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Structurally stable families of periodic solutions in sweeping processes of networks of elastoplastic springs

Ivan Gudoshnikov, Oleg Makarenkov

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Abstract Networks of elastoplastic springs (elastoplastic systems) have been linked to differential equations with polyhedral constraints in the pioneering paper by Moreau (1974). Periodic loading of an elastoplastic system, therefore, corresponds to a periodic motion of the polyhedral constraint. According to Krejci (1996), every solution of a sweeping process with a periodically moving constraint asymptotically converges to a periodic orbit. Understanding whether such an asymptotic periodic orbit is unique or there can be an entire family of asymptotic periodic orbits (that form a periodic attractor) has been an open problem since then. Since suitable small perturbation of a polyhedral constraint seems to be always capable to destroy a potential family of periodic orbits, it is expected that none of potential periodic attractor is structurally stable. In the present paper we give a simple example to prove that even though the periodic attractor (of non-stationary periodic solutions) can be destroyed by a little perturbation of the moving constraint, the periodic attractor resists perturbations of the physical parameters of the mechanical model (i.e. the parameters of the network of elastoplastic springs).

Keywords Elastoplastic springs · lattice spring model · sweeping process · structural stability · cyclic loading · uniqueness of periodic response

Mathematics Subject Classification (2000) 34A36 · 37G15 · 74C15

O. Makarenkov
Department of Mathematical Sciences, University of Texas at Dallas, 75080 Richardson, USA
E-mail: makarenkov@utdallas.edu

1 Introduction

Networks of elastoplastic springs are increasingly used in the modeling of the distribution of stresses in elastopastic media [4,5], swarming of mobile router networks [2,14], and other physical phenomena. According to Moreau [12], the stresses of springs of such a network can be described by a differential inclusion (*Moreau sweeping process*)

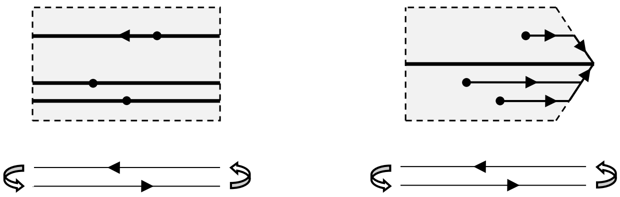
$$-y'(t) \in N_{C(t)}(y(t)), \quad y(t) \in \mathbb{R}^m, \tag{1}$$

where $C(t) \subset \mathbb{R}^m$ is a closed polyhedron that plays the role of a constraint,

$$N_C(x) = \begin{cases} \{\zeta \in \mathbb{R}^n : \langle \zeta, c - x\rangle \leqslant 0, \;\; c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

and the dimension m equals or smaller than the number of springs in the network.

Periodicity of the constraint $C(t)$ corresponds to periodicity of the external loading applied to the given network of springs. The fundamental result by Krejci [10, Theorem 3.14] says that for $C(t)$ of the form $C(t) = \mathcal{C} + c(t)$, where $\mathcal{C}$ is a convex closed bounded set and $t \mapsto c(t)$ is a T-periodic vector-function, any solution of sweeping process (1) converges to some T-periodic regime. For a class of continuum elastoplastic media subjected to a T-periodic loading the uniqueness of T-periodic response is established in Frederick-Armstrong [7, p. 159]. Sufficient conditions for the uniqueness of the response in sweeping processes can be drawn based on Adly et al [1]. The non-uniqueness of the response for sweeping processes can of course be easily designed, see Fig. 1a, where one gets a family of periodic solutions by moving a rectangle back and forth orthogonal to its sides. However, as shown at Fig. 1b, small perturbation of such a rectangle destroys the attracting family of orbits of Fig. 1a leaving only a single attracting solution.



(a) (b)

Fig. 1 Sample trajectories (solid curves) of Moreau sweeping process with a moving constraint (dashed rectangle) that moves back and forth. The bold points are the initial conditions of sample trajectories. The figure illustrates the type of attractor (solid black curves) when (a) the moving constraint is just a rectangle, (b) the moving constraint is a pentagon with a corner that accumulates all the trajectories.

That is why a natural question arises:

> whether or not any network of elastoplastic springs can always be slightly perturbed in a way that destroys any potential family of periodic orbits in the respective sweeping process (1)?

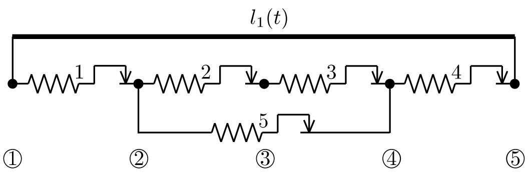


Fig. 2 A one-dimensional network of 5 springs on 5 nodes with one displacement-controlled loading. The circled digits stand for indexes of nodes. The regular digits are the indexes of springs. The thick bar is the displacement-controlled loading $l_1(t)$. The stress-controlled loadings $f_1(t), ..., f_5(t)$ are applied at nodes.

As uniqueness of the response lies in the core of reliability of modeling prediction (see e.g. [3,15]), the above-stated question is not of merely academic value. We introduce a simple example that answers this question negatively. Specifically, we show that the cyclically loaded network of elastoplastic springs of Fig. 2 leads to a sweeping process with a family of attracting periodic orbits that persists under perturbations of the physical parameters of the network.

The paper is organized as follows. In the next section we define a network of elastoplastic springs formally. In section 3 we derive a sweeping process (1) that governs the quasi-static evolution of such a network. Section 4 establishes a condition for periodic loading that rules out the existence of stationary solutions (i.e. solutions that are constant in time). Section 5 is based on Moreau [12] and Gudoshnikov-Makarenkov [9]. It compiles a guide for closed-form computation of the quantities required for construction of a sweeping process of a given network of elastoplastic springs. This guide is then used in Section 6 to construct the sweeping process of the network of elastoplastic springs of Fig. 2. We rigorously prove (Proposition 2 and Corollary 1) that such a sweeping process admits a family of periodic orbits that persists under perturbations of the mechanical parameters of the network.

2 A concise definition of a general network of elastoplastic springs

We consider a network of m elastoplastic springs on n nodes that are connected according to a directed graph given by the $n \times m$ incidence matrix $-D^T$. In other words, the (i, j)-element of matrix D^T is 1 or -1 according to whether node i is the right end of spring j or the left end. If none of these two cases takes place, then the (i, j)-element of matrix D^T is 0. The Hooke's coefficients $a_1, ..., a_m$ of the springs are arranged into an $m \times m$-matrix $A = \text{diag}\,\{a_1, ..., a_m\}$. The elastic limits $[c_i^-, c_i^+]$ of springs are used to introduce a parallelepiped $C \subset \mathbb{R}^m$ as $C = [c_1^-, c_1^+] \times ... \times [c_m^-, c_m^+]$. In addition the network comes with a collection of stress-controlled and displacement-controlled loadings $\{f_i(t)\}_{i=1}^n$ and $\{l_i(t)\}_{i=1}^q$ respectively. The stress-controlled loadings are applied at the n nodes of the network and are supposed to satisfy the equation of static balance

$$f_1(t) + ... + f_n(t) = 0. \tag{2}$$

With each displacement-controlled loading $l_k(t)$, $k \in \overline{1, q}$, we associate a path of springs and nodes which connects the left node I_k of the constraint k with its right node J_k. In the simplest case, the coordinates of the nodes in the path monotonically increase when the path is followed from node I_k to node J_k, and the length $l_j(t)$ is just the sum of the lengths of the springs of the path (as it happens e.g. in Fig. 2). Such a path can be described by a so-called *incidence vector* $R^k \in \mathbb{R}^m$ whose i-th component is 0 or 1 according on whether spring i is a part of the path or not. For example, the displacement-controlled loading $l_1(t)$ in Fig. 2 admits a path of springs 1, 2, 3, 4 (whose incidence vector is $(1, 1, 1, 1, 0)^T$) and a path of springs 1, 5, 4 (whose incidence vector is $(1, 0, 0, 1, 1)^T$) connecting nodes ① and ⑤. Displacement-controlled loadings of more complex networks may admit a path where the coordinates of the nodes do not increase monotonically (see e.g. [13, Fig. 2]). In general, if one follows a path of a displacement-controlled loading $l_k(t)$ beginning its left node I_k and heading towards its right node J_k, and if ξ_* and ξ_{**} are two successive nodes on this way connected through spring i, then i-th component of the incident vector R^k equals -1 or 1 according to whether $\xi_* > \xi_{**}$ or $\xi_* < \xi_{**}$, see Fig. 3.

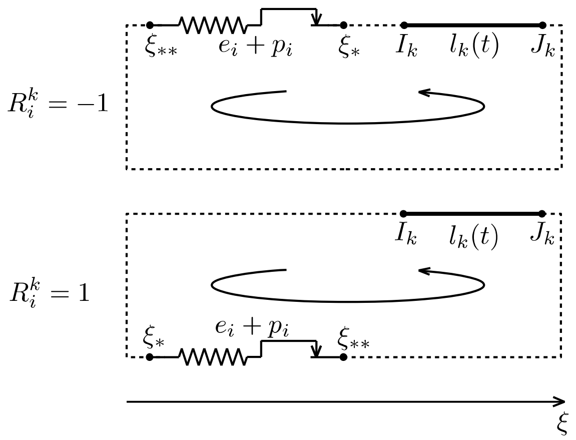


Fig. 3 Illustration of the signs of the components of the incidence vector $R^k \in \mathbb{R}^m$. The dotted contour stays for the chain of the springs associated with the vector R^k.

We assume that the displacement-controlled loadings $\{l_i(t)\}_{i=1}^m$ are independent in the sense that

$$\operatorname{rank}\left(D^T R\right) = q. \tag{3}$$

Mechanically, condition (3) ensures that the displacement-controlled loadings don't contradict one another. For example, (3) rules out the situation where two different displacement-controlled loadings connect same pair of nodes.

3 A concise formulation of the sweeping process of a general network of elastoplastic springs

In this section we follow Moreau [12] (see also Gudoshnikov-Makarenkov [9]). If condition (2) holds, then there exists a function $\bar{h} : \mathbb{R} \to \mathbb{R}^m$, such that

$$f(t) = -D^T \bar{h}(t). \tag{4}$$

Furthermore, condition (3) ensures the existence of an $n \times q-$matrix L, such that

$$R^T D L = I_{q \times q}. \tag{5}$$

Introducing

$$U = \left\{x \in D\mathbb{R}^n : R^T x = 0\right\}, \qquad V = A^{-1} U^\perp, \tag{6}$$

where $U^\perp = \{y \in \mathbb{R}^m : \langle x, y\rangle = 0,\ x \in U\}$, the space V becomes an orthogonal complement of the space U in the sense of the scalar product

$$(u, v)_A = \langle u, Av\rangle. \tag{7}$$

Therefore, any element $x \in \mathbb{R}^m$ can be uniquely decomposed as

$$x = P_U x + P_V x,$$

where P_U and P_V are linear (orthogonal in sense of (7)) projection maps on U and V respectively. Define

$$g(t) = P_V D L l(t), \tag{8}$$
$$h(t) = P_U A^{-1} \bar{h}(t), \tag{9}$$
$$N_C^A(x) = \tag{10}$$
$$= \begin{cases} \{\xi \in \mathbb{R}^m : \langle \xi, A(c - x)\rangle \le 0,\ c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases}$$
$$\Pi(t) = A^{-1} C + h(t) - g(t), \tag{11}$$

Assuming that both $f : \mathbb{R} \to \mathbb{R}^n$ and $l : \mathbb{R} \to \mathbb{R}^q$ are Lipschitz continuous, we get that $h(t)$ and $g(t)$ are Lipschitz continuous as well, so that the function

$$y(t) = A^{-1} s(t) + h(t) - g(t)$$

is Lipschitz continuous for any Lipschitz continuous $t \mapsto s(t)$.

Theorem 1 [12] (see also [9]) *Assume that the network of elastoplastic springs $(D, A, C, R, f(t), l(t))$ of section 2 satisfies conditions (2) and (3). Assume that $h : \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R} \to \mathbb{R}^m$ given by (8)-(9) are Lipschitz continuous. Assume that safe load condition*

$$(C + Ah(t)) \cap U^\perp \neq \emptyset \tag{12}$$

holds on some time interval $[0, T]$. Then, the function $s(t) = (s_1(t), ..., s_m(t))$ defines the evolution of stresses of the network $(D, A, C, R, f(t), l(t))$ for $t \in [0, T]$ if and only if the function

$$y(t) = A^{-1} s(t) + h(t) - g(t)$$

satisfies the differential inclusion (called sweeping process)

$$-\dot{y} \in N^A_{\Pi(t)\cap V}(y), \quad \text{for a.a. } t \in [0, T], \tag{13}$$
$$y(0) \in \Pi(0) \cap V. \tag{14}$$

It remains to note that, for Lipschitz continuous $h : \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R} \to \mathbb{R}^m$, sweeping process (2)-(3) has a unique Lipschitz-continuous solution for any initial condition (see e.g. Kunze and Monteiro Marques [11, sect. 3]).

4 The shakedown condition

The following conditions will rule out the existence of constant solutions.

Proposition 1 [9, Proposition 3] *Assume that conditions of Theorem 1 hold. If*

$$\|A^{-1}c^- - A^{-1}c^+\|_A < \|g(t_1) - g(t_2)\|_A\,, \tag{15}$$

for some $0 \le t_1 < t_2$*, where*

$$\begin{aligned}\|x\|_A &= \sqrt{\langle x, Ax\rangle},\\ c^- &= (c_1^-, ..., c_m^-)^T,\\ c^+ &= (c_1^+, ..., c_m^+)^T.\end{aligned}$$

then sweeping process (13) doesn't have any solutions that are constant on $[t_1, t_2]$.

Proposition 1 is proved in Gudoshnikov-Makarenkov [9], but since [9] has not been published yet, we include a proof of Proposition 1 in the Appendix.

Remark 1 Note, the left-hand-side in inequality (15) can be computed from

$$\|A^{-1}c^- - A^{-1}c^+\|_A^2 = \left\langle c^- - c^+, A^{-1}(c^- - c^+)\right\rangle.$$

5 A step-by-step guide to compute the quantities of the sweeping process from a network of elastoplastic springs

In this section we again follow Moreau [12], but use notations and additional properties as established in Gudoshnikov-Makarenkov [9]. In particular, [9, Lemma 1] and [9, formula (49)] say that

$$\dim U = n - q - 1, \tag{16}$$

$$\dim V = m - n + q + 1, \tag{17}$$

provided that (3) is satisfied.

Step 1. The matrix M. According to (16), there should exist an $n \times (n - q - 1)$−matrix M such that

$$R^T DM = 0 \quad \text{and} \quad \operatorname{rank}(DM) = n - q - 1 \tag{18}$$

which allows to introduce U_{basis} as

$$U_{basis} = DM. \tag{19}$$

Step 2. The matrix V_{basis}. According to (6), V_{basis} is an arbitrary matrix of $m-n+q+1 = \dim V$ linearly independent columns that solves

$$(U_{basis})^T AV_{basis} = 0. \tag{20}$$

Step 3. The matrix $D^\perp$. Define $D^\perp$ to be an $m \times (m-n+1)$−matrix of full rank that solves the equation

$$(D^\perp)^T D = 0_{(m-n+1)\times(m-n+1)}. \tag{21}$$

Step 4. Other quantities. Using Steps 2 and 3, we can define an $(m - n + q + 1) \times q$-matrix $\bar{L}$ as

$$\bar{L} = \left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} V_{basis}\right)^{-1} \begin{pmatrix} I_{q\times q} \\ 0_{(m-n+1)\times q}\end{pmatrix}. \tag{22}$$

It turns out that formula (8) can now be rewritten in closed-form as

$$g(t) = V_{basis}\bar{L}l(t). \tag{23}$$

To account for all possible functions $h(t)$ from (9) we will simply take $h(t)$ as

$$h(t) = U_{basis}H(t), \tag{24}$$

where $H(t)$ is an arbitrary Lipschitz continuous control input. It is possible to compute $H(t)$ in terms of $f(t)$, but it is not of added value here.

Finally, for $\Pi(t) \cap V$ we have

$$\Pi(t) \cap V = \bigcap_{i=1}^{m} V_i(t), \tag{25}$$

with

$$\begin{aligned} V_i(t) = \{x \in V : c_i^- + a_i h_i(t) \le \\ \le \langle n_i, Ax + Ag(t)\rangle \le c_i^+ + a_i h_i(t)\}\,,\end{aligned} \tag{26}$$

where $n_i = P_V e_i$, and $e_i \in \mathbb{R}^m$ is the vector with 1 in the i-th component and zeros elsewhere (standard basis vector in $\mathbb{R}^m$). From $e_i = P_V e_i + P_U e_j$, one has $e_i - n_i \in U$ and since

$$x \in U \quad \text{if and only if} \quad \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} x = 0,$$

the following property holds

$$\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} e_i = \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} n_i, \quad i \in \overline{1,m}.$$

Therefore, n_i, $i \in \overline{1,m}$, can be computed as

$$\begin{aligned} n_i &= V_{basis}\bar{e}_i,\\ \bar{e}_i &= \left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} V_{basis}\right)^{-1} \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} e_i. \end{aligned} \tag{27}$$

6 The sweeping process of the network of elastoplastic springs of Figure 2

The network of elastoplastic springs of Fig. 2 can be described by

$$D\xi = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}, \quad R = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad (28)$$

along with some 5×5 diagonal matrix A of Hooke's coefficients and some intervals $[c_i^-, c_i^+]$, $i \in \overline{1,5}$, of elasticity bounds.

Formulas (16) and (17) lead to

$$\dim U = 5 - 1 - 1 = 3,$$
$$\dim V = 5 - 5 + 1 + 1 = 2.$$

Following Step 3 of section 5, we compute dimensions of matrix $D^\perp$ as $5 \times (5 - 5 + 1) = 5 \times 1$ and the 5×1-dimensional full rank solution of (21) is

$$D^\perp = (0, 1, 1, 0, -1)^T. \quad (29)$$

Using (22), (23), and (27) we get

$$g(t) = V_{basis} \left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} V_{basis} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} l(t), \quad (30)$$

$$n_i = V_{basis} \left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} V_{basis} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix} e_i. \quad (31)$$

Note, formulas (30) and (31) hold for any a_i, $i \in \overline{1,5}$, and any c_i^-, c_i^+, $i \in \overline{1,5}$. Therefore, we see from formulas (30) and (31) that $n_1 \parallel g(t)$ and $n_4 \parallel g(t)$ for any values of the physical parameters of the network of Fig. 2 (the shortcut "$\|$" stays for "parallel"). However, at this point we don't know whether or not the normals n_1 and n_4 have anything to do with the sides of the shape $\Pi(t) \cap V$ given by (25), as it may happen that the constraints of (25) provided by n_1 and n_4 become redundant for a particular $h(t)$, see Fig. 4.

Proposition 2 *There exist an open set of the parameters a_i, c_i^-, c_i^+, $i \in \overline{1,5}$, and an open set of Lipschitz-continuous functions $H : [0,T] \mapsto \mathbb{R}^3$, for which the vectors n_1 and n_4 are the normal vectors of the two opposite sides of the two-dimensional shape $\Pi(t) \cap V$ and the shape $\Pi(t) \cap V$ is a trapezoid. In particular, this open set of the parameters contains the point*

$$c_i^- = -1, \; c_i^+ = 1, \; a_i = 1,$$
$$H(t) \equiv (-0.5, -0.8, -1)^T. \quad (32)$$

Here $[0,T]$ is an arbitrary chosen domain of the functions $t \mapsto H(t)$.

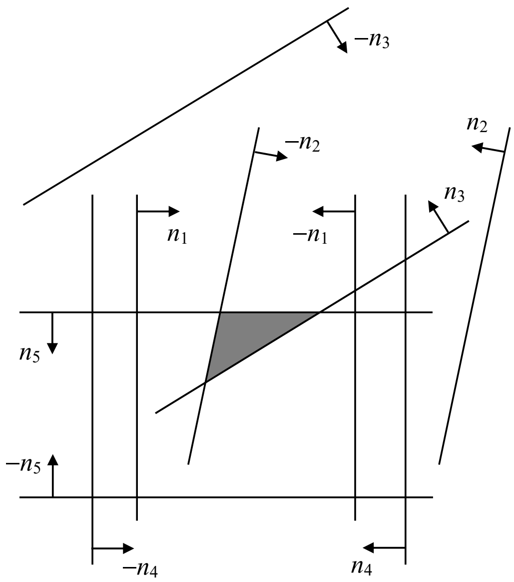


Fig. 4 Possible 2-dimensional shape $\Pi(t) \cap V$ (shaded triangle) as defined by (25)-(26) for the network of elastoplastic springs given by (28). The vectors n_i and $-n_i$ indicate the half-planes $\left\{x \in V : \langle n_i, Ax + Ag(t)\rangle \le c_i^+ + a_i h(t)\right\}$ and $\left\{x \in V : c_i^- + a_i h(t) \le \langle n_i, Ax + Ag(t)\rangle\right\}$ respectively. The figure illustrate a hypothetic choice of $h(t)$ for which normal vectors n_1 and n_4 are not normal vectors of any of the sides of $\Pi(t) \cap V$, even though n_1 and n_4 participate in the formula (26).

Proof. Without loss of generality we can consider $g(t) \equiv 0$. Indeed, since $g(t)$ acts along V, $g(t)$ simply translates $\Pi(t) \cap V$ within V, so that $g(t)$ doesn't change the shape of $\Pi(t) \cap V$.

The $5 \times 3-$matrix M that solves (18) and the respective $5 \times 3-$matrix (19) are found as

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_{basis} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad (33)$$

so that (24) and (33) yield

$$h(t) \equiv (-0.5, -0.3, -0.2, 1, -0.5)^T$$

for $H(t)$ given by (32).

A simple basis V_{basis} that solves (20) can be taken as

$$V_{basis} = \begin{pmatrix} 1/a_1 & 0 \\ 1/a_2 & -1/a_2 \\ 1/a_3 & -1/a_3 \\ 1/a_4 & 0 \\ 0 & 1/a_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (34)$$

that we plug to (31) and obtain the following normal vectors for the parameters (32)

$$
\begin{aligned}
&n_1 = n_4 = \frac{1}{d}\left(-3,-1,-1,-3,-2\right)^T, \\
&n_2 = n_3 = \frac{1}{d}\left(-1,-3,-3,-1,2\right)^T, \\
&n_5 = \frac{1}{d}\left(-2,2,2,-2,-4\right)^T, \\
&d = -8.
\end{aligned}
\tag{35}
$$

Substituting our findings for $h(t)$ and n_i into formula (25) we conclude that for $x \in V$ one has $x \in \Pi(t) \cap V$ if and only if

$$
\begin{cases}
-1-0.5 \leq \langle n_1, x\rangle \leq 1-0.5, \\
-1-0.3 \leq \langle n_2, x\rangle \leq 1-0.3, \\
-1-0.2 \leq \langle n_3, x\rangle \leq 1-0.2, \\
-1+1 \quad \leq \langle n_4, x\rangle \leq 1+1, \\
-1-0.5 \leq \langle n_5, x\rangle \leq 1-0.5.
\end{cases}
\tag{36}
$$

In order to visualize the shape of the polyhedron $\Pi(t) \cap V$ on the plane, we will make the changes of the variables

$$
x = \widetilde{V}_{basis} v, \quad v \in \mathbb{R}^2 \tag{37}
$$

where $\widetilde{V}_{basis}$ is an orthogonal basis in V. To come up with an orthogonal basis $\widetilde{V}_{basis}$ we will amend the first column of basis V_{basis} obtaining

$$
\widetilde{V}_{basis} = \begin{pmatrix}
1/a_1 & 0 \\
a_2a_3^2/(a_2^2a_3^2+a_2^2a_5^2+a_3^2a_5^2) & -1/a_2 \\
a_2^2a_3/(a_2^2a_3^2+a_2^2a_5^2+a_3^2a_5^2) & -1/a_3 \\
1/a_4 & 0 \\
a_5(a_2^2+a_3^2)/(a_2^2a_3^2+a_2^2a_5^2+a_3^2a_5^2) & 1/a_5
\end{pmatrix},
$$

that yields

$$
\widetilde{V}_{basis} = \begin{pmatrix}
1 & 0 \\
1/3 & -1 \\
1/3 & -1 \\
1 & 0 \\
2/3 & 1
\end{pmatrix}, \tag{38}
$$

for parameters (32). Making the change of the variables (37)-(38) in (36) and deleting redundant inequalities, we get

$$
\begin{array}{lrcl}
\text{normal } n_1: & & v_1 & \leq 0.5 \\
\text{normal } n_2: & & (1/3)v_1 - v_2 & \leq 0.7 \\
\text{normal } n_3: & -1.2 \leq & (1/3)v_1 - v_2 & \\
\text{normal } n_4: & 0 \leq & v_1 & \\
\text{normal } n_5: & -1.5 \leq & (2/3)v_1 + v_2 & \leq 0.5.
\end{array}
\tag{39}
$$

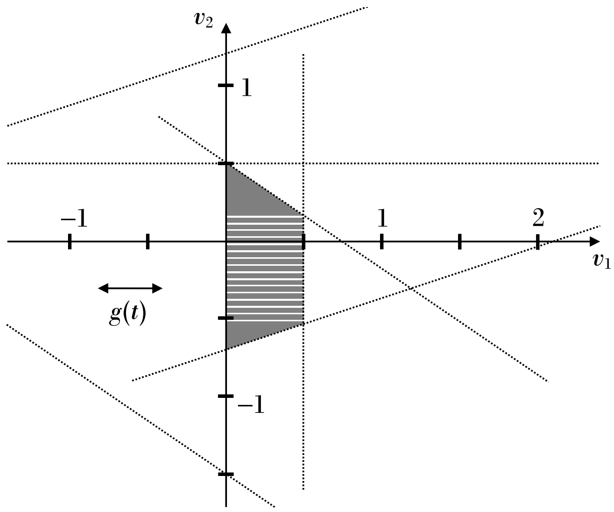


Fig. 5 The shaded region stays for the set of (v_1, v_2) given by inequalities (39). The dotted lines denote the sets of (v_1, v_2) where equalities of (39) are attained (the dotted line $v_1 = 0$ coincides with the vertical axis). The subregion of the shaded region that is textured in stripes is the family of T-periodic solutions of sweeping process (13)-(14) (discussed in Corollary 1).

Fig. 5 and its formulas (39) illustrate that (i) the two constraints from (25) corresponding to normal vectors n_1 and n_4 constitute the opposite sides of shape $\Pi(t) \cap V$, (ii) the shape $\Pi(t) \cap V$ is a trapezoid. These properties persist under small perturbations of the parameters (32) because same inequalities of (36) will stay redundant. The proof of the proposition is complete. □

In order to obtain the existence of a structurally stable family of non-stationary periodic solutions it now remains to apply the displacement-controlled loading (30) of sufficiently large amplitude. We will use Proposition 1 to give an estimate for the required amplitude. In the case of a 5-spring network, condition (15) of Proposition 1 takes the form

$$
\sum_{i=1}^{5} \frac{1}{a_i}\left(c_i^+ - c_i^-\right)^2 < \left\|V_{basis}\bar{L}\right\|_A^2 \cdot \left(l(t_1) - l(t_2)\right)^2. \tag{40}
$$

And in the case of parameters (32), formula (40) further reduces to

$$
\sum_{i=1}^{5} 2^2 < \|n_1\|^2 \cdot \left(l(t_1) - l(t_2)\right)^2,
$$

where n_1 is given by (35), or simply

$$
\frac{160}{3} < \left(l(t_1) - l(t_2)\right)^2.
$$

Since $\sqrt{\frac{160}{3}} \approx 7.3$, we introduce $l(t)$ as follows

$$
l(t) = \begin{cases}
t, & \text{if } t \in [0,8], \\
-t+16, & \text{if } t \in [8,16],
\end{cases}
\tag{41}
$$

extended to $[0,\infty)$ by 16-periodicity.

Corollary 1 *Consider the network of elastoplastic springs of Fig. 2 with the parameters (32). Assume that the displacement-controlled loading is given by (41), so that* $T = 16$. *Then, for any parameters* a_i, c_i^-, c_i^+, $i \in \overline{1,5}$, *and any Lipschitz-continuous functions* T*-periodic* $H : [0,T] \mapsto \mathbb{R}^3$, *that are close to those in (32), and for any Lipschitz-continuous* T*-periodic* $l(t)$ *that is close to (41), the sweeping process (13)-(14) admits a structurally stable family of non-stationary* T*-periodic solutions (created by the parallel sides of Fig. 5 and highlighted in Fig. 5 with a striped rectangle). Accordingly, the mechanical model of Fig. 2 admits a structurally stable family of co-existing stress distributions that evolves* T*-periodically in time.*

7 Conclusions

In this paper we showed that sweeping processes of networks of elastoplastic springs (elastoplastic systems) inherit a designated structure that restrict possible dynamic transitions. Specifically, we gave an example of an elastoplastic system whose sweeping process admits a structurally stable family of non-stationary periodic solutions. Specifically, the structure given by the elastoplastic system locks the family of periodic solutions of the associated sweeping process, so that it persists under all such small perturbations of the sweeping process that come from small perturbations of the physical parameters of the elastoplastic system.

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9 Appendix

The following proof is taken from Gudoshnikov-Makarenkov [9] (see Proposition 3), which is not yet published.

Proof of Proposition 1. The claim follows by showing that

$$(\Pi(t_1) \cap V) \cap (\Pi(t_2) \cap V) = \emptyset$$

Since $h(t) \in U$, then, for any $t \in [t_1, t_2]$,

$$\Pi(t) \cap V = \big(A^{-1}C + h(t) - g(t)\big) \cap V \subset P_V A^{-1}C - g(t),$$

and it is sufficient to prove that the sets

$P_V A^{-1}C - g(t_1)$ and $P_V A^{-1}C - g(t_2)$ don't intersect. The latter will hold, if the diameter of the set $P_V A^{-1}C$ is smaller than the distance between $g(t_1)$ and $g(t_2)$, which fact will now be established.

Since P_V is the orthogonal projection in the sense of the scalar product $(x,y)_A = \langle x, Ay \rangle$, we have (see e.g. Conway [6, Theorem 2.7 b)])

$$\|P_V x\|_A \le \|x\|_A, \quad x \in \mathbb{R}^m.$$

Therefore, for any $c_1, c_2 \in C$,

$$\left\|P_V \left(A^{-1}c_1 - A^{-1}c_2\right)\right\|_A \le \left\|A^{-1}c_1 - A^{-1}c_2\right\|_A \le$$
$$\le \|A^{-1}c^- - A^{-1}c^+\|_A < \|g(t_1) - g(t_2)\|_A .$$

The proof of the proposition is complete. □

Compliance with Ethical Standards

Conflict of Interest: The authors have no conflict of interest.

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