Recent progress on well-quasi-ordering graphs

Chun-Hung Liu

Abstract Graphs are arguably the first objects studied in the field of well-quasi-ordering. Giant successes in research on well-quasi-ordering graphs and fruitful extensions of them have been obtained since Vázsonyi proposed the conjecture about well-quasi-ordering trees by the topological minor relation in the 1940's. In this article, we survey recent development of well-quasi-ordering on graphs and directed graphs by various graph containment relations, including the relations of topological minor, minor, immersion, subgraph, and their variants.

1 Introduction

A *quasi-ordering* on a set X is a reflexive and transitive binary relation on X. A quasi-ordering $\preceq$ on X is a *well-quasi-ordering* if for every infinite sequence $x_1, x_2, \ldots$, there exist $i < i'$ such that $x_i \preceq x_{i'}$. The concept of well-quasi-ordering was discovered from different aspects. One problem that stimulates the development of this concept was raised by Vázsonyi in the 1940's about well-quasi-ordering graphs. Precisely, he conjectured that any infinite collection of trees contains some pair of trees such that one is homeomorphically embeddable in the other (see [31]). This conjecture together with another conjecture of Vázsonyi, which states that subcubic graphs are well-quasi-ordered by the topological minor relation, motivate the study of well-quasi-ordering on graphs. During past decades, giant successes and fruitful extensions were obtained in this direction.

Chun-Hung Liu
Department of Mathematics, Princeton University, Princeton, NJ 08544-1000, USA.
e-mail: chliu@math.princeton.edu.
Partially supported by the National Science Foundation under Grant No. DMS-1664593.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1664593.

One benefit of well-quasi-ordering is the existence of finite characterization of properties closed under well-quasi-orderings. Given a quasi-ordering $Q = (V(Q), \preceq_Q)$, a *Q-ideal* $\mathscr{I}$ is a set of graphs such that if $G \in \mathscr{I}$ and $H \preceq_Q G$, then $H \in \mathscr{I}$. For any graph property that is closed under Q, the set of graphs satisfying this property is a Q-ideal. Let $\mathscr{F}$ be the family of graphs consisting of the minimal graphs that do not belong to $\mathscr{I}$ with respect to Q. Then a graph G belongs to $\mathscr{I}$ if and only if $H \not\preceq_Q G$ for every $H \in \mathscr{F}$. Hence, to describe $\mathscr{I}$, it is sufficient to describe $\mathscr{F}$. Since $\mathscr{F}$ is an antichain with respect to Q, $\mathscr{F}$ must be finite if Q is a well-quasi-ordering. If for any fixed graph $H \in \mathscr{F}$, one can test whether any input graph G satisfies $H \preceq_Q G$ or not in time polynomial in $|V(G)|$, then $\mathscr{I}$ and hence the corresponding graph property can be tested in polynomial time.

The purpose of this article is to survey recent development on well-quasi-ordering on graphs. Though some notions and results mentioned in this article were extended to other combinatorial objects, such as matroids, permutations or words, we focus on results about graphs only for the simplicity.

This paper is organized as follows. We will discuss the topological minor relation, which is the relation stated in Vázsonyi's conjectures, in Section 2. Then we will discuss minor and immersion relations, which are two graph containments closely related to the topological minor relation and attract wide attention, in Sections 3 and 4, respectively. Finally, we will discuss the subgraph relation, which is the most natural containment on graphs, in Section 5.

We start with some formal definitions about graphs. Graphs are finite and possibly have parallel edges and loops in this article. That is, an *(undirected) graph* G consists of a finite set $V(G)$ of vertices and a finite multiset $E(G)$ of 2-element multisubsets of $V(G)$. Each member e of $E(G)$ is called an *edge* of G, and its two elements are called the *ends* of e. Any edge with no two distinct ends is called a *loop*. Two distinct edges are *parallel* if they have the same ends. Loops and parallel edges are considered as cycles of length 1 and 2, respectively. Two vertices are *adjacent* if they are the ends of the same edge. A vertex is *incident* with an edge if it is an end of this edge. The *degree* of a vertex is the number of edges incident with it, where any loop is counted twice. A graph is *subcubic* if every vertex has degree at most three.

A graph is *simple* if it does not contain any loop or parallel edges. A *directed graph* is a graph equipped with an orientation of its edges. Formally, a directed graph consists of a finite set $V(G)$ of vertices and a finite multiset $E(G)$ of ordered pairs of vertices. If $(x,y) \in E(G)$, then we say that x is the *tail* of (x,y) and y is the *head* of (x,y). The *in-degree* (or *out-degree*, respectively) of a vertex v is the number of edges with head (or tail, respectively) v. The *underlying graph* of a directed graph D is a graph obtained from D by removing the direction of the edges. That is, replacing each ordered pair in the edge-set by a 2-element mulitset.

For any positive integer n, $[n]$ denotes the set $\{1,2,...,n\}$. The *complete graph* on n vertices, denoted by K_n, is the simple graph on n vertices with vertices pairwise adjacent. We also call K_3 a *triangle*. And K_n^- denotes the simple graph obtained from K_n by deleting an edge. A *stable set* of a graph is a subset of pairwise non-adjacent vertices. A graph is *bipartite* if it is simple and its vertex-set can be partitioned into two stable sets, and we call this partition a *bipartition*. The *complete bipartite graph*, denoted by $K_{m,n}$ for some positive integers m,n, is the simple bipartite graph with a bipartition whose one part has m vertices and the other part has n vertices such that any pair of vertices belonging to different parts of this bipartition is adjacent. The *4-wheel* W_4 is the simple graph obtained from the cycle of length four by adding a new vertex adjacent to all other vertices. The graph W_4^- is the simple graph obtained from W_4 by deleting an edge not incident with the vertex of degree four. The path and cycle on k vertices are denoted by P_k and C_k, respectively. Given a simple graph G, the *complement of* G is the simple graph with vertex-set $V(G)$ such that any pair of distinct vertices are adjacent if and only if they are non-adjacent in G. A *clique* is a set of pairwise adjacent vertices. A *split graph* is a simple graph whose vertex-set can be partitioned into a clique and a stable set. Given a collection $\mathscr{X}$ of sets, the *intersection graph* of $\mathscr{X}$ is the simple graph with vertex-set $\mathscr{X}$, and two distinct vertices $S,T \in \mathscr{X}$ are adjacent if and only if $S \cap T \neq \emptyset$. Given two graphs G,H, $G \cup H$ denotes the graph that is a disjoint union of a copy of G and a copy of H.

We refer readers to [13] for other undefined standard terminologies about graphs.

2 Topological minors

We focus on the topological minor relation in this section. It is the graph containment that is involved in Vázsonyi's conjectures, so it is arguably the oldest graph containment that is considered for well-quasi-ordering.

Let G be a graph and v a vertex of degree two in G. By *suppressing* v we mean deleting v and all its incident edges from G, and then adding an edge with ends x,y, where the two edges of G incident with v are $\{x,v\}$ and $\{y,v\}$, if v is not incident with a loop; and we simply delete v, if v is incident with a loop. Note that suppressing a vertex of degree two is equivalent with contracting an edge incident with it. (Edge-contraction is an operation that will be defined in Section 3.)

A graph G contains another graph H as a *topological minor* if H can be obtained from G by repeatedly deleting vertices and edges and suppressing vertices of degree two.

A equivalent way to define the topological minor relation is through the notion of homeomorphic embeddings. For graphs G and H, we say that a function π with domain $V(H) \cup E(H)$ is a homeomorphic embedding from H into G if the following hold.

- π maps vertices of H injectively to vertices of G.
- For each non-loop e of H with ends x,y, $\pi(e)$ is a path in G with ends $\pi(x)$ and $\pi(y)$.
- For each loop e of H with end v, $\pi(e)$ is a cycle in G containing $\pi(v)$.
- If e_1,e_2 are distinct edges of H, then $\pi(e_1) \cap \pi(e_2) \subseteq \{\pi(t) : t \in e_1 \cap e_2\}$.

It is easy to see that G contains H as a topological minor if and only if there exists a homeomorphic embedding from H into G.

Vázsonyi in the 1940's conjectured that trees are well-quasi-ordered by the topological minor relation. This conjecture was proved by Kruskal [30] and independently by Tarkowski [54]. Nash-Williams [42] later introduced the "minimal bad sequence" argument to provide an elegant proof of this conjecture. The minimal bad sequence argument has had a profound impact on proving well-quasi-ordering results since then. Indeed, they proved Vázsonyi's conjecture is true even when vertices are labelled by a well-quasi-ordering.

Theorem 1 ([30, 54, 42]). *Let $Q = (V(Q), \preceq_Q)$ be a well-quasi-ordering. For each positive integer i, let T_i be a tree and let $\phi_i : V(T) \to V(Q)$ be a function. Then there exist $1 \leq j < j'$ and a homeomorphic embedding π from T_j into $T_{j'}$ such that $\phi_j(v) \preceq_Q \phi_{j'}(\pi(v))$ for every $v \in V(T_j)$.*

One might expect that Theorem 1 can be generalized in a way that the homeomorphic embedding mentioned in Theorem 1 also preserves the ancestor-descendant relation if we make those trees be rooted trees. But in fact, this stronger version for rooted trees is equivalent with Theorem 1 as one can add a new incomparable element into Q to obtained a new well-quasi-ordering and add this new element into the labels of the roots of those trees.

Theorem 1 is a generalization of a very useful result of Higman [20], which is now known as the Higman's Lemma. Higman's Lemma states that every well-quasi-ordering Q on a set $V(Q)$ can be extended to a well-quasi-ordering on the set of finite sequences over $V(Q)$ by the natural "sequence embedding" relation.

Theorem 2 ([20]). *If $Q = (V(Q), \preceq_Q)$ is a well-quasi-ordering, then the set of finite sequences over $V(Q)$ is well-quasi-ordered by $\preceq$, where two finite sequences $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_n)$ over $V(Q)$ satisfy $a \preceq b$ if and only if there exist $1 \leq i_1 < i_2 < ... < i_m \leq n$ such that $a_j \preceq_Q b_{i_j}$ for every $j \in [m]$.*

Higman's Lemma is equivalent with the case when every tree T_i is a path in Theorem 1. In addition, by using Higman's Lemma, Theorem 1 can be extended to the case that each T_i is a forest.

Theorem 1 was later generalized by Mader [39] and Fellows, Hermelin and Rosamond [18] as follows.

Theorem 3. *Let t be a positive integer.*

1. *[39] Graphs that do not contain t disjoint cycles are well-quasi-ordered by the topological minor relation.*
2. *[18] Graphs that have feedback vertex sets with size at most t are well-quasi-ordered by the topological minor relation.*

A *feedback vertex set* in a graph G is a subset of $V(G)$ intersecting all cycles in G. In fact, Statement 2 of Theorem 3 can be easily derived from the forest-version of Theorem 1 by appropriately labelling the vertices; Statement 1 of Theorem 3 is equivalent with Statement 2 due to a classical result of Erdős and Pósa [17] stating that a graph has only a bounded number of disjoint cycles if and only if it has a feedback vertex set with bounded size. We remark that Statement 1 was proved much earlier than Statement 2.

Another class of graphs that is known to be well-quasi-ordered by the topological minor relation is the set of subcubic graphs. It was originally conjectured by Vázsonyi and proved by Robertson and Seymour [51] via the Graph Minor Theorem. (The Graph Minor Theorem will be described in Section 3.) We remark that the proof of the Graph Minor Theorem is very difficult, and it remains unknown how to prove Vázsonyi's conjecture on subcubic graphs without using the Graph Minor Theorem.

However, the topological minor relation does not well-quasi-order graphs in general. For every positive integer k, let R_k be the graph obtained from a path of length k by doubling each edge. The *ends* of R_k are the ends of the original path. We call R_k the *Robertson chain* of length k. Let R'_k be the graph obtained from R_k by attaching two leaves to each end of R_k. It is easy to see that $\{R'_k : k \geq 1\}$ is an antichain with respect to the topological minor relation.

There is another infinite antichain. Let R''_k be the graph obtained from a cycle of length k by duplicating each edge. Then for any subdivision R^*_k of R''_k, $\{R^*_k : k \geq 1\}$ is also an antichain with respect to the topological minor relation. More antichains were known, and all of them contain arbitrarily long Robertson chain as a topological minor.

Robertson in the 1980's conjectured that the Robertson chain is the only obstruction. That is, he conjectured that for every positive integer k, the set of graphs that do not contain R_k as a topological minor is well-quasi-ordered by the topological minor relation. We remark that Robertson's conjecture is strong. Though R_k has quite simple structures, the class of graphs with no R_k topological minor is still broad. In particular, every subcubic graph does not contain R_2 as a topological minor. So Robertson's conjecture for the case $k = 2$ contains Vázsonyi's subcubic graph conjecture.

Ding [15] proved that a weakening of Robertson's conjecture is true: the set of graphs that do not contain R_k as a minor is well-quasi-ordered by the topological minor relation.

Robertson's conjecture was recently completely solved by Liu and Thomas [34, 36], even when vertices are labeled.

Theorem 4 ([34, 36]). *For every well-quasi-ordering $Q = (V(Q), \preceq_Q)$ and for every positive integer k, if for each $i \geq 1$, G_i is a graph with no R_k topological minor and $f_i : V(G) \to V(Q)$ is a function, then there exist $1 \leq j < j'$ and a homeomorphic embedding $\pi : G_j \to G_{j'}$ such that $f_j(v) \preceq_Q f_{j'}(\pi(v))$ for every $v \in V(G_j)$.*

Theorem 4 implies all known results about well-quasi-ordering graphs by the topological minor relation. The case $k = 1$ of Theorem 4 implies Kruskal's Tree Theorem (Theorem 1); the case $k = 2t - 1$ implies Mader's theorem for graphs with no t disjoint cycles (Theorem 3) and hence for graphs having feedback vertex sets of bounded size; the case $k = 2$ implies Vázsonyi's conjecture on subcubic graphs. Theorem 4 also implies a well-known result about well-quasi-ordering bounded diamenter graphs by the subgraph relation (see Theorem 24 in Section 5).

The proof of Theorem 4 is long and difficult. The first step is to prove the case for graphs with bounded treewidth. It turns out to be harder than expected for graphs with bounded treewidth, which is a case that is not very hard to deal with in the proofs of the Graph Minor Theorem and several algorithmic results in the literature. We remark that the bounded treewidth case of Theorem 4 implies Ding's result, since graphs that do not contain R_k as a minor do not contain a $3 \times (k+1)$ grid as a minor and hence have bounded treewidth. Though the proof of the bounded treewidth case is not simple, the proof is self-contained and does not require the Graph Minor Theorem. One key ingredient is a technique to convert vertex-cuts realized by bags in the tree-decomposition into edge-cuts.

The second step of the proof of Theorem 4 is to study the structure of graphs with large treewidth but with no R_k topological minor. Liu and Thomas [34, 36] prove that such graphs are "nearly subcubic" by extensively applying techniques developed in Robertson and Seymour's Graph Minors series and earlier work of Liu and Thomas [35]. (The formal description of nearly subcubic graphs is complicated, so we skip the details in this paper.)

The third step for proving Theorem 4 is to prove that nearly subcubic graphs are well-quasi-ordered by the topological minor relation. This step requires non-trivial applications of the Graph Minor Theorem, and it is the only step that uses the Graph Minor Theorem in the entire proof.

We remark that Theorem 4 is best possible as long as the vertices are labelled by a well-quasi-ordered set Q with $|V(Q)| \geq 2$. Let a be a maximal element of Q, and let $b \in V(Q) - \{a\}$. If we label the ends of R_k by a and label other vertices by b, then $\{R_k : k \geq 1\}$ is an antichain with respect to the topological minor relation that preserves ordering on the labels of the vertices. This shows that the converse of Theorem 4 is also true.

However, Robertson's conjecture can be strengthened if the vertices are unlabelled (or equivalently, labelled by Q with $|V(Q)| = 1$), since $\{R_k : k \geq 1\}$ is not an antichain if the vertices are unlabelled. Liu and Thomas [36] also provide a complete characterization for the family of unlabelled graphs that are well-quasi-ordered by the topological minor relation. Such a characterization involves a notion of Robertson family that is defined as follows.

For a positive integer k and an end v of R_k, by *planting* on v we mean the operation that either adds a new vertex adjacent to v, or adds a new loop incident with v; a *thickening* on v is the operation that adds a new edge incident with v and its neighbor; a *strong planting* on v is the operation that either applies planting on v twice, or applies thickening on v once. Let k be a positive integer, the *Robertson cycle* of length k is the graph that can be obtained from the cycle of length k by duplicating each edge.

For each positive integer $k \geq 3$, the *Robertson family* of length k is the set of graphs consisting of the Robertson cycle of length k and the graphs that can be obtained from R_k by either

- strong planting on each end of R_k once, or
- planting on each end of R_k once and adding an edge incident with both ends of R_k, or

- planting on one end of R_k once, thickening on the other end once, and adding an edge incident with both ends of R_k.

So for each $k \geq 3$, the Robertson family of length k consists of 16 non-isomorphic graphs. Note that the graph R'_k mentioned earlier in the infinite antichain $\{R'_k : k \geq 1\}$ can be obtained from R_k by strong planting on each end of R_k. Clearly, the union of the Robertson families of length k over all integers k can be partitioned into 16 infinite antichains with respect to the topological minor relation.

Liu and Thomas [36] prove that Robertson's conjecture can be strengthened to graphs with no topological minor isomorphic to members of Robertson families.

Theorem 5 ([36]). *For every positive integer $k \geq 3$, if $G_1, G_2, \ldots$ are graphs that do not contain any member of the Robertson family of length at least k as a topological minor, then there exist $1 \leq j < j'$ such that $G_{j'}$ contains G_j as a topological minor.*

Theorem 5 is best possible since to obtain a well-quasi-ordered set, we can only allow finitely many members in each of the 16 infinite disjoint antichains whose union is the union of Robertson families of all lengths. This theorem also provides a characterization of well-quasi-ordered topological minor ideals.

A family $\mathscr{I}$ of graphs is a *topological minor ideal* if every topological minor of any member of $\mathscr{I}$ belongs to $\mathscr{I}$.

Theorem 6 ([36]). *Let $\mathscr{I}$ be a topological minor ideal. Let $\mathscr{R}$ be the union of the Robertson family of length k over all positive integers $k \geq 3$. Then $\mathscr{I}$ is well-quasi-ordered by the topological minor relation if and only if $\mathscr{I}$ contains only finitely many members of $\mathscr{R}$.*

We remark that Theorems 4 and 6 show a significant difference between well-quasi-ordered topological minor ideals for labelled graphs and for unlabelled graphs. Furthermore, if a topological minor ideal is well-quasi-ordered with a set of two labels, then it cannot contain arbitrarily long Robertson chain, so Theorem 4 shows that it is also well-quasi-ordered with a set of labels of any cardinality. Hence, the cardinality of the set of labels does not affect whether a topological minor ideal is well-quasi-ordered or not, as long as at least two labels are allowed. This fact could be viewed as a possible support for a conjecture of Pazout (Conjecture 7) about a similar situation for the induced subgraph relation, though it could also be viewed as a support for a similar but false conjecture of Kříž and Thomas [29] on QO-categories disproved by Kříž and Sgall [28].

2.1 Directed graphs

Now we consider topological minors for directed graphs. The notion of homeomorphic embedding of undirected graphs naturally extends to directed graphs. A function π is a *homeomorphic embedding* from a directed graph H into a directed graph G if the following hold.

- π maps vertices of H injectively to vertices of G.
- For each non-loop e of H with tail x and head y, $\pi(e)$ is a directed path in G from $\pi(x)$ to $\pi(y)$.
- For each loop e of H with end v, $\pi(e)$ is a directed cycle in G containing $\pi(v)$.
- If e_1, e_2 are distinct edges of H, then $\pi(e_1) \cap \pi(e_2) \subseteq \{\pi(t) : t \in e_1 \cap e_2\}$.

We say that a directed graph G contains another directed graph H as a *topological minor* if there exists a homeomorphic embedding from H into G.

It is easy to see that the topological minor relation does not well-quasi-order directed graphs, as any orientation of the graphs in $\{R'_k : k \geq 1\}$ form an infinite antichain. Indeed, it is still not a well-quasi-ordering even if we restrict the problem to a specific kind of directed graphs.

A directed graph G is a *tournament* if its underlying graph is a simple graph, and for every pair u, v of distinct vertices of G, exactly one of (u, v) and (v, u) belongs to $E(G)$. It seems well-known that tournaments are not well-quasi-ordered by the topological minor relation, but we were not able to find any example of an infinite antichain in the literature. So we provide an example of an infinite antichain here.

For any positive integer n, we say a tournament is a *transitive tournament on* $[n]$ if its vertex-set is $[n]$ and every edge is of the form (i, j) with $1 \leq i < j \leq n$. For any positive integer k, let G_k be the tournament obtained from the transitive tournament on $[2k+13]$ by reversing the direction of the edges in $\{(1,2),(3,4),(5,6),(2,7),(4,7),(6,7),(2k+7,2k+8),(2k+8,2k+9),(2k+7,2k+10),(2k+10,2k+11),(2k+7,2k+12),(2k+12,2k+13)\} \cup \{(2i+5,2i+6),(2i+6,2i+7),(2i+5,2i+7) : i \in [k]\}$. Note that the undirected graph formed by the reversed edges is the simple graph obtained from R_k by attaching three leaves to each end of R_k and then subdividing all except one edge in each pair of parallel edges once.

Theorem 7. *$\{G_k : k \geq 1\}$ is an antichain of tournaments with respect to the topological minor relation.*

Proof. Suppose to the contrary that there exist $1 < i < j$ and a homeomorphic embedding π from G_i to G_j. Let $u_1, u_2, ..., u_{2i+13}$ be the vertices $1, 2, ..., 2i+13$ of G_i, respectively; let $v_1, v_2, ..., v_{2j+13}$ be the vertices $1, 2, ..., 2j+13$ of G_j, respectively.

We first show that $\pi(u_7) = v_7$. For $t \in [3]$, let H_t be the directed cycles $\pi((u_7, u_{8-2t})) \cup \pi((u_{8-2t}, u_{7-2t})) \cup \pi((u_{7-2t}, u_7))$ in G_j. Suppose that $\pi(u_7) = v_{2r+7}$ for some $r \in [j]$. Since there exists no edge from $\{v_\ell : \ell > 2r+7\}$ to $\{v_\ell : \ell < 2r+7\}$ in G_j, at most one of H_1, H_2, H_3, say H_1, contains an edge of the form (v_{2r+7}, v_x) with $x > 2r+7$. Since (v_{2r+7}, v_{2r+5}) and (v_{2r+7}, v_{2r+6}) are the only two edges of the form (v_{2r+7}, v_y) with $y < 2r+7$, one of H_2, H_3 contains (v_{2r+7}, v_{2r+5}) and the other contains (v_{2r+7}, v_{2r+6}). But then H_2, H_3 must share v_{2r+5}, a contradiction. A similar argument shows that $\pi(u_7) \not\in \{v_\ell : \ell \in [6]\} \cup \{v_{2\ell+6} : \ell \in [j]\}$. Since the out-degree of u_7 equals $2i+7$, which is greater than the out-degree of any vertex in $\{v_\ell : 2j+8 \leq \ell \leq 2j+13\}$. Hence $\pi(u_7) = v_7$. Similarly, $\pi(u_{2i+7}) = v_{2j+7}$.

Since u_7 has in-degree five in G_i, in order to accommodate H_1, H_2, H_3, $\pi((u_8, u_7))$ and $\pi((u_9, u_7))$, we have that $\{\pi(u_\ell) : \ell \in [7]\} = \{v_\ell : \ell \in [7]\}$. Since u_{2i+5} has out-degree six in G_i, $\pi(u_{2i+5}) \not\in \{v_\ell : 2j+8 \leq \ell \leq 2j+13\}$. Then we have $\{\pi(u_\ell) :$

$2i+8 \leq \ell \leq 2i+13\} = \{v_\ell : 2j+8 \leq \ell \leq 2j+13\}$. Then it is easy to show that $\pi(u_{2\ell+7}) = v_{2\ell+7}$ for each $\ell \in [i] \cup \{0\}$ by induction on ℓ. In particular, $\pi(u_{2i+7}) = v_{2i+7}$. So $j = i$, a contradiction. This proves the theorem. ■

3 Minors

Let G be a graph, and e be an edge with ends x, y. By *contracting* e we mean deleting x, y from $V(G)$ and adding a new element w into $V(G)$, and deleting e from $E(G)$ and replacing any appearance of x or y in edges by w. Note that contracting an edge contained in a triangle will create parallel edges; contracting an edge in a pair of parallel edges will create loops. We say that G contains a graph H as a *minor* if H can be obtained from G by repeatedly deleting vertices and edges and contracting edges.

Wagner [56] conjectured that the minor relation is a well-quasi-ordering. Note that Wagner's conjecture contains Vázsonyi's conjecture on subcubic graphs since the minor relation and the topological minor relation are equivalent for subcubic graphs. Robertson and Seymour [51] proved Wagner's conjecture and hence Vázsonyi's conjecture on subcubic graphs. Note that deriving from Wagner's conjecture is the only currently known proof of Vázsonyi's conjecture on subcubic graphs.

Theorem 8 ([51]). *Graphs are well-quasi-ordered by the minor relation.*

Theorem 8 is now known as the Graph Minor Theorem. The Graph Minor Theorem is one of the most difficult theorems in graph theory. It is proved in the 20th paper of the famous Graph Minors series by extensively applying the structural theorems developed in other papers of the same series. Robertson and Seymour's groundbreaking work in this series of paper not only solves well-quasi-ordering problems but also opens a new research field in structural graph theory.

Indeed, Robertson and Seymour proved that Theorem 8 is true even when the edges of the graphs are labelled. Formal descriptions for the version of labelled graphs are involved, so we omit the details. We refer interested readers to [51, 52]. A sketch of a proof of Theorem 8 can be found in [13].

A *minor ideal* $\mathscr{I}$ is a set of graphs such that every minor of a member of $\mathscr{I}$ belongs to $\mathscr{I}$. Theorem 8 implies that for every minor ideal $\mathscr{I}$, there exists a finite set of graphs $\mathscr{F}$ such that any graph belongs to $\mathscr{I}$ if and only if it does not contain any member of $\mathscr{F}$ as a minor. In other words, any minor ideal (or any minor closed property) can be characterized by finitely many graphs. Since minor testing is fixed-parameter tractable [50], any minor closed property can be tested in polynomial time.

3.1 Directed minors

There are different notions of minors for directed graphs, and it is unclear which one is the better than others.

One possible way to define minors for directed graphs is the same as for undirected graphs: just deleting vertices, edges or contracting edges. Robertson and Seymour [51] also showed that the Graph Minor Theorem is true for this notion of minors for directed graphs.

Theorem 9 ([51]). *Given infinitely many directed graphs $G_1, G_2, \ldots$, there exist $1 \leq i < j$ such that G_i can be obtained from G_j by deleting vertices and edges and contracting edges.*

One drawback for allowing contracting any edges in directed graphs is about an issue of connectivity. Observe that contracting edges in undirected graphs does not create new connected components. A natural analog of the connectivity for directed graphs is strong connectivity. A directed graph is *strongly connected* if for any pair of vertices u, v, there exist a directed path from u to v and a directed path from v to u. A *strong connected component* in a directed graph is a maximal strongly connected subdigraph. Note that contracting edges in directed graph might create new strongly connected components. Hence, people seek notions of minors for directed graphs that preserve strong connectivity. In this subsection we discuss two such notions.

The first one is called the butterfly minor. A directed graph G contains another directed graph H as a *butterfly minor* if H can be obtained from a subdigraph of G by repeatedly contracting edges e satisfying the property that either the tail of e has out-degree one, or the head of e has in-degree one. Note that contracting such edges will not create new strongly connected components.

However, the butterfly minor relation is not a well-quasi-ordering on directed graphs. For any positive integer k, let G_k be the directed graph obtained from a cycle of length $2k$ by orienting the edges clockwise or counterclockwise alternately. Hence every vertex of G_k has either in-degree 0 and out-degree 2, or in-degree 2 and out degree 0. So no edge of G_k can be contracted according to the requirement for butterfly minors. Therefore, $\{G_k : k \geq 1\}$ is an antichain with respect to the butterfly minor relation.

Another antichain with respect to the butterfly minor relation is as follows. For any positive integer k, let G_k be the directed graph obtained by a path of length $2k$ by orienting edges alternately such that the ends of the path have in-degree 0, and attaching two leaves to each end of the original path and direct the edges such that the ends of the original paths have out-degree 3. Similarly as the previous example, no edge in G_k can be contracted. Therefore, $\{G_k : k \geq 1\}$ is an antichain with respect to the butterfly minor relation.

Each of these two antichains contains arbitrarily long paths with edges oriented alternately. Chudnovsky, Muzi, Oum, Seymour and Wollan (see [41]) proved that such long alternating paths are the only obstructions for butterfly minor ideals being well-quasi-ordered by the butterfly minor relation.

A set of directed graphs $\mathscr{I}$ is called a *butterfly minor ideal* if any butterfly minor of any member of $\mathscr{I}$ belongs to $\mathscr{I}$. An *alternating path* of length k is a directed graph that is obtained from a path of length k by orienting edges such that no directed subpath has length two.

Theorem 10 ([41]). *Let $\mathscr{I}$ be a butterfly minor ideal. If there exists a positive integer k such that $\mathscr{I}$ does not contain any alternating path of length k, then $\mathscr{I}$ is well-quasi-ordered by the butterfly minor relation.*

Now we discuss another notion of minors for directed graphs. An equivalent way to define minors for undirected graphs is by contracting connected subgraphs instead of contracting edges. Here we consider such an analog for directed graphs. More precisely, this containment allows vertex-deletions, edge-deletions and contracting directed cycles. Note that as contracting special edges for butterfly minors, contracting directed cycles does not create new strongly connected components, either. We are not aware of any formal term in the literature describing this type of minor containment besides of simply calling it "minors". But to avoid confusion, we do not call it minors in this paper.

Note that this new containment is incomparable with the butterfly minor relation. There exist directed graphs G_1, G_2, H such that G_1 contains H as a butterfly minor, but H cannot be obtained from a subdigraph of G_1 by contracting directed cycles, and G_2 does not contain H as a butterfly minor, but H can be obtained from a subdigraph of G_2 by contracting directed cycles.

Clearly, directed graphs are not well-quasi-ordered by this containment relation since the set of directed cycles is an antichain with respect to this containment. But Kim and Seymour [23] proved that the set of semi-complete directed graphs are well-quasi-ordered by this containment. A directed graph D is *semi-complete* if $E(D)$ is a set of ordered pairs of distinct vertices, and for any distinct vertices u, v of D, at least one of (u, v) and (v, u) belongs to $E(D)$.

Theorem 11 ([23]). *If $G_1, G_2, \ldots$ are semi-complete directed graphs, then there exist $1 \leq i < j$ such that G_i can be obtained from a subdigraph of G_j by repeatedly contracting directed cycles.*

3.2 Induced minors

In this subsection we consider minors where edge-deletions are not allowed. This notion is a combination of the minor relation and the induced subgraph relation. We remark that most of the statements in this subsection address simple graphs. One reason is that for any graph H, the set $\{H_i : i \geq 1\}$ is an infinite antichain with respect to the induced minor relation or the induced subgraph relation, where H_i is the graph obtained from H by duplicating each edge i times. Hence, to keep graphs simple, we have to delete all resulting loops and parallel edges when we contract an edge.

Formally, we say a simple graph G contains another simple graph H as an *induced minor* if H can be obtained from G by repeatedly deleting vertices, contracting edges, and deleting resulting loops and parallel edges.

Several infinite antichains with respect to the induced minor relation are known in the literature.

Theorem 12. *The following sets are antichains with respect to the induced minor relation.*

1. *[55] The set of "alternating double wheels".*
2. *[40] A specific set of simple graphs of maximum degree at most eight with no K_5^- minor.*
3. *[16] A specific set of interval graphs.*
4. *[5] The set of anti-holes with length at least six.*
5. *[33] A specific set of simple graphs that do not contain W_4 or K_5^- as an induced minor.*

An *alternating double wheel* is the simple graph obtained from a cycle $v_1v_2...v_{2k}v_1$ of even length with $k \geq 6$ by adding two non-adjacent vertices x,y and adding the edges $\{x,v_{2i}\},\{y,v_{2i-1}\}$ for $i \in [k]$. Recall that K_n^- denotes the graph obtained from K_n by deleting an edge, and W_4 is the simple graph obtained from the cycle of length four by adding a new vertex adjacent to all other vertices. A graph is an *interval graph* if it is the intersection graph of intervals of $\mathbb{R}$. A graph is an *anti-hole* of length k if it is the complement of a cycle of length k.

Since simple graphs are not well-quasi-ordered by the induced minor relation, questions about graphs in more restricted sets were proposed. Thomas [55] first proved the following.

Theorem 13 ([55]). *The set of simple series-parallel graphs are well-quasi-ordered by the induced minor relation.*

A graph is *series-parallel* if it does not contain K_4 as a minor. Note that a simple graph contains K_4 as a minor if and only if it contains K_4 as an induced minor. Thomas [55] also asked whether Theorem 13 can be generalized to the set of simple graphs with no K_5^- minor. Matoušek, Nešetřtil and Thomas [40] and Lewchalermvongs [33] provided negative answers of this question as indicated in Statements 2 and 5 of Theorem 12.

Even though Theorem 13 cannot be generalized to graphs with no K_5^- minor, people keep looking for specific classes of simple graphs that are well-quasi-ordered by the induced minor relation.

For any set $\mathscr{F}$ of graphs, define $\mathrm{Forb}_{im}^s(\mathscr{F})$ to be the set of simple graphs that do not contain any member of $\mathscr{F}$ as an induced minor. When the set $\mathscr{F}$ consists of only one graph, say H, we write $\mathrm{Forb}_{im}^s(\mathscr{F})$ as $\mathrm{Forb}_{im}^s(H)$. Well-quasi-ordered $\mathrm{Forb}_{im}^s(\mathscr{F})$ are characterized by Błasiok, Kamiński, Raymond and Trunck, when $|\mathscr{F}| = 1$.

Theorem 14 ([5]). *Let H be a simple graph. Then $\mathrm{Forb}_{im}^s(H)$ is well-quasi-ordered by the induced minor relation if and only if H is $\hat{K}_4$ or W_4^-.*

Here $\hat{K}_4$ is the simple graph obtained from K_4 by adding a new vertex v adjacent to exactly two vertices of K_4; W_4^- is the graph obtained from W_4 by deleting an edge not incident with the vertex of degree four.

Furthermore, Lewchalermvongs [33] characterizes all induced minor ideals $\mathscr{I}$ that are contained in $\mathrm{Forb}^s_{im}(\{W_4, K_5^-\})$ and well-quasi-ordered by the induced minor relation. Formal descriptions of this result are involved, so we omit the details.

Another result about induced minors was proved by Ding [16] as follows. (A graph is *chordal* if it does not contain any cycle of length at least four as an induced subgraph.)

Theorem 15 ([16]). *If t is a positive integer, then simple chordal graphs with no clique of size $t+1$ are well-quasi-ordered by the induced minor relation.*

Other classes of simple graphs are also concerned. Lozin and Mayhill [37] proposed the following conjecture. (A *unit interval graph* is an intersection graph of a collection of intervals of $\mathbb{R}$ of length one; a *permutation graph* is a simple graph such that its vertex-set is $\{v_1, v_2, ..., v_n\}$ for some positive integer n, and there exists a permutation σ on $[n]$ such that v_i is adjacent to v_j if and only if $(i-j)(\sigma(i)-\sigma(j)) < 0$.)

Conjecture 1 ([37]). Unit interval graphs and bipartite permutation graphs are well-quasi-ordered by the induced minor relation.

Note that the set of interval graphs is not well-quasi-ordered by the induced minor relation by Statement 3 in Theorem 12.

Another positive result about induced minors is proved by Fellows, Hermelin and Rosamond [18].

Theorem 16 ([18]). *If k is a positive integer, then the set of simple graphs with no cycle of length greater than k is well-quasi-ordered by the induced minor relation.*

In the rest of the subsection, we consider containment relations that only allow edge-contractions. Clearly, graphs with different number of components form an antichain if only edge-contractions are allowed. Hence one should limit the number of components when considering this containment.

We say that a simple graph (or loopless graph, respectively) G contains another simple graph (or loopless graph, respectively) H as a *simple-contraction* (or *loopless-contraction*, respectively) if H can be obtained from G by contracting edges and deleting resulting loops and parallel edges (or deleting resulting loops, respectively). For every positive integer k, define Θ_k to be the 2-vertex loopless graph with k parallel edges. It is easy to see that $\{\Theta_k : k \geq 1\}$ is an antichain with respect to loopless-contraction.

For a positive integer p and a family of graphs $\mathscr{F}$, let $\mathrm{Forb}^{s,p}_{sc}(\mathscr{F})$ (or $\mathrm{Forb}^{\ell,p}_{lc}(\mathscr{F})$, respectively) be the set of simple (or loopless, respectively) graphs with at most p components containing no member of $\mathscr{F}$ as a simple-contraction (or loopless-contraction, respectively). The following are proved by Kamiński, Raymond and Trunck [22, 21].

Theorem 17. *Let k, p be positive integers.*

1. *[22] Let H be a simple graph. Then* $\text{Forb}_{sc}^{s,1}(\{H\})$ *is well-quasi-ordered by the simple-contraction relation if and only if K_4^- contains H as a simple-contraction.*
2. *[21]* $\text{Forb}_{lc}^{\ell,p}(\{\Theta_i : i \geq k\})$ *is well-quasi-ordered by the loopless-contraction relation.*

3.3 Vertex-minors and pivot-minors

Let G be a simple graph. The simple graph obtained from G by applying *local complementation* on a vertex v of G is the simple graph $G*v$ with vertex-set $V(G)$ and two distinct vertices x, y are adjacent in $G*v$ if and only if either v is adjacent in G to both x, y and $\{x,y\} \notin E(G)$, or at least one of x, y is not adjacent in G to v and $\{x,y\} \in E(G)$. A simple graph H is a *vertex-minor* of G if H can be obtained from G by repeatedly deleting vertices and applying local complementations.

It is straightforward to verify that for any edge $\{x,y\}$ of a simple graph G, $G*x*y*x = G*y*x*y$. The simple graph, denoted by $G \wedge \{u,v\}$, obtained from G by applying *pivoting* an edge $\{u,v\}$ of G is the graph $G*u*v*u$. A simple graph H is a *pivot-minor* of G if H can be obtained from G by repeatedly deleting vertices and applying pivotings.

Clearly, if H is a pivot-minor of G, then H is a vertex-minor of G. Oum [46] asks whether the pivot-minor relation is a well-quasi-ordering on simple graphs or not.

Question 1. Are simple graphs well-quasi-ordered by the pivot-minor relation?

Proving a positive answer of Question 1 is expected to be very difficult, since even a positive answer of this question on bipartite graphs implies the Graph Minor Theorem.

Now we discuss the relationship between pivot-minors and minors. Note that if G is a graph and T is a spanning forest in G, then for every edge $e \in E(G) - E(T)$, there uniquely exists a cycle in $T+e$ containing e. This cycle is called the *fundamental cycle for e with respect to T*. For a graph G and a spanning forest T of G, the *fundamental graph of G with respect to T*, denoted by $F(G;T)$, is a simple bipartite graph with (ordered) bipartition $(E(T), E(G)-E(T))$ such that for any $e \in E(T)$ and $f \in E(G)-E(T)$, e is adjacent to f in $F(G;T)$ if and only if e belongs to the fundamental cycle for f with respect to T.

Deleting vertices from $F(G;T)$ corresponds to deleting or contracting edges of G. Let $e \in V(F(G;T))$. It is straightforward to see that if $e \in E(T)$, then deleting e from $F(G;T)$ results in the graph $F(G/e;T/e)$, where G/e and T/e denote the graphs obtained from G and T by contracting e, respectively; if $e \notin E(T)$, then deleting e from $F(G;T)$ results in the graph $F(G-e;T)$.

Pivoting an edge in $F(G;T)$ corresponds to switching to a new spanning forest. Let $\{e,f\} \in E(F(G;T))$, where $e \in E(T)$ and $f \in E(G)-E(T)$. Then $F(G;T) \wedge \{e,f\} = F(G;(T-e)+f)$.

Therefore, if G_1, G_2 are graphs and T_1, T_2 are spanning forests in G_1, G_2, respectively, such that $F(G_1;T_1)$ is a pivot-minor of $F(G_2;T_2)$, then G_1 is a minor of G_2. This shows that if simple bipartite graphs are well-quasi-ordered by the pivot-minor relation, then graphs are well-quasi-ordered by the minor relation, which is what the Graph Minor Theorem states.

Oum [45] proved that Question 1 has a positive answer for simple graphs with bounded "rank-width". Rank-width is a graph parameter that does not increase by taking vertex-minors or pivot-minors, which is an analog of the relationship between treewidth and the minor containment. Oum's theorem can be viewed as a step toward a potential answer of Question 1 as proving the bounded treewidth case serves the first step of the proofs of the Graph Minor Theorem and Robertson's conjecture. We omit the formal definition of rank-width in this article.

Theorem 18 ([45]). *For every positive integer k, simple graphs with rank-width at most k are well-quasi-ordered by the pivot-minor relation.*

One can ask whether a weakening of Question 1 for the vertex-minor relation holds.

Conjecture 2. Simple graphs are well-quasi-ordered by the vertex-minor relation.

The vertex-minor relation is a weakening of the induced topological minor relation. We say that a simple graph H is an *induced topological minor* of another simple graph G if H can be obtained from G by repeatedly deleting vertices and suppressing vertices of degree two not contained in triangles. Note that we only allow suppressing vertices not contained in triangles since we focus on simple graphs here. Furthermore, one can also define induced topological minors that allow suppressing any vertex of degree two and deleting parallel edges. It is equivalent with the earlier definition since suppressing a vertex of degree two contained in a triangle and deleting resulting parallel edges is equivalent with the operation that simply deletes this degree two vertex. It is easy to see that the simple graph obtained from a simple graph G by suppressing a vertex v of degree two not contained in a triangle can be obtained from $G*v$ by deleting v. Therefore, if a simple graph G contains another simple graph H as an induced topological minor, then G contains H as a vertex-minor. It is easy to see that the topological minor relation and the induced topological minor relation are the same for trees. Hence Theorem 1 indeed shows that trees are well-quasi-ordered by the induced topological minor relation.

Conjecture 2 is known to be true for circle graphs. A simple graph is a *circle graph* if it is the intersection graph of a set of chords of a circle. Bouchet [6] proved that the following theorem follows from Theorem 20 on 4-regular graphs.

Theorem 19 ([6]). *Circle graphs are well-quasi-ordered by the vertex-minor relation.*

4 Immersions

Immersions are graph containments that are closely related to the topological minor relation. A *weak immersion* of a graph H in another graph G is a function π with domain $V(H) \cup E(H)$ such that the following hold.

- π maps vertices of H to vertices of G injectively.
- For each non-loop e of H with ends x,y, $\pi(e)$ is a path in G with ends $\pi(x)$ and $\pi(y)$.
- For each loop e of H with end v, $\pi(e)$ is a cycle in G containing $\pi(v)$.
- If e_1, e_2 are distinct edges of H, then $E(\pi(e_1) \cap \pi(e_2)) = \emptyset$.

A *strong immersion* of H in G is a weak immersion π of H in G such that for every $e \in E(H)$ and vertex v of H not incident with e, $\pi(v) \notin V(\pi(e))$. We say that G contains H as a *weak immersion* (or *strong immersion*, respectively) if there exists a weak (or strong, respectively) immersion of H in G.

Clearly, any homeomorphic embedding from H into G is a strong immersion of H in G, and every strong immersion of H in G is a weak immersion of H in G. Hence if G contains H as a topological minor, then G contains H as a strong immersion and a weak immersion. However, the immersion relations and minor relation are incomparable. There exist graphs G,H such that G contains H as a minor, but G does not contain H as a weak immersion; there exist graphs G',H' such that G' contains H' as a strong immersion, but G' does not contain H' as a minor. It is worthwhile mentioning that the minor relation, topological minor relation and weak and strong immersion relations are equivalent for subcubic graphs.

Nash-Williams in the 1960's conjectured that the weak immersion relation [43] and the strong immersion relation [44] are well-quasi-ordering. The weak immersion conjecture was proved by Robertson and Seymour [52] in the currently last paper in their Graph Minors Series. Indeed, they proved that it is true even when graphs are labelled.

Theorem 20 ([52]). *Let $Q = (V(Q), \preceq_Q)$ be a well-quasi-ordering. For each positive integer i, let G_i be a graph and $\phi_i : V(G_i) \to V(Q)$ be a function. Then there exist $1 \leq j < j'$ and a weak immersion π of G_j in $G_{j'}$ such that $\phi_j(v) \preceq_Q \phi_{j'}(\pi(v))$ for every $v \in V(G_j)$.*

The strong immersion conjecture remains open. Robertson and Seymour believe that they had a proof of the strong immersion conjecture at one time, but even if it was correct, it was very complicated, and it is unlikely that they will write it down (see [52]).

Conjecture 3 ([44]). Graphs are well-quasi-ordered by the strong immersion relation.

It is not hard to prove Conjecture 3 for graphs with bounded maximum degree by using Theorem 20.

Theorem 21. *Let k be a nonnegative integer, and let $Q=(V(Q),\preceq_Q)$ be a well-quasi-ordering. For each positive integer i, let G_i be a graph with maximum degree at most k, and let $\phi_i : V(G_i) \to V(Q)$. Then there exist $1 \leq j < j'$ and a strong immersion π of G_j in $G_{j'}$ such that $\phi_j(v) \preceq_Q \phi_{j'}(\pi(v))$ for every $v \in V(G_j)$.*

Proof. Define Q' to be the well-quasi-ordering $(V(Q)\times([k]\cup\{0\}),\preceq)$, where for any $(x_1,y_1),(x_2,y_2) \in V(Q)\times([k]\cup\{0\})$, $(x_1,y_1) \preceq (x_2,y_2)$ if and only if $x_1 \preceq_Q x_2$ and $y_1=y_2$. For each $i \geq 1$, define $f_i : V(G_i) \to V(Q)\times([k]\cup\{0\})$ to be the function such that $f_i(v)=(\phi_i(v),d(v))$ for each $v \in V(G_i)$, where $d(v)$ is the degree of v in G_i. By Theorem 20, there exist $1 \leq j < j'$ and a weak immersion π of G_j in $G_{j'}$ such that $f_j(v) \preceq f_{j'}(\pi(v))$ for every $v \in V(G'_j)$. In particular, for every $v \in V(G_j)$, the degree of $\pi(v)$ in $G_{j'}$ equals the degree of v in G_j. So for each $v \in V(G_j)$, all edges of $G_{j'}$ incident with $\pi(v)$ are contained in $\bigcup \pi(e)$, where the union is over all edges e of G_j incident with v. Hence π is a strong immersion of G_j in $G_{j'}$. This proves the theorem. ■

Andreae [2] made some progress on Conjecture 3.

Theorem 22 ([2]). *The following classes of simple graphs are well-quasi-ordered by the strong immersion relation.*

1. *Simple graphs that do not contain $K_{2,3}$ as a strong immersion.*
2. *Simple graphs whose blocks are either complete graphs, cycles, or balanced complete bipartite graphs.*

4.1 Directed graphs

The notion of weak immersion and strong immersion naturally extend to directed graphs. A *weak immersion* of a directed graph H in another directed graph G is a function π with domain $V(H)\cup E(H)$ such that the following hold.

- π maps vertices of H to vertices of G injectively.
- For each non-loop e of H with head x and tail y, $\pi(e)$ is a directed path in G with from $\pi(x)$ to $\pi(y)$.
- For each loop e of H with end v, $\pi(e)$ is a directed cycle in G containing $\pi(v)$.
- If e_1,e_2 are distinct edges of H, then $E(\pi(e_1)\cap\pi(e_2))=\emptyset$.

A *strong immersion* of H in G is a weak immersion π of H in G such that for every $e \in E(H)$ and vertex v of H not incident with e, $\pi(v) \notin V(\pi(e))$.

Directed graphs are not well-quasi-ordered by the immersion relations, even for weak immersion. Consider the cycles of length $2k$ with edges oriented clockwise and counterclockwise alternately. It is easy to see that these orientated cycles form an infinite antichain with respect to weak immersion.

But Chudnovsky and Seymour [7] proved that tournaments are well-quasi-ordered by the strong immersion relation. Recall that tournaments are not well-quasi-ordered by the topological minor relation (Theorem 7).

Theorem 23 ([7]). *Tournaments are well-quasi-ordered by the strong immersion relation.*

5 Subgraphs

In this section we discuss the subgraph relation. A graph H is a *subgraph* of another graph G if H can be obtained from G by deleting vertices and edges. A*subgraph embedding* from H into G is an injective function $f : V(H) \cup E(H) \to E(H) \cup E(G)$ such that $f(V(H)) \subseteq V(G)$, $f(E(H)) \subseteq E(G)$, and for any edge $\{x,y\}$ of H, $f(e) = \{f(x), f(y)\}$. Clearly, H is a subgraph of G if and only if there exists a subgraph embedding from H into G.

The subgraph relation does not well-quasi-order graphs. The set of all cycles is an infinite antichain with respect to the subgraph relation. There is another antichain. For every positive integer k, the *fork* of length k, denoted by F_k, is the simple graph obtained from a path of length k by attaching two leaves to each end of the original path. Clearly, the set of all forks is an infinite antichain with respect to the subgraph relation. This situation is similar with the topological minor case. Indeed, Ding [14] proved an analog of Robertson's conjecture with respect to the subgraph relation.

Theorem 24 ([14]). *Let k be a positive integer, and let $Q = (V(Q), \preceq)$ be a well-quasi-ordering. For any positive integer i, let G_i be a graph that does not contain a path of length k as a subgraph, and let $\phi_i : V(G_i) \to V(Q)$. Then there exist $1 \leq j < j'$ and a subgraph embedding ϕ from G_j into $G_{j'}$ such that $\phi_j(v) \preceq \phi_{j'}(\pi(v))$ for every $v \in V(G_j)$.*

Ding's proof of Theorem 24 is nice and short based on the simple fact that every connected graph that does not contain a path of length k as a subgraph can be modified into a graph that does not contain a path of length $k-1$ as a subgraph by deleting at most k vertices. Theorem 24 can also be derived from Theorem 4. Let G'_i be the graph obtained from G_i by subdividing every edge once and then duplicating all edges. Define a new well-quasi-ordering Q' by adding a new element into Q incomparable to all other elements of Q. Further label all vertices of G'_i obtained by subdividing edges of G_i by this new element. Then $G'_{j'}$ contains G'_j as a topological minor with respect to the labelling if and only if $G_{j'}$ contains G_j as a subgraph with respect to the labelling. And it is easy to see that if G_i does not contain a path of length k as a subgraph, then G'_i does not contain R_{2k+2} as a topological minor. So Theorem 24 follows from Theorem 4.

A set $\mathscr{I}$ of graphs is a *subgraph ideal* if every subgraph of a member of $\mathscr{I}$ belongs to $\mathscr{I}$. Ding [14] characterized all well-quasi-ordered subgraph ideals of simple graphs.

Theorem 25 ([14]). *Let $\mathscr{I}$ be a subgraph ideal of simple graphs. Then the following are equivalent.*

1. *$\mathscr{I}$ is well-quasi-ordered by the subgraph relation.*

2. *$\mathcal{I}$ is well-quasi-ordered by the induced subgraph relation.*
3. *There exists a positive integer k such that $\mathcal{I}$ does not contain any cycle or fork of length at least k.*

5.1 Subdigraphs

Now we discuss the subdigraph relation on directed graphs. As shown in Section 3, there exists an infinite antichain of directed graphs with respect to the butterfly minor relation. This antichain is also an antichain with respect to the subdigraph relation. Note that directed graphs in this antichain do not contain a directed path of length at least two. So Theorem 24 does not extend to directed graphs with no long directed paths. But Ding [14] points out that his proof of Theorem 24 can be easily modified to prove that directed graphs whose underlying graphs do not contain a path of length k are well-quasi-ordered by the subdigraph relation.

Recall that Theorem 7 shows that there exists an infinite antichain of tournaments with respect to the topological minor relation. So tournaments are not well-quasi-ordered by the subdigraph relation. More examples of infinite antichains of tournaments were proved by Latka [32]. For any positive integer $n \geq 9$, let A_n be the tournament obtained from the transitive tournament on $[n]$ by reversing the edges in $\{(i,i+1),(1,3),(n-2,n) : 1 \leq i \leq n-1\}$. For any positive integer $n \geq 4$, let B_n be the tournament with $V(B_n) = \mathbb{Z}/((2n+1)\mathbb{Z})$ such that with $E(B_n) = \{(i,j) : j-i \in \{1,2,...,n-1,n+1\}\}$, where the computation is in $\mathbb{Z}/((2n+1)\mathbb{Z})$.

Theorem 26 ([32]). *$\{A_n : n \geq 9\}$ and $\{B_n : n \geq 4\}$ are infinite antichains with respect to the subdigraph relation.*

5.2 Induced subgraphs

A graph H is an *induced subgraph* of G if H can be obtained from G by deleting vertices. It is required to focus on simple graphs only when considering well-quasi-ordering by induced subgraph relation, since for any graph G, the set $\{G_i : i \geq 1\}$ is an infinite antichain with respect to the induced subgraph relation, where G_i is obtained from G by duplicating each edge i times. So we only focus on simple graphs in this subsection.

Let $\mathcal{F}$ be a set of graphs. Define $\text{Forb}_s^s(\mathcal{F})$ (and $\text{Forb}_{is}^s(\mathcal{F})$, respectively) to be the set of simple graphs that do not contain any member of $\mathcal{F}$ as a subgraph (and an induced subgraph, respectively). When $\mathcal{F}$ consists of one graph H, we write $\text{Forb}_s^s(\{H\})$ and $\text{Forb}_{is}^s(\{H\})$ as $\text{Forb}_s^s(H)$ and $\text{Forb}_{is}^s(H)$, respectively, for short. Theorem 24 can be restated as: $\text{Forb}_s^s(P_n)$ is well-quasi-ordered by the subgraph relation. However, Damaschke [11] showed that $\text{Forb}_{is}^s(P_n)$ is not well-quasi-ordered by the induced subgraph relation for $n \geq 5$, though it is true if $n \leq 4$.

For every positive integer k, the *k-sun*, denoted by S_k, is the simple graph obtained from a complete graph with vertex-set $\{x_i : 1 \le i \le k\}$ by adding k vertices $y_1, y_2, \ldots, y_k$ such that y_i is adjacent to x_{i-1} and x_i for each i with $1 \le i \le k$, where $x_0 = x_k$. Define $2K_2$ to be the graph that consists of a disjoint union of two copies of K_2. Clearly, for every $k \ge 4$, S_k does not contain $2K_2$ as an induced subgraph, and hence does not contain P_5 as an induced subgraph. Damaschke [11] showed that $\{S_{2k} : k \ge 2\}$ is an antichain with respect to the induced subgraph relation.

Theorem 27 ([11]). *The following statements are true.*

1. *Let H be a simple graph. Then* $\mathrm{Forb}^s_{is}(H)$ *is well-quasi-ordered by the induced subgraph relation if and only if H is an induced subgraph of P_4.*
2. $\{S_{2k} : k \ge 2\}$ *is an infinite antichain with respect to the induced subgraph relation. In particular, for every $k \ge 5$,* $\mathrm{Forb}^s_{is}(P_k)$ *is not well-quasi-ordered by the induced subgraph relation.*
3. $\mathrm{Forb}^s_{is}(\{K_3, P_5\})$ *and* $\mathrm{Forb}^s_{is}(\{K_3, K_2 \cup 2K_1\})$ *are well-quasi-ordered by the induced subgraph relation.*

Answering a question of Damaschke, Ding [14] showed that $\mathrm{Forb}^s_{is}(\{2K_2, C_4, C_5, S_4\})$ is not well-quasi-ordered by the induced subgraph relation. Ding also proved that several other families $\mathscr{F}$ in which $\mathrm{Forb}^s_{is}(\mathscr{F})$ are well-quasi-ordered by the induced subgraph relation. We refer interested readers to [14]. In the same paper, Ding [14] proposed the following conjecture about permutation graphs.

Conjecture 4 ([14]). For every positive integer $k \ge 5$, permutation graphs that do not contain P_k or the complement of P_k as a induced subgraph are well-quasi-ordered by the induced subgraph relation.

Note that Conjecture 4 concerns special classes of graphs. This special class is actually an induced subgraph ideal. There are more results concerning special classes of graphs. For example, Atminas, Brignall, Korpelainen, Lozin and Vatter [4] determined whether permutation graphs in $\mathrm{Forb}^s_{is}(\mathscr{F})$ for some families $\mathscr{F}$ with small size are well-quasi-ordered or not.

Another special class of graphs is the set of k-letter graphs introduced by Petkovšek [47]. For a positive integer k, a simple graph G is a *k-letter graph* if $V(G)$ can be partitioned into $V_1, V_2, \ldots, V_p$ for some $p \le k$, where each V_i is a clique or a stable set, such that there exists a linear ordering σ of $V(G)$ such that for each pair of distinct indices $i, j \in [p]$, either every vertex in V_i is adjacent to every vertex in V_j, or every vertex in V_i is non-adjacent to every vertex in V_j, or for every vertex x in V_i, its neighbors in V_j are the veritces y in V_j with $\sigma(x) < \sigma(y)$, or for every vertex x in V_i, its neighbors in V_j are the vertices y in V_j with $\sigma(x) > \sigma(y)$.

Theorem 28 ([47]). *For every positive integer k, the set of k-letter graphs is well-quasi-ordered by the induced subgraph relation.*

Using Theorem 28, Lozin and Mayhill [37] proved results related to unit interval graphs and bipartite permutation graphs. Note that the class of unit interval graphs and the class of bipartite permutation graphs are induced subgraph ideals. Recall that F_k is the fork of length k. Every F_k is a bipartite permutation graph, but $\{F_k : k \geq 1\}$ is an antichain with respect to the induced subgraph relation. The graph F_k^+ is defined to be the simple graph obtained from F_k by adding an edge to each pair of leaves sharing a common neighbor. Every F_k^+ is a unit interval graph, but $\{F_k^+ : k \geq 1\}$ is an antichain with respect to the induced subgraph relation.

Theorem 29 ([37]). *The following are true.*

1. *Let $\mathscr{I}$ be an induced subgraph ideal of unit interval graphs. Then $\mathscr{I}$ is well-quasi-ordered by the induced subgraph relation if and only if $\mathscr{I}$ contains finitely many members of $\{F_k^+ : k \geq 1\}$.*
2. *Let $\mathscr{I}$ be an induced subgraph ideal of bipartite permutation graphs. Then $\mathscr{I}$ is well-quasi-ordered by the induced subgraph relation if and only if $\mathscr{I}$ contains finitely many members of $\{F_k : k \geq 1\}$.*

Now let us consider $\mathrm{Forb}^s_{is}(\mathscr{F})$ in terms of the size of $\mathscr{F}$. As mentioned in Theorem 27, the family $\mathscr{F}$ with size one in which $\mathrm{Forb}^s_{is}(\mathscr{F})$ is well-quasi-ordered by the induced subgraph relation is characterized in [11]. For families $\mathscr{F}$ with $|\mathscr{F}| \geq 2$, the complete characterization for $\mathscr{F}$ such that $\mathrm{Forb}^s_{is}(\mathscr{F})$ is well-quasi-ordered by the induced subgraph relation is not known. But numerous families with size two were studied. For example, see [3, 4, 25, 24].

Following this direction, people study what the minimal non-well-quasi-ordered sets $\mathscr{S}$ of simple graphs such that $\mathscr{S} = \mathrm{Forb}^s_{im}(\mathscr{F})$ for some family $\mathscr{F}$ of simple graphs with $|\mathscr{F}| \leq k$ are. For every positive integer k, we say that a set $\mathscr{S}$ of simple graphs is *k-bad* if $\mathscr{S} = \mathrm{Forb}^s_{is}(\mathscr{F})$ for some $|\mathscr{F}| = k$, $\mathscr{S}$ is not well-quasi-ordered by the induced subgraph relation, and $\mathscr{S}$ is minimal among the sets satisfying the previous two properties. Korpelainen and Lozin [24] conjectured that for every positive integer k, there are only finitely many k-bad sets. Using Theorem 27, Korpelainen, Lozin and Razgon [26] showed that it is true when $k = 1$. Korpelainen and Lozin [24] proved the case $k = 2$. However, the case $k \geq 3$ was disproved by Korpelainen, Lozin and Razgon [26].

Theorem 30. *The following are true.*

1. *[26] The 1-bad sets are* $\mathrm{Forb}^s_{is}(C_3)$, $\mathrm{Forb}^s_{is}(C_4)$, $\mathrm{Forb}^s_{is}(C_5)$, $\mathrm{Forb}^s_{is}(3K_1)$ *and* $\mathrm{Forb}^s_{is}(2K_2)$.
2. *[24] There are only finitely many 2-bad sets.*
3. *[26] There are infinitely many k-bad sets for any $k \geq 3$. In particular, for any positive integer t with $t > k$,* $\mathrm{Forb}^s_{is}(\{K_{1,3}, C_i, C_t : 3 \leq i \leq k\})$ *is a k-bad set.*

Whether $\mathrm{Forb}^s_{is}(\mathscr{F})$ is well-quasi-ordered by the induced subgraph relation has been determined for almost all families $\mathscr{F}$ with $|\mathscr{F}| = 2$. A summary can be founded in [9]. The remaining undetermined classes are the following.

Question 2 ([9]). Let $\mathscr{F} = \{H_1, H_2\}$ for some simple graphs H_1, H_2. Determine whether $\mathrm{Forb}^s_{is}(\mathscr{F})$ is well-quasi-ordered by the induced subgraph relation for the following cases.

1. $H_1 = K_3$ and $H_2 \in \{P_1 \cup 2P_2, P_1 \cup P_5, P_2 \cup P_4\}$.
2. $H_1 = K_4^-$ and $H_2 \in \{P_1 \cup 2P_2, P_1 \cup P_4\}$.
3. $H_1 = W_4^-$ and $H_2 \in \{P_1 \cup P_4, 2P_2, P_2 \cup P_3, P_5\}$.

"Clique width" is a well-known graph parameter that is used for measuring how "homogenous" its vertices are. So graphs with smaller clique width are less complicated. Moreover, any induced subgraph H of a graph G has clique width no more than G. Hence, for any positive integer k, the set of simple graphs of clique width at most k is an induced subgraph ideal. However, every cycle has clique width at most four, so the set of simple graphs of bounded clique width is not well-quasi-ordered by the induced subgraph relation, even when the bound is four. On the other hand, intuitively, graphs in any induced subgraph ideal that can be well-quasi-ordered by the induced subgraph relation are expected not to be too "complicated". Daligault, Rao and Thomassé [10] asked whether it is true that every induced subgraph ideal containing graphs with arbitrarily large clique width cannot be well-quasi-ordered by the induced subgraph relation. However, Lozin, Razgon and Zamaraev [38] provide a negative answer of this question.

For every positive integer k, define D_k to be the simple graph with $V(D_k) = [k]$ and where two vertices i, j are adjacent if and only if either $|i-j| = 1$, or $q(i) = q(j)$, where for any $x \in [k]$, $q(x)$ is the largest number of the form 2^n (for some positive integer n) dividing x.

Theorem 31 ([38]). *Let $\mathscr{I}$ be the set of simple graphs consisting of $\{D_k : k \geq 1\}$ and all induced subgraphs of D_k for some k. Then $\mathscr{I}$ is well-quasi-ordered by the induced subgraph relation, but for every number n, there exists n' such that the clique width of $D_{n'}$ is greater than n.*

As indicated in [9], the ideal $\mathscr{I}$ mentioned in Theorem 31 cannot be written as $\text{Forb}^s_{is}(\mathscr{F})$ for some finite family $\mathscr{F}$. Dabrowski, Lozin and Paulusma [9] conjecture that the finiteness of $\mathscr{F}$ can ensure a positive answer of the question of Daligault, Rao and Thomassé mentioned above. In fact, the question of Daligault, Rao and Thomassé is motivated by another weaker conjecture of theirs (see Conjecture 9 below), and the finiteness is ensured in the setting of that weaker conjecture.

Conjecture 5 ([9]). If $\mathscr{F}$ is a finite set of simple graphs, and $\text{Forb}^s_{is}(\mathscr{F})$ is well-quasi-ordered by the induced subgraph relation, then there exists a number N such that every graph in $\text{Forb}^s_{is}(\mathscr{F})$ has clique width at most N.

Conjecture 5 is true when $|\mathscr{F}| = 1$. It follows from the fact that $\{P_4\}$ is the only family $\mathscr{F}$ with size one with $\text{Forb}^s_{is}(\mathscr{F})$ well-quasi-ordered, and the fact that every graph in $\text{Forb}^s_{is}(P_4)$ has clique width at most three. Almost all cases for $\mathscr{F}$ with $|\mathscr{F}| = 2$ are verified (see [9]), except the following.

Question 3 ([9]). Let $\mathscr{F} = \{H_1, H_2\}$ for some simple graphs H_1, H_2. Determine whether $\mathscr{F}$ satisfies Conjecture 5 or not for the following cases.

1. $H_1 = K_3$ and $H_2 = P_2 \cup P_4$.

2. $H_1 = W_4^-$ and $H_2 = P_2 \cup P_3$.

In addition, some sets that were proved to be well-quasi-ordered by the induced subgraph relation are also well-quasi-ordered even when the vertices are labelled by a well-quasi-ordering [3]. A *induced subgraph embedding* π from a graph H into a graph G is a subgraph embedding such that the image of π is an induced subgraph of G. When the vertices of G and H are labelled by a quasi-ordering Q, we say that G contains H as a *Q-labelled induced subgraph* if there exists an induced subgraph embedding π from H into G such that the label of v is less than or equal to the label of $\pi(v)$ with respect to Q, for every $v \in V(H)$. We say that a set of simple graphs is well-quasi-order by the *labelled induced subgraph relation* if for every well-quasi-ordering Q and any infinite sequence $G_1, G_2, ...$ of Q-labelled graphs in this set, there exist $1 \leq i < j$ such that G_j contains G_i as a Q-labelled-induced subgraph. Inspired by the known examples of ideals that are well-quasi-ordered by the labelled induced subgraph relation in the literature, Atminas and Lozin conjectured the following.

Conjecture 6 ([3]). Let $\mathscr{I}$ be an induced subgraph ideal that is well-quasi-ordered by the induced subgraph relation. Then $\mathscr{I}$ is well-quasi-ordered by the labelled induced subgraph relation if and only if $\mathscr{I} = \text{Forb}_{is}^{s}(\mathscr{F})$ for some finite set $\mathscr{F}$.

Conjecture 6 implies a long-standing conjecture of Pouzet [48], which we describe as follows.

Let n be a positive integer, and let Q be the quasi-ordering $([n], =)$. Let G, H be simple graphs and let f_G, f_H be functions with $f_G : V(G) \to [n]$ and $f_H : V(H) \to [n]$. We say that (G, f_G) contains (H, f_H) as an *n-induced subgraph* if there exists an induced subgraph embedding π from H into G such that $f_H(v) = f_G(\pi(v))$ for every $v \in V(H)$. A set $\mathscr{S}$ of simple graphs is *n-well-quasi-ordered* if for any infinite sequence of simple graphs $G_1, G_2, ...$ in $\mathscr{S}$ and for all functions $f_i : V(G_i) \to [n]$ for all $i \geq 1$, there exist $1 \leq j < j'$ such that $(G_{j'}, f_{j'})$ contains (G_j, f_j) as an n-induced subgraph.

Clearly, being 1-well-quasi-ordered is equivalent to being well-quasi-ordered by the induced subgraph relation. But 2-well-quasi-ordering is very different from 1-well-quasi-ordering. One evidence is that any 2-well-quasi-ordered induced subgraph ideal of simple graphs cannot contain arbitrarily long paths, but some 1-well-quasi-ordered induced subgraph ideals can. Another evidence is shown by Daligault, Rao and Thomassé [10], that every 2-well-quasi-ordered induced subgraph ideal of simple graphs can be expressed as $\text{Forb}_{is}^{s}(\mathscr{F})$ for some finite family $\mathscr{F}$. However, having more than two labels seems not different from simply having two labels. The following is conjectured by Pazout [48] and Fraïssé [19].

Conjecture 7 ([19, 48]). Let $\mathscr{I}$ be an induced subgraph ideal of simple graphs. Then $\mathscr{I}$ is 2-well-quasi-ordered if and only if $\mathscr{I}$ is n-well-quasi-ordered for all positive integers n.

We remark that Conjecture 6 implies Conjecture 7. Since any 2-well-quasi-ordered ideal is a 1-well-quasi-ordered induced ideal which is of the form $\text{Forb}_{is}^{s}(\mathscr{F})$

for some finite $\mathscr{F}$, Conjecture 6 implies that it is well-quasi-ordered by the labelled induced subgraph relation, so it is n-well-quasi-ordered for all n.

When Daligault, Rao and Thomassé [10] tried to solve Conjecture 7, they found a special kind of induced subgraph ideal, denoted by $\mathrm{NLC}_k^{\mathscr{F}}$, in which 1-well-quasi-ordering is equivalent with n-well-quasi-ordering for any n. Roughly speaking, given a positive integer k and a family of functions $\mathscr{F}$ from $[k]$ to $[k]$, the class $\mathrm{NLC}_k^{\mathscr{F}}$ consists of the simple graphs that can be generated by using k symbols and relabelling functions in $\mathscr{F}$. When $\mathscr{F}$ is the family that consists of all functions from $[k]$ to $[k]$, any graph in $\mathrm{NLC}_k^{\mathscr{F}}$ has "NLC-width" at most k. The NLC-width is equivalent with the clique width in terms of boundedness. Namely, a class of graphs has bounded NLC-width if and only if it has bounded clique width. We refer readers to [10] for formal definitions of $\mathrm{NLC}_k^{\mathscr{F}}$ and the NLC-width.

Theorem 32 ([10]). *Let k be a positive integer and let $\mathscr{F}$ be a family of functions from $[k]$ to $[k]$. Then the following are equivalent.*

1. *For any $f,g \in \mathscr{F}$, either the image of $f \circ g$ equals the image of f, or the image of $g \circ f$ equals the image of g.*
2. *$\mathrm{NLC}_k^{\mathscr{F}}$ is well-quasi-ordered.*
3. *$\mathrm{NLC}_k^{\mathscr{F}}$ is n-well-quasi-ordered for all positive integers n.*
4. *There exists M such that $P_M \notin \mathrm{NLC}_k^{\mathscr{F}}$.*

Daligault, Rao and Thomassé [10] proposed the following conjecture, which implies Conjecture 7 by using Theorem 32.

Conjecture 8 ([10]). If $\mathscr{I}$ is a 2-well-quasi-ordered induced subgraph ideal of simple graphs, then there exist a positive integer k and a family of functions from $[k]$ to $[k]$ such that $\mathscr{I} \subseteq \mathrm{NLC}_k^{\mathscr{F}}$ and $\mathrm{NLC}_k^{\mathscr{F}}$ is n-well-quasi-ordered for every positive integer n.

As a potential step to prove Conjecture 8, Daligault, Rao and Thomassé proposed a weaker conjecture in which the restriction for the relabelling functions is not concerned. (Recall that having bounded NLC-width is equivalent with having bounded clique width.)

Conjecture 9 ([10]). Let $\mathscr{I}$ be an induced subgraph ideal of simple graphs. If $\mathscr{I}$ is 2-well-quasi-ordered, then there exists M such that every graph in $\mathscr{I}$ has clique width at most M.

Recall that any 2-well-quasi-ordered induced subgraph ideal can be written as $\mathrm{Forb}_{is}^{s}(\mathscr{F})$ for some finite set of simple graphs $\mathscr{F}$. Hence Conjecture 5 implies Conjecture 9.

5.3 Rao-containments

Recall that the induced subgraph relation does not well-quasi-order simple graphs. Rao proposed a way to tweak this relation to be possibly a well-quasi-ordering.

Let n be a positive integer. We say a finite sequence $(a_1, a_2, ..., a_n)$ over nonnegative integers is *graphic* if there exists a simple graph G with $V(G) = [n]$ such that for each $i \in [n]$, the degree of i in G equals a_i. We call such a simple graph G a *realization* of $(a_1, a_2, ..., a_n)$. Rao [49] proposed the following conjecture.

Conjecture 10 ([49]). Given infinitely many graphic sequences $s_1, s_2, ...$, there exist $1 \leq j < j'$ such that some realization of $s_{j'}$ contains some realization of s_j as an induced subgraph.

Conjecture 10 was completely solved by Chudnovsky and Seymour [8] in a stronger sense (Theorem 33 below). Their proof is complicated. Altomare [2] and Sivaraman [53] gave short proofs of Conjecture 10 when there exists a number M such that every entry of every sequence is at most M.

We say that a simple graph G is *degree equivalent* with another simple graph G' if $V(G) = V(G')$ and for every vertex, its degree in G equals its degree in G'. We say that a simple graph G *Rao-contains* another simple graph H if H is an induced subgraph of a simple graph G' that is degree equivalent to G. Chudnovsky and Seymour [8] proved the following.

Theorem 33 ([8]). *If $G_1, G_2, ...$ are simple graphs, then there exist $1 \leq j < j'$ such that $G_{j'}$ Rao-contains G_j.*

It is clear that Theorem 33 implies that Conjecture 10. The concept of Rao containment can be extended to directed graphs. In fact, this extension to directed graphs plays an important role in the proof of Theorem 33. In the proof of Theorem 33, Chudnovsky and Seymour [8] reduced the problem to split graphs, and then further reduced the problem to "complete bipartite directed graphs" with respect to the directed version of Rao-containment.

We say that two directed graphs G and G' are *degree-equivalent* if their underlying graphs are the same, and every vertex has the same out-degree in G and in G'. A directed graph G *switching-contains* another directed graph H if there exists a directed graph G' degree-equivalent to G, and H is isomorphic to an induced subdigraph of G'.

The switching-containment is not a well-quasi-ordering on directed graphs. For example, the set of directed cycles is an infinite antichain with respect to the switching-containment relation. However, Chudnovsky and Seymour [7, 8] proved that switching-containment well-quasi-orders tournaments. It follows from Theorem 23 and the observation that if a tournament G contains another tournament H as a strong immersion, then G switching-contains H. Chunnovsky and Seymour [8] also proved this for the directed graphs whose underlying graphs are complete bipartite graphs, and used this fact to prove Theorem 33.

Theorem 34 ([8]). *Tournaments and directed graphs whose underlying graphs are complete bipartite graphs are well-quasi-ordered by the switching-containment relation.*

Acknowledgements The author thanks Sang-il Oum, Paul Seymour and Robin Thomas for providing some references and comments for an early draft of this article. The author also thanks the anonymous reviewer for his or her careful reading and pointing out many typos.

References

1. Altomare, C. J.: *A semigroup proof of the bounded degree case of S.B. Rao's conjecture on degree sequences and a bipartite analogue*, J. Combin. Theory Ser. B **102**, 756–759 (2012).
2. Andreae, T.: *On well-quasi-ordering-finite graphs by immersion*, Combinatorica **6**, 287–298 (1986).
3. Atminas, A., Lozin, V. V.: *Labelled induced subgraphs and well-quasi-ordering*, Order **32**, 313–328 (2015).
4. Atminas, A., Brignall, R., Korpelainen, N., Lozin, V., Vatter, V.: *Well-quasi-order for permutation graphs omitting a path and a clique*, Eletron. J. Combin. **22**, #P2.20 (2015).
5. Błasiok, J., Kamiński, M., Raymond, J.-F., Trunck, T.: *Induced minors and well-quasi-ordering*, arXiv:1510.07135.
6. Bouchet, A.: *Circle graph obstructions*, J. Combin. Theory Ser. B **60**, 107–144 (1994).
7. Chudnovsky, M., Seymour, P.: *A well-quasi-order for tournaments*, J. Combin. Theory Ser. B **101**, 47–53 (2011).
8. Chudnovsky, M., Seymour, P.: *Rao's degree sequence conjecture*, J. Combin. Theory Ser. B **105**, 44–92 (2014).
9. Dabrowski, K. K., Lozin, V. V., Paulusma, D.: *Well-quasi-ordering versus clique-width: new results on bigenic classes*, arXiv:1611.03671.
10. Daligault, J., Rao, M., Thomassé, S.: *Well-quasi-order of relabel functions*, Order **27**, 301–315 (2010).
11. Damaschke, P.: *Induced subgraphs and well-quasi-ordering*, J. Graph Theory **14**, 427–435 (1990).
12. de Fraysseix, H.: *Local complementation and interlacement graphs*, Discrete Math. **33**, 29–35 (1981).
13. Diestel, R.: Graph Theory, Springer, 2006. 3rd Edition.
14. Ding, G. *Subgraphs and well-quasi-ordering*, J. Graph Theory **16**, 489–502 (1992).
15. Ding, G. *Excluding a long double path minor*, J. Combin. Theory Ser. B **66**, 11–23 (1996).
16. Ding, G.: *Chordal graphs, interval graphs, and wqo*, J. Graph Theory **28**, 105–114 (1998).
17. Erdős, P., Pósa, L.: *On independent circuits contained in a graph*, Canad. J. Math. **17**, 347–352 (1965).
18. Fellows, M. R., Hermelin, D., Rosamond, F. A.: *Well-quasi-orders in subclasses of bounded treewidth graphs and their algorithmic applications*, Algorithmica **64**, 3–18 (2012).
19. Fraïssé, R. *Theory of relations*, in: Studies in Logic **118**, North Holland (1986).
20. Higman, G.: *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. **2**, 326–336 (1952).
21. Karmiński, M., Raymond, J.-F., Trunck, T.: *Multgraphs without large bonds are wqo by contraction*, arXiv:1412.2407.
22. Karmiński, M., Raymond, J.-F., Trunck, T.: *Well-quasi-ordering H-contraction-free graphs*, arXiv:1602.00733.
23. Kim, I., Seymour, P. *Tournament minors*, J. Combin. Theory Ser. B **112**, 138–153 (2015).
24. Korpelainen, N., Lozin, V.: *Two forbidden induced subgraphs and well-quasi-ordering*, Discrete Math **311**, 1813–1822 (2011).
25. Korpelainen, N., Lozin, V. V.: *Bipartite induced subgraphs and well-quasi-ordering*, J. Graph Theory **67**, 235–249 (2011).
26. Korpelainen, N., Lozin, V. V., Razgon, I.: *Boundary properties of well-quasi-ordered sets of graphs*, Order **30**, 723–735 (2013).

27. Kotzig, A.: *Quelques remarques sur les transformations* κ, séminaire Paris (1977).
28. Kříž, I., Sgall, J.: *Well-quasiordering depends on the labels*, Acta Sci. Math. **55**, 59–65 (1991).
29. Kříž, I., Thomas, R.: *On well-quasi-ordering finite structures with labels*, Graphs and Combin. **6**, 41–49 (1990).
30. Kruskal, J. B.: *Well-quasi-ordering, the tree theorem, and Vászonyi's conjecture*, Trans. Amer. Math. Soc. **95**, 210–225 (1960).
31. Kruskal, J. B.: *The theory of well-quasi-ordering: a frequently discovered concept*, J. Combin. Theory Ser. A **13**, 297–305 (1972).
32. Latka, B. J.: *Finitely constrained classes of homogeneous directed graphs*, J. Symbol. Logic **59**, 124–139 (1994).
33. Lewchalermvongs, C.: *Well-quasi-ordering by the induced-minor relation*, PhD Dissertation, Louisiana State University and Agricultural and Mechanical College (2015).
34. Liu, C.-H.: *Graph Structures and well-quasi-ordering*, PhD Dissertation, Georgia Institute of Technology (2014).
35. Liu, C.-H., Thomas, R.: *Excluding subdivisions of bounded degree graphs*, arXiv:1407.4428.
36. Liu, C.-H., Thomas, R.: *Robertson's conjecture. I-IV.*, in preparation.
37. Lozin, V. V., Mayhill, C.: *Canonical antichains of unit interval and bipartite permutation graphs*, Order **28**, 513–522 (2011).
38. Lozin, V. V., Razgon, I., Zamaraev, V.: *Well-quasi-ordering does not imply bounded clique-width*, in: Mayr E. (eds) Graph-Theoretic Concepts in Computer Science. WG 2015. Lecture Notes in Computer Science **9224**, 351–359 (2016).
39. Mader, W.: *Wohlquasigeordnete klassen endlicher graphen*, J. Combin. Theory, Ser. B **12**, 105–122 (1972).
40. Matoušek, J., Nešetřtil, J., Thomas, R.: *On polynomial time decidability of induced-minor-closed classes*, Commentationes Mathematicae Universitatis Carolinae **29**, 703–710 (1988).
41. Muzi, I.: *Paths and topological minors in directed and undireced graphs*, PhD Dissertation, Sapienza Universita Di Roma (2017).
42. Nash-Williams, C. St. J. A.: *On well-quasi-ordering finite trees*, Proc. Camb. Philos. Soc. **59**, 833–835 (1963).
43. Nash-Williams, C. St. J. A.: *On well-quasi-ordering trees*, in: Theory of Graphs and Its Applications (Proc. Symp. Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., 83–84 (1964).
44. Nash-Williams, C. St. J. A.: *On well-quasi-ordering infinite trees*, Proc. Comb. Philos. Soc. **61**, 697–720 (1965).
45. Oum, S.: *Rank-width and well-quasi-ordering*, SIAM J. Discrete Math. **22**, 666-682 (2008).
46. Oum, S.: *Rank-width: Algorithmic and structural results*, Discrete Appl. Math., http://dx.doi.org/10.1016/j.dam.2016.08.006.
47. Petkovšek, M.: *Letter graphs and well-quasi-order by induced subgraphs*, Discrete Math **244**, 375–388 (2002).
48. Pouzet, M.: *Un bel ordre d'abritement et ses rapports avec les bornes d'une multirelation*, C. R. Acad. Sci., Paris Sér. A-B **274**, 1677-1680 (1972).
49. Rao, S. B.: *Towards a theory of forcibly hereditary P-graphic sequences*, in: S. B. Rao (Ed.), Combinatorics and Graph Theory, Proc. Int. Conf., Kolkata, India, Feb. 1980, in: Lecture Notes in Math. **885**, Springer Verlag, 441–458 (1981).
50. Robertson, N., Seymour, P. D.: *Graph minors. XIII. The disjoint paths problem*, J. Combin. Theory Ser. B **63**, 65–110 (1995).
51. Robertson, N., Seymour, P. D.: *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory, Ser. B **92** 325–357 (2004).
52. Robertson, N., Seymour, P. D.: *Graph minors. XXIII. The Nash-Williams immersion conjecture*, J. Combin. Theory, Ser. B **100**, 181–205 (2010).
53. Sivaraman, V.: *Two short proofs of the bounded case of S.B. Rao's degree sequence conjecture*, Discrete Math. **313**, 1500–1501 (2013).
54. Tarkowski, S.: *On the comparability of dendrites*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **8**, 39–41 (1960).
55. Thomas, R.: *Graphs without* K_4 *and well-quasi-ordering*, J. Combin. Theory Ser. B **38**, 240–247 (1985).
56. Wagner, K.: Graphentheorie, 248/248a, B. J. Hochschultaschenbucher, Mannheim, 61 (1970).