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EFFECTIVE EQUIDISTRIBUTION AND PROPERTY (τ)

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1. Introduction

1.1. **Homogeneous sets and measures.** Number theoretical problems often relate to orbits of subgroups (periods) and so can be attacked by dynamical methods. To be more specific let us recall the following terminology.

Let $X = \Gamma\backslash G$ be a homogeneous space defined by a lattice $\Gamma < G$ in a locally compact group G. Note that any subgroup $H < G$ acts naturally by right multiplication on X, sending $h \in H$ to the map $x \in X \mapsto xh^{-1}$. We will refer to H as the acting subgroup. A *homogeneous* (*probability*) *measure* on X is, by definition, a probability measure μ that is supported on a single closed orbit $Y = \Gamma g H_Y$ of its stabilizer $H_Y = \mathrm{Stab}(\mu)$. A *homogeneous set* is the support of some homogeneous probability measure. In what follows, we shall deal only with probability measures and shall consequently simply refer to them as homogeneous measures.

Ratner's celebrated measure classification theorem [54] and the so-called linearization techniques (cf. [18] and [45]) imply in the case where G is a real Lie group that, given a sequence of homogeneous probability measures $\{\mu_i\}$ with the property that $H_i = \mathrm{Stab}(\mu_i)$ contain "enough" unipotents, any weak* limit of $\{\mu_i\}$

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is also homogeneous, where often the stabilizer of the weak* limit has bigger dimension than H_i for every i. This has been extended also to quotients of S-algebraic groups (see [55], [44], [26, App. A], and [31, Sect. 6]) for any finite set S of places. We note that the latter allow similar corollaries (see [31]) for adelic quotients *if* the acting groups H_i contain unipotents at one and the same place for all i – let us refer to this as a *splitting condition*. These theorems have found many applications in number theory (see, e.g., [28], [26], and [31] to name a few examples), but are (in most cases) ineffective.

Our aim in this paper is to present one instance of an adelic result which is entirely quantitative in terms of the "volume" of the orbits, and is in many cases not accessible, even in a non-quantitative form, by the measure classification theorem and linearization techniques (as we will dispense with the splitting condition). A special case of this result will recover "property (τ)" (but with weaker exponents) from the theory of automorphic forms.

1.2. **Construction of homogeneous measures.** In the following F will always denote a number field, $\mathbb{A}$ will denote the ring of adeles over F, and $\mathbf{G}$ will be a connected semisimple algebraic F-group. We will consider the homogeneous space $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$ defined by the group $G = \mathbf{G}(\mathbb{A})$ of $\mathbb{A}$-points of $\mathbf{G}$.

We normalize the Haar measure vol_G on G so that the induced measure on X (again denoted by vol_G) is a probability measure. Let us fix the following data $\mathscr{D} = (\mathbf{H}, \iota, g)$ consisting of

(1) an F-algebraic group $\mathbf{H}$ such that $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ has finite volume,
(2) an algebraic homomorphism $\iota : \mathbf{H} \to \mathbf{G}$ defined over F with finite kernel, and
(3) an element $g \in G$.

To this data, we may associate a homogeneous set

$$Y_{\mathscr{D}} := \iota(\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}))g \subset X$$

and the *algebraic* homogeneous measure $\mu_{\mathscr{D}}$ given by the push-forward, under the map $x \mapsto \iota(x)g$, of the normalized Haar measure on $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$. We refer to such a set Y as an *algebraic* homogeneous set; we say it is simple, semisimple, simply connected, etc., according to whether the algebraic group $\mathbf{H}$ is so, and we say it is maximal if $\iota(\mathbf{H}) \subset \mathbf{G}$ is a maximal[1] proper subgroup.

Our main theorem will discuss the equidistribution of maximal semisimple simply connected homogeneous sets. The assumption that $\mathbf{H}$ is simply connected can be readily removed, as we explain in §7.11.

1.3. **The intrinsic volume of a homogeneous set.** What does it mean for a homogeneous set to be "large"?

If $H = \mathrm{Stab}(\mu)$ is fixed, then one may define the volume of an H-orbit xH using a fixed Haar measure on H. However, as we will allow the acting group H to vary we give another reasonably intrinsic way of measuring this, as we now explain.

Let $Y = Y_{\mathscr{D}}$ be an algebraic homogeneous set with corresponding probability measure $\mu_{\mathscr{D}}$ and associated group $H_{\mathscr{D}} = g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$. We shall always consider $H_{\mathscr{D}}$ as equipped with that measure, denoted by $m_{\mathscr{D}}$, which projects to $\mu_{\mathscr{D}}$ under the orbit map.

[1]Here by maximality we mean maximal as an algebraic group over the algebraic closure of F.

Fix an open subset $\Omega_0 \subset \mathbf{G}(\mathbb{A})$ that contains the identity and has compact closure. Set

$$\mathrm{vol}(Y) := m_{\mathscr{D}}(H_{\mathscr{D}} \cap \Omega_0)^{-1}; \tag{1.1}$$

this should be regarded as a measure of the "volume" of Y. It depends on Ω_0, but the notions arising from two different choices of Ω_0 are comparable to each other, in the sense that their ratio is bounded above and below; see §2.3. Consequently, we do not explicate the choice of Ω_0 in the notation.

The above notion of the volume of an adelic orbit is strongly related to the discriminant of the orbit; see Appendix B. The theorem below could also be phrased using this notion of arithmetic height or complexity instead of the volume.

1.4. **Notation for equidistribution in X.** If in addition $\mathbf{G}$ is simply connected and $\iota(\mathbf{H})$ is a maximal subgroup of $\mathbf{G}$ we will show in this paper that a homogeneous measure $\mu_{\mathscr{D}}$ as above is almost equidistributed if it already has large volume. Dropping the assumption that $\mathbf{G}$ is simply connected we need the following notation: Let $\mathbf{G}(\mathbb{A})^+$ denote the image of the simply connected cover (see also §2.1). Using this we define the decomposition

$$L^2(X, \mathrm{vol}_G) = L^2_0(X, \mathrm{vol}_G) \oplus L^2(X, \mathrm{vol}_G)^{\mathbf{G}(\mathbb{A})^+}, \tag{1.2}$$

where $L^2(X, \mathrm{vol}_G)^{\mathbf{G}(\mathbb{A})^+}$ denotes the space of $\mathbf{G}(\mathbb{A})^+$-invariant functions and $L^2_0(X, \mathrm{vol}_G)$ is the orthogonal complement of $L^2(X, \mathrm{vol}_G)^{\mathbf{G}(\mathbb{A})^+}$. Note that if $\mathbf{G}$ is simply connected, then $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{A})^+$ and $L^2(X, \mathrm{vol}_G)^{\mathbf{G}(\mathbb{A})^+}$ is the space of constant functions.

The group $\mathbf{G}(\mathbb{A})^+$ is a closed, normal subgroup of $\mathbf{G}(\mathbb{A})$; see, e.g., [49, p. 451]. Therefore, the subspaces introduced in (1.2) are $\mathbf{G}(\mathbb{A})$-invariant. Let

$$\pi^+ : L^2(X, \mathrm{vol}_G) \to L^2(X, \mathrm{vol}_G)^{\mathbf{G}(\mathbb{A})^+}$$

denote the orthogonal projection. Let $C_c^\infty(X)$ denote the space of smooth compactly supported functions on X; see §7.5 for a discussion. Finally let us note that given $f \in C_c^\infty(X)$

$$\pi^+ f(x) = \int_X f \, \mathrm{d}\mu_{x\mathbf{G}(\mathbb{A})^+}.$$

It is worth noting that $\pi^+ f$ is a finite-valued function for all $f \in C_c^\infty(X)$.

1.5. **Theorem** (Equidistribution of adelic periods)**.** *Let $Y_{\mathscr{D}}$ be a maximal algebraic semisimple homogeneous set arising from $\mathscr{D} = (\mathbf{H}, \iota, g)$. Furthermore, assume that $\mathbf{H}$ is simply connected. Then*

$$\left| \int_{Y_{\mathscr{D}}} f \, \mathrm{d}\mu_{\mathscr{D}} - \pi^+ f(y) \right| \ll \mathrm{vol}(Y_{\mathscr{D}})^{-\kappa_0} \mathcal{S}(f) \quad \textit{for all } f \in C_c^\infty(X),$$

where $y \in Y_{\mathscr{D}}$ is arbitrary, $\mathcal{S}(f)$ denotes a certain adelic Sobolev norm (see §7.5 and Appendix A), and κ_0 is a positive constant which depends only on $[F : \mathbb{Q}]$ and $\dim \mathbf{G}$.

Below we will abbreviate the assumption that $Y_{\mathscr{D}}$ is a **m**aximal **a**lgebraic **s**emisimple **h**omogeneous set in the theorem by saying that $Y_{\mathscr{D}}$ is a MASH set (resp., $\mu_{\mathscr{D}}$ is a MASH measure). We stated the above theorem under the natural assumption that $\mathbf{H}$ is simply connected (but note that $\iota(\mathbf{H})$ may not be simply connected). In §7.11 we discuss a formulation of the theorem without that assumption.

Let us highlight two features of this theorem. Our method relies on a uniform version of Clozel's property (τ) (see [12], [30, Thm. 1.11], and §4.2 for a summary of the history). However, it also allows us to give an independent proof of Clozel's part of the proof of property (τ) except for groups of type A_1 – i.e., if we only suppose property (τ) for groups of type A_1, we can deduce property (τ) in all other cases as well as our theorem. We will discuss this in greater detail in §4. The theorem also allows $\mathbf{H}$ to vary without any splitting condition (as, e.g., in [31, Thm. 1.7]); an application of this to quadratic forms is given in §3.

1.6. **An overview of the argument.** To overcome the absence of a splitting condition we make crucial use of Prasad's volume formula in [50] to find a small place where the acting group has good properties (see §§5 and 6.1 for a summary). This is needed to make the dynamics at this place useful.

The dynamical argument uses unipotent flows (but we note that one could also give an argument using the mixing property). Assuming that the volume is large, we find by a pigeon-hole principle nearby points that have equidistributing orbits. Using polynomial divergence of the unipotent flow we obtain almost invariance under a transverse direction. By maximality and spectral gap on the ambient space we conclude the equidistribution; see §7.

The first difficulty is to ensure that one really can choose a place which is "sufficiently small", relative to the size of the orbit. Using [50] we establish a logarithmic bound for the first useful ("good") prime in terms of the volume – see §5. We also need to use [6] if $\iota(\mathbf{H})$ is not simply connected (as in that case the stabilizer of $\mu_{\mathscr{D}}$ is larger than $H_{\mathscr{D}} = g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$ which affects the notion of volume).

The second difficulty is that we also need to know that there are many points for which the unipotent orbit effectively equidistributes with respect to the measure in question. This effectivity also relies on spectral gap, but as the measure $\mu_{\mathscr{D}}$ (and so its L^2-space) varies we need uniformity for this spectral gap. This is a uniform version of Clozel's property (τ) (see §4.2).

After completion of this project the first author, R. Rühr, and P. Wirth worked out a special case [25] that goes slightly beyond the setting of this paper. However, due to the concrete setting many of the difficult ingredients of this paper were not needed in [25], which may make it more accessible for some readers.

1.7. **Uniform non-escape of mass.** We also note the following corollary of the above which does not seem to follow from the standard non-divergence results alone.[2]

Corollary. *Let $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. Then for every $\epsilon > 0$ there exists some compact $X_\epsilon \subset X$ such that $\mu(X_\epsilon) > 1 - \epsilon$ for every MASH measure μ on X.*

We will prove the corollary in §7.12.

2. Notation and preliminary statements

2.1. **Notation.** Let us recall that F denotes a number field. Throughout the paper Σ denotes the set of places on F; similarly let Σ_f and Σ_∞ denote the set of finite and infinite (archimedean) places, respectively.

[2]The non-divergence estimates for unipotent flows enter our proof (see Lemma 7.3) but removing all effects from the above-mentioned "splitting condition", resp., its absence, seems to require the equidistribution theorem (Theorem 1.5).

For each $v \in \Sigma$, we denote by F_v the completion of F at v. For $v \in \Sigma_f$, we denote by $\mathfrak{o}_v$ the maximal compact subring of the completion F_v and let ϖ_v be a uniformizer of $\mathfrak{o}_v$. We let $\mathbb{A} = \prod'_{v\in\Sigma} F_v$ be the ring of adeles over F and define $\mathbb{A}_f = \prod'_{v\in\Sigma_f} F_v$, where $\prod'$ denotes the restricted direct product with respect to the compact open subgroups $\mathfrak{o}_v < F_v$ for $v \in \Sigma_f$.

For any finite place $v \in \Sigma_f$ we let $k_v = \mathfrak{o}_v/\varpi_v\mathfrak{o}_v$ be the residue field, and we set $q_v = \#k_v$. Let $|x|_v$ denote the absolute value on F_v normalized so that $|\varpi_v|_v = 1/q_v$. Finally let $\widehat{F_v}$ denote the maximal unramified extension of F_v. We let $\widehat{\mathfrak{o}_v}$ denote the ring of integers in $\widehat{F_v}$ and we let $\widehat{k_v}$ denote the residue field of $\widehat{\mathfrak{o}_v}$. We note that $\widehat{k_v}$ is the algebraic closure of k_v.

Fix $\mathbf{G}$ and $\mathbf{H}$ as in the introduction, and let $\mathfrak{g}$ (resp., $\mathfrak{h}$) denote the Lie algebra of $\mathbf{G}$ (resp., $\mathbf{H}$); they are equipped with compatible F-structures. We define $G = \mathbf{G}(\mathbb{A})$ and $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$.

In this paper rank of an algebraic group refers to its absolute rank. If we want to refer to the rank of an algebraic group over a not necessarily algebraically closed field E, we will use *relative rank* or *E-rank*.

For any $v \in \Sigma$ let $\mathbf{G}(F_v)^+$ be the image of $\tilde{\mathbf{G}}(F_v)$, and let $\mathbf{G}(\mathbb{A})^+$ be the image of $\tilde{\mathbf{G}}(\mathbb{A})$, where $\tilde{\mathbf{G}}$ is the simply connected cover of $\mathbf{G}$. If each F_v-almost simple factor of $\mathbf{G}$ is F_v-isotropic, then $\mathbf{G}(F_v)^+$ is the subgroup generated by all unipotent elements; it is worth mentioning that our notation is different from the usual notation in the anisotropic case.

Let $\rho : \mathbf{G} \to \mathrm{SL}_N$ be an embedding defined over F. For any $v \in \Sigma_f$, we let $K_v := \rho^{-1}(\mathrm{SL}_N(\mathfrak{o}_v))$ and $K_f = \prod_{v\in\Sigma_f} K_v$. Set also

$$K_v[m] := \ker(K_v \to \mathrm{SL}_N(\mathfrak{o}_v/\varpi_v^m\mathfrak{o}_v)) \tag{2.1}$$

for $m \geq 1$. It is convenient to write $K_v[0] := K_v$.

We also set up the corresponding notions at the level of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. For any $v \in \Sigma$ we let $\mathfrak{g}_v$ be the Lie algebra of $\mathbf{G}$ over F_v. For $v \in \Sigma_f$ we write $\mathfrak{g}_v[0]$ for the preimage of the $\mathfrak{o}_v$-integral $N \times N$ matrices under the differential $D\rho : \mathfrak{g} \to \mathfrak{sl}_N$. More generally, we write $\mathfrak{g}_v[m]$ for the preimage of the matrices all of whose entries have valuation at least m.

Throughout, $\mathrm{red}_v : \mathrm{SL}_N(\mathfrak{o}_v) \to \mathrm{SL}_N(k_v)$ denotes the reduction mod ϖ_v map; similarly we consider reduction mod ϖ_v for the Lie algebras; see [49, Ch. 3] for a discussion of reduction maps.

For $g \in \mathbf{G}(F_v)$, we write $\|g\|$ for the largest absolute value of the matrix entries of $\rho(g)$ and $\rho(g)^{-1}$.

We let vol_G denote the volume measure on G which is normalized so that it assigns mass 1 to the quotient $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. We will also use the same notation for the induced Haar measure on X.

The notation $A \ll B$, meaning "there exists a constant $c_1 > 0$ so that $A \leq c_1 B$", will be used; the implicit constant c_1 is permitted to depend on F, $\mathbf{G}$, and ρ, but (unless otherwise noted) not on anything else. We write $A \asymp B$ if $A \ll B \ll A$. We will use $c_1, c_2, \ldots$ to denote constants depending on F, $\mathbf{G}$, and ρ (and their numbering is reset at the end of a section). If a constant (implicit or explicit) depends on another parameter or only on a certain part of $(F, \mathbf{G}, \rho)$, we will make this clear by writing, e.g., $\ll_\epsilon$, $c_3(N)$, etc.

We also adopt the $\star$-notation from [24]: We write $B = A^{\pm\star}$ if $B = c_4 A^{\pm\kappa_1}$, where $\kappa_1 > 0$ depends only on $\dim_F \mathbf{G}$ and $[F : \mathbb{Q}]$. Similarly one defines $B \ll A^\star$,

$B \gg A^\star$. Finally we also write $A \asymp B^\star$ if $A^\star \ll B \ll A^\star$ (possibly with different exponents).

We fix a MASH set $Y_{\mathscr{D}}$, arising from the data $\mathscr{D} = (\mathbf{H}, \iota, g)$ and with corresponding measure $\mu_{\mathscr{D}}$ as in §1.2. By [49, p. 451] we have $\iota(\mathbf{H}(\mathbb{A})) \subset \mathbf{G}(\mathbb{A})$ is a closed subgroup and by [52, Thm. 1.13] we also have that $\mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))g = \operatorname{supp}(\mu_{\mathscr{D}})$ is closed. We will also write $g_{\mathscr{D}} = g$ for the element $g = (g_v)_{v\in\Sigma} \in \mathbf{G}(\mathbb{A})$ determining the MASH set $Y_{\mathscr{D}}$. Let $H_v = g_v^{-1}\iota(\mathbf{H}(F_v))g_v$; it is contained in $\mathbf{G}(F_v)$ and stabilizes $\mu_{\mathscr{D}}$. The subgroup H_v is a Zariski-dense subset of the F_v-algebraic group $g_v^{-1}\iota(\mathbf{H})g_v$. Of course, H_v need not be the set of all F_v-points of $g_v^{-1}\iota(\mathbf{H})g_v$.

We recall that $\mathbf{H}$ is assumed to be simply connected in Theorem 1.5. Therefore, except for §7.11, the standing assumption is that $\mathbf{H}$ is simply connected.

2.2. **Lemma** (Stabilizer lemma). *Let $\mathbf{N}$ be the normalizer of $\iota(\mathbf{H})$ in $\mathbf{G}$. Then the stabilizer* $\operatorname{stab}(\mu_{\mathscr{D}}) = \{h \in G : h \text{ preserves } \mu_{\mathscr{D}}\}$ *of $\mu_{\mathscr{D}}$ consists of $g^{-1}\iota(\mathbf{H}(\mathbb{A}))\mathbf{N}(F)g$ and contains $g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$ as an open subgroup.*

Proof. Without loss of generality we may and will assume $g = e$ is the identity element. Suppose that $h \in \mathbf{N}(F)$. Then since $\mathbf{H}$ is simply connected and the simply connected cover is unique up to isomorphism, the automorphism $x \mapsto h^{-1}xh$ of $\iota(\mathbf{H})$ may be lifted to an F-automorphism of $\mathbf{H}$, and in particular preserves adelic points; so

$$h^{-1}\iota(\mathbf{H}(\mathbb{A}))h = \iota(\mathbf{H}(\mathbb{A})).$$

Also note that the Haar measure on $\iota(\mathbf{H}(\mathbb{A}))$ is not changed by conjugation by h as $\mathbf{H}$ is semisimple. Therefore, $\mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))h = \mathbf{G}(F)h^{-1}\iota(\mathbf{H}(\mathbb{A}))h = \mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))$ and h preserves $\mu_{\mathscr{D}}$.

Suppose now that $h \in \operatorname{stab}(\mu_{\mathscr{D}})$; then $h \in G(F)\iota(\mathbf{H}(\mathbb{A}))$ because it must preserve the support of $\mu_{\mathscr{D}}$. Adjusting h by an element in $\iota(\mathbf{H}(\mathbb{A}))$, we may assume $h = \gamma \in \mathbf{G}(F)$ and $\gamma^{-1}\iota(\mathbf{H}(\mathbb{A}))\gamma \subset \mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))$. We note that the connected component of $\mathbf{H}(\mathbb{A})$ of the identity with respect to the Hausdorff topology is the subgroup $\prod_{v\in\Sigma_\infty} \mathbf{H}(F_v)^\circ$ and the connected component of the countable union $\mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))$ of cosets equals $\prod_{v\in\Sigma_\infty} \iota(\mathbf{H}(F_v))$. Therefore, $\gamma^{-1}\iota(\mathbf{H}(F_v))\gamma \subset \iota(\mathbf{H}(F_v))$ for every $v \in \Sigma_\infty$. However, by taking Zariski closure this implies that γ normalizes $\iota(\mathbf{H})$, i.e., $\gamma \in \mathbf{N}(F)$.

For the final claim of the lemma, suppose $\gamma_i \in \mathbf{N}(F)$ and $h_i \in \mathbf{H}(\mathbb{A})$ are such that $\gamma_i\iota(h_i) \to e$ as $i \to \infty$. We need to show that $\gamma_i \in \iota(\mathbf{H}(\mathbb{A}))$ for all large enough i. Without loss of generality we may and will assume that $\gamma_i \notin \iota(\mathbf{H}(\mathbb{A}))$ for all i and derive a contradiction. If $\mathbf{H}(F)h_i \to \mathbf{H}(F)h$ for some $h \in \mathbf{H}(\mathbb{A})$, then there exists $\eta_i \in \mathbf{H}(F)$ so that $\eta_i h_i \to h$. Applying ι we obtain

$$\gamma_i\iota(\eta_i^{-1})\iota(\eta_i h_i) \to \iota(h^{-1})\iota(h)$$

which forces $\gamma_i\iota(\eta_i^{-1}) = \iota(h^{-1}) \in \mathbf{G}(F) \cap \iota(\mathbf{H}(\mathbb{A}))$ for all large enough i. This is a contradiction to our assumption even if the assumed convergence holds only along some subsequence. Using the compactness criterion [52, Thm. 1.12] on the finite volume homogeneous spaces $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ we now obtain that there exists a sequence $e \neq \eta_i \in \mathbf{H}(F)$ with $h_i^{-1}\eta_i h_i \to e$. Note that as the center of $\mathbf{H}$ is finite we see that $\iota(\eta_i) \neq e$ for all large enough i. This contradicts $\mathbf{G}(F)\iota(h_i) \to \mathbf{G}(F)$ by the same compactness criterion [52, Thm. 1.12] applied to X. □

2.3. **Volume of homogeneous sets.** Let us discuss the definition of the volume of a homogeneous set in a general context. Let G be a locally compact group and let $\Gamma < G$ be a discrete subgroup. Let μ be a homogeneous probability measure on $X = \Gamma\backslash G$ so that μ is supported on a single closed orbit $Y = xH_Y$ of the stabilizer $H_Y = \mathrm{stab}(\mu) = \{g \in G : g \text{ preserves } \mu\}$. Recall that $Y = xH_Y$ is called a homogeneous subset of X.

We normalize the Haar measure m_Y on the stabilizer group $H_Y = \mathrm{stab}(\mu)$ so that m_Y projects to μ. I.e., if we choose $z \in \mathrm{supp}(\mu)$ and any subset $S \subset H_Y$ for which the map $h \in S \mapsto zh \in Y$ is injective, then we require that $m_Y(S) = \mu(zS)$. Equivalently we identify Y with $\Gamma_Y\backslash H_Y$ (where $\Gamma_Y = \mathrm{stab}_{H_Y}(z)$) and normalize m_Y so that μ is identified with the quotient measure of m_Y by the counting measure on Γ_Y.

We fix some open neighborhood Ω_0 of the identity in G with compact closure and use it to normalize a general definition of the volume of a homogeneous subset: If $Y \subset X$ is a homogeneous set and m_Y is the Haar measure on its stabilizer subgroup H_Y (normalized as above), then $\mathrm{vol}_{\Omega_0}(Y) = m_Y(\Omega_0)^{-1}$.

We claim that $\mathrm{vol}_\Omega(Y) \ll \mathrm{vol}_{\Omega_0}(Y) \ll \mathrm{vol}_\Omega(Y)$ if Ω is another such neighborhood. Consequently we will drop the mention of Ω_0 in the notation of the volume. To prove the claim it suffices to assume that $\Omega \subset \Omega_0$, which immediately implies $m_Y(\Omega) \le m_Y(\Omega_0)$. To prove the opposite, choose some open neighborhood O of the identity with $O^{-1}O \subset \Omega$ and find some $g_1, \ldots, g_n \in \overline{\Omega}_0$ with $\Omega_0 \subset \bigcup_i g_i O$. This gives $m_Y(\Omega_0) \le \sum_i m_Y(g_i O)$. If $m_Y(g_i O) > 0$ for $i \in \{1, \ldots, n\}$, then there exists some $h_i = g_i\epsilon \in H_Y \cap g_i O$ which gives $m_Y(g_i O) = m_Y(h_i^{-1} g_i O) = m_Y(\epsilon^{-1}O) \le m_Y(\Omega)$. Consequently $m_Y(\Omega_0) \le n m_Y(\Omega)$ as required.

In the context of this paper we will work with algebraic homogeneous sets $Y_{\mathscr{D}}$ and algebraic homogeneous measures $\mu_{\mathscr{D}}$ as in §§1.2 and 2.1. By Lemma 2.2 we have that $H_{\mathscr{D}} = g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$ is an open subgroup of the stabilizer $H_{Y_{\mathscr{D}}}$. Therefore, the Haar measure on $H_{\mathscr{D}}$ is obtained from the Haar measure on $H_{Y_{\mathscr{D}}}$ by restriction (and this is compatible with the above normalization of the Haar measures). Also, the volume defined by using the Haar measure on $H_{\mathscr{D}}$ (as done in §1.3) is bigger than the volume defined using the Haar measure on the full stabilizer subgroup (as done here). In most of the paper (with the exception of §§5.12 and 7.7) we will work with the volume defined using the Haar measure on $H_{\mathscr{D}}$ (as in §1.3).

We will assume that $\Omega_0 = \prod_{v\in\Sigma} \Omega_v$, where Ω_v is an open neighborhood of the identity in $\mathbf{G}(F_v)$ for all infinite places $v \in \Sigma_\infty$ and $\Omega_v = K_v$ for all finite places $v \in \Sigma_f$.

We will make crucial use of the notion of volume in §5.10, where we will construct a good place, and again in §7.7.

3. An application to quadratic forms

We now give an example of an equidistribution result that follows from our theorem but – even in non-quantitative form – does not appear to follow directly from the (ineffective) measure classification theorems for the action of unipotent or semisimple groups.

Let $\mathcal{Q} = \mathrm{PGL}(n,\mathbb{Z})\backslash\mathrm{PGL}(n,\mathbb{R})/\mathrm{PO}(n,\mathbb{R})$ be the space of positive definite quadratic forms on $\mathbb{R}^n$ up to the equivalence relation defined by scaling and equivalence over $\mathbb{Z}$. We equip $\mathcal{Q}$ with the push-forward of the normalized Haar measure on $\mathrm{PGL}(n,\mathbb{Z})\backslash\mathrm{PGL}(n,\mathbb{R})$.

Let Q be a positive definite integral quadratic form on $\mathbb{Z}^n$, and let $\mathrm{genus}(Q)$ (resp., spin $\mathrm{genus}(Q)$) be its *genus* (resp., spin genus).

For the rest of this section, we assume that $n \geq 3$.

3.1. **Theorem.** *Suppose Q_i varies through any sequence of pairwise inequivalent, integral, positive definite quadratic forms. Then the genus* (*and also the spin genus*) *of Q_i, considered as a subset of $\mathcal{Q}$, equidistributes as $i \to \infty$* (*with speed determined by a power of* $|\operatorname{genus}(Q_i)|$).

Similar theorems have been proved elsewhere (see, e.g., [27] where the splitting condition is made at the archimedean place). What is novel here, besides the speed of convergence, is the absence of any type of splitting condition on the Q_i. This is where the quantitative result of the present paper becomes useful. We also note that it seems plausible that one could remove the splitting assumptions of [26] in the borderline cases where $m - n \in \{3, 4\}$ by means of the methods of this paper. However, for this the maximality assumption in Theorem 1.5 would need to be removed.

3.2. **Setup for the proof.** We set $F = \mathbb{Q}$, $\mathbf{G} = \mathrm{PGL}_n$, and define the quotient $X = \mathrm{PGL}_n(\mathbb{Q}) \backslash \mathrm{PGL}_n(\mathbb{A})$. Let us recall some facts about the genus and spin genus in order to relate the above theorem with Theorem 1.5. For every rational prime p put $K_p = \mathrm{PGL}(n, \mathbb{Z}_p)$ and note that K_p is a maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_p)$. We also define $K = \prod_p K_p$. With this notation we have

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{Q})\mathbf{G}(\mathbb{R})K = \mathbf{G}(\mathbb{Q})\mathrm{PGL}(n, \mathbb{R})K. \tag{3.1}$$

It is worth mentioning that (3.1) gives a natural identification between $L^2(X, \mathrm{vol}_G)^K$, the space of K-invariant functions, and

$$L^2(\mathrm{PGL}(n, \mathbb{Z}) \backslash \mathrm{PGL}(n, \mathbb{R}), \mathrm{vol}_{\mathrm{PGL}(n,\mathbb{R})});$$

this identification maps smooth functions to smooth functions.

Given a positive definite integral quadratic form Q in n variables, the isometry group $\mathbf{H}' = \mathrm{SO}(Q)$ is a $\mathbb{Q}$-group; it actually comes equipped with a model over $\mathbb{Z}$. This group naturally embeds in $\mathbf{G}$, and this embedding is defined over $\mathbb{Z}$. We define $\mathbf{H} = \mathrm{Spin}(Q)$ and let $\pi : \mathbf{H} \to \mathbf{H}'$ be the covering map.

Put $K'_p = \mathbf{H}'(\mathbb{Q}_p) \cap K_p$, $K' = \prod_p K'_p$, and $K'(\infty) = \mathbf{H}'(\mathbb{R})K'$, the latter being a compact open subgroup of $\mathbf{H}'(\mathbb{A})$. Note that $\operatorname{genus}(Q)$ is identified with the finite set $\mathbf{H}'(\mathbb{Q}) \backslash \mathbf{H}'(\mathbb{A}_f)/K'$, which may also be rewritten as $\mathbf{H}'(\mathbb{Q}) \backslash \mathbf{H}'(\mathbb{A})/K'(\infty)$. Similarly the spin genus of Q is given by $\mathbf{H}'(\mathbb{Q}) \backslash \mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}_f))K'/K'$, which may also be written as

$$\mathbf{H}'(\mathbb{Q}) \backslash \mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))K'(\infty)/K'(\infty).$$

Let $g_Q \in \mathrm{PGL}(n, \mathbb{R})$ be so that $g_Q^{-1}\mathbf{H}'(\mathbb{R})g_Q = g_Q^{-1}\iota(\mathbf{H}(\mathbb{R}))g_Q = \mathrm{SO}(n, \mathbb{R})$, the standard compact isometry group. We define the associated MASH set $Y := Y_Q = \pi(\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}))(g_Q, e, \ldots)$.

3.3. **Lemma.** *The volume of the MASH set Y, the spin genus of Q, the genus of Q, and the discriminant of Y* (*as defined in Appendix* B) *are related to each other via*[3]

$$\mathrm{vol}(Y) \asymp |\operatorname{spin\ genus}(Q)|^\star \asymp |\operatorname{genus}(Q)|^\star \asymp \mathrm{disc}(Y)^\star.$$

We postpone the proof to Appendix B.1.

[3] See §2.1 for the $\star$-notation.

3.4. **Proof of Theorem 3.1.** Let Q be a positive definite integral quadratic form in n variables as above. Let $f \in C_c^\infty(X)^K$ be a smooth, compactly supported and K-invariant function. Denote by $\pi^+ : L^2(X, \mathrm{vol}_G) \to L^2(X, \mathrm{vol}_G)^{\mathrm{PSL}(n,\mathbb{A})}$ the projection onto the space of $\mathrm{PSL}(n,\mathbb{A})$-invariant functions; this is a $\mathbf{G}(\mathbb{A})$-equivariant map. Therefore, $\pi^+(f)$ is K-invariant as f is K-invariant. Thus by (3.1) we have $\pi^+(f)$ is $\mathbf{G}(\mathbb{A})$-invariant which implies

$$\pi^+(f) = \int_X f \,\mathrm{dvol}_G \tag{3.2}$$

for all K-invariant $f \in C_c^\infty(X)$.

Let $f \in C_c^\infty(X)^K$. Applying Theorem 1.5, with the homogeneous space Y and in view of (3.2), we get

$$\left| \int_Y f \,\mathrm{d}\mu_{\mathscr{D}} - \int_X f \,\mathrm{dvol}_G \right| \ll \mathrm{vol}(Y)^{-\kappa_0} \mathcal{S}(f).$$

Using $\mathrm{vol}(Y) \asymp |\,\mathrm{spin\ genus}(Q)|^\star \gg |\,\mathrm{genus}(Q)|^\star$ (see Lemma 3.3) this implies Theorem 3.1.

4. A proof of property (τ)

The following theorem was established in full generality through works of Selberg [56], Každan [38], Burger-Sarnak [11], and the work of Clozel [12, Thm. 3.1] completed the proof.[4]

4.1. **Theorem** (Property (τ))**.** *Let v be a place of F and let $\mathbf{G}_v$ be an F_v-algebraic semisimple group which is isotropic over F_v. Let $\mathbf{G}$ be an algebraic F-group such that $\mathbf{G}$ is isomorphic to $\mathbf{G}_v$ over F_v. Then the representation $L^2_0(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ – the orthogonal complement of $\mathbf{G}(\mathbb{A})^+$-invariant functions – is isolated from the trivial representation as a representation of $\mathbf{G}_v(F_v)$. Moreover, this isolation* (*spectral gap*) *is independent of* $\mathbf{G}$.

A corollary of the above (dropping the crucial uniformity in $\mathbf{G}$) is that, if Γ is any S-arithmetic lattice in the group $\mathbf{G}$, and $\{\Gamma_N\}_{N\geq 1}$ is the family of all congruence lattices, then the Γ-action on $L^2(\Gamma/\Gamma_N)$ possesses a uniform spectral gap.

Our main result offers an alternative to Clozel's part of the proof. (Besides the groups of type A_1, this is the most "non-formal" part, as it relies on a special instance of the functoriality principle of Langlands.)

4.2. **Short history of the problem.** Let us describe some of the history of Theorem 4.1. First, it is not difficult to reduce to the case of an *absolutely almost simple*, *simply connected* group $\mathbf{G}$. This being so, it follows by combining the following distinct results and principles:

(1) Property (T): If the F_v-rank of $\mathbf{G}(F_v)$ is ≥ 2, it follows from Kazhdan's "property (T)", which furnishes the stronger statement that *any* representation not containing the identity is isolated from it; see [38] and the work of Oh [47] for a more uniform version that is of importance to us.

[4]Clozel states this theorem for a fixed F-group $\mathbf{G}$, however, his proof also gives Theorem 4.1. Our proof of Theorem 4.1 will include uniformity in the F-group $\mathbf{G}$.

(2) Groups of type A_1: If the rank of $\mathbf{G}$ over the algebraic closure $\bar{F}$ is equal to 1, i.e., $\mathbf{G} \times_F \bar{F}$ is isogenous to SL_2, then $\mathbf{G}$ is necessarily the group of units in a quaternion algebra over F. In that case, the result can be established by the methods of Kloosterman or by the work of Jacquet-Langlands and Selberg; see [36, 56].
(3) Burger-Sarnak principle: Let $\rho : \mathbf{G} \to \mathbf{G}'$ be a homomorphism with finite central kernel, let $\mathbf{G}'$ be absolutely almost simple and simply connected, and suppose that property (τ) is known for groups that are isomorphic to $\mathbf{G}$ over $\bar{F}$. Then property (τ) is known for $\mathbf{G}'$ *at any place where* $\mathbf{G}(F_v)$ *is isotropic*; see [11].
(4) Groups of type A_n: Property (τ) is true for groups of the form $\mathrm{SL}(1, D)$, where D is a division algebra over F whose dimension is the square of a prime; or for groups of the form $\mathrm{SU}(D, \star)$, where D is a division algebra over a quadratic extension E of F, and $\star$ is an "involution of the second kind" on D, i.e., inducing the Galois automorphism on E; see Clozel's work [12].
(5) For us the uniformity of the spectral gap across all types of groups and across all places is crucial. This is obtained by combining the above results and was done by Gorodnik, Maucourant, and Oh [30, Thm. 1.11].

The hardest of these results is arguably the fourth step. It is established in [12] and uses a comparison of trace formulae. In addition to these results, Clozel [12, Thm. 1.1] proves that any absolutely almost simple, simply connected group defined over F admits a morphism from a group $\mathrm{Res}_{F'/F}\mathbf{G}$, where $\mathbf{G}$ is an algebraic F'-group isomorphic to one of the types described in (4).

4.3. **Effective decay of matrix coefficients.** Let us also note that by the work of Cowling, Haagerup, and Howe [16] and others the conclusion in Theorem 4.1 is equivalent to the existence of a uniform decay rate for matrix coefficients on the orthogonal complement of the $\mathbf{G}(\mathbb{A})^+$-invariant functions. Once more, for groups with property (T) this statement is true for any representation; see [30, 47]. Due to the assumption that $\mathbf{G}$ is simply connected, we are reduced to studying functions in $f \in L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ with $\int f = 0$ see [30, Lemma 3.22].

More precisely, Theorem 4.1 is equivalent to the existence of some $\kappa_2 > 0$ such that for all K_v-finite functions $f_1, f_2 \in L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ with $\int f_1 = \int f_2 = 0$, the matrix coefficient can be estimated as follows:

$$\left|\left\langle \pi_{g_v} f_1, f_2 \right\rangle\right| \le \dim\langle K_v \cdot f_1\rangle^{\frac{1}{2}} \dim\langle K_v \cdot f_2\rangle^{\frac{1}{2}} \; \|f_1\|_2 \; \|f_2\|_2 \; \Xi_{\mathbf{G}_v}(g_v)^{\kappa_2}, \tag{4.1}$$

where $g_v \in \mathbf{G}_v(F_v)$, π_{g_v} is its associated unitary operator, K_v is a good maximal compact open subgroup of $\mathbf{G}_v(F_v)$, $\langle K_v \cdot f\rangle$ is the linear span of $K_v \cdot f$, and $\Xi_{\mathbf{G}_v}$ is a Harish-Chandra spherical function of $\mathbf{G}_v(F_v)$.

As noted in Theorem 4.1 the constant κ_2 is independent of the precise F-structure of $\mathbf{G}$. What we did not mention before (as we did not have the notation) is that κ_2 is also independent of the place v. For groups with property (T) these statements are proven in [47].

We are able to give a direct proof of (4.1) (relying on [47]) which avoids the third and fourth[5] points of §4.2 (but leads to weaker exponents). Indeed, using

[5] By only avoiding the fourth point we may restrict ourselves to compact quotients and obtain in these cases a constant κ_2 that only depends on $\dim_F \mathbf{G}$, the type of $\mathbf{G}_v$ over F_v and the dimension of F_v over $\mathbb{Q}_p$ where $p|v$, but not on F or even the degree $[F : \mathbb{Q}]$. If we wish to avoid the third and fourth the constant depends on $[F : \mathbb{Q}] \dim_F \mathbf{G}$.

the second point, we are left with the case where $\mathbf{G}$ is an absolutely almost simple, simply connected group over F of absolute rank ≥ 2. In that case, one applies Theorem 1.5 to translates of the diagonal copy of $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$ inside $X \times X$ by elements from $\mathbf{G}(F_v)$ to establish a uniform decay rate for matrix coefficients and so Theorem 4.1.

We will explain this step first in a special case and then in §4.5 in general.

4.4. **A purely real instance of transportation of spectral gap.** Let G_1, G_2 be almost simple, connected Lie groups, and suppose G_2 has (T) but G_1 has not. Let Γ be an irreducible lattice in $G = G_1 \times G_2$, e.g., this is possible for $G_1 = \mathrm{SU}(2,1)(\mathbb{R})$ and $G_2 = \mathrm{SL}_3(\mathbb{R})$.

We wish to bound the matrix coefficients of G_1 acting on $X = \Gamma\backslash G$. Let $G_\Delta = \{(h,h) : h \in G\} < G \times G$ and notice that the diagonal orbit $(\Gamma \times \Gamma)G_\Delta \subset X \times X$ is responsible for the inner product in the sense that the integral of $f_1 \otimes \bar{f}_2$ over this orbit equals the inner product $\langle f_1, f_2 \rangle$. In the same sense is the deformed orbit $(\Gamma \times \Gamma)G_\Delta(g, e)$ responsible for the matrix coefficients of g. The volume of this deformed orbit is roughly speaking a power of $\|g\|$, hence effective equidistribution of this orbit gives effective decay of matrix coefficients.

We note that the main theorem[6] of [24] does not apply to this situation as the acting group giving the closed orbit has been conjugated and does not remain fixed. However, if $g = (g_1, e)$, then the almost simple factor of G_Δ corresponding to G_2 remains (as a subgroup of $G \times G$) fixed and this is the part with known effective decay (due to property (T)). In this case the method of [24] (which is also applied in this paper in the adelic context) can be used to show effective equidistribution and so decay of matrix coefficients for the G_1-action. In all of this, the rate of (i.e., the exponent for) the decay of matrix coefficients for G_1 only depends on the spectral gap for G_2 and the dimension of G (but not on Γ).

4.5. **The general case with absolute rank at least two.** Let F be a number field and let $\mathbf{G}$ be an absolutely almost simple, simply connected F-group whose absolute rank is at least two. Let v be a place of the number field F such that $\mathbf{G}(F_v)$ is non-compact. For any $g \in \mathbf{G}(F_v)$ put $X_g = \{(xg, x) : x \in X\}$, where we identify g with an element of $\mathbf{G}(\mathbb{A})$. Then X_g is a MASH set, and in view of our definition of volume of a homogeneous set there exist two positive constants κ_3, κ_4 (depending only on the root system of $\mathbf{G}(F_v)$) such that

$$\|g\|^{\kappa_3} \ll \mathrm{vol}(X_g) \ll \|g\|^{\kappa_4}.$$

As mentioned before we want to apply Theorem 1.5 to $X_g \subset X \times X$. However, we want the proof of that theorem to be independent of (3) and (4) of §4.2. We note that in the proof of Theorem 1.5 that spectral gap will be used for a "good place" w. In §5 (see also §5.11 and the summary in §6.1) the following properties of a good place will be established:

(i) $\mathrm{char}(k_w) \gg 1$ is large compared to $\dim \mathbf{G}$,
(ii) both $\mathbf{G}$ and $\iota(\mathbf{H})$ are quasisplit over F_w, and split over $\widehat{F_w}$,
(iii) both K_w and K'_w are hyperspecial subgroups of $\mathbf{G}(F_w)$ and the subgroup H_w (the component of the acting group at the place w), respectively.

[6] In that theorem the implicit constant in the rate of equidistribution is allowed to depend on the acting group and so implicity in this instance also on g.

Indeed almost all places satisfy these conditions. We also will show the effective estimate $q_w \ll \log(\mathrm{vol}(\text{homogeneous set}))^2$; this needs special care when the F-group $\mathbf{H}$ changes. In our application to Theorem 4.1, however, the algebraic subgroup $\mathbf{H} = \{(h,h) : h \in \mathbf{G}\} < \mathbf{G}\times\mathbf{G}$ is fixed and X_g changes with the element $g \in \mathbf{G}(F_v)$. In this case we find a place $w \neq v$ (independent of g) which satisfies (i), (ii), and (iii) so that in addition $\mathbf{G}$ is F_w-split.[7] Note that by the Chebotarev density theorem, [49, Thm. 6.7] there are infinitely many such places. Then by our assumption that the *absolute rank of* $\mathbf{G}$ *is at least two* we have the required spectral gap for $\mathbf{G}(F_w)$ since this group has property (T). Therefore we get: there exists a constant $\kappa_5 > 0$ (which depends only on the type of $\mathbf{G}$) so that for all K_w-finite functions $f_1, f_2 \in L^2_0(X)$ we have

$$|\langle \pi_{h_w} f_1, f_2\rangle| \ll \dim\langle K_w \cdot f_1\rangle^{1/2} \dim\langle K_w \cdot f_2\rangle^{1/2} \|f_1\|_2 \|f_2\|_2 \|h_w\|^{-\kappa_5}, \tag{4.2}$$

where the implicit constant depends on $\mathbf{G}(F_w)$; see [47].

Fix such a place, then using (4.2) as an input in the proof of Theorem 4.1 and taking g large enough so that $q_w \leq \log(\|g\|)$ we get from Theorem 1.5 the conclusion of the theorem. In particular, if $f = f_1 \otimes \bar{f}_2$ with $f_i \in C_c^\infty(X)$ for $i = 1, 2$, then

$$\begin{aligned} \Big| \int_{X\times X} f \, \mathrm{d}\mu_{X_g} - \int_{X\times X} f_1 \otimes \bar{f}_2 \, \mathrm{dvol}_{G\times G} \Big| \\ = \Big| \langle \pi_g f_1, f_2\rangle_X - \int_X f_1 \, \mathrm{dvol}_G \int_X \bar{f}_2 \, \mathrm{dvol}_G \Big| \ll \|g\|^{-\kappa_0} \mathcal{S}(f); \end{aligned}$$

where $\kappa_0 > 0$ depends on $\dim \mathbf{G}$, $[F : \mathbb{Q}]$ (if X is non-compact), and κ_5 as in (4.2). The implied multiplicative constant depends on X and so also on the F-structure of $\mathbf{G}$. We note however, that this constant is irrelevant due to [16], which upgrades the above to a uniform effective bound on the decay of the matrix coefficients as in (4.1) with κ_2 independent of $\mathbf{G}$. This implies Theorem 4.1.

5. Construction of good places

See §2 for general notation. In particular, $\mathscr{D} = (\mathbf{H}, \iota, g)$ consists of a simply connected semisimple F-group $\mathbf{H}$, an F-homomorphism $\iota : \mathbf{H} \to \mathbf{G}$, and an element $g = (g_v) \in \mathbf{G}(\mathbb{A})$ determining a homogeneous set; the stabilizer of this set contains the acting group $H_{\mathscr{D}} = g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$. We will not assume within this section (or the related Appendix B) that $\iota(\mathbf{H})$ is a maximal subgroup of $\mathbf{G}$.

The purpose of this section is to show that we may always choose a place w with the property that $H_w = g_w^{-1}\iota(\mathbf{H}(F_w))g_w \subset \mathbf{G}(F_w)$ is not too "distorted". The precise statement is the proposition in §5.11, but *if the reader is interested in the case of Theorem* 1.5 *where* Y *varies through a sequence of sets where* w *and* H_w *are fixed* (*e.g., the argument in* §4) *the reader may skip directly to* §6. This section relies heavily on the results established in [50] and [6] which in turn relies on Bruhat-Tits theory.

[7] In fact we may also ensure that F splits over $\mathbb{Q}_p$ by applying this argument for $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{G}$. In this case $F_w = \mathbb{Q}_p$ for $w|p$ and so $\mathbf{G}(F_w)$ is a simple group over $\mathbb{Q}_p$ from a finite list that is independent of F and even of $[F : \mathbb{Q}]$. Using this one can establish the earlier noted independence of κ_2 from $[F : \mathbb{Q}]$.

5.1. **Bruhat-Tits theory.** We recall a few facts from Bruhat-Tits theory; see [60] and the references there for the proofs. Let $\mathbf{G}$ be a connected semisimple group defined over F. Let v be a finite place. Then

(1) For any point x in the Bruhat-Tits building of $\mathbf{G}(F_v)$, there exists a smooth affine group scheme $\mathfrak{G}_v^{(x)}$ over $\mathfrak{o}_v$, unique up to isomorphism, such that: its generic fiber is $\mathbf{G}(F_v)$, and the compact open subgroup $\mathfrak{G}_v^{(x)}(\mathfrak{o}_v)$ is the stabilizer of x in $\mathbf{G}(F_v)$; see [60, 3.4.1].
(2) If $\mathbf{G}$ is split over F_v and x is a *special* point, then the group scheme $\mathfrak{G}_v^{(x)}$ is a Chevalley group scheme with generic fiber $\mathbf{G}$; see [60, 3.4.2].
(3) $\mathrm{red}_v : \mathfrak{G}_v^{(x)}(\mathfrak{o}_v) \to \underline{\mathfrak{G}_v}^{(x)}(k_v)$, the reduction mod ϖ_v map, is surjective, which follows from the smoothness above; see [60, 3.4.4].
(4) $\underline{\mathfrak{G}_v}^{(x)}$ is connected and semisimple if and only if x is a *hyperspecial* point. Stabilizers of hyperspecial points in $\mathbf{G}(F_v)$ will be called hyperspecial subgroups; see [60, 3.8.1] and [50, 2.5].

If $\mathbf{G}$ is quasisplit over F_v, and splits over $\widehat{F_v}$, then hyperspecial vertices exist, and they are compact open subgroups of maximal volume. Moreover a theorem of Steinberg implies that $\mathbf{G}$ is quasisplit over $\widehat{F_v}$ for all v; see [60, 1.10.4].

It is known that for almost all v the groups K_v are hyperspecial; see [60, 3.9.1] (and §2.1 for the definition of K_v). We also recall that: for almost all v the group $\mathbf{G}$ is quasisplit over F_v; see [49, Thm. 6.7].

5.2. **Passage to absolutely almost simple case.** We will first find the place w under the assumption that $\mathbf{H}$ is F-almost simple; the result for semisimple groups will be deduced from this case.

In this section we will need to work with finite extensions of F as well. To avoid confusion we will denote $\mathbb{A}_E$ for the ring of adeles of a number field E. This notation is used only here in §5 and in Appendix B.

Suppose for the rest of this section, until specifically mentioned otherwise, that $\mathbf{H}$ is F-almost simple. Let F'/F be a finite extension so that $\mathbf{H} = \mathrm{Res}_{F'/F}(\mathbf{H}')$, where $\mathbf{H}'$ is an absolutely almost simple F'-group; note that $[F' : F] \le \dim \mathbf{H}$. We use the notation $v' \in \Sigma_{F'}$ for the places of F'. For any $v \in \Sigma_F$, there is a natural isomorphism between $\mathbf{H}(F_v)$ and $\prod_{v'|v} \mathbf{H}'(F'_{v'})$; this induces an isomorphism between $\mathbf{H}(\mathbb{A}_F)$ and $\mathbf{H}'(\mathbb{A}_{F'})$.

5.3. **Adelic volumes and the Tamagawa number.** Fix an algebraic volume form ω' on $\mathbf{H}'$ defined over F'. The form ω' determines a Haar measure on each vector space $\mathfrak{h}'_{v'} := \mathrm{Lie}(\mathbf{H}') \otimes F'_{v'}$ which also gives rise to a normalization of the Haar measure on $\mathbf{H}'(F'_{v'})$. Let us agree to refer to both these measures as $|\omega'_{v'}|$. We denote by $|\omega'_{\mathbb{A}}|$ the product measure on $\mathbf{H}'(\mathbb{A}_{F'})$; then

$$|\omega'_{\mathbb{A}}|(\mathbf{H}'(F')\backslash\mathbf{H}'(\mathbb{A}_{F'})) = D_{F'}^{\frac{1}{2}\dim \mathbf{H}'}\tau(\mathbf{H}'), \tag{5.1}$$

where $\tau(\mathbf{H}')$ is the *Tamagawa number* of $\mathbf{H}'$, and $D_{F'}$ is the discriminant of F'. In the case at hand $\mathbf{H}'$ is simply connected, thus, it is known that $\tau(\mathbf{H}') = 1$; see [41] and [50, Sect. 3.3] for historic remarks and references.

The volume formula (5.1) is for us just a starting point. It relates the Haar measure on Y to the algebraic volume form ω' (and the field F'). However, the volume of our homogeneous set Y as a subset of X depends heavily on the amount of distortion (coming from the precise F-structure of $\mathbf{H}$ and g).

5.4. **The quasisplit form.** Following [50, Sect. 0.4] we let $\mathcal{H}'$ denote a simply connected algebraic group defined and quasisplit over F' which is an inner form of $\mathbf{H}'$. Let L be the field associated[8] to $\mathcal{H}'$ as in [50, Sect. 0.2], it has degree $[L : F'] \leq 3$. We note that $\mathcal{H}'$ should be thought of as the least distorted version of $\mathbf{H}'$; it and the field L will feature in all upcoming volume considerations.

Let ω^0 be a differential form on $\mathcal{H}'$ corresponding to ω'. This can be described as follows: Let $\varphi : \mathbf{H}' \to \mathcal{H}'$ be an isomorphism defined over some Galois extension of F'. We choose ω^0 so that $\omega' = \varphi^*(\omega^0)$; it is defined over F'. It is shown in [50, Sect. 2.0–2.1] that, up to a root of unity of order at most 3, this is independent of the choice of φ.

As was done in [50] we now introduce local normalizing parameters $\lambda_{v'}$ which scale the volume form ω^0 to a more canonical volume form on $\mathcal{H}'(F'_{v'})$.

5.5. **Normalization of the Riemannian volume form.** Let us start the definition of these parameters at the archimedean places.

Let $\mathfrak{g}$ be any d-dimensional semisimple real Lie algebra. We may normalize an inner product on $\mathfrak{g}$ as follows: Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{g}_0$ be a maximal compact subalgebra. The negative Killing form gives rise to an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_0$. This can be complexified to a Hermitian form on $\mathfrak{g}_{\mathbb{C}}$ and then restricted to a (real) inner product on $\mathfrak{g}$.

As usual, the choice of an inner product on a real vector space determines a non-zero $\nu \in \wedge^d \mathfrak{g}^*$ up to sign. We refer to this as the Riemannian volume form on $\mathfrak{g}$, and again write $|\nu|$ for the associated Riemannian volume on $\mathfrak{g}$ or on a real Lie group with Lie algebra $\mathfrak{g}$. Note that the Hermitian form depends on the choice of the maximal compact subalgebra, but the Riemannian volume is independent of this choice.

For any archimedean place, let $\lambda_{v'} > 0$ be such that $\lambda_{v'} |\omega^0_{v'}|$ coincides with the Riemannian volume on $\mathcal{H}'(F'_{v'})$ (using the above normalization).

5.6. **Normalization of the Haar measure at the finite places.** For any finite place v' of F', we choose an $\mathfrak{o}'_{v'}$-structure on $\mathcal{H}'$, i.e., a smooth affine group scheme over $\mathfrak{o}'_{v'}$ with generic fiber $\mathcal{H}'$. To define $\lambda_{v'}$ at the finite places we have to choose the $\mathfrak{o}'_{v'}$-structure more explicitly, as in [50, Sect. 1.2].

We let $\{\mathcal{P}_{v'} \subset \mathcal{H}'(F'_{v'})\}$ denote a coherent collection of parahoric subgroups of “maximal volume”; see [50, Sect. 1.2] for an explicit description. Let us recall that by a coherent collection we mean that $\prod_{v' \in \Sigma_{F',f}} \mathcal{P}_{v'}$ is a compact open subgroup of $\mathcal{H}'(\mathbb{A}_{F',f})$.

Note that any F'-embedding of $\mathcal{H}'$ into $\mathrm{GL}_{N'}$ gives rise to a coherent family of compact open subgroups of $\mathcal{H}'(\mathbb{A}_{F'})$ which at almost all places satisfies the above requirements on $\mathcal{P}_{v'}$; see §5.1. At the other places we may choose $\mathcal{P}_{v'}$ as above and then use (1) in §5.1 to define the $\mathfrak{o}'_{v'}$-structure on $\mathcal{H}'(F'_{v'})$. Let us also remark that maximality of the volume implies that the corresponding parahoric is either hyperspecial (if $\mathcal{H}'$ splits over an unramified extension) or special with maximum volume (otherwise).

[8]In most cases L is the splitting field of $\mathcal{H}'$ except in the case where $\mathcal{H}'$ is a triality form of ${}^6\mathsf{D}_4$ where it is a degree 3 subfield of the degree 6 Galois splitting field with Galois group S_3. Note that there are three such subfields which are all Galois-conjugate.

This allows us, in particular, to speak of "reduction modulo $\varpi'_{v'}$". If v' is a finite place of F', we let $\overline{\mathcal{M}}_{v'}$ denote the reductive quotient of $\mathrm{red}_{v'}(\mathcal{P}_{v'})$; this is a reductive group over the residue field.

For any non-archimedean place, let $\ell_{v'} \in F'_{v'}$ be so that $\ell_{v'}\omega^0_{v'}$ is a form of maximal degree, defined over $\mathfrak{o}'_{v'}$, whose reduction mod $\varpi'_{v'}$ is non-zero, and let $\lambda_{v'} = |\ell_{v'}|_{v'}$.

5.7. **Product formula.** Let us use the abbreviation $D_{L/F'} = D_L D_{F'}^{-[L:F']}$ for the norm of the relative discriminant of L/F'; see [50, Thm. A]. It is shown in [50, Thm. 1.6] that

$$\prod_{v'\in\Sigma_{F'}} \lambda_{v'} = D_{L/F'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}')} \cdot A, \tag{5.2}$$

where $A > 0$ depends only on $\mathbf{H}$ over $\bar{F}$ and $[F' : \mathbb{Q}]$, $\mathfrak{s}(\mathcal{H}') = 0$ when $\mathcal{H}'$ splits over F' in which case $L = F'$, and $\mathfrak{s}(\mathcal{H}') \geq 5$ otherwise; these constants depend only on the root system of $\mathcal{H}'$.

It should be noted that the parameters $\lambda_{v'}$ were defined using $\mathcal{H}'$ and ω^0 but will be used to renormalize $\omega'_{v'}$ on $\mathbf{H}'$.

5.8. **Local volume contributions.** Recall that we fixed an open subset $\Omega_0 = \prod_{v\in\Sigma_\infty} \Omega_v \times \prod_{v\in\Sigma_f} K_v \subset \mathbf{G}(\mathbb{A})$ and defined $\mathrm{vol}(Y)$ of an algebraic semisimple homogeneous set Y using this subset; see (1.1) and §2.3.

For every $v \in \Sigma_\infty$ we may assume that Ω_v is constructed as follows. Fix a bounded open subset $\Xi_v \subset \mathfrak{g}_v$ which is symmetric around the origin such that exp is diffeomorphic on it and so that

$$\text{every eigenvalue } \sigma \text{ of } \mathrm{ad}(u) \text{ for } u \in \Xi_v \text{ satisfies } |\sigma|_v < \frac{1}{10}, \tag{5.3}$$

where we regard the norm $|\cdot|_v$ as being extended to an algebraic closure of F_v. With this we define $\Omega_v = \exp(\Xi_v)$. We will also require that properties similar to (5.3) hold for finitely many finite-dimensional representations which will be introduced in the proof below; see the discussion leading to (5.5).

To compare the Haar measure on $\mathbf{H}(F_v)$ with the Haar measure on the Lie algebra $\mathfrak{h}_v$ in the following proof we also recall that the derivative of the exponential map $\exp : \mathfrak{h}_v \to \mathbf{H}(F_v)$ at $u \in \mathfrak{h}_v$ is given by[9]

$$\frac{1-\exp(-\mathrm{ad}u)}{\mathrm{ad}u} = 1 - \frac{\mathrm{ad}u}{2} + \frac{(\mathrm{ad}u)^2}{3!} - + \cdots. \tag{5.4}$$

For $v \in \Sigma_{F,f}$, set $K_v^* = \iota^{-1}(g_v K_v g_v^{-1})$, and put $K_v^* = \iota^{-1}(g_v \Omega_v g_v^{-1})$ for $v \in \Sigma_{F,\infty}$. Note that, for each finite place v, the group K_v^* is an open compact subgroup of $\mathbf{H}(F_v)$. For any place $v \in \Sigma_{F,f}$ we can write

$$K_v^* \subseteq \prod_{v'|v} K_{v'}^* \quad \text{where } K_{v'}^* \text{ is the projection of } K_v^* \text{ into } \mathbf{H}'(F'_{v'}).$$

Let $J_{v'}$ be the measure of $K_{v'}^*$ under $\lambda_{v'}|\omega'_{v'}|$.

We define $p_{v'} = \mathrm{char}(k_{v'})$ and note that $q_{v'} = \#k_{v'} = p_{v'}^l$ for some $l \leq [F' : \mathbb{Q}]$.

[9] We think of the derivative as a map from $\mathfrak{h}_v$ to itself by using left-mutiplication by the inverse of $\exp(u)$ to identify the tangent plane at the point $\exp(u)$ with the tangent plane at the identity. As the latter is measure preserving, this identification does not affect the estimates for the Jacobian of the exponential map.

Proposition. *The local terms $J_{v'}$ as above satisfy the following properties:*

(1) *For v' a finite place of F', $J_{v'} \leq 1$.*
(2) *Let v' be a finite place of F' such that L/F' is unramified at v'. Suppose that $\mathbf{H}'$ is not quasisplit over $F'_{v'}$ or $K^*_{v'}$ is not hyperspecial; then $J_{v'} \leq 3/4$. If in addition $q_{v'} > 13$, then $J_{v'} \leq \max\{\frac{1}{p_{v'}}, \frac{q_{v'}+1}{q^2_{v'}}\} \leq 1/2$.*
(3) *For an archimedean place v of F, $\Big(\prod_{v'|v} \lambda_{v'}|\omega'_{v'}|\Big)(K^*_v)$ is bounded above by a constant depending only on $\mathbf{G}$ and Ω.*

Proof (Case of v' finite). Let $P_{v'}$ be a minimal parahoric subgroup containing $K^*_{v'}$. Let $\overline{\mathrm{M}}_{v'}$ be the reductive quotient of the corresponding $k_{v'}$-group $\mathrm{red}_{v'}(\mathfrak{P}_{v'})$ where $\mathfrak{P}_{v'}$ is the smooth affine $\mathfrak{o}_{v'}$-group scheme whose $\mathfrak{o}_{v'}$-points are $P_{v'}$; existence of such is guaranteed by Bruhat-Tits theory; see (1) of §5.1.

This gives

$$J_{v'} = \lambda_{v'}|\omega'_{v'}|(K^*_{v'}) \leq \lambda_{v'}|\omega'_{v'}|(P_{v'}) = \frac{\#\overline{\mathrm{M}}_{v'}(k_{v'})}{q_{v'}^{(\dim \overline{\mathrm{M}}_{v'} + \dim \overline{\mathcal{M}}_{v'})/2}}$$

where the last equality is [50, Prop. 2.10]. The same proposition also shows that the right-hand side is at most 1 as claimed in (1).

Using [50, Prop. 2.10] one more time we have: if $\mathbf{H}'$ is not quasisplit over $F'_{v'}$, or $\mathbf{H}'$ splits over the maximal unramified extension $\widehat{F'_{v'}}$ but $P_{v'}$ is not hyperspecial, then

$$\frac{\#\overline{\mathrm{M}}_{v'}(k_{v'})}{q_{v'}^{(\dim \overline{\mathrm{M}}_{v'} + \dim \overline{\mathcal{M}}_{v'})/2}} \leq \frac{q_{v'}+1}{q^2_{v'}}.$$

Therefore, we assume now (as we may), that $\mathbf{H}'$ is quasisplit over $F'_{v'}$.

As the quasisplit inner form is unique we obtain that $\mathcal{H}'$ and $\mathbf{H}'$ are isomorphic over $F'_{v'}$.

Note that if v' does not ramify in L, then $\mathcal{H}'$ splits over the maximal unramified extension $\widehat{F'_{v'}}$. Indeed, by the footnote on page 236 in most cases L is the splitting field of $\mathcal{H}'$ which gives the remark immediately. In the case of the triality form of ${}^6\mathsf{D}_4$ the splitting field of $\mathcal{H}'$ is a degree 6 Galois extension E/F' with Galois group S_3 which is generated by $L \subset E$ and its Galois images. As v' does not ramify in L, this also implies that v' does not ramify in E. As we may assume $\mathbf{H}'$ and $\mathcal{H}'$ are isomorphic over $F'_{v'}$, the group $\mathbf{H}'$ also splits over $\widehat{F'_{v'}}$.

In view of this the only case which needs extra argument is when $\mathbf{H}'$ is $F'_{v'}$-quasisplit, split over $\widehat{F'_{v'}}$, the only parahoric subgroup containing $K^*_{v'}$ is a hyperspecial parahoric subgroup $P_{v'}$, and $K^*_{v'} \subsetneq P_{v'}$. Note that (1) and the fact that $K^*_{v'} \subsetneq P_{v'}$ in particular imply that[10] $J_{v'} \leq 1/2$. It remains to show that the stronger bound holds in this case as well.

We will use the notation and statements recalled in §5.1. Let $\mathfrak{P}_{v'}$ be the smooth group scheme associated to $P_{v'}$ by Bruhat-Tits theory. Since $P_{v'}$ is hyperspecial we have $\mathrm{red}_{v'}\,\mathfrak{P}_{v'}$ is an almost simple group. The natural map $P'_v \to \mathrm{red}_{v'}\,\mathfrak{P}_{v'}(k_{v'})$ is surjective. We also recall that since $\mathbf{H}'$ is simply connected $\mathrm{red}_{v'}\,\mathfrak{P}_{v'}$ is connected; see [60, 3.5.3]. Let $P^{(1)}_{v'}$ denote the first congruence subgroup of $P_{v'}$, i.e., the kernel of the natural projection.

[10] Let us mention that this bound is sufficient for finding a "good place" in §5.10. However, the stronger estimate in (2) will be needed in §5.12 if $\iota(\mathbf{H})$ is not simply connected and in Appendix B.

First note that if $P_{v'}^{(1)} \not\subset K_{v'}^*$, then the finite set $P_{v'}^{(1)}/P_{v'}^{(1)} \cap K_{v'}^*$ injects into $P_{v'}/K_{v'}^*$. But $P_{v'}^{(1)}$ is a pro-$p_{v'}$ group and hence any subgroup of it has an index which is a power of $p_{v'}$. Therefore, we get the claim under this assumption. In view of this observation we assume $P_{v'}^{(1)} \subset K_{v'}^*$. Therefore, since $K_{v'}^* \subsetneq P_{v'}$ we have $K_{v'}^*/P_{v'}^{(1)}$ is a proper subgroup of $P_{v'}/P_{v'}^{(1)} = \mathrm{red}_{v'}\,\mathfrak{P}_{v'}(k_{v'})$. The latter is a connected almost simple group of Lie-type and the smallest index of its subgroups is well understood. In particular, by [39, Prop. 5.2.1] the question reduces to the case of simple groups of Lie-type. Then by the main Theorem in [43], for the exceptional groups, and [15] for the classical groups, see also [39, Thm. 5.2.2] for a discussion, we have $[P_{v'}^* : K_{v'}^*] \geq q_{v'}$ so long as $q_{v'} \geq 13$. The conclusion in part (2) follows from these bounds.

(Case of v infinite:).[11] Note that up to conjugation by $\mathbf{G}(F_v)$ there are only finitely many homomorphisms from $\mathbf{H}(F_v)$ to $\mathbf{G}(F_v)$ with finite (central) kernel. We fix once and for all representatives for these maps. We will refer to these representatives as *standard homomorphisms* (and the list depends only on $\mathbf{G}(F_v)$). We fix a compact form $\mathfrak{h}_{v,0}$ for the group $\mathbf{H}(F_v)$. Taking the negative of the restriction of the Killing form to the compact form we extend it to a Hermitian form on $\mathfrak{h}_v^{\mathbb{C}}$ and restrict it to a Euclidean structure on $\mathfrak{h}_v$, which we will denote by $\mathfrak{q}$. For each standard homomorphism we fix a *standard Euclidean structure* on $\mathfrak{g}_v$ as follows: Let j_0 be the derivative of a standard homomorphism. Let $\mathfrak{g}_{v,0} \subset \mathfrak{g}_v^{\mathbb{C}}$ be a compact form of $\mathfrak{g}_v$ so that $\mathrm{j}_0(\mathfrak{h}_{v,0}) \subset \mathfrak{g}_{v,0}$. As above for the Euclidean structure $\mathfrak{q}$ on $\mathfrak{h}_v$, we use the compact forms $\mathfrak{g}_{v,0}$ to induce standard Euclidean structures associated to j_0 which we will denote by $\mathfrak{p}_0$.

We also let $\rho_0 : \mathbf{G} \to \mathrm{SL}_D$ be a representation given by Chevalley's theorem applied to the semisimple algebraic group $\mathrm{j}_0(\mathbf{H})$ considered over F_v, and let $w_{\mathrm{j}_0} \in F_v^D$ be so that $\mathrm{j}_0(\mathbf{H}) = \mathrm{Stab}_{\mathbf{G}}(w_{\mathrm{j}_0})$. As there are only finitely many standard homomorphisms we may require that the analogue of (5.3) holds also for these representations. In particular, we obtain that for $u \in \Xi_v$ there is a one-to-one correspondence between the eigenvalues and eigenvectors of $D\rho_0 u$ and the eigenvalues and eigenvectors of $\rho_0(\exp u)$. Hence for any $u \in \Xi_v$ and $w \in F_v^D$ we have

$$\rho_0(\exp u)w = w \text{ implies that } \rho_0(\exp(tu))w = w \text{ for all } t \in F_v. \tag{5.5}$$

Let $D\iota : \mathfrak{h}_v \to \mathfrak{g}_v$ denote the derivative of the homomorphism $\iota : \mathbf{H}(F_v) \to \mathbf{G}(F_v)$. Then the map $\mathrm{Ad}(g_v^{-1}) \circ D\iota$ induces an inclusion of real Lie algebras $\mathrm{j} : \mathfrak{h}_v \to \mathfrak{g}_v$, and a corresponding inclusion of complexifications $\mathrm{j}_{\mathbb{C}} : \mathfrak{h}_v^{\mathbb{C}} \to \mathfrak{g}_v^{\mathbb{C}}$. Let $g_0 \in \mathbf{G}(F_v)$ be so that $\mathrm{j}_0 = \mathrm{Ad}(g_0) \circ \mathrm{j}$ is the derivative of one of the standard homomorphisms. Then $\mathfrak{g}_{v,\mathrm{j}} = \mathrm{Ad}(g_0^{-1})\mathfrak{g}_{v,0} \subset \mathfrak{g}_v^{\mathbb{C}}$ is a compact form of $\mathfrak{g}_v$ so that $\mathrm{j}(\mathfrak{h}_{v,0}) \subset \mathfrak{g}_{v,\mathrm{j}}$. The compact form $\mathfrak{g}_{v,\mathrm{j}}$ induces a Euclidean structure on $\mathfrak{g}_v$, which we refer to as $\mathfrak{p}_{\mathrm{j}}$ and satisfies $\|\mathrm{Ad}(g_0)u\|_{\mathfrak{p}_0} = \|u\|_{\mathfrak{p}_{\mathrm{j}}}$ for all $u \in \mathfrak{g}_v$.

Recall the definition of Ξ_v from (5.3). We now will analyze the preimage K_v^* of $g_v\Omega_v g_v^{-1}$ in $\mathbf{H}(F_v)$ under the map ι, and show that it equals $\exp(\mathrm{j}^{-1}(\Xi_v))$. Clearly the latter is contained in K_v^* and we only have to concern ourselves with the opposite implication. So let $h \in K_v^*$ satisfy $g_v^{-1}\iota(h)g_v = \exp u \in \Omega_v$ for some $u \in \Xi_v$. We need to show that $u \in \mathrm{j}(\mathfrak{h}_v)$. Note that $g_v^{-1}\iota(\mathbf{H})(F_v)g_v$ is the stabilizer of $w = \rho_{\mathrm{j}_0}(g_0^{-1})w_{\mathrm{j}_0}$ in $\mathbf{G}(F_v)$. So $\exp(u)$ fixes w and by the property of Ξ_v in (5.5) we obtain $\exp(F_v u) \subset g_v^{-1}\iota(\mathbf{H})(F_v)g_v$. Thus we have $u \in \mathrm{j}(\mathfrak{h}_v)$ as we wanted.

[11] See also [50, Sect. 3.5].

Let us write $E(u)$ for the Jacobian of the exponential map and use the abbreviation $\mu_{\mathfrak{q}} = \prod_{v'|v} \lambda_{v'} |\omega'_{v'}|$ for the normalized Riemannian volume on $\mathfrak{h}_v$. We define

$$J_v := \Big(\prod_{v'|v} \lambda_{v'} |\omega'_{v'}|\Big)(K_v^*) = \int_{u\in \mathrm{j}^{-1}(\Xi_v)} E(u)\, \mathrm{d}\mu_{\mathfrak{q}}(u), \tag{5.6}$$

where we used also the definitions above. In view of (5.3) (which pulls back to an analogous claim for $u \in \mathrm{j}^{-1}(\Xi_v)$ and its adjoint on $\mathfrak{h}_v$) and (5.4), $E(u)$ is bounded above and below for all $u \in \mathrm{j}^{-1}(\Xi_v)$. Therefore

$$J_v \asymp \int_{u\in \mathrm{j}^{-1}(\Xi_v)} \mathrm{d}\mu_{\mathfrak{q}}(u) = \mu_{\mathfrak{q}}(\mathrm{j}^{-1}(\Xi_v)).$$

Now note that for the derivative j_0 of a standard homomorphism we have

$$\|u\|_{\mathfrak{q}} \asymp \|\,\mathrm{j}_0(u)\|_{\mathfrak{p}_0} = \|\,\mathrm{j}(u)\|_{\mathfrak{p}_{\mathrm{j}}},$$

which also gives

$$J_v \asymp \mu_{\mathfrak{q}}(\mathrm{j}^{-1}(\Xi_v)) \asymp \mu_{\mathrm{j}}(\Xi_v \cap \mathrm{j}(\mathfrak{h}_v)),$$

where μ_{j} is the ℓ-dimensional Riemannian volume (induced by $\mathfrak{p}_{\mathrm{j}}$) on the subspace $\mathrm{j}(\mathfrak{h}_v) \subset \mathfrak{g}_v$ with $\ell = \dim \mathfrak{h}_v$.

Let $u_1, \ldots, u_\ell \in \mathrm{j}(\mathfrak{h}_v)$ be an orthonormal basis, with respect to the standard Euclidean structure $\mathfrak{p}_0$, for $\mathrm{j}(\mathfrak{h}_v)$. Then there exists a constant c_1 (which depends only on Ξ_v and so on $\mathbf{G}$) such that

$$\Big\{ \sum_{|\lambda_r|\le 1/c_1} \lambda_r u_r \in \mathrm{j}(\mathfrak{h}_v)\Big\} \subset \Xi_v \cap \mathrm{j}(\mathfrak{h}_v) \subset \Big\{ \sum_{|\lambda_r|\le c_1} \lambda_r u_r \in \mathrm{j}(\mathfrak{h}_v)\Big\}$$

which gives

$$J_v \asymp \mu_{\mathrm{j}}(\Xi_v \cap \mathrm{j}(\mathfrak{h}_v)) \asymp \|u_1 \wedge \cdots \wedge u_\ell\|_{\mathfrak{p}_{\mathrm{j}}} = \frac{\|u_1 \wedge \cdots \wedge u_\ell\|_{\mathfrak{p}_{\mathrm{j}}}}{\|u_1 \wedge \cdots \wedge u_\ell\|_{\mathfrak{p}_0}}.$$

However, the last expression is independent of the choice of the basis of $\mathrm{j}(\mathfrak{h}_v)$. Let us now choose it so that $u_i = \mathrm{Ad}(g_0^{-1})(u_{0,i})$ for $i = 1, \ldots, \ell$ and a fixed orthonormal basis $u_{0,1}, \ldots, u_{0,\ell}$ of $\mathrm{j}_0(\mathfrak{h}_v)$ w.r.t. $\mathfrak{p}_0$. This gives

$$J_v \asymp \frac{1}{\| \wedge^\ell \mathrm{Ad}(g_0^{-1})(u_{0,1} \wedge \cdots \wedge u_{0,\ell})\|_{\mathfrak{p}_0}}$$

and part (3) of the proposition will follow if we show that

$$\| \wedge^\ell \mathrm{Ad}(g_0^{-1})(u_{0,1} \wedge \cdots \wedge u_{0,\ell})\|_{\mathfrak{p}_0}$$

is bounded away[12] from 0 (independently of j).

To see this claim recall that the Killing form $B := B_{\mathbf{G}(F_v)}$ is a $\mathbf{G}(F_v)$ invariant non-degenerate bilinear form on $\mathfrak{g}_v$ whose restriction to $\mathrm{j}_0(\mathfrak{h}_v)$ is non-degenerate. Let Q_B be the quadratic form on $\wedge^\ell \mathfrak{g}_v$ induced by B. Then $|Q_B(\cdot)|$ is bounded from above by a multiple of $\|\cdot\|^2$. Our claim follows from the fact that the value of Q_B at the vector $\wedge^\ell \mathrm{Ad}(g_0^{-1})(u_{0,1} \wedge \cdots \wedge u_{0,\ell})$ is non-zero and independent of g_0. $\square$

[12] This could also be seen using the more general fact that $\wedge^\ell \mathrm{Ad}(\mathbf{G}(F_v))(u_{0,1} \wedge \cdots \wedge u_{0,\ell})$ is a closed subset of $\wedge^\ell \mathfrak{g}_v$ which does not contain 0.

5.9. **Finite index in volume normalization.** Let $\mathbf{C}$ be the central kernel of $\mathbf{H} \to \iota(\mathbf{H})$. We may identify $g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$ with the quotient of $\mathbf{H}(\mathbb{A})$ by the *compact* group $\mathbf{C}(\mathbb{A})$ – it is a product of infinitely many finite groups.

The associated homogeneous space $Y = \mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))g$ is identified with

$$\iota^{-1}\Delta\backslash\mathbf{H}(\mathbb{A}),$$

where

$$\Delta = \iota(\mathbf{H})(F) \cap \iota(\mathbf{H}(\mathbb{A})).$$

Note that Δ is a discrete subgroup of $\iota(\mathbf{H}(\mathbb{A}))$, which is a closed subgroup of $\mathbf{G}(\mathbb{A})$. We need to compare the Haar measure on $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ (studied in this section) with the Haar measure on $\iota^{-1}\Delta\backslash\mathbf{H}(\mathbb{A})$ (used to define the volume of Y).

Now $\mathbf{H}(F)\mathbf{C}(\mathbb{A}) \subset \iota^{-1}\Delta$. The quotient $\iota^{-1}\Delta/\mathbf{H}(F)\mathbf{C}(\mathbb{A}) \cong \Delta/\iota(\mathbf{H}(F))$ is isomorphic to a subgroup $\mathcal{S}'$ of the kernel

$$\mathcal{S} := \ker\Big(H^1(F,\mathbf{C}) \to \prod_v H^1(F_v,\mathbf{C})\Big).$$

This can be seen from the exact sequence of pointed sets

$$\mathbf{H}(F) \xrightarrow{\iota} \iota(\mathbf{H})(F) \xrightarrow{\delta} H^1(F,\mathbf{C}),$$

arising from Galois cohomology, whereby Δ is identified with the preimage under δ of $\mathcal{S}$. The group $\mathcal{S}$ is finite by [49, Thm. 6.15] and so is $\mathcal{S}'$.

We endow $\mathbf{H}(\mathbb{A})$ with the measure for which $H(F)\backslash\mathbf{H}(\mathbb{A})$ has volume 1, use on $\iota(\mathbf{H}(\mathbb{A}))$ the quotient measure by the Haar probability measure on $\mathbf{C}(\mathbb{A})$, and use counting measure on Δ. With these choices the homogeneous space $\Delta\backslash\iota(\mathbf{H}(\mathbb{A})) \cong \iota^{-1}\Delta\backslash\mathbf{H}(\mathbb{A})$ has total mass

$$\frac{1}{\#\mathcal{S}'}\text{mass of}\left(\mathbf{H}(F)\mathbf{C}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})\right),$$

where the Haar measure on $\mathbf{H}(F)\mathbf{C}(\mathbb{A})$ is such that each coset of $\mathbf{C}(\mathbb{A})$ has measure 1. Together this gives that the mass of $\Delta\backslash\iota(\mathbf{H}(\mathbb{A}))$ equals $\frac{\#\mathbf{C}(F)}{\#\mathcal{S}'}$.

Finally, the size of $\mathbf{C}(F)$ is certainly bounded above and below in terms of $\dim(\mathbf{H})$, by the classification of semisimple groups. As for $\mathcal{S} \supset \mathcal{S}'$, it is finite[13] by [49, Thm. 6.15]. Indeed we can give an explicit upper bound for it in terms of $\dim\mathbf{H}$; see the proof of [49, Lemma 6.11]. We outline the argument. The absolute Galois group of F acts on $\mathbf{C}(\bar{F})$, by "applying Galois automorphisms to each coordinate"; we may choose a Galois extension E/F such that the Galois group of E acts trivially on $\mathbf{C}(\bar{F})$. Then $[E:F]$ can be chosen to be bounded in terms of $\dim(\mathbf{H})$. By the inflation-restriction sequence in group cohomology, the kernel of $H^1(F,\mathbf{C}) \to H^1(E,\mathbf{C})$ is isomorphic to a quotient of $H^1(\mathrm{Gal}(E/F),\mathbf{C}(\bar{F}))$, whose size can be bounded in terms of $\dim(\mathbf{H})$. On the other hand, the image of $\mathcal{S}$ consists of classes in $H^1(E,\mathbf{C})$ – i.e., homomorphisms from the Galois group of E to the abelian group $\mathbf{C}(\bar{F})$ – which are trivial when restricted to the Galois group of each completion of E. Any such homomorphism is trivial, by the Chebotarev density theorem.

Let us summarize the above discussions.

[13]It need not itself be trivial, because of Wang's counterexample related to the Grunwald–Wang theorem.

Lemma. *Normalize the Haar measure on $\mathbf{H}(\mathbb{A})$ so that the induced measure on $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ is a probability measure. Then the induced measure on the homogeneous set $\mathbf{G}(F)\iota(\mathbf{H}(\mathbb{A}))g$ equals $\frac{\#\mathbf{C}(F)}{\#\mathcal{S}'} \in [\frac{1}{M}, M]$, where $M \geq 1$ only depends on* $\dim \mathbf{H}$.

5.10. **The volume of a homogeneous set.** In view of our definition of volume and taking into account the choice of Ω_0, the equation (5.1) implies that

$$\operatorname{vol}(Y) = \frac{\#\mathbf{C}(F)}{\#\mathcal{S}'} D_{F'}^{\frac{1}{2}\dim \mathbf{H}'} \prod_{v\in\Sigma_F} (|\omega_v|(K_v^*))^{-1}, \tag{5.7}$$

where $|\omega_v| := \prod_{v'|v} |\omega'_{v'}|$.

Since $K_v^* \subseteq \prod_{v'|v} K_{v'}^*$

$$\begin{aligned}\operatorname{vol}(Y) &= \frac{\#\mathbf{C}(F)}{\#\mathcal{S}'} A D_{L/F'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}')} D_{F'}^{\frac{1}{2}\dim \mathbf{H}'} \prod_{v\in\Sigma_F} \left(|\omega_v|(K_v^*)\textstyle\prod_{v'|v}\lambda_{v'}\right)^{-1} \\ &\gg D_{L/F'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}')} D_{F'}^{\frac{1}{2}\dim \mathbf{H}'} \prod_{v'\in\Sigma_{F',f}} (\lambda_{v'}|\omega'_{v'}|(K_{v'}^*))^{-1} \prod_{v\in\Sigma_{F,f}} \left[\textstyle\prod_{v'|v} K_{v'}^* : K_v^*\right],\end{aligned} \tag{5.8}$$

where we used (5.7) and (5.2) in the first line and part (3) of the proposition in §5.8 in the second line. Let us note the rather trivial consequence[14] $\operatorname{vol}(Y) \gg 1$ of (5.8). Below we will assume implicitly $\operatorname{vol}(Y) \geq 2$ (which we may achieve by replacing Ω_v by a smaller neighborhood at one infinite place in a way that depends only on $\mathbf{G}$).

Let $\Sigma_{\mathrm{ur}}^\flat$ be the set of finite places v such that L/F is unramified at v but at least one of the following holds:

- $K_v^* \subsetneq \prod_{v'|v} K_{v'}^*$, or
- there is some $v'|v$ such that $\mathbf{H}'$ is not quasisplit over $F'_{v'}$, or
- there is some $v'|v$ such that $K_{v'}^*$ is not hyperspecial.

Then, in view of the proposition in §5.8 we find some $\kappa_6 > 0$ such that

$$\operatorname{vol}(Y) \gg D_{L/F'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}')} D_{F'}^{\frac{1}{2}\dim \mathbf{H}'} 2^{\#\Sigma_{\mathrm{ur}}^\flat} \gg D_L^{\kappa_6}\, 2^{\#\Sigma_{\mathrm{ur}}^\flat}; \tag{5.9}$$

where $\mathfrak{s}(\mathcal{H}') \geq 0$ as in §5.7. We note that (5.9) and the prime number theorem imply the existence of a good place in the case at hand.

5.11. **Existence of a good place in general.** Recall that the discussion in this section, so far, assumed $\mathbf{H}$ is F-almost simple. For the details of the proof of the existence of a good place we return to the general case. Thus, let $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_k$ be a direct product decomposition of $\mathbf{H}$ into F-almost simple factors. Let F'_j/F be a finite extension so that $\mathbf{H}_j = \operatorname{Res}_{F'_j/F}(\mathbf{H}'_j)$ where $\mathbf{H}'_j$ is an absolutely almost simple F'_j-group for all $1 \leq j \leq k$. As above $[F'_j : F]$ is bounded by $\dim \mathbf{H}$. Let $\mathcal{H}'_j$ and L_j/F'_j be the corresponding algebraic group and number field defined as in §5.4.

For any place $v \in \Sigma_F$ let K_v^* be as above. We have $K_v^* \subset \prod_{j=1}^k \prod_{v'|v} K_{j,v'}^*$ where $K_{j,v'}^*$ is the projection of K_v^* into $\mathbf{H}'_j(F'_{j,v'})$ and, in particular, it is a compact open subgroup of $\mathbf{H}'_j(F'_{j,v'})$ when v is finite.

[14] This would also follow trivially from the definition if only we would know that the orbit intersects a fixed compact subset.

Proposition (Existence of a good place). *There exists a place w of F such that*

(i) $\mathbf{G}$ *is quasisplit over* F_w *and split over* $\widehat{F_w}$, *and* K_w *is a hyperspecial subgroup of* $\mathbf{G}(F_w)$,
(ii) L_j/F *is unramified at* w *for every* $1 \le j \le k$,
(iii) $\mathbf{H}'_{j,w'}$ *is quasisplit over* $F'_{j,w'}$ (*and split over* $\widehat{F'_{j,w'}}$) *for every* $1 \le j \le k$ *and every* $w'|w$,
(iv) $K^*_w = \prod_{j=1}^k \prod_{w'|w} K^*_{j,w'}$, *and* $K^*_{j,w'}$ *is hyperspecial for all* $1 \le j \le k$ *and all* $w'|w$, *and finally*
(v) $q_w \ll (\log(\operatorname{vol} Y))^2$.

Proof. First note that similar to (5.7)–(5.9) we have

$$\operatorname{vol}(Y) \gg 2^I \prod_{j=1}^k \left(D_{L_j}^{\kappa_6} \textstyle\prod_{v' \in \Sigma_{F'_j}} \left(\lambda_{v'} |\omega'_{v'}|(K^*_{j,v'})\right)^{-1} \right),$$

where I is the number of finite places where the first assertion in (iv) does not hold. Note that at the archimedean places replacing K^*_v by $\prod_{j,v'|v} K^*_{j,v'}$ leads to a lower bound of the volume to which we may again apply part (3) of the proposition in §5.8.

As was done prior to (5.9), let $\Sigma^\flat_{\mathrm{ur}}$ be the set of finite places v of F where L_j/F is unramified at v for all $1 \le j \le k$ but (iii) or (iv) does not hold. Then

$$\operatorname{vol}(Y) \ge c_2 2^{\#\Sigma^\flat_{\mathrm{ur}}} \prod_{j=1}^k D_{L_j}^{\kappa_6}.$$

This implies the proposition in view of the prime number theorem. More concretely, suppose $T = q_w$ is the smallest norm of the prime ideal of a good place (satisfying (i)–(iv)) and recall that by Landau's prime ideal theorem the number of prime ideals in F with norm below T is asymptotic to $\frac{T}{\log T}$. Recall that (i) only fails at finitely many places $w \in \Sigma_f$ so we restrict ourselves to places w with $q_w \ge c_3$. Hence if $T \ge c_4 = c_4(F, c_3)$ we may assume that there are more than $\frac{T}{2\log T}$ places with norm between c_3 and T where (ii), (iii), or (iv) fails. Combining this with the above estimate gives

$$\sqrt{q_w} = \sqrt{T} \ll \frac{\kappa_6}{[F:\mathbb{Q}]} \frac{T}{2\log T} \le \log_2 \operatorname{vol}(Y) - \log_2 c_2,$$

which implies (v). □

5.12. **Comparison of two notions of volume.** Let $\mathbf{N}$ be the normalizer of $\iota(\mathbf{H})$ in $\mathbf{G}$. By Lemma 2.2 we have

$$\operatorname{stab}(\mu_{\mathscr{D}}) = g^{-1}\iota(\mathbf{H}(\mathbb{A}))\mathbf{N}(F)g.$$

It will be essential for our argument in §7.7 to control the "interplay" between the volume defined using $H_{\mathscr{D}} = g^{-1}\iota(\mathbf{H}(\mathbb{A}))g$ (as done so far) and the volume defined using $g^{-1}\iota(\mathbf{H}(\mathbb{A}_F))\mathbf{N}(F)g$ (which contains $H_{\mathscr{D}}$ as an open subgroup). In fact it will not be too difficult to reduce from $\gamma \in \mathbf{N}(F)$ to the case where $\gamma \in \iota(\mathbf{H}(F_v))$ at finitely many places. Note that $\gamma \in \iota(\mathbf{H}(F_v))$ at one place implies that $\gamma \in \iota(\mathbf{H})(F)$.

Let us make this more precise; recall that $\operatorname{vol}(Y) = m_{\mathscr{D}}(H_{\mathscr{D}} \cap \Omega_0)^{-1}$ where $\Omega_0 = \left(\prod_{v \in \Sigma_{F,\infty}} \Omega_v\right) K_f$ and $K_f = \prod_{v \in \Sigma_{F,f}} K_v$. We write $\mathrm{j}(\cdot) = g^{-1}\iota(\cdot)g$ and use this map also for one or several local factors. Note, in particular, that $H_v = \mathrm{j}(\mathbf{H}(F_v))$.

Let w be a good place given by the proposition in §5.11. We define $S = \Sigma_\infty \cup \{w\}$, $F_S = \prod_{v\in S} F_v$, and

$$\Psi_S = \mathrm{j}(\mathbf{H}(F_S)) \times \prod_{v\in\Sigma\setminus S} K_v.$$

Let $\widetilde{H_{\mathscr{D}}} = \mathrm{j}(\mathbf{H}(\mathbb{A}_F))N_S$ where

$$N_S = g^{-1}\big\{\gamma \in \iota(\mathbf{H})(F) : \gamma \in \iota(\mathbf{H}(F_v)) \text{ for all } v \in S\big\}g,$$

Note that $H_{\mathscr{D}} \subset \widetilde{H_{\mathscr{D}}} \subset \mathrm{stab}(\mu_{\mathscr{D}})$. We will see in §7.7 that we need to compare $\mathrm{vol}(Y)$ with

$$\widetilde{\mathrm{vol}}(Y) = \widetilde{m_{\mathscr{D}}}(\widetilde{H_{\mathscr{D}}} \cap \Omega_0)^{-1},$$

where $\widetilde{m_{\mathscr{D}}}$ is the unique Haar measure induced on $\mathrm{stab}(\mu_{\mathscr{D}})$ from $m_{\mathscr{D}}$.

Define $\Lambda := \Psi_S \cap \mathrm{j}(\mathbf{H}(F))$ and $\widetilde{\Lambda} := \Psi_S \cap N_S$.

Lemma (Volume and index)**.** *The index of Λ in $\widetilde{\Lambda}$ controls the ratio of the above notions of volume, i.e., we have*

$$\mathrm{vol}(Y) \ll [\widetilde{\Lambda} : \Lambda]\, \widetilde{\mathrm{vol}}(Y), \tag{5.10}$$

where the implicit constant depends on $\mathbf{G}(F_v)$ for $v \in \Sigma_\infty$.

Proof. Set

$$B := \big\{\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma : \gamma \in N_S \text{ and } \big(\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma\big) \cap \Omega_0 \neq \emptyset\big\}.$$

We will first prove that $\#B \le [\widetilde{\Lambda} : \Lambda]$.

The properties of the good place w, in particular, guarantee that using the strong approximation theorem for $\mathbf{H}$ we have

$$\mathbf{H}(\mathbb{A}_F) = \mathbf{H}(F_S)\Big(\prod_{v\in\Sigma\setminus S} K_v^*\Big)\mathbf{H}(F).$$

Let now $\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma \in B$; then there exists some $g_\gamma \in \mathrm{j}(\mathbf{H}(F_S)\prod_{v\in\Sigma\setminus S} K_v^*)$ and some $\delta \in \mathrm{j}(\mathbf{H}(F))$ so that $g_\gamma\delta\gamma \in \Omega_0$. Hence for all $v \in \Sigma \setminus S$ we have $(\delta\gamma)_v \in K_v$. This says

$$\delta\gamma \in N_S \cap \Big(\mathrm{j}(\mathbf{H}(F_S)) \times \prod_{v\in\Sigma\setminus S} K_v\Big) = \widetilde{\Lambda}.$$

Suppose now $\delta' \in \mathrm{j}(\mathbf{H}(F))$ is so that $\delta'\gamma \in \widetilde{\Lambda}$; then

$$\delta'\delta^{-1} \in \widetilde{\Lambda} \cap \mathrm{j}(\mathbf{H}(F)) = \Lambda \trianglelefteq \widetilde{\Lambda}.$$

Hence we get a map from $B \to \widetilde{\Lambda}/\Lambda$.

This map is injective. Indeed let $\gamma, \gamma' \in N_S$ be as in the definition of B, suppose that $\delta_\gamma, \delta_{\gamma'} \in \mathrm{j}(\mathbf{H}(F))$ are as above, and $\delta_\gamma\gamma$ and $\delta_{\gamma'}\gamma'$ map to the same coset in $\widetilde{\Lambda}/\Lambda$. Then $\Lambda\delta_\gamma\gamma = \Lambda\delta_{\gamma'}\gamma'$, and in particular, $\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma = \mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma'$. In other words we have shown that $\#B \le [\widetilde{\Lambda} : \Lambda]$.

With this we now have the estimate

$$\widetilde{m_{\mathscr{D}}}(\widetilde{H_{\mathscr{D}}} \cap \Omega_0) = \sum_{\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma\in B} m_{\mathscr{D}}((h_\gamma\gamma)^{-1}\Omega_0) \le [\widetilde{\Lambda} : \Lambda] m_{\mathscr{D}}(\Omega_0^{-1}\Omega_0),$$

where we use for every $\mathrm{j}(\mathbf{H}(\mathbb{A}_F))\gamma \in B$ some $h_\gamma \in \mathrm{j}(\mathbf{H}(\mathbb{A}))$ with $h_\gamma\gamma \in \Omega_0$. The claim now follows from the independence, up to a multiplicative scalar, of the notion of volume from the neighborhood Ω_0; see §2.3. □

The landmark paper [6] by Borel and Prasad deals with questions similar to bounding the above index, $[\widetilde{\Lambda} : \Lambda]$. The setup in [6] is that Λ is defined using a coherent family of *parahoric* subgroups at every place. However, our group Λ is defined using $\{K_v^*\}$, and K_v^* may only be a parahoric subgroup for almost all v. We will use the strong approximation theorem to address this issue and then use [6] to estimate the above index.

We will again need some reductions due to the fact that our group $\mathbf{H}$ is not necessarily absolutely almost simple. Recall that $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_k$ is a product of F-almost simple groups where $\mathbf{H}_i = \mathrm{Res}_{F_i'/F}(\mathbf{H}_i')$ with $\mathbf{H}_i'$ an absolutely almost simple F_i'-group.

Let $v \in \Sigma \setminus S$; that is: v is a finite place and $v \neq w$. The Bruhat-Tits building $\mathcal{B}_v$ of $\mathbf{H}(F_v)$ is the product of the corresponding buildings $\mathcal{B}_{i,v}$ for $1 \le i \le k$. The group H_v is naturally identified with $\mathrm{j}(\prod_{i,v'|v} \mathbf{H}_i'(F_{i,v'}'))$ and acts on $\mathcal{B}_v$; this action is identified with the action of $\prod_{i,v'|v} \mathbf{H}_i'(F_{i,v'}')$ on the product of the corresponding buildings $\mathcal{B}_{i,v'}$. Our group $K_{i,v'}^*$ (which by definition is the group obtained by projecting K_v^* into $\mathbf{H}_i'(F_{i,v'}')$) is a compact open subgroup of $\mathbf{H}_i'(F_{i,v'}')$ for all places v' of F_i' over v. Hence by [60, §3.2] the fixed point set $\mathrm{Fix}_{i,v'}$ of $K_{i,v'}^*$ in $\mathcal{B}_{i,v'}$ is a compact and non-empty subset.

Let $\overline{\mathbf{H}}$ denote the adjoint form of $\mathbf{H}$ and let $\varphi : \mathbf{H} \to \overline{\mathbf{H}}$ be the universal covering map. The adjoint form $\overline{\mathbf{H}}$ is identified with $\prod_i \mathrm{Res}_{F_i'/F}(\overline{\mathbf{H}_i'})$ where $\overline{\mathbf{H}_i'}$ is the adjoint form of $\mathbf{H}_i'$. Recall that $\widetilde{\Lambda} \subset g^{-1}\iota(\mathbf{H})(F)g$, and let $\varphi_v' : \mathrm{j}(\mathbf{H}) \to \overline{\mathbf{H}}$ be so that $\varphi = \varphi_v' \circ \mathrm{j}$. Then $\varphi_v'(\widetilde{\Lambda}) \subset \overline{\mathbf{H}}(F)$. In particular, $\widetilde{\Lambda}$ naturally acts on $\mathcal{B}_{i,v'}$ for all i and all places v' of F_i' above v.

Lemma. *The fixed point set* $\widetilde{\mathrm{Fix}_{i,v'}}$ *of* $\widetilde{\Lambda}$ *in* $\mathcal{B}_{i,v'}$ *is a non-empty compact subset which satisfies* $\widetilde{\mathrm{Fix}_{i,v'}} \subset \mathrm{Fix}_{i,v'}$.

Proof. Let Λ_v (resp., $\widetilde{\Lambda}_v$) be the closure (in the Hausdorff topology) of the projection of Λ (resp., $\widetilde{\Lambda}$) in K_v. By the strong approximation theorem, we have

$$\Lambda_v = H_v \cap K_v = \mathrm{j}(K_v^*).$$

Moreover, taking projections, we may identify both Λ and $\widetilde{\Lambda}$ as lattices in $\mathrm{j}(\mathbf{H}(F_S))$. Therefore, we have $[\widetilde{\Lambda}_v : \Lambda_v] \le [\widetilde{\Lambda} : \Lambda] < \infty$.

Hence, using [60, §3.2], the fixed point set $\widetilde{\mathrm{Fix}_{i,v'}}$ of $\widetilde{\Lambda}$ in $\mathcal{B}_{i,v'}$ is a non-empty compact subset which satisfies $\widetilde{\mathrm{Fix}_{i,v'}} \subset \mathrm{Fix}_{i,v'}$ as claimed. $\square$

Let us fix, for every $v \in \Sigma \setminus S$, one point in $\mathcal{B}_v$ which is fixed by $\widetilde{\Lambda}$. This determines a subset $\Phi_{i,v'}$ of the affine root system $\Delta_{i,v'}$. The collection $\{\Phi_{i,v'}\}$ gives us a coherent collection of parahoric subgroups $P_{i,v'} \subset \mathbf{H}_i'(F_{i,v'}')$. For every $v \in \Sigma \setminus S$, let $\widetilde{P}_v$ denote the stabilizer of $\prod_{v'|v} \Phi_{i,v'}$ in $\mathrm{j}(\mathbf{H})(F_v)$. We define two subgroups

$$\Lambda' = \mathrm{j}\Big(\prod_i \mathbf{H}_i'(F_i') \cap \big(H_S' \times \prod_{i,v' \nmid w} P_{i,v'}\big)\Big),$$

$$\widetilde{\Lambda}' = N_{\mathrm{j}(H_S') \times \prod_{v \neq w} \widetilde{P}_v}(\Lambda'),$$

where $H_S' = \prod_i \prod_{v'|v, v \in S} \mathbf{H}_i'(F_{i,v}')$.

Note that $\Lambda \subset \Lambda'$ and $\widetilde{\Lambda} \subset \widetilde{\Lambda}'$ by the construction of the parahoric subgroups.[15] Moreover, Λ' is a finite index subgroup of $\widetilde{\Lambda}'$; see [6, Prop. 1.4].

Recall the definition of the fields L_i/F_i' from §5.11. As we have done before we define a subset $\Sigma^\flat \subset \Sigma_{F,f}$ as follows. Let $\Sigma^\flat_{\mathrm{ur}}$ be the set of finite places v of F where L_j/F is unramified at v for all $1 \leq i \leq k$ but at least one of the following fails:

(1) $\mathbf{H}'_{i,v'}$ is quasisplit over $F'_{i,v'}$ (and split over $\widehat{F'_{i,v'}}$) for every $1 \leq i \leq k$ and every $v'|v$,
(2) $K^*_{i,v'}$ is hyperspecial for all $1 \leq i \leq k$ and all $v'|v$ and $K^*_v = \prod_{i,v'|v} K^*_{i,v'}$.

Define $\Sigma^\flat_{\mathrm{rm}}$ to be the set of places $v \in \Sigma_{F,f}$ so that L_i/F is ramified at v for some $1 \leq i \leq k$. Put $\Sigma^\flat = \Sigma^\flat_{\mathrm{ur}} \cup \Sigma^\flat_{\mathrm{rm}}$; note that $\Sigma^\flat \cap S = \emptyset$.

Let us note that if $K^*_{i,v'}$ is hyperspecial for all $1 \leq i \leq k$ and all $v'|v$ but $K^*_v \neq \prod_{i,v'|v} K^*_{i,v'}$, then

$$\Big[\prod_{i,v'|v} K^*_{i,v'} : K^*_v\Big] \geq p_v. \tag{5.11}$$

Indeed the reduction mod v' of the group scheme corresponding to $K^*_{i,v'}$ is an almost simple group and $K^*_{i,v'}$ maps onto the $k_{i,v'}$ points of this group. Let R be the semisimple group obtained from $\prod_{i,v'|v} K^*_{i,v'}$ by taking it modulo the first congruence subgroup. By construction of $K^*_{i,v'}$ the image of K^*_v modulo the first congruence subgroup of $\prod_{i,v'|v} K^*_{i,v'}$, let us call it R', projects onto each factor of R. If R' does not equal R, then (5.11) follows. If these two equal each other, then an argument as in the proof of part (2) of the proposition in §5.8 implies (5.11).

This observation together with parts (1) and (2) of the proposition in §5.8 implies that for all $v \in \Sigma^\flat_{\mathrm{ur}}$ we have

$$\Big(\prod_{i,v'|v} \lambda_{i,v'}|\omega'_{i,v'}|\Big)(K^*_v) \leq \frac{p_v+1}{p_v^2} \tag{5.12}$$

if $q_v > 13$.

Lemma (Bound on index)**.** *The index of Λ in $\widetilde{\Lambda}$ satisfies the bound*

$$[\widetilde{\Lambda} : \Lambda] \leq N^{\kappa_7 + \kappa_8(\#\Sigma^\flat)} \prod_i 2h^a_{L_i}(D_{L_i/F'_i})^b \tag{5.13}$$

where

- $\kappa_7 = \sum_i [L_i : \mathbb{Q}]$ *and* $\kappa_8 = 2\sum_i [F'_i : F]$,
- h_{L_i} *is the class number of* L_i,
- $a = 2$ *if* $\mathbf{H}'_i$ *is an inner form of a split group of type* D_r *with* r *even, resp.,* $a = 1$ *otherwise, and finally*
- $b = 1$ *if* $\mathbf{H}'_i$ *is an outer form of type* D_r *with* r *even, resp.,* $b = 0$ *otherwise.*

[15] To verify the second inclusion, for example, we first verify that $\widetilde{\Lambda}$ belongs to $j(H'_S) \times \prod_{v \notin S} \widetilde{P}_v$: it projects to $j(H'_S)$ at places in S because $\widetilde{\Lambda} \subset \Psi_S$, and it projects to the $\widetilde{P}_v$ by the way they were chosen. We then verify $\widetilde{\Lambda}$ normalizes Λ'. Because of the inclusion $\widetilde{\Lambda} \subset N_S$ we can regard $\widetilde{\Lambda}$ as acting on $\mathbf{H}(F) = \prod_i \mathbf{H}'_i(F'_i)$. It preserves the subset of this defined by intersecting with $(H'_S \times \prod P_{i,v'})$ because each $\tilde{P}_v$ normalizes $\prod_{i,v'|v} P_{i,v'}$.

Proof. We first consider the map $\widetilde{\Lambda}/\Lambda \to \widetilde{\Lambda}'/\Lambda'$. This is an injective map. Indeed, if $\gamma \in \Lambda' \cap \widetilde{\Lambda}$, then $\gamma \in \widetilde{\Lambda} \subset \Psi_S$ and $\gamma \in \Lambda' \subset \mathrm{j}(\mathbf{H}(F))$. Hence

$$\gamma \in \Psi_S \cap \mathrm{j}(\mathbf{H}(F)) = \Lambda.$$

We, thus, get that $[\widetilde{\Lambda} : \Lambda] \leq [\widetilde{\Lambda}' : \Lambda']$.

Bounding $[\widetilde{\Lambda}' : \Lambda']$ is rather non-trivial. This is done in [6, §2 and §5], and we have

$$[\widetilde{\Lambda}' : \Lambda'] \leq \prod_i 2h_{L_i}^a N^{[L_i:\mathbb{Q}]+2[F_i':F]\#\Sigma^\flat} (D_{L_i/F_i'})^b,$$

with a, b and h_{L_i} as in the statement of the lemma. □

The following is crucial in the application of the volume for the pigeon-hole argument in §7.7.

Proposition (Equivalence of volume definitions)**.** *The above two notions of volume are related in the sense that there exists some $\kappa_9 > 0$ so that*

$$\mathrm{vol}(Y)^{\kappa_9} \leq \widetilde{\mathrm{vol}}(Y) \leq \mathrm{vol}(Y) \tag{5.14}$$

if $\mathrm{vol}(Y)$ *is sufficiently large depending only on the dimensions* $\dim \mathbf{G}$ *and* $[F : \mathbb{Q}]$.

Proof. For any number field E we have

$$h_E \leq 10^2 \left(\tfrac{\pi}{12}\right)^{[E:\mathbb{Q}]} D_E;$$

see, e.g., equation (7) in the proof of [6, Prop. 6.1]. Also recall that

$$D_{L_i/F_i'} = D_{L_i}/D_{F_i'}^{[L_i:F_i']} \geq 1.$$

These imply the following[16] estimates (for the field related quantities coming in part from (5.13) and (5.8)). If $\mathcal{H}_i'$ is an outer form, then $a = 1$ and $b \leq 1$. Hence in this case we have

$$h_{L_i}^{-a} D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')-b} \gg D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'-c} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')-2},$$

where $c = [L_i : F_i']$ equals 3 if $\mathcal{H}_i'$ is a triality form of D_4, resp., 2 otherwise.

Suppose $\mathbf{H}_i'$ is an inner form of type other than A_1. Then $L_i = F_i'$, $a \leq 2$, and $b = 0$. Together we get

$$h_{L_i}^{-a} D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')-b} \gg D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'-2} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')} = D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'-2}.$$

Finally let $\mathbf{H}_i'$ be an inner form of type A_1,; then $L_i = F_i'$, $a = 1$, $b = 0$, and we have

$$h_{L_i}^{-a} D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')-b} \gg D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'-1} = D_{F_i'}^{1/2}.$$

These estimates together with $\mathfrak{s}(\mathcal{H}_i') \geq 5$ when $\mathcal{H}_i'$ is an outer form and $L_i = F_i'$ when $\mathcal{H}_i'$ is an inner form give

$$h_{L_i}^{-a} D_{F_i'}^{\frac{1}{2}\dim \mathbf{H}_i'} D_{L_i/F_i'}^{\frac{1}{2}\mathfrak{s}(\mathcal{H}_i')-b} \gg \left(D_{F_i'} D_{L_i/F_i'}\right)^{1/2}. \tag{5.15}$$

We now prove (5.14) and note that $\widetilde{\mathrm{vol}}(Y) \leq \mathrm{vol}(Y)$ follows directly from the definition.

[16] See [6, Prop. 6.1], and also [2, Prop. 3.3], for more general statements.

For the opposite inequality we argue as follows:

$$\widetilde{\mathrm{vol}}(Y) \geq \frac{1}{[\widetilde{\Lambda}:\Lambda]}\,\mathrm{vol}(Y) \qquad \text{by (5.10)}$$

$$\geq \Big(N^{\kappa_7+\kappa_8(\#\Sigma^\flat)}\prod_i 2h^a_{L_i}(D_{L_i/F_i'})^b\Big)^{-1}\mathrm{vol}(Y) \qquad \text{by (5.13).} \tag{5.16}$$

Now by (5.15) we have

$$\Big(\prod_i 2h^a_{L_i}(D_{L_i/F_i'})^b\Big)^{-1} \gg \prod_i D_{F_i'}^{-\frac12\dim\mathbf{H}_i'+\frac12} D_{L_i/F_i'}^{-\frac12\mathfrak{s}(\mathcal{H}_i')+\frac12}. \tag{5.17}$$

Moreover, by (5.12) we have

$$N^{-\kappa_7-\kappa_8(\#\Sigma^\flat)} \gg \Big(\prod_{\Sigma^\flat}\big(\prod_{i,v'|v}\lambda_{i,v'}|\omega'_{i,v'}|\big)(K_v^*)\Big)^{-\frac12}. \tag{5.18}$$

Note that for the few bad places with $q_v < 13$ the power of N simply becomes an implicit multiplicative constant.

In view of (5.17) and (5.18), the lower bound in (5.14) follows from (5.16) and the asymptotic in (5.8). □

6. Algebraic properties at a good place

As in §§1.2 and 2.1 we let $Y = Y_{\mathscr{D}}$ be the MASH set for the data $\mathscr{D} = (\mathbf{H}, \iota, g_{\mathscr{D}})$ and let $\mathbf{G}$ be the ambient algebraic group; in particular we are assuming that $\mathbf{H}$ is simply connected and that $\iota(\mathbf{H})$ is maximal in $\mathbf{G}$. In this section we collect algebraic properties of the MASH set Y and its associated groups at a good place w. These properties may be summarized as saying that the acting group is not distorted at w and will be needed in the dynamical argument of the next section.

6.1. **Good places.** We say a place $w \in \Sigma_f$ is *good* (for Y) when

- w satisfies (i)–(iv) in the proposition concerning the existence of good places in §5.11,
- in particular $\mathbf{G}$ and $\iota(\mathbf{H})$ are quasisplit over F_w and split over $\widehat{F_w}$, the maximal unramified extension, and[17]
- $\mathrm{char}(k_w) \gg_{N,F} 1$, where $\rho(\mathbf{G}) \subset \mathrm{SL}_N$ as before.

We note that the last property of a good place as above allows us, e.g., to avoid difficulties arising from the theory of finite-dimensional representations of algebraic groups over fields with "small" characteristic.

By the proposition in §5.11 we have: there is a good place w satisfying[18]

$$q_w \ll (\log(\mathrm{vol}\,Y))^2.$$

Let $g_{\mathscr{D},w} \in \mathbf{G}(F_w)$ denote the component of $g_{\mathscr{D}}$ at w. For simplicity in notation we write $\mathrm{j}_w : \mathbf{H} \to \mathbf{G}$ for the homomorphism defined by $\mathrm{j}_w(\cdot) = g_{\mathscr{D},w}^{-1}\iota(\cdot)g_{\mathscr{D},w}$ at the good place w. We define the group $H_w^* = \mathbf{H}(F_w)$ and recall from §5.8 the notation $K_w^* = \mathrm{j}_w^{-1}(K_w)$. It is worth mentioning again that $\mathrm{j}_w(H_w^*)$ does not necessarily equal $\mathrm{j}_w(\mathbf{H})(F_w)$ or the group of F_w-points of any algebraic group.

[17]For the last claim increase in the proof of §5.11 the value of c_3 accordingly.

[18]The good place for the proof of Theorem 4.1 is found as in §4.5: There are infinitely many places where $\mathbf{G}$ splits, and all properties of a good place for the maximal subgroup $\mathbf{H} = \{(h,h) : h \in \mathbf{G}\} < \mathbf{G}\times\mathbf{G}$ are satisfied for almost all places.

6.2. **Compatibility of hyperspecial subgroups.** By the properties of the good place $\mathbf{G}$ and $\mathbf{H}$ are quasisplit over F_w and split over $\widehat{F_w}$. Furthermore, K_w and K_w^* are hyperspecial subgroups of $\mathbf{G}(F_w)$ and $H_w^* = \mathbf{H}(F_w)$, respectively.

Let vert and vert* denote the vertices corresponding to K_w and K_w^* in the respective buildings. As was recalled in §5.1, Bruhat-Tits theory associates smooth group schemes $\mathfrak{G}_w$ and $\mathfrak{H}_w$ to vert and vert* in $\mathbf{G}(F_w)$ and $\mathbf{H}(F_w)$, respectively, so that $K_w = \mathfrak{G}_w(\mathfrak{o}_w)$ and $K_w^* = \mathfrak{H}_w(\mathfrak{o}_w)$. Since j_w is a homomorphism, $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ acts on the building of $\mathbf{H}(\widehat{F_w})$.

Let p_w be the prime number so that $w \mid p_w$, i.e., $p_w = \mathrm{char}(k_w)$.

Lemma. *For $p_w \gg 1$ the following hold. The stabilizer of* vert* *in* $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ *equals* $\mathrm{j}_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w}))$, *i.e., the image of the stabilizer of* vert* *in* $\mathbf{H}(\widehat{F_w})$ *under the map* j_w. *Moreover, the homomorphism* j_w *extends to a closed immersion from* $\mathfrak{H}_w$ *to* $\mathfrak{G}_w$ *which we continue to denote by* j_w.

Proof. Let $h \in \mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ be in the stabilizer of vert* in $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$. In the following paragraph we will use similar arguments to that of [6, §2] and we refer to that for unexplained notions. Then the induced action of h on the affine root system fixes the vertex corresponding to vert*; this implies h acts trivially on the affine root system; see [35, 1.8]. It follows from [35, 1.8] and [6, Prop. 2.7] that $h \in \mathrm{j}_w(\mathbf{H}(\widehat{F_w}))$ (i.e., represents the trivial cohomology class with reference to [6, §2.5(1)]), at least for p_w large enough. Now by [60, 3.4.3] the smooth scheme structure corresponding to the stabilizer of vert* in $\mathbf{H}(\widehat{F_w})$ is deduced from $\mathfrak{H}_w$ by base change from $\mathfrak{o}_w$ to $\widehat{\mathfrak{o}_w}$. Therefore, $\mathrm{j}_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w}))$ equals the stabilizer of vert* in $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ which is the first claim of the lemma.

We now claim

$$\mathrm{j}_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w})) \subset \mathfrak{G}_w(\widehat{\mathfrak{o}_w}). \tag{6.1}$$

Assuming the claim, let us finish the proof. By the criterion described in [10, 1.7.3, 1.7.6], the homomorphism j_w extends to an $\mathfrak{o}_w$-morphism $\tilde{\mathrm{j}}_w : \mathfrak{H}_w \to \mathfrak{G}_w$ which by [51, Cor. 1.3] is a closed immersion.

Let us now turn to the proof of (6.1). It suffices to prove

$$\rho \circ \mathrm{j}_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w})) \subset \mathrm{SL}_N(\widehat{\mathfrak{o}_w}).$$

Put $\rho_w := \rho \circ \mathrm{j}_w$. Then $\rho_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w}))$ is a bounded subgroup of $\mathrm{SL}_N(\widehat{F_w})$, hence it is contained in a maximal parahoric subgroup P of $\mathrm{SL}_N(\widehat{F_w})$ – we may even suppose that P is a hyperspecial parahoric subgroup. Let us assume $P \neq \mathrm{SL}_N(\widehat{\mathfrak{o}_w})$ as there is nothing to prove otherwise.

Inside the building of SL_N over $\widehat{F_w}$, let v_0 be the vertex corresponding to P and let v be the vertex corresponding to $\mathrm{SL}_N(\widehat{\mathfrak{o}_w})$. Choose a geodesic, inside this building, connecting the vertex v_0 with the vertex v. Consider the collection $\mathcal{C}$ of all facets whose interior meets this geodesic path. Any element of $P \cap \mathrm{SL}_N(\widehat{\mathfrak{o}_w})$ fixes all facets in $\mathcal{C}$ – recall that, in the building for SL_N, fixing a facet setwise implies fixing it pointwise. Therefore, $\rho_w(\mathfrak{H}_w(\mathfrak{o}_w))$ fixes all the facets in $\mathcal{C}$.

In this language, we must show that $\rho_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w}))$ fixes the vertex v. The union of facets in $\mathcal{C}$ is connected, and so its 1-skeleton is connected; thus we may choose a path $v_0, v_1, \ldots, v_\ell = v$ starting from the vertex v_0 and ending at v, where any two adjacent vertices belong to a common chamber. Let P_i be the stabilizer of v_i. We have seen above that $\rho_w(\mathfrak{H}_w(\mathfrak{o}_w)) \subset P_i$ and we will argue inductively that $\rho_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w})) \subset P_i$ for all $0 \le i \le \ell$.

For each $0 \le i < \ell$ there is an element $g_i \in \mathrm{PGL}_N(\widehat{F_w})$ so that $P_i = g_i \mathrm{SL}_N(\widehat{\mathfrak{o}_w}) g_i^{-1}$. Denote by SL_{N,g_i} the corresponding scheme structure, that is, $P_i = \mathrm{SL}_{N,g_i}(\widehat{\mathfrak{o}_w})$. Assume $\rho_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w})) \subset \mathrm{SL}_{N,g_i}(\widehat{\mathfrak{o}_w})$. By [10, 1.7.3, 1.7.6] the homomorphism ρ_w extends to an $\widehat{\mathfrak{o}_w}$-morphism $\tilde\rho_w : \mathfrak{H}_w \to \mathrm{SL}_{N,g_i}$ which by [51, Cor. 1.3] is a closed immersion. Let $\mathrm{red}_w(\tilde\rho_w) : \underline{\mathfrak{H}_w}(\widehat{k_w}) \to \mathrm{SL}_{N,g_i}(\widehat{k_w})$ be the corresponding homomorphism on special fibers. Then the finite group $\mathrm{red}_w(\tilde\rho_w)\big(\underline{\mathfrak{H}_w}(k_w)\big)$ is contained in $\mathrm{red}_w(P_i \cap P_{i+1})$ which is a proper parabolic subgroup of SL_{N,g_i}; and we must show that the same is true with k_w replaced by $\widehat{k_w}$.

Since each proper parabolic subgroup of SL_N can be expressed as the intersection of certain subspace stabilizers, our assertion reduces to the following: Regarding $\underline{\mathfrak{H}_w}$ as acting on an N-dimensional representation via $\mathrm{red}_w(\tilde\rho_w)$, and if $p_w \gg_N 1$, the following holds:

$$\text{If } \underline{\mathfrak{H}_w}(k_w) \text{ fixes a subspace } W \subset \widehat{k_w}^N, \text{ then } \underline{\mathfrak{H}_w}(\widehat{k_w}) \text{ also fixes } W.$$

Passing to exterior powers, and using the semisimplicity we reduce to the same statement with the subspace W replaced by a vector v. But $\underline{\mathfrak{H}_w}$ is generated by unipotent one-parameter subgroups, i.e., by closed immersions $u : \mathbb{G}_a \to \underline{\mathfrak{H}_w}$. Because the map $\underline{\mathfrak{H}_w} \to \mathrm{SL}_{N,g_i}/\widehat{k_w}$ is a closed immersion, we can regard u as a closed immersion $\mathbb{G}_a \to \mathrm{SL}_{N,g_i}/\widehat{k_w}$ also, and from that we see that the coordinates of $u(t)v$ are polynomials in t whose degree is bounded in terms of N. Since these polynomials vanish identically for $t \in k_w$, we see that, for $p_w \gg 1$, they vanish identically on $\widehat{k_w}$, too. $\square$

In view of the above lemma, and abusing the notation, $\mathrm{j}_w(\mathfrak{H}_w)$ is a smooth subgroup scheme of $\mathfrak{G}_w$. Taking reduction mod w on $\mathfrak{G}_w$, which induces the reduction map on $\mathrm{j}_w(\mathfrak{H}_w)$, we have $\underline{\mathrm{j}_w(\mathfrak{H}_w)} \subset \underline{\mathfrak{G}_w}$ for the corresponding algebraic groups over $\widehat{k_w}$ (the residue field of $\widehat{F_w}$, i.e., the algebraic closure of k_w).

6.3. **Lemma** (Inheritance of maximality)**.** *Let $\iota(\mathbf{H}) < \mathbf{G}$, the place $w \in \Sigma_f$ and $g_{\mathscr{D},w} \in G_w$ be as above. Then $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ is a maximal connected algebraic subgroup of $\underline{\mathfrak{G}_w}$ provided that p_w is large enough.*

Proof. First note that the subgroups $\mathfrak{G}_w(\widehat{\mathfrak{o}_w})$ and $\mathrm{j}_w(\mathfrak{H}_w)(\widehat{\mathfrak{o}_w})$ are hyperspecial subgroups of $\mathbf{G}(\widehat{F_w})$ and $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$; see §5.1 (1) as well as [60, 2.6.1 and 3.4.3]. In particular, $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ and $\underline{\mathfrak{G}_w}$ are connected by §5.1 (4).

Let us also recall our assumption that both $\mathbf{G}$ and $\mathbf{H}$ split over $\widehat{F_w}$. Therefore, by §5.1 (2) we have: $\mathrm{j}_w(\mathfrak{H}_w)$ and $\mathfrak{G}_w$ are $\widehat{\mathfrak{o}_w}$-Chevalley group schemes with generic fibers $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ and $\mathbf{G}(\widehat{F_w})$, respectively.

We now show that this and maximality of $\iota(\mathbf{H})$ in $\mathbf{G}$ implies that the subgroup $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ is a maximal subgroup of $\underline{\mathfrak{G}_w}$.

We first claim that $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ is not contained in any proper parabolic subgroup of $\underline{\mathfrak{G}_w}$.

To see this, let H (resp., G) denote the split Chevalley group over $\mathbb{Z}$ which has the same type as $\mathbf{H}$ (resp., $\mathbf{G}$). These are affine schemes. For an arbitrary ring R, we denote by H_R the base change of H to R, and similarly for G.

We want to make an argument involving the "scheme of homomorphisms from H to G". Such a scheme, whose R-points are canonically in bijection with homomorphisms of R-group schemes $\mathsf{H}_R \to \mathsf{G}_R$ is constructed in [19], but it is too big for us, because it has many components corresponding to Frobenius twists of morphisms. Thus we use a homemade variant.

Let $\mathcal{O}_H$ and $\mathcal{O}_G$ be the ring of global sections of the structure sheafs of H and G, respectively. Fix generators $f_1, \dots, f_r$ for $\mathcal{O}_G$ as a $\mathbb{Z}$-algebra. There are finitely many conjugacy classes of homomorphisms $\mathsf{H}_{\overline{\mathbb{Q}}} \to \mathsf{G}_{\overline{\mathbb{Q}}}$. Therefore, there exists a finite-dimensional $\mathsf{H}(\overline{\mathbb{Q}})$-stable subvector space $\widetilde{M} \subset \mathcal{O}_H \otimes \overline{\mathbb{Q}}$ with the following property: For any homomorphism $\rho : \mathsf{H}_{\overline{\mathbb{Q}}} \to \mathsf{G}_{\overline{\mathbb{Q}}}$, the pullback $\rho^* f_i$ belongs to $\widetilde{M}$. Write $M = \widetilde{M} \cap \mathcal{O}_H$.

Let $\mathcal{S}$ be the affine scheme defined thus: an R-point of $\mathcal{S}$ is a homomorphism of Hopf algebras $\rho^* : \mathcal{O}_G \otimes R \to \mathcal{O}_H \otimes R$ such that $\rho(f_i) \subset M \otimes R$. Said differently, $\mathcal{S}(R)$ parameterizes homomorphisms of R-group schemes $\rho : \mathsf{H}_R \to \mathsf{G}_R$ with the finiteness property just noted, i.e.,

$$\rho^* f_i \in M \otimes R, \quad 1 \leq i \leq r. \tag{6.2}$$

It is easy to see by writing out equations that this functor is indeed represented by a scheme of finite-type over $\mathbb{Z}$.

If R is an integral domain whose quotient field E has characteristic zero, then $\mathcal{S}(R)$ actually classifies arbitrary homomorphisms $\mathsf{H}_R \to \mathsf{G}_R$ (i.e., there is no need to impose the condition (6.2)). This is because an arbitrary homomorphism $\mathsf{H}_E \to \mathsf{G}_E$ has the property (6.2), since we can pass from $\overline{\mathbb{Q}}$ to E by means of the Lefschetz principle.

Next, let $\mathcal{M}$ be the projective smooth $\mathbb{Z}$-scheme of parabolic subgroups of G (see [19, Theorem 3.3, Exposé XXVI]) and let $\mathcal{Y} \subset \mathcal{S} \times \mathcal{M}$ be a scheme of finite-type over $\mathbb{Z}$ defined as follows:

$$\mathcal{Y} := \{(\rho, \mathsf{P}) : \rho(\mathsf{H}) \subset \mathsf{P}\},$$

where the condition "$\rho(\mathsf{H}) \subset \mathsf{P}$" means, more formally, that the pull-back of the ideal sheaf of P under ρ^* is identically zero.

In view of the main theorem in [32] we have for all $p_w \gg 1$ the reduction map $\mathcal{Y}(\widehat{\mathfrak{o}_w}) \to \mathcal{Y}(\widehat{k_w})$ is surjective. This together with our assumption that $\underline{\mathrm{j}_w(\mathfrak{H}_w)}(\widehat{k_w})$ is contained in a proper parabolic subgroup of $\underline{\mathfrak{G}_w}(\widehat{k_w})$ implies that there exists some $f : \mathfrak{H}_w \to \mathfrak{G}_w$ and some parabolic $\mathbf{P}$ of $\mathbf{G}$ so that $f(\mathbf{H}) \subset \mathbf{P}$, moreover, the reductions of f and j_w coincide. A contradiction will follow if we verify that $f(\mathbf{H})$ is conjugate to $j_w(\mathbf{H})$ (which is conjugate to $\iota(\mathbf{H})$ by definition).

Both f and j_w define $\widehat{\mathfrak{o}_w}$-points of $\mathcal{S}$ and their reductions to $\widehat{k_w}$-points coincide. We will deduce from this that $f(\mathbf{H})$ and $j_w(\mathbf{H})$ must actually be conjugate, as follows.

By an infinitesimal computation (which we omit) the geometric generic fiber $\mathcal{S}_{\overline{\mathbb{Q}}}$ (i.e., the base-change of $\mathcal{S}$ to $\overline{\mathbb{Q}}$) is smooth, and moreover the orbit map of $\mathsf{G}_{\overline{\mathbb{Q}}}$ is surjective on each tangent space. Therefore, each connected component of $\mathcal{S}_{\overline{\mathbb{Q}}}$ is a single orbit of $\mathsf{G}_{\overline{\mathbb{Q}}}$.

Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be these geometric connected components. We can choose a finite extension $E \supset \mathbb{Q}$ so that every $\mathcal{S}_i$ is defined over E and also has an E-point, call it x_i. For simplicity, we suppose that $E = \mathbb{Q}$, the general case being similar, but notationally more complicated.

By "spreading out", there is an integer A and a decomposition into disjoint closed subschemes:

$$\mathcal{S} \times_{\mathbb{Z}} \mathrm{Spec}\ \mathbb{Z}[\tfrac{1}{A}] = \coprod \mathcal{S}_i,$$

i.e., $\mathcal{S}_i$ is a closed subscheme, the different $\mathcal{S}_i$ are disjoint, and the union of $\mathcal{S}_i$ is the left-hand side. Note that each $\mathcal{S}_i$ is both open and closed inside the left-hand side.

In particular, if $p_w > A$, any $\widehat{\mathfrak{o}_w}$-point of $\mathcal{S}$ will necessarily factor through some $\mathcal{S}_i$. Therefore, if two $\widehat{\mathfrak{o}_w}$-points of $\mathcal{S}$ have the same reduction, they must factor through the same $\mathcal{S}_i$. In particular, the associated $\widehat{F_w}$-points of $\mathcal{S}$ belong to the same $\mathcal{S}_i$, and therefore to the same geometric $\mathbf{G}$-orbit (i.e., the same orbit over $\overline{\mathbb{Q}}$). This implies that $f(\mathbf{H})$ and $j_w(\mathbf{H})$ were conjugate inside $\mathbf{G}(\overline{\mathbb{Q}})$, giving the desired contradiction.

Let now $\underline{\mathfrak{S}}$ be a maximal, proper, connected subgroup of $\underline{\mathfrak{G}_w}$ so that $\underline{\mathrm{j}_w(\mathfrak{H}_w)} \subset \underline{\mathfrak{S}} \subset \underline{\mathfrak{G}_w}$. Then by [7, Cor. 3.3] either $\underline{\mathfrak{S}}$ is a parabolic subgroup or it is reductive. In view of the above discussion $\underline{\mathfrak{S}}$ must be reductive, and by the above claim in fact semisimple. Hence there is an isomorphism $f : \mathsf{S} \times_{\mathbb{Z}} \widehat{k_w} \to \underline{\mathfrak{S}}$ where S denotes the Chevalley group scheme over $\mathbb{Z}$ of the same type as $\underline{\mathfrak{S}}$.

We are now in a similar situation to the prior argument, i.e., we will lift the offending subgroup $\underline{\mathfrak{S}}$ to characteristic zero using [32]. Let $\mathsf{H}, \mathsf{S}, \mathsf{G}$ be split Chevalley groups over $\mathbb{Z}$ of the same type as $\mathfrak{H}_w, \underline{\mathfrak{S}}, \mathfrak{G}_w$. Consider the $\mathbb{Z}$-scheme parameterizing pairs of homomorphisms

$$(\rho_1 : \mathsf{H} \to \mathsf{G}, \rho_2 : \mathrm{Lie}(\mathsf{S}) \to \mathrm{Lie}(\mathsf{G})) \text{ with } \mathrm{image}(d\rho_1) \subset \mathrm{image}(\rho_2),$$

where we impose the same finiteness conditions of ρ_1 as in the prior argument.

The pair (j_w, f), together with identifications of $\underline{\mathfrak{H}_w}$ and $\underline{\mathfrak{S}}$ with H and S, gives rise to a $\widehat{k_w}$-point of this scheme with the maps $d\rho_1$ and ρ_2 injective; for large enough p_w this lifts, again by [32], to an $\widehat{\mathfrak{o}_w}$-point $(\widetilde{\rho}_1, \widetilde{\rho}_2)$, and still with $d\widetilde{\rho_1}$ and $\widetilde{\rho_2}$ injective.

But, then, $\widetilde{\rho}_1(\mathsf{H}_{\widehat{F_w}})$ cannot be maximal, e.g., by examining its derivative. As in the previous argument we deduce that $\mathrm{j}_w(\mathbf{H})$ is not maximal, and this contradiction finishes the proof. □

6.4. **A Lie algebra complement.** We regard $\mathfrak{g}$ as a sub-Lie-algebra of $\mathfrak{sl}_N$. Let B be the Killing form of $\mathfrak{sl}_N$ whose restriction to $\mathfrak{g}$ we will still denote by B. The properties of the good place w give us, in particular, that the restriction of B on the Lie algebra $\mathfrak{h}_w = \mathrm{Lie}(H_w)$ has the following property.

Lemma. *Assuming p_w is larger than an absolute constant depending only on the dimension the following holds. If we choose an $\mathfrak{o}_w$-basis $\{e_1, \ldots, e_{\dim \mathbf{H}}\}$ for $\mathfrak{h}_w \cap \mathfrak{sl}_N(\mathfrak{o}_w)$, then $\det(B(e_i, e_j))_{ij}$ is a unit in $\mathfrak{o}_w^\times$. That is:*

(6.3) *B restricted to $\mathfrak{h}_w \cap \mathfrak{sl}_N(\mathfrak{o}_w)$ is anon-degenerate bilinear form over $\mathfrak{o}_w$.*

Proof. Let the notation be as in the previous section, in particular, abusing the notation we denote the derivative of j_w with j_w as well. Let $\mathfrak{H}_w$ denote the smooth $\mathfrak{o}_w$-group scheme whose generic fiber is $\mathbf{H}(F_w)$ and $\mathfrak{H}_w(\mathfrak{o}_w) = K_w^*$ given by Bruhat-Tits theory. The Lie algebra $\mathrm{Lie}(\mathrm{j}_w(\mathfrak{H}_w))$ of $\mathrm{j}_w(\mathfrak{H}_w)$ is an $\mathfrak{o}_w$-algebra. The Lie algebra $\mathfrak{h}_w$ is isomorphic to $\mathrm{Lie}(\mathrm{j}_w(\mathfrak{H}_w)) \otimes_{\mathfrak{o}_w} F_w$. Fix an $\mathfrak{o}_w$-basis $\{e_i\}$ for $\mathrm{Lie}(\mathrm{j}_w(\mathfrak{H}_w))$, this gives a basis for $\mathfrak{h}_w$. Now since $\mathbf{H}$ splits over $\widehat{F_w}$ and K_w^* is a hyperspecial subgroup of H_w^*, we get: $\mathfrak{H}_w(\widehat{\mathfrak{o}_w})$ is a hyperspecial subgroup of $\mathbf{H}(\widehat{F_w})$; see [60, 2.6.1 and 3.4.1]. Fix a Chevalley $\widehat{\mathfrak{o}_w}$-basis $\{\widehat{e_i}\}$ for $\mathrm{Lie}(\mathrm{j}_w(\mathfrak{H}_w)) \otimes_{\mathfrak{o}_w} \widehat{\mathfrak{o}_w}$ which is a Chevalley basis for $\mathfrak{h}_w \otimes_{F_w} \widehat{F_w}$; see [60, §3.4.2 and 3.4.3].

Recall that $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ denotes the reduction mod ϖ_w of $\mathrm{j}_w(\mathfrak{H}_w)$. This is a semisimple k_w-subgroup, therefore, in view of our assumption on characteristic[19] of k_w we get

$$\det B(\underline{\widehat{e_i}}, \underline{\widehat{e_j}}) \neq 0,$$

hence, $\det B(\widehat{e_i}, \widehat{e_j}) \in \widehat{\mathfrak{o}_w}^\times$. This implies that $\det B(e_i, e_j) \in \mathfrak{o}_w^\times$, as $\{e_i\}$ is another $\widehat{\mathfrak{o}_w}$-basis for $\mathrm{Lie}(\mathrm{j}_w(\mathfrak{H}_w)) \otimes_{\mathfrak{o}_w} \widehat{\mathfrak{o}_w}$. □

It follows from (6.3), and our assumption on $\mathrm{char}(k_w)$ that there exists an $\mathfrak{o}_w$-module $\mathfrak{r}_w^{\mathfrak{sl}_N}[0]$ which is the orthogonal complement of $\mathfrak{h}_w[0]$ in $\mathfrak{sl}_N(\mathfrak{o}_w)$ with respect to B; see, e.g., [1]. Let $\mathfrak{r}_w[0] = \mathfrak{r}_w^{\mathfrak{sl}_N}[0] \cap \mathfrak{g}$ and write $\mathfrak{r}_w$ for its F_w-span. Then we have

$$\mathfrak{g}_w[m] = (\mathfrak{h}_w \cap \mathfrak{g}_w[m]) \oplus (\mathfrak{r}_w \cap \mathfrak{g}_w[m]) \quad \text{for all} \quad m \geq 0 \tag{6.4}$$

(see discussion after (2.1) for notation).

6.5. **The implicit function theorem at the good place.** Recall that $p_w = \mathrm{char}(k_w)$. The first congruence subgroup of $\mathrm{SL}_N(\mathfrak{o}_w)$ is a pro-p_w group; see, e.g., [49, Lemma 3.8]. Moreover, a direct calculation shows that if the w-adic valuation of p_w is at most $p_w - 2$, then the first congruence subgroup of $\mathrm{SL}_N(\mathfrak{o}_w)$ is torsion free. The condition on the valuation comes from estimating the radius of convergence of the exponential map on $N \times N$ matrices with entries in F_w; we just use the estimate that the p-adic valuation of $n!$ is bounded by $n/(p-1)$. This condition is satisfied in particular if w is unramified over p_w or if $p_w \geq [F:\mathbb{Q}] + 2$.

In what follows we assume w is so that $p_w \geq \max\{N^3, [F:\mathbb{Q}]+2\}$. In view of the above discussion, for such w we have $\exp : \mathfrak{g}_w[m] \to K_w[m]$ is a diffeomorphism for any $m \geq 1$,; see, e.g., [20, Ch. 9] for a discussion.

Let us also put $H_w' = g_{\mathscr{D},w}^{-1} \iota(\mathbf{H})(F_w) g_{\mathscr{D},w} = \mathrm{j}_w(\mathbf{H})(F_w)$.

Lemma. *For any $m \geq 1$ we have*

(i) $H_w \cap K_w[m] = \exp(\mathfrak{h}_w \cap \mathfrak{g}_w[m])$.

(ii) *Moreover, every element of $K_w[m]$ can be expressed as $\exp(z)h$ where $h \in H_w \cap K_w[m]$, and $z \in \mathfrak{r}_w \cap \mathfrak{g}_w[m]$.*

Proof. We shall use the following characterization of the Lie algebra of H_w: u belongs to $\mathfrak{h}_w$ if and only if $\exp(tu) \in H_w$ for all sufficiently small t; see, e.g., [9] or [29, Lemma 1.6].

For $z \in \mathfrak{h}_w \cap \mathfrak{g}_w[m]$, $\exp(tz)$ defines a p-adic analytic function of t for $t \in \mathfrak{o}_w$. If f is a polynomial function vanishing on H_w', we see that $f(\exp(tz))$ vanishes for t in a sufficiently small neighborhood of zero, and so also for $t \in \mathfrak{o}_w$. Therefore, $\exp(tz) \in K_w[m] \cap H_w'$. Recall that $K_w[m]$ is a pro-p_w group, hence, $K_w[m] \cap H_w'$ is also a pro-p_w group. This, in view of our assumption that $p > N$, implies that

[19] The requirement here is that $\mathrm{char}(k_w)$ is big enough so that the following holds. The restriction of B to each simple factor of $\mathrm{Lie}(\mathrm{j}_w(H_w))$ is a multiple of the Killing form on that factor, and this multiple is bounded in terms on N. We take $\mathrm{char}(k_w)$ to be bigger than all the primes appearing in these factors.

$K_w[m] \cap H'_w \subset H_w$, indeed $[H'_w : H_w] \leq F_w^\times/(F_w^\times)^N$ which is bounded by N^2; see, e.g., [49, Ch. 8].

Conversely, taking $h \in H_w \cap K_w[m]$, there is some $z \in \mathfrak{g}_w[m]$ with $\exp(z) = h$. Then $h^\ell = \exp(\ell z) \in H_w \cap K_w[m]$ for $\ell = 1, \dots$. The map $t \mapsto \exp(tz)$ is p_w-adic analytic, so $\exp(\ell z) \in H_w \cap K_w[m]$ for all t in a p_w-adic neighborhood of zero. It follows that z in fact belongs to the Lie algebra of H_w.

The second assertion is a consequence of the first and the implicit function theorem, thanks to the fact that exp is a diffeomorphism on $\mathfrak{g}_w[m]$ (see the discussion before the lemma). $\square$

6.6. **Adjustment lemma.** As usual we induce a measure in H_w using a measure on its Lie algebra. Then,

$$\exp : \mathfrak{h}_w \cap \mathfrak{g}_w[m] \to H_w \cap K_w[m], \quad m \geq 1$$

is a *measure preserving* map. To see this, it is enough to compute the Jacobian of this map; after identifying the tangent spaces at different points in H_w via left translation the derivative may be thought of as a map $\mathfrak{h}_w \to \mathfrak{h}_w$. We again apply (5.4) for $u \in \mathfrak{h}_w$. If now $u \in \mathfrak{h}_w \cap \mathfrak{g}_w[m]$, then $\operatorname{ad} u$ preserves the lattice $\mathfrak{h}_w \cap \mathfrak{g}_w[0]$ and induces an endomorphism of it that is congruent to 0 modulo the uniformizer. It follows that (5.4) is congruent to the identity modulo ϖ_w, and in particular the Jacobian is a unit which implies the claim.

The following is useful in acquiring two measure theoretically generic points to be algebraically "in transverse position" relative to each other; see the lemma regarding nearby generic points in §7.8.

Lemma (Adjustment lemma)**.** *Let $m \geq 1$ be an integer, and $g \in K_w[m]$. Given subsets $A_1, A_2 \in K_w[1] \cap H_w$ of relative measure $> 1/2$, there exists $\alpha_i \in A_i$ so that $\alpha_1^{-1} g \alpha_2 = \exp(z)$ for some $z \in \mathfrak{r}_w, \|z\| \leq q_w^{-m}$.*

Proof. Write, using the previous lemma, $g = \exp(z)h$ where $z \in \mathfrak{r}_w, \|z\| \leq q_w^{-m}, h \in H_w \cap K_w[m]$. If $\alpha \in K_w[1] \cap H_w \subset \mathrm{SL}_N(\mathfrak{o}_w)$ we have:

$$\alpha^{-1} g = \exp(\mathrm{Ad}(\alpha^{-1})z)(\alpha^{-1}h).$$

The map $f : \alpha \mapsto \alpha^{-1}h$ is measure preserving. In view of our assumption on the relative measures of A_1 and A_2, we may choose $\alpha \in A_1$ with $f(\alpha)^{-1} \in A_2$; the conclusion follows. $\square$

6.7. **The principal** SL_2**.** In the dynamical argument we will use spectral gap properties and dynamics of a unipotent flow. The following lemma will provide us with an *undistorted* copy of SL_2. Here undistorted refers to the property that the "standard" maximal compact subgroup of SL_2 is mapped into K_w which will be needed to relate our notion of Sobolev norm with the representation theory of SL_2.

As before we let $\mathfrak{H}_w$ be a smooth $\mathfrak{o}_w$-group scheme whose generic fiber is $\mathbf{H}(F_w)$ and so that $\mathfrak{H}_w(\mathfrak{o}_w) = K_w^*$.

Lemma. *There exists a homomorphism of $\mathfrak{o}_w$-group schemes*

$$\theta : \mathrm{SL}_2 \longrightarrow \mathfrak{H}_w,$$

such that the projection of $\theta_w(\mathrm{SL}_2(F_w))$ into each F_w-almost simple factor of H_w is non-trivial where $\theta_w = \mathrm{j}_w \circ \theta$.

The following proof is due to Brian Conrad. We are grateful for his permission to include it here.[14]

Proof. By our assumption $\mathfrak{H}_w$ is semisimple. Letting $R = \mathfrak{o}_w$ for ease of notation, pick a Borel R-subgroup $\mathcal{B}$ in $\mathfrak{H}_w$ and a maximal R-torus $\mathcal{T}$ in $\mathcal{B}$ (which exist by Hensel's lemma and Lang's theorem [49, §6.2]). Let $R \to R'$ be a finite unramified extension that splits $\mathcal{T}$, so $(\mathfrak{H}_w, \mathcal{T}, \mathcal{B})_{R'}$ is R'-split.

By the existence and isomorphism theorems for reductive groups over rings [13, Thm. 6.1.16] this R'-split triple descends to a $\mathbb{Z}_{(p_w)}$-split triple $(\mathsf{H}, \mathsf{T}, \mathsf{B})$. By [13, Thm. 7.1.9(3)], $(\mathfrak{H}_w, \mathcal{T}, \mathcal{B})$ is obtained from $(\mathsf{H}, \mathsf{T}, \mathsf{B})$ by twisting through an R'/R-descent datum valued in the finite group of pinned R'-automorphisms of $(\mathsf{H}, \mathsf{T}, \mathsf{B})$ (all of which are defined over $\mathbb{Z}_{(p_w)}$). A specific $\mathbb{Z}_{(p_w)}$-homomorphism $\theta : \mathrm{SL}_2 \to \mathsf{H}$ is constructed in [58, Prop. 2] that carries the diagonal torus into T and carries the strictly upper triangular subgroup into B. Though it is assumed in [58] that the target group is of adjoint-type, semisimplicity is all that is actually used in the construction.

We claim that for any local extension of discrete valuation rings $\mathbb{Z}_{(p_w)} \to A$ (such as $\mathbb{Z}_{(p)} \to R'$), the map θ_A is invariant under the finite group Γ of pinned automorphisms of H. It suffices to check this invariance over the fraction field E of A, and then even on $\mathrm{Lie}(\mathsf{H}_E)$ since $\mathrm{char}(E) = 0$. This in turn follows from the explicit description of $\mathrm{Lie}(\theta_{\mathbb{Q}})$ in Serre's paper because pinned automorphisms permute simple positive root lines respecting the chosen bases for each.

Thus, $\theta_{R'}$ is compatible with any Γ-valued R'/R-descent datum (such as the one obtained above), so $\theta_{R'}$ descends to an R-homomorphism $\mathrm{SL}_2 \to \mathfrak{H}_w$. This has the desired property relative to almost simple factors of the generic fiber over F_w because (by design) the composition of $\theta_{\mathbb{Q}}$ with projection to every simple factor of the split isogenous quotient $\mathsf{H}^{\mathrm{ad}}_{\mathbb{Q}}$ is non-trivial. □

We will refer to $\theta_w(\mathrm{SL}_2)$ as the *principal* SL_2 in what follows. We define the one-parameter unipotent subgroup $u : F_w \to \theta_w(\mathrm{SL}_2(F_w))$ by

$$u(t) = \theta_w\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right),$$

and define the diagonalizable element

$$a = \theta_w\left(\begin{pmatrix} p_w^{-1} & 0 \\ 0 & p_w \end{pmatrix}\right) \in \theta_w(\mathrm{SL}_2(F_w)).$$

6.8. **Divergence of unipotent flows.** In the dynamical argument of the next section we will study the unipotent orbits of two nearby typical points. As is well known the fundamental property of unipotent flows is their polynomial divergence. We now make a few algebraic preparations regarding this behavior at the good place w.

Recall that $\mathfrak{r}_w$ is invariant under the adjoint action of $\theta_w(\mathrm{SL}_2)$. Let $\mathfrak{r}_w^{\mathrm{trv}}$ denote the sum of all *trivial* $\theta_w(\mathrm{SL}_2)$ components of $\mathfrak{r}_w$, and put $\mathfrak{r}_w^{\mathrm{nt}}$ to be the sum of all *non-trivial* $\theta_w(\mathrm{SL}_2)$ components of $\mathfrak{r}_w$. In particular $\mathfrak{r}_w = \mathfrak{r}_w^{\mathrm{trv}} + \mathfrak{r}_w^{\mathrm{nt}}$ as a $\theta_w(\mathrm{SL}_2)$ representation.

Even though we will not use this fact, let us remark that $\mathfrak{r}_w^{\mathrm{trv}}$ does not contain any H_w-invariant subspace. To see this let V be such a subspace. Let $V[1] = V \cap \mathfrak{g}_w[1]$, in particular, $V[1]$ is a compact open (additive) subgroup of V and the exponential map is defined on $V[1]$. Then the Zariski closure of the group generated by $\exp(V[1])$ is a *proper* subgroup of $\mathbf{G}$ which is normalized by H_w and centralized by $\theta_w(\mathrm{SL}_2)$.

In particular, it is normalized by $\iota(\mathbf{H})$, the Zariski closure of H_w. This however contradicts the maximality of $\iota(\mathbf{H})$ in view of the fact that $\mathbf{G}$ is semisimple.

Let $\mathfrak{r}_w^{\mathrm{hwt}}$ be the sum of all of the highest weight spaces with respect to the diagonal torus of $\theta_w(\mathrm{SL}_2)$ in $\mathfrak{r}_w^{\mathrm{nt}}$. Note that $\mathfrak{r}_w^{\mathrm{hwt}}$ is the space of $u(F_w)$-fixed vectors in $\mathfrak{r}_w^{\mathrm{nt}}$. Let $\mathfrak{r}_w^{\mathrm{mov}}$ be the sum of all of the remaining weight spaces in $\mathfrak{r}_w^{\mathrm{nt}}$ where mov stands for moving.

Using this decomposition we write $\mathfrak{r}_w = \mathfrak{r}_w^{\mathrm{hwt}} + \mathfrak{r}_w^{\mathrm{mov}} + \mathfrak{r}_w^{\mathrm{trv}}$; therefore, given $z_0 \in \mathfrak{r}_w$ we have $z_0 = z_0^{\mathrm{hwt}} + z_0^{\mathrm{mov}} + z_0^{\mathrm{trv}}$. In view of the construction of $\theta_w(\mathrm{SL}_2)$ and $\mathrm{char}(k_w) \gg 1$ we also have

$$\mathfrak{r}_w[m] = \mathfrak{r}_w^{\mathrm{hwt}}[m] + \mathfrak{r}_w^{\mathrm{mov}}[m] + \mathfrak{r}_w^{\mathrm{trv}}[m] \quad \text{for all } m \geq 0. \tag{6.5}$$

Note that elements in $\mathfrak{r}_w^{\mathrm{hwt}}$ are *nilpotent*.[20]

In the following we understand $\mathbf{G}$ as a subvariety of the N^2-dimensional affine space via $\rho : \mathbf{G} \to \mathrm{Mat}_N$. We call a polynomial $p : F_w \to \mathbf{G}(F_w)$ *admissible* if it has the following properties:

(1) The image of p is centralized by $u(F_w)$ and contracted by a^{-1}, i.e., for every t we have $a^{-N}p(t)a^N \to e$ as $N \to \infty$; in particular the image of p consists of unipotent elements.
(2) $\deg(p) \leq N^3$.
(3) $p(0) = e$, the identity element.
(4) All coefficients of p belong to $\mathfrak{o}_w$.
(5) $p(F_w) \subset \exp(\mathfrak{r})$.
(6) There exists some $t_0 \in \mathfrak{o}_w$ such that $p(t_0)$ is not small. More precisely, we have $p(t_0) = \exp(\varpi_w^r z)$, where $0 < r \leq N^2$, and z is a nilpotent element of $\mathfrak{r}_w[0] \setminus \mathfrak{r}_w[1]$.

Note that $p(F_w) \subset \exp(\mathfrak{r}_w^{\mathrm{hwt}})$ for all admissible polynomials.

The following construction and its dynamical significance is one of the driving tools in unipotent dynamics; we refer the reader, e.g., to [53] and [21, Lemma 4.7] in the real case.

Lemma (Admissible polynomials). *Let $z_0 \in \mathfrak{r}_w[1]$ with $z_0^{\mathrm{mov}} \neq 0$. There exists $T \in F_w$ with $|T| \gg \|z_0^{\mathrm{mov}}\|^{-\star} q_w^{-1}$, and an admissible polynomial function p so that:*

$$\exp(\mathrm{Ad}(u(t))z_0) = p(t/T)g_t,$$

where $g_t \in \mathbf{G}(F_w)$ satisfies $d(g_t, 1) \leq \|z_0\|^\star q_w$ whenever $|t| \leq |T|$.

Proof. By the above we may write $z_0 = z_0^{\mathrm{hwt}} + z_0^{\mathrm{mov}} + z_0^{\mathrm{trv}}$ with $z_0^\bullet \in \mathfrak{r}_w^\bullet[1]$. By (6.5) we have $\|z_0^\bullet\| \leq \|z_0\|$ for $\bullet = \mathrm{hwt}, \mathrm{mov}, \mathrm{trv}$.

Let us now decompose $\mathrm{Ad}(u(t))z_0 = p^{\mathrm{hwt}}(t) + p^{\mathrm{mov}}(t) + z_0^{\mathrm{trv}}$ according to the above splitting of $\mathfrak{r}_w$. Since $z_0 \notin \mathfrak{r}_w^{\mathrm{hwt}} + \mathfrak{r}_w^{\mathrm{trv}}$, the polynomial p^{hwt} is non-constant, has degree $\leq N^2$, and $p^{\mathrm{hwt}}(0) = z_0^{\mathrm{hwt}}$. Let $p_0(t) = p^{\mathrm{hwt}}(t) - z_0^{\mathrm{hwt}}$, and choose $T \in F_w$ of maximal norm so that the polynomial $p_0(Ts)$ has coefficients of norm less than one. Then $p(s) := \exp(p_0(Ts))$ defines a polynomial of degree at most N^3. In fact $p_0(Ts)$ is nilpotent for every s and $\exp(\cdot)$ evaluated on nilpotent elements is a polynomial of degree at most N with values in Mat_N. Moreover, it still has integral coefficients so long as $\mathrm{char}(k_w) > N$. This polynomial satisfies conditions (1)–(5) of admissibility by definition.

[20] This can be seen, e.g., because they can be contracted to zero by the action of the torus inside $\theta_w(\mathrm{SL}_2)$.

Note that each coefficient of $p_0(t)$ is bounded from above by a constant multiple of $\|z_0^{\mathrm{mov}}\|$ and so the rth coefficient of $p_0(tT)$ is bounded from above by a constant multiple of $\|z_0^{\mathrm{mov}}\||T|^r$. By our choice of T we have that for some $r \in \{1, \dots, N^2\}$ we have $\|z_0^{\mathrm{mov}}\||T|^r q_w^r \gg 1$. Therefore we obtain that $|T| \gg \|z_0^{\mathrm{mov}}\|^{-\star} q_w^{-1}$.

Suppose that condition (6) fails, i.e., we have $p_0(Ts) \in \mathfrak{g}_w[N^2+1]$ for all $s \in \mathfrak{o}_w$. As $\mathrm{char}(k_w) \gg_N 1$ we then may choose N^2 points in $\mathfrak{o}_w$ with distance 1. Using Lagrange interpolation for the polynomial $p_0(Ts)$ and these points we see that the coefficients of $p_0(Ts)$ all belong to $\mathfrak{o}_w[N^2+1]$. However, this contradicts our choice of T and proves (6).

Finally we define the function $t \mapsto g_t$ by the formula

$$p(t/T)g_t = \exp(\mathrm{Ad}(u(t))z_0) = \exp\bigl(p^{\mathrm{hwt}}(t) + p^{\mathrm{mov}}(t) + z_0^{\mathrm{trv}}\bigr)$$

for all $t \in F_w$ with $|t| \leq |T|$. We note that the polynomial $p^{\mathrm{mov}}(t)$ corresponds to the weight spaces that are not of highest weight. We will now use the description of the SL_2-representation $\mathfrak{r}_w$ in terms of a basis consisting of weight vectors obtained from a list of highest weight vectors in $\mathfrak{r}_w^{\mathrm{hwt}}$. In fact, our choice of the place w implies that this basis can be chosen integrally and also over $\mathfrak{o}_w$. Using this basis we see that the coefficients of $p^{\mathrm{mov}}(t)$ appear also as coefficients in $p^{\mathrm{hwt}}(t)$ up to some constant factors of norm one – recall that $p_w \gg N$. Moreover, terms in $p^{\mathrm{mov}}(t)$ always have smaller degree than the corresponding terms with the same coefficient (up to a norm one factor) in $p^{\mathrm{hwt}}(t)$. Together with our choice of T this implies that

$$\|p^{\mathrm{mov}}(t)\| \leq |T|^{-1} \ll \|z_0^{\mathrm{mov}}\|^{\star} q_w$$

for all $t \in F_w$ with $|t| \leq |T|$. In fact, this holds initially for each of the monomials in the various weight spaces appearing in p^{mov}, but then by the ultrametric triangle inequality and integrality of the weight decomposition also for their combination p^{mov}. Since $p(t/T) = \exp(p^{\mathrm{hwt}}(t) - z_0^{\mathrm{hwt}})$ and $\|z_0^{\mathrm{hwt}}\|, \|z_0^{\mathrm{trv}}\| \leq \|z_0\|$, the estimate concerning $d(g_t, e)$ for all $t \in F_w$ with $|t| \leq |T|$ now follows since the map $\exp : \mathfrak{g}_w[1] \to K_w[1]$ is 1-Lipschitz. $\square$

6.9. **Efficient generation of the Lie algebra.** In the dynamical argument of the next section the admissible polynomial constructed above will give us elements of the ambient group that our measure will be almost invariant under. We now study how effectively this new element together with the maximal group H_w generate some open neighborhood of the identity in $\mathbf{G}(F_w)$.

Lemma. *There exist constants ℓ and $L \geq 1$, depending on N, such that, for any $z \in \mathfrak{r}_w^{\mathrm{hwt}}[0] \setminus \mathfrak{r}_w^{\mathrm{hwt}}[1]$, the following holds: Every $g \in K_w[L]$ can be written as*

$$g = g_1 g_2 \dots g_\ell, \quad \textit{where} \quad g_i \in K_w \cap (H_w \cup \exp(z) H_w \exp(-z)).$$

Note that z as in the above lemma is a nilpotent element (because it belongs to the highest weight space), and its exponential $\exp(z)$ belongs to $K_w \setminus H_w$. It turns out that the latter statement continues to hold even reduced modulo w, and this is what is crucial for the proof.

It was mentioned in §6.5 that our choice of w implies that for all $m \geq 1$ the group $K_w[m]$ is a torsion free pro-p_w group; we also recall that $K_w[m] \subset \mathbf{G}(F_w)^+$.

Proof. Let $\mathfrak{G}_w$ (resp., $\mathfrak{H}_w$) be smooth $\mathfrak{o}_w$-group schemes with generic fiber G_w (resp., $\mathbf{H}(F_w)$) so that $K_w = \mathfrak{G}_w(\mathfrak{o}_w)$ (resp., $K_w^* = \mathfrak{H}_w(\mathfrak{o}_w)$). Recall the notation: for any $\mathfrak{o}_w$-group scheme $\mathfrak{M}$ we let $\underline{\mathfrak{M}}$ denote the reduction mod ϖ_w. As was shown in the lemma of §6.2 $\mathrm{j}_w(\mathfrak{H}_w(\widehat{\mathfrak{o}_w}))$ and $\mathfrak{G}_w(\widehat{\mathfrak{o}_w})$ are hyperspecial subgroups of $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ and $\mathbf{G}(\widehat{F_w})$, respectively. Furthermore, they are $\widehat{\mathfrak{o}_w}$ Chevalley group schemes with generic fibers $\mathrm{j}_w(\mathbf{H})(\widehat{F_w})$ and $\mathbf{G}(\widehat{F_w})$; see, e.g., the discussion following (6.3).

Since the group $\underline{\mathfrak{H}_w}$ is quasisplit over k_w we may choose one-dimensional unipotent subgroups $\underline{\mathfrak{U}}_i$ for $1 \leq i \leq \dim \mathbf{H}$, with the property that the product map $\prod_{i=1}^{\dim \mathbf{H}} \underline{\mathfrak{U}}_i \to \underline{\mathfrak{H}_w}$ is dominant. In fact, these can be taken to be the reduction mod ϖ_w of smooth closed $\mathfrak{o}_w$-subgroup schemes $\mathfrak{U}_i$ of $\mathfrak{H}_w$; see [60, §3.5].

To see why, fix a collection $\underline{\mathfrak{U}}_\alpha$ of one-dimensional unipotent groups which generate $\underline{\mathfrak{H}_w}$ as an algebraic group. We prove inductively on r that we may choose $\alpha_1, \ldots, \alpha_r$ such that the Zariski closure Z_r of $\prod_{i=1}^r \underline{\mathfrak{U}}_{\alpha_i}$ is r-dimensional. Suppose this has been done for a given r. Then for any β the closure of $Z_r \cdot \underline{\mathfrak{U}}_\beta$ is an irreducible algebraic set; if it is r-dimensional, it must therefore coincide with Z_r. If $r < \dim \mathbf{H}$ this cannot be true for all choices of β, by the generation hypothesis, and we deduce that we can increase r by taking $\alpha_{r+1} = \beta$.

By Lemma 6.3, $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ is a maximal connected algebraic subgroup of $\underline{\mathfrak{G}_w}$. Now let g be the reduction modulo ϖ_w of $\exp(z)$. We claim that

$$g \notin \underline{\mathrm{j}_w(\mathfrak{H}_w)}, \tag{6.6}$$

from where it follows that $g\underline{\mathrm{j}_w(\mathfrak{H}_w)}g^{-1}$ together with $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ generates $\underline{\mathfrak{G}_w}$.

Indeed, if (6.6) failed, the elements g^t for $t \in \mathbb{Z}$ belong to $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$. Let $\bar{z}$ be the reduction of z to the Lie algebra of $\underline{\mathfrak{G}_w}$. Now $t \in \mathbb{A}^1 \mapsto \exp(t\bar{z}) \in \mathrm{SL}_N$ defines a one-parameter subgroup of SL_N over k_w. Consider the associated homomorphism

$$\mathbb{A}^1 \to \mathrm{End}(\wedge^{\dim \mathbf{H}} \mathrm{Lie}(\mathrm{SL}_N)). \tag{6.7}$$

The degree of this map is bounded only in terms of $N, \dim(\mathbf{G})$. The value of (6.7) at each $t \in \mathbb{Z}$ preserves the line in $\wedge^{\dim \mathbf{H}} \mathrm{Lie}(\mathrm{SL}_N)$ associated to the Lie algebra of $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$. In a suitable basis, this assertion amounts to the vanishing of various matrix coefficients of (6.7). But if a matrix coefficient of the map (6.7) vanishes for all $t \in \mathbb{Z}$, it vanishes identically – possibly after increasing the implicit bound for $\mathrm{char}(k_w)$ in §6.1 if necessary. Therefore the one-parameter subgroup $t \mapsto \exp(t\bar{z})$ of SL_N normalizes the Lie algebra of $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$. Therefore

$$[\bar{z}, \mathrm{Lie}\, \underline{\mathrm{j}_w(\mathfrak{H}_w)}] \subset \mathrm{Lie}\, \underline{\mathrm{j}_w(\mathfrak{H}_w)}.$$

But this contradicts the assumption on z – e.g., we can find an element H in the Lie algebra of $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$, arising from the SL_2 in §6.7, such that $[\bar{z}, \mathrm{H}]$ is a non-zero multiple of $\bar{z}$, and $\bar{z}$ is not in Lie $\underline{\mathrm{j}_w(\mathfrak{H}_w)}$ by (6.4).

For simplicity in the notation, put $\underline{\mathfrak{U}'_i} = \exp(z)\, \mathrm{j}_w(\mathfrak{U}_i) \exp(-z)$. Arguing just as above, we see that we may choose $\mathfrak{X}_i$ (for $i = 1, \ldots, \dim \mathbf{G}$), each equal to either $\mathrm{j}_w(\mathfrak{U}_i)$ or $\mathfrak{U}'_i$ for some i, with $\mathfrak{X}_i = \mathrm{j}_w(\mathfrak{U}_i)$ for $i = 1, \ldots, \dim \mathbf{H}$ and such that if we define φ via

$$\varphi : \mathfrak{X} := \prod_{i=1}^{\dim \mathbf{G}} \mathfrak{X}_i \xrightarrow{(\iota_i)} \prod_{i=1}^{\dim \mathbf{G}} \mathfrak{G}_w \xrightarrow{\text{mult}} \mathfrak{G}_w\ ,$$

then the map $\underline{\varphi}$ is dominant.

The above definition implies that $\underline{\varphi}$ is a polynomial map on the $\dim \mathbf{G}$ dimensional affine space with $\deg(\underline{\varphi}) \leq N^4$.

Therefore, in view of our assumption on the characteristic of k_w, one gets that $\underline{\varphi}$ is a separable map. We recall the argument: note that $\underline{\varphi}$ is a map from the $\dim \overline{\mathbf{G}}$ dimensional affine space $\underline{\mathfrak{X}}$ into the affine variety $\underline{\mathfrak{G}_w}$. Let $E = \widehat{k_w}(\underline{\mathfrak{X}})$ and let E' be the quotient field of $\widehat{k_w}[\underline{\varphi}^*(\underline{\mathfrak{G}_w})]$ in E. In view of the above construction, E is an algebraic extension of E'. By Bezout's theorem the degree of this finite extension is bounded by a constant depending on $\deg(\underline{\varphi})$; see, e.g., [61] or [22, App. B]. The claim follows in view of $\mathrm{char}(k_w) \gg_N 1$.

In particular, $\Phi = \det(\mathrm{D}(\underline{\varphi}))$ is a non-zero polynomial. This implies that we can find a finite extension k'_w of k_w and a point $\underline{a'} \in \underline{\mathfrak{X}}(k'_w)$ so that $\Phi(\underline{a'}) = \det(\mathrm{D}_{\underline{a'}}(\underline{\varphi})) \neq 0$. Note that in fact under our assumption on $\mathrm{char}(k_w)$ and since $\deg(\underline{\varphi}) \leq N^4$ there is some point $\underline{a} \in \underline{\mathfrak{X}}(k_w)$ so that $\Phi(\underline{a}) \neq 0$. This can be seen by an inductive argument on the number of variables in the polynomial Φ. For polynomials in one variable the bound one needs is $\mathrm{char}(k_w) > \deg(\Phi)$; now we write $\Phi(a_1, \ldots, a_{\dim \mathbf{G}}) = \sum_j \Phi^j(a_2, \ldots, a_{\dim \mathbf{G}}) a_1^j$ and get the claim from the inductive hypothesis.

All together we get: there is a point $a \in \mathfrak{X}(\mathfrak{o}_w)$ so that $\det(\mathrm{D}_a(\varphi))$ is a unit in $\mathfrak{o}_w^\times$. The implicit function theorem thus implies that there is some $b \in K_w$ so that $\varphi\big(\mathfrak{X}(\mathfrak{o}_w)\big)$ contains $bK_w[L]$, where L is an absolute constant. Therefore

$$\Big(\varphi\big(\mathfrak{X}(\mathfrak{o}_w)\big)\Big)^{-1}\Big(\varphi\big(\mathfrak{X}(\mathfrak{o}_w)\big)\Big)$$

contains $K_w[L]$ as we wanted to show. $\square$

The following is an immediate corollary of the above discussion; this statement will be used in what follows.

6.10. **Proposition** (Efficient generation). *Let $p : F_w \to \mathbf{G}(F_w)$ be an admissible polynomial map as defined in §6.8. Then there exist constants $L \geq 1$ and ℓ, depending on N, so that: each $g \in K_w[L]$ may be written as a product $g = g_1 g_2 \ldots g_\ell$, where*

$$g_i \in \big\{h \in H_w : \|h\| \leq q_w^L\big\} \cup \big\{p(t)^{p_w^s} : t \in \mathfrak{o}_w \text{ and } 0 \leq s \leq 2N\big\}^{\pm 1}.$$

Proof. Let $t_0 \in \mathfrak{o}_w$ be as in property (6) of admissibility so that $p(t_0) = \exp(\varpi_w^r z)$, where $0 < r \leq N^2$, and $z \in \mathfrak{r}_w^{\mathrm{hwt}}[0] \setminus \mathfrak{r}_w^{\mathrm{hwt}}[1]$.

Let $a \in \theta_w(\mathrm{SL}_2(F_w))$ be the element corresponding to the diagonal element with[21] eigenvalues p_w^{-1}, p_w. Since $r > 0$, we will use conjugation by $a \in H_w$ to produce again an element which we may use in the previous lemma. Indeed, let $j \geq 1$ be minimal for which $a^j p(t_0) a^{-j} = \exp(\varpi_w^r \mathrm{Ad}_a^j z) \notin K_w[1]$ and note that $j \leq N^2$. If $z' = \varpi_w^r \mathrm{Ad}_a^j z \in \mathfrak{r}_w[0]$ (and $z' \notin \mathfrak{r}_w[1]$ by choice of j), we set $z'' = z'$ and will use this element below. However, if $z' \notin \mathfrak{r}_w[0]$, then $\|z'\| = q_w^i$ for some $i \in \mathbb{N}$ with $i \leq 2N$. In this case we find that the element $z'' = p_w^i z' \in \mathfrak{r}_w[0] \setminus \mathfrak{r}_w[1]$ satisfies

$$a^j p(t_0)^{p_w^i} a^{-j} = a^j \exp(p_w^i \varpi_w^r z) a^{-j} = \exp(p_w^i z') = \exp(z''),$$

and can be used in the previous lemma. For this also note that we may assume that F is unramified at w, since there are only finitely many ramified places for F

[21] We apologize for the notational clash between p_w, which is the residue characteristic of F_w, and the polynomials p, p_0.

and the implicit constants in the definition of "good place" are permitted to depend on F; this gives that p_w is a uniformizer for F at w.

Increasing ℓ to accommodate the change in the formulation of the statements, the proposition follows from the previous lemma. $\square$

7. The dynamical argument

Throughout this section we let $w \in \Sigma_f$ denote a good place for the MASH $Y = Y_{\mathscr{D}}$ with $\mathscr{D} = (\mathbf{H}, \iota, g_{\mathscr{D}})$. Moreover, we let $\theta_w(\mathrm{SL}_2)$ be the principal SL_2 as in §6.7 satisfying that $\theta_w(\mathrm{SL}_2(F_w))$ is contained in the acting subgroup H_w at the place w and $\theta_w(\mathrm{SL}_2(\mathfrak{o}_w)) < K_w$.

7.1. **Non-compactness.** As usual, when X is not compact some extra care is required to control the behavior near the "cusp"; using the well-studied non-divergence properties of unipotent flows we need to show that "most" of the interesting dynamics takes place in a "compact part" of X. We will also introduce in this subsection the height function $\mathrm{ht} : X \to \mathbb{R}_{>0}$, which is used in our definition of the Sobolev norms.

For the discussion in this subsection we make the following reduction: put $\mathbf{G}' = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$ and $\mathbf{H}' = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{H})$; then $\mathbf{G}'$ and $\mathbf{H}'$ are semisimple $\mathbb{Q}$-groups and we have the $\mathbb{Q}$-homomorphism $\mathrm{Res}_{F/\mathbb{Q}}(\iota) : \mathbf{H}' \to \mathbf{G}'$. Moreover

$$\mathbf{L}'(\mathbb{Q})\backslash\mathbf{L}'(\mathbb{A}_{\mathbb{Q}}) = \mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F) \text{ for } \mathbf{L} = \mathbf{G}, \mathbf{H}$$

and we also get a natural isomorphism between $\mathbf{L}'(\mathbb{Z}_p)$ and $\prod_{v|p} \mathbf{L}(\mathfrak{o}_v)$; see [49] for a discussion of these facts. Similarly we write $\mathbf{H}'_j = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{H}_j)$ for any F-simple factor $\mathbf{H}_j$ of $\mathbf{H}$.

As is well known (e.g., see [5]) there exists a finite set $\Xi \subset \mathbf{G}'(\mathbb{A}_{\mathbb{Q}})$ so that

$$\mathbf{G}'(\mathbb{A}_{\mathbb{Q}}) = \bigsqcup_{\xi \in \Xi} \mathbf{G}'(\mathbb{Q})\xi G'_\infty K'_f,$$

where $G'_\infty = \mathbf{G}'(\mathbb{R})$ and K'_f is the compact open subgroup of $\mathbf{G}'(\mathbb{A}_{\mathbb{Q},f})$ corresponding to $K_f < \mathbf{G}(\mathbb{A}_{F,f})$. We define G'_q and K'_q similarly for every rational prime q. We let $S_0 = S_0(\mathbf{G})$ be the union of $\{\infty\}$ and a finite set of primes so that $\Xi \subset \prod_{v \in S_0} G'_v K'_f$.

We now recall the standard terminology for S-arithmetic quotients (S is a finite set). Set $K'(S) = \prod_{q \notin S} K'_q$ and put $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{q} : q \in S]$, $\mathbb{Q}_S = \mathbb{R} \times \prod_{q \in S} \mathbb{Q}_q$ and $G'_S = \mathbf{G}'(\mathbb{Q}_S)$. We let $\mathfrak{g}'$ be the Lie algebra of $\mathbf{G}'$. We choose an integral lattice $\mathfrak{g}'_{\mathbb{Z}}$ in the $\mathbb{Q}$-vector space $\mathfrak{g}'$, with the property that $[\mathfrak{g}'_{\mathbb{Z}}, \mathfrak{g}'_{\mathbb{Z}}] \subset \mathfrak{g}'_{\mathbb{Z}}$. Let $\mathfrak{g}'_{\mathbb{Z}_S} = \mathbb{Z}_S \mathfrak{g}'_{\mathbb{Z}}$ be the corresponding $\mathbb{Z}_S$-module. We also define $\|u\|_S = \prod_{v \in S} \|u\|_v$ for elements u of the Lie algebra $\mathfrak{g}'_S = \mathbb{Q}_S \otimes_{\mathbb{Q}} \mathfrak{g}'$ over $\mathbb{Q}_S$.

Our choice of S_0 now implies $\mathbf{G}'(\mathbb{A}_{\mathbb{Q}}) = \mathbf{G}'(\mathbb{Q}) G'_S K'(S)$ whenever $S \supseteq S_0$, which also gives

$$\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_{\mathbb{Q}})/K'(S) \cong X_S = \Gamma_S\backslash G'_S = \Gamma_{S_0}\backslash\Big(G'_{S_0} \times \prod_{q \in S\setminus S_0} K'_q\Big)$$

where $\Gamma_S = \mathbf{G}'(\mathbb{Q}) \cap K'(S)$. In that sense we have a projection map $\pi_S(x) = xK'(S)$ from X to X_S.

Similar to [24], for every $x \in X$ we put

$$\mathrm{ht}(x) := \mathrm{ht}(\pi_S(x)) = \sup\Big\{\|\mathrm{Ad}(g^{-1})u\|_S^{-1} : u \in \mathfrak{g}'_{\mathbb{Z}_S} \setminus \{0\} \text{ and } g \in G'_S \text{ with } \pi_S(x) = \Gamma_S g\Big\}.$$

We note that in the definition of $\mathrm{ht}(\pi_S(x))$ we may also fix the choice g of the representative for a given $\pi_S(x)$, the supremum over all $u \in \mathfrak{g}'_{\mathbb{Z}_S}$ will be independent of the choice. If $S \supsetneq S_0$, we may choose g such that $g_q \in K'_q$ for $q \in S \setminus S_0$. This in turn implies that the definition of $\mathrm{ht}(x)$ is also independent of $S \supseteq S_0$. Define

$$\mathfrak{S}(R) := \{x \in X : \mathrm{ht}(x) \leq R\}.$$

Note that

$$\mathrm{ht}(xg) \ll \|g\|^2 \mathrm{ht}(x) \text{ for any } g \in G_v, v \in \Sigma. \tag{7.1}$$

If $v \in \Sigma_f$ the implicit constant is 1 and moreover

$$\mathrm{ht}(xg) = \mathrm{ht}(x) \text{ for any } g \in K_v. \tag{7.2}$$

Finally, we need the following.

7.2. **Lemma.** *There exists constants $\kappa_{10} > 1$ and $c_1 > 0$ such that for all $x \in X$ the map*

$$g \mapsto xg \textit{ is injective on } \big\{g = (g_\infty, g_f) : d(g_\infty, 1) \leq c_1 \mathrm{ht}(x)^{-\kappa_{10}}, g_f \in K'_f\big\}. \tag{7.3}$$

Proof. Suppose that $xg_1 = xg_2$ for g_1, g_2 belonging to the set above. In what follows take $S = S_0$. Let $g_{1,S}$ and $g_{2,S}$ be the S component of g_1 and g_2.

Fix $g \in G'_S$ such that $\pi_S(x) = \Gamma_S g$. Then $\Gamma_S g g_{1,S} = \Gamma_S g g_{2,S}$, and so $g_{1,S} g_{2,S}^{-1}$ fixes $g^{-1}\mathfrak{g}'_{\mathbb{Z}_S}$. In particular, $g_{1,\infty} g_{2,\infty}^{-1}$ fixes

$$L_x := g^{-1}\mathfrak{g}'_{\mathbb{Z}_S} \cap \mathfrak{g}'_{\mathbb{Z}},$$

the intersection being taken inside of $\mathfrak{g}'$; this can also be described as those elements $u \in g^{-1}\mathfrak{g}'_{\mathbb{Z}_S}$ that satisfy $\|u\|_v \leq 1$ for all non-archimedean $v \in S$.

We consider L_x as a $\mathbb{Z}$-lattice inside the real vector space $\mathfrak{g}' \otimes \mathbb{R}$. For every $\lambda \in L_x$ we have $\|\lambda\| \geq \mathrm{ht}(x)^{-1}$. The covolume of L_x inside $\mathfrak{g}' \otimes \mathbb{R}$ is the same as the covolume of $g^{-1}\mathfrak{g}'_{\mathbb{Z}_S}$ inside $\mathfrak{g}' \otimes \mathbb{Q}_S$, and this latter covolume is independent of x. By lattice reduction theory, then, L_x admits a basis $\lambda_1, \ldots, \lambda_d$ such that $\|\lambda_i\| \ll \mathrm{ht}(x)^{(d-1)}$.

Thus, if we choose the constant κ_{10} sufficiently large and c_1 suitably small, we have

$$\|(g_{1,\infty} g_{2,\infty}^{-1})\lambda_i - \lambda_i\| < \mathrm{ht}(x)^{-1} \text{ for all } 1 \leq i \leq d,$$

and thus the fact that $g_{1,\infty} g_{2,\infty}^{-1}$ fixes the lattice L_x setwise implies that it in fact fixes L_x pointwise. This forces $g_{1,\infty} g_{2,\infty}^{-1}$ to belong to the center of G'_∞, and this will be impossible if we choose c_1 small enough. $\square$

Let $w \in \Sigma_f$ be the good place as above, which gives that $\mathbf{H}_j(F_w)$ is not compact for all j, and let p_w be the prime so that $w \mid p_w$. Then $\mathbf{H}'_j(\mathbb{Q}_{p_w})$ is not compact for all j.

We have the following analogue[22] of [24, Lemma 3.2].

[22] We note that due to the dependence on p we do not obtain at this stage a fixed compact subset that contains 90% of the measure for all MASH; see the corollary in §1.7.

7.3. **Lemma** (Non-divergence estimate)**.** *There are positive constants κ_{11} and κ_{12}, depending on $[F:\mathbb{Q}]$ and* $\dim \mathbf{G}$*, so that for any* MASH *set Y we have*

$$\mu_{\mathscr{D}}\left(X \setminus \mathfrak{S}(R)\right) \ll p_w^{\kappa_{11}} R^{-\kappa_{12}},$$

where p_w is a rational prime with $w \mid p_w$ for a good place $w \in \Sigma_F$ for Y.

Proof. The proof is similar to the proof of [24, Lemma 3.2], using the S-arithmetic version of the quantitative non-divergence of unipotent flows which is proved in [40], for which we set $S = S_0 \cup \{p_w\}$. We recall parts of the proof.

Recall that $\mathbf{H}'_j(\mathbb{Q}_{p_w})$ is not compact for any F-almost simple factor $\mathbf{H}_j$ of $\mathbf{H}$ and that $\mathbf{H}(F_w)$ is naturally identified with the group of $\mathbb{Q}_{p_w}$-points of $\mathrm{Res}_{F_w/\mathbb{Q}_{p_w}}(\mathbf{H})$; see, e.g., [49]. Let $H_w = g_{\mathscr{D},w}^{-1}\iota(\mathbf{H}(F_w))g_{\mathscr{D},w}$ be the component of the acting group at the place w, where $g_{\mathscr{D}} \in \mathbf{G}(\mathbb{A})$ is the group element from the data $\mathscr{D} = (\mathbf{H}, \iota, g_{\mathscr{D}})$ determining the MASH set $Y = Y_{\mathscr{D}}$.

Let us note that discrete $\mathbb{Z}_S$-submodules of $\mathbb{Q}_S^k$ are free [40, Prop. 8.1]. Furthermore, by [40, Lemma 8.2] if $\Delta = \bigoplus_{i=1}^{\ell} \mathbb{Z}_S \mathbf{v}_i$ is a discrete $\mathbb{Z}_S$-module, then the covolume of Δ in $V = \bigoplus_{i=1}^{\ell} \mathbb{Q}_S \mathbf{v}_i$ is defined by $\mathrm{cov}(\Delta) = \prod_{v\in S} \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_\ell\|_v$ and we will refer to Δ as an S-arithmetic lattice in V.

Let $h \in G'_S$. A subspace $V \subset \mathfrak{g}'_S$ is called $\Gamma_S h$-rational if $V \cap \mathrm{Ad}_h^{-1}\mathfrak{g}'_{\mathbb{Z}_S}$ is an S-arithmetic lattice in V; the covolume of V with respect to $\Gamma_S h$ is defined to be $\mathrm{cov}(V \cap \mathrm{Ad}_h^{-1}\mathfrak{g}'_{\mathbb{Z}_S})$ (and is independent of the representative). One argues as in the proof of [24, Lemma 3.2] (given in Appendix B of [24]) and gets: there exist positive constants c_2 and κ_{13} such that

(7.4) there is no x-rational, H_w-invariant

proper subspace of covolume $\leq c_2 p_w^{-\kappa_{13}}$

where we fix some $x \in \pi_S(Y)$. We define $\rho = c_2 p_w^{-\kappa_{13}}$.

Let now $U = \{u(t)\}$ be a one-parameter $\mathbb{Q}_{p_w}$-unipotent subgroup of H_w which projects non-trivially into all $\mathbb{Q}_{p_w}$-simple factors of $\mathrm{Res}_{F_w/\mathbb{Q}_{p_w}}(\mathbf{H})$. Then, since the number of x-rational proper subspaces of covolume $\leq \rho = c_2 p_w^{-\kappa_{13}}$ is finite and by the choice of U above, a.e. $h \in H_w$ has the property that hUh^{-1} does not leave invariant any proper x-rational subspace of covolume $\leq \rho$. Alternatively, we may also conclude for a.e. $h \in H_w \cap K_w$ that U does not leave invariant any proper xh-rational subspace of covolume $\leq \rho$.

Since $\mathbf{H}$ is simply connected, it follows from the strong approximation theorem and the Mautner phenomenon that $\mu_{\mathscr{D}}$ is ergodic for the action of $\{u(t)\}$. This also implies that the U-orbit of xh equidistributes with respect to $\mu_{\mathscr{D}}$ for a.e. h. We choose $h \in H_w \cap K_w$ so that both of the above properties hold true for $x' = xh$.

Let $x' = \Gamma_S h'$. Hence, for any $\Gamma_S h'$-rational subspace V, if we let

$$\psi_V(t) = \mathrm{cov}(\mathrm{Ad}_{u(t)}(V \cap \mathrm{Ad}_{h'}^{-1}\mathfrak{g}'_{\mathbb{Z}_S})),$$

then either ψ_V is unbounded or equals a constant $\geq \rho$. Thus, by [40, Thm. 7.3] there exists a positive constant κ_{14} so that

$$|\{t : |t|_w \leq r, x'u(t) \notin \mathfrak{S}(\epsilon^{-1})\}| \ll p_w^{\kappa_{14}} (\tfrac{\epsilon}{\rho})^{\alpha} |\{t : |t|_w \leq r\}| \tag{7.5}$$

for all large enough r and $\epsilon > 0$, where $\alpha = \kappa_{12}$ only depends on the degree of the polynomials appearing in the matrix entries for the elements of the one-parameter unipotent subgroup U (see [40, Lemma 3.4]). The lemma now follows as the U-orbit equidistributes with respect to $\mu_{\mathscr{D}}$.

We note that the proof of (7.4) also uses non-divergence estimates and induction on the dimension, which is the reason why the right-hand side contains a power of p. $\square$

7.4. **Spectral input.** As in §6.7 we let $\theta_w(\mathrm{SL}_2) < g_{\mathscr{D},w}^{-1}\iota(\mathbf{H})g_{\mathscr{D},w}$ be the principal SL_2 and also recall the one-parameter unipotent subgroup

$$u(t) := \theta_w\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right).$$

In the following we will assume that the representations of $\mathrm{SL}_2(F_w)$, via θ_w, both on

$$L_0^2(\mu_{\mathscr{D}}) := \left\{f \in L^2(X, \mu_{\mathscr{D}}) : \int f \,\mathrm{d}\mu_{\mathscr{D}} = 0\right\},$$

and on $L_0^2(X, \mathrm{vol}_G)$ are $1/M$-tempered (i.e., the matrix coefficients of the M-fold tensor product are in $L^{2+\epsilon}(\mathrm{SL}_2(F_w))$ for all $\epsilon > 0$). (Recall again here that $\mathbf{H}$ is simply connected.) As was discussed in §4 this follows directly in the case when $\mathbf{H}(F_w)$ has property (T), see [47, Thm. 1.1–1.2], and in the general case we apply property (τ) in the strong form; see [12], [30], and [24, §6].[23]

7.5. **Adelic Sobolev norms.** Let $C^\infty(X)$ denote the space of functions which are invariant by a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and are smooth at all infinite places. There exist a system of norms $\mathcal{S}_d$ on $C_c^\infty(X)$ with the following properties; see Appendix A, in particular, see (A.3) and (A.4).

S0. (*Norm on $C_c(X)$*). Each $\mathcal{S}_d$ is a pre-Hilbert norm on $C_c^\infty(X) = C^\infty(X) \cap C_c(X)$ (and so in particular finite there).

S1. (*Sobolev embedding*). There exists some d_0 depending on $\dim \mathbf{G}$ and $[F : \mathbb{Q}]$ such that for all $d \geq d_0$ we have $\|f\|_{L^\infty} \ll_d \mathcal{S}_d(f)$.

S2. (*Trace estimates*). Given d_0, there are $d > d' > d_0$ and an orthonormal basis $\{e_k\}$ of the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$ which is orthogonal with respect to $\mathcal{S}_{d'}$ so that

$$\sum_k \mathcal{S}_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \tfrac{\mathcal{S}_{d_0}(e_k)^2}{\mathcal{S}_{d'}(e_k)^2} < \infty.$$

S3. (*Continuity of representation*). Let us write $g \cdot f$ for the action of $g \in \mathbf{G}(\mathbb{A})$ on $f \in C_c^\infty(X)$. For all $d \geq 0$ we have

$$\mathcal{S}_d(g \cdot f) \ll \|g\|^{4d}\mathcal{S}_d(f)$$

for all $f \in C_c^\infty(X)$ and where

$$\|g\| = \prod_{v \in \Sigma} \|g_v\|.$$

Moreover, we have $\mathcal{S}_d(g \cdot f) = \mathcal{S}_d(f)$ if in addition $g \in K_f$. For the unipotent subgroup $u(\cdot)$ in the principal SL_2 at the good place w we note that $\|u(t)\| \leq (1 + |t|_w)^N$ for all $t \in F_w$.

S4. (*Lipshitz constant* at w). There exists some d_0 depending on $\dim \mathbf{G}$ and $[F : \mathbb{Q}]$ such that for all $d \geq d_0$ the following holds. For any $r \geq 0$ and any

[23] Note that the F_w-rank of the almost simple factors of $\mathbf{H}$ are never zero, since $\mathbf{H}$ is F_w quasisplit.

$g \in K_w[r]$ we have

$$\|g \cdot f - f\|_\infty \le q_w^{-r} \mathcal{S}_d(f)$$

for all $f \in C_c^\infty(X)$.

S5. (*Convolution on ambient space*). Recall from §§1.4 and 2.1 that π^+ is the projection onto the space of $\mathbf{G}(\mathbb{A})^+$ invariant functions and that $L_0^2(X, \mathrm{vol})$ is the kernel of π^+. Let Av_L be the operation of averaging over $K_w[L]$, where L is given by Proposition 6.10. For $t \in F_w$ we define $\mathbb{T}_t = \mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L$ by convolution. For all $x \in X$, all $f \in C_c^\infty(X)$, and $d \ge d_0$ we have

$$|\mathbb{T}_t(f - \pi^+ f)(x)| \ll q_w^{(d+2)L} \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0} \mathcal{S}_d(f),$$

where $\|\mathbb{T}_t\|_{2,0}$ denotes the operator norm of $\mathbb{T}_t$ on $L_0^2(X, \mathrm{vol}_G)$. Once more d_0 depends on $\dim \mathbf{G}$ and $[F : \mathbb{Q}]$.

S6. (*Decay of matrix coefficients*). For all $d \ge d_0$ we have

$$\left| \langle u(t) f_1, f_2 \rangle_{L^2(\mu_{\mathscr{D}})} - \int f_1 \,\mathrm{d}\mu_{\mathscr{D}} \int \bar{f}_2 \,\mathrm{d}\mu_{\mathscr{D}} \right| \ll (1 + |t|_w)^{-1/2M} \mathcal{S}_d(f_1) \mathcal{S}_d(f_2), \tag{7.6}$$

where d_0 depends on $\dim \mathbf{G}$ and $[F : \mathbb{Q}]$; recall that $\mathbf{H}$ is simply connected.

7.6. **Discrepancy along v-adic unipotent flows.** We let M be as in §7.4 and choose the depending parameter $m = 100M$.

We say a point $x \in X$ is *T_0-generic w.r.t. the Sobolev norm $\mathcal{S}$* if for any ball of the form $J = \{t \in F_w : |t - t_0|_w \le |t_0|_w^{1-1/m}\}$, with its center satisfying $\mathrm{n}(J) = |t_0|_w \ge T_0$, we have

$$D_J(f)(x) = \left| \frac{1}{|J|} \int_{t \in J} f(xu(t)) \,\mathrm{d}t - \int f \,\mathrm{d}\mu_{\mathscr{D}} \right| \le \mathrm{n}(J)^{-1/m} \mathcal{S}(f) \tag{7.7}$$

for all $f \in C_c^\infty(X)$. Here $|J|$ denotes the Haar measure of J and we note that the definition of $\mathrm{n}(J) = |t_0|_w$ is independent of the choice of the center $t_0 \in J$.

Lemma (T_0-generic points)**.** *For a suitable d_0 depending only on $\dim \mathbf{G}$ and $[F : \mathbb{Q}]$ and all $d \ge d_0$ the measure of points that are not T_0-generic w.r.t. $\mathcal{S}_d$ is decaying polynomially with T_0. More precisely,*

$$\mu_{\mathscr{D}}\big(\{y \in Y : y \textit{ is not } T_0\textit{-generic}\}\big) \ll T_0^{-1/4M}$$

for all $T_0 > q_w^\star$.

Proof. We let $\mathcal{S} = \mathcal{S}_{d_0}$ and will make requirements on $d_0 \ge 1$ during the proof. We first consider a fixed f in $L^2(X)$ which is in the closure of $C_c^\infty(X)$ with respect to $\mathcal{S}$. Since $\mathbf{H}$ is simply connected, by (7.6) we have

$$\left| \langle u(t) f, f \rangle_{L^2(\mu_{\mathscr{D}})} - \left| \int f \,\mathrm{d}\mu_{\mathscr{D}} \right|^2 \right| \ll (1 + |t|_w)^{-1/2M} \mathcal{S}(f)^2, \tag{7.8}$$

where we assume d_0 is sufficiently big for S6. to hold.

For a fixed J let $D_J(f)(x)$ be defined in (7.7). Then we have

$$\int_X \big|D_J(f)(x)\big|^2 \,\mathrm{d}\mu_{\mathscr{D}} = \frac{1}{|J|^2} \int_{J \times J} \langle u(t) f, u(s) f \rangle \,\mathrm{d}s \,\mathrm{d}t - \left(\int f \,\mathrm{d}\mu_{\mathscr{D}} \right)^2.$$

Split $J \times J$ into $|t - s|_w \le \mathrm{n}(J)^{\frac{1}{2}(1-1/m)}$ and $|t - s|_w > \mathrm{n}(J)^{\frac{1}{2}(1-1/m)}$; in view of (7.8) we thus get

$$\int_X \big|D_J(f)(x)\big|^2 \,\mathrm{d}\mu_{\mathscr{D}} \ll \mathrm{n}(J)^{-\frac{1}{4M}(1-\frac{1}{m})} \mathcal{S}(f)^2.$$

Still working with a fixed function f, this implies in particular that

$$\mu_{\mathscr{D}}\Big(\Big\{x \in X : D_J f(x) \geq \mathrm{n}(J)^{-1/m}\lambda\Big\}\Big) \ll \lambda^{-2}\mathrm{n}(J)^{\frac{8M+1}{4Mm}-\frac{1}{4M}}\mathcal{S}(f)^2$$

for any $\lambda > 0$. We note that by our choice of m the second term $-\frac{1}{4M}$ in the exponent is more significant than the first fraction.

Given $n \in \mathbb{N}$, the number of disjoint balls J as above with $\mathrm{n}(J) = q_w^n$ is bounded above by $q_w \mathrm{n}(J)^{1/m}$. Consequently, summing over all possible values of $n \in \mathbb{N}$ with $q_w^n \geq T_0$ and all possible subsets J as above, we see that

$$\mu_{\mathscr{D}}\Big(\Big\{x \in X : D_J f(x) \geq \mathrm{n}(J)^{-1/m}\lambda\, \mathcal{S}(f) \text{ and } \mathrm{n}(J) \geq T_0\Big\}\Big) \ll \lambda^{-2} q_w \sum_{q_w^n \geq T_0} q_w^{n\left(\frac{12M+1}{4Mm}-\frac{1}{4M}\right)} \ll \lambda^{-2} T_0^{-1/8M}, \tag{7.9}$$

where we used $T_0 > q_w^\star$.

To conclude, we use property S2. of the Sobolev norms. Therefore, there are $d > d' > d_0$ and an orthonormal basis $\{e_k\}$ of the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$ which is orthogonal with respect to $\mathcal{S}_{d'}$ so that

$$\sum_k \mathcal{S}_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \tfrac{\mathcal{S}(e_k)^2}{\mathcal{S}_{d'}(e_k)^2} < \infty. \tag{7.10}$$

Put $c = (\sum_k \mathcal{S}_{d'}(e_k)^2)^{-1/2}$ and let B be the set of points so that for some k and some J with $\mathrm{n}(J) \geq T_0$ we have[24]

$$D_J e_k(x) \geq c\, \mathrm{n}(J)^{-1/m}\mathcal{S}_{d'}(e_k).$$

In view of (7.9), applied for $f = e_k$ with $\lambda_k = c\frac{\mathcal{S}_{d'}(e_k)}{\mathcal{S}(e_k)}$, and (7.10) the measure of this set is $\ll T_0^{-1/8M}$.

Let $f \in C_c^\infty(X)$ and write $f = \sum f_k e_k$ and suppose $x \notin B$. Let J be a ball with $\mathrm{n}(J) \geq T_0$. Then using the triangle inequality for D_J we obtain

$$\begin{aligned} D_J(f)(x) &\leq \sum_k |f_k| D_J(e_k)(x) \leq c\mathrm{n}(J)^{-1/m}\sum_k |f_k|\mathcal{S}_{d'}(e_k) \\ &\leq c\mathrm{n}(J)^{-1/m}\left(\sum_k |f_k|^2\right)^{1/2}\left(\sum_k \mathcal{S}_{d'}(e_k)^2\right)^{1/2} \\ &= \mathrm{n}(J)^{-1/m}\mathcal{S}_d(f). \end{aligned}$$

□

7.7. **Pigeon-hole principle.** We now use a version of the pigeon-hole principle to show that if $\mathrm{vol}(Y)$ is large, then in some part of the space and on certain "small but not too small" scales Y is not aligned along $\mathrm{stab}(\mu)$. This gives the first step to producing nearby generic points to which we may apply the effective ergodic theorem, discussed above.

With the notation as in §7.1 put

$$X_{\mathrm{cpt}} = \mathfrak{S}\left(p_w^{(\kappa_{11}+20)/\kappa_{12}}\right);$$

then by Lemma 7.3 we have $\mu_{\mathscr{D}}(X_{\mathrm{cpt}}) \geq 1 - 2^{-20}$.

[24]Thanks to S1. and assuming d_0 is big enough, all expressions considered here are continuous w.r.t. $\mathcal{S}_d$ for all $d \geq d_0$.

Let us also assume that the analogue of (5.3) holds for $\wedge^\ell \mathrm{Ad}$ for $1 \le \ell \le \dim \mathbf{G}$ where as usual Ad denotes the adjoint representation. Therefore, we have that the analogue of (5.5) holds for $\wedge^\ell \mathrm{Ad}$. More precisely, for any infinite place v, any $u \in \Xi_v$, and all $z \in \bigwedge^\ell \mathfrak{g}_v$ we have

$$\wedge^\ell \mathrm{Ad}(\exp u) z = z \text{ implies that } \mathrm{Ad}(\exp(tu)) z = z \text{ for all } t \in F_v \tag{7.11}$$

for all $1 \le \ell \le \dim \mathbf{G}$.

We now fix $\Theta^* = \prod_{v \in \Sigma_\infty} \Theta_v^* \times K_f \subset \mathbf{G}(\mathbb{A})$ with $\Theta_v^* \subset \exp(\Xi_v)$ open for all infinite places v so that the map $g' \in \Theta^* \mapsto xg' \in X$ is injective for all $x \in X_{\mathrm{cpt}}$. Note that in view of our choice of X_{cpt} and (7.3) we may and will choose Θ^* with $\mathrm{vol}_G(\Theta^*) \gg p_w^{-\kappa_{15}}$ for some $\kappa_{15} > 0$. We will also use the notation

$$\Theta^*[w^m] = \{g \in \Theta^* : g_w \in K_w[m]\}$$

for all $m \ge 0$.

Recall from the stabilizer lemma (Lemma 2.2) that the stabilizer of our MASH set is given by $\mathrm{Stab}(\mu_{\mathscr{D}}) = g_{\mathscr{D}}^{-1} \iota(\mathbf{H}(\mathbb{A})) \mathbf{N}(F) g_{\mathscr{D}}$ where $\mathbf{N}$ denotes the normalizer of $\iota(\mathbf{H})$ in $\mathbf{G}$. In the following we will use §5.12 and in particular the notation $S = \Sigma_\infty \cup \{w\}$, $\widetilde{H_{\mathscr{D}}}$, and N_S introduced there.

We claim that

$$\mathrm{stab}(\mu_{\mathscr{D}}) \cap \Theta^*[w^1] \subset \widetilde{H_{\mathscr{D}}} = N_S H_{\mathscr{D}}. \tag{7.12}$$

To see this let $g' = \gamma h \in \mathrm{stab}(\mu_{\mathscr{D}}) \cap \Theta^*[w^1]$ with $\gamma \in g_{\mathscr{D}}^{-1} \mathbf{N}(F) g_{\mathscr{D}}$ and $h \in H_{\mathscr{D}}$. At all $v \in \Sigma_\infty$ apply (7.11) with $\ell = \dim \mathbf{H}(F_v)$, with the vector z belonging to $\wedge^\ell \mathrm{Ad}(g_v^{-1}) \mathrm{Lie}(\iota(\mathbf{H})(F_v))$, and taking u such that $\exp(u) = g'_v$. The quoted statement shows that a one-parameter subgroup containing g'_v normalizes H_v, and since the connected component of the normalizer of the Lie group H_v equals H_v this implies $\gamma_v \in H_v$. In particular we get $\gamma \in g_{\mathscr{D}}^{-1} \iota(\mathbf{H})(F) g_{\mathscr{D}}$. At the place w we use the fact that

$$K_w[1] \cap \iota(\mathbf{H})(F_w) \subset \iota(\mathbf{H}(F_w))$$

to establish the claim.

For a subset $\mathcal{N} \subset \mathbf{G}(\mathbb{A})$ denote the "doubled sets" by $\mathcal{N}_2 = \mathcal{N} \cdot \mathcal{N}^{-1}$ and $\mathcal{N}_4 = \mathcal{N}_2 \cdot \mathcal{N}_2$.

Lemma. *Suppose a measurable subset $E \subset Y$ satisfies $\mu_{\mathscr{D}}(E) > 3/4$. Let $\mathcal{N} \subset G$ be open with $\mathcal{N}_4 \subset \Theta^*[w^1]$ and $\mathrm{vol}_G(\mathcal{N}) > 2\widetilde{\mathrm{vol}}(Y)^{-1}$. Then there exist $x, y \in E$ so that $x = y g_0$ with $g_0 \in \mathcal{N}_4 \setminus \mathrm{stab}(\mu_{\mathscr{D}})$.*

Proof. Let $\{x_i : 1 \le i \le I\}$ be a maximal set of points in X_{cpt} such that $x_i \mathcal{N}$ are disjoint. By our choice of Θ^* (as a function of X_{cpt} and so of q_w) we have $I \le \mathrm{vol}_G(\mathcal{N})^{-1}$. By maximality of I we also have that $\{x_i \mathcal{N}_2 : 1 \le i \le I\}$ covers X_{cpt}. This observation implies in particular that there exists some i_0 so that

$$\mu_{\mathscr{D}}(x_{i_0} \mathcal{N}_2 \cap E) \ge \frac{1}{2I}.$$

Fix some $y_1 \in x_{i_0} \mathcal{N}_2 \cap E$; then any $y_2 \in x_{i_0} \mathcal{N}_2 \cap E$ is of the form $y_1 g$, where $g \in \mathcal{N}_4$.

Suppose, contrary to our claim, that every $y_2 \in x_{i_0} \mathcal{N}_2 \cap E$ were actually of the form $y_1 h$ with $h \in \mathrm{stab}(\mu_{\mathscr{D}}) \cap \mathcal{N}_4$. Recall that $\mathrm{stab}(\mu_{\mathscr{D}}) \cap \mathcal{N}_4 \subset \widetilde{H_{\mathscr{D}}}$. The orbit map $h \mapsto y_1 h$, upon restriction to $\mathcal{N}_4$, is injective by assumption (on Θ^*) and

$y_1 \in \operatorname{supp}(\mu_{\mathscr{D}})$, we thus get $\mu_{\mathscr{D}}(x_{i_0}\mathcal{N}_2 \cap E) \le \widetilde{m_{\mathscr{D}}}(\mathcal{N}_4 \cap \widetilde{H_{\mathscr{D}}})$. The definition of the volume of a homogeneous set together with the above discussion now gives

$$\operatorname{vol}_G(\mathcal{N}) \le \frac{1}{I} \le 2\mu_{\mathscr{D}}(x_{i_0}\mathcal{N}_2 \cap E) \le 2\widetilde{m_{\mathscr{D}}}(\mathcal{N}_4 \cap \widetilde{H_{\mathscr{D}}}) \le 2\widetilde{\operatorname{vol}}(Y)^{-1}$$

which contradicts our assumption. □

7.8. **Combining pigeon-hole and adjustment lemmas.** For any $v \in \Sigma_\infty$ let $\Theta_v \subset \Theta_v^*$ be so that $(\Theta_v)_4 \subset \Theta_v^*$ and put $\Theta = \prod_{v\in\Sigma_\infty} \Theta_v \times K_f$. We may assume that $\operatorname{vol}_G(\Theta) \ge c_3 \operatorname{vol}(Y)^{-1}$ where c_3 depends only on $\mathbf{G}(F_v)$ for $v \in \Sigma_\infty$. We define $\Theta[w^m] = \Theta \cap \Theta^*[w^m]$. We will use the notation $\nu^g(f) := \nu(g \cdot f)$ (with $f \in C_c(X)$) for the action of $g \in G$ on a probability measure ν on X.

Put $\mathcal{S} = \mathcal{S}_d$ for some $d > d_0$ so that the conclusion of the *generic points lemma* of §7.6 holds true.

Lemma (Nearby generic points)**.** *Let $r \ge 0$ be so that*

$$2\operatorname{vol}_G(\Theta[w^r])^{-1} \le \operatorname{vol}(Y)^{\kappa_9}.$$

There exists $x_1, x_2 \in X_{\mathrm{cpt}} \cap Y$ and $g \in G$ so that $x_2 = x_1 g$ and

(1) *x_1, x_2 are both T-generic for $\mu_{\mathscr{D}}$ for some $T > q_w^\star$;*
(2) *$g \in \Theta^*[w^r]$;*
(3) *we may write[25] $g_w = \exp(z)$, where $z \in \mathfrak{r}_w$ is not fixed by $\operatorname{Ad}(u(t))$ and in particular $z \ne 0$. Moreover $\|z\| \le q_w^{-r}$.*

Proof. Let us call $x \in X_{\mathrm{cpt}}$ a *T-good* point if the fraction of $h \in K_w[1] \cap H_w$ for which xh is T-generic exceeds $3/4$, with respect to the Haar measure on H_w. Note that by the defintion $xh \in X_{\mathrm{cpt}}$ for all $h \in K_w$ and $x \in X_{\mathrm{cpt}}$. We apply the *generic points lemma* in §7.6 and obtain that for $T \ge q_w^\star$ the $\mu_{\mathscr{D}}$-measure of the set of T-generic points exceeds 0.99. Using Fubini's theorem, and our choice of X_{cpt} we conclude that the measure of the set $E = \{y \text{ is a } T\text{-good point}\}$ exceeds $3/4$.

By our assumption on r and (5.14) we have $\operatorname{vol}_G(\Theta[w^r]) \ge 2\widetilde{\operatorname{vol}}(Y)^{-1}$. Let $\mathcal{N} = \Theta[w^r]$. Applying the lemma in §7.7 there are T-good points $y_1, y_2 \in X$ such that

$$y_1 = y_2 g_0 \quad \text{where } g_0 \in \Theta^*[w^r] \setminus \operatorname{stab}(\mu).$$

By the *adjustment lemma* in §6.6 and definition of T-good points, there exists $g_1, g_2 \in K_w[1] \cap H_w$ so that $x_i := y_i g_i$ are T-generic, and so that $g := g_1^{-1} g_0 g_2$ satisfies $g_w = \exp(z)$, where $z \in \mathfrak{r}_w$ and $\|z\| \le q_w^{-r}$.

Now let us show that z is not centralized by $u(t)$. Suppose to the contrary. Because $x_2 = x_1 g$ and x_1, x_2 are T-generic, for any $f \in C_0^\infty(X)$ and any $t_0 \in F_w$ with $|t_0| > T$, we have

$$|\mu_{\mathscr{D}}(f) - \mu_{\mathscr{D}}^g(f)| \le D_J(f)(x_2) + D_J(g\cdot f)(x_1) \le |t_0|^{-1/m}(\mathcal{S}(f) + \mathcal{S}(g\cdot f))$$

which implies $\mu_{\mathscr{D}}$ is invariant under g. But we assumed $g_0 \notin \operatorname{stab}(\mu_{\mathscr{D}})$ which also implies $g \notin \operatorname{stab}(\mu_{\mathscr{D}})$. □

[25]As before g_w denotes simply the w-component of $g \in \mathbf{G}(\mathbb{A})$.

7.9. **Combining generic point and admissible polynomial lemmas.** We refer to §6.8 for the definition of admissible polynomials.

Lemma (Polynomial divergence)**.** *There exists an admissible polynomial* $p : F_w \to \mathbf{G}(F_w)$ *so that:*

$$\left|\mu_{\mathscr{D}}^{p(t)}(f) - \mu_{\mathscr{D}}(f)\right| \leq \mathrm{vol}(Y)^{-\star}\mathcal{S}(f) \quad \textit{for all } \ t \in \mathfrak{o}_w. \tag{7.13}$$

Proof. We maximize r in the *nearby generic points lemma* of §7.8. This gives $p_w^{\kappa_{15}} q_w^{(r+1)\dim \mathbf{G}} \gg \mathrm{vol}(Y)^{\kappa_9}$. Using $q_w \ll (\log \mathrm{vol}(Y))^2$ we may simply write $q_w^r \gg \mathrm{vol}(Y)^\star$.

Let x_1, x_2 be two T_0-generic points given by this lemma, in particular, there is $g \in \Theta^*[w^r]$ so that $x_2 = x_1 g$ where $g_w = \exp(z_0)$, and $z_0 \in \mathfrak{r}_w$ is not fixed by $\mathrm{Ad}(u(t))$. In the notation of §6.8 we have $z_0^{\mathrm{mov}} \neq 0$. Then by the *admissible polynomials lemma* in §6.8 there exists $T \in F_w$ with

$$|T| \gg \|z_0^{\mathrm{mov}}\|^{-\star} \geq \|z_0\|^{-\star} \gg \mathrm{vol}(Y)^\star,$$

and an admissible polynomial p so that:

$$\exp(\mathrm{Ad}(u(t))z_0) = p(t/T)g_t \tag{7.14}$$

where $d(g_t, 1) \leq \|z_0\|^\star$ when $|t| \leq |T|$.

Suppose $t_0 \in F_w$ with $|t_0| \leq |T|$ and as in §7.6 put $J = \{t \in F_w : |t - t_0|_w \leq |t_0|_w^{1-1/m}\}$. Fix some arbitrary $f \in C_c^\infty(X)$. Then by the *generic point lemma* in §7.6 and assuming $|t_0|_w \geq T_0$ we have

$$\left|\frac{1}{|J|}\int_{t\in J} f(x_i u(-t))\,\mathrm{d}t - \int f\,\mathrm{d}\mu_{\mathscr{D}}\right| \leq |t_0|^{-1/m}\mathcal{S}(f), \quad i = 1, 2. \tag{7.15}$$

Let $\tilde{p} : F_w \to \mathbf{G}(\mathbb{A})$, be a polynomial given by $\tilde{p}(t/T)_w = p(t/T)$ with p as above and[26] $\tilde{p}(t)_v = g_v$ for all $v \neq w$. Using property S4. and (7.14) this polynomial satisfies

$$\begin{aligned} f(x_2 u(-t)) &= f(x_1 u(-t)\tilde{p}(t/T)) + O(\|z_0\|^\star \mathcal{S}(f)) \\ &= f(x_1 u(-t)\tilde{p}(t_0/T)) + O(|T|^{-\star}\mathcal{S}(f)) + O(\|z_0\|^\star \mathcal{S}(f)) \end{aligned} \tag{7.16}$$

for $|t_0| \geq |T|^{1/2}$ and $t \in J$ (defined by t_0), where we used S4. and the definition of J in the last step.

All together we get

$$\left|\mu_{\mathscr{D}}^{\tilde{p}(t/T)}(f) - \mu_{\mathscr{D}}(f)\right| \leq \mathrm{vol}(Y)^{-\star}\mathcal{S}(f) \quad \text{for } \ |T|^{1/2} \leq |t|_w \leq |T|. \tag{7.17}$$

Indeed this follows from (7.15) and (7.16).

Now choose $t_1 \in \mathfrak{o}_w$ with $|t_1| \in [|T|^{-1/2}, q_w|T|^{-1/2}]$, this implies that (7.17) holds for $t = t_1 T$. Also note that with this choice $p(t_1) \in K_w[\kappa \log_{q_w}(\mathrm{vol}(Y))]$ for some constant $\kappa > 0$ (that only depends on the parameters appearing in the definition of an admissible polynomial). The latter implies that (7.17) holds for $p(t_1)$ instead of $\tilde{p}(t/T)$ trivially as a consequence of S4. Since $p(t/T) = \tilde{p}(t/T)\tilde{p}(0)^{-1} = \tilde{p}(t/T)\tilde{p}(t_1)^{-1}p(t_1)$, we get (7.13) from (7.17) in view of property S3. – note that, if $|t| \leq |T|^{-1/2}$, (7.17) holds for $p(t/T)$ instead of $\tilde{p}(t/T)$ trivially as a consequence of S4. $\square$

[26]The element g above need not have "small" v components for $v \neq w$.

7.10. **Proof of Theorem 1.5.** For simplicity in notation we write μ for $\mu_{\mathscr{D}}$. Let $p: F_w \to \mathbf{G}(F_w)$ be the admissible polynomial given by the polynomial divergence lemma in §7.9. Let Av_L be the operation of averaging over $K_w[L]$, where L is given by Proposition 6.10. Then, it follows from that proposition and property S3. that

$$|\mu(f) - \mu(\mathrm{Av}_L * f)| \ll q_w^\star (\mathrm{vol}\, Y)^{-\star} \mathcal{S}(f) \ll \mathrm{vol}(Y)^{-\star}\mathcal{S}(f)$$

for all $f \in C_c^\infty(X)$. Note that in Proposition 6.10 any $g \in K_w[L]$ is written as a bounded product of two types of elements. The first type of elements belong to $\{h \in H_w : \|h\| \le q_w^L\}$, preserve μ, and distort the Sobolev norm by a power of q_w. The second type of elements are powers of the values of the admissible polynomial at $\mathfrak{o}_w$, preserve the Sobolev norm, and almost preserve the measure.

Let $t \in F_w$. Denote by $\delta_{u(t)}$ the delta-mass at $u(t)$, and let $\star$ denote convolution of measures. Using the fact that μ is $u(t)$-invariant the above gives

$$|\mu(f) - \mu(\mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L * f)| \le \mathrm{vol}(Y)^{-\star}(\mathcal{S}(\delta_{u(t)} \star \mathrm{Av}_L * f) + \mathcal{S}(f)). \tag{7.18}$$

Recall that we are assuming $\mathbf{H}$ is simply connected, thus $\iota(\mathbf{H}(\mathbb{A})) \subset \mathbf{G}(\mathbb{A})^+$ and in particular $\pi^+ f$ is H_w-invariant. Also since $K_w[L] \subset \mathbf{G}(\mathbb{A})^+$, the support of $\mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L$ is contained in $\mathbf{G}(\mathbb{A})^+$. These observations together with (7.18) imply

$$\begin{aligned}&|\mu(f - \pi^+ f)| \\ &\quad \ll |\mu(\mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L * (f - \pi^+ f))| + \mathrm{vol}(Y)^{-\star}(\mathcal{S}(\delta_{u(t)} \star \mathrm{Av}_L * f) + \mathcal{S}(f)).\end{aligned}$$

But Av_L reduces Sobolev norms, and by property S3. the application of $u(t)$ multiplies them by at most $(1 + |t|_w)^{4Nd}$. Therefore

$$|\mu(f - \pi^+ f)| \ll \int_X |\mathbb{T}_t(f - \pi^+ f)(x)| \,\mathrm{d}\mu(x) + \mathrm{vol}(Y)^{-\star}(1 + |t|_w)^{4dN}\mathcal{S}(f),$$

where we write $\mathbb{T}_t$ for the "Hecke operator" $\mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L$.

By property S5. we have for any $x \in X$

$$|\mathbb{T}_t(f - \pi^+ f)(x)| \ll q_w^{(d+2)L} \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0} \mathcal{S}(f);$$

moreover, by (A.12) we have $\|\mathbb{T}_t\|_{2,0} \ll |t|_w^{-1/2M} q_w^{2dL}$.

Let $R > 0$ be a (large) parameter; writing $\int_X |\mathbb{T}_t f(x)| \,\mathrm{d}\mu(x)$ as integrals over $\mathfrak{S}(R)$ and $X \setminus \mathfrak{S}(R)$, in view of the lemma in §7.1, we get

$$\begin{aligned}&|\mu(f - \pi^+ f)| \\ &\quad \ll \left(|t|_w^{-1/2M} q_w^{(3d+2)L} R^d + p_w^{\kappa_{11}} R^{-\kappa_{12}} + \mathrm{vol}(Y)^{-\star}(1 + |t|_w)^{4dN}\right)\mathcal{S}(f).\end{aligned}$$

Optimizing $|t|_w$ and R, using the fact that $q_w \ll (\log \mathrm{vol}(Y))^2$, we get the theorem. We note that the power of $\mathrm{vol}(Y)$ depends only on the parameter M from S6., $\dim_F \mathbf{G}$ and $[F : \mathbb{Q}]$. □

7.11. **Beyond the simply connected case.** The proof of the main theorem above assumed that $\mathbf{H}$ is simply connected. In this section, using the discussion in the simply connected case, we will relax this assumption. It is worth mentioning that for most applications the theorem in the simply connected case already suffices.

Let $\widetilde{\mathbf{H}}$ denote the simply connected covering of $\mathbf{H}$, and let $\pi : \widetilde{\mathbf{H}} \to \mathbf{H}$ denote the covering map. We define $H' = \mathbf{H}(F)\pi(\widetilde{\mathbf{H}}(\mathbb{A}))$, which is closed since it corresponds to a finite volume orbit of $\pi(\widetilde{\mathbf{H}}(\mathbb{A}))$ in $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$. By the properties of the simply connected cover H' is a normal subgroup of $\mathbf{H}(\mathbb{A})$ and $\mathbf{H}(\mathbb{A})/H'$ is abelian; see, e.g., [49, p. 451]. As $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ has finite volume the same applies to $\mathbf{H}(\mathbb{A})/H'$ which implies this quotient is compact. Let ν be the probability Haar measure on this compact abelian group.

Suppose the data $\mathscr{D}$ is fixed as in the introduction, dropping the assumption that $\mathbf{H}$ is simply connected and let $Y = Y_{\mathscr{D}}$ be as before. We also define the MASH set and measure

$$(\widetilde{Y}, \tilde{\mu}) = \Big(\iota\big(\mathbf{H}(F)\backslash\mathbf{H}(F)\pi(\widetilde{\mathbf{H}}(\mathbb{A}))\big)g, \tilde{\mu}\Big),$$

which is defined by the simply connected group $\widetilde{\mathbf{H}}$, the homomorphism $\iota \circ \pi$, and the same element $g \in \mathbf{G}(\mathbb{A})$ as for Y.

Then, we have $\mu = \int_{\mathbf{H}(\mathbb{A})/H'} \tilde{\mu}^h d\nu(h)$ where $\tilde{\mu}^h$ is the probability Haar measure on $\iota\left(\mathbf{H}(F)\backslash\mathbf{H}(F)\pi(\widetilde{\mathbf{H}}(\mathbb{A}))h\right)g$.

Moreover, in view of our definition of volume and the fact that $\mathbf{H}(\mathbb{A})/H'$ is abelian we have $\mathrm{vol}(\widetilde{Y}) = \mathrm{vol}(\widetilde{Y}g^{-1}\iota(h)g)$ (as those orbits have the same stabilizer group). Applying Theorem 1.5 we get

$$|\tilde{\mu}^h(f) - \tilde{\mu}^h(\pi^+(f))| \le \mathrm{vol}(\widetilde{Y})^{-\kappa_0}\mathcal{S}(f) \quad \text{for all } h \in \mathbf{H}(\mathbb{A})/H'.$$

All together we thus have $|\int_X (f - \pi^+(f))d\mu| \le \mathrm{vol}(\widetilde{Y})^{-\kappa_0}\mathcal{S}(f)$.

It seems likely that the argument in §5.12 could be used to show that $\mathrm{vol}(\widetilde{Y}) \asymp \mathrm{vol}(Y)^\star$. We will not pursue this here.

7.12. **Proof of Corollary in §1.7.** We will first consider MASH measures for which the algebraic group $\mathbf{H}$ is simply connected.

Let $\epsilon > 0$ be arbitrary. Choose some compact $Z \subset X$ with $\mu_{x\mathbf{G}(\mathbb{A})^+}(Z) > 1 - \frac{\epsilon}{2}$ for every $x \in X$. Now choose some $f_\epsilon \in C_c^\infty(X)$ with $1_Z \le f_\epsilon \le 1$. Applying Theorem 1.5 to f_ϵ and any MASH measure $\mu_{\mathscr{D}}$ with $\mathscr{D} = (\mathbf{H}, \iota, g)$ and $\mathbf{H}$ simply connected we find some c_4 with

$$\int f_\epsilon \, \mathrm{d}\mu_{\mathscr{D}} > \int_X f_\epsilon \, \mathrm{dvol}_G - c_4\mathcal{S}(f_\epsilon)\,\mathrm{vol}(Y)^{-\kappa_0}.$$

In particular, there exists some $c_5 = c_5(\epsilon)$ such that if $\mathrm{vol}(Y) > c_5$, then

$$\mu_{\mathscr{D}}(\mathrm{supp}(f_\epsilon)) \ge \int f_\epsilon \, \mathrm{d}\mu_{\mathscr{D}} > 1 - \epsilon.$$

In the case where $\mathrm{vol}(Y) \le c_5$ we first find a good place w as in §6.1 with $q_w \ll_\epsilon 1$ and then apply Lemma 7.3 to find another compact set Z' with $\mu_{\mathscr{D}}(Z') > 1 - \epsilon$. The set $X_\epsilon = \mathrm{supp}(f_\epsilon) \cup Z'$ now satisfies the corollary for all MASH measures with $\mathbf{H}$ simply connected.

If $\mu_{\mathscr{D}}$ is a MASH measure and $\mathbf{H}$ is not simply connected, then we can repeat the argument from the previous subsection to obtain $\mu_{\mathscr{D}}(X_\epsilon) > 1 - \epsilon$ also.

Appendix A. Adelic Sobolev norms

We begin by defining, for each finite place v, a certain system of projections $\mathrm{pr}_v[m]$ of any unitary $\mathbf{G}(F_v)$-representation; these have the property that

$$\sum_{m \ge 0} \mathrm{pr}_v[m] = 1.$$

The definitions in the archimedean place likely can be handled in a similar fashion using spectral theory applied to a certain unbounded self-adjoint differential operator (e.g., by splitting the spectrum into intervals $[e^m, e^{m+1})$). However, we will work instead more directly with differential operators in the definition of the norm.

A.1. **Finite places.** Let v be a finite place. Let $\mathrm{Av}_v[m]$ be the averaging projection on $K_v[m]$-invariant vectors, put $\mathrm{pr}_v[0] = \mathrm{Av}_v[0]$ and $\mathrm{pr}_v[m] = \mathrm{Av}_v[m] - \mathrm{Av}_v[m-1]$ for $m \geq 1$.

We note that, if μ is any spherical (=$K_v[0]$-bi-invariant) probability measure on $\mathbf{G}(F_v)$, then convolution with μ commutes with $\mathrm{pr}_v[m]$ for all m. Indeed, the composition (in either direction) of μ with $\mathrm{pr}_v[m]$ is *zero* for $m \geq 1$, and equals μ for $m = 0$.

A.2. **Adelization.** We denote by $\underline{m}$ any function on the set of finite places of F to non-negative integers, which is zero for almost all v. Write $\|\underline{m}\| = \prod_v q_v^{m_v}$. Note that

$$\|\underline{m}\| \geq 1 \quad \text{and} \quad \#\{\underline{m} : \|\underline{m}\| \leq N\} = O_\epsilon(N^{1+\epsilon}), \tag{A.1}$$

which follows since ℓ^ϵ bounds the number of ways in which ℓ can be written as a product of $[F : \mathbb{Q}]$ factors.

For such $\underline{m}$, we set $K[\underline{m}] := \prod_{v \in \Sigma_f} K_v[m_v]$, and $\mathrm{pr}[\underline{m}] := \prod_v \mathrm{pr}_v[m_v]$. Then $\mathrm{pr}[\underline{m}]$ acts on any unitary $\mathbf{G}(\mathbb{A})$-representation, and $\sum_{\underline{m}} \mathrm{pr}[\underline{m}] = 1$. We remark that if $f \in C^\infty(X)$, then $\sum_{\underline{m}} \mathrm{pr}[\underline{m}]f = f$ and the left-hand side is actually a finite sum. We may refer to this as the decomposition of f into pure level components.

If we fix a Haar measure on $\mathbf{G}(\mathbb{A}_f)$, then

$$\mathrm{vol}(K[\underline{m}])^{-1} \ll \|\underline{m}\|^{1+\dim(\mathbf{G})}, \tag{A.2}$$

where the implicit constant depends on $\mathbf{G}, \rho$ (cf. §2.1) and the choice of Haar measure. Here one uses a local calculation in order to control $[K_v[0] : K_v[m_v]]$ for a finite place v; see, e.g., [46, Lemma 3.5].

A.3. **Definition of the Sobolev norms.** For any archimedean place v we fix a basis $\{X_{v,i}\}$ for $\mathfrak{g}_v = \mathfrak{g} \otimes_F F_v$. Let $V = L^2(X)$, where, as in the text, $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. Given an integer $d \geq 0$ we define a degree d Sobolev norm by

$$\mathcal{S}_d(f)^2 := \sum_{\underline{m}} \left(\|\underline{m}\|^d \sum_{\mathcal{D}} \| \mathrm{pr}[\underline{m}](1 + \mathrm{ht}(x))^d \mathcal{D} f(x)\|_2^2 \right), \tag{A.3}$$

where the inner sum is over all monomials $\mathcal{D} = \prod_{v \in \Sigma_\infty} \mathcal{D}_v$ with $\mathcal{D}_v \in U(\mathfrak{g}_v)$ of degree at most d_v in the given basis $\{X_{v,i}\}$ and $\deg \mathcal{D} = \sum d_v \leq d$. For a compactly supported smooth function on X any of these Sobolev norms is finite. It is easy to see that

$$\mathcal{S}_d(f) \leq \mathcal{S}_{d'}(f) \text{ if } d < d'. \tag{A.4}$$

Note that since $\mathrm{ht}(\cdot)$ is $K_f = K[0]$-invariant, we see that $\mathrm{pr}[\underline{m}]$ commutes with multiplication by $(1 + \mathrm{ht}(x))$ and with the differential operators $\mathcal{D}_v$.

We note that the contribution of the finite places to the above is related to the "level" of f, since for a finite place v a function of the form $\mathrm{pr}_v[m_v]f$ should be thought of as having pure level m_v at v.

Also note that, if X is compact, then $\mathrm{ht}(\cdot)$ is uniformly bounded and may be dropped from the definition.

A.4. **Property S1. – Upper bound for L^∞-norms.** We shall now verify property S1. of the Sobolev norms. Let us recall from (7.3) that the map $g \mapsto xg$ is an injection for all $g = (g_\infty, g_f)$ with $g_\infty \in G_\infty = \mathbf{G}(F_{\Sigma_\infty})$ with $d(g_\infty, 1) \le c_1 \mathrm{ht}(x)^{-\kappa_{10}}$ and $g_f \in K_f$.

Let f belong to the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$. Suppose first that f is invariant under $K[\underline{m}]$ for some $\underline{m}$. For any $x \in X$ define the function $g \mapsto f(xg)$ on

$$\Omega_\infty(x) = \{g \in G_\infty : d(g,1) \le c_1 \mathrm{ht}(x)^{-\kappa_{10}}\}.$$

Then by the usual Sobolev inequality, see, e.g., [24, Lemma 5.1.1], there is some integer $d_0 > [F:\mathbb{Q}] \dim \mathbf{G}$ so that we have

$$|f(x)|^2 \ll \sum_{\mathcal{D}} \tfrac{1}{\mathrm{vol}(\Omega_\infty(x))} \int_{\Omega_\infty(x)} |\mathcal{D} f|^2,$$

where the sum is taken over all $\mathcal{D}$ of degree at most d_0.

Let $d \ge 1 + \kappa_{10} d_0$,. If we integrate the above over $K[\underline{m}]$, then in view of the fact that f is invariant under $K[\underline{m}]$ we get from (A.2) and the estimate $\mathrm{vol}(\Omega_\infty(x))^{-1} \ll \mathrm{ht}(x)^{\kappa_{10}[F:\mathbb{Q}] \dim \mathbf{G}}$ that

$$\begin{aligned}
|f(x)|^2 &\ll \mathrm{vol}(K[\underline{m}])^{-1}\, \mathrm{vol}(\Omega_\infty(x))^{-1} \sum_{\mathcal{D}} \int_{\Omega_\infty(x) \times K[\underline{m}]} |\mathcal{D} f|^2 \\
&\ll \|\underline{m}\|^d \sum_{\mathcal{D}} \int_{\Omega_\infty(x) \times K[\underline{m}]} |(1+\mathrm{ht}(x))^d \mathcal{D} f|^2 \\
&\ll \|\underline{m}\|^d \sum_{\mathcal{D}} \|(1+\mathrm{ht}(x))^d \mathcal{D} f\|_2^2,
\end{aligned} \tag{A.5}$$

where again the sum is over all $\mathcal{D}$ of degree at most d.

Let us now drop the assumption that f is invariant under a fixed compact subgroup of K_f. In this case we may decompose f into a converging sum $f = \sum_{\underline{m}} \mathrm{pr}[\underline{m}] f$, and obtain

$$\begin{aligned}
|f(x)|^2 = |\sum_{\underline{m}} \mathrm{pr}[\underline{m}] f(x)|^2 &\le \sum_{\underline{m}} \|\underline{m}\|^{-2} \sum_{\underline{m}} \|\underline{m}\|^2 |\mathrm{pr}[\underline{m}] f(x)|^2 \\
&\ll \sum_{\underline{m}} \|\underline{m}\|^{-2} \sum_{\underline{m}, \mathcal{D}} \|\underline{m}\|^{d+2} \|\mathrm{pr}[\underline{m}](1+\mathrm{ht}(x))^d \mathcal{D} f\|_2^2 \ll \mathcal{S}_{d+2}(f)^2,
\end{aligned} \tag{A.6}$$

where we used Cauchy-Schwarz, the above, the definition in (A.3), and the estimate

$$\sum_{\underline{m}} \|\underline{m}\|^{-2} = \sum_{k \ge 1} \sum_{\underline{m} : \|\underline{m}\| = k} k^{-2} \ll_\epsilon \sum_k k^{-2+\epsilon} < \infty.$$

A.5. **Property S2. – Trace estimates.** Let $r \ge 0$, let $\mathcal{D}_0$ be a monomial of degree at most r, and let $\underline{m}$ be arbitrary. Furthermore, let $f \in C_c^\infty(X)$, and apply (A.5) to the function $\mathcal{D}_0 \,\mathrm{pr}[\underline{m}] f$, multiplying the inequality by $\|\underline{m}\|^r (1+\mathrm{ht}(x))^r$ we get

$$\begin{aligned}
&\|\underline{m}\|^r |(1+\mathrm{ht}(x))^r \mathcal{D}_0 \,\mathrm{pr}[\underline{m}] f(x)|^2 \\
&\qquad \le c \|\underline{m}\|^{d+r} \sum_{\mathcal{D}} \int_{\Omega_\infty(x) \times K[\underline{m}]} |(1+\mathrm{ht}(x))^{d+r} \mathcal{D} \,\mathrm{pr}[\underline{m}] f|^2,
\end{aligned}$$

where the sum is over $\mathcal{D}$ of degree at most $d+r$ and $d \geq \kappa_{10} d_0$ is as above. Moreover, this also gives

$$\|\underline{m}\|^r |(1+\mathrm{ht}(x))^r \mathcal{D}_0 \operatorname{pr}[\underline{m}] f(x)|^2 \leq c \|\underline{m}\|^{-s} \mathcal{S}_{d+r+s}(f)$$

for all d as above and $s \geq 0$.

For $x \in X$ put $L_{x,\underline{m}}(f) = \|\underline{m}\|^r (1+\mathrm{ht}(x))^r \mathcal{D}_0 \operatorname{pr}[\underline{m}] f(x)$. Then the above implies

$$\operatorname{Tr}(|L_{x,\underline{m}}|^2 | \mathcal{S}_{d'}^2) \leq c \|\underline{m}\|^{-s} \quad \text{for all } d' \geq d+r+s \text{ and any } s \geq 0;$$

see [3] and [24] for a discussion of relative traces.

Integrating over $x \in X$, using (A.1) to sum over $\underline{m}$, and summing over $\mathcal{D}_0$ with $\deg \mathcal{D}_0 \leq r$ we get $\operatorname{Tr}(\mathcal{S}_r^2 | \mathcal{S}_{d'}^2) \ll 1$, again for all $d' \geq d+r+s$ and $s \geq 2$.

Let us now use the notation of S2. Given d_0, the above shows that there exists $d' > d_0$ and $d > d'$ with $\operatorname{Tr}(\mathcal{S}_{d_0}^2 | \mathcal{S}_{d'}^2) < \infty$ and $\operatorname{Tr}(\mathcal{S}_{d'}^2 | \mathcal{S}_d^2) < \infty$. To find an orthonormal basis with respect to $\mathcal{S}_{d'}$ which is orthogonal with respect to $\mathcal{S}_d$ as in S2. one may argue as follows. Recall that $\mathcal{S}_{d'}(f) \leq \mathcal{S}_d(f)$ and therefore, by Riesz representation theorem, there exists some positive definite operator $\operatorname{Op}_{d',d}$ so that

$$\langle f_1, f_2 \rangle_{\mathcal{S}_{d'}} = \langle \operatorname{Op}_{d',d} f_1, f_2 \rangle_{\mathcal{S}_d} \quad \text{for } f_1, f_2 \in C_c^\infty(X).$$

This operator satisfies $\operatorname{Tr}(\operatorname{Op}_{d',d}) = \operatorname{Tr}(\mathcal{S}_{d'}^2 | \mathcal{S}_d^2)$ and so it is compact. Now choose an orthonormal basis with respect to $\mathcal{S}_d$ consisting of eigenvectors for $\operatorname{Op}_{d',d}$. Therefore, this basis is still orthogonal with respect to $\mathcal{S}_{d'}$, and S2. follows from the definition of the relative trace.

A.6. **Property S3. – Bounding the distortion by $g \in \mathbf{G}(F_v)$.** Let $v \in \mathbf{G}(F_v)$ for some $v \in \Sigma_f$. Note that g commutes with any differential operator $\mathcal{D}$ used above as well as with the averaging and projection operators $\operatorname{Av}_{v'}[\cdot]$ and $\operatorname{pr}_{v'}[\cdot]$ for $v' \in \Sigma_f \setminus \{v\}$.

So if $g \in K_v$ (or more generally $g \in K_f$), then $g K_v[m_v] g^{-1} = K_v[m_v]$ for all $m_v \geq 0$. This implies that the action of g commutes also with the decomposition of $f \in C_c^\infty(X)$ into pure level components at v, and so $\mathcal{S}_d(g \cdot f) = \mathcal{S}(f)$ by (7.2) and (A.3).

Let now $g \notin K_v$ and $f \in C_c^\infty(X)$. This also implies

$$g K_v[2 \log_{q_v} \|g\| + m] g^{-1} \subseteq K_v[m].$$

Using this, that $\operatorname{pr}_v[\ell] f$ is invariant under $K_v[\ell]$ for $\ell \geq 0$, and that

$$\operatorname{Av}_v[\ell - 1](\operatorname{pr}_v[\ell] f) = 0$$

for $\ell \geq 1$, we get for all $m, \ell \geq 0$ that

$$\operatorname{pr}_v[m](g \cdot (\operatorname{pr}_v[\ell] f)) = 0 \text{ unless } |m - \ell| \leq 2 \log_{q_v} \|g\|.$$

Applying this and defining $R = 2 \log_{q_v} \|g\|$ we get

$$\begin{aligned}
&\|(1+\mathrm{ht})^d \operatorname{pr}_v[m](g \cdot f)\|_2 \\
&\qquad = \Big\| \operatorname{pr}_v[m] (1+\mathrm{ht})^d \Big(g \cdot \sum_{|\ell - m| \leq R} \operatorname{pr}_v[\ell] f \Big) \Big\|_2 \\
&\qquad \leq \sum_{|\ell - m| \leq R} \|(1 + g^{-1} \cdot \mathrm{ht})^d \operatorname{pr}_v[\ell] f\|_2 \\
&\qquad \ll (2R+1) \|g\|^{2d} \max_{|\ell - m| \leq R} \|(1+\mathrm{ht})^d \operatorname{pr}_v[\ell] f\|_2,
\end{aligned}$$

where we also used (7.1). Fixing $f \in C_c^\infty(X)$ we now apply this for the functions $\prod_{v' \in \Sigma_f \setminus \{v\}} \mathrm{pr}_{v'}[m_{v'}]\mathcal{D}f$ and sum over all $\underline{m}$ and $\mathcal{D}$ to get

$$\begin{aligned}
&\mathcal{S}_d(g \cdot f)^2 \\
&\quad \ll (2R+1)^2 \|g\|^{4d} \sum_{\underline{m},\mathcal{D}} \|\underline{m}\|^d \sum_{|\ell - m_v| \le R} \Big\| (1+\mathrm{ht})^d \,\mathrm{pr}_v[\ell] \prod_{v' \in \Sigma_f \setminus \{v\}} \mathrm{pr}_{v'}[m_{v'}]\mathcal{D}f \Big\|_2^2 \\
&\qquad\qquad \le (2R+1)^3 \|g\|^{6d} \mathcal{S}_d(f)^2 \ll \|g\|^{8d} \mathcal{S}_d(f)^2,
\end{aligned}$$

which gives S3.

For $v \in \Sigma_\infty$ the argument consists of expressing the element $\mathrm{Ad}_g(\mathcal{D})$ in terms of the basis elements considered in the definition of $\mathcal{S}_d(\cdot)$, and bounding the change of the height as above.

Let now $u(\cdot)$ be the unipotent subgroup as in Property S3. In view of the definition of K_w we get $\rho \circ \theta_w(\mathrm{SL}_2(\mathfrak{o}_w)) \subset \mathrm{SL}_N(\mathfrak{o}_w)$. Therefore, $\|u(t)\| \le |t|_w^N$ as was claimed.

A.7. **Property S4. – Estimating the Lipshitz constant at w.** Let f belong to the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$. First note that if f is invariant under $K[\underline{m}]$ for some $\underline{m}$, then $g \cdot f$ is also invariant under $K[\underline{m}]$ for all $g \in K_w$. Therefore, $\mathrm{pr}[\underline{m}]g \cdot f = g \cdot \mathrm{pr}[\underline{m}]f$ for all $g \in K_w$. Also note that if $g \in K_w[r]$ and f is $K[\underline{m}]$ invariant with $m_w \le r$, then $g \cdot f = f$.

Let now $g \in K_w[r]$ and let f be in the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$. Therefore, as in (A.6) we can use (A.5) and get

$$\begin{aligned}
|(g \cdot f - f)(x)|^2 &= \Big| \sum_{\underline{m}} \mathrm{pr}[\underline{m}](g \cdot f - f)(x) \Big|^2 = \Big| \sum_{\underline{m}} (g \cdot \mathrm{pr}[\underline{m}]f - \mathrm{pr}[\underline{m}]f)(x) \Big|^2 \\
&= \Big| \sum_{\underline{m}: m_w > r} (g \cdot \mathrm{pr}[\underline{m}]f - \mathrm{pr}[\underline{m}]f)(x) \Big|^2 \\
&\le \sum_{\underline{m}: m_w > r} \|\underline{m}\|^{-2} \sum_{\underline{m}: m_w > r} \|\underline{m}\|^2 |(g \cdot \mathrm{pr}[\underline{m}]f - \mathrm{pr}[\underline{m}]f)(x)|^2 \\
&\le \sum_{\underline{m}: m_w > r} \|\underline{m}\|^{-2} \sum_{\underline{m}} \|\underline{m}\|^2 |\,\mathrm{pr}[\underline{m}](g \cdot f - f)(x)|^2 \\
&\ll \sum_{m_w > r} \|\underline{m}\|^{-2} \sum_{\underline{m},\mathcal{D}} \|\underline{m}\|^{d+2} \|\,\mathrm{pr}[\underline{m}](1+\mathrm{ht}(x))^d \mathcal{D}(g \cdot f - f)\|_2^2 \\
&\ll q_w^{-2r} \mathcal{S}_{d+2}(g \cdot f - f)^2 \ll q_w^{-2r} \mathcal{S}_{d+2}(f)^2,
\end{aligned}$$

where in the last inequality we used property S3.

A.8. **Property S6. – Bounds for matrix coefficients.** Recall that at the good place w there exists a non-trivial homomorphism $\theta : \mathrm{SL}_2(F_w) \to H_w \subset \mathbf{G}(F_w)$ such that $K_{\mathrm{SL}_2} = \theta(\mathrm{SL}_2(\mathfrak{o}_w)) \subset K_f$. We also write

$$u(t) = \theta\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right).$$

Let ν be a MASH measure on X which is invariant and ergodic by $\theta(\mathrm{SL}_2(F_w))$. Recall from §7.4 that the $\mathrm{SL}_2(F_w)$-representation on

$$L_0^2(X,\nu) = \left\{ f \in L^2(X,\nu) : \int f d\nu = 0 \right\}$$

is $1/M$-tempered.

Let $f_1, f_2 \in C_c^\infty(X)$. Consider $I := \langle u(t) f_1, f_2 \rangle_{L^2(\nu)} - \int f_1 d\nu \int \bar{f}_2 d\nu$. By [16] (see also §4.3) $|I|$ can be bounded above by

$$\text{(A.7)} \quad (1 + |t|_w)^{-1/2M} \dim(K_{\mathrm{SL}_2} \cdot f_1)^{1/2} \dim(K_{\mathrm{SL}_2} \cdot f_2)^{1/2} \|f_1\|_{L^2(\nu)} \|f_2\|_{L^2(\nu)}.$$

Suppose that f_1 is fixed by $K[\underline{m}]$, the dimension of $K_{\mathrm{SL}_2} \cdot f_1$ is bounded above by the number of $K[\underline{m}] \cap K_{\mathrm{SL}_2}$-cosets in K_{SL_2}, which in turn is bounded by

$$\text{(A.8)} \qquad \left[K_w[0] : K_w[m_w] \right] \ll q_w^{m_w \dim \mathbf{G}}.$$

Decomposing $f_1 := \sum \mathrm{pr}_v[\underline{m}] f_1$ and similarly for f_2, we see that in general:

$$|I| \ll (1 + |t|_w)^{-1/2M} \prod_{i \in \{1,2\}} \left(\sum_{\underline{m}} \|\underline{m}\|^{\frac{1}{2} \dim \mathbf{G}} \| \mathrm{pr}[\underline{m}] f_i \|_{L^2(\nu)} \right).$$

We now apply Cauchy-Schwarz inequality and (A.5) to the expression in the parenthesis to get

$$\begin{aligned} &\left(\sum_{\underline{m}} \|\underline{m}\|^{\frac{1}{2} \dim \mathbf{G}} \| \mathrm{pr}[\underline{m}] f_i \|_{L^2(\nu)} \right)^2 \\ &\qquad \ll \sum_{\underline{m}} \|\underline{m}\|^{\dim \mathbf{G} + 2} \| \mathrm{pr}[\underline{m}] f_i \|_{L^2(\nu)}^2 \\ &\qquad \ll \sum_{\underline{m}} \|\underline{m}\|^{\dim \mathbf{G} + 2 + d} \sum_{\mathcal{D}} \| (1 + \mathrm{ht}(x))^d \mathcal{D}(\mathrm{pr}[\underline{m}] f_i) \|_2^2 \ll \mathcal{S}_{\dim \mathbf{G} + 2 + d}(f_i). \end{aligned}$$

This gives property S6.

A.9. **Property S5. – The operator $\mathbb{T}_t$ and Sobolev norms.** We will use the same notation as above. Recall that we defined the operator $\mathbb{T}_t$ to be $\mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L$ where Av_L is the operation of averaging over $K_w[L]$, with $L > 0$ as in Proposition 6.10.

Here we will modify the argument in the proof of property S1. to get the desired property. Let us note again that $\mathrm{ht}(\cdot)$ is invariant under $K[0]$.

Let $d \geq \kappa_{10} d_0$ and let f be an arbitrary smooth compactly supported function. Then

$$\text{(A.9)} \qquad \mathcal{D}\, \mathrm{pr}[\underline{m}] \mathbb{T}_t \pi^+ f = \mathrm{pr}[\underline{m}] \mathcal{D} \mathbb{T}_t \pi^+ f = 0, \quad \text{whenever } \deg \mathcal{D} \geq 1.$$

Indeed $\pi^+(f)$ is invariant under $\mathbf{G}(\mathbb{A})^+$ and the latter contains $\exp(\mathfrak{g}_v)$ for all $v \in \Sigma_\infty$. We also note that $\mathbb{T}_t \pi^+ = \pi^+$ because $\mathbf{G}(\mathbb{A})^+$ contains $\{u(t)\}$ and $L \geq 1$ satisfies $K_w[L] \subset \mathbf{G}(\mathbb{A})^+$. Given $\underline{m}$ let us put $\mathrm{pr}^{(w)}[\underline{m}] = \prod_{v \neq w} \mathrm{pr}[m_v]$. Note that

$$\text{(A.10)} \qquad \mathcal{D} \mathbb{T}_t = \mathbb{T}_t \mathcal{D} \quad \text{and} \quad \mathrm{pr}^{(w)}[\underline{m}] \mathbb{T}_t = \mathbb{T}_t \, \mathrm{pr}^{(w)}[\underline{m}].$$

For any $\underline{m}$ put $\Phi_{\underline{m}}(x) = \mathrm{pr}[\underline{m}]\mathbb{T}_t(f - \pi^+ f)(x)$; note that $\Phi_{\underline{m}}$ is $K[\underline{m}]$ invariant. Arguing as in (A.5) for the function $\Phi_{\underline{m}}(\cdot)$ and using (A.10) we get

$$
\begin{aligned}
(\mathrm{A.11}) \qquad |\Phi_{\underline{m}}(x)|^2 &\ll \mathrm{vol}(K[\underline{m}])^{-1}\,\mathrm{vol}(\Omega_\infty(x))^{-1} \sum_{\mathcal{D}} \int_{\Omega_\infty(x)\times K[\underline{m}]} |\mathcal{D}\Phi_{\underline{m}}|^2 \\
&\ll \|\underline{m}\|^d \mathrm{ht}(x)^d \sum_{\mathcal{D}} \|\,\mathrm{pr}[\underline{m}]\mathbb{T}_t(\mathcal{D}(f - \pi^+ f))\|_2^2 \\
&\ll \|\underline{m}\|^d \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0}^2 \sum_{\mathcal{D}} \|\,\mathrm{pr}^{(w)}[\underline{m}]\mathcal{D}f\|_2^2,
\end{aligned}
$$

where in the last step we used the fact that both $\mathrm{pr}_w[m_w]$ and π^+ are projections, together with (A.9) and (A.10).

Since $\mathbb{T}_t = \mathrm{Av}_L \star \delta_{u(t)} \star \mathrm{Av}_L$, we have: $\mathrm{pr}[\underline{m}]\mathbb{T}_t = 0$ for all $m_w > L$. Also recall that $\sum_{\underline{m}} \mathrm{pr}[\underline{m}] = 1$. Therefore,

$$
\mathbb{T}_t(f - \pi^+ f)(x) = \sum_{\underline{m}: m_w \le L} \underbrace{\mathrm{pr}[\underline{m}]\mathbb{T}_t(f - \pi^+ f)(x)}_{\Phi_{\underline{m}}(x)}.
$$

Arguing as in the last paragraph in §A.4, using (A.11) and the above identity we get

$$
\begin{aligned}
|\mathbb{T}_t(f - \pi^+ f)(x)|^2 &\ll \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0}^2 \sum_{\substack{\underline{m}: m_w \le L \\ \mathcal{D}}} \|\underline{m}\|^{d+2} \|\,\mathrm{pr}^{(w)}[\underline{m}]\mathcal{D}f\|_2^2 \\
&\ll \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0}^2 (L+1) q_w^{(d+2)L} \sum_{\substack{\underline{m}: m_w = 0 \\ \mathcal{D}}} \|\underline{m}\|^{d+2} \|\,\mathrm{pr}^{(w)}[\underline{m}]\mathcal{D}f\|_2^2 \\
&\ll \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0}^2 q_w^{(d+2)L} \sum_{\substack{\underline{m}: m_w = 0 \\ \mathcal{D}}} \|\underline{m}\|^{d+2} \Big\| \sum_{m_w} \mathrm{pr}[m_w](\mathrm{pr}^{(w)}[\underline{m}]\mathcal{D}f) \Big\|_2^2 \\
&\ll \mathrm{ht}(x)^d \|\mathbb{T}_t\|_{2,0}^2 q_w^{(d+2)L} \sum_{\underline{m}, \mathcal{D}} \|\underline{m}\|^{d+2} \|\,\mathrm{pr}[\underline{m}]\mathcal{D}f\|_2^2
\end{aligned}
$$

which implies S5.

Let us note that the argument in §A.8 applies to the representation of $\mathrm{SL}_2(F_w)$ on $L^2_0(X, \mathrm{vol}_G)$, i.e., the orthogonal complement of $\mathbf{G}(\mathbb{A})^+$-invariant functions. Suppose this representation is $1/M$-tempered; then similar to (A.7) we get

$$
\begin{aligned}
&\big| \langle u(t)\cdot f_1, f_2\rangle - \langle \pi^+ f_1, \pi^+ f_2\rangle \big| \\
&\qquad \ll (1 + |t|_w)^{-1/2M} \dim(K_{\mathrm{SL}_2}\cdot f_1)^{1/2} \dim(K_{\mathrm{SL}_2}\cdot f_2)^{1/2} \|f_1\|_2 \|f_2\|_2.
\end{aligned}
$$

This estimate and (A.8), in view of the definition of $\mathbb{T}_t$, imply

$$
(\mathrm{A.12}) \qquad \|\mathbb{T}_t\|_{2,0} \ll (1 + |t|_w)^{-1/2M} q_w^{2dL}.
$$

Appendix B. The discriminant of a homogeneous set

The paper [23] defined the *discriminant* of a homogeneous set in the case when the stabilizer is a torus. Here we shall adapt this definition to the case at hand; see also [24, Sec. 17].

Let $\mathbf{H}$ be a semisimple, simply connected group defined over F. As in §1.1 we fix a non-trivial F-homomorphism $\iota : \mathbf{H} \to \mathbf{G}$ with central kernel and let $(g_v)_v \in \mathbf{G}(\mathbb{A})$. Put $Y = \iota(\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}))(g_v)_v$.

We choose a differential form ω of top degree on $\mathbf{H}$ and an F-basis $\{f_1, \dots, f_r\}$ for $\mathrm{Lie}(\mathbf{H})$ such that $\omega(z) = 1$ for

$$z = f_1 \wedge \cdots \wedge f_r \in \wedge^r \mathrm{Lie}(\mathbf{H}).$$

Using $\rho : \mathbf{G} \to \mathrm{SL}_N$ and ι we have $\rho \circ \iota(z) \in \wedge^r \mathfrak{sl}_N$. We put $z_v := \rho \circ \mathrm{Ad}(g_v)^{-1} \circ \iota(z)$. For each $v \in \Sigma$, we denote by ω_v the form of top degree on $\mathbf{H}(F_v)$ induced by ω.

Let $\| \, \|_v$ be a compatible system of norms on the vector spaces $\wedge^r \mathfrak{sl}_N \otimes F_v$. In particular, we require at all the finite places v that the norm $\| \, \|_v$ is the max norm. Denote by B the bilinear form on $\wedge^r \mathfrak{h}$ induced by the Killing form. We define the *discriminant* of the homogeneous set Y by

$$\mathrm{disc}(Y) = \tfrac{\mathcal{D}(\mathbf{H})}{\mathcal{E}(\mathbf{H})} \prod_v \mathrm{disc}_v(Y) = \tfrac{\mathcal{D}(\mathbf{H})}{\mathcal{E}(\mathbf{H})} \prod_v \|z_v\|_v, \tag{B.1}$$

where

(1) $\mathrm{disc}_v(Y) = |B(\omega_v, \omega_v)|_v^{1/2} \|z_v\|_v$ is the *local discriminant* at v, and is independent of the choice of the F-basis,
(2) $\mathcal{D}(\mathbf{H}) \geq 1$ is defined in (B.13), and
(3) $0 < \mathcal{E}(\mathbf{H}) \leq 1$ is defined in (B.15).

The second equality in (B.1) uses the fact that $\prod_v |B(\omega_v, \omega_v)|_v = 1$ which is a consequence of the product formula and the equality $B(\omega_v, \omega_v) = B(\omega, \omega)$.

One key feature of this definition is that it is closely related to the volume in the sense that

$$\mathrm{vol}(Y)^\star \ll \mathrm{disc}(Y) \ll \mathrm{vol}(Y)^\star. \tag{B.2}$$

We outline a proof of this in this section.

Before doing that let us use (B.2) to complete the discussion from §3.

B.1. **Proof of Lemma 3.3.** We use the notation from §3. In particular, $F = \mathbb{Q}$, Q is a positive definite integral quadratic form in n variables, $\mathbf{H}' = \mathrm{SO}(Q)$, and $\mathbf{H} = \mathrm{Spin}(Q)$.

Let $g_Q \in \mathrm{PGL}(n, \mathbb{R})$ be so that $g_Q^{-1} \mathbf{H}'(\mathbb{R}) g_Q = \mathrm{SO}(n, \mathbb{R})$ and put

$$Y = Y_Q = \pi(\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}))(g_Q, e, \dots).$$

We recall that K' and $K'(\infty)$ are compact open subgroups of $\mathbf{H}'(\mathbb{A}_f)$ and $\mathbf{H}'(\mathbb{A})$, respectively. Also recall the notation $K^* = K' \cap \pi(\mathbf{H}(\mathbb{A}_f))$, and $K^*(\infty) = \pi(\mathbf{H}(\mathbb{R}))K^*$. Finally put

$$K_Q^*(\infty) = g_Q^{-1} K^*(\infty) g_Q = g_Q^{-1} K'(\infty) g_Q \cap H_{\mathscr{D}}.$$

Lemma. *We have the following:*

$$\mathrm{vol}(Y)^\star \ll \text{spin genus}(Q) \ll \mathrm{vol}(Y). \tag{B.3}$$

Proof. Using the definition of the volume as in (1.1), we have up to a multiplicative constant depending on Ω_0, that $\mathrm{vol}(Y) \asymp m_Y(K_Q^*(\infty))^{-1}$. On the other hand we have

$$1 = \mu_{\mathscr{D}}(Y) = \sum_h \frac{m_Y\big(K_Q^*(\infty)\big)}{\ell_h},$$

where $1 \leq \ell_h \leq \#\big(\mathbf{H}'(\mathbb{Q}) \cap hK'(\infty)h^{-1}\big)$ for every double coset representative

$$\mathbf{H}'(\mathbb{Q})hK^*(\infty) \in \mathbf{H}'(\mathbb{Q}) \backslash \mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))/K^*(\infty).$$

Since ℓ_h is bounded by the maximum of orders of finite subgroups of $\mathrm{PGL}_n(\mathbb{Q})$, see, e.g., [57, LG, Ch. IV, App. 3, Thm. 1] we get that, up to a constant depending on Ω_0, we have

$$\mathrm{vol}(Y) \asymp \#\big(\mathbf{H}'(\mathbb{Q})\backslash\mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))/K^*(\infty)\big). \tag{B.4}$$

Recall that

$$\text{spin genus}(Q) = \#\big(\mathbf{H}'(\mathbb{Q})\backslash\mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))K'(\infty)/K'(\infty)\big).$$

Hence (B.4) implies the claimed upper bound in the lemma.

We now turn to the proof of the lower bound. The idea is to use strong approximation and discussion in §5.12 to relate the orbit space appearing on the right side of (B.4) to the spin genus.

Let p be a good prime for Y given by the proposition in §5.11. In view of the strong approximation theorem applied to the simply connected group $\mathbf{H}$, the choice of p, and the definition of $K^*(\infty)$ we have

$$\pi(\mathbf{H}(\mathbb{A})) = \pi(\mathbf{H}(\mathbb{Q}))\pi(\mathbf{H}(\mathbb{Q}_p))K^*(\infty), \tag{B.5}$$

where we have identified $\mathbf{H}(\mathbb{Q}_p)$ as a subgroup of $\mathbf{H}(\mathbb{A})$.

Therefore, every double coset $\mathbf{H}'(\mathbb{Q})\pi(h)K'(\infty)$ has a representative in $\pi(\mathbf{H}(\mathbb{Q}_p))$. That is: we use (B.5) and write $\pi(h) = \pi(\delta,\delta,\ldots)\pi\big((h'_p,e,\ldots)\big)k^*$, where $\delta \in \mathbf{H}(\mathbb{Q})$, $h'_p \in \mathbf{H}(\mathbb{Q}_p)$, and $k^* \in K^*(\infty)$.

Let now $h_p^{(1)}, h_p^{(2)} \in \mathbf{H}(\mathbb{Q}_p)$ be so that

$$\pi\big((h_p^{(2)},e,\ldots)\big) = (\gamma,\gamma,\ldots)\pi\big((h_p^{(1)},e,\ldots)\big)k, \tag{B.6}$$

where $\gamma \in \mathbf{H}'(\mathbb{Q})$, $k \in K'(\infty)$. Let us write $k = \big(k_p,(k_q)_{q\neq p}\big)$; we note that $q=\infty$ is allowed. Then we have

$$\gamma k_q = 1 \quad \text{for all } q \neq p.$$

Hence we get $k = (k_p,\gamma^{-1},\gamma^{-1},\ldots)$. This, in particular, implies $\gamma \in K'_q$ for all $q \neq p$, that is,

$$(\gamma,\gamma,\ldots) \in \mathbf{H}'(\mathbb{Q}) \cap \mathbf{H}'(\mathbb{Q}_p)K'(\infty).$$

Put $\Lambda' := \mathbf{H}'(\mathbb{Q}) \cap \mathbf{H}'(\mathbb{Q}_p)K'(\infty)$ and $\Lambda := \pi(\mathbf{H}(\mathbb{Q})) \cap \mathbf{H}'(\mathbb{Q}_p)K'(\infty)$. Taking their projections into $\mathbf{H}'(\mathbb{Q}_p)$, we identify Λ and Λ' as two lattices in $\mathbf{H}'(\mathbb{Q}_p)$. Note that Λ is a normal subgroup of Λ'. We write

$$\Lambda'/\Lambda = \cup_{i=1}^r \Lambda\gamma_i.$$

Also write $K'_p/K^*_p = \cup_{j=1}^s k_jK^*_p$. The above discussion thus implies

$$\pi(h_p^{(2)}) \in \bigsqcup_{i,j} \Lambda\gamma_i\pi(h_p^{(1)})k_jK^*_p. \tag{B.7}$$

Define the natural surjective map from $\pi(\mathbf{H}(\mathbb{Q}))\backslash\pi(\mathbf{H}(\mathbb{A}))/K^*(\infty)$ to $\mathbf{H}'(\mathbb{Q})\backslash\mathbf{H}'(\mathbb{Q})$ $\pi(\mathbf{H}(\mathbb{A}))K'(\infty)/K'(\infty)$ by

$$\pi(\mathbf{H}(\mathbb{Q}))\big(\pi(h_p),e,\ldots\big)K^*(\infty) \mapsto \mathbf{H}'(\mathbb{Q})\big(\pi(h_p),e,\ldots\big)K'(\infty).$$

Then since $\Lambda \subset \pi(\mathbf{H}(\mathbb{Q}))$ and $K^*_p \subset K^*(\infty)$ we get from (B.7) that the preimage of $\mathbf{H}'(\mathbb{Q})\big(\pi(h_p),e,\ldots\big)K'(\infty)$ is contained in $\bigcup_{i,j} D_{i,j}$ where

$$D_{i,j} := \big\{\pi(\mathbf{H}(\mathbb{Q}))\big(\pi(h'_p),e,\ldots\big)K^*(\infty) : \gamma_i\pi(h_p)k_j \in \Lambda\pi(h'_p)K^*_p\big\}.$$

Note also that $D_{i,j}$ has at most one element.

Putting this all together, we get the following:

$$\begin{aligned}
\text{spin genus}(Q) &= \#\big(\mathbf{H}'(\mathbb{Q})\backslash\mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))K'(\infty)/K'(\infty)\big)\\
{\scriptstyle\text{the above discussion}} \rightsquigarrow &\gg \frac{\#\big(\mathbf{H}'(\mathbb{Q})\backslash\mathbf{H}'(\mathbb{Q})\pi(\mathbf{H}(\mathbb{A}))/K^*(\infty)\big)}{[\Lambda':\Lambda][K'_p:K^*_p]}\\
{\scriptstyle(\text{B.4})} \rightsquigarrow &\gg \frac{\mathrm{vol}(Y)}{[\Lambda':\Lambda][K'_p:K^*_p]}. \qquad (\text{B.8})
\end{aligned}$$

We now bound the denominator in (B.8). First note that

$$[K'_p : K^*_p] \le [\mathbf{H}'(\mathbb{Q}_p):\pi(\mathbf{H}(\mathbb{Q}_p))] \le M$$

where $M \ll 1$ is an absolute constant.

Bounding the term $[\Lambda':\Lambda]$ is far less trivial and relies on results in §5.12. Put $\widetilde{\Lambda} = \mathbf{H}'(\mathbb{Q}) \cap \pi(\mathbf{H}(\mathbb{Q}_p))K'(\infty)$. Then

$$[\Lambda':\Lambda] = [\Lambda':\widetilde{\Lambda}][\widetilde{\Lambda}:\Lambda] \le [\mathbf{H}'(\mathbb{Q}_p):\pi(\mathbf{H}(\mathbb{Q}_p))][\widetilde{\Lambda}:\Lambda] \le M[\widetilde{\Lambda}:\Lambda].$$

Finally the index $[\widetilde{\Lambda} : \Lambda]$ is controlled as in (5.13). We note that the quantities appearing on the right side of (5.13), in particular $\Sigma^\flat$, are the same for the group $\pi(\mathbf{H}(\mathbb{A}))$, which we used to obtain the bound in (B.8), as well as for $(g_Q, e, \ldots)^{-1}\pi(\mathbf{H}(\mathbb{A}))(g_Q, e, \ldots)$, which is used to define Y. Therefore, in view of the *equivalence of volume definitions* proposition in §5.12 we get

$$\text{spin genus}(Q) \gg \mathrm{vol}(Y)^\star$$

which is the claimed lower bound. □

By (B.2) we also know $\mathrm{vol}(Y) \asymp \mathrm{disc}(Y)^\star$. So it remains to discuss the genus of Q. For this we are making the following claim.

Claim. For any given T the number of equivalence classes of quadratic forms Q with $|\,\text{spin genus}(Q)| < T$ is $\ll T^\star$.

Proof. Let $X' \subset X$ be a compact set so that $\mu(X') > 0.9$ for any MASH measure μ; this exists by the corollary in §1.7. Suppose Q is a quadratic form with $|\,\text{spin genus}(Q)| < T$. We see that

$$\mathbf{G}(\mathbb{Q})(g_Q, e, \ldots)H_{\mathscr{D}} \cap X' \text{ is non-empty.}$$

We may assume that X' is invariant under $\mathrm{SO}(n,\mathbb{R})$; then the above also gives some $\iota(h) \in \iota(\mathbf{H}(\mathbb{A}_f))$ with $\mathbf{G}(\mathbb{Q})(g_Q, \iota(h)) \in X'$. By the correspondence between the spin genus of Q and the $g_Q^{-1}K'(\infty)g_Q$-orbits in Y we have found a quadratic form Q_h in the spin genus of Q for which the conjugating matrix g_{Q_h} can be chosen from a fixed compact subset $L \subset \mathrm{PGL}(n,\mathbb{R})$.

Let us now note that the spin genus of Q_h equals the spin genus of Q. So in view of (B.3) the MASH set Y_{Q_h} associated to the quadratic form Q_h has volume which is bounded above and below by powers of the volume of the MASH set Y_Q.

We will use the assumption $g_{Q_h} \in L$ in order to relate the size of the spin genus with the volume of the rational MASH set $Y'_h = \pi(\mathbf{H}_{Q_h}(\mathbb{Q})\backslash\mathbf{H}_{Q_h}(\mathbb{A}))$, where $\mathbf{H}_{Q_h}$ is the spin cover of the orthogonal group of Q_h (and we intentionally did not include g_{Q_h} in the definition). Indeed, since $g_{Q_h} \in L$, we get from (B.2) and (B.3) that

$$|\,\text{spin genus}(Q)| \asymp \mathrm{vol}(Y_{Q_h})^\star \asymp \mathrm{vol}(Y'_h)^\star \asymp \mathrm{disc}(Y'_h)^\star. \qquad (\text{B.9})$$

The definition of $\mathrm{disc}(Y'_h)$, see (B.1), gives

$$\mathrm{disc}(Y'_h) = \tfrac{\mathcal{D}(\mathbf{H}_{Q_h})}{\mathcal{E}(\mathbf{H}_{Q_h})} \prod_{v\in\Sigma} \|z_v\|_v = \tfrac{\mathcal{D}(\mathbf{H}_{Q_h})}{\mathcal{E}(\mathbf{H}_{Q_h})} \|z(h)\|_\infty,$$

where $z(h) \in \wedge^r \mathfrak{sl}_N$ is the primitive integral vector which is a rational multiple of $(z_v)_v$; and in the second equality we used the product formula. Note that $z(h) = f_1 \wedge \cdots \wedge f_r$ determines the group $\mathbf{H}_{Q_h}$ and hence the form Q_h (up to homotheties). In particular, since $\mathcal{D}(\mathbf{H}_{Q_h})/\mathcal{E}(\mathbf{H}_{Q_h}) \geq 1$ we get $\|z(h)\|_\infty \ll T^\star$. As there are only $\ll T^\star$ many integral vectors of norm $\ll T^\star$ in $\wedge^r \mathfrak{sl}_N$ we obtain the claimed estimate. □

We can now finish the proof of Lemma 3.3. Let us recall from the definitions that

$$\mathrm{genus}(Q) = \bigsqcup_i \mathrm{spin\ genus}(Q_i)$$

where $Q_i \in \mathrm{genus}(Q)$. Let Y_{Q_i} denote the MASH's which correspond to Q_i as above. Then by (B.3) we have spin genus$(Q_i) \asymp \mathrm{vol}(Y_i)^\star$. Note however that $\mathrm{vol}(Y_i) = \mathrm{vol}(Y_j)$ for all i, j since the corresponding algebraic stabilizers are the same; indeed $Y_i = Y_j h$ for some $h \in \mathrm{SO}(Q)$ and $\mathrm{SO}(Q)$ normalizes the algebraic stabilizer of Y_i. Suppose now genus$(Q) = S$. Then the above MASH sets all have the same volume $\mathrm{vol}(Y_i) = V$ which, in view of the above claim, gives $S \ll V^\star$ and finishes the proof of Lemma 3.3.

B.2. **Expressing the discriminant in terms of the volume.** Recall the notation from the proof of part (3) in the proposition[27] in §5.8. In particular, we fixed finitely many standard homomorphisms of $\mathbf{H}(F_v)$ into $\mathbf{G}(F_v)$ for any archimedean place v. Recall also that corresponding to the standard homomorphism $\mathrm{j}_0 = \mathrm{Ad}(g_0) \circ \mathrm{j}$ we have a Euclidean structure $\mathfrak{p}_0$ and that $\| \; \|_{\mathfrak{p}_0}$ denotes the corresponding Euclidean norm.

In this section we have fixed a compatible family of norms $\| \; \|_v$. Note that for any archimedean place v we have $\| \; \|_v \asymp \| \; \|_{\mathfrak{p}_0}$ with constants depending only on the dimension. Therefore, without loss of generality we may and will assume that $\{f_1, \ldots, f_r\}$ are chosen so that

$$1/c \leq \| \wedge^r \mathrm{j}_0(f_1 \wedge \cdots \wedge f_r)\|_{\mathfrak{p}_0} \leq c \tag{B.10}$$

for any archimedean place v where c is a universal constant.

Fix an archimedean place v, as in the proof of part (3) in the proposition in §5.8 we have

$$\begin{aligned} &u_1 \wedge \cdots \wedge u_r \\ &\quad = \tfrac{\|u_1\wedge\cdots\wedge u_r\|_{\mathfrak{p}_0}}{\|\wedge^r \mathrm{Ad}(g_0^{-1})(\wedge^r \mathrm{j}_0(f_1\wedge\cdots\wedge f_r))\|_{\mathfrak{p}_0}} \wedge^r \mathrm{Ad}(g_0^{-1})(\wedge^r \mathrm{j}_0(f_1 \wedge \cdots \wedge f_r)), \end{aligned} \tag{B.11}$$

where $\{u_1, \ldots, u_r\}$ is chosen as in there.

Note that $1/c \leq \| \wedge^r \mathrm{Ad}(g_0^{-1})(\wedge^r \mathrm{j}_0(f_1 \wedge \cdots \wedge f_r))\|_{\mathfrak{p}_\mathrm{j}} \leq c$, by (B.10). Hence we get

$$J_v \asymp \|u_1 \wedge \cdots \wedge u_r\|_{\mathfrak{p}_\mathrm{j}} \asymp \tfrac{1}{\|\wedge^r \mathrm{Ad}(g_0^{-1})(\wedge^r \mathrm{j}_0(f_1\wedge\cdots\wedge f_r))\|_{\mathfrak{p}_0}} \asymp \tfrac{1}{\|z_v\|_v} \tag{B.12}$$

where the implied constants are absolute.

[27] We note that the standing assumption in §5.8 was that $\mathbf{H}$ is F-simple. However, the proof of part (3) in the proposition in §5.8 works for the case of semisimple groups.

Therefore, in view of (5.6) and (5.7), in order to prove (B.2) we need to control the contribution from finite places.

We will use the notation from §5. Since $\mathbf{H}$ is simply connected we can write $\mathbf{H}$ as the direct product $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_k$ of its F-almost simple factors. Let F'_j/F be a finite extension so that $\mathbf{H}_j = \mathrm{Res}_{F'_j/F}(\mathbf{H}'_j)$ where $\mathbf{H}'_j$ is an absolutely almost simple F'_j-group for all $1 \le j \le k$. Then $[F'_j : F]$ is bounded by $\dim \mathbf{H}$. Let $\mathcal{H}'_j$ and L_j/F'_j be defined as in §5.4. Put

$$\mathcal{D}(\mathbf{H}) = \Big(\prod_j D_{L_j/F_j}^{\mathfrak{s}(\mathcal{H}'_j)} D_{F'_j}^{\dim \mathbf{H}'_j}\Big)^{1/2}. \tag{B.13}$$

For each j let ω'_j denote a differential form of top degree on $\mathbf{H}'_j$ and choose an F'_j-basis $\{f_1^{(j)}, \ldots, f_{r_j}^{(j)}\}$ for $\mathrm{Lie}(\mathbf{H}'_j)$ so that $\omega'_j(z^{(j)}) = 1$ where

$$z^{(j)} = f_1^{(j)} \wedge \cdots \wedge f_{r_j}^{(j)} \in \wedge^{r_j} \mathrm{Lie}(\mathbf{H}'_j).$$

We may and will work with the F-basis $\{f_1, \ldots, f_r\}$ for $\mathrm{Lie}(\mathbf{H})$ obtained from $\{f_i^{(j)}\}$ using the restriction of scalars, i.e., we assume fixed a basis $\{e_l^{(j)}\}$ for F'_j/F and write $f_i^{(j)}$ in this basis for each $1 \le i \le r_j$. As before put $z = f_1 \wedge \cdots \wedge f_r$ and let ω be a form of top degree on $\mathbf{H}$ so that $\omega(z) = 1$.

For each $v' \in \Sigma_{F'_j}$, let $\omega'_{j,v'}$ denote the form of top degree on $\mathbf{H}'_j(F'_{j,v'})$ induced by ω'_j. Similarly, for any $v \in \Sigma$ let ω_v denote the form of top degree on $\mathbf{H}(F_v)$ induced by ω.

Given $v \in \Sigma$ we define a form of top degree on $\mathbf{H}(F_v)$ by $\widetilde{\omega}_v := \big((\omega'_{j,v'})_{v'|v}\big)_j$. Since $\mathbf{H}(F_v)$ is naturally isomorphic to $\prod_j \prod_{v'|v} \mathbf{H}'_j(F'_{j,v'})$, it follows from the definitions that $\widetilde{\omega}_v(z) = 1$. Therefore, for every $v \in \Sigma$ we have $\widetilde{\omega}_v = \omega_v$.

Let $v \in \Sigma_{F,f}$; following our notation in §5.3, we denote by $|\omega_v|$ the measure induced on $\mathrm{Lie}(\mathbf{H}) \otimes F_v$ and abusing the notation the measure on $\mathbf{H}(F_v)$. Since $\omega_v(z) = 1$, the $\mathfrak{o}_v$-span of the $\{f_i\}$ has volume 1 with respect to $|\omega_v|$. Applying a suitable change of basis to the $\{f_i\}$, we may assume that there is an integral basis, $\{e_1, \ldots, e_{N^2-1}\}$, for $\mathfrak{sl}_N(\mathfrak{o}_v)$ with the property that each $\rho \circ \mathrm{Ad}(g_v^{-1}) \circ \iota(f_i) = c_i e_i$ for $1 \le i \le r$. Then, $\|z_v\|_v = \prod_i |c_i|_v$ where $\|\ \|_v$ denotes the compatible family of norms which we fixed before and $z_v = \rho \circ \mathrm{Ad}(g_v^{-1}) \circ \iota(z)$.

For every $v \in \Sigma_f$ let H'_v be the scheme theoretic closure of $\rho(g_v^{-1}\iota(\mathbf{H})g_v)$ in $\mathrm{SL}_N/\mathfrak{o}_v$. Then for each $v \in \Sigma_f$ we have

$$\mathsf{H}'_v(\mathfrak{o}_v) = \rho(g_v^{-1}\iota(\mathbf{H})g_v) \cap \mathrm{SL}_N(\mathfrak{o}_v).$$

Put $\mathsf{H}_v := \iota^{-1}(g_v\rho^{-1}(\mathsf{H}'_v)g_v^{-1})$ for any $v \in \Sigma_f$.

Recall that $K_v^* = \iota^{-1}(g_v\rho^{-1}(\mathrm{SL}_N(\mathfrak{o}_v))g_v^{-1})$; using the above notation we have $K_v^* = \mathsf{H}_v(\mathfrak{o}_v)$.

We write $\mathrm{Lie}(K_v^*) = D\rho \circ \mathrm{Ad}(g_v^{-1}) \circ \iota(\mathrm{Lie}(\mathbf{H}) \otimes F_v) \cap \mathfrak{sl}_N(\mathfrak{o}_v)$.

Let red_v denote reduction mod ϖ_v with respect to the scheme structure induced by H_v. In particular $\mathrm{red}_v K_v^* = K_v^*/(K_v^*)^{(1)}$ where $(K_v^*)^{(1)}$ is the first congruence subgroup of $\mathsf{H}_v(\mathfrak{o}_v)$. For $p_v \gg 1$, $(K_v^*)^{(1)}$ is the image under the exponential map of the first congruence subalgebra of $\mathrm{Lie}(K_v^*)$.

Let us recall that k_v is the residue field of F_v with $\mathrm{char}(k_v) = p_v$ and $\#k_v = q_v = p_v^l$ for some $l \le [F : \mathbb{Q}]$. Similarly $k'_{j,v'}$, is the residue field of $F'_{j,v'}$ and $\#k'_{j,v'} = q'_{j,v'} = p_v^{l_j}$.

With this notation, the above discussion implies that for $p_v \gg 1$ we have

$$\begin{aligned}\|z_v\|_v &= \textstyle\prod_i |c_i|_v = \left|\omega_v(c_1^{-1}f_1 \wedge \cdots \wedge c_r^{-1}f_r)\right|^{-1}\\ &= \left(|\omega_v|(\{u \in \mathrm{Lie}(\mathbf{H}) \otimes F_v : D\rho \circ \mathrm{Ad}(g_v^{-1}) \circ \iota(u) \in \mathfrak{sl}_N(\mathfrak{o}_v)\})\right)^{-1}\\ &= \left(|\omega_v|(\mathrm{Lie}(K_v^*))\right)^{-1}\\ &= \left(|\omega_v|(K_v^*)\right)^{-1}(\#(\mathrm{red}_v\, K_v^*) \cdot q_v^{-\dim \mathbf{H}}).\end{aligned}$$

In the last equality, we used the fact that exp is a measure preserving diffeomorphism on $\mathfrak{sl}_N^{(1)}(\mathfrak{o}_v)$ and $\mathfrak{h}[1]$ for all $p_v \gg 1$.

For small p_v, not covered above, exp is a measure preserving diffeomorphism on $\mathfrak{h}[m]$ for large enough m; see §6.5, in particular (5.4) and the discussion in that paragraph. Hence the contribution of these small primes is $\ll 1$.

Recall that $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_k$ is a direct product and $|\omega_v| = \prod_j \prod_{v'|v} |\omega'_{j,v'}|$. Therefore, the above, in view of (5.7), (B.1), and (B.12), implies

$$\prod_v \|z_v\| \asymp \tfrac{1}{D(\{F'_j\})}\,\mathrm{vol}(Y) \prod_{v \in \Sigma_f} \#(\mathrm{red}_v\, K_v^*) \cdot q_v^{-\dim \mathbf{H}}, \tag{B.14}$$

where $D(\{F'_j\}) = \prod_j D_{F'_j}^{\dim \mathbf{H}'_j/2}$.

B.3. **The upper bound.** Let the notation be as in §5.6, in particular, for all j and all $v' \in \Sigma_{F'_j,f}$ the parahoric subgroup $\mathcal{P}'_{j,v'}$ of maximum volume in $\mathcal{H}'_j(F'_{j,v'})$ is fixed as in that section. Abusing the notation, we denote the corresponding smooth $\mathfrak{o}'_{j,v'}$ group scheme by $\mathcal{P}'_{j,v'}$. Given a $v \in \Sigma_{F,f}$ put $\mathcal{P}_v := \prod_j \prod_{v'|v} \mathcal{P}_{j,v'}$. Define

$$\begin{aligned}\mathcal{E}(\mathbf{H}) : &= \textstyle\prod_{v \in \Sigma_{F,f}} \#\underline{\mathcal{P}_v}(k_v) q_v^{-\dim \mathbf{H}}\\ &= \textstyle\prod_j \prod_{v' \in \Sigma_{F'_j,f}} \#\underline{\mathcal{P}'_{j,v'}}(k'_{j,v'}) \cdot (q'_{j,v'})^{-\dim \mathbf{H}'_j}.\end{aligned} \tag{B.15}$$

Using Prasad's volume formula and the order of almost simple finite groups of Lie-type, see [50, Rmk. 3.11], we have the following. The quantity $\mathcal{E}(\mathbf{H})$ is a product of the values of the Dedekind zeta functions of F'_j and certain Dirichlet L-functions attached to L_j/F'_j at some integer points. In particular, $\mathcal{E}(\mathbf{H})$ is a positive constant depending on F'_j, L_j, and $\mathcal{H}'_j$. Moreover, [50, §2.5, §2.9] imply

$$\#\underline{\mathcal{P}'_{j,v'}}(k'_{j,v}) \cdot (q'_{j,v})^{-\dim \mathbf{H}'_j} < 1.$$

All together we have shown

$$0 < \mathcal{E}(\mathbf{H}) \leq 1. \tag{B.16}$$

Let $\Sigma_{\mathrm{ur}}^{\flat}$ denote the set of places $v \in \Sigma_{F,f}$ so that

- for all j and all $v' \in \Sigma_{F'_j,f}$ with $v'|v$ we have v' is unramified in L_j, and
- one of the following holds:
  - there is some j and some $v'|v$ so that the group $\mathbf{H}'_j$ is not quasisplit over F'_j, or
  - all $\mathbf{H}'_j$'s are quasisplit over $F'_{j,v'}$, hence $\mathbf{H}'_j$ is isomorphic to $\mathcal{H}'_j$ over $F'_{j,v'}$ for all j and all $v'|v$, but K_v^* is not hyperspecial.

It was shown in (5.12) that we have

$$\lambda_v|\omega_v|(K_v^*) \leq \tfrac{p_v}{p_v^2+1} \quad \text{for all } v \in \Sigma_{\mathrm{ur}}^\flat \tag{B.17}$$

if $q_v > 13$.

Let $\Sigma_{\mathrm{r}}^\flat$ be the set of places $v \in \Sigma_{F,f}$ so that there exists some j and some $v'|v$ in F'_j which ramified in L_j. Put $\Sigma^\flat := \Sigma_{\mathrm{ur}}^\flat \cup \Sigma_{\mathrm{r}}^\flat$.

Put $D(\{L_j\},\{F'_j\}) := \prod_j D_{L_j/F'_j}^{\mathfrak{s}(\mathcal{H}'_j)/2}$. Then, as we got (5.8) from (5.7), in view of (B.17) we get

$$\mathrm{vol}(Y) \gg D(\{L_j\},\{F'_j\}) D(\{F'_j\}) \prod_{v\in\Sigma_{\mathrm{ur}}^\flat} p_v. \tag{B.18}$$

Combining (B.14) and (B.18) we get the upper bound as follows:

$$\begin{aligned}
\mathrm{disc}(Y) &= \tfrac{\mathcal{D}(\mathbf{H})}{\mathcal{E}(\mathbf{H})} \prod_v \|z_v\|_v \\
&\asymp D(\{L_j\},\{F'_j\})\,\mathrm{vol}(Y) \frac{\prod_{v\in\Sigma_f} \#(\mathrm{red}_v\, K_v^*)\cdot q_v^{-\dim \mathbf{H}}}{\mathcal{E}(\mathbf{H})} \\
&\ll \mathrm{vol}(Y)^\star \frac{\prod_{v\in\Sigma_f} \#(\mathrm{red}_v\, \mathsf{H}_v(k_v))\cdot q_v^{-\dim \mathbf{H}}}{\mathcal{E}(\mathbf{H})} \\
&\ll \mathrm{vol}(Y)^\star \prod_{v\in\Sigma^\flat} \tfrac{\#(\mathrm{red}_v\, \mathsf{H}_v(k_v))}{\#\underline{\mathcal{P}_v}(k_v)} \quad \ll \quad \mathrm{vol}(Y)^\star,
\end{aligned}$$

where in the last inequality we used $\#(\mathrm{red}_v\, \mathsf{H}_v(k_v)) \leq q_v^\star$, and also the fact that for any $v \in \Sigma_{\mathrm{r}}^\flat$ we have $p_v | D(\{L_j\},\{F_j\})$.

B.4. **The lower bound.** We now turn to the lower bound. For this part we work with normalized volume forms. Fix the notation

$$\lambda_v|\omega_v| := \textstyle\prod_j \prod_{v'|v} \lambda'_{j,v'}\omega'_{j,v'};$$

see §5.6 for the notation on the right side of the above.

We will need the following. If M is a connected linear algebraic group over k_v, then

$$(q_v-1)^{\dim \mathsf{M}} \leq \#\mathsf{M}(k_v) \leq (q_v+1)^{\dim \mathsf{M}}; \tag{B.19}$$

see, e.g., [46, Lemma 3.5].

Given a parahoric subgroup P_v of $\mathbf{H}(F_v)$, let $\mathfrak{P}_v$ denote the smooth $\mathfrak{o}_v$ group scheme associated to it by Bruhat-Tits theory. Recall from §5.1 that P_v maps onto $\underline{\mathfrak{P}_v}(k_v)$. We also remark that since $\mathbf{H}$ is simply connected the k_v-group scheme $\underline{\mathfrak{P}_v}$ is connected; see [60, 3.5.3].

Let $v \in \Sigma_f$ for any j and any $v'|v$ we choose a parahoric subgroup $P'_{j,v'}$ which is minimal among those parahoric subgroups containing $\pi_{j,v'}K_v^*$; here $\pi_{j,v'}$ denotes the natural projection. Put

$$P_v := \prod_j \prod_{v'|v} P'_{j,v'}.$$

Then $K_v^* \subset P_v$.

We will prove the lower bound in a few steps. First we prove some local estimates, i.e., we bound terms appearing in the product on the right side of (B.14) for all $v \in \Sigma_f$. Taking the product of these estimates then we will get the lower bound.

1. *Step.* We have

$$\begin{aligned}(\text{B.20}) \qquad &\big(\lambda_v|\omega_v|(K_v^*)\big)^{-1}\big(\#(\mathrm{red}_v\, K_v^*)\cdot q_v^{-\dim \mathbf{H}}\big)\\ &= \big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v : K_v^*]\big(\#(\mathrm{red}_v\, K_v^*)\cdot q_v^{-\dim \mathbf{H}}\big)\\ &= \big(\lambda_v|\omega_v|(P_v)\big)^{-1}\tfrac{[P_v:P_v^{(1)}][P_v^{(1)}:(K_v^*)^{(1)}]}{[K_v^*:(K_v^*)^{(1)}]}\big(\#\big(\mathrm{red}_v\, K_v^*\big)\cdot q_v^{-\dim \mathbf{H}}\big)\\ &\overset{\text{by (3) in §5.1}}{=}\big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v^{(1)} : (K_v^*)^{(1)}](\#\underline{\mathfrak{P}_v}(k_w))\cdot q_v^{-\dim \mathbf{H}}),\end{aligned}$$

where $P_v^{(1)}$ and $(K_v^*)^{(1)}$ denote the first congruence subgroups defined using the $\mathfrak{o}_v$-scheme structures $\mathfrak{P}_v$ and H_v, respectively.

2. *Step.* In this step we will estimate the contribution coming from the product $\prod_{v\in\Sigma_f}\big(\#\underline{\mathfrak{P}_v}(k_v)\big)\cdot q_v^{-\dim\mathbf{H}}$.

The fact that for any $v\in\Sigma_{\mathrm{r}}^\flat$ we have $p_v|D(\{L_j\},\{F_j\})$ together with (B.18) implies $\#\Sigma^\flat \ll \log(\mathrm{vol}(Y))$.

Now since $\underline{\mathfrak{P}_v}$ is connected, we can use (B.19) together with the definition of $\mathcal{E}(\mathbf{H})$ and get

$$\begin{aligned}\prod_{v\in\Sigma_f}\Big(\#\underline{\mathfrak{P}_v}(k_v)\cdot q_v^{-\dim\mathbf{H}}\Big) &= \prod_{v\in\Sigma_f}\big(\#\underline{\mathcal{P}_v}(k_v)\cdot q_v^{-\dim\mathbf{H}}\big)\prod_{v\in\Sigma^\flat}\tfrac{\#\underline{\mathfrak{P}_v}(k_v)\cdot q_v^{-\dim\mathbf{H}}}{\#\underline{\mathcal{P}_v}(k_v)\cdot q_v^{-\dim\mathbf{H}}}\\ (\text{B.21}) \qquad &\gg \mathcal{E}(\mathbf{H})\big(\log\mathrm{vol}(Y)\big)^{-\kappa_{16}}\end{aligned}$$

for some $\kappa_{16}>0$ depending only on F and $\mathbf{G}$.

3. *Step.* We will now get a control over $\big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v^{(1)} : (K_v^*)^{(1)}]$.

We claim that there exists some $0<\kappa_{17}<1$, depending only on $\dim_F\mathbf{G}$, so that for all $v\in\Sigma_f$ at least one of the following holds: either

$$\begin{aligned}&\big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v^{(1)} : (K_v^*)^{(1)}] \geq \big(\lambda_v|\omega_v|(K_v^*)\big)^{-\kappa_{17}}\\ (\text{B.22}) \qquad &= \big(\lambda_v|\omega_v|(P_v)\big)^{-\kappa_{17}}\left(\tfrac{(\#\underline{\mathfrak{P}_v}(k_v))[P_v^{(1)}:(K_v^*)^{(1)}]}{\#(\mathrm{red}_v\, K_v^*)}\right)^{\kappa_{17}},\end{aligned}$$

or $v\in\Sigma_{\mathrm{r}}^\flat$ and

$$(\text{B.23}) \qquad \big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v^{(1)} : (K_v^*)^{(1)}] \geq 1 \geq \big(\lambda_v|\omega_v|(K_v^*)\big)^{-\kappa_{17}}p_v^{-1/2}.$$

Let us first recall that $\lambda_v|\omega_v|(P_v)\leq 1$; see [50, Prop. 2.10]. Therefore, if $P_v = K_v^*$, then (B.22) holds for any $0<\kappa_{17}<1$. In particular, if $v\notin\Sigma_f\setminus\Sigma^\flat$, then (B.22) holds for any $0<\kappa_{17}<1$.

Recall that $K_v^*\subset P_v$. Assume first that $(K_v^*)^{(1)}\subsetneq P_v^{(1)}$.

Then, since $P_v^{(1)}$ is a pro-p_v group we have $[P_v^{(1)} : (K_v^*)^{(1)}]\geq p_v$. We again note that by [50, Prop. 2.10] we have

$$(\text{B.24}) \qquad q_v^{-\dim\mathbf{H}} \leq \lambda_v|\omega_v|(P_v) \leq 1.$$

Therefore, (B.22) follows if we show

$$\begin{aligned}\#(\mathrm{red}_v\, K_v^*)^{\kappa_{17}}[P_v^{(1)} : (K_v^*)^{(1)}]^{1-\kappa_{17}} &\geq \big(\#\underline{\mathfrak{P}_v}(k_v)\big)^{\kappa_{17}}\\ &\geq \big(\lambda_v|\omega_v|(P_v)\big)^{1-\kappa_{17}}\big(\#\underline{\mathfrak{P}_v}(k_v)\big)^{\kappa_{17}}.\end{aligned}$$

The second inequality holds for any $0<\kappa_{17}<1$ in view of the upper bound in (B.24). The first inequality follows from the upper bound in (B.19) and our assumption $[P_v^{(1)} : (K_v^*)^{(1)}]\geq p_v$ if we take $0<\kappa_{17}<1$ to be small enough.

Similarly, if $\lambda_v|\omega_v|(P_v) \leq 2/p_v$, then (B.22) becomes

$$\begin{aligned}\big(\lambda_v|\omega_v|(P_v)\big)^{-1+\kappa_{17}}[P_v^{(1)} : (K_v^*)^{(1)}]^{1-\kappa_{17}} &\geq (p_v/2)^{-1+\kappa_{17}}\\ &\geq \left(\tfrac{\#\underline{\mathfrak{P}_v}(k_v)}{\#(\mathrm{red}_v\, K_v^*)}\right)^{\kappa_{17}}.\end{aligned}$$

Since $\frac{\#\underline{\mathfrak{P}_v}(k_v)}{\#(\mathrm{red}_v\, K_v^*)} = p_v^\star$, the above estimate, and hence (B.22), hold for all small enough κ_{17} provided that $\lambda_v|\omega_v|(P_v) \leq 2/p_v$.

In view of these observations and [50, Prop. 2.10] we get that (B.22) holds unless $P_v^{(1)} = (K_v^*)^{(1)}$ and we are in one of the following cases:

- $v \in \Sigma_{\mathrm{ur}}^\flat$, $\mathbf{H}$ is F_v-quasisplit, and P_v is a hyperspecial parahoric subgroup, or
- $v \in \Sigma_{\mathrm{r}}^\flat$ and P_v is a special parahoric subgroup.

First note that under the assumption $P_v^{(1)} = (K_v^*)^{(1)}$, the estimate in (B.23) follows from (B.24) so long as we choose κ_{17} small enough. This establishes the claim for $v \in \Sigma_{\mathrm{r}}^\flat$.

Therefore, we may now assume that $P_v^{(1)} = (K_v^*)^{(1)}$, $v \in \Sigma_{\mathrm{ur}}^\flat$, and P_v is hyperspecial. We claim that these imply $P_v = K_v^*$ if $p_v \gg 1$ which then implies that (B.22) holds for any $0 < \kappa_{17} < 1$.

Assume $p_v \gg 1$ is large enough, so that the exponential map is a diffeomorphism from $\mathfrak{sl}_N^{(1)}(\mathfrak{o}_v)$ onto $\mathrm{SL}_N^{(1)}(\mathfrak{o}_v)$. Our assumption $P_v^{(1)} = (K_v^*)^{(1)}$ implies that $\mathrm{Lie}(K_v^*) = \mathrm{Lie}(\mathfrak{P}_v)$. Since P_v is hyperspecial, we have $P_v/P_v^{(1)}$ is the k_v-points of a semisimple group which is generated by unipotent subgroups. These unipotent subgroups are reduction mod ϖ_v of unipotent subgroups of P_v [60, §3.5.1]. In view of our assumption $p_v \gg 1$, unipotent subgroups of P_v are obtained using the exponential map from $\mathrm{Lie}(\mathfrak{P}_v) = \mathrm{Lie}(K_v^*)$. Hence K_v^* surjects onto $P_v/P_v^{(1)}$. Since $P_v^{(1)} = (K_v^*)^{(1)}$, this implies $P_v = K_v^*$ as we claimed.

4. *Step.* We now conclude the proof of the lower bound. We use the notation

$$J_\infty = \Big(\prod\nolimits_{v\in\Sigma_\infty} J_v\Big)^{-1}.$$

Recall that in view of part (iii) of the proposition in §5.8 we have $J_v \ll 1$ for all the archimedean places v. Taking the product of (B.22) over all $v \in \Sigma_f$, using the fact that $p_v|D(\{L_j\},\{F_j'\})$ for all $v \in \Sigma_{\mathrm{r}}^\flat$ and (B.23), and arguing as in §5.10 we get the lower bound as follows:

$$\begin{aligned}
\mathrm{disc}(Y) &= \tfrac{\mathcal{D}(\mathbf{H})}{\mathcal{E}(\mathbf{H})}\prod_v \|z_v\|_v\\
&\asymp D(\{L_j\},\{F_j'\})\,\mathrm{vol}(Y)\,\frac{\prod_{v\in\Sigma_f}\#(\mathrm{red}_v\,K_v^*)\cdot q_v^{-\dim\mathbf{H}}}{\mathcal{E}(\mathbf{H})} && \text{(B.14)}\\
&\gg \mathcal{D}(\mathbf{H})\prod_{v\in\Sigma}\big(\lambda_v|\omega_v|(K_v^*)\big)^{-1}\frac{\prod_{v\in\Sigma_f}\#(\mathrm{red}_v\,K_v^*)\cdot q_v^{-\dim\mathbf{H}}}{\mathcal{E}(\mathbf{H})} && \text{(5.8)}\\
&\gg \mathcal{D}(\mathbf{H})J_\infty\frac{\prod_{v\in\Sigma_f}\big(\lambda_v|\omega_v|(K_v^*)\big)^{-1}\big(\#(\mathrm{red}_v\,K_v^*)\cdot q_v^{-\dim\mathbf{H}}\big)}{\mathcal{E}(\mathbf{H})}\\
&\gg \tfrac{\mathcal{D}(\mathbf{H})}{(\log\mathrm{vol}(Y))^{\kappa_{16}}}J_\infty\prod_{v\in\Sigma_f}\big(\lambda_v|\omega_v|(P_v)\big)^{-1}[P_v^{(1)} : (K_v^*)^{(1)}] && \text{(B.20), (B.21)}
\end{aligned}$$

$$\gg \tfrac{\mathcal{D}(\mathbf{H})^{1/2}}{(\log \operatorname{vol}(Y))^{\kappa_{16}}} J_\infty \prod_{v\in\Sigma_f} \big(\lambda_v|\omega_v|(K_v^*)\big)^{-\kappa_{17}} \qquad \text{(B.22), (B.23)}$$

$$\gg \tfrac{\mathcal{D}(\mathbf{H})^{\kappa_{17}}}{(\log \operatorname{vol}(Y))^{\kappa_{16}}} \prod_{v\in\Sigma} \big(\lambda_v|\omega_v|(K_v^*)\big)^{-\kappa_{17}} \qquad J_\infty, \mathcal{D}(\mathbf{H}) \gg 1$$

$$\gg \operatorname{vol}(Y)^{\kappa_{18}} \qquad \text{(5.8).}$$

The proof of the lower bound in (B.2) is now complete.

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