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ON EFFECTIVE EQUIDISTRIBUTION FOR QUOTIENTS OF SL$(d, \mathbb{R})$

BY

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ABSTRACT

We prove the first case of polynomially effective equidistribution of closed orbits of semisimple groups with nontrivial centralizer. The proof relies on uniform spectral gap, builds on, and extends work of Einsiedler, Margulis, and Venkatesh.

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1. Introduction

Rigidity results for dynamics of group actions on homogeneous spaces, i.e., quotients of Lie groups by discrete subgroups of finite covolume, have been a subject of great interest with several striking results and applications. Indeed the solution of Oppenheim's conjecture by Margulis in [14] opened up a new chapter in the dialogue between homogeneous dynamics and number theory. The landmark results in [16, 17, 18, 20, 19] of Ratner on unipotent dynamics became quite quickly the engines to many interesting applications of homogeneous dynamics. The proofs of these rigidity results use techniques from ergodic theory and are often not quantitative. It is a challenging problem, with several applications, to provide effective and quantitative accounts of these rigidity results.

In [6] a polynomially effective equidistribution theorem for closed orbits of semisimple group H is proven under the assumption that the Lie algebra $\mathfrak{h}$ of H has trivial centralizer in the Lie algebra $\mathfrak{g}$ of the ambient group G. As explained in [6] this centralizer assumption does not seem to be truly essential to the method. We consider a first case where similar results can be obtained in the presence of a one-dimensional centralizer.

Let $k, l \in \mathbb{N}$ and assume $\mathbb{G}$ is a $\mathbb{Q}$-form of SL_{k+l} that splits over $\mathbb{R}$. Consider a $\mathbb{Q}$-embedding

$$\rho : \mathbb{G} \to \mathrm{SL}_N \tag{1.1}$$

for some $N \in \mathbb{N}$. Set $G = \mathbb{G}(\mathbb{R}) \cong \mathrm{SL}_{k+l}(\mathbb{R})$. By a theorem of Borel and Harish-Chandra $\Gamma := \rho^{-1}(\mathrm{SL}_N(\mathbb{Z})) \cap G$ is a lattice in G. We define $X = G/\Gamma$. Throughout the paper we divide matrices in Mat_{k+l} into blocks consisting of the first k or last l rows and columns. Moreover, we consider the algebraic group

$$\mathbb{H} = \mathrm{SL}_k \times \mathrm{SL}_l = \left[\begin{array}{c|c} \mathrm{SL}_k & 0 \\ \hline 0 & \mathrm{SL}_l \end{array}\right]$$

over $\mathbb{R}$; let $H = \mathbb{H}(\mathbb{R})$.

As the centralizer of H is not trivial, H-orbits may lie far from any given compact set, e.g., this happens for the $\mathbb{Q}$-split group $\mathbb{G} = \mathrm{SL}_{k+l}$ and orbits $Ha\,\mathrm{SL}_{k+l}(\mathbb{Z}) \subset \mathrm{SL}_{k+l}(\mathbb{R})/\mathrm{SL}_{k+l}(\mathbb{Z})$ for a large $a \in C_G(H)$. This is obviously an obstruction to equidistribution, and we take this possibility into account via a height function $\mathrm{ht}(\cdot)$ on X whose definition is given in (2.4) and the function $\mathrm{mht}(\mathcal{Q}) = \inf\{\mathrm{ht}(x) : x \in \mathcal{Q}\}$ for subsets $\mathcal{Q} \subset X$.

A closed subgroup $S \subset G$ containing H we will call an **intermediate** subgroup; we will show in §3 that there are only finitely many such subgroups. Similar to [6], we define the notion of volume of a closed S-orbit using a Haar measure on S. More precisely, we fix a Haar measure m_S on S and define the **volume** $\mathrm{vol}(Sx)$ of a closed S-orbit Sx to be $m_S(F)$ where $F \subset S$ is a Borel fundamental domain for the quotient map $S \to Sx$. In contrast, μ_{Sx} denotes the normalized Haar probability measure on the orbit Sx.

Theorem 1.1: *Assume that $(k,l) \neq (2,2)$ and fix G, H, Γ as above. There exist $d \in \mathbb{N}$ and $\kappa_1, \kappa_2 > 0$ depending only on G and H, and a constant $V_0 = V_0(G,H,\Gamma) > 0$ such that for all $V > V_0$ there exists an intermediate subgroup $H \subset S \subset G$ such that for any $f \in C_c^\infty(X)$ we have*

$$|\mu_{Hx_0}(f) - \mu_{Sx_0}(f)| \ll \mathcal{S}_d(f)\,\mathrm{mht}(Hx_0)^{\kappa_2} V^{-\kappa_1} \text{ and } \mathrm{vol}(Sx_0) \leq V,$$

for any closed orbit Hx_0, where the implicit constant depends only on G, H, Γ and $\mathcal{S}_d$ is a Sobolev norm (defined in §2.5).

The general strategy of the proof is similar to that of [6]. We use spectral gap to show in an effective way that most points are generic in an effective manner for the dynamics of a unipotent subgroup. In fact, we are relying on uniform spectral as provided by the so-called property (τ), which in general is the combination of results of Selberg [21] for congruence quotients of the split form of SL_2, Jaquet–Langlands [9] for other forms of SL_2, extensions of this by Burger and Sarnak [3], Kazhdan's property (T) from [10], and Clozel's work [4]. We refer also to [7, §4] for more details and a more dynamical proof of that fact. In the context of this paper Kazhdan's property (T) is sufficient once $\min(k,\ell) \geq 3$.

Using the effectively generic points and an effective version of the polynomial divergence property, that also played a big role in the work of Margulis and Ratner mentioned before, we effectively produce almost invariance under new elements transversal to H. This is then upgraded to establish almost invariance under a subgroup $S \supsetneq H$. The special choice of H, in particular, the fact that the centralizer of H is one dimensional, simplifies the proof in several places. This makes the possibilities of S quite restricted, see §3, which allows us to use well-known facts regarding effective equidistribution of horospherical orbits in our proof. The special case at hand also allows us to utilize known results from nondivergence of unipotent flows and get a much simplified version of a closing lemma—an effective closing lemma is one of the main technical ingredients in [6], which is used to handle the intermediate orbits. Indeed, we show that Sx_0 is closed unless the orbit Hx_0 is far in the cusp; see §4.

The case of $(k,l) = (2,2)$ is slightly more complicated due to the presence of further intermediate subgroups S (see §3) and we avoid it in the current paper.

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2. Notation and Preliminaries

As this work builds heavily on [6] we borrow much notation and conventions from loc. cit.

2.1. Constants and their dependency and Landau's notation. The notation $A \ll B$, meaning "there exists a constant $c_1 > 0$ so that $A \leq c_1 B$", will be used; the implicit constant c_1 is permitted to depend on $\mathbb{G}$ and ρ, but (unless otherwise noted) not on anything else. We write $A \asymp B$ if $A \ll B \ll A$. We will use $c_1, c_2, \ldots$ to denote positive constants depending on $\mathbb{G}$ and ρ (and their numbering is reset at the end of a section). If a constant (implicit or explicit) depends on another parameter or only on a certain part of $(\mathbb{G}, \rho)$, we will make this clear by writing, e.g., $\ll_\epsilon$, $c_3(N)$, etc.

We will use $\kappa_1, \kappa_2, \kappa_3, \ldots$ to denote positive constants depending only on $\dim \mathbb{G}$. We also adopt the $\star$-notation from [6]: we write $B = A^{\pm\star}$ if $B = c_4 A^{\pm\kappa_4}$. Similarly one defines $B \ll A^\star$, $B \gg A^\star$. Finally we also write $A \asymp B^\star$ if $A^\star \ll B \ll A^\star$ (possibly with different exponents).

2.2. Setup. Much of the notation below will depend on the choice of k, l and N in (1.1) which are fixed throughout the paper.

We recall the definition of congruence lattices in our setting. A congruence subgroup of $\mathrm{SL}_N(\mathbb{R})$ is a subgroup commensurable to $\mathrm{SL}_N(\mathbb{Z})$ containing a principal congruence subgroup, i.e., a kernel of the reduction map $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/D\mathbb{Z})$ for some $D \in \mathbb{N}$. We assume that $\Gamma \supset \rho^{-1}(\Gamma')$ where Γ'

is a congruence subgroup of SL_N and ρ was defined in (1.1). By the same argument as in [6, §1.6.1] Theorem 1.1 also holds for arithmetic subgroup at the cost of allowing the exponent to depend on Γ.

Given an element $g \in G$ we let

$$|g| = \max\{\|g\|_\infty, \|g^{-1}\|_\infty\}.$$

We fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{g} := \mathrm{Lie}(G)$ such that $\|[v_1, v_2]\| \leq \|v_1\|\|v_2\|$. The embedding $\rho : \mathbb{G} \to \mathrm{SL}_N$ induces a $\mathbb{Q}$-structure on $\mathfrak{g}$ and we may choose a Γ-stable lattice $\mathfrak{g}_\mathbb{Z}$ such that $[\mathfrak{g}_\mathbb{Z}, \mathfrak{g}_\mathbb{Z}] \subset \mathfrak{g}_\mathbb{Z}$. We also let $\mathrm{dist}(\cdot,\cdot)$ denote a right invariant Riemannian metric on G.

The choice of the inner product on $\mathfrak{g}$ induces a normalization of the Haar measure on any closed subgroup of G and therefore a notion of volume for orbits of these subgroups in X. Given a subgroup P and a point $x \in X$ we denote this **volume measure** as dvol and the **volume** of the orbit Px as $\mathrm{vol}(Px)$. In contrast, μ_{Px} denotes the normalized Haar probability measure on the orbit Px. Let us write

$$g.f(x) := f(g^{-1}x)$$

for $g \in G$ and $f \in C(X)$ and $x \in X$. Similarly, for any measure ν on X we let $g_*\nu$ be the measure defined by $g_*\nu(A) = \nu(g^{-1}A)$ for any Borel set $A \subset X$.

We choose $\mathfrak{r}$, a particular $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h} = \mathrm{Lie}(H) \subset \mathfrak{g}$, by letting $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1$ where, using block notation, $\mathfrak{r}_1 = \mathfrak{r}_1^+ \oplus \mathfrak{r}_1^-$ and

$$\mathfrak{r}_0 = \mathrm{span}\left[\begin{array}{c|c} l \cdot I_k & 0 \\ \hline 0 & -k \cdot I_l \end{array}\right], \quad \mathfrak{r}_1^+ = \left[\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right], \quad \mathfrak{r}_1^- = \left[\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right].$$

Let $u_k(t) \in \mathrm{SL}_k(\mathbb{R})$ denote the unipotent element

$$\begin{bmatrix} 1 & \frac{t^1}{1!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} = \exp \begin{bmatrix} 0 & t & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

and let

$$U = \{u(t) : t \in \mathbb{R}\}, \quad u(t) = \left[\begin{array}{c|c} u_k(t) & 0 \\ \hline 0 & u_l(t) \end{array}\right].$$

For $\mathfrak{s} \in \{\mathfrak{r}_1, \mathfrak{r}_1^+, \mathfrak{r}_1^-\}$ we put

$$\mathrm{Fix}_U(\mathfrak{s}) := \{w \in \mathfrak{s} : \mathrm{Ad}(u)w = w \text{ for all } u \in U\}.$$

We also define the one-parameter subgroup

$$a_t = \left[\begin{array}{c|c} e^{lt}I_k & 0 \\ \hline 0 & e^{-kt}I_l \end{array}\right], \quad A = \{a_t : t \in \mathbb{R}\}$$

whose Lie algebra is $\mathfrak{r}_0$.

For a diagonalizable element a we define the **expanding horospherical** subgroup

$$W_G^+(a) = \{g \in G : a^n g a^{-n} \to e, \text{ as } n \to -\infty\},$$

and the **contracting horospherical** subgroup

$$W_G^-(a) = W_G^+(a^{-1}).$$

Put $W := W_G^+(a_1)$ for $a_1 \in A$ as above, and note that

$$W = \left[\begin{array}{c|c} I_k & * \\ \hline 0 & I_l \end{array}\right]. \tag{2.1}$$

Finally we let $P^{\pm}$ denote the connected subgroups of G with

$$\mathrm{Lie}(P^+) = \mathrm{Lie}(H) \oplus \mathfrak{r}_1^+, \quad \mathrm{Lie}(P^-) = \mathrm{Lie}(H) \oplus \mathfrak{r}_1^-.$$

2.3. Height, discriminant and volume. Given a lattice $\mathfrak{s}_{\mathbb{Z}}$ in a vector space $\mathfrak{s}$ and a subspace $\mathfrak{l}$ that intersects $\mathfrak{s}_{\mathbb{Z}}$ in a lattice we define the covolume (or discriminant) of $\mathfrak{l}$ by setting

$$\mathrm{covol}(\mathfrak{l}) := \mathrm{disc}(\mathfrak{l}) := \|p_{\mathfrak{l}}\| \tag{2.2}$$

where $p_{\mathfrak{l}}$ is a primitive vector in

$$\textstyle\bigwedge^{\dim \mathfrak{l}} \mathfrak{l} \cap \bigwedge^{\dim \mathfrak{l}} \mathfrak{s}_{\mathbb{Z}}. \tag{2.3}$$

For an element $g \in G$ we say that a subspace $\mathfrak{l} \subset \mathfrak{g}$ is g**-rational** if $\mathfrak{l} \cap \mathrm{Ad}(g)(\mathfrak{g}_{\mathbb{Z}})$ is a lattice in $\mathfrak{l}$. It will be called simply **rational** if it is g-rational for $g = e$. Given a g-rational subspace $\mathfrak{l}$ we define the covolume of $\mathfrak{l}$ using (2.2) with $\mathfrak{s} = \mathfrak{g}$ and $\mathfrak{s}_{\mathbb{Z}} = \mathrm{Ad}(g)\mathfrak{g}_{\mathbb{Z}}$.

For a $\mathbb{Q}$-subgroup $\mathbb{L}$ of $\mathbb{G}$ we put $\mathrm{disc}(\mathbb{L}) := \mathrm{disc}(\mathrm{Lie}(\mathbb{L}))$.

Recall that the lattice $\mathfrak{g}_{\mathbb{Z}}$ is Γ-stable. Hence we may define the **height** of a point $x \in X$ by

$$\mathrm{ht}(x) = \sup\{\|\mathrm{Ad}(g).v\|^{-1} : x = g\Gamma,\ v \in \mathfrak{g}_{\mathbb{Z}} \setminus \{0\}\}. \tag{2.4}$$

The height of x in $\mathrm{SL}_N(\mathbb{R})/\mathrm{SL}_N(\mathbb{Z})$ is defined similarly. This defines the term $\mathrm{mht}(Hx_0)$ appearing in the statement of Theorem 1.1.

Let

$$\mathfrak{S}(R) = \{x \in X : \mathrm{ht}(x) \le R\}.$$

By Mahler's compactness criterion, the sets $\{\mathfrak{S}(R) : R > 0\}$ are all compact and

$$\bigcup_{R>0} \mathfrak{S}(R) = X.$$

2.4. Spectral input. Let us denote by L the group generated by U and its transpose; it is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and we call it the **principal** $\mathrm{SL}_2(\mathbb{R})$. Also let $P^{\pm}$ be defined as in §2.2.

We will use the following as a blackbox: The representations of L, the principal $\mathrm{SL}_2(\mathbb{R})$, on

$$L^2_0(\nu) = \left\{ f \in L^2(X, \nu) : \int f \,\mathrm{d}\nu = 0 \right\},$$

are $1/M_0$-tempered (i.e., the matrix coefficients of the M_0-fold tensor product are in $L^{2+\epsilon}(\mathrm{SL}_2(\mathbb{R}))$ for all $\epsilon > 0$), where ν is the S-invariant probability measure on a closed S-orbit with $S = H, P^{\pm}$ or $S = G$.

Since $H \subset S$ for all choices of S above, $1/M_0$-temperedness follows directly in the case when H has property (T), see [15, Thm. 1.1–1.2], and in the general case we may apply property (τ) in the strong form, see [4], [8] and [6, §6].

2.5. Sobolev norms. We now recall the definition of a certain family of Sobolev norms and their main properties (see [6, §3.7]). For any integer $d \ge 0$ we let $\mathcal{S}_d$ be the Sobolev norm on X defined by

$$\mathcal{S}_d(f)^2 = \sum_{\mathcal{D}} \|\mathrm{ht}(\cdot)^d \mathcal{D} f\|_2^2, \tag{2.5}$$

where $f \in C_c^\infty(X)$ and the sum is taken over all $\mathcal{D} \in U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, which are monomials in a chosen basis of $\mathfrak{g}$ of degree at most d. We will need the following properties of $\mathcal{S}_d$. There exists a constant κ_5 such that for any $d \ge \kappa_5$ and any $g \in G$ and $f \in C_c^\infty(X)$, we have:

($\mathcal{S}$-1) For any $g \in G$ and $f \in C_c^\infty(X)$ we have

$$\mathcal{S}_d(g.f) \ll_d |g|^{3d} \mathcal{S}_d(f).$$

($\mathcal{S}$-2) For any $f \in C_c^\infty(X)$ we have

$$\|f\|_\infty \ll_d \mathcal{S}_d(f).$$

($\mathcal{S}$-3) For any $g \in G$ and $f \in C_c^\infty(X)$ we have

$$\|g.f - f\|_\infty \ll_d \operatorname{dist}(e,g)\mathcal{S}_d(f).$$

($\mathcal{S}$-4) For any $f_1, f_2 \in C_c^\infty(X)$ we have

$$\mathcal{S}_d(f_1 f_2) \ll_d \mathcal{S}_{d+\kappa_5}(f_1)\mathcal{S}_{d+\kappa_5}(f_2).$$

($\mathcal{S}$-5) Let ν and M_0 be as in §2.4. We have

$$|\langle u(t).f_1, f_2\rangle_{L^2(\nu)} - \nu(f_1)\nu(\bar{f}_2)| \ll_d (1+|t|)^{-\frac{1}{2M_0}}\mathcal{S}_d(f_1)\mathcal{S}_d(f_2). \tag{2.6}$$

For a discussion of the Sobolev norm, the reason for introducing the factor $\operatorname{ht}(\cdot)^d$, and the proofs of the above properties we refer to [6, §5].

Let $\mathcal{S}_d$ be as above and let $\epsilon > 0$. We say that a measure σ is ϵ-almost invariant under $g \in G$ (w.r.t. $\mathcal{S}_d$) if

$$|\sigma(g.f) - \sigma(f)| \le \epsilon\, \mathcal{S}_d(f) \quad \text{for all } f \in C_c^\infty(X).$$

We say that σ is ϵ-almost invariant under a subgroup $L \subset G$ if it is ϵ-almost invariant under all $g \in L$ with $|g| \le 2$. Similarly, given $w \in \mathfrak{g}$ we say σ is ϵ-almost invariant under w if σ is ϵ-almost invariant under $\exp(tw)$ for all $|t| \le 2$.

3. Structure of intermediate subgroups

Any finite-dimensional representation of H decomposes into irreducible sub-representations as H is semisimple. In order to study the connected intermediate subgroups we may work with the Lie algebra of G; see [1, §7]. Consider the adjoint representation of H on $\operatorname{Lie}(G)$. It decomposes as

$$\operatorname{Lie}(H) \oplus \mathfrak{r}_1^+ \oplus \mathfrak{r}_1^- \oplus \mathfrak{r}_0.$$

Indeed, it is easily verified that each of these factors are sub-representations and a dimension count shows that it is a complete decomposition. The analysis of the possible intermediate subgroups follows simply from noting that for any intermediate closed subgroup with $H \subset S \subset G$, $\operatorname{Lie} S$ will be a sub-representation, which is also a Lie subalgebra, of $\operatorname{Lie}(G)$.

Proposition 3.1: *Fix (k,l) with $\max\{k,l\} \geq 3$ and let S be a closed connected subgroup with $H \subset S \subset G$. Then*

$$S \in \{H, P^+, P^-, AH, AP^+, AP^-, G\},$$
$$AP^+ = \exp(\mathrm{Lie}(H) \oplus \mathfrak{r}_1^+ \oplus \mathfrak{r}_0), \quad AP^- = \exp(\mathrm{Lie}(H) \oplus \mathfrak{r}_1^- \oplus \mathfrak{r}_0)$$
$$AH = \exp(\mathrm{Lie}(H) \oplus \mathfrak{r}_0).$$

Proof. Note that $\mathfrak{r}_1^+$ and $\mathfrak{r}_1^-$ are both irreducible and are dual to each other. If $\max\{k,l\} \geq 3$, then we claim that the representations $\mathfrak{r}_1^+$ and $\mathfrak{r}_1^-$ are non-isomorphic. Indeed, if they were isomorphic, then they will be isomorphic also as a representation of the larger block, say of $\mathrm{SL}_k < H$. Note that as an SL_k representation $\mathfrak{r}_1^+$ is a direct sum of the standard representation of SL_k on $\mathbb{R}^k$ and $\mathfrak{r}_1^-$ is a direct sum of its dual. If they were isomorphic then the standard representation on SL_k on $\mathbb{R}^k$ is isomorphic to its dual, which is a contradiction when $k \geq 3$ (e.g., because $\mathrm{diag}(t, \ldots, t, t^{-(k-1)}) \in \mathrm{SL}_k$ and its inverse cannot be conjugated to each other when $k \geq 3$).

The proposition now follows as the possible subrepresentations of $\mathrm{Lie}(G)$ which contain $\mathrm{Lie}(H)$ correspond exactly to the Lie algebras of the groups listed above. ∎

For the cases $(k,l) \in \{(2,2),(2,1),(1,2)\}$ we have that $\mathfrak{r}_1^+$ and $\mathfrak{r}_1^-$ are isomorphic as representations of H. When $k = l = 2$ this isomorphism gives rise to a family of subgroups, which are isomorphic to $\mathrm{Sp}(4)$. This case will probably also yield to the methods of this paper, but requires a special treatment in each step, and therefore we avoid it in the current paper. In contrast, we have:

Proposition 3.2: *Proposition 3.1 holds also when $k=2, l=1$ (or $k=1, l=2$).*

Proof. Fix an isomorphism $\phi : \mathfrak{r}_1^+ \to \mathfrak{r}_1^-$ and let

$$\mathfrak{s}_p := \{(v, p\phi(v)) : v \in \mathfrak{r}_1^+\}$$

for $p \in \mathbb{R}$. The proposition will follow once we will show that the H-subrepresentation $\mathrm{Lie}(H) \oplus \mathfrak{s}_p$ or $\mathrm{Lie}(H) \oplus \mathfrak{s}_p \oplus \mathfrak{r}_0$ for $p \in \mathbb{R} \setminus \{0\}$ are never Lie subalgebras. This follows just by calculations of Lie brackets. Indeed, for concreteness, let e_1, e_2 (resp. f_1, f_2) denote the standard basis of $\mathfrak{r}_1^+$ (resp. $\mathfrak{r}_1^-$) and fix ϕ to be the isomorphism sending

$$\alpha e_1 + \beta e_2 \mapsto -\beta f_1 + \alpha f_2.$$

Now, a direct calculation shows that the commutator of $(e_1, p\phi(e_1)) \in \mathfrak{s}_p$ and $(e_2, p\phi(e_2)) \in \mathfrak{s}_p$ is a nontrivial element of $\mathfrak{r}_0$ (whenever $p \neq 0$). Another calculation shows that the Lie algebra generated by $\mathfrak{r}_0$ and $\mathfrak{s}_p$ contains $\mathfrak{r}_1^+ \oplus \mathfrak{r}_1^- \oplus \mathfrak{r}_0$, so the Lie subalgebra generated by $\mathrm{Lie}(H) \oplus \mathfrak{s}_p$ is always $\mathrm{Lie}(G)$. ∎

4. Applying nondivergence of unipotent flows

When $\mathrm{rank}_{\mathbb{Q}}\mathbb{G} = 0$ the quotient X is compact. In this case the addition of the height function in the definition of the Sobolev norm in not needed and several other analytic arguments become simpler. In particular, this section is only important in the case that $\mathrm{rank}_{\mathbb{Q}}\mathbb{G} > 0$.

Let us recall the definition of certain functions $d_\alpha : G \to \mathbb{R}$. These functions were considered by Dani and Margulis in [5] in order to study the recurrence properties of unipotent flows on homogeneous spaces. Let $\mathbb{S}$ be a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbb{G}$. Let $\mathbb{P} \supset \mathbb{S}$ be a minimal $\mathbb{Q}$-parabolic subgroup and let Δ be the associated simple roots relative to $\mathbb{S}$; see [2, Sec. 12]. For $\alpha \in \Delta$, let $\mathbb{P}_\alpha$ be the corresponding maximal $\mathbb{Q}$-parabolic subgroup. Let $\mathbb{U}_\alpha = R_u(\mathbb{P}_\alpha)$ be the unipotent radical and let $\mathfrak{u}_\alpha$ denote the Lie algebra of $\mathbb{U}_\alpha$. Put $\ell_\alpha := \dim \mathfrak{u}_\alpha$ and let $\vartheta_\alpha = \wedge^{\ell_\alpha}\mathrm{Ad}$ denote the ℓ_α-th exterior power of the adjoint representation. Note that $\wedge^{\ell_\alpha}\mathfrak{u}_\alpha$ defines a $\mathbb{Q}$-rational one-dimensional subspace of $\wedge^{\ell_\alpha}\mathfrak{g}$. Fix a unit vector $v_\alpha \in \wedge^{\ell_\alpha}\mathfrak{u}_\alpha$. Note that if $g \in \mathbb{P}_\alpha(\mathbb{R})$, then

$$\vartheta_\alpha(g)v_\alpha = \det(\mathrm{Ad}(g)|_{\mathfrak{u}_\alpha})v_\alpha.$$

Define $d_\alpha : G \to \mathbb{R}$ by

$$d_\alpha(g) = \|\vartheta_\alpha(g)v_\alpha\| \quad \text{for all } g \in G.$$

For each $\alpha \in \Delta$ put $\mathbb{P}_\alpha^{(1)} = \{g \in \mathbb{P}_\alpha : \vartheta_\alpha(g)v_\alpha = v_\alpha\}$. Put $P_\alpha = \mathbb{P}_\alpha(\mathbb{R})$ and $P_\alpha^{(1)} = \mathbb{P}_\alpha^{(1)}(\mathbb{R})$. Since $P_\alpha^{(1)}$ is a $\mathbb{Q}$-group without any $\mathbb{Q}$-characters, it follows from a theorem of Borel and Harish-Chandra that $P_\alpha^{(1)}\Gamma/\Gamma$ is a closed orbit with a finite $P_\alpha^{(1)}$-invariant measure.

Theorem 4.1 (Cf. [5], Theorem 2): *There exist a finite subset $\Xi \subset \mathbb{G}(\mathbb{Q})$, and some $R_0 > 0$ with the following property. For every $x = g\Gamma \in X$ there exists T_x so that one of the following holds:*

1. $|\{|t| \le T : u(t)g\Gamma \in \mathfrak{S}(R_0)\}| \ge (1 - 2^{-20})T$ *for all* $T > T_x$.
2. *There exist $\lambda \in \Gamma\Xi$ and $\alpha \in \Delta$ such that $g^{-1}Ug \subset \lambda P_\alpha^{(1)}\lambda^{-1}$ and moreover $d_\alpha(g\lambda) < 1$.*

We note that if $\mathbb{G}$ has $\mathbb{Q}$-rank zero, then X is compact and the height function is bounded. In particular, the first case in the theorem and its corollary below hold trivially.

Corollary 4.2: *Let $y = g\Gamma$ be so that Hy is a closed orbit. Then one of the following holds:*

(1) $\mu_{Hy}(Hy \setminus \mathfrak{S}(R_0)) \leq 2^{-10}$.
(2) *There exist $\lambda \in \Gamma\Xi$, $\alpha \in \Delta$ and $h \in H$ with $|h| \leq 2$ such that $g^{-1}Hg \subset \lambda P_\alpha^{(1)} \lambda^{-1}$, and $d_\alpha(hg\lambda) < 1$.*

Proof. Let $h \in H$ with $|h| \leq 2$ be so that $hg\Gamma \in Hy$ is a generic point for the action of $U = \{u(t) : t \in \mathbb{R}\}$ in the sense of the Birkhoff ergodic theorem; that is

$$\lim_T \frac{1}{T} \int_0^T f(u(t)hg\Gamma)\,\mathrm{d}t = \int f\,\mathrm{d}\mu_{Hy} \tag{4.1}$$

for all $f \in C_c(X)$.

We consider two possibilities. First let us assume that Theorem 4.1(1) holds true for $x = hg\Gamma$. Then the conclusion (1) of Corollary 4.2 holds by (4.1).

Therefore, let us assume that Theorem 4.1(2) holds true for $x = hg\Gamma$ and we show that the conclusion (2) of Corollary 4.2 must hold true. Then there exist some $\lambda \in \Gamma\Xi$ and some $\alpha \in \Delta$ so that $d_\alpha(hg\lambda) < 1$ and

$$g^{-1}h^{-1}Uhg \subset \lambda P_\alpha^{(1)} \lambda^{-1}.$$

We claim that $g^{-1}Hg \subset \lambda P_\alpha^{(1)}\lambda^{-1}$. Since $\lambda \in \Gamma\Xi$ and $\Xi \subset \mathbb{G}(\mathbb{Q})$ we have that the orbit $\lambda P_\alpha^{(1)}\lambda^{-1}\Gamma/\Gamma$ is a closed orbit. Hence,

$$\overline{g^{-1}h^{-1}Uhg\Gamma/\Gamma} \subset \lambda P_\alpha^{(1)}\lambda^{-1}\Gamma/\Gamma. \tag{4.2}$$

However, by (4.1) we have

$$\overline{g^{-1}h^{-1}Uhg\Gamma/\Gamma} = g^{-1}Hg\Gamma/\Gamma.$$

This together with (4.2) implies that $g^{-1}Hg \subset \lambda P_\alpha^{(1)}\lambda^{-1}$ as we claimed. ∎

Lemma 4.3 (Mass in the cusp): *There exist constants κ_6, κ_7 depending only on G and c_1 such that for any periodic H-orbit Hx, we have*

$$\mu_{Hx}(Hx \setminus \mathfrak{S}(R)) \leq c_1 \mathrm{mht}(Hx)^{\kappa_7} R^{-\kappa_6}.$$

Proof. Let $z \in Hx$ be the point of the smallest height, namely $\mathrm{ht}(z) = \mathrm{mht}(Hx)$. By the Birkhoff ergodic theorem, there exists some $h \in H$ with $|h| \leq 1$ such that

$$\lim_{T\to\infty} \frac{1}{T}|\{0 < t < T : \mathrm{ht}(u(t)hz) > L\}| = \mu_{Hx}(Hx \setminus \mathfrak{S}(L)) \tag{4.3}$$

for all $L \in \mathbb{N}$.

Let $hz = g\Gamma$ for some element $g \in G$. Let $\mathfrak{s}$ be a g-rational subspace of $\mathfrak{g}$. As was done in [6, App. B], define the function

$$\psi_{\mathfrak{s}}(t) = \mathrm{covol}(\mathrm{Ad}(u_t)\mathfrak{s})^2.$$

Note that $\psi_{\mathfrak{s}}$ is a polynomial whose degree is bounded in terms of $\dim G$ only. On the other hand, since $|h| \leq 1$ we have

$$\psi_{\mathfrak{s}}(0) = \mathrm{covol}(\mathfrak{s})^2 \gg \mathrm{ht}(z)^{-m}$$

for some absolute constant m depending on N. We have, by [12, Thm. 5.2], for any $T > 0$

$$\frac{1}{T}|\{0 < t < T : \mathrm{ht}(u_t hz) > R\}| \ll \left(\frac{\mathrm{ht}(z)^m}{R}\right)^{\kappa_6} = \frac{\mathrm{mht}(Hx)^{\kappa_7}}{R^{\kappa_6}},$$

where κ_6 depends on the degree of $\psi_{\mathfrak{s}}$ and $\kappa_7 := m\kappa_6$. This together with (4.3) implies the claim. ∎

Given a closed orbit Hx, define

$$R_{Hx} := \frac{2^{20/\kappa_6}\mathrm{mht}(Hx)^{\kappa_7/\kappa_6}}{{c_1}^{1/\kappa_6}}. \tag{4.4}$$

This choice in view of Lemma 4.3 implies the following:

$$\mu_{Hx}(Hx \setminus \mathfrak{S}(R_{Hx})) \leq 2^{-20}. \tag{4.5}$$

5. From generic points to new almost invariants

Let ν and M_0 be as in §2.4; we continue to denote by μ the H-invariant probability measure on Hx_0. Let $M = 20M_0$ and let $T \geq 1$ be a parameter. Following [6] we define for a function f

$$D_{T,\nu}(f)(x) = \frac{1}{(T+1)^M - T^M}\int_{T^M}^{(T+1)^M} f(u(t)x)\,\mathrm{d}t - \int_X f\,\mathrm{d}\nu. \tag{5.1}$$

We write D_T for $D_{T,\mu}$. A point $x \in X$ is called T_0**-generic** for the measure ν

w.r.t. a Sobolev norm $\mathcal{S}$ if for all integers $n > T_0$ and all $f \in C_c^\infty(X)$ we have

$$|D_{n,\nu}(f)(x)| \leq n^{-1}\mathcal{S}(f). \tag{5.2}$$

Furthermore, a point $x \in X$ is called $[T_0, T_1]$**-generic** if (5.2) is satisfied for all $n \in [T_0, T_1]$.

Let $\mathrm{d}w$ be the Lebesgue measure on $W \cong \mathbb{R}^{kl}$, see (2.1), and for any $\tau > 0$ put

$$W[\tau] := \{w \in W : \|w\|_\infty \leq \tau\}.$$

Lemma 5.1 (Cf. [6], §9.1): (1) *For $d \gg 1$, depending only on G, the ν-measure of the points that are not T_0-generic for ν with respect to $\mathcal{S}_d$ is $\ll T_0^{-1}$.*

(2) *There exists $d' > d$ depending on d and G with the following property. Suppose*

$$|w_*\mu(f) - \mu(f)| \ll \epsilon\mathcal{S}_d(f) \quad \textit{for all } w \in W[\tau].$$

Then there exists some κ_8 so that the proportion of points $(w, x) \in W[\tau] \times X$ such that wx is not $[T_0, \epsilon^{-\kappa_8}]$-generic w.r.t. $\mathcal{S}_{d'}$ is $\ll T_0^{-1}$.

Proof. First note that using (2.6) we have the following estimate on the L^2-norm of $D_{T,\nu}(f)$:

$$\int_X |D_{T,\nu}(f)|^2 \,\mathrm{d}\nu \ll T^{-4}\mathcal{S}_d(f)^2. \tag{5.3}$$

The deduction of part (1) from (5.3) is identical to that of [6, Prop. 9.1].

For (2), consider the integral

$$\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X |D_T(f)(wx)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}w. \tag{5.4}$$

The inner integral of (5.4) satisfies

$$\int_X |D_T f(wx)|^2 \,\mathrm{d}\mu(x) = w_*\mu(|D_T(f)|^2) \ll \epsilon\mathcal{S}_d(|D_T(f)|^2) + \mu(|D_T(f)|^2).$$

By properties of Sobolev norm in §2.5

$$\mathcal{S}_d(|D_T(f)|^2) \ll T^{\star d}\mathcal{S}_{d+\kappa_5}(f)^2.$$

Combining this with (5.3) we have

$$\int_X |D_T f(wx)|^2 \,\mathrm{d}\mu(x) \ll \epsilon T^{\star d}\mathcal{S}_{d+\kappa_5}(f)^2 + T^{-4}\mathcal{S}_d(f)^2.$$

Therefore, if we choose T so that $\epsilon T^{\star d} = T^{-4}$, then

$$\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X |D_T(f)(wx)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}w \ll T^{-4}\mathcal{S}_{d+\kappa_5}(f)^2. \tag{5.5}$$

The deduction of part (2) from (5.5) is identical to that of part (1); see also [6, Prop. 9.1–9.2]. ∎

The following lemma provides us with generic points which differ in directions transversal to H. The proof is based on a pigeonhole principle argument. We note that in this lemma the existence of the centralizer $\mathfrak{r}_0$ starts to play a more significant role.

Lemma 5.2 (Cf. [6], Prop. 14.1): *There exist κ_9 and κ_{10} with the following property. Let Hx_0 be a closed orbit so that*

$$\mathrm{vol}(Hx_0) \geq \mathrm{mht}(Hx_0)^{\kappa_9}. \tag{5.6}$$

Then there exist $w \in \mathfrak{r}\setminus\{0\}$ and $x, y \in \mathfrak{S}(R_{Hx_0}) \cap Hx_0$ so that the following hold:

1. $\|w\| \leq (\mathrm{vol}(Hx_0))^{-\kappa_{10}}$. *Moreover, if we decompose $w = w_0 + w_1$ into a sum of $w_0 \in \mathfrak{r}_0$ and $w_1 \in \mathfrak{r}_1$, then $\|w_1'\| \gg \|w_1\|$ where*

$$w_1 = w_1' + w_1'' \in \mathrm{Fix}_U(\mathfrak{r}_1)^{\perp} \oplus \mathrm{Fix}_U(\mathfrak{r}_1)$$

and the decomposition is with respect to the Euclidean structure on $\mathfrak{r}_1$ which is induced by $\|\cdot\|$.

2. $\exp(w)x = y$.

Further, given T_0 large enough, x and y can be chosen to be T_0-generic.

Proof. Let $T_0 > 1$, and let E' be the set of T_0-generic points. Put

$$E := E' \cap \mathfrak{S}(R_{Hx_0}).$$

In view of Lemma 5.1(1) and the choice of R_0, see (4.4), we have $\mu(X\setminus E) \leq 2^{-10}$ assuming T_0 is sufficiently large depending on G, H and Γ.

For any $\delta > 0$ let $\mathfrak{r}_\delta$ (resp. $\mathfrak{h}_\delta$) denote the ball of radius δ in $\mathfrak{r}$ (resp. $\mathfrak{h}$) around the origin with respect to the norm $\|\cdot\|$ on $\mathfrak{g}$. Throughout the proof we will put more and more restrictions on δ. To begin with, let $\delta > 0$ be smaller than $1/20$ of the minimum of the injectivity radii at all $z \in \mathfrak{S}(R_{Hx_0})$. This constraint amounts to an inequality of the form

$$\delta \ll R_{Hx_0}^{-\star}. \tag{5.7}$$

Moreover, assume δ is small enough so that the map

$$(r,v)\in\mathfrak{r}_\delta\times\mathfrak{h}_\delta\mapsto\exp(r)\exp(v)$$

is a diffeomorphism onto its image in G. Therefore, for any $z\in\mathfrak{S}(R_{Hx_0})$ the natural map from $\pi_z:\mathfrak{r}_\delta\times\mathfrak{h}_\delta\to X$ defined by $\pi_z(r,v):=\exp(r)\exp(v)z$ is a diffeomorphism. Let $\Omega=\exp(\mathfrak{h}_{2\delta})$. Define the following function on X:

$$\phi(z)=\frac{1}{\mathrm{vol}(\Omega)}\int_\Omega\chi_E(hz)\,\mathrm{dvol}(h). \tag{5.8}$$

We have $\int_X\phi(z)d\mu=\mu(E)\geq 1-2^{-10}$. Put

$$F:=\{z\in E:\phi(z)>0.99\}. \tag{5.9}$$

Then $\mu(F)\geq 0.9$.

For δ chosen as above define

$$\mathsf{B}(z,\delta):=\pi_z(\mathfrak{r}_\delta\times\mathfrak{h}_\delta).$$

We may cover F by $\ll\delta^{-\dim G}$-many sets of the form $\mathsf{B}(z,\delta)$ with $z\in F$ and with finite multiplicity depending only on G. Using the pigeonhole principle we have the following: So long as

$$\mathrm{vol}(Hx_0)\gg\delta^{\dim H-\dim G}, \tag{5.10}$$

there exist $x',y'\in F\cap\mathsf{B}(z,\delta)$ for some $z\in F$ so that $x'\neq hy'$ for any $h\in\Omega$.

We now want to perturb x',y' to obtain elements of E that satisfy the above claimed properties. First, note that since $\mathfrak{r}_1^+$ and $\mathfrak{r}_1^-$ are irreducible representations of H, there is a constant $\iota>0$ such that

$$\mathrm{vol}\{h\in\Omega:\|\mathrm{Ad}(h)(r)'\|\leq\iota\|r\|\}<\mathrm{vol}(\Omega)/2 \tag{5.11}$$

for all $r\in\mathfrak{r}_1$ where r' is the component of r in $\mathrm{Fix}_U(\mathfrak{r}_1)^\perp$. Now, if we apply the Implicit Function Theorem and use the fact that $\phi(x')>0.99$ and $\phi(y')>0.99$, we can find $h_1,h_2\in\Omega$ such that

- $h_1x',h_2y'\in E$ and
- $h_2y'=\exp(w)h_1x'$ where $w\in\mathfrak{r}$ and $\|w\|\ll\delta$,
- $\|w_1'\|\gg\|w_1\|$.

Therefore, w, $x=h_1x'$, and $y=h_2y'$ satisfy the conclusion of the proposition so long as we choose $\delta=\mathrm{vol}(Hx_0)^{-\star}$ so that (5.10) holds. In light of our assumption (5.6) it is possible to satisfy this and our earlier requirement in (5.7). ∎

Lemma 5.3: *Let $w \in \mathfrak{r}$ be the "difference" found in Lemma 5.2. Then*

$$\exp(w) \notin C_G(H).$$

Proof. Let us write $\exp(w)x = hx$ for some $h \in H$ and $x = g\Gamma$. We also let $\mathbb{H}$ be the connected, simply connected, algebraic group such that $\mathbb{H}(\mathbb{R}) = g^{-1}Hg$. Note that $\mathbb{H}$ is defined over $\mathbb{Q}$ as Hx is a closed orbit. With this notation we have

$$\exp(-w)hg\Gamma = g\Gamma,$$

and the claim is equivalent to showing $g^{-1}\exp(-w)g \notin C_{\mathbb{G}}(\mathbb{H})(\mathbb{R})$.

Assume this is not the case. Then

$$\gamma = g^{-1}\exp(-w)gg^{-1}hg \in \mathbb{L} := C_{\mathbb{G}}(\mathbb{H}) \cdot \mathbb{H}.$$

Note that $C_{\mathbb{G}}(\mathbb{H})$ is one-dimensional. We define a set of characters on $\mathbb{L}$, $\Delta = \{\chi_1, \chi_2\}$, as follows. First note that for $\ell \in \mathbb{L}(\mathbb{R})$, $g\ell g^{-1}$ has a block structure, that is, it has the form

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right].$$

We define $\{\chi_1(\ell), \chi_2(\ell)\}$ to be the determinants of the diagonal blocks of $g\ell g^{-1}$. This determines Δ. Further note that since $\mathbb{H}$ is semisimple, any character on $\mathbb{H}$ is trivial. Moreover, for $a \in \mathbb{L}(\mathbb{R})$ we have that $a \in \mathbb{H}(\mathbb{R})$ if and only if a is in the kernels of χ_i, $i = 1, 2$.

Furthermore, we have that $\chi_2 = \chi_1^{-1}$ and that Δ is stable under the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so either χ_1 and χ_2 are both defined over $\mathbb{Q}$ or over a real quadratic extension (as the centralizer splits over $\mathbb{R}$). In both cases, the integrality of γ implies that $\chi_i(\gamma)$ is either 1 or uniformly bounded away from 1. Indeed in the second case $\chi_i(\gamma) \in \mathcal{O}^\times \subset \mathbb{R}$ where $\mathcal{O}$ is an order in a real quadratic extension. And if a unit u in a quadratic field is close to 1, then $u + u^{-1}$ is close to 2 and an integer which implies that $u = 1$. In particular this argument is independent of the quadratic extension.

On the other hand, as χ_i are trivial on $\mathbb{H}$, we have for $i = 1, 2$

$$\chi_i(\gamma) = \chi_i(g^{-1}\exp(-w)g).$$

By definition the characters χ_i are defined by conjugating elements of $\mathbb{L} = C_{\mathbb{G}}(\mathbb{H}) \cdot \mathbb{H}$ to $C_G(H)H$ and taking the determinants of one of the blocks. In other words, we are taking the determinants of the blocks of the matrix

$\exp(-w)$, which is at distance $\ll (\mathrm{vol}(Hx_0))^{-\kappa_{10}}$ from the identity. However, with the above it follows that

$$\chi_i(g^{-1}\exp(-w)g) = 1 \quad \text{for } i = 1, 2.$$

Since the kernel of these characters is $\mathbb{H}(\mathbb{R})$, this contradict the fact that $g^{-1}\exp(-w)g \notin \mathbb{H}(\mathbb{R})$. ∎

We now use the effective ergodic theorem, Lemma 5.1(1), and the above results to prove the following.

Lemma 5.4 (Cf. [6], Prop. 10.1): *Assume that* (5.6) *holds. There exists some* $v \in \mathrm{Fix}_U(\mathfrak{r}_1)$ *with* $\|v\| = 1$ *so that*

$$|\exp(tv)_*\mu(f) - \mu(f)| \le \mathrm{vol}(Hx_0)^{-\star}\mathcal{S}(f), \quad \textit{for all } |t| \ll 1.$$

Proof. We will show that there exists some $v \in \mathrm{Fix}_U(\mathfrak{r}_1)$ with $\|v\| = 1$ so that

$$|\exp(v)_*\mu(f) - \mu(f)| \le \mathrm{vol}(Hx_0)^{-\star}\mathcal{S}(f). \tag{5.12}$$

The claim for all $|t| \ll 1$ will follow from this using conjugation by elements in H as in Lemma 6.2, or alternatively, by an argument as in the proof of [6, Prop. 10.1].

Let T_0 and $x, y \in Hx_0$ be as in Lemma 5.2. In particular, x and y are T_0-generic, $y = \exp(w)x$ with $w \in \mathfrak{r}$, $\|w\| \ll \mathrm{vol}(Hx_0)^{-\star}$, and $\|w_1'\| \gg \|w_1\|$. Recall also that by Lemma 5.3 we have $w_1 \neq 0$.

Let us write $w = w_0 + w_1'' + w_1'$ as in Lemma 5.2. Then

$$\mathrm{Ad}(u(t))w = w_0 + w_1'' + \mathrm{Ad}(u(t))w_1'.$$

Therefore, there exists $T_1 \gg \|w_1'\|^{-\star}$ and a polynomial $p : \mathbb{R} \to \mathrm{Fix}_U(\mathfrak{r}_1)$ with degree $\ll N$ and $\sup\{\|p(t)\| : 0 \le t \le 1\} = \|p(1)\| = 1$ so that

$$\mathrm{Ad}(u(t))w = p(t/T_1) + O(\|w\|^\star) \tag{5.13}$$

for all $t \in [0, 2T_1]$. This is the polynomial divergence property of unipotent flows relying on the fact that $\mathrm{Ad}(u(t))w$ is a $\mathfrak{g}$-valued polynomial whose terms of highest degree belong to $\mathrm{Fix}_U(\mathfrak{r})$.

Let now $n > T_0$ be so that $T_1 \in [n^M, (n+1)^M]$ where M is as in (5.1). Then by (5.2) we have

$$|D_n(f)(z)| \le n^{-1}\mathcal{S}(f) \quad \text{for } z = x, y.$$

In view of (5.13), property ($\mathcal{S}$-3), and the fact that $f \in C_c^\infty(X)$ we have

$$f(u(t)y) = f(\exp(p(t/T_1))u(t)x) + O(\|w\|^\star)\mathcal{S}(f)$$

for all $t \in [0, 2T_1]$. Moreover, for any $t \in [n^M, (n+1)^M]$ we have

$$|t - T_1| \ll T_1^{1-1/M}.$$

Hence all together we get

$$f(u(t)y) = f(\exp(p(1))u(t)x) + O(T_1^{-1/M}\mathcal{S}(f)) + O(\|w\|^\star)\mathcal{S}(f).$$

Therefore, $\mu(f) = \mu(\exp(p(1)).f) + O(T_1^{-1/M} + \|w\|^\star)\mathcal{S}(f)$.

Since $\|p(1)\| = 1$, $T_1 \gg \|w_1'\|^{-\star}$, and $\|w\| \ll \mathrm{vol}(Hx_0)^{-\star}$ we get (5.12). ∎

6. Effective generation of a bigger group

We continue to use the previous notation. Let us first recall the following.

Proposition 6.1 (Cf. [6], Proposition 8.1): *Let $\mathcal{S}_d$ be a fixed Sobolev norm. Suppose that μ is ϵ-almost invariant w.r.t. $\mathcal{S}_d$ under $w \in \exp(\mathfrak{r}_1)$ with $\|w\| = 1$. Then there exists $\kappa_{11} > 0$ so that μ is $c(d)\epsilon^{\kappa_{11}}$-almost invariant w.r.t. $\mathcal{S}_d$ under at least one of the groups P^+, P^-, or G.*

Proof. We note that [6, Prop. 8.1] is proved in the general setting that applies to our situation, i.e., the assumption on triviality of the centralizer is not used in the proof of [6, Prop. 8.1]. The claim thus follows from the results in §3. ∎

We also record the following.

Lemma 6.2 (Cf. [6], Lemma 8.2): *There exists $\kappa_{12} > 0$ with the following property. Let $S = P^+$, P^-, or G and suppose that μ is ϵ-almost invariant under S w.r.t. $\mathcal{S}_d$. Then*

$$|\mu(q.f) - \mu(f)| \ll \epsilon|q|^{\kappa_{12}}\mathcal{S}_d(f), \quad q \in S.$$

Proof. For $S = G$, this is proved in [6, Lemma 8.2]. Let us assume $S = P^+$. In view of our assumption we have

$$|\mu(q.f) - \mu(f)| \ll \epsilon\mathcal{S}_d(f), \quad \text{for all } |q| \le 2. \tag{6.1}$$

In particular, (6.1) holds true for $q_{ij} = 1 + E_{ij}$ with $j > i$. Let now $a \in H$ be a diagonal element with $|a| \ll t^\star$ so that

$$aq_{ij}a^{-1} = 1 + tE_{ij} =: q_{ij}(t).$$

Since μ is invariant under $a_{ij}(t)$, the above, (6.1) and properties of the Sobolev norm, see §2.5, imply that

$$|\mu(q_{ij}(t).f) - \mu(f)| \ll t^\star \epsilon \mathcal{S}_d(f) \ll |q_{ij}(t)|^\star \epsilon \mathcal{S}_d(f).$$

Since W is abelian we obtained the lemma for elements in W. Since μ is H-invariant this gives the claim for $S = P^+$. The proof for $S = P^-$ is similar. ∎

7. Proof of Theorem 1.1

Recall that dw is the Lebesgue measure on $W \cong \mathbb{R}^{kl}$, and for any $\tau > 0$ we put $W[\tau] := \{w \in w : \|w\|_\infty \leq \tau\}$. We let m denote the G-invariant probability measure on X.

Lemma 7.1: *There exists a constant κ_{13} satisfying the following property. Let $s \geq 1$, put $\tau = e^{s(k+l)}$. Suppose that*

$$a_{-s} z \in \mathfrak{S}(R),$$

for $z \in X$. Then for any $f \in C_c^\infty(X)$ we have

$$\left| \frac{1}{|W[\tau]|} \int_{W[\tau]} f(wz)\, \mathrm{d}w - \int_X f \, \mathrm{d}m \right| \ll R^\star e^{-\kappa_{13} s} \mathcal{S}_d(f). \tag{7.1}$$

Proof. By the definition of τ, note that $W[\tau] = a_s W[1] a_{-s}$. Denote

$$y := a_{-s} z \in \mathfrak{S}(R).$$

We have

$$\frac{1}{|W[\tau]|} \int_{W[\tau]} f(wz)\, \mathrm{d}w = \frac{1}{|W[1]|} \int_{W[1]} f(a_s w y)\, \mathrm{d}w.$$

Now using [11, Prop. 2.4.8] (see also [13, Thm. 2.3] for the dependence on the height R) there exists a $\kappa > 0$ so that the following holds:

$$\left| \frac{1}{|W[1]|} \int_{W[1]} f(a_s w y)\, \mathrm{d}w - \int_X f \, \mathrm{d}m \right| \ll R^\star e^{-\kappa s} \mathcal{S}_d(f). \tag{7.2}$$ ∎

We will also need the following for the proof.

Lemma 7.2: *Suppose there exists some $\kappa > 0$ so that*

$$\mathrm{mht}(Hx_0) \gg \mathrm{vol}(Hx_0)^\kappa.$$

Then Theorem 1.1 holds (trivially).

Proof. If $\mathrm{vol}(Hx_0) < V$, the theorem holds trivially with $S = H$.

On the other hand, our assumption implies the theorem with $S = G$ if $V \le \mathrm{vol}(Hx_0)$. Indeed, we may assume $\kappa_2 \ge \kappa_1/\kappa$ so that

$$\mathrm{mht}(Hx_0)^{\kappa_2} V^{-\kappa_1} \gg \mathrm{vol}(Hx_0)^{\kappa\kappa_2-\kappa_1} \gg 1.$$

This implies the conclusion of the theorem because of ($\mathcal{S}$-2) in §2.5. ∎

Proof of Theorem 1.1. Let μ denote the H-invariant probability measure on Hx_0 and $x_0 = g_0\Gamma$. By Lemma 7.2 we may assume (5.6). Using Lemma 5.4 we get almost invariance under an element in $\mathfrak{r}_1$. Then by Proposition 6.1 and Lemma 6.2 we get almost invariance under a subgroup $S = G, P^+$, or P^-. Since the case $S = P^-$ is similar, we may assume $S = G$ or P^+. Therefore, we assume throughout the argument that for any $f \in C_c^\infty(X)$ the following holds:

$$|w_*\mu(f) - \mu(f)| \ll \epsilon S_d(f) \quad \text{for all } w \in W[\tau], \tag{7.3}$$

with $\tau = \mathrm{vol}(Hx_0)^{\kappa_{14}}$ and $\epsilon = \mathrm{vol}(Hx_0)^{-\star}$. Here κ_{14} needs to be small enough to get (7.3); we will need to optimize κ_{14} further in the argument below.

We investigate

$$\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X f(wx)\,\mathrm{d}\mu(x)\,\mathrm{d}w. \tag{7.4}$$

First note that (7.3) implies

$$\left|\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X f(wx)\,\mathrm{d}\mu(x)\,\mathrm{d}w - \int_X f\,\mathrm{d}\mu\right| \ll \epsilon\mathcal{S}_d(f). \tag{7.5}$$

Let s be a parameter so that $\tau = e^{s(k+l)}$ and put $\mu_s = a_{-s}\mu$. Apply Corollary 4.2 to the measure μ_s and the closed orbit

$$a_{-s}Hx_0 = Ha_{-s}x_0.$$

By the conclusion of that corollary there are two cases to consider.

Case 1. Assume Corollary 4.2(1) holds for μ_s. That is

$$\mu_s(Ha_{-s}x_0 \setminus \mathfrak{S}(R_0)) \le 2^{-10}. \tag{7.6}$$

For every R put

$$B_{s,R} := Hg_0\Gamma \setminus a_s\mathfrak{S}(R).$$

Then by Lemma 4.3, for any $R > 1$ we have

$$\mu(B_{s,R}) = \mu_s(Ha_{-s}x_0 \setminus \mathfrak{S}(R)) \ll R^{-\kappa_6}; \tag{7.7}$$

note that in view of (7.6), we have $\mathrm{mht}(Ha_{-s}x_0) \ll R_0$ and R_0 as chosen in Theorem 4.1 satisfies $R_0 \ll 1$.

Let $R > R_0$; using Fubini's theorem we can now rewrite (7.4) in the form

$$\begin{aligned}\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X f(wx)\,\mathrm{d}\mu(x)\,\mathrm{d}w =& \frac{1}{|W[\tau]|}\int_X\int_{W[\tau]} f(wx)\,\mathrm{d}w\,\mathrm{d}\mu(x)\\ =& \frac{1}{|W[\tau]|}\int_{a_s\mathfrak{S}(R)}\int_{W[\tau]} f(wx)\,\mathrm{d}w\,\mathrm{d}\mu(x)\\ &+ \frac{1}{|W[\tau]|}\int_{B_{s,R}}\int_{W[\tau]} f(wx)\,\mathrm{d}w\,\mathrm{d}\mu(x).\end{aligned}$$

By (7.7) and property ($\mathcal{S}$-2) of the Sobolev norm the second term above is $\ll \mathcal{S}_d(f)R^{-\star}$. For the first term, note that by Lemma 7.1 for any

$$z \in Hx_0 \setminus B_{s,R} = Hx_0 \cap a_s\mathfrak{S}(R)$$

we have

$$\left|\frac{1}{|W[\tau]|}\int_{W[\tau]} f(wz)\,\mathrm{d}w - \int_X f\,\mathrm{d}m\right| \ll R^\star e^{-\kappa_{13}s}\mathcal{S}_d(f).$$

Hence using (7.7) one more time we get

$$\begin{aligned}&\left|\frac{1}{|W[\tau]|}\int_{a_s\mathfrak{S}(R)}\int_{W[\tau]} f(wx)\,\mathrm{d}w\,\mathrm{d}\mu(x) - \int_X f\,\mathrm{d}m\right|\\ &\qquad\ll R^\star e^{-\kappa_{13}s}\mathcal{S}_d(f) + R^{-\star}\mathcal{S}_d(f).\end{aligned}$$

Putting these together and recalling that $\tau = e^{\star s}$ we get

$$\begin{aligned}&\left|\frac{1}{|W[\tau]|}\int_{W[\tau]}\int_X f(wx)\,\mathrm{d}\mu(x)\,\mathrm{d}w - \int_X f\,\mathrm{d}m\right|\\ &\qquad\ll R^\star \tau^{-\star}\mathcal{S}_d(f) + R^{-\star}\mathcal{S}_d(f).\end{aligned} \tag{7.8}$$

Recall that so far our constraint on τ was only $\tau = \mathrm{vol}(Hx_0)^{\kappa_{14}}$ as in (7.3). If we now choose $R = \mathrm{vol}(Hx_0)^\star$ and a small enough exponent we obtain $R^\star\tau^{-\star} \ll \mathrm{vol}(Hx_0)^{-\star}$; then (7.5) and (7.8) imply

$$|\mu(f) - m(f)| \ll \mathrm{vol}(Hx_0)^{-\star}\mathcal{S}_d(f),$$

and hence the theorem in this case.

We note that if the $\mathbb{Q}$-rank of $\mathbb{G}$ is zero, then we are always in case 1 of Corollary 4.2.

Case 2. Recall again that μ_s is supported on $Ha_{-s}x_0 = Ha_{-s}g_0\Gamma$. We now assume that Corollary 4.2(2) holds for μ_s. In view of this assumption, there exist $\lambda \in \Gamma\Xi$ and $\alpha \in \Delta$ such that

$$g_0^{-1}a_sHa_{-s}g_0 = g_0^{-1}Hg_0 \subset \lambda P_\alpha^{(1)}\lambda^{-1}; \tag{7.9}$$

moreover, there exists some $h_0 \in H$ with $|h_0| \leq 2$ so that

$$d_\alpha(h_0a_{-s}g_0\lambda) < 1. \tag{7.10}$$

Recall from §4 that $P_\alpha^{(1)} = \{g \in G : \vartheta_\alpha(g)v_\alpha = v_\alpha\}$, where v_α corresponds to a rational subspace of $\mathfrak{g}$. In view of (7.9), we have $Hg_0\lambda \subset g_0\lambda P_\alpha^{(1)}$; hence,

$$\vartheta_\alpha(hg_0\lambda)v_\alpha = \vartheta_\alpha(g_0\lambda)v_\alpha \quad \text{for all } h \in H. \tag{7.11}$$

Using the definition of ϑ_α and v_α again, we get from the above that

$$\mathrm{Ad}(g_0\lambda)\mathrm{Lie}(R_u(P_\alpha))$$

is an invariant subspace for adjoint action of H. Since $R_u(P_\alpha)$ is a unipotent group, this and the discussion in §3 imply that

$$g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^+ \quad \text{or} \quad g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^-. \tag{7.12}$$

We will consider these two subcases separately.

Subcase 1. Assume first that

$$g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^-.$$

We claim that under this assumption we have

$$\mathrm{mht}(Hx_0) \gg \tau^{\kappa_{15}} = \mathrm{vol}(Hx_0)^{\kappa_{14}\kappa_{15}} \tag{7.13}$$

for some $\kappa_{15} > 0$ depending only on G. Note that (7.13) implies the theorem in view of Lemma 7.2.

We now turn to the proof of (7.13). First note that since we deal with the case $g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^-$ and both $\{a_s\}$ and H normalize W^-, we get that

$$\vartheta_\alpha(a_{-s}h_0g_0\lambda)v_\alpha \in \wedge^{\dim W^-}\mathrm{Lie}(W^-) \tag{7.14}$$

and $\|\vartheta_\alpha(a_{-s}h_0g_0\lambda)v_\alpha\| = d_\alpha(a_{-s}h_0g_0\lambda)$.

We now have

$$\begin{aligned}
\|\vartheta_\alpha(hg_0\lambda)v_\alpha\| =&\|\vartheta_\alpha(h_0g_0\lambda)v_\alpha\| && \text{by (7.11)}\\
=&\|\vartheta_\alpha(a_sa_{-s}h_0g_0\lambda)v_\alpha\|\\
=&e^{-\star s}\|\vartheta_\alpha(a_{-s}h_0g_0\lambda)v_\alpha\| && \text{by (7.14)}\\
=&e^{-\star s}d_\alpha(a_{-s}h_0g_0\lambda)\\
\ll& e^{-\star s} && \text{by (7.10).}
\end{aligned}$$

Recall from §4 that $\vartheta(\lambda)v_\alpha$ corresponds to a rational subspace of $\mathfrak{g}$. Therefore, from the above we get that $\mathrm{Ad}(hg_0)\mathfrak{g}_{\mathbb{Z}}$ has a nontrivial sublattice with volume $\ll e^{-\star s}$ for every $h \in H$. The claim in (7.13) thus follows from Minkowski's theorem on successive minima in the geometry of numbers and since

$$e^{s(k+l)} = \tau = \mathrm{vol}(Hx_0)^{\kappa_{14}}.$$

Subcase 2. Assume now that

$$g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^+.$$

This assumption together with (7.9) and the definitions of P^+, implies that

$$\lambda^{-1}g_0^{-1}P^+g_0\lambda \subset P_\alpha^{(1)} = \{g \in G : \vartheta_\alpha(g)v_\alpha = v_\alpha\}.$$

Hence we have

(7.15) $\lambda^{-1}g_0^{-1}P^+g_0\lambda$ is the connected component of the identity in $P_\alpha^{(1)}$.

In particular, with $\lambda \in \Gamma\Xi \subset \mathbb{G}(\mathbb{Q})$ this implies that $P^+g_0\Gamma/\Gamma$ is a closed orbit.

Claim: *We have*

$$\mathrm{vol}(P^+g_0\Gamma/\Gamma) \ll \tau^\star. \tag{7.16}$$

Let us assume the claim and finish the proof. We will show that

$$|\mu(f) - \mu_{P^+x_0}(f)| \ll \mathrm{vol}(Hx_0)^{-\star}\mathcal{S}_d(f)$$

for any $f \in C_c^\infty(X)$, which will finish the proof.

The argument is based on finding a point which is generic both for μ and $\mu_{P^+x_0}$ unless we are in a situation where Lemma 7.2 is applicable. We first fix some notation.

Recall the definition of R_{Hx_0} from (4.4). We apply the first part of the proof of Lemma 5.2, up to and including the estimate for the set F. For this part of the argument we only have to choose T_0 sufficiently large and δ has to satisfy the constraint $\delta \ll R_{Hx_0}^{-\star}$. So we define

$$\delta = R_{Hx_0}^{-\star}$$

appropriately and obtain the set $F \subset \mathfrak{S}(R_{Hx_0})$ as in (5.9) for this T_0.

Define

$$T_1 := \epsilon^{-\kappa_8/2} = \mathrm{vol}(Hx_0)^\star \tag{7.17}$$

where ϵ is as in (7.3). Recall that we may assume (5.6) by Lemma 7.2, so that in particular $T_1 > T_0$.

In view of the above claim, we will further assume that κ_{14} in $\tau = \mathrm{vol}(Hx_0)^{\kappa_{14}}$ is small enough so that

$$T_1^{-1}\,\mathrm{vol}(P^+x_0) < T_1^{-\star}. \tag{7.18}$$

We now turn to the construction of a generic point for μ and μ_{P^+x}.

Let

$$P_\delta^+ := \{\exp(v) : v \in \mathrm{Lie}(P^+), \|v\| \le \delta\}.$$

Equip $P_\delta^+ \times X$ with the product measure $m_{P^+} \times \mu$ where m_{P^+} is the Haar measure on P^+. Now (7.3) together with an argument as in Lemma 5.1(2) (see also [6, Prop. 9.1]), implies that the portion of points in $P_\delta^+ \times X$, so that gx is not $[T_1, \epsilon^{-\kappa_8}]$-generic for μ w.r.t. $\mathcal{S}_{d'}$, is $\ll T_1^{-1}$.

Recall that $\mu(F) \ge 0.9$. Therefore, using Fubini's Theorem, we get the following. There exists a point $x_1 \in F$, which is T_1-generic for μ so that

$$m_{P^+}(|\{g \in P_\delta^+ : gx_1 \text{ is not } [T_1, \epsilon^{-\kappa_8}]\text{-generic for } \mu\}) \ll T_1^{-1/2} m_{P^+}(P_\delta^+). \tag{7.19}$$

We may assume that the upper bound in (7.19) is $< 0.1 m_{P^+}(P_\delta^+)$ (for otherwise $\mathrm{vol}(Hx_0)$ is bounded).

Recall that $\mu_{P^+x_0}$ is the normalized probability P^+-invariant measure on P^+x_0. Applying Lemma 5.1(1) with $\mu_{P^+x_0}$ we get that the $\mu_{P^+x_0}$ measure of the set of points which are not T_1-generic for $\mu_{P^+x_0}$ w.r.t. $\mathcal{S}_d$ is $\ll T_1^{-1}$. Therefore, taking the restriction of $\mu_{P^+x_0}$ to $P_\delta^+ x_1 \subset P^+x_1 = P^+x_0$ we get the following. If

$$\mu_{P^+x_0}(P_\delta^+ x_1) \gg T_1^{-1}$$

(with a suitably chosen implicit multiplicative constant), then

$$m_{P^+}(\{g \in P_\delta^+ : gx_1 \text{ is } T_1\text{-generic for } \mu_{P^+x_0} \text{ w.r.t. } \mathcal{S}_d\}) \ge 0.9 m_{P^+}(P_\delta^+). \tag{7.20}$$

In consequence, either (7.20) holds or δ is very small in the sense that $\mu_{P^+x_0}(P^+_\delta x_1) \ll T_1^{-1}$. Let us first assume that the former holds. Then, in view of (7.19) and (7.20), we may replace x_1 by gx_1 for some $g \in P^+_\delta$ so that gx_1 is $[T_1, \epsilon^{-\kappa_8}]$-generic for μ, moreover, gx_1 is T_1-generic for $\mu_{P^+x_0}$. This, in particular, implies that

$$|\mu(f) - \mu_{P^+x_0}(f)| \ll T^{-1}\mathcal{S}_d(f) \tag{7.21}$$

for all $T \in [T_1, \epsilon^{-\kappa_8}]$.

In view of (7.17), (7.21) completes the proof in this case.

It thus remains to consider that $\mu_{P^+x_0}(P^+_\delta x_1) \ll T_1^{-1}$. First note that this is to say $m_{P^+}(P^+_\delta) \ll T_1^{-1}\operatorname{vol}(P^+x_0)$. This and (7.18) imply that there exists some κ_{16} so that

$$T_1^{\kappa_{16}} = \operatorname{vol}(Hx_0)^\star \ll \operatorname{mht}(Hx_0) = \delta^{-\star}.$$

This again implies the theorem by Lemma 7.2.

The proof thus is complete modulo the claim in (7.16). ∎

Proof of Claim. Recall from (7.15) that $\lambda^{-1}g_0^{-1}P^+g_0\lambda = P^+_\alpha$ is the connected component of the identity in $P^{(1)}_\alpha$ and that $\Xi \subset \mathbb{G}(\mathbb{Q})$ is a finite set. For any $\xi \in \Xi$, define

$$P_{\alpha,\xi} := \xi P_\alpha \xi^{-1}, \quad P^{(1)}_{\alpha,\xi} = \xi P^{(1)}_\alpha \xi^{-1}, \quad P^+_{\alpha,\xi} = \xi P^+_\alpha \xi^{-1}, \quad \text{and} \quad v_{\alpha,\xi} = \vartheta_\alpha(\xi)v_\alpha.$$

Let $A_{\alpha,\xi}$ denote the center of the Levi component of $P_{\alpha,\xi}$. Then $A_{\alpha,\xi}P^+_{\alpha,\xi}$ has finite index in $P_{\alpha,\xi}$. Let $B_{\alpha,\xi} \subset P_{\alpha,\xi}$ denote a set of representatives for $P_{\alpha,\xi}/A_{\alpha,\xi}P^+_{\alpha,\xi}$.

Write $\lambda = \gamma\xi \in \Gamma\Xi$. Since $G = \operatorname{SO}(k+\ell)P_{\alpha,\xi}$ the above discussion implies that we may write

$$g_0\gamma = cba_\gamma p^+$$

where $c \in \operatorname{SO}(k+\ell)$, $b \in B_{\alpha,\xi}$, $a_\gamma \in A_{\alpha,\xi}$, and $p^+ \in P^+_{\alpha,\xi}$. In particular, since $P^+g_0\Gamma = g_0\gamma P^+_{\alpha,\xi}\Gamma$, we have

$$\operatorname{vol}(P^+g_0\Gamma/\Gamma) \asymp \operatorname{vol}(a_\gamma P^+_{\alpha,\xi}\Gamma/\Gamma). \tag{7.22}$$

Therefore, it suffices to bound $\operatorname{vol}(a_\gamma P^+_{\alpha,\xi}\Gamma/\Gamma)$.

The argument is similar to the one in Subcase 1. Note first that by (7.10) we have

$$\|\vartheta_\alpha(a_{-s}h_0g_0\lambda)v_\alpha\| = d_\alpha(a_{-s}h_0g_0\lambda) < 1$$

for some $|h_0| \leq 2$. Also recall that $g_0\lambda R_u(P_\alpha)\lambda^{-1}g_0^{-1} = W^+$. Therefore arguing as in Subcase 1, with W^+ in place of W^-, we have

$$\|\vartheta_\alpha(h_0g_0\lambda)v_\alpha\| = \|\vartheta_\alpha(a_sa_{-s}h_0g_0\lambda)v_\alpha\| \ll \tau^\star.$$

Now, in view of the fact that $P^+_{\alpha,\xi} \subset P^{(1)}_{\alpha,\xi}$, the above discussion implies

$$\begin{aligned}\|\vartheta_\alpha(a_\gamma)v_{\alpha,\xi}\| &\asymp \|\vartheta_\alpha(g_0\gamma)v_{\alpha,\xi}\| \\ &= \|\vartheta_\alpha(g_0\gamma\xi)v_\alpha\| \\ &= \|\vartheta_\alpha(h_0g_0\lambda)v_\alpha\| \ll \tau^\star,\end{aligned}$$

where (7.12) and $\lambda = \gamma\xi$ were used in the last equality. This implies that $|a_\gamma| \ll \tau^\star$. Now since the distortion of the volume when applying g is bounded by $|g|^\star$ we get (7.16) from this bound and (7.22). ∎

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