{j+\frac{1}{2}}-s}^{x_{j+\frac{1}{2}}}|\gamma_\delta(s)E_s^+u|\,\mathrm{d}x\mathrm{d}s\\
&\le C\,\|u\|_{C^{0,1}(\mathcal{T}_h)}\|v_h\|_{C^{0,1}(\mathcal{T}_h)}+C\,\|u\|_{C^{0,1}(\mathcal{T}_h)}h^{-1}\Big(\sum_j[\![v_h]\!]^2_{j+\frac{1}{2}}\Big)^{\frac{1}{2}}\\
&\le C\,\|u\|_{C^{0,1}(\mathcal{T}_h)}\|v_h\|_{C^{0,1}(\mathcal{T}_h)}+C\,\|u\|_{C^{0,1}(\mathcal{T}_h)}|||v_h|||<\infty,
\end{aligned}
$$

with $\mu_j=\eta h^{-1}$ and η is sufficiently large. For $B_3(u,v_h)$, if $\hat{h}=\delta$, then $B_3(u,v_h)=0$. When $\hat{h}=\min_j h_j$, then we have

$$
\begin{aligned}
B_3(u,v_h) &\le 2\int_{\hat{h}}^{\delta}\sum_j\int_{I_j}|\gamma_\delta(s)E_s^+uE_s^+v_h|\,\mathrm{d}x\mathrm{d}s\\
&\le 4\,\|u\|_{C^{0,1}(\mathcal{T}_h)}\|v_h\|_{L^2}\int_{\hat{h}}^{\delta}s\gamma_\delta(s)\mathrm{d}s\ \le 2\,\|u\|_{L^2}\|v_h\|_{L^2}\hat{h}^{-1}<\infty.
\end{aligned}
$$

Thus, $B(u,v_h)$ is well defined.

In the following, we show that $B(u,v_h)=(\mathcal{L}_\delta u,v_h)=(f,v_h)$ for any $v_h\in V_h$ when u is smooth enough. Indeed,

$$
\begin{aligned}
B(u, v_h) &= 2\int_0^{\delta} \gamma_\delta(s) \sum_j \int_{I_j} E_s^+ u E_s^+ v_h \,\mathrm{d}x\mathrm{d}s \\
&= 2\int_0^{\delta} \gamma_\delta(s)\Big(\sum_j \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u(x+s)-u(x))v_h(x+s)\,\mathrm{d}x \\
&\qquad - \sum_j \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u(x+s)-u(x))v_h(x)\,\mathrm{d}x\Big)\mathrm{d}s \\
&= 2\int_0^{\delta} \gamma_\delta(s)\Big(\sum_j \int_{x_{j-\frac{1}{2}}+s}^{x_{j+\frac{1}{2}}+s} (u(x)-u(x-s))v_h(x)\,\mathrm{d}x \\
&\qquad - \sum_j \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u(x+s)-u(x))v_h(x)\,\mathrm{d}x\Big)\mathrm{d}s \\
&= -2\sum_j \int_0^{\delta}\int_{I_j} \gamma_\delta(s)(u(x+s)-2u(x)+u(x-s))v_h(x)\,\mathrm{d}x\mathrm{d}s \\
&= -2\sum_j \int_{-\delta}^{\delta}\int_{I_j} \gamma_\delta(s) v_h E_s^+ u \,\mathrm{d}x\mathrm{d}s \\
&= \sum_j \int_{I_j}\Big(-2\int_{-\delta}^{\delta} \gamma_\delta(s) E_s^+ u\,\mathrm{d}s\Big) v_h\,\mathrm{d}x = (\mathcal{L}_\delta u, v_h) = (f_\delta, v_h).
\end{aligned} \tag{35}
$$

As we can see, even when the solution u is smooth enough, the DG approximation of $B(u, v)$ is generically not consistent, i.e.,

$$
B_h(u, v_h) = B(u, v_h) + C(u, v_h), \qquad \forall\, v_h \in V_h,
$$

where $C(u, v_h)$ is the inconsistent term given in (16). Since the DG scheme (11) is not consistent, to derive the error estimates, we then need to estimate the inconsistent errors. The so-called superpenalty technique is adopted to control the inconsistent term $C(u, v_h)$. We present the error estimates of the DG scheme (11) in the following theorem.

Theorem 4.1 *For the DG scheme* (11) *with the finite element space V_h defined in* (7) *with $k \geq 1$ and $\mu_j = O(h_j^{-2k-1})$, there exists a unique approximate solution $u_h \in V_h$. Assume that the exact solution u of the problem* (1) *is smooth enough, then we have the following a priori error estimate*:

$$
|||u - u_h||| \leq Ch^k \,\|u\|_{H^{k+1}}.
$$

Proof We firstly consider the estimate of $C(u, v_h)$. With the Cauchy–Schwarz inequality and Sobolev's inequality, we obtain

$$
\begin{aligned}
C(u, v_h) &\leq 2\|u_x\|_{L^\infty} \int_0^{\hat{h}} s^2\gamma_\delta(s)\mathrm{d}s \sum_j \frac{1}{\sqrt{\mu_j}} \sqrt{\mu_j}\big|[\![v_h]\!]_{j+\frac{1}{2}}\big| \\
&\leq \beta(\hat{h}, \delta)\Big(\int_0^{\hat{h}} s^2\gamma_\delta(s)\mathrm{d}s \sum_j \mu_j [\![u_h]\!]^2_{j+\frac{1}{2}} \Big)^{\frac{1}{2}} \\
&\leq \beta(\hat{h}, \delta)\, |||v_h|||,
\end{aligned} \tag{36}
$$

$$\beta(\hat{h},\delta) = C\,\|u\|_{H^2}\Big(\int_0^{\hat{h}} s^2\gamma_\delta(s)\mathrm{d}s \sum_j \mu_j^{-1}\Big)^{\frac{1}{2}}. \tag{37}$$

Therefore, we have

$$\begin{aligned} |||u_I - u_h|||^2 &= B_h(u_I - u_h, u_I - u_h) \\ &= B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) \\ &\le |||u_I - u|||\,|||u_I - u_h||| + \mathcal{C}(u, u_I - u_h) \\ &\le |||u_I - u|||\,|||u_I - u_h||| + \beta(\hat{h},\delta)|||u_I - u_h|||. \end{aligned} \tag{38}$$

In (38), we have used the relation in Lemma 4.3 that

$$B(u, v_h) = (f_\delta, v_h) = B_h(u_h, v_h), \quad \forall\, v_h \in V_h.$$

Then, by (38) and the triangle inequality, we obtain

$$\begin{aligned} |||u - u_h||| &\le |||u - u_I||| + |||u_I - u_h||| \\ &\le 2\,|||u_I - u||| + \beta(\hat{h},\delta). \end{aligned} \tag{39}$$

As $\mu_j = O(h_j^{-2k-1})$, then it implies $\beta(\hat{h},\delta) \le Ch^k\,\|u\|_{H^2}$. This completes the proof.

From (37), it is easy to find that different μ_j will lead to different estimates for $\beta(\hat{h},\delta)$. We state it in the following corollary.

Corollary 4.2 *Under the conditions in Theorem 4.1, if the kernel $\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$ and δ be fixed, we have*

$$\beta(\hat{h},\delta) \le C(\delta)h^{k+1},$$

and consequently plugging it into (39) gives

$$|||u - u_h||| \le C(\delta)h^{k+1}.$$

If $s\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$, we have

$$\beta(\hat{h},\delta) \le C(\delta)h^{k+\frac{1}{2}},$$

which then gives

$$|||u - u_h||| \le C(\delta)h^{k+\frac{1}{2}}.$$

The asymptotic compatibility is a nice property to have, since the solutions of some numerical methods may converge to the wrong limits if one let $h_j, \delta \to 0$ [33]. For the scheme that is asymptotically compatible, the numerical solution will converge to the exact solution of the local problem when $h_j, \delta \to 0$ simultaneously, i.e.,

$$\|u_h - u_{\text{loc}}\|_{L^2} \to 0, \quad \text{as } h_j, \delta \to 0,$$

where u_{loc} is the exact solution to the local counterpart of the problem (1). For more details, one may refer to, e.g., [34]. To obtain the asymptotic compatibility of the DG scheme (11), we need a known result (e.g., see [13]) that

$$\|u - u_{\text{loc}}\|_{L^2} \to 0, \quad \text{as } \delta \to 0. \tag{40}$$

Meanwhile, we further assume that there exists $C_0 > 0$ is a constant such that

$$\lim_{\delta\to 0} \|u\|_{H^{k+1}} \text{ exists and} \lim_{\delta\to 0} \|u\|_{H^{k+1}} \le C_0. \tag{41}$$

Corollary 4.3 *Assume* (41) *holds. Under the conditions in Theorem* 4.1, *the scheme* (11) *is asymptotically compatible.*

Proof From (38), we have

$$|||u_I - u_h||| \le |||u_I - u||| + \beta(\hat{h}, \delta) \le Ch^k \|u\|_{H^{k+1}},$$

where the constant C is independent of δ and h. By the discrete Poincaré's inequality (31), we have $\|u_I - u_h\|_{L^2} \le C\,|||u_I - u_h||| \le Ch^k \|u\|_{H^{k+1}}$. Together with the approximation property (34), we have

$$\|u - u_h\|_{L^2} \le \|u - u_I\|_{L^2} + |||u_I - u_h||| \le Ch^k \|u\|_{H^{k+1}}.$$

Together with (40) and (41), we then obtain the desired result.

Remark 4.1 In the local case, to obtain the error estimates in the L^2 norm, we need to utilize the dual problem in the derivation. However, since the nonlocal problem does not have the elliptic regularity theory as in the local case, any extra regularity of the exact solution in (1) is not expected; thus the duality argument may not work for general nonlocal diffusion problems. On the other hand, when δ is fixed and $\gamma_\delta(s)$ is integrable, the energy norm $|||\cdot|||$ is equivalent to the L^2 norm, which leads to the error estimates in the L^2 norm.

4.3 L^2-Stability and a Priori Error Estimates for the Semi-discrete DG Scheme (12)

Without loss of generality, let us set $f_\delta = 0$ in (12). By taking $v_h = u_h$ in (12) and using (31), we then obtain

$$((u_h)_t, u_h) = -B_h(u_h, u_h) = -|||u_h(\cdot, t)|||^2 \le -C\|u_h(\cdot, t)\|_{L^2}^2,$$

which implies

$$\|u_h(\cdot, T)\|_{L^2}^2 \le \mathrm{e}^{-CT}\|u_h(\cdot, 0)\|_{L^2}^2, \quad \forall\, T > 0.$$

Therefore, the numerical solution $u_h \to 0$ when the final time $T \to \infty$.

Next, we study a priori error estimates under the assumption that u is smooth enough. For simplicity, we use the notations as follows:

$$\bar{e}_h = e_h + \varepsilon_h, \quad e_h = u_I - u_h, \quad \varepsilon_h = u - u_I,$$

where $u_I \in V_h$ is an approximation to the exact solution u. With (5) and (12), we get the following error equation:

$$((\bar{e}_h)_t, v_h) + B_h(\bar{e}_h, v_h) + C(u, v_h) = 0, \quad \forall v_h \in V_h, \tag{42}$$

where $C(u, v_h)$ is the inconsistent term given in (16).

By taking $v_h = e_h$ in (42), we have

$$((e_h)_t, e_h) + B_h(e_h, e_h) = -((\varepsilon_h)_t, e_h) - B_h(\varepsilon_h, e_h) - C(u, e_h). \tag{43}$$

Let $u_I \in V_h$ be a suitable interpolant of the exact solution u so that the approximation property (34) holds. By the Cauchy–Schwarz inequality and approximation property, we then have

$$((\varepsilon_h)_t, e_h) \le \|(\varepsilon_h)_t\|_{L^2} \, \|e_h\|_{L^2}, \tag{44}$$

$$B_h(\varepsilon_h, e_h) \le |||\varepsilon_h||| \, |||e_h|||. \tag{45}$$

By plugging (36), (44), and (45) into (43) and using the Cauchy–Schwarz inequality and (31), we can obtain

$$\begin{aligned} \frac{\mathrm{d}}{2\mathrm{d}t}\|e_h\|_{L^2}^2 &\le -B_h(e_h, e_h) + \|(\varepsilon_h)_t\|_{L^2}\,\|e_h\|_{L^2} + |||\varepsilon_h|||\,|||e_h||| + \beta(\hat{h}, \delta)\,|||e_h||| \\ &\le -C\|e_h\|_{L^2}^2 + C(\|(\varepsilon_h)_t\|_{L^2}^2 + |||\varepsilon_h|||^2 + \beta(\hat{h}, \delta)^2). \end{aligned} \tag{46}$$

Thus, we obtain by applying Gronwall's inequality with (46)

$$\|e_h\|_{L^2} \le Ch^k \, \|u\|_{H^{k+1}}.$$

By the triangle inequality, finally we have

$$\|\bar{e}_h\|_{L^2} \le \|\varepsilon_h\|_{L^2} + \|e_h\|_{L^2} \le Ch^k \, \|u\|_{H^{k+1}}.$$

Table 1 L^2 errors and convergence orders produced by the DG scheme (11) when $k = 1$ in Example 1

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha = 1/2$	8	7.03E−02	–	2.53E−02	–	7.88E−02	–	2.54E−02	–
	16	1.74E−02	2.01	3.22E−03	2.97	1.98E−02	1.99	3.91E−03	2.70
	32	4.34E−03	2.00	7.32E−04	2.13	4.97E−03	2.00	7.76E−04	2.33
	64	1.08E−03	2.00	1.80E−04	2.02	1.24E−03	2.00	1.82E−04	2.09
	128	2.71E−04	2.00	4.50E−05	2.00	3.11E−04	2.00	4.50E−05	2.02
	256	6.78E−05	2.00	1.12E−05	2.00	7.77E−05	2.00	1.12E−05	2.00
	512	1.69E−05	2.00	2.81E−06	2.00	1.94E−05	2.00	2.81E−06	2.00
$\alpha = 5/2$	8	7.03E−02	–	5.83E−02	–	7.48E−02	–	5.84E−02	–
	16	1.74E−02	2.01	1.06E−02	2.46	1.86E−02	2.01	1.25E−02	2.23
	32	4.34E−03	2.00	1.96E−03	2.44	4.64E−03	2.00	2.66E−03	2.23
	64	1.08E−03	2.00	3.70E−04	2.41	1.16E−03	2.00	5.68E−04	2.23
	128	2.71E−04	2.00	7.28E−05	2.35	2.90E−04	2.00	1.22E−04	2.22
	256	6.78E−05	2.00	1.51E−05	2.27	7.24E−05	2.00	2.63E−05	2.21
	512	1.69E−05	2.00	3.33E−06	2.18	1.81E−05	2.00	5.74E−06	2.20

Table 2 L^2 errors and convergence orders produced by the DG scheme (11) when $k = 2$ in Example 1

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha = 1/2$	8	1.28E−02	–	4.83E−03	–	1.23E−02	–	4.85E−03	–
	16	8.27E−04	3.96	3.03E−04	3.99	7.90E−04	3.96	3.13E−04	3.95
	32	5.95E−05	3.80	3.51E−05	3.11	5.73E−05	3.79	3.52E−05	3.15
	64	5.26E−06	3.50	4.32E−06	3.02	5.16E−06	3.47	4.32E−06	3.02
	128	5.70E−07	3.21	5.38E−07	3.01	5.66E−07	3.19	5.38E−07	3.01
	256	6.83E−08	3.06	6.72E−08	3.00	6.81E−08	3.06	6.72E−08	3.00
	512	8.43E−09	3.02	8.40E−09	3.00	8.43E−09	3.01	8.40E−09	3.00
$\alpha = 5/2$	8	1.28E−02	–	6.67E−03	–	1.23E−02	–	6.69E−03	–
	16	8.27E−04	3.96	3.26E−04	4.36	7.93E−04	3.96	3.66E−04	4.19
	32	5.95E−05	3.80	3.52E−05	3.21	5.75E−05	3.78	3.59E−05	3.35
	64	5.26E−06	3.50	4.32E−06	3.03	5.17E−06	3.48	4.33E−06	3.05
	128	5.70E−07	3.21	5.38E−07	3.01	5.67E−07	3.19	5.38E−07	3.01
	256	6.83E−08	3.06	6.72E−08	3.00	6.81E−08	3.06	6.72E−08	3.00
	512	8.43E−09	3.02	8.40E−09	3.00	8.43E−09	3.01	8.40E−09	3.00

Table 3 Errors and convergence orders of $|||u_I - u_h|||$ produced by the DG scheme (11) when $k = 1$ in Example 1

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		$\vert\vert\vert u_I - u_h\vert\vert\vert$	Order	$\vert\vert\vert u_I - u_h\vert\vert\vert$	Order	$\vert\vert\vert u_I - u_h\vert\vert\vert$	Order	$\vert\vert\vert u_I - u_h\vert\vert\vert$	Order
$\alpha = 1/2$	8	4.84E−02	–	5.03E−02	–	7.39E−02	–	5.04E−02	–
	16	1.20E−02	2.02	1.15E−02	2.97	1.93E−02	1.94	1.19E−02	2.08
	32	2.98E−03	2.00	2.85E−03	2.13	4.87E−03	1.98	2.90E−03	2.04
	64	7.45E−04	2.00	7.11E−04	2.02	1.22E−03	2.00	7.17E−04	2.02
	128	1.86E−04	2.00	1.78E−04	2.00	3.05E−04	2.00	1.78E−04	2.01
	256	4.65E−05	2.00	4.45E−05	2.00	7.64E−05	2.00	4.45E−05	2.00
	512	1.16E−05	2.00	1.11E−05	2.00	1.91E−05	2.00	1.11E−05	2.00
$\alpha = 5/2$	8	4.84E−02	–	5.52E−02	–	6.10E−02	–	5.53E−02	–
	16	1.20E−02	2.02	1.29E−02	2.10	1.56E−02	1.97	1.35E−02	2.04
	32	2.98E−03	2.00	3.08E−03	2.07	3.91E−03	1.99	3.26E−03	2.05
	64	7.45E−04	2.00	7.46E−04	2.04	9.79E−04	2.00	7.90E−04	2.04
	128	1.86E−04	2.00	1.83E−04	2.03	2.45E−04	2.00	1.93E−04	2.03
	256	4.65E−05	2.00	4.53E−05	2.02	6.12E−05	2.00	4.74E−05	2.03
	512	1.16E−05	2.00	1.12E−05	2.01	1.53E−05	2.00	1.17E−05	2.02

Theorem 4.2 *Let $u_h(\cdot, t) \in V_h$ be the approximate solution generated from the semi-discrete DG scheme (12), with the finite element space V_h defined in (7) and $k \geq 1$. Then, we have the following stability result:*

$$\|u_h(\cdot, T)\|_{L^2} \leq \mathrm{e}^{-CT}\|u_h(\cdot, 0)\|_{L^2}, \quad \forall\, T \geq 0.$$

Table 4 Errors and convergence orders of $|||u_I - u_h|||$ produced by the DG scheme (11) when $k = 2$ in Example 1

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		$|||u_I - u_h|||$	Order	$|||u_I - u_h|||$	Order	$|||u_I - u_h|||$	Order	$|||u_I - u_h|||$	Order
$\alpha = 1/2$	8	2.93E−03	–	1.27E−02	–	2.30E−02	–	1.27E−02	–
	16	3.74E−04	2.97	1.19E−03	3.41	4.81E−03	2.26	1.69E−03	2.90
	32	4.70E−05	2.99	7.47E−05	3.99	7.37E−04	2.71	1.73E−04	3.29
	64	5.88E−06	3.00	4.73E−06	3.98	9.85E−05	2.90	1.27E−05	3.77
	128	7.35E−07	3.00	2.70E−07	4.13	1.26E−05	2.97	1.12E−06	3.50
	256	9.19E−08	3.00	1.69E−08	4.00	1.59E−06	2.99	1.25E−07	3.17
	512	1.15E−08	3.00	9.74E−10	4.11	1.99E−07	3.01	9.42E−09	3.73
$\alpha = 5/2$	8	2.93E−03	–	1.12E−02	–	1.16E−02	–	1.12E−02	–
	16	3.74E−04	2.97	1.51E−03	2.89	1.94E−03	2.58	1.63E−03	2.78
	32	4.70E−05	2.99	1.72E−04	3.14	2.67E−04	2.86	2.04E−04	3.00
	64	5.88E−06	3.00	1.86E−05	3.21	3.44E−05	2.96	2.39E−05	3.09
	128	7.35E−07	3.00	1.97E−06	3.24	4.35E−06	2.99	2.78E−06	3.11
	256	9.19E−08	3.00	2.08E−07	3.25	5.45E−07	2.99	3.20E−07	3.12
	512	1.15E−08	3.00	2.19E−08	3.25	6.83E−08	3.00	3.68E−08	3.12

Moreover, assume that the exact solution $u(\cdot, t)$ of the problem (2) is smooth enough, then we have the following error estimate:

$$\|u(\cdot, T) - u_h(\cdot, T)\|_{L^2} \leq Ch^k \, \|u\|_{H^{k+1}}.$$

Table 5 L^2 errors at the final time $T = 1$ and convergence orders produced by the semi-discrete DG scheme (12) when $k = 1$ in Example 2

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha = 1/2$	8	9.46E−03	–	5.07E−03	–	9.69E−03	–	5.08E−03	–
	16	2.47E−03	1.94	1.08E−03	2.23	2.51E−03	1.95	1.11E−03	2.20
	32	6.24E−04	1.99	2.65E−04	2.02	6.34E−04	1.98	2.67E−04	2.05
	64	1.56E−04	2.00	6.61E−05	2.00	1.59E−04	1.99	6.62E−05	2.01
	128	3.91E−05	2.00	1.65E−05	2.00	3.99E−05	2.00	1.65E−05	2.00
	256	9.77E−06	2.00	4.13E−06	2.00	1.00E−05	2.00	4.13E−06	2.00
	512	2.44E−06	2.00	1.03E−06	2.00	2.50E−06	2.00	1.03E−06	2.00
$\alpha = 5/2$	8	9.46E−03	–	7.59E−03	–	9.01E−03	–	7.60E−03	–
	16	2.47E−03	1.94	1.54E−03	2.30	2.25E−03	2.00	1.69E−03	2.17
	32	6.24E−04	1.99	3.30E−04	2.22	5.64E−04	2.00	3.83E−04	2.14
	64	1.56E−04	2.00	7.46E−05	2.14	1.41E−04	2.00	8.81E−05	2.12
	128	3.91E−05	2.00	1.76E−05	2.08	3.53E−05	2.00	2.06E−05	2.10
	256	9.77E−06	2.00	4.27E−06	2.05	8.83E−06	2.00	4.87E−06	2.08
	512	2.44E−06	2.00	1.05E−06	2.02	2.21E−06	2.00	1.17E−06	2.06

Table 6 L^2 errors at the final time $T = 1$ and convergence orders produced by the semi-discrete DG scheme (12) when $k = 2$ in Example 2

	N	$\delta = 10^{-12}\pi$		$\delta = \pi/5$		$\delta = h$		$\delta = \sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha = 1/2$	8	1.64E−03	–	7.74E−04	–	1.46E−03	–	7.74E−04	–
	16	1.34E−04	3.61	9.82E−05	2.98	1.22E−04	3.58	9.82E−04	2.98
	32	1.37E−05	3.29	1.27E−05	2.96	1.31E−05	3.22	1.27E−05	2.96
	64	1.62E−06	3.09	1.60E−06	2.99	1.59E−06	3.05	1.60E−06	2.99
	128	2.00E−07	3.01	2.00E−07	2.99	1.99E−07	3.00	2.00E−07	2.99
	256	2.50E−08	3.00	2.51E−08	3.00	2.50E−08	2.99	2.51E−08	3.00
	512	3.13E−09	3.00	3.14E−09	3.00	3.13E−09	3.00	3.14E−09	3.00
$\alpha = 5/2$	8	1.64E−03	–	1.27E−03	–	1.52E−03	–	1.27E−03	–
	16	1.34E−04	3.61	1.07E−04	3.56	1.27E−04	3.58	1.11E−04	3.52
	32	1.37E−05	3.29	1.27E−05	3.08	1.34E−05	3.06	1.28E−05	3.12
	64	1.62E−06	3.09	1.59E−06	3.00	1.60E−06	3.06	1.59E−06	3.01
	128	2.00E−07	3.01	2.00E−07	2.99	1.99E−07	3.01	2.00E−07	2.99
	256	2.50E−08	3.00	2.50E−08	3.00	2.50E−08	3.00	2.50E−08	3.00
	512	3.13E−09	3.00	3.13E−09	3.00	3.13E−09	3.00	3.12E−09	3.00

Corollary 4.4 *Under the conditions in Theorem 4.2 and δ is fixed, then it holds that when the kernel $\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$,*

$$\|u(\cdot,T) - u_h(\cdot,T)\|_{L^2} \le C(\delta)\, h^{k+1},$$

and when $s\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$,

$$\|u(\cdot,T) - u_h(\cdot,T)\|_{L^2} \le C(\delta)\, h^{k+\frac{1}{2}}.$$

5 Numerical Experiments

In this section, we present some numerical experiments to verify the theoretical results. The kernel function for all examples is chosen to be

$$\gamma_\delta(s) = \frac{3-\alpha}{2\delta^{3-\alpha}}|s|^{-\alpha}, \quad 0 < \alpha < 3,$$

which gives $\int_{-\delta}^{\delta} s^2\gamma_\delta(s)\mathrm{d}s = 1$. For simplicity, we use uniform partitions for the problem domains, i.e., $h_j = h$, and $\hat{h} = h$. The penalty parameter is taken to be $\mu_j \equiv \mu = 3/h^{2k+1}$ if not noted otherwise. For all examples, we consider four cases for δ:

$$\delta = 10^{-12}\pi, \quad \delta = \pi/5, \quad \delta = h, \quad \delta = \sqrt{h}.$$

The latter two cases are used to test the asymptotic compatibility of the proposed DG schemes. We also test two different values for α: $\alpha = 1/2$ and $\alpha = 5/2$. It is easy to see $\gamma_\delta(s) \in L^1_{\mathrm{loc}}(\mathbb{R})$ when $\alpha = 1/2$.

Table 7 L^2 errors and convergence orders produced by the DG scheme (11) when $k=1$, $\mu_j \equiv \mu = 3h^{-1}$ in Example 3

	N	$\delta=10^{-12}\pi$		$\delta=\pi/5$		$\delta=h$		$\delta=\sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha=1/2$	8	1.63E−01	–	4.64E−02	–	1.72E−01	–	4.67E−02	–
	16	1.71E−01	− 0.07	1.16E−02	2.00	1.74E−01	− 0.02	1.92E−02	1.28
	32	1.74E−01	− 0.02	4.37E−03	1.41	1.74E−01	− 0.00	7.99E−03	1.27
	64	1.74E−01	− 0.00	1.83E−03	1.26	1.74E−01	− 0.00	3.36E−03	1.25
	128	1.74E−01	− 0.00	7.60E−04	1.27	1.74E−01	− 0.00	1.44E−03	1.22
	256	1.74E−01	− 0.00	3.07E−04	1.31	1.74E−01	− 0.00	6.23E−04	1.21
	512	1.74E−01	− 0.00	1.21E−04	1.35	1.74E−01	− 0.00	2.71E−04	1.20
$\alpha=5/2$	8	1.63E−01	–	1.26E−01	–	1.68E−01	–	1.26E−01	–
	16	1.71E−01	− 0.07	8.52E−02	0.56	1.73E−01	− 0.04	1.04E−01	0.28
	32	1.74E−01	− 0.02	5.77E−02	0.56	1.74E−01	− 0.01	8.44E−02	0.30
	64	1.74E−01	− 0.00	3.95E−02	0.55	1.74E−01	− 0.00	6.90E−02	0.29
	128	1.74E−01	− 0.00	2.73E−02	0.53	1.74E−01	− 0.00	5.67E−02	0.28
	256	1.74E−01	− 0.00	1.90E−02	0.52	1.74E−01	− 0.00	4.68E−02	0.28
	512	1.74E−01	− 0.00	1.33E−02	0.52	1.74E−01	− 0.00	3.88E−02	0.27

Table 8 L^2 errors and convergence orders produced by the DG scheme (11) when $k=2$, $\mu_j \equiv \mu = 3h^{-1}$ in Example 1

	N	$\delta=10^{-12}\pi$		$\delta=\pi/5$		$\delta=h$		$\delta=\sqrt{h}$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\alpha=1/2$	8	1.77E−01	–	4.77E−02	–	1.74E−01	–	4.80E−02	–
	16	1.75E−01	0.02	1.16E−02	2.04	1.74E−01	− 0.00	1.93E−02	1.31
	32	1.74E−01	0.00	4.37E−03	1.41	1.74E−01	− 0.00	7.99E−03	1.27
	64	1.74E−01	0.00	1.83E−03	1.26	1.74E−01	− 0.00	3.36E−03	1.25
	128	1.74E−01	0.00	7.59E−04	1.27	1.74E−01	− 0.00	1.44E−03	1.22
	256	1.74E−01	0.00	3.07E−04	1.31	1.74E−01	− 0.00	6.23E−04	1.21
	512	1.74E−01	0.00	1.21E−04	1.35	1.74E−01	− 0.00	2.71E−04	1.20
$\alpha=5/2$	8	1.77E−01	–	1.32E−01	–	1.76E−01	–	1.32E−01	–
	16	1.75E−01	0.02	8.61E−02	0.61	1.75E−01	0.01	1.05E−01	0.33
	32	1.74E−01	0.00	5.78E−02	0.57	1.74E−01	0.00	8.46E−02	0.31
	64	1.74E−01	0.00	3.95E−02	0.55	1.74E−01	0.00	6.90E−02	0.29
	128	1.74E−01	0.00	2.73E−02	0.53	1.74E−01	0.00	5.67E−02	0.28
	256	1.74E−01	0.00	1.90E−02	0.52	1.74E−01	0.00	4.68E−02	0.28
	512	1.74E−01	0.00	1.33E−02	0.52	1.74E−01	0.00	3.88E−02	0.27

Example 1 For the steady-state problem (1), we take the source term as

$$f_\delta(x) = -2\int_{-\delta}^{\delta} \gamma_\delta(s)(g(x+s)-g(x))\,\mathrm{d}s, \quad x \in (0,\pi),$$

where $g(x)$ is defined by

$$g(x) = \begin{cases} \sin^4(x), & x \in (0, \pi), \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, the exact solution is $u(x) = g(x)$.

Numerical results for Example 1 computed by the DG scheme (11) for the steady-state problem are reported in Tables 1, 2, 3, and 4. It is observed that the DG scheme achieves the optimal convergence of order $k + 1$ for the P^k-element ($k = 1, 2$) in the L^2 norm for all cases, even though the kernel function γ_δ is not integrable when $\alpha = 5/2$. Besides, we also compute $|||u_I - u_h|||$ where u_I is the continuous interpolation of the exact solution u and report the errors and convergence orders in Tables 3 and 4. In particular, the results with $\delta = h$ and $\delta = \sqrt{h}$ verify that the DG scheme is asymptotically compatible.

Example 2 For the time-dependent problem (2), we set the initial condition to be

$$u_0(x) = g(x), \quad x \in (0, \pi),$$

and the source term f_δ to be

$$f_\delta(x, t) = -\mathrm{e}^{-t} g(x) - 2\mathrm{e}^{-t} \int_{-\delta}^{\delta} \gamma_\delta(s)(g(x + s) - g(x))\mathrm{d}s, \qquad x \in (0, \pi), \quad t > 0,$$

where $g(x)$ is defined in Example 1. Thus, the exact solution is $u(x, t) = \mathrm{e}^{-t} g(x)$. We adopt the third-order singly diagonal implicit Runge–Kutta (SDIRK) method with two stages as the time-stepping method [22, 23]. We take the time step size $\tau = h/\pi$ and the final time is $T = 1$.

Numerical results for Example 2 computed by the semi-discrete DG scheme (12) for the time-dependent problem are reported in Tables 5 and 6. We again observe the similar convergence behavior as that in Example 1.

Example 3 Our last example is to reconsider Example 1 with the penalty parameter $\mu_j = O(h^{-1})$, instead of the $\mu_j = O(h^{-2k-1})$.

Tables 7 and 8 report the numerical results produced by the DG scheme (11) with the penalty parameter $\mu_j = O(h^{-1})$. From the results, we can see that the L^2 errors heavily depend on the penalty terms and also depend on the choices of δ and α in $\gamma_\delta(s)$ given in the beginning of this section. It can be explained by the estimates of $\beta(\hat{h}, \delta)$ in (37) when different $\gamma_\delta(s)$ and μ_j are chosen. Therefore, although we have the stability of the DG scheme (11) when $\mu_j > 0$, the accuracy could degenerate if μ_j is not large enough.

6 Concluding Remarks

In this paper, we propose and analyze a new discontinuous Galerkin (DG) method for one-dimensional nonlocal diffusion problems. We firstly identify the term in the nonlocal integral that requires particular treatment. We then introduce a jump into this term to overcome the singularity of the kernel function. Since the resulting formulation is not consistent, we add a penalty term in the proposed DG scheme to control the inconsistency error. Based on the dual problem, we prove the Poincaré's inequality at the discrete level for the proposed scheme, which plays an important role in further stability and error analysis. For the steady-state ND problem, we obtain the stability, boundedness, and a priori error estimates of the DG scheme. In particular, the error estimates imply the DG scheme is asymptotically compatible. For the time-dependent ND problem, we establish the L^2-stability and a priori error estimates of the semi-discrete DG scheme. Numerical experiments show that the proposed DG schemes achieve the optimal order of convergence.

Considering the new contributions in the current work and studies in the past, we have developed two types of DG schemes for the ND problems: one is to use auxiliary variables [14] and the other is the one with penalty developed here. Since these two DG schemes both have their respective limitations, it is desirable to develop a unified framework for the DG schemes for the ND problem in the future work, as in [1] for local models. In addition, the present work is focused on one-dimensional ND problems, and its extension to multidimensional problems remains an interesting ongoing research work.

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