DIAGONAL ACTIONS IN POSITIVE CHARACTERISTIC

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Abstract

We prove positive-characteristic analogues of certain measure rigidity theorems in characteristic 0*. More specifically, we give a classification result for positive entropy measures on quotients of* SL_d *and a classification of joinings for higher-rank actions on simply connected, absolutely almost simple groups.*

Contents

1. Introduction

Let G be a locally compact, second countable group, and let Γ be a lattice in G. Put $X = G/\Gamma$. A subset $S \subset X$ is called *homogeneous* if there exists a closed subgroup $\Sigma < G$ and some $x \in X$ such that Σx is closed and supports a Σ-invariant probability measure. A probability measure μ on X is called *homogeneous* if $\operatorname{supp}\mu$ is homogeneous and μ is the Σ-invariant probability measure on $\operatorname{supp}\mu$.

Let A be a closed abelian subgroup of G. An A-invariant probability measure μ on G/Γ will be called *almost homogeneous* if

$$\mu = \int_{A/A\cap\Sigma} a_*\nu \,\mathrm{d}a, \tag{1.1}$$

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where

(1) $\Sigma \subset G$ is a closed subgroup such that $A/A \cap \Sigma$ is compact,

(2) ν is a homogeneous measure stabilized by Σ, and

(3) $\mathrm{d}a$ is the Haar probability measure on the group $A/A \cap \Sigma$.

Let K be a global function field, that is, a finite extension of the field of rational functions in one variable over a finite field $\mathbb{F}_p$. For any place w of K, we let K_w denote the completion of K at w, and we let $\mathfrak{o}_w$ be the ring of integers in K_w. As in the case of number fields, the field K embeds diagonally in the restricted product $\prod_w' K_w$. Given a place v, we put

$$\mathcal{O}_v = K \cap \prod_{w \neq v} \mathfrak{o}_w$$

to be the ring of v-integers in K.

For the rest of this paper, we will assume that a place v of K is fixed and we will put

$$k := K_v, \qquad \mathfrak{o} := \mathfrak{o}_v, \qquad \text{and} \qquad \mathcal{O} := \mathcal{O}_v.$$

Recall that we may and will identify k with $\mathbb{F}_q((\theta^{-1}))$, the field of Laurent series over the finite field $\mathbb{F}_q$; after this identification, we have $\mathfrak{o} = \mathbb{F}_q[[\theta^{-1}]]$ (see [38, Chapter 1]).

The most familiar case is the one where $K = \mathbb{F}_q(\theta)$, the field of rational functions in one variable with coefficients in $\mathbb{F}_q$. Then if we choose the valuation v coming from θ^{-1}, we have that $\mathcal{O}_v = \mathbb{F}_q[\theta]$ is the polynomial ring.

1.1. Positive entropy classification for measures on quotients of SL_d

Let $G = \mathrm{SL}(d,k)$, and let $\Gamma < G$ be an *inner-type* lattice in G (see Section 2.4 for the definition and discussion of inner-type lattices; for an explicit example, the reader may let $\Gamma = \mathrm{SL}(d,\mathcal{O})$). Let $X := G/\Gamma$. Furthermore, we let A be the full diagonal subgroup of $\mathrm{SL}(d,k)$. *Throughout the present article, we always assume that $d > 2$.*

Given an A-invariant probability measure μ, we let $\mathsf{h}_\mu(a)$ denote the measure-theoretic entropy of $a \in A$. (We note that the following theorem is a positive-characteristic analogue of the result of [11].)

THEOREM 1.1

Suppose that μ is an A-invariant ergodic probability measure on X, and further assume that $\mathsf{h}_\mu(a) > 0$ for some $a \in A$. Then μ is almost homogeneous.

The conclusion of Theorem 1.1 cannot be strengthened to say that μ is homogeneous. In fact, $K = \mathbb{F}_q(\theta)$ has many subfields K' (without a bound on $[K : K']$).

Defining k' to be the closure of K' in k, one could take the measure ν to be the Haar measure on the closed orbit $\Sigma\Gamma$ for $\Sigma = \mathrm{SL}(d, k')$, and μ could be as in (1.1) since $A/(A \cap \Sigma)$ is compact.

1.2. Joining classification

In 1967, Furstenberg [19] introduced the following notion that has since become a central tool in ergodic theory. Suppose that we are given two measure-preserving systems for a group S acting on Borel probability spaces (X_i, m_i) for $i = 1, 2$. A *joining* is a Borel probability measure μ on $X_1 \times X_2$ such that the pushforwards satisfy $(\pi_i)_*\mu = m_i$ for $i = 1, 2$ and are invariant under the diagonal action on $X_1 \times X_2$—that is, $s.(x_1, x_2) = (s.x_1, s.x_2)$ for all $s \in S$ and $(x_1, x_2) \in X_1 \times X_2$.

We give a classification of ergodic joinings in the following setting. Let $\mathbf{G}_i$ be connected, simply connected, absolutely almost simple groups defined over k for $i = 1, 2$. Put $G_i = \mathbf{G}_i(k)$, let Γ_i be a lattice in G_i, and define $X_i = G_i/\Gamma_i$ for $i = 1, 2$. Denote by m_i the Haar measure on X_i. Let $\lambda_i : \mathbf{G}_m^2 \to \mathbf{G}_i$ be two algebraic homomorphisms with finite kernel defined over k, and put $\mathbf{A}_i = \lambda_i(\mathbf{G}_m^2)$. We define the notion of joining as above using these monomorphisms. Let $\mathbf{A} = \{(\lambda_1(t), \lambda_2(t)) : t \in \mathbf{G}_m^2\}$, and let $A = \mathbf{A}(k)$. (The following theorem is a positive-characteristic analogue of the work of the first and second authors [12]; see also [16] for stronger results in the characteristic 0 setting.)

THEOREM 1.2

Assume that $\mathrm{char}(k) \neq 2, 3$. *Suppose that* $\mathbf{G}_i$, $\mathbf{A}_i$, *and* X_i *are as above for* $i = 1, 2$. *Let* μ *be an ergodic joining of the action of* A_i *on* (X_i, m_i) *for* $i = 1, 2$. *Then* μ *is an algebraic joining. That is, one of the following holds:*

(1) $\mu = m_1 \times m_2$ *is the trivial joining, or*

(2) μ *is almost homogeneous, and moreover, the group* Σ *appearing in the definition of an almost homogeneous measure satisfies the following:*
- $\pi_i(\Sigma) = G_i$ *for* $i = 1, 2$, *and*
- $\ker(\pi_i|_\Sigma)$ *is contained in the finite group* $Z(G_1 \times G_2)$ *for* $i = 1, 2$.

It is also worth mentioning that even for joinings, in general, virtual homogeneity cannot be improved to homogeneity. Indeed, let k/k' be a Galois extension of degree 2 with the nontrivial Galois automorphism τ. Let $\mathbf{G}_1 = \mathbf{G}_2 = \mathrm{SL}_3$, and let $\Gamma_1 = \Gamma$ and $\Gamma_2 = \tau(\Gamma)$ for a lattice $\Gamma \subset \mathrm{SL}(3, k)$. Let $\lambda_1 = \lambda_2$ be the monomorphism $(t, s) \mapsto \mathrm{diag}(t, s, (ts)^{-1})$. The measure ν could be the Haar measure on the closed orbit $\Sigma(\Gamma_1 \times \Gamma_2)$ of $\Sigma = \{(g, \tau(g)) : g \in \mathrm{SL}(3, k)\}$ and μ could be as in (1.1).

1.3. Main difference to the characteristic 0 *setting*

In the present article, we apply the high entropy method that was developed in the characteristic 0 setting in a series of papers (see, e.g., [9]–[11], [14]), and for Theorem 1.1 we also apply the low entropy method (see, e.g., [11], [13], [25]). These arguments crucially use leafwise measures for the root subgroups (or more generally the coarse Lyapunov subgroups), which are locally finite measures on unipotent subgroups (for a comprehensive treatment of leafwise measures, see [14]).

Suppose that we were able, using the above tools, to show that the leafwise measures on the coarse Lyapunov subgroups have some invariance. Then, using Poincaré recurrence along A, one could show that the invariance group has arbitrarily large and arbitrarily small elements. The key difference lies in the next step of the argument. In the characteristic 0 setting, a closed subgroup of a unipotent group containing arbitrarily small and arbitrarily large elements has to contain a 1-parameter subgroup—and hence the leafwise measures for the 1-parameter subgroup have to be Haar, which gives unipotent invariance for the measure under consideration.

In the positive-characteristic world this is very far from being true. In fact, using a fairly direct adaptation of the methods used in [11], [12] and elsewhere, one can find almost surely an unbounded subgroup of a unipotent group that has positive Hausdorff dimension which again preserves the leafwise measure. However, as there are uncountably many such subgroups and since these may vary from one point to another, it is not clear how to continue from this by purely dynamical methods.

Decomposing the measure μ according to the Pinsker σ-algebra $\mathcal{P}_a$ (for some $a \in A$), we find a subgroup of G that preserves the conditional measure on an atom for $\mathcal{P}_a$ and has a semisimple Zariski closure. To classify such subgroups, we use a result of Pink [29] (see also [23] for related results by Larsen and Pink). This allows us to deduce invariance under the group of points of a semisimple subgroup for some local subfield. After this, we use a measure classification result in [28] by Golsefidy and the third author as a replacement of Ratner's measure classification theorem in [32] and [33], extended to the S-arithmetic setting by Ratner [33] (resp., Margulis and Tomanov [27]).

We note that analogues of Ratner's measure rigidity theorems for general unipotent flows in positive-characteristic settings are not yet known. Some special cases have been investigated, specifically in [28], which we use in our proof, and an earlier work [8]. Finally, we note that ideally one would like to have a result similar to [15] in the setting at hand. A general treatment as in [15] will likely require more subtle algebraic considerations.

2. Notation

2.1

Throughout this article, K denotes a global function field. We let v be a place in K, fixed once and for all. Denote by $\mathcal{O}$ the ring of v-integers in K. Put $k := K_v$, the completion of K at v. Then k is identified with $\mathbb{F}_q((\theta^{-1}))$, the field of Laurent series over the finite field $\mathbb{F}_q$ where q is a power of the prime number $p = \mathrm{char}(K)$. We denote by $\mathfrak{o}$ the ring of integers in k. Then $\mathfrak{o} = \mathbb{F}_q[[\theta^{-1}]]$ and the maximal ideal $\mathfrak{m}$ in $\mathfrak{o}$ equals $\theta^{-1}\mathfrak{o}$. The norm on k will be denoted by $|\cdot|_v$, or simply by $|\cdot|$; note that with our notation we have $|\theta|_v > 1$. With our normalizations, $\log_q(|r|)$ is the v-valuation of $r \in k$. Unless explicitly mentioned otherwise, a subfield $k' \subset k$ is always an infinite and closed subfield of k; hence, k/k' is a finite extension.

2.2

Let $\mathbf{G}$ be a connected, simply connected, semisimple k-algebraic group. Put $G = \mathbf{G}(k)$. We always assume that $\mathbf{G}$ is k-isotropic. Next, fix a maximal, k-split, k-torus $\mathbf{S}$ of $\mathbf{G}$. We will always assume that $\mathbf{A} = \mathbf{S}$, in the case of Theorem 1.1, and that $\mathbf{A}_i$ is contained in $\mathbf{S}_i$, for $i = 1, 2$, in the case of Theorem 1.2.

Let ${}_k\Phi$ denote the set of relative roots ${}_k\Phi(\mathbf{S}, \mathbf{G})$; this is a (possibly not reduced) root system (see [1, Theorem 21.6]). Let ${}_k\Phi^{\pm}$ denote positive and negative roots with respect to a fixed ordering on ${}_k\Phi$. Recall from [1, Remark 2.17, Proposition 21.9, Theorem 21.20] that for any $\alpha \in {}_k\Phi$ there exists a unique affine k-split unipotent k-subgroup $\mathbf{U}_{(\alpha)}$ which is normalized by $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$, the centralizer of $\mathbf{S}$, and its Lie algebra is $\mathfrak{g}_{(\alpha)} := \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. Here, as usual, for a root $\beta \in {}_k\Phi$ we let $\mathfrak{g}_\beta$ be the subspace in the Lie algebra on which $\mathbf{S}$ acts by the root β.

A subset $\Psi \subset {}_k\Phi$ is said to be *closed* if $\alpha \in \Psi$ and $\frac{1}{2}\alpha \in \Phi$ imply that $\frac{1}{2}\alpha \in \Psi$, and if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in {}_k\Phi$ imply that $\alpha + \beta \in \Psi$. A subset $\Psi \subset {}_k\Phi$ is said to be *positively closed* if it is closed and is contained in ${}_k\Phi^+$ for some ordering of the root system. For any positively closed subset $\Psi \subset {}_k\Phi$ there exists a unique affine k-split unipotent k-subgroup $\mathbf{U}_\Psi$ which is normalized by $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$, and its Lie algebra is the sum of $\{\mathfrak{g}_{(\alpha)} : \alpha \in \Psi\}$. Moreover, $\mathbf{U}_\Psi$ is generated by $\{\mathbf{U}_{(\alpha)} : \alpha \in \Psi \setminus 2\Psi\}$—that is, $\mathbf{U}_\Psi$ is k-isomorphic as a k-variety to $\prod_{\alpha \in \Psi \setminus 2\Psi} \mathbf{U}_{(\alpha)}$, where the product can be taken in any order (see [1, Proposition 21.9 and Theorem 21.20].

If $\Psi = \{\alpha\}$ and no multiple of α is a root, then we simply write $\mathbf{U}_\alpha$ for $\mathbf{U}_\Psi$. We also write $U_\Psi = \mathbf{U}_\Psi(k)$ for a positively closed subset $\Psi \subset {}_k\Phi$. Given a subset $E \subset G$, we let $\langle E \rangle$ denote the closed (in the Hausdorff topology) group generated by E. For each $\alpha \in {}_k\Phi$, we fix a collection of 1-parameter subgroups $\{u_{\alpha,i} : 1 \le i \le d_\alpha\}$ generating $U_{(\alpha)}$ and we define $U_{(\alpha)}[R]$ to be the compact group generated by $\{u_{\alpha,i}(r) : |r|_v < R, 1 \le i \le d_\alpha\}$. For any positively closed $\Psi \subset \Phi$, we put

$$U_\Psi[R] = \big\langle \big\{ U_{(\alpha)}[R] : (\alpha) \subset \Psi \big\} \big\rangle.$$

Given $a \in A$, we put

$$W_G^{\pm}(a) = \big\{ g \in G : \lim_{k \to \pm\infty} a^{-k} g a^k = \mathrm{id} \big\} \tag{2.1}$$

to be the expanding (resp., contracting) horospherical subgroup corresponding to a.

2.3

Let ${}_k\Phi(\mathbf{A}, \mathbf{G})$ denote the set of roots of $\mathbf{A}$, that is, the characters for the adjoint action of $\mathbf{A}$ on the Lie algebra of $\mathbf{G}$. We consider $\Psi \subset {}_k\Phi(\mathbf{A}, \mathbf{G})$ to be positively closed if

$$\big\{ \alpha \in \Phi(\mathbf{S}, \mathbf{G}) : \alpha | A \in \Psi \big\} \tag{2.2}$$

is positively closed in the sense of Section 2.2, and we set

$$V_\Psi := \prod_{\alpha|_A \in \Psi} U_{(\alpha)}$$

for any positively closed subset $\Psi \subset {}_k\Phi(\mathbf{A}, \mathbf{G})$. We also let $\mathbf{V}_\Psi$ denote the underlying algebraic group. An important special case is when $\Psi = [\alpha] = \{ r\alpha \in {}_k\Phi(\mathbf{A}, \mathbf{G}) : r > 0 \}$ for some $\alpha \in {}_k\Phi(\mathbf{A}, \mathbf{G})$. In this case, $V_{[\alpha]}$ is called a *coarse Lyapunov subgroup*.

2.4. Inner-type lattices in $\mathrm{SL}(d, k)$

Recall that in Theorem 1.1 we assumed that Γ is an inner-type lattice in $\mathrm{SL}(d, k)$; we recall the definition here. Let D be a division algebra of dimension s^2 over K, and let $B = \mathrm{Mat}_r(D)$ be a central simple algebra over K; we assume that $d = rs$. Let Ω be any field extension of K so that $B \otimes_K \Omega \simeq \mathrm{Mat}_d(\Omega)$—one can always find a finite separable extension of K with this property. Define the *reduced norm* $\mathrm{Nrd}_B : B \to \Omega$ of B by $\mathrm{Nrd}_B(g) := \det(g \otimes 1)$. Then $\mathrm{Nrd}_B(g) \in K$ for all $g \in B$ and $\mathrm{Nrd}_B(g)$ is independent of the choice of the splitting field Ω and the implicit isomorphism which we fixed. More generally (see, e.g., [7, Section 22]),

$$\det(g \otimes 1 - \xi \mathrm{id}) \in K[\xi] \quad \text{for every } g \in B. \tag{2.3}$$

We now use B to define a K-group which is isomorphic to SL_d over the algebraic closure $\bar{K}$ of K. Fix a K-basis $\mathcal{C}$ for D, and consider the (left) regular representation ρ of D into $\mathrm{Mat}_{s^2}(K)$; that is, $g \in D$ is sent to the matrix corresponding to $y \mapsto gy$. If we express ρ in the basis $\mathcal{C}$, we get a system $\{ f_\ell(g_{ij}) = 0 \}$ of *linear* equations in entries g_{ij} with coefficients in K that together define the image of ρ. We identify $\mathrm{Mat}_{rs^2}(K)$ with $\mathrm{Mat}_r(\mathrm{Mat}_{s^2}(K))$ and we let B' be the subset of $\mathrm{Mat}_{rs^2}(K)$ consisting of elements g_{ij}^{cd} for $1 \le i, j \le s^2$ and $1 \le c, d \le r$ satisfying $\{ f_\ell(g_{ij}^{cd}) = 0 \}$ for all $1 \le c, d \le r$. Then ρ identifies B and B'. Moreover, in view of the above discussion on Nrd_B, there exists a polynomial h with coefficients in K so that $\mathrm{Nrd}_B(g) = h(\rho(g^{cd}))$ for all $g \in B$ (see [36] and [30, Chapter 2] for a similar discussion and construction).

For any K-algebra Υ, define

$$\mathrm{SL}_{1,B}(\Upsilon) := \{g \in \mathrm{Mat}_{rs^2}(\Upsilon) : f_\ell(g_{ij}^{cd}) = 0, h(g_{ij}^{cd}) = 1\}.$$

If Ω is any field extension of K so that $B \otimes_K \Omega \simeq \mathrm{Mat}_d(\Omega)$, then $\mathrm{SL}_{1,B}(\Omega)$ is isomorphic to $\mathrm{SL}(d, \Omega)$. In particular, $\mathrm{SL}_{1,B}(\bar{K})$ is isomorphic to $\mathrm{SL}(d, \bar{K})$. A group so defined is called an *inner K-form* of SL_d.

Assume now that B is a central simple algebra over K as above; further, assume that it satisfies $B \otimes_K k \simeq \mathrm{Mat}_d(k)$. For every place w of K, define

$$\mathrm{SL}_{1,B}(\mathfrak{o}_w) := \mathrm{SL}_{1,B}(K_w) \cap \mathrm{GL}_{rs^2}(\mathfrak{o}_w).$$

Recall that $\mathrm{SL}_{1,B}(K)$ diagonally embeds in the restricted (with respect to $\mathrm{SL}_{1,B}(\mathfrak{o}_w)$) product $\prod'_w \mathrm{SL}_{1,B}(K_w)$. Put

$$\Lambda_B = \{\gamma \in \mathrm{SL}_{1,B}(K) : \gamma \in \mathrm{SL}_{1,B}(\mathfrak{o}_w) \text{ for all } w \neq v\}. \tag{2.4}$$

Then Λ_B is a lattice in $\mathrm{SL}(d, k)$ (see, e.g., [26, Chapter I, Section 3]). We will call a subgroup $\Gamma < \mathrm{SL}(d, k)$ a lattice of *inner type* if there exists a central simple algebra B over K so that Γ is commensurable to Λ_B.

3. Preliminary results

3.1. Algebraic structure of compact subgroups of semisimple groups

Given a variety $\mathbf{M}$ which is defined over k. there are two topologies on $\mathbf{M}(k)$, the set of k-points of $\mathbf{M}$; namely, the Zariski topology and the topology arising from the local field k. We will refer to the latter as the *Hausdorff topology*.

The following theorems are very special cases of the work of Pink [29] which play an important role in our study. Roughly speaking, they assert that compact and Zariski-dense subgroups of semisimple groups have an algebraic description.

THEOREM A.1 ([29, Theorem 0.2, Theorem 7.2])
Suppose that $\mathcal{Q} \subset \mathrm{SL}(2, k)$ is a compact and Zariski-dense subgroup. Further, assume that

$$\mathcal{Q} = \langle \{g \in \mathcal{Q} : g \text{ is a unipotent element}\} \rangle. \tag{3.1}$$

Let k'' be the closed field of quotients generated by $\{\mathrm{tr}(\rho(g)) : g \in \mathcal{Q}\}$, where ρ is the unique irreducible subquotient of the adjoint representation of PGL_2, and set

$$k' := \begin{cases} k'' & \textit{if } \operatorname{char}(k) \neq 2, \\ \{c : c^2 \in k''\} & \textit{if } \operatorname{char}(k) = 2. \end{cases} \tag{3.2}$$

Then there is a k-isomorphism (unique up to unique isomorphism)

$$\varphi : \mathrm{SL}_2 \times_{k'} k \to \mathrm{SL}_2,$$

so that $\mathcal{Q}$ is an open subgroup of $\varphi(\mathrm{SL}(2, k'))$.

Proof
Denote by $\bar{\mathcal{Q}}$ the image of $\mathcal{Q}$ under the natural map from SL_2 to PGL_2. Then $\bar{\mathcal{Q}}$ is Zariski-dense in PGL_2. By [29, Theorem 0.2], there exist

- a subfield $k' \subset k$,
- an absolutely simple adjoint group $\mathbf{L}$ defined over k', and
- a k-isogeny $\phi : \mathbf{L} \times_{k'} k \to \mathrm{PGL}_2$, whose derivative vanishes nowhere,

where k' is unique, and where $\mathbf{L}$ and ϕ are unique up to unique isomorphism, so that the following conditions hold.

- We have $\bar{\mathcal{Q}} \subset \varphi(\mathbf{L}(k'))$ (see [29, Theorem 3.6]).
- Let $\widetilde{\mathbf{L}}$ denote the simply connected cover of $\mathbf{L}$, and let $\widetilde{\phi}$ be the induced isogeny from $\widetilde{\mathbf{L}} \times_{k'} k$ to SL_2. Then any compact subgroup $\mathcal{Q}' \subset \widetilde{\phi}(\widetilde{\mathbf{L}}(k'))$ which is Zariski-dense and normalized by $[\bar{\mathcal{Q}}, \bar{\mathcal{Q}}]$ is an open subgroup of $\widetilde{\phi}(\widetilde{\mathbf{L}}(k'))$ (see [29, Theorem 7.2]).

The fact that k' can be taken as in (3.2) follows from the proof of [29, Proposition 0.6(a)] (see, in particular, [29, Proposition 3.14])—in particular, since we are dealing with groups of type A_1, we only need the exceptional definition of k' in characteristic 2. Moreover, [29, Proposition 1.6] implies that there are no nonstandard isogenies for groups of type A_1. Hence, by [29, Theorem 1.7(b)], the isogeny ϕ above is an isomorphism.

We now prove the other claims. First, let us recall from [20, Théorème 2] that since SL_2 is simply connected, for every unipotent element $u \in \mathrm{SL}(2, k)$ there exists a parabolic k-subgroup, $\mathbf{P}$, of SL_2 so that $u \in R_u(\mathbf{P}(k))$. Hence, (3.1) implies that

$$\mathcal{Q} = \big\langle \mathcal{Q} \cap R_u(P) : P \text{ is a parabolic subgroup of } \mathrm{SL}(2, k) \big\rangle. \tag{3.3}$$

Let P be a parabolic subgroup so that $\mathcal{Q} \cap R_u(P) \neq \{1\}$. Let a be a diagonalizable matrix in $\mathrm{PSL}(2, k) \subset \mathrm{PGL}(2, k)$ whose conjugation action contracts $R_u(P)$. Then a contracts $\phi(h)$ for any $h \in \mathbf{L}(k')$, where $\phi(h) \in R_u(P)$. Put $a' = \phi^{-1}(a)$. The above implies that h can be contracted to identity using conjugation by a'. In particular, h is a unipotent element. In view of (3.1) and the above discussion, $\widetilde{\mathbf{L}}(k')$ contains nontrivial unipotent elements. Thus, we get from [3, Corollaire 3.8] (see also

[20]) that $\widetilde{\mathbf{L}}$ is k'-isotropic. Since $\widetilde{\mathbf{L}}$ is simply connected and ϕ is an isomorphism, we get $\widetilde{\mathbf{L}} = \mathrm{SL}_2$.

Finally, using [26, Chapter I, Theorem 2.3.1], we have

$$\mathcal{Q} \cap R_u(P) \subset \widetilde{\phi}\big(\widetilde{\mathbf{L}}(k')\big)$$

for any parabolic subgroup P of $\mathrm{SL}(2,k)$. Hence, $\mathcal{Q} \subset \widetilde{\phi}(\widetilde{\mathbf{L}}(k'))$ by (3.3). This finishes the proof in case (a). $\square$

For the second theorem we need some more terminology. By a linear algebraic group $\mathbf{G}$ over $k \oplus k$, we mean $\mathbf{G}_1 \coprod \mathbf{G}_2$, where each $\mathbf{G}_i$ is a linear algebraic group over k. The adjoint representation of $\mathbf{G}$ on $\mathrm{Lie}(\mathbf{G}) = \mathrm{Lie}(\mathbf{G}_1) \oplus \mathrm{Lie}(\mathbf{G}_2)$ is the direct sum of the adjoint representations of $\mathbf{G}_i$ on $\mathrm{Lie}(\mathbf{G}_i)$, and the group of $(k \oplus k)$-points of $\mathbf{G}$ is $\mathbf{G}(k \oplus k) = \mathbf{G}_1(k) \times \mathbf{G}_2(k)$.

Suppose that $\mathbf{G} = \mathbf{G}_1 \coprod \mathbf{G}_2$ is a fiberwise absolutely almost simple, connected, simply connected $(k \oplus k)$-group. Let $\rho = (\rho_1, \rho_2)$, where ρ_i is the unique irreducible subquotient of the adjoint representation of $\mathbf{G}_i^{\mathrm{ad}}$ (see [29, Section 1]). The trace $\mathrm{tr}(\rho(g))$ for an element $g = (g_1, g_2)$ in $\mathbf{G}(k \oplus k)$ is defined by

$$\mathrm{tr}\big(\rho(g)\big) = \big(\mathrm{tr}\big(\rho_1(g_1)\big), \mathrm{tr}\big(\rho_2(g_2)\big)\big) \in k \oplus k.$$

Given a subfield $k' \subset k$ and a continuous embedding $\tau : k' \to k$ of fields, we put

$$\Delta_\tau(k') := \big\{\big(c, \tau(c)\big) : c \in k'\big\}. \tag{3.4}$$

As in [29, pp. 16–17], by a semisimple subring $k'' \subset k \oplus k$, we mean one of the following:

(k''-1) $k'' = k_1 \oplus k_2$, where $k_i \subset k$ is a closed subfield for $i = 1, 2$, or

(k''-2) $k'' = \Delta_\tau(k')$ for a subfield $k' \subset k$ and a continuous embedding $\tau : k' \to k$.

If $k'' = \Delta_\tau(k')$ and $\mathbf{H}$ is a k'-group, then we write, by abuse of notation, also $\mathbf{H}$ for the corresponding $\tau(k')$-group as well as the $\Delta_\tau(k')$-group obtained from $\mathbf{H}$. The base change of $\mathbf{H}$ from $\Delta_\tau(k)$ to $k \oplus k$ is then defined by

$$\mathbf{H} \times_{\Delta_\tau(k')} (k \oplus k) = (\mathbf{H} \times_{k'} k) \coprod (\mathbf{H} \times_{\tau(k')} k).$$

THEOREM A.2 ([29, Theorem 0.2, Theorem 7.2])
Assume that $\mathrm{char}(k) \neq 2, 3$, *and let* $\mathbf{G}_i$, $i = 1, 2$ *be absolutely almost simple, connected, simply connected* k*-groups. Let* $\mathcal{Q} \subset \mathbf{G}_1(k) \times \mathbf{G}_2(k)$ *be a compact subgroup so that* $\pi_i(\mathcal{Q})$ *is Zariski-dense in* $\mathbf{G}_i$ *for* $i = 1, 2$. *Further, assume that*

$$\mathcal{Q} = \big\langle\{g \in \mathcal{Q} : g \textit{ is a unipotent element}\}\big\rangle. \tag{3.5}$$

Let $k'' \subset k \oplus k$ *be defined as follows:*

$$k'' := \textit{the closed ring of quotients generated by } \{\mathrm{tr}(\rho(g)) : g \in \mathcal{Q}\}. \tag{3.6}$$

Then one of the following holds.

(1) *There are*
- (i) *closed subfields $k_i \subset k$ so that $k'' = k_1 \oplus k_2$,*
- (ii) *k_i-groups $\mathbf{H}_i$, and*
- (iii) *a k-isomorphism $\varphi_i : \mathbf{H}_i \times_{k_i} k \to \mathbf{G}_i$,*

so that $\mathcal{Q}$ contains an open subgroup of the form

$$\mathcal{Q}_1 \times \mathcal{Q}_2 \subset \varphi_1\big(\mathbf{H}_1(k_1)\big) \times \varphi_2\big(\mathbf{H}_2(k_2)\big).$$

(2) *There are*
- (i′) *a closed subfield $k' \subset k$ and a continuous embedding $\tau : k' \to k$ so that $k'' = \Delta_\tau(k')$,*
- (ii′) *a k'-group $\mathbf{H}$, and*
- (iii′) *a $(k \oplus k)$-isomorphism $\varphi : \mathbf{H} \times_{k''} (k \oplus k) \to \mathbf{G}_1 \coprod \mathbf{G}_2$,*

so that $\mathcal{Q}$ is an open subgroup of $\varphi(\mathbf{H}(k''))$.

Moreover, k'' is unique, and $\mathbf{H}$ and φ are unique up to unique isomorphisms.

Proof

Similar to Theorem A.1, these assertions are special cases of results in [29], as we now explain. Let $\mathbf{G}_i^{\mathrm{ad}}$ denote the adjoint form of $\mathbf{G}_i$ for $i = 1, 2$. Denote by $\bar{\mathcal{Q}}$ the image of $\mathcal{Q}$ under the natural map from $\mathbf{G}_1 \coprod \mathbf{G}_2$ to $\mathbf{G}_1^{\mathrm{ad}} \coprod \mathbf{G}_2^{\mathrm{ad}}$. Then $\pi_i(\bar{\mathcal{Q}})$ is Zariski-dense in $\mathbf{G}_i^{\mathrm{ad}}$ for $i = 1, 2$.

By [29, Theorem 0.2], we have the following. There exist
- a semisimple subring $k'' \subset k \oplus k$,
- a fiberwise absolutely simple adjoint group $\mathbf{L}$ defined over k'', and
- a $(k \oplus k)$-isogeny $\phi : \mathbf{L} \times_{k''} (k \oplus k) \to \mathbf{G}_1^{\mathrm{ad}} \coprod \mathbf{G}_2^{\mathrm{ad}}$ whose derivative vanishes nowhere,

where k'' is unique, and $\mathbf{L}$ and ϕ are unique up to unique isomorphism, so that the following hold.
- We have $\bar{\mathcal{Q}} \subset \phi(\mathbf{L}(k''))$ (see [29, Theorem 3.6]).
- Let $\widetilde{\mathbf{L}}$ denote the simply connected cover of $\mathbf{L}$, and let $\widetilde{\phi}$ be the induced isogeny from $\widetilde{\mathbf{L}} \times_{k''} (k \oplus k)$ to $\mathbf{G}_1 \coprod \mathbf{G}_2$. Then any compact subgroup $\mathcal{Q}' \subset \widetilde{\phi}(\widetilde{\mathbf{L}}(k''))$ which is fiberwise Zariski-dense and normalized by $[\bar{\mathcal{Q}}, \bar{\mathcal{Q}}]$ is an open subgroup of $\widetilde{\phi}(\widetilde{\mathbf{L}}(k''))$ (see [29, Theorem 7.2]).

Recall our assumption that $\mathrm{char}(k) \neq 2, 3$. Therefore, $\mathbf{G}_1$ and $\mathbf{G}_2$ have no nonstandard isogenies (see [29, Proposition 1.6]). This also implies that k'' can be taken as in (3.6) (see [29, Propositions 3.13 and 3.14]). Moreover, by [29, Theorem 1.7(b)], the isogeny ϕ above is an isomorphism.

The preceding discussion thus implies that if $k'' = \Delta_\tau(k')$ (see (k''-2)), then (i$'$), (ii$'$), and (iii$'$) hold. Similarly, if $k'' = k_1 \oplus k_2$ (see (k''-1)), then (i), (ii), and (iii) hold, in view of the above discussion and the description of algebraic groups and their isogenies over $k_1 \oplus k_2$ and $k \oplus k$. Finally, recall from (3.5) that $\mathcal{Q}$ is generated by unipotent elements; therefore, $\mathcal{Q} \subset \widetilde{\phi}(\widetilde{\mathbf{L}}(k''))$ (see [26, Chapter I, Theorem 2.3.1]). This finishes the proof of case (b). $\square$

We will also need the following lemma. Let $\mathbf{U}^+$ (resp., $\mathbf{U}^-$) denote the group of upper (resp., lower) triangular unipotent matrices in SL_2. Also, let $\mathbf{T}$ denote the group of diagonal matrices in SL_2. Put $U^\pm := \mathbf{U}^\pm(k)$ and $T := \mathbf{T}(k)$.

LEMMA 3.1
Let the notation be as in Theorem A.1. Put $E = \varphi(\mathrm{SL}(2,k'))$. Then
(1) $\quad E = \langle E \cap U^+, E \cap U^- \rangle$,
(2) $\quad E \cap T$ *is unbounded.*

Proof
We showed in the course of the proof of Theorem A.1 that there are nontrivial unipotent elements $h^\pm \in \mathrm{SL}(2,k')$ so that $\varphi(h^\pm) \in U^\pm$, respectively. Since SL_2 is simply connected, it follows from [20, Théorème 2] that there are k'-parabolic subgroups $\mathbf{P}^\pm$ of SL_2 so that $h^\pm \in R_u(\mathbf{P}^\pm)$. The groups $R_u(\mathbf{P}^\pm)$ are 1-dimensional k'-split unipotent subgroups; hence, $\varphi(R_u(\mathbf{P}^\pm)(k')) \subset \varphi(\mathrm{SL}_2)$ is an infinite group. Note that $\varphi(\mathrm{SL}_2) = \mathrm{SL}_2$ in Theorem A.1. Let $\mathbf{U}'_\pm$ denote the Zariski closure of $\varphi(R_u(\mathbf{P}^\pm)(k'))$. Then $\mathbf{U}'_\pm$ is a nontrivial connected unipotent subgroup of $\varphi(\mathrm{SL}_2)$ which intersects $\mathbf{U}^\pm \cap \varphi(\mathrm{SL}_2)$ nontrivially. Therefore, $\mathbf{U}'_\pm = \mathbf{U}^\pm \cap \varphi(\mathrm{SL}_2)$, which implies that

$$\varphi\big(R_u(\mathbf{P}^\pm)(k')\big) \subset U^\pm \cap E. \tag{3.7}$$

Using the fact that SL_2 is simply connected one more time, we note that $\mathrm{SL}(2,k')$ is generated by $R_u(\mathbf{P}^\pm)(k')$ (see [26, Chapter 1, Theorem 2.3.1]). This and (3.7) imply (1) in the lemma.

We now show (2) in the lemma. Let $\mathbf{S} = \mathbf{P}^+ \cap \mathbf{P}^-$. Then $\mathbf{S}$ is a 1-dimensional k'-split k'-torus; put $S = \mathbf{S}(k')$. Now

$$T' := \varphi(S) \subset TU^+ \cap TU^- = T$$

satisfies the claim in (2). $\square$

3.2. Measures invariant under semisimple groups
We will state in this section the measure classification result by Golsefidy and the third author in [28] for probability measures that are invariant under noncompact semisimple groups in the positive-characteristic setting. For this we need some notation and

definitions to help us generalize the notions defined in (2.1) to a general connected group. Let k be a local field. Suppose that $\mathbf{M}$ is a connected k-algebraic group, and let $\lambda : \mathbf{G}_m \to \mathbf{M}$ be a noncentral homomorphism defined over k. Define $-\lambda(\cdot) = \lambda(\cdot)^{-1}$.

Recall that a morphism from $\mathbf{G}_m$ to $\mathbf{M}$ is said to have a limit at 0 when it can be extended to a morphism from $\mathbf{A}^1$ to $\mathbf{M}$. As in [35, Section 13.4] and [6, Chapter 2 and Appendix C], we let $\mathbf{P}_{\mathbf{M}}(\lambda)$ denote the smooth closed subgroup of $\mathbf{M}$ defined over k so that

$$\mathbf{P}_{\mathbf{M}}(\lambda)(R) = \big\{r \in \mathbf{M}(R) : \lambda r \lambda^{-1} \text{ from } \mathbf{G}_m \text{ to } \mathbf{M} \text{ has a limit at } 0\big\}$$

for any algebra R/k.

Let $\mathbf{W}^+_{\mathbf{M}}(\lambda)$ be the closed normal subgroup of $\mathbf{P}_{\mathbf{M}}(\lambda)$ so that

$$\mathbf{W}^+_{\mathbf{M}}(\lambda)(R) = \big\{r \in \mathbf{M}(R) : \lambda r \lambda^{-1} \text{ from } \mathbf{G}_m \text{ to } \mathbf{M} \text{ has a limit at } 0\big\}$$

for any algebra R/k. Similarly, define $\mathbf{W}^+_{\mathbf{M}}(-\lambda)$, which we will denote by $\mathbf{W}^-_{\mathbf{M}}(\lambda)$. The centralizer of the image of λ is denoted by $\mathbf{Z}_{\mathbf{M}}(\lambda)$. The subgroups $\mathbf{W}^+_{\mathbf{M}}(\lambda)$, $\mathbf{Z}_{\mathbf{M}}(\lambda)$, and $\mathbf{W}^-_{\mathbf{M}}(\lambda)$ are *smooth* closed subgroups (see [6, Chapter 2 and Appendix C]).

The multiplicative group $\mathbf{G}_m$ acts on Lie($\mathbf{M}$) via λ, and the weights are integers. The Lie algebras of $\mathbf{Z}_{\mathbf{M}}(\lambda)$ and $\mathbf{W}^{\pm}_{\mathbf{M}}(\lambda)$ may be identified with the weight subspaces of this action corresponding to the zero, positive, and negative weights. It is shown in [6, Chapter 2 and Appendix C] that $\mathbf{P}_{\mathbf{M}}(\lambda)$, $\mathbf{Z}_{\mathbf{M}}(\lambda)$, and $\mathbf{W}^{\pm}_{\mathbf{M}}(\lambda)$ are k-subgroups of $\mathbf{M}$. Moreover, $\mathbf{W}^+_{\mathbf{M}}(\lambda)$ is a normal subgroup of $\mathbf{P}_{\mathbf{M}}(\lambda)$ and the product map

$$\mathbf{Z}_{\mathbf{M}}(\lambda) \times \mathbf{W}^+_{\mathbf{M}}(\lambda) \to \mathbf{P}_{\mathbf{M}}(\lambda) \text{ is a } k\text{-isomorphism of varieties.}$$

A pseudoparabolic k-subgroup of $\mathbf{M}$ is a group of the form $\mathbf{P}_{\mathbf{M}}(\lambda) R_{u,k}(\mathbf{M})$ for some λ as above, where $R_{u,k}(\mathbf{M})$ denotes the maximal connected normal unipotent k-subgroup of $\mathbf{M}$ (see [6, Definition 2.2.1]). We also recall from [6, Proposition 2.1.8(3)] that the product map

$$\mathbf{W}^-_{\mathbf{M}}(\lambda) \times \mathbf{Z}_{\mathbf{M}}(\lambda) \times \mathbf{W}^+_{\mathbf{M}}(\lambda) \to \mathbf{M} \text{ is an open immersion of } k\text{-schemes.} \tag{3.8}$$

It is worth mentioning that these results are generalizations to arbitrary groups of analogous and well-known statements for reductive groups.

Let $M = \mathbf{M}(k)$, and put

$$W^{\pm}_M(\lambda) = \mathbf{W}^{\pm}_{\mathbf{M}}(\lambda)(k) \qquad \text{and} \qquad Z_M(\lambda) = \mathbf{Z}_{\mathbf{M}}(\lambda)(k).$$

From (3.8) we conclude that $W^-_M(\lambda) Z_M(\lambda) W^+_M(\lambda)$ is a Zariski-open dense subset of M, which contains a neighborhood of identity with respect to the Hausdorff topology. For any λ as above, define

$$M^+(\lambda) := \langle W_M^+(\lambda), W_M^-(\lambda)\rangle. \tag{3.9}$$

LEMMA 3.2

(1) *For any λ as above, $M^+(\lambda)$ is a normal and unimodular subgroup of M.*

(2) *There are only countably many subgroups of the form $M^+(\lambda)$ in M.*

Combining results in [6, Appendix C] together with part (1) in the lemma, one can actually conclude that there are only finitely many such subgroups. We will only make use of the weaker statement above.

Proof

Part (1) is proved in [28, Lemma 2.1]. We now prove (2). First, note that if $\lambda_1, \lambda_2 : \mathbf{G}_m \to \mathbf{M}$ are two homomorphisms so that $\lambda_1 = g\lambda_2 g^{-1}$ for some $g \in M$, then $M^+(\lambda_1) = gM^+(\lambda_2)g^{-1}$. Therefore, by part (1) we have

$$M^+(\lambda_1) = M^+(\lambda_2) \quad \text{whenever } \lambda_1 = g\lambda_2 g^{-1} \text{ for some } g \in M. \tag{3.10}$$

Now let $\mathbf{S}$ be a maximal, k-split, k-torus in $\mathbf{M}$. By [6, Theorem C.2.3], there is some $g \in M$ so that $g\lambda g^{-1} : \mathbf{G}_m \to \mathbf{S}$. The claim now follows from this, (3.10), and the fact that the finitely generated abelian group $X_*(\mathbf{S}) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{S})$ is countable. $\square$

Given any subfield $l \subset k$ so that k/l is a finite extension, we let $\mathcal{R}_{k/l}$ denote the Weil's restriction of scalars (see [6, Section A.5]).

In the following, let $\mathbf{G}$ be a connected k-group, and let $\Gamma \subset G$ be a discrete subgroup in $G = \mathbf{G}(k)$. Furthermore, let $k' \subset k$ be a closed subfield, and let $\mathbf{H}$ be an absolutely almost simple k'-isotropic k'-group. Assume that $\varphi : \mathbf{H} \times_{k'} k \to \mathbf{G}$ is a nontrivial k-homomorphism, and put $E = \varphi(\mathbf{H}(k'))$. We use in an essential way the following measure classification result by Golsefidy and the third author.

THEOREM B ([28, Theorem 6.9, Corollary 6.10])

Let ν be a probability measure on G/Γ which is E-invariant and ergodic. Then there exist

(1) *some $l = (k')^q \subset k$, where $q = p^n$, $p = \mathrm{char}(k)$, and n is a nonnegative integer,*

(2) *a connected l-subgroup $\mathbf{M}$ of $\mathcal{R}_{k/l}(\mathbf{G})$ so that $\mathbf{M}(l) \cap \Gamma$ is Zariski-dense in $\mathbf{M}$,*

(3) *an element $g_0 \in G$*

such that ν is the $g_0 L g_0^{-1}$-invariant probability Haar measure on the closed orbit $g_0 L\Gamma/\Gamma$ with

$$L = \overline{M^+(\lambda)\big(\mathbf{M}(l) \cap \Gamma\big)},$$

where

- *the closure is with respect to the Hausdorff topology, and*
- $\lambda : \mathbf{G}_m \to \mathbf{M}$ *is a noncentral* l*-homomorphism,* $M^+(\lambda)$ *is defined in (3.9), and* $E \subset g_0 M^+(\lambda) g_0^{-1}$.

3.3. A version of the Borel density theorem

Let $k' \subset k$ be an infinite closed subfield. We recall from [34, Proposition 1.4] that the *discompact radical* of a k'-group is the maximal k'-subgroup which does not have any nontrivial compact k'-algebraic quotients. It is shown in [34, Proposition 1.4] that this subgroup exists and the quotient of the k'-points of the original group by the k'-points of the discompact radical is compact. Let $\mathbf{A}$ be a k-split torus. Let $A_{k'}^{\mathrm{sp}} \subset \mathcal{R}_{k/k'}(\mathbf{A})(k') = A$ denote the k'-points of the maximal, k'-split, subtorus of $\mathcal{R}_{k/k'}(\mathbf{A})$. Suppose that $\mathbf{V}$ is a variety defined over k', and assume that $\mathcal{R}_{k/k'}(\mathbf{A})$ acts on $\mathbf{V}$ via k'-morphisms. In particular, $A = \mathcal{R}_{k/k'}(\mathbf{A})(k')$ acts on $V = \mathbf{V}(k')$ via k'-morphisms.

LEMMA 3.3 ([34, Theorem 1.1])

Let (X, η) *be an* A*-invariant ergodic probability space. Let* $f : X \to V$ *be an* A*-equivariant Borel map. Then there exists some* $v_0 \in \mathrm{Fix}_{A_{k'}^{\mathrm{sp}}}(V)$ *so that* $f_*\eta$ *is the* A*-invariant measure on the compact orbit* Av_0*. In particular,* $f(x) \in Av_0$ *for* η*-a.e.* x.

Proof

This follows from [34, Theorem 1.1] in view of the fact that $A_{k'}^{\mathrm{sp}}$ is the *discompact* radical of $\mathcal{R}_{k/k'}(\mathbf{A})$ as defined in [34] (see also [34, Theorem 3.6]). ☐

3.4. Pinsker σ-algebra and unstable leaves

Throughout this section we assume that $\mathbf{G}$ is a k-isotropic, semisimple k-group, and we let $\mathbf{A}$ be a k-split k-torus in $\mathbf{G}$. Put $G = \mathbf{G}(k)$ and $A = \mathbf{A}(k)$. Let Γ be a discrete subgroup of G, and put $X = G/\Gamma$. Let $a \in A$ be a nontrivial element. Recall that, for an a-invariant measure μ, we define the Pinsker σ-algebra as

$$\mathcal{P}_a := \big\{ B \in \mathcal{B} : \mathsf{h}_\mu\big(a, \{B, X \setminus B\}\big) = 0 \big\}.$$

It is the largest σ-algebra with respect to which μ has zero entropy (see [37] for further discussion). Let us recall the following important and well-known proposition; we outline the proof for the sake of completeness.

PROPOSITION 3.4

The Pinsker σ*-algebra,* $\mathcal{P}_a$*, is equivalent to the* σ*-algebra of Borel sets foliated by* $W_G^+(a)$ *leaves.*

Note that the Pinsker σ-algebra for a equals the Pinsker σ-algebra for a^{-1}, which shows that the proposition also applies similarly for $W_G^-(a)$.

Proof
Suppose that $\mathcal{C}$ is any σ-algebra whose elements are foliated by $W_G^+(a)$ leaves. Let $\mathrm{p}:(X,\mu)\to(Y,\mathrm{p}_*\mu)$ be the corresponding factor map. Using the Abramov–Rokhlin conditional entropy formula and the relationship between entropy and leafwise measures (see [14]), we get

$$\mathsf{h}\big(a,(Y,\mathrm{p}_*\mu)\big)=0.$$

The definition of the Pinsker σ-algebra then implies that $\mathcal{C}\subset\mathcal{P}_a$.

For the converse, we recall from [27, Section 9] (see also [14]) that there is a finite entropy generator (i.e., a countable partition ξ of finite entropy) such that $\bigvee_{n=-\infty}^{\infty}a^{-n}\xi$ is equivalent to the full Borel σ-algebra, and so that in addition the past is subordinate with respect to $W_G^+(a)$. That is to say, that on the complement of a null set, every atom of $\bigvee_{n=-\infty}^{0}a^{-n}\xi$ is an open subset of a $W_G^+(a)$-orbit. Hence, after removing a null set, any set measurable with respect to the tail $\bigcap_{k\in\mathbb{N}}\bigvee_{n=-\infty}^{-k}a^{-n}\xi$ is a union of $W_G^+(a)$-orbits. Since $\mathcal{P}_a$ is equivalent to the tail of ξ modulo μ, the claim follows. $\square$

The following will be used in the course of the proof of Theorem 1.2.

LEMMA 3.5
Let $X_i=G_i/\Gamma$ be as in Theorem 1.2. In particular, $G_i=\mathbf{G}_i(k)$, where $\mathbf{G}_i$ is a connected, simply connected, absolutely almost simple group defined over k for $i=1,2$. Let $a=(a_1,a_2)\in A$ be such that a generates an unbounded group, and suppose that μ is an ergodic joining of the A_i-action on (X_i,m_i), for $i=1,2$. Let $\mu=\int_{X_1\times X_2}\mu_x^{\mathcal{P}_a}\,\mathrm{d}\mu(x)$, where $\mu_x^{\mathcal{P}_a}$ denotes the conditional measure for μ-a.e. x with respect to the Pinsker σ-algebra $\mathcal{P}_a$. Then there exists a subset $X'\subset X_1\times X_2$ with $\mu(X')=1$ so that

$$\pi_{i*}(\mu_x^{\mathcal{P}_a})=m_i\quad\textit{for all } x\in X' \textit{ and } i=1,2.$$

Proof
Let P denote the Pinsker factor of X, and let $\Upsilon:X\to\mathrm{P}$ be the corresponding factor map. This is a zero entropy factor of X.

Put $Z=X_1\times X_2\times\mathrm{P}$, and let

$$\nu=\int\mu_x^{\mathcal{P}_a}\times\delta_{\Upsilon(x)}\,\mathrm{d}\Upsilon_*\mu(x).$$

Let $\mathrm{p}_i : Z \to X_i \times \mathrm{P}$ be the natural projection. Then $\mathrm{p}_{i*}\nu$ is a measure on $X_i \times \mathrm{P}$ which projects to m_i and $\Upsilon_*\mu$ for $i = 1, 2$. Now (X_i, m_i) is a system with completely positive entropy. This follows, for example, from Proposition 3.4 and the ergodicity of the action of $W^{\pm}(a_i)$; note that the latter holds since $\mathbf{G}_i$ is connected, simply connected, and absolutely almost simple (see [26, Chapter 1, Theorem 2.3.1], [26, Chapter 2, Theorem 2.7]). However, $(\mathrm{P}, \Upsilon_*\mu)$ is a zero entropy system; therefore, by the disjointness theorem of Furstenberg [19] (see also [21, Theorem 18.16]), we obtain

$$\mathrm{p}_{i*}\nu = m_i \times \Upsilon_*\mu. \tag{3.11}$$

Let us now decompose $\mathrm{p}_{i*}\nu$ as

$$\mathrm{p}_{i*}\nu = \int (\mathrm{p}_{i*}\nu)_{(x_i,p)}^{X_i \times \mathcal{B}_\mathrm{P}} \, d\mathrm{p}_{i*}\nu.$$

Then (3.11) implies that, for $\mathrm{p}_{i*}\nu$-a.e. (x_i, p), we have

$$(\mathrm{p}_{i*}\nu)_{(x_i,p)}^{X_i \times \mathcal{B}_\mathrm{P}} = m_i \times \delta_p.$$

This in view of the definition of ν implies the claim. □

3.5. Leafwise measures

Recall that $\mathbf{G}$ is a k-isotropic, semisimple k-group, and let $\mathbf{A}$ be a k-split k-torus in $\mathbf{G}$. Let $\mathbf{S}$ be a maximal, k-split, k-torus of $\mathbf{G}$ which contains $\mathbf{A}$. Let ${}_k\Phi(\mathbf{S}, \mathbf{G})$ be the relative root system of $\mathbf{G}$, and let ${}_k\Phi(\mathbf{A}, \mathbf{G})$ denote the set of roots of $\mathbf{A}$ as in Section 2.

Definition

Let U be an A-normalized unipotent k-subgroup of G contained in some $W_G^-(a)$. The leafwise measure μ_x^U along U is defined for μ-a.e. $x \in X$. For all such x, we put

$$\mathcal{S}_x^U = \mathrm{supp}(\mu_x^U) \qquad \text{and} \qquad \mathcal{I}_x^U = \{v \in U : v\mu_x^U = \mu_x^U\}.$$

The leafwise measures are canonically defined up to proportionality, and we write $\propto$ to denote proportionality. The main case we are interested in is when $V_\Psi := U_{\vartheta(\Psi)}$ is the associated unipotent subgroup of a positively closed set $\Psi \subset {}_k\Phi(\mathbf{A}, \mathbf{G})$, in which case we will use μ_x^Ψ, $\mathcal{S}_x^\Psi$, $\mathcal{I}_x^\Psi$ to denote $\mu_x^{V_\Psi}$, $\mathcal{S}_x^{V_\Psi}$, $\mathcal{I}_x^{V_\Psi}$, respectively.

LEMMA 3.6

Under the above assumptions, almost surely $\mathcal{I}_x^U = \{v \in U : v\mu_x^U \propto \mu_x^U\}$.

Proof
This is true in general, but is particularly easy in the positive-characteristic case. Suppose that $u \in U$ is such that $u\mu_x^U \propto \mu_x^U$. Then $u\mu_x^U = \kappa\mu_x^U$ for some $\kappa > 0$. Since U is unipotent, u is torsion of exponent p^n for some n, and hence $\kappa^{p^n} = 1$, which implies (since $\kappa > 0$) that $\kappa = 1$. □

We recall some properties of leafwise measures which will be used throughout this article. Our formulation is taken from [13] (see [25]; see also [14] and the references therein).

LEMMA 3.7
Let U be an A-normalized, unipotent, k-subgroup of G contained in some $W_G^-(a)$. Then there is a conull subset $X' \subset X$ with the following properties.
(1) *For all $x \in X'$, the map $x \mapsto \mu_x^U$ from X to the space of Radon measures on U is normalized so that $\mu_x^U([1]) = 1$ is a measurable map. In particular, μ_x^U is defined for all $x \in X'$.*
(2) *For every $x \in X'$ and every $u \in U$ so that $ux \in X'$, we have $\mu_x^U \propto (\mu_{ux}^U)u$, where $(\mu_{ux}^U)u$ denotes the pushforward of μ_{ux}^U under the map $v \mapsto vu$.*
(3) *For every $x \in X'$, we have $\mu_x^U(U[1]) = 1$ and $\mu_x^U(U[\epsilon]) > 0$ for all $\epsilon > 0$.*
(4) *Suppose that μ is a-invariant under some $a \in A$. Then for μ-a-e. $x \in X$, we have $\mu_{ax}^U \propto (a\mu_x^U a^{-1})$.*

LEMMA 3.8 ([13, Section 6])
Let $a \in A$ be so that the Zariski closure of $\langle a \rangle$, $\mathbf{A}'$ say, is k-isomorphic to $\mathbf{G}_m$ and so that $\mathbf{A}'(k)/\langle a \rangle$ is compact. Suppose that μ is a-invariant, and let U be an A-normalized, unipotent, k-subgroup of G contained in $W_G^-(a)$. Let Q be any compact open subgroup of U. Then for μ-a.e. x, the Zariski closure of $\mathcal{I}_x^U \cap Q$ is normalized by a and contains $\mathcal{I}_x^U$.

Proof
Let $\mathcal{E}$ denote a countably generated σ-algebra that is equivalent to the σ-algebra of a-invariant sets. Then $(\mu_x^{\mathcal{E}})_y^U = \mu_y^U$ for $\mu_x^{\mathcal{E}}$-a.e. y and μ-a.e. x (see, e.g., [14]). Therefore, we may assume that μ is a-ergodic. Let $\mathfrak{U}_0$ denote a fixed compact open subgroup of U. For any $n \in \mathbb{Z}$, define

$$\mathfrak{U}_n = a^n \mathfrak{U}_0 a^{-n}.$$

Then $\mathfrak{U}_n \subset Q$ for large enough n; hence, it suffices to prove the lemma for $Q = \mathfrak{U}_n$. Let $X' \subset X$ be a conull set where Lemma 3.7 holds. For any $x \in X'$ and any $n \in \mathbb{Z}$, define

$$\mathbf{F}_{x,n} = \text{the Zariski closure of } \mathfrak{U}_n \cap \mathcal{I}_x^U.$$

Then $\mathbf{F}_{x,n}$ is a k-group (see, e.g., [35, Lemma 11.2.4(ii)]).

Note also that $\mathbf{F}_{x,n} \subset \mathbf{F}_{x,m}$ whenever $n \geq m$. Therefore, there exists some $n_0 = n_0(x)$ so that $\dim \mathbf{F}_{x,n} = \dim \mathbf{F}_{x,n_0}$ for all $n \geq n_0$, where dim is the dimension as a k-group. Since the number of connected components of $\mathbf{F}_{x,n_0}$ is finite, there exists $n_1 = n_1(x)$ so that $\mathbf{F}_{x,n} = \mathbf{F}_{x,n_1}$ for all $n \geq n_1$. Put $\mathbf{F}_x := \mathbf{F}_{x,n_1}$.

The definition of $\mathbf{F}_{x,n}$, in view of Lemma 3.7(4), implies that

$$\mathbf{F}_{ax,n+1} = a\mathbf{F}_{x,n}a^{-1}.$$

Therefore, we have

$$\mathbf{F}_{ax} = a\mathbf{F}_x a^{-1}. \tag{3.12}$$

Let $k[\mathbf{G}]$ denote the ring of regular functions of $\mathbf{G}$. For every $x \in X'$, let $J_x \subset k[\mathbf{G}]$ be the ideal of regular functions vanishing on $\mathbf{F}_x$. Let $m(x)$ be the minimum integer so that J_x is generated by polynomials of degree at most $m(x)$. In view of (3.12), we have $m(x) = m(ax)$. Since μ is a-ergodic, we have that $x \mapsto m(x)$ is essentially constant. Replacing X' by a conull subset if necessary, we assume that $m(x) = m$ for all $x \in X'$.

Let $\Upsilon = \{h \in k[\mathbf{G}] : \deg(h) \leq m\}$. Using a similar argument as above, we may assume that $\dim(J_x \cap \Upsilon) = \ell$ for all $x \in X'$.

Let $f : X \to \mathrm{Grass}(\ell)$, the Grassmannian of ℓ-dimensional subspaces of Υ, be the map defined by $f(x) = J_x \cap \Upsilon$ for all $x \in X'$. Then f is an A-equivariant Borel map. Therefore, $\nu = f_*\mu$ is a probability measure on $\mathrm{Grass}(\ell)$ which is invariant and ergodic for a k-algebraic action of a on $\mathrm{Grass}(\ell)$. Hence,

$$\bar{\nu} = \int_{\mathbf{A}'(k)/\langle a\rangle} b_*\nu \,\mathrm{d}b$$

is an $\mathbf{A}'(k)$-invariant, ergodic probability measure on $\mathrm{Grass}(\ell)$ equipped with an algebraic action of $\mathbf{A}'(k)$. By [34, Theorem 3.6], $\bar{\nu}$ is the delta mass at an $\mathbf{A}'(k)$-fixed point, which implies that $\nu = \bar{\nu}$ is the delta mass at an $\mathbf{A}'(k)$-fixed point. Therefore, f is essentially constant. Using the definition of f, we get that $a\mathbf{F}_x a^{-1} = \mathbf{F}_x$ for μ-a.e. x. This, (3.12), and the ergodicity of μ imply that $\mathbf{F}_x = \mathbf{F}$ for μ-a.e. x.

Now let $C \subset X'$ be a compact subset with $\mu(C) > 1 - \epsilon$ so that

- $n_1(x) \leq N_1$ for all $x \in C$,
- $\mathbf{F}_x = \mathbf{F}$ for all $x \in C$.

By the pointwise ergodic theorem, for almost every $x \in X$, there is a sequence $m_i \to \infty$ so that $a^{m_i}x \in C$ for all i. Now let x be such a point, and let $u \in \mathcal{I}_x^U$. By Lemma 3.7(4), we have

$$a^{m_i} u a^{-m_i} \in \mathfrak{U}_{N_1} \cap \mathcal{I}^U_{a^{m_i} x} \subset \mathbf{F}(k)$$

for all large enough i. Since $\mathbf{F}(k)$ is normalized by a, we get $u \in \mathbf{F}(k)$. $\square$

From this point on, we will assume that μ is A-invariant. We recall the product structure for leafwise measures (see [10]). Our formulation is taken from [14, Proposition 8.5 and Corollary 8.8].

LEMMA 3.9
Fix some $a \in A$. Let $H = T \ltimes U$, where $U < W_G^-(a)$ and $T < Z_G(a)$. Then there exists a conull subset $X' \subset X$ with the following properties.

(1) *For every $x \in X'$ and $h \in H$ such that $hx \in X'$, we have $\mu_x^T \propto (\mu_{hx}^T)t$, where $h = ut = tu'$ for $t \in T$ and $u, u' \in U$.*
(2) *For every $x \in X'$, we have $\mu_x^H \propto \iota_*(\mu_x^T \times \mu_x^U)$, where $\iota(t,u) = tu$ is the product map.*
(3) *Assume further that T centralizes U. Then for all $x \in X'$ and $t \in T$ so that $tx \in X'$, we have $\mu_x^U \propto \mu_{tx}^U$.*

By induction, as in [14, Section 8], this lemma implies a product structure for the conditional measures μ_x^Ψ.

PROPOSITION 3.10 ([10, Theorem 8.4])
Let $\Psi \subset {}_k\Phi(\mathbf{A}, \mathbf{G})$ be a positively closed subset of Lyapunov exponents. Let $[\alpha_1], [\alpha_2], \ldots, [\alpha_k]$ be any ordering of the course Lyapunov weights contained in Ψ. Then for μ-a.e. $x \in X$,

$$\mu_x^\Psi \propto \iota_*(\mu_x^{[\alpha_1]} \times \cdots \times \mu_x^{[\alpha_k]}).$$

(For the proof, see, e.g., [10] or [14, Section 8].)

LEMMA 3.11
Suppose that μ is an A-invariant, ergodic probability measure. Let $\Psi \subset {}_k\Phi(\mathbf{A}, \mathbf{G})$ be a positively closed subset, and assume that $\alpha, \beta \in \Psi$ are linearly independent roots. Let $\Psi' \subset \Psi$ be those elements of Ψ that can be expressed as a linear combination of α and β with strictly positive coefficients. Then Ψ' is also closed, and for μ-a.e. x we have

$$[\mathcal{S}_x^{[\alpha]}, \mathcal{S}_x^{[\beta]}] \subset \mathcal{I}_x^\Psi \qquad \text{and} \qquad [\mathcal{S}_x^{[\alpha]}, \mathcal{S}_x^{[\beta]}] \subset \mathcal{I}_x^{\Psi'}.$$

Proof
By [2, Section 2.5], for example, both Ψ' and $\Psi' \cup \{\alpha, \beta\}$ are positively closed subsets

of ${}_k\Phi(\mathbf{A},\mathbf{G})$. Let $[\gamma_1],\ldots,[\gamma_\ell]$ be an enumeration of all course Lyapunovs in $\Psi\setminus(\Psi'\cup\{\alpha,\beta\})$. Then by Proposition 3.10,

$$\begin{aligned}\mu_x^\Psi &\propto \iota_*(\mu_x^{[\alpha]}\times\mu_x^{[\beta]}\times\mu_x^{\Psi'}\times\mu_x^{[\gamma_1]}\times\cdots\times\mu_x^{[\gamma_\ell]})\\ &\propto \iota_*(\mu_x^{[\beta]}\times\mu_x^{[\alpha]}\times\mu_x^{\Psi'}\times\mu_x^{[\gamma_1]}\times\cdots\times\mu_x^{[\gamma_\ell]}),\end{aligned}\tag{3.13}$$

where ι is the product map. Now let $f\in C_c(V^\Psi)$. Then (3.13) and Fubini's theorem imply that

$$\begin{aligned}&\int f(g)\,\mathrm{d}\mu_x^W\\ &\quad=\kappa\int f(v_\alpha v_\beta v_{\Psi'}v_{\gamma_1}\cdots v_{\gamma_\ell})\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{V_{[\beta]}}\,\mathrm{d}\mu_x^{\Psi'}\,\mathrm{d}\mu_x^{V_{[\gamma_1]}}\cdots\mathrm{d}\mu_x^{V_{[\gamma_\ell]}}\\ &\quad=\kappa'\int f(v_\beta v_\alpha v_{\Psi'}v_{\gamma_1}\cdots v_{\gamma_\ell})\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{V_{[\beta]}}\,\mathrm{d}\mu_x^{\Psi'}\,\mathrm{d}\mu_x^{V_{[\gamma_1]}}\cdots\mathrm{d}\mu_x^{V_{[\gamma_\ell]}}\\ &\quad=\kappa'\int f\big(v_\alpha v_\beta[v_\beta,v_\alpha]v_{\Psi'}v_{\gamma_1}\cdots v_{\gamma_\ell}\big)\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{V_{[\beta]}}\,\mathrm{d}\mu_x^{\Psi'}\,\mathrm{d}\mu_x^{V_{[\gamma_1]}}\cdots\mathrm{d}\mu_x^{V_{[\gamma_\ell]}}\end{aligned}$$

for κ, κ' independent of f. From this we get for $\mu_x^{[\alpha]}$-a.e. $v_\alpha\in V_{[\alpha]}$ and $\mu_x^{[\beta]}$-a.e. $v_\beta\in V_{[\beta]}$,

$$\mu_x^{\Psi'}\propto[v_\beta,v_\alpha]\mu_x^{\Psi'};$$

hence, applying Lemma 3.6, we deduce that $[v_\beta,v_\alpha]\mu_x^{\Psi'}=\mu_x^{\Psi'}$. Applying Proposition 3.10 again, we conclude that also $[v_\beta,v_\alpha]\mu_x^\Psi=\mu_x^\Psi$. Since $\mathcal{I}_x^\Psi$ is a (Hausdorff) closed subgroup of V^Ψ, it follows that, almost surely,

$$[\mathcal{S}_x^{[\alpha]},\mathcal{S}_x^{[\beta]}]\subset\mathcal{I}_x^\Psi.\tag{3.14}$$

□

LEMMA 3.12 ([10, Section 8])
Let μ be an A-invariant probability measure on X. There is a conull subset $X'\subset X$ with the following property. Let $\Psi\subset{}_k\Phi(\mathbf{A},\mathbf{G})$ be a positively closed subset such that $V_\Psi\subset W_G^-(a)$ for some a. Then for all $x\in X'$, if $v=\prod v_\alpha\in\mathcal{I}_x^\Psi$, with $v_\alpha\in V_{[\alpha]}$ for all $[\alpha]\subset\Psi$, then $v_\alpha\in\mathcal{I}_x^{[\alpha]}$ for all $[\alpha]$.

Proof
We say that a root $\alpha\in\Psi$ is *exposed* (see [14]) if there exists an element $b\in A$ so that $\alpha(b)=1$ and $|\beta(b)|<1$ for all $\beta\in\Psi\setminus[\alpha]$. If Ψ is as above, then clearly it has at least one exposed Lyapunov weight α, and that $\Psi'=\Psi\setminus[\alpha]$ is also positively closed. Moreover, for any $v_\alpha\in V_{[\alpha]}$ and $v'\in V_{\Psi'}$, it holds that $[v_\alpha,v']\in V_{\Psi'}$. Suppose that $v_\alpha v'\in\mathcal{I}_x^\Psi$ with $v_\alpha\in V_{[\alpha]}$ and $v'\in V_{\Psi'}$. Then

$$\begin{aligned}\int f(g)\,\mathrm{d}\mu_x^{\Psi} &= \kappa \int f(g_\alpha g')\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{\Psi'}\\ &= \int f(v_\alpha v' g)\,\mathrm{d}\mu_x^{\Psi}\\ &= \kappa \int f(v_\alpha v' g_\alpha g')\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{\Psi'}\\ &= \kappa \int f\big(v_\alpha g_\alpha v'[v', g_\alpha] g'\big)\,\mathrm{d}\mu_x^{V_{[\alpha]}}\,\mathrm{d}\mu_x^{\Psi'}\end{aligned}$$

for some κ independent of f.

It follows by uniqueness of decomposition that for $\mu_x^{V_{[\alpha]}}$-a.e. g_α,

$$v'[v', g_\alpha]\mu_x^{\Psi'} \propto \mu_x^{\Psi'};$$

hence, by Lemma 3.6 we have that $v'[v', g_\alpha] \in \mathcal{I}_x^{\Psi'}$. It follows that $v_\alpha \mu_x^{V_{[\alpha]}} = \mu_x^{V_{[\alpha]}}$ and $v_\alpha \in \mathcal{I}_x^{[\alpha]}$. Moreover, as for x a.e., the identity is in support of $\mu_x^{V_{[\alpha]}}$ by Lemma 3.7(3), we have that $v' \in \mathcal{I}_x^{\Psi'}$. The lemma now easily follows by induction on the cardinality of Ψ. □

For any $W_G^{\pm}(a)$, we fix some increasing sequence of compact open subgroups K_n with $W_G^{\pm}(a) = \bigcup_n K_n$ and some decreasing sequence of compact open subgroups $O_n \subset K_1$ with $\{e\} = \bigcap_n O_n$. Then any closed subgroup $\mathcal{I} < W_G^{\pm}(a)$ is determined by the finite subgroups $\mathcal{I} \cap K_n/O_n < K_n/O_n$, which allows us to speak of measurability of a subgroup depending on $x \in X$.

LEMMA 3.13
Let $a \in A$. Then $\mathcal{I}_x^{W_G^{\pm}(a)}$ is $\mathcal{P}_a$-measurable.

Proof
We prove this for $W_G^-(a)$; the proof in the other case is similar. There is a full measure set $X' \subset X$ so that, whenever $x, wx \in X'$, for some $w \in W_G^-(a)$, then we have

$$\mu_x^{W_G^-(a)} \propto \mu_{wx}^{W_G^-(a)} w.$$

This implies that $\mathcal{I}_x^{W_G^-(a)} = \mathcal{I}_{wx}^{W_G^-(a)}$. The lemma now follows from Proposition 3.4. □

LEMMA 3.14
Let $\alpha \in {}_k\Phi(\mathbf{A}, \mathbf{G})$ be such that $V_{[\alpha]} < W_G^-(a)$. Then the subgroup $\mathcal{I}_x^{[\alpha]}$ is $\mathcal{P}_a$-measurable.

Proof
In view of Proposition 3.4, it suffices to show that $x \mapsto \mathcal{J}_x^{[\alpha]}$ is constant along $W_G^-(a)$-leaves almost surely, which is an immediate corollary of Lemmas 3.13 and 3.12. $\square$

4. High entropy part of Theorem 1.1

We now start the proof of Theorem 1.1. Recall that A is the full diagonal subgroup of $G = \mathrm{SL}(d,k)$. Throughout Sections 4–6, μ denotes an ergodic A-invariant measure on G/Γ.

For any $\alpha \in \Phi$, there exists a k-embedding $\varphi_\alpha : \mathrm{SL}_2 \to \mathrm{SL}_d$ so that $U_\alpha = \varphi_\alpha(U^+)$ and $U_{-\alpha} = \varphi_\alpha(U^-)$, where $U^\pm$ denote the upper and lower triangular unipotent subgroups of SL_2. We let $H_\alpha := \mathrm{Im}(\varphi_\alpha)$. Let T denote the diagonal subgroup of SL_2. Let $t_\alpha = \left(\begin{smallmatrix}\theta & 0\\ 0 & \theta^{-1}\end{smallmatrix}\right) \in T$ be an element so that $\alpha(\varphi_\alpha(t_\alpha)) = \theta^2$ and $\beta(\varphi_\alpha(t_\alpha)) = \theta^\varepsilon$ with $\varepsilon \in \{-1,0,1\}$ for all $\beta \in \Phi \setminus \{\pm\alpha\}$, where θ is as in Section 2.1. Put

$$a_\alpha := \varphi_\alpha(t_\alpha).$$

Then $U_\alpha \subset W_G^+(a_\alpha)$.

Given a root $\alpha \in \Phi$, we define

$$\Phi_\alpha^+ := \{\beta \in \Phi : U_\beta \subset W^+(a_\alpha)\},$$

and put $\Phi_\alpha^- = -\Phi_\alpha^+$.

LEMMA 4.1
Let $\alpha \in \Phi$, and let $\beta \in \Phi_\alpha^- \setminus \{-\alpha\}$. The following hold:
(1) $\beta + \alpha \in \Phi_\alpha^+$,
(2) *if $\beta + n\alpha \in \Phi$ for some integer $n \geq 1$, then $n = 1$,*
(3) $\alpha \in \Phi_\beta^-$.

Proof
Assertions (1) and (3) are general facts, which follow from the definitions and hold for any root system. Part (2) is a special feature of root systems of type A, which is the case we are concerned with here. $\square$

A well-known theorem by Ledrappier and Young [24] relates the entropy, the dimension of conditional measures along invariant foliations, and Lyapunov exponents, for a general C^2 map on a compact manifold, and [27, Section 9] provides an adaptation of the general results to flows on locally homogeneous spaces.

The following is taken from [9, Lemma 6.2] (see also [11, Proposition 3.1] and [14]). For any root $\alpha \in \Phi$, there exists $s_\alpha(\mu) \in [0,1]$ so that, for any $a \in A$ with

$|\alpha(a)| \geq 1$, we have

$$\mathsf{h}_\mu(a, U_\alpha) = s_\alpha(\mu) \log\bigl|\alpha(a)\bigr|,$$

where $\mathsf{h}_\mu(a, U_\alpha)$ denotes the entropy contribution of U_α. Indeed, $s_\alpha(\mu)$ is defined as the local dimension of the leafwise measure along α as we now recall. Define

$$D_\mu(a_\alpha, U_\alpha)(x) = \lim_{|n|\to\infty} \frac{\log(\mu_x^{U_\alpha}(a_\alpha^n U_\alpha[1] a_\alpha^{-n}))}{n},$$

where the limit exists by [10, Lemma 9.1], and define $\mathsf{h}_\mu(a_\alpha, U_\alpha) = \int D_\mu(a_\alpha, U_\alpha) \, \mathrm{d}\mu$, the entropy contribution of U_α. Since $D_\mu(a_\alpha, U_\alpha)(x)$ is A-invariant and μ is A-ergodic, we have

$$\mathsf{h}_\mu(a_\alpha, U_\alpha) = D_\mu(a, U)(x) \quad \text{for } \mu\text{-a.e. } x.$$

Therefore, $s_\alpha(\mu) = \frac{1}{2} D_\mu(a, U)(x)$ for μ-a.e. x. Moreover, the following properties hold:

(s_α-1) $s_\alpha(\mu) = 0$ if and only if μ_x^α is the delta mass at the identity,

(s_α-2) $s_\alpha(\mu) = 1$ if and only μ_x^α is the Haar measure on U_α,

(s_α-3) for any $a \in A$ we have

$$\mathsf{h}_\mu(a) = \sum s_\alpha(\mu) \log^+\bigl|\alpha(a)\bigr|,$$

where $\log^+(\ell) = \max\{0, \log \ell\}$.

The following is the main result of this section.

PROPOSITION 4.2 ([15, Theorem 5.1])

Let $\alpha \in \Phi$ be so that μ_x^α is nontrivial for μ-a.e. x. Then at least one of the following holds.

(1) *We have $\mu_x^\beta = \delta_{\mathrm{id}}$ for all $\beta \in \Phi_\alpha^- \setminus \{-\alpha\}$ and μ-a.e. x.*

(2) *$\mathcal{I}_x^{\pm\alpha}$ are nondiscrete subgroups of $U_{\pm\alpha}$ for μ-a.e. x.*

Proof

Recall that, for $\mathrm{SL}(d)$, the roots α can be identified with ordered tuples of indices $(i, j) \in \{1, \ldots, d\}$ satisfying $i \neq j$. We use the local dimensions $s_\alpha = s_{(i,j)}$ to define a relation on $\{1, \ldots, d\}$. In fact, we write $i \precsim j$ if $i = j$ or $s_{(i,j)} > 0$, and we write $i \sim j$ if $i \precsim j \precsim i$. Lemma 3.11 implies that $\precsim$ is transitive; that is, if $i \precsim j \precsim k$, then also $i \precsim k$ for $i, j, k \in \{1, \ldots, d\}$.

It follows that $\sim$ is an equivalence relation on $\{1, \ldots, d\}$ and that $\precsim$ descends to a partial order on the quotient by $\sim$. Let us write $[i]$ for the equivalence classes with respect to $\sim$. To simplify matters, we may assume (by applying a suitable element of the Weyl group) that, for every i, the equivalence class $[i] = \{m, m+1, m+2, \dots, n\}$ consists of consecutive indices for some $m \le i$ and $n \ge i$. Moreover, we may assume that $i \precsim j$ for two indices implies that either $i \sim j$ or $i \le j$.

We now prove that $i \precsim j$ implies that $i \sim j$. Otherwise, we claim that we can choose a diagonal matrix a with two different eigenvalues (equal to powers of θ; see Section 2.1) such that the leafwise measures of the stable horospherical subgroup $W_G^-(a)$ are nontrivial and such that the leafwise measures of the unstable horospherical subgroup $W_G^+(a)$ are trivial almost surely. More precisely, assuming $[i] = \{m, m+1, m+2, \dots, n\}$ (so that by the indirect assumption, $j > n$), we define a to be the diagonal matrix with the first m eigenvalues equal to $\theta^{(d-m)}$ and the last $d-m$ eigenvalues equal to θ^{-m}. By assumption, $s_{(i,j)} > 0$, which implies that $h_\mu(a) > 0$ by (s_α-3), the choice of a, and since $i \le n < j$. However, for all $k \le n < \ell$ we have $s_{\ell,k} = 0$ (by our ordering of the indices) and hence $h_\mu(a^{-1}) = 0$ also by (s_α-3). This contradiction proves the claim that $i \precsim j$ implies that $i \sim j$.

Given a root $\alpha = (i,j)$ with $s_\alpha > 0$, there are now two options: either $[i] = \{i, j\}$ or the cardinality of $[i]$ is at least 3. In the first case, we have $s_{(i,\ell)} = s_{(j,\ell)} = s_{(\ell,i)} = s_{(\ell,j)} = 0$ for all $\ell \notin \{i, j\}$, and translating this to the language of roots, we obtain (1). In the second case, let $\ell \in [i] \setminus \{i, j\}$ and apply Lemma 3.11 for the roots (i,ℓ), (ℓ, j) to see that $\mathcal{I}_x^{(i,j)}$ (and similarly also $\mathcal{I}_x^{(j,i)}$) is a nondiscrete group almost surely. $\square$

5. Low entropy part of Theorem 1.1

We use the notation introduced in Section 4. In view of Proposition 4.2, the following is the standing assumption for the rest of this section. There is a root $\alpha \in \Phi$ so that

$$s_\alpha = s_{-\alpha} > 0 \qquad \text{and} \qquad s_\beta = 0 \tag{5.1}$$

for any $\beta \in \Phi_\alpha^\pm \setminus \{\alpha, -\alpha\}$. Let us put

$$Z_\alpha := Z_G(U_\alpha) \cap Z_G(U_{-\alpha}) = Z_G(H_\alpha).$$

We have the following.

LEMMA 5.1 ([11, Lemma 4.4(1)])
There is a null set N so that, for all $x \in X \setminus N$, we have

$$W_G^+(a_\alpha)x \cap (X \setminus N) \subset U_\alpha x.$$

In particular, for all $x \in X \setminus N$ if $u \in W_G^+(a_\alpha)$ is so that $ux \in W_G^+(a_\alpha)x \cap (X \setminus N)$ and $\mu_x^\alpha = \mu_{ux}^\alpha$, then $u \in \mathcal{I}_x^\alpha$.

Proof
In view of Lemma 3.9, there is a null set N_1 so that, for all $x \in X \setminus N_1$, we have that

$\mu_x^{W_G^+(a_\alpha)}$ is a product of the leafwise measures μ_x^β for all $U_\beta \subset W_G^+(a_\alpha)$. By (5.1), it follows that

$$\operatorname{supp}(\mu_x^{W_G^+(a_\alpha)}) = \operatorname{supp}(\mu_x^\alpha) \quad \text{for all } x \in X \setminus N_1. \tag{5.2}$$

Recall also that there is a null set N_2 so that if $x, ux \in X \setminus N_2$ for some $u \in W_G^+(a_\alpha)$, then

$$\mu_x^{W_G^+(a_\alpha)} \propto \mu_{ux}^{W_G^+(a_\alpha)} u. \tag{5.3}$$

Let $x \in X \setminus (N_1 \cup N_2)$. Then, by (5.2), we have $\operatorname{supp}(\mu_x^{W_G^+(a_\alpha)}) \subset U_\alpha$. Therefore, by (5.3), we get $u \in U_\alpha$. This finishes the proof of the first claim if we require that $N \supsetneq N_1 \cup N_2$.

To see the last assertion, let $N_3 \subset X$ be a null subset so that $\mu_{ux}^\alpha u \propto \mu_x^\alpha$ for all $x \notin N_3$. Set $N = N_1 \cup N_2 \cup N_3$. Let $x \in X \setminus N$, and let u be as in the statement. In view of the first part in the lemma, we have $u \in U_\alpha$. Our assumption and the fact that U_α is a commutative group give

$$u\mu_x^\alpha = \mu_{ux}^\alpha u \propto \mu_x^\alpha.$$

Now one argues as in the proof Lemma 3.11 and gets $u \in \mathcal{I}_x^\alpha$. □

We also recall the following definition from [13].

Definition 5.2
Let $H, Z \subset G$ be closed subgroups of G. We say that the leafwise measures μ_x^H are *locally Z-aligned modulo μ* if, for every $\varepsilon > 0$ and neighborhood $\mathsf{B}_{\mathrm{id}}^Z \subset Z$ of the identity, there exists a compact set Q with $\mu(\mathsf{Q}) > 1 - \varepsilon$ and some $\delta > 0$ so that for every $x \in \mathsf{Q}$ we have

$$\{y \in \mathsf{Q} : \mu_x^H = \mu_y^H\} \cap \mathsf{B}_x(\delta) \subset \mathsf{B}_{\mathrm{id}}^Z x.$$

The following is a direct corollary of the main result of [13], proved there explicitly also for the positive-characteristic case.

THEOREM 5.3 ([13, Theorem 1.4])
Under the assumption (5.1), one of the following holds.
(LE-1) *We have that μ_x^α is locally Z_α-aligned modulo μ.*
(LE-2) *There exists an a_α-invariant subset $X_{\mathrm{inv}}(\alpha) \subset X$ with $\mu(X_{\mathrm{inv}}(\alpha)) > 0$ so that for all $x \in X_{\mathrm{inv}}(\alpha)$ there is an unbounded sequence $\{u_{x,m}\} \subset W_G^+(a_\alpha)$ such that $\mu_x^\alpha = \mu_{u_{x,m}x}^\alpha$.*

6. Proof of Theorem 1.1

Recall the notation in Section 2.1 and in particular, $k = K_v$, where K is a global function field and v is a place of K and we work with the maximal torus $\mathbf{A}$. Throughout our discussion, $\Gamma \subset \mathrm{SL}(d,k)$ is a lattice of inner type (see Section 2.4).

Put $\mathrm{GL}(n,\mathfrak{o})_m = \ker(\mathrm{GL}(n,\mathfrak{o}) \to \mathrm{GL}(n,\mathfrak{o}/\theta^{-m}\mathfrak{o}))$.

LEMMA 6.1 ([11, Lemma 5.3])
For any positive integer n there exists some $m = m(n) \geq 1$ with the following property. Let $a = \mathrm{diag}(a_1,\dots,a_n)$ with

$$\left|v(a_i) - v(a_j)\right| > m \quad \textit{for all } i \neq j.$$

Then ga is diagonalizable over k, for all $g \in \mathrm{GL}(n,\mathfrak{o})_m$. Moreover, if $a_1',\dots,a_n'$ are the eigenvalues of ga, then it is possible to order them so that $v(a_i) = v(a_i')$ for all i.

Proof
Let $\tilde{k}_n$ be the composite of all field extensions of k of degree at most $n!$. Then the characteristic polynomial of any element in $\mathrm{GL}(n,k)$ splits over $\tilde{k}_n$. Moreover, $\tilde{k}_n$ is a local field; that is, $\tilde{k}_n/k$ is a finite extension. We let v denote the unique extension of v to $\tilde{k}_n$.

We begin with the following observation. There is some $m_n \geq 1$ so that every $g \in \mathrm{GL}(n,\mathfrak{o})_{m_n}$ can be decomposed as $g = g^- g^0 g^+$ with $g^\pm \in W^\pm \cap \mathrm{GL}(n,\mathfrak{o})_1$ and $g^0 \in A \cap \mathrm{GL}(n,\mathfrak{o})_1$, where W^+ (resp., W^-) is the group of upper (resp., lower) triangular unipotent matrices. Indeed, in view of (3.8), the product map is a diffeomorphism from

$$\left(W^- \cap \mathrm{GL}(n,\mathfrak{o})_1\right) \times \left(A \cap \mathrm{GL}(n,\mathfrak{o})_1\right) \times \left(W^+ \cap \mathrm{GL}(n,\mathfrak{o})_1\right)$$

onto its image. Therefore the claim follows from the inverse function theorem.

We show that the lemma holds with $m = m_n$. First, note that after conjugating by a permutation matrix, we can assume that $v(a_1) > \cdots > v(a_n)$. Let $g \in \mathrm{GL}(n,\mathfrak{o})_m$, and let $b_1,\dots,b_n$ be the eigenvalues of ga listed with multiplicity and ordered so that $v(b_1) \geq \cdots \geq v(b_n)$. Note that $b_i \in \tilde{k}_n$ for all $1 \leq i \leq n$. Let $\|\ \|$ be the max norm on the ith exterior power $\wedge^i \tilde{k}_n^n$ with respect to the standard basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i}\}$. Denote by $\|\ \|$ the operator norm of the action of $\mathrm{GL}(n,\tilde{k}_n)$ on $\wedge^i \tilde{k}_n^n$ for $1 \leq i \leq n$. Choosing a basis of $\tilde{k}_n^n$ consisting of the generalized eigenvectors for ga, we get

$$\lim_\ell \left\|\wedge^i (ga)^\ell\right\|^{1/\ell} = |b_1 \cdots b_i| \quad \text{for all } i. \tag{6.1}$$

We now claim that

$$\left\|\wedge^i (ga)^\ell\right\| = \|\wedge^i a^\ell\| = |a_1 \cdots a_i|^\ell \quad \text{for all } \ell. \tag{6.2}$$

The second equality in the claim is immediate. To see the first equality, note that if $g_1, g_2 \in \mathrm{GL}(n, \mathfrak{o})_m$, then

$$g_1 a g_2^- g_2^0 g_2^+ a = g_1 (a g_2^- a^{-1}) a^2 g_2^0 (a^{-1} g_2^+ a).$$

Moreover, since $g^{\pm} \in \mathrm{GL}(n, \mathfrak{o})_1$ and $v(a_i) - v(a_{i+1}) > m$ for all i, we have that $a g_2^- a^{-1}$ and $g_2^0 a^{-1} g_2^+ a$ belong to $\mathrm{GL}(n, \mathfrak{o})_m$. Using this, we get

$$(ga)^\ell = g_\ell a^\ell a'_\ell g'_\ell,$$

where $g_\ell, g'_\ell \in \mathrm{GL}(n, \mathfrak{o})_m$ and $a'_\ell \in \mathrm{GL}(n, \mathfrak{o})_1$ for all ℓ. This implies (6.2).

Now (6.1) and (6.2) imply that $v(a_i) = v(b_i)$ for all $1 \le i \le n$, in particular, $v(b_i) \neq v(b_j)$ whenever $i \neq j$. This implies that the b_i's are distinct and hence ga is a semisimple element. We now show that $b_i \in k$ for all i. Recall that $b_1, \dots, b_n$ are roots of the characteristic polynomial of ga which is a polynomial with coefficients in k. For every $1 \le i \le n$, let $\mathrm{Gal}(b_i) = \{b_j : b_j \text{ is a Galois conjugate of } b_i\}$. Then $\{b_1, \dots, b_n\}$ is a disjoint union of $\bigsqcup_{j=1}^r \mathrm{Gal}(b_{i_j})$ for some $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$. Since $v(b_i) \neq v(b_j)$ whenever $i \neq j$ and Galois automorphisms preserve the valuation, we get that $\mathrm{Gal}(b_i) = \{b_i\}$ for all i. This establishes the final claim in the lemma. ☐

PROPOSITION 6.2

Recall that Γ is an inner type lattice. Then μ_x^α is not locally Z_α-aligned modulo μ. In particular, under the assumption (5.1), we have that (LE-2) in Theorem 5.3 holds.

Proof

We recall the argument from the proof of Theorem 5.1 in [11]. Let m be large enough so that the conclusion of Lemma 6.1 holds with $n = d - 2$. Without loss of generality, we may assume that $\alpha(\mathrm{diag}(a_1, \dots, a_d)) = a_1 a_2^{-1}$. Define

$$\tilde{\mathfrak{B}} = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & C \end{pmatrix} : r \in 1 + \theta^{-2}\mathfrak{o}, C \in \mathrm{GL}(d-2, \mathfrak{o})_m \right\} \subset \mathrm{GL}(d, \mathfrak{o}).$$

Put $\mathfrak{B} := \tilde{\mathfrak{B}} \cap Z_\alpha$; we note that $\mathfrak{B}$ is a compact open subgroup of Z_α. Let $a = \mathrm{diag}(a_2, a_2, a_3, \dots, a_d) \in A \cap Z_\alpha$ with $v(a_2) \neq 0$, and $|v(a_i) - v(a_j)| > m$ for all $i > j \ge 2$. In particular, we have $\alpha(a) = 1$.

Suppose that (LE-1) holds. Then, by Poincaré recurrence for μ-a.e. $g\Gamma \in G/\Gamma$, there exists a sequence $\ell_i \to \infty$ so that

$$a^{\ell_i} g\Gamma \in \mathfrak{B} g\Gamma \quad \text{for all } i.$$

Hence, for all i there exist some $\gamma_i \in \Gamma$ and some $h_i \in \mathfrak{B}$ so that $h_i a^{\ell_i} = g\gamma_i g^{-1}$. Now Lemma 6.1 implies the following. If ℓ_i is large enough and we write

$$g\gamma_i g^{-1} = h_i a^{\ell_i} = \begin{pmatrix} r_i & 0 & 0 \\ 0 & r_i & 0 \\ 0 & 0 & D_i \end{pmatrix}, \tag{6.3}$$

then D_i is diagonalizable whose eigenvalues have the same valuation as $a_j^{\ell_i}$ for all $3 \le j \le d$. Dropping the few first terms if necessary, we assume that (6.3) holds for all i.

Since Γ is an inner-type lattice, there exists a central simple algebra B over K so that Γ is commensurable with Λ_B (see Section 2.4). There exists some i (which we fix) and infinitely many j's so that $\hat{\gamma}_j := \gamma_j \gamma_i^{-1} \in \Lambda_B$. We have

$$g\hat{\gamma}_j g^{-1} = h_j a^{\ell_j - \ell_i} h_i^{-1};$$

hence if $\ell_j - \ell_i$ is large enough, we get from $h_j, h_i^{-1} \in \mathfrak{B} \subset \mathrm{GL}(n, \mathfrak{o})_1$ that

$$g\hat{\gamma}_j g^{-1} = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & D \end{pmatrix},$$

where D is diagonalizable and whose eigenvalues have the same valuation as $a_j^{\ell_j - \ell_i}$ for all $3 \le j \le d$. Indeed, after conjugation by h_i^{-1}, we may apply Lemma 6.1. Altogether, (LE-1) in Theorem 5.3 implies that there exists an element $\gamma \in \Lambda_B$ with the following properties:

- γ is a semisimple element,
- no eigenvalue of γ is a root of unity,
- all of the eigenvalues of γ are simple except exactly one eigenvalue which has multiplicity 2.

We now claim that none of the eigenvalues of γ lies in K. To see this, assume that γ has an eigenvalue $\sigma \in K$. Recall from the definition of Λ_B in Section 2.4 that Λ_B is bounded in $\mathrm{SL}_{1,B}(K_w)$ for all $w \neq v$. In particular, $w(\sigma) = 0$; otherwise, the group generated by γ in $\mathrm{SL}_{1,B}(K_w)$ would be unbounded. This in view of the product formula implies that $v(\sigma) = 0$. Hence, σ is a root of unity, which is a contradiction.

Since $\gamma \in \Lambda_B$, by (2.3) we have that the coefficients of the characteristic polynomial of γ are in K. This and the fact that γ is semisimple imply that there exists a finite separable extension $\tilde{K}$ of K which contains the eigenvalues of γ (see [1, Section 4.1(c)]). Thus, using the above claim, we get that the eigenvalue with multiplicity 2 is not in K and is separable over K. Since any Galois conjugate of this eigenvalue is also an eigenvalue of γ with the same multiplicity, we get a contradiction with the fact that γ has only one nonsimple eigenvalue. $\square$

6.1. Pinsker components have nontrivial invariance

We begin with the following corollary of the results in Sections 4 and 5.

COROLLARY 6.3

Under the assumptions of Theorem 1.1, we have the following: there exists some $\alpha \in \Phi$ and a μ-conull subset $X_{\mathrm{inv}}(\alpha) \subset X$ so that $\mathcal{I}_x^{\pm\alpha}$ are nondiscrete for all $x \in X_{\mathrm{inv}}(\alpha)$.

Proof

Since $h_\mu(a) > 0$, for some $a \in A$ there exists some $\alpha \in \Phi$ with $s_\alpha > 0$. In view of Proposition 4.2, the claim in the corollary holds true almost surely unless α satisfies (5.1).

However, in this case Theorem 5.3 and Proposition 6.2 imply that (LE-2) must hold true. Put $X' = \{x \in X : \mathcal{I}_x^{\pm\alpha} \text{ is nontrivial}\}$. By (LE-2) and Lemma 5.1, we get that X' has positive measure. Moreover, X' is A-invariant in view of Lemma 3.7(4). Since μ is A-ergodic, we get that $\mu(X') = 1$. Now choose $\ell \in \mathbb{Z}$ such that $X'_\ell = \{x \in X' : \mathcal{I}_x^{\pm\alpha} \cap U_{\pm\alpha}[\ell] \text{ is nontrivial}\}$ satisfies $\mu(X'_\ell) > 0$. Applying ergodicity and the pointwise ergodic theorem, we see that $x \in X$ a.e. satisfies that there exist some $a \in A$ and infinitely many $n \geq 0$ and infinitely many $n \leq 0$ such that $a_\alpha^n a x \in X'_\ell$. Using Lemma 3.7(4), this implies the corollary. □

Throughout the rest of this section, we fix some root α so that the conclusion of Corollary 6.3 holds true, and we put $X_{\mathrm{inv}} := X_{\mathrm{inv}}(\alpha)$. For any root β, let A_β denote the 1-parameter diagonal subgroup which is the group of k-points of the Zariski closure of the group generated by a_β. For the sake of notational convenience, we will denote $A_\beta = \{\check{\beta}(t) : t \in k^\times\}$, where $a_\beta = \check{\beta}(\theta)$. Recall that $V_{[\alpha]}$ is contained in $W_G^+(a_\alpha)$. For the rest of this section, we denote the Pinsker σ-algebra $\mathcal{P}_{a_\alpha}$ for a_α simply by $\mathcal{P}$. We further take a decomposition

$$\mu = \int_X \mu_x^{\mathcal{P}} \, \mathrm{d}\mu(x), \tag{6.4}$$

where $\mu_x^{\mathcal{P}}$ denotes the $\mathcal{P}$ conditional measure for μ-a.e. $x \in X$.

Since μ is A-invariant and A commutes with a_α, the σ-algebra $\mathcal{P}$ is A-invariant. Hence, we get

$$a\mu_x^{\mathcal{P}} = \mu_{ax}^{\mathcal{P}} \quad \text{for } \mu\text{-a.e. } x \in X. \tag{6.5}$$

Recall the definition of $H_\alpha = \varphi_\alpha(\mathrm{SL}(2,k))$ from the beginning of Section 4. For every $x \in X$, we put

$$\mathcal{H}_x := \{g \in H_\alpha : g\mu_x^{\mathcal{P}} = \mu_x^{\mathcal{P}}\}. \tag{6.6}$$

It follows from (6.5) that

$$\mathcal{H}_{ax} = a\mathcal{H}_x a^{-1} \tag{6.7}$$

for all $a \in A$ and μ-a.e. x.

COROLLARY 6.4
We have that $\langle \mathcal{I}_x^{\alpha}, \mathcal{I}_x^{-\alpha} \rangle$ is Zariski-dense in H_α as a k-group for μ-a.e. $x \in X_{\mathrm{inv}}$. Moreover, $\langle \mathcal{I}_x^{\alpha}, \mathcal{I}_x^{-\alpha} \rangle \subset \mathcal{H}_x$.

Proof
The first claim follows from Corollary 6.3. To see the second claim, note that by Lemma 3.14, we know that $\mathcal{I}_x^{\pm\alpha}$ is measurable with respect to $\mathcal{P}$. Equivalently, the groups $\mathcal{I}_x^{\pm\alpha}$ are (almost surely) constant on the atoms of a countably generated σ-algebra $\mathcal{P}'$ that is equivalent to $\mathcal{P}$. We now decompose μ as in (6.4) into conditional measures for the σ-algebra $\mathcal{P}'$, and we take the leafwise measures of $\mu_x^{\mathcal{P}'}$ for the subgroup U_α.

However, Proposition 3.4 implies that we may assume that the atoms with respect to $\mathcal{P}'$ are unions of U_α-orbits. This implies in turn for the leafwise measure that $(\mu_x^{\mathcal{P}'})_y^{U_\alpha} = \mu_y^{\alpha}$ for $\mu_x^{\mathcal{P}'}$-a.e. y and μ-a.e. x (see [18, Proposition 5.20] and [14, Proposition 7.22] for a similar argument). Fixing one such x, we obtain that $(\mu_x^{\mathcal{P}'})_y^{U_\alpha}$ is almost surely invariant under $\mathcal{I}_y^{\alpha} = \mathcal{I}_x^{\alpha}$. However, this implies by the relationship between the measure and its leafwise measures that $\mu_x^{\mathcal{P}'}$ is invariant under $\mathcal{I}_x^{\alpha}$. Since $\mu_x^{\mathcal{P}'} = \mu_x^{\mathcal{P}}$ almost surely, we may apply the same argument for $\mathcal{I}_x^{-\alpha}$. Therefore, $\mathcal{I}_x^{\pm\alpha} \subset \mathcal{H}_x$ for μ-a.e. x. $\square$

6.2. *Algebraic structure of $\mathcal{H}_x$*

Recall from the beginning of Section 4 that $H_\alpha = \varphi_\alpha(\mathrm{SL}_2(k))$. Put $U_{\pm\alpha}(\mathfrak{o}_v) = \varphi_\alpha(\mathbf{U}^{\pm}(\mathfrak{o}_v))$, where $\mathbf{U}^+$ (resp., $\mathbf{U}^-$) denotes the group of upper (resp., lower) triangular unipotent matrices in SL_2. Note that $H_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$. By Corollary 6.4, for μ-a.e. x we have $\langle \mathcal{I}_x^{\alpha}, \mathcal{I}_x^{-\alpha} \rangle \subset \mathcal{H}_x$. Define

$$\mathcal{Q}_x := \big\langle \mathcal{H}_x \cap U_\alpha(\mathfrak{o}_v), \mathcal{H}_x \cap U_{-\alpha}(\mathfrak{o}_v) \big\rangle. \tag{6.8}$$

Put

$$X_{\wp} := \{x \in X : \mathcal{Q}_x \text{ is Zariski-dense in } H_\alpha \text{ and } \mathcal{Q}_x \cap U_{\pm\alpha} \text{ are infinite}\}. \tag{6.9}$$

Corollary 6.4 and the above definitions imply that $X_{\wp} \cap X_{\mathrm{inv}}$ is conull in X_{inv}. In particular, Corollary 6.3 implies that $\mu(X_{\wp}) = 1$.

Note that for all $x \in X_{\wp}$, the group $\mathcal{Q}_x$ satisfies the conditions of Theorem A.1 in Section 3.1. For any $x \in X_{\wp}$, define

$$k'_x := \text{the field generated by } \big\{\mathrm{tr}\big(\rho(g)\big) : g \in \mathcal{Q}_x\big\},$$

and put

$$k_x := \begin{cases} k'_x & \text{if } \mathrm{char}(k) \neq 2, \\ \{c : c^2 \in k'_x\} & \text{if } \mathrm{char}(k) = 2. \end{cases} \tag{6.10}$$

Theorem A.1 then implies that there exist

(C-1) a unique (up to a unique isomorphism) k-isogeny $\varphi_x : \mathrm{SL}_2 \times_{k_x} k \to \mathrm{SL}_2$ whose derivative vanishes nowhere, and

(C-2) some nonnegative integer m_x

so that

$$\varphi_x\big(\mathrm{SL}(2,\mathfrak{o}_x)_{m_x}\big) \subset \mathcal{Q}_x \subset \varphi_x\big(\mathrm{SL}(2,k_x)\big), \tag{6.11}$$

where $\mathfrak{o}_x$ is the ring of integers in k_x and

$$\mathrm{SL}(2,\mathfrak{o}_x)_m := \ker\big(\mathrm{SL}(2,\mathfrak{o}_x) \to \mathrm{SL}(2,\mathfrak{o}_x/\varpi_x^m\mathfrak{o}_x)\big),$$

with ϖ_x a uniformizer in $\mathfrak{o}_x$. Let us put

$$E_x := \varphi_x\big(\mathrm{SL}(2,k_x)\big). \tag{6.12}$$

We will use without further remark the following lemma, which is a consequence of the implicit function theorem. The group generated by $\mathbf{U}^{\pm}(\varpi_x^m\mathfrak{o}_x)$ is an open subgroup of $\mathrm{SL}(2,\mathfrak{o}_x)_m$; for example, a direct computation yields that this group contains $\mathrm{SL}(2,\mathfrak{o}_x)_{2m}$.

LEMMA 6.5

Consider the Borel σ-algebra arising from the Chabauty topology on closed subgroups of $(k,+)$ and $\mathrm{SL}(d,k)$.

(1) *The map $x \mapsto k_x$ is a Borel map on $X_{\mathcal{P}}$.*

(2) *The equation (6.12) defines a Borel map, $x \mapsto E_x$, on $X_{\mathcal{P}}$.*

Proof

The map $x \mapsto \mathcal{Q}_x$ is a Borel map from a conull subset of X into the set of closed subgroups of $H_\alpha(\mathfrak{o}_v)$. This and (6.10) imply that $x \mapsto k_x$ is a Borel map on the conull set $X_{\mathcal{P}}$, as we claimed in (1).

By part (1), the map $x \mapsto k_x$ is a Borel map. Also recall from Lemma 3.1(1) that $E_x = \langle E_x \cap U_\alpha, E_x \cap U_{-\alpha}\rangle$. Therefore, part (2) follows if we show that the map $x \mapsto E_x \cap U_{\pm\alpha}$ is a Borel map. Note, however, that if we realize $U_{\pm\alpha} = \{u_r : r \in k\}$

as a k_x-vectors space, then $E_x \cap U_{\pm\alpha} = \{u_r : r \in k_x\}$ is a 1-dimensional k_x-subspace of $U_{\pm\alpha}$, respectively. Hence,

$$E_x \cap U_{\pm\alpha} = \{u_{rr'} : r \in k_x, u_{r'} \in \mathcal{Q}_x \cap U_{\pm\alpha}\},$$

which implies the claim. $\square$

LEMMA 6.6

We have the following.

(1) *The map $x \mapsto k_x$ is essentially constant.*

(2) *The map $x \mapsto E_x$ is an A-equivariant Borel map on a conull subset of X.*

Proof

We claim that $k_x \subset k_{ax}$ for all $a \in A$. First, let us note that, by symmetry, this also implies that $k_{ax} \subset k_x$. Therefore, it implies that the map $x \mapsto k_x$ is A-invariant; since μ is A-ergodic, we get part (1).

We now show the claim. Let m_x be as in (C-2). Recall from (6.7) that there is a full measure set $X' \subset X$ so that, for all $x \in X'$ and all $a \in A$, we have $\mathcal{H}_{ax} = a\mathcal{H}_x a^{-1}$. Now, for any a there exists some $m_{x,a} \geq m_x$ so that if $m \geq m_{x,a}$, then

$$a\varphi_x\big(\mathrm{SL}(2, \mathfrak{o}_x)_m\big)a^{-1} \subset \mathcal{Q}_{ax}. \tag{6.13}$$

Define $l_x(m)$ to be the field generated by $\{\mathrm{tr}(\rho(g)) : g \in \varphi_x(\mathrm{SL}(2, \mathfrak{o}_x)_m)\}$. Then

$$l_x(m) = k_x \quad \text{for all } m \geq m_x. \tag{6.14}$$

Indeed, this is true for the field generated by $\{\mathrm{tr}(\rho(g)) : g \in \mathrm{SL}(2, \mathfrak{o}_x)_m\}$. Since φ_x has nowhere vanishing derivative and there are no nonstandard isogenies for type A_1 (see [29, Proposition 1.6]), we get $\rho_1 = \rho_2 \circ \varphi_x$, where ρ_1 and ρ_2 are the adjoint representation on the source and the target of φ_x. This implies (6.14). It follows from (6.13) and (6.14) that $k_x \subset k_{ax}$, as we claimed.

Let us now prove part (2). By part (1), there is an A-invariant conull set X' and a subfield k' so that $k_x = k'$ for all $x \in X'$. Let $\mathfrak{o}'$ denote the ring of integers in k'. We note that the same proof as in the proof of Lemma 6.5(2) implies that $E_x \cap U_{\pm\alpha}$ is the Zariski closure of $C \cap U_{\pm}$ in $\mathcal{R}_{k/k'}(\mathrm{SL}_d)$ for any nontrivial open subgroup C of $\mathcal{Q}_x$.

Now let $a \in A$ and $x \in X'$. Then by (6.13), we have

$$a\varphi_x\big(\mathrm{SL}(2, \mathfrak{o}')_m\big)a^{-1} \subset \mathcal{Q}_{ax}$$

for all $m \geq m_{x,a}$. Since $a\mathcal{H}_x a^{-1} = \mathcal{H}_{ax}$ and $\varphi_x(\mathrm{SL}(2, \mathfrak{o}')_m)$ is open in $\mathcal{Q}_x$ by (6.11), we thus get that $a\varphi_x(\mathrm{SL}(2, \mathfrak{o}')_m)a^{-1}$ is open in $\mathcal{Q}_{ax}$ for all $m \geq m_{x,a}$. Since $U_{\pm\alpha}$ are normalized by A, for all $a \in A$ and all $m \geq m_{x,a}$ we have

$$a\varphi_x\big(\mathrm{SL}(2,\mathfrak{o}')_m\big)a^{-1} \cap U_{\pm\alpha} = a\big(\varphi_x\big(\mathrm{SL}(2,\mathfrak{o}')_m\big) \cap U_{\pm\alpha}\big)a^{-1}.$$

Taking the Zariski closure in $\mathcal{R}_{k/k'}(\mathrm{SL}_d)$, we get that

$$a(E_x \cap U_{\pm\alpha})a^{-1} = E_{ax} \cap U_{\pm\alpha}.$$

This and Lemma 3.1(1) imply the claim. □

PROPOSITION 6.7
For μ-a.e. $x \in X_{\mathcal{P}}$, we have $E_x \subset \mathcal{H}_x$.

Proof
Let $x \in X_{\mathcal{P}}$, and put $A'_x := E_x \cap A$. In view of Lemma 6.6(2), we have

$$A'_{ax} = E_{ax} \cap A = aE_xa^{-1} \cap A = a(E_x \cap A)a^{-1} = A'_x \tag{6.15}$$

for μ-a.e. x and all $a \in A$. Since μ is A-ergodic, we get that $x \mapsto A'_x$ is essentially constant. Let us denote by A' this essential value.

Then by Lemma 3.1(2), we have that A' is an unbounded subgroup of $A_\alpha = H_\alpha \cap A$. The group A_α is a 1-dimensional, k-split, k-torus; therefore, A_α/A' is compact. For any $s \in k$, we let $\check{\alpha}(s) \in A_\alpha$ be the cocharacter associated to α and evaluated at s—that is, $\check{\alpha}(s)$ is the diagonal matrix with eigenvalues s, s^{-1} and 1 with multiplicity $d-2$ so that $\alpha(\check{\alpha}(s)) = s^2$. This implies that there exist some $\ell > 0$ and some $r \in \mathfrak{o}_v^\times$, so that if we put $s := \theta^\ell r$, then $\check{\alpha}(s) \in E_x$. In particular, $\check{\alpha}(s)$ normalizes both $E_x \cap U_\alpha$ and $E_x \cap U_{-\alpha}$.

For every $\varepsilon > 0$, there is subset $X_{\mathcal{P}}(\varepsilon) \subset X_{\mathcal{P}}$ with $\mu(X_{\mathcal{P}}(\varepsilon)) > 1-\varepsilon$ so that the map

$$x \mapsto \mu_x^{\mathcal{P}}$$

is continuous on $X_{\mathcal{P}}(\varepsilon)$. Now by Poincaré recurrence, for μ-a.e. $x \in X_{\mathcal{P}}(\varepsilon)$ there is a sequence $n_{x,i} \to \infty$ so that $\check{\alpha}(s^{n_{x,i}}) \in X_{\mathcal{P}}(\varepsilon)$ for all i and $\check{\alpha}(s^{n_{x,i}})x \to x$. Then

$$\lim_{i\to\infty} \mathcal{H}_{\check{\alpha}(s^{n_{x,i}})x} \subset \mathcal{H}_x.$$

Recall from (6.11) that $\mathcal{Q}_x \cap U_\alpha$ contains an open compact subgroup of $E_x \cap U_\alpha$. Therefore, using (6.7) we get that

$$E_x \cap U_\alpha \subset \lim_{i\to\infty} \check{\alpha}(s^{n_{x,i}})(\mathcal{Q}_x \cap U_\alpha)\check{\alpha}(s^{-n_{x,i}}) \subset \lim_i \mathcal{H}_{\check{\alpha}(s^{n_{x,i}})x} \subset \mathcal{H}_x$$

for μ-a.e. $x \in X_{\mathcal{P}}(\varepsilon)$. Choosing a sequence $\varepsilon_n \to 0$, we get that $E_x \cap U_\alpha \subset \mathcal{H}_x$ for μ-a.e. $x \in X_{\mathcal{P}}$. Similarly, we get $E_x \cap U_{-\alpha} \subset \mathcal{H}_x$ for μ-a.e. $x \in X_{\mathcal{P}}$. Recall from Lemma 3.1(1) that E_x is generated by $E_x \cap U_{\pm\alpha}$. Therefore, $E_x \subset \mathcal{H}_x$ for μ-a.e. $x \in X_{\mathcal{P}}$. □

6.3. Applying the measure classification for semisimple groups

We now apply the measure classification theorem due to Golsefidy and the third author (Theorem B from Section 3.2).

LEMMA 6.8

Let μ be as in Theorem 1.1. Then there exist a closed infinite subfield $l < k$ and a smooth algebraic l-subgroup $\mathbf{M} < \mathcal{R}_{k/l}(\mathrm{SL}_d)$ such that $\mathbf{M}(l) \cap \Gamma$ is Zariski-dense in $\mathbf{M}$ over l, and a noncentral cocharacter $\lambda : \mathbf{G}_m \to \mathbf{M}$ over l so that the topological group

$$L = \overline{M^+(\lambda)\big(\mathbf{M}(l) \cap \Gamma\big)}$$

satisfies that $L/(L \cap \Gamma)$ has finite volume. Moreover, for μ-a.e. x, the E_x-ergodic component of $\mu_x^{\mathcal{P}}$ equals $h\nu_L$ for some $h \in \mathrm{SL}(d,k)$ so that $x = h\Gamma$, and ν_L is the homogeneous measure on $L/(L \cap \Gamma)$.

Proof

Let k' denote the essential value of the map $x \mapsto k_x$ (see Lemma 6.6(1)). In view of Proposition 6.7, for μ-a.e. x the measure $\mu_x^{\mathcal{P}}$ is invariant under E_x.

Since the σ-algebra $\mathcal{P}$ is A-invariant, we have $a\mu_x^{\mathcal{P}} = \mu_{ax}^{\mathcal{P}}$ for all $a \in A$ and μ-a.e. x. Moreover, by Lemma 6.6(2), we have $E_{ax} = aE_x a^{-1}$ for μ-a.e. x. Therefore, if we let

$$\mu_x^{\mathcal{P}} = \int \nu_z \,\mathrm{d}\mu_x^{\mathcal{P}}(z) \tag{6.16}$$

be the ergodic decomposition of $\mu_x^{\mathcal{P}}$ with respect to E_x (where for $\mu_x^{\mathcal{P}}$-a.e. z we let ν_z denote the E_x-ergodic components of $\mu_x^{\mathcal{P}}$), then

$$\mu_{ax}^{\mathcal{P}} = \int a_*\nu_z \,\mathrm{d}\mu_x^{\mathcal{P}}(z) \tag{6.17}$$

is the ergodic decomposition of $\mu_{ax}^{\mathcal{P}}$ with respect to E_{ax}.

Applying Theorem B in Section 3.2, we conclude that for $\mu_x^{\mathcal{P}}$-a.e. z the measure ν_z is described as follows. There exist

(B-1) $l_z = (k')^{q_z} \subset k$, where $q_z = p^{n_z}$, $p = \mathrm{char}(k)$ and $n_z \ge 1$,

(B-2) a connected l_z-subgroup $\mathbf{M}_z$ of $\mathcal{R}_{k/l_z}(\mathrm{SL}_d)$ so that $\mathbf{M}_z(l_z) \cap \Gamma$ is Zariski-dense in $\mathbf{M}_z$,

(B-3) an element $g_z \in G$,

such that ν_z is the $g_z L_z g_z^{-1}$-invariant probability Haar measure on the closed orbit $g_z L_z \Gamma/\Gamma$ with

$$L_z = \overline{M_z^+(\lambda_z)\big(\mathbf{M}_z(l_z) \cap \Gamma\big)},$$

where

- the closure is with respect to the Hausdorff topology, and
- $\lambda_z : \mathbf{G}_m \to \mathbf{M}_z$ is a noncentral l_z-homomorphism, $M_z^+(\lambda_z)$ is defined in (3.9), and $E_x \subset M_z^+(\lambda_z)$.

Note that $\mathbf{M}_z$ in (B-2) is l_z-smooth—indeed, $\mathbf{M}_z(l_z)$ is Zariski-dense in $\mathbf{M}_z$ (see [35, Lemma 11.2.4(ii)]).

For any z where ν_z is described as above, let $(l_z, [\mathbf{M}_z], [M_z^+(\lambda_z)])$ be the corresponding triple where $[\bullet]$ denotes the Γ conjugacy class. This is well defined and we will refer to it as the *triple associated to* z. Given a triple $(l, [\mathbf{M}], [M^+(\lambda)])$, put

$$\mathfrak{S}\big(l, [\mathbf{M}], \big[M^+(\lambda)\big]\big) = \big\{z \in X : \big(l, [\mathbf{M}], \big[M^+(\lambda)\big]\big) \text{ is associated to } z\big\}.$$

Note that there are only countably many such triples. Indeed, there are only countably many closed subfields $l \subset k'$ as in Theorem B(1); also, there are only countably many $\mathbf{M}$'s as in Theorem B(2). For any such l and $\mathbf{M}$, there are only countably many choices of $M^+(\lambda)$ by Lemma 3.2(2). Therefore, there exists a triple $(l, [\mathbf{M}], [M^+(\lambda)])$ such that

$$\mu\big(\mathfrak{S}\big(l, [\mathbf{M}], \big[M^+(\lambda)\big]\big)\big) > 0.$$

Note, however, that in view of (6.17), $\mathfrak{S}(l, [\mathbf{M}], [M^+(\lambda)])$ is A-invariant. This, together with the fact that μ is A-ergodic, implies that

$$\mu\big(\mathfrak{S}\big(l, [\mathbf{M}], \big[M^+(\lambda)\big]\big)\big) = 1.$$

This finishes the proof of the lemma. $\square$

We let l, $\mathbf{M}$, and $L := \overline{M^+(\lambda)(\mathbf{M}(l) \cap \Gamma)}$ be as in Lemma 6.8. Define

$$\mathbf{N} := \text{the Zariski closure of } N_{G'}\big(\mathbf{M}(l)\big) \cap \Gamma \text{ in } \mathbf{G}', \tag{6.18}$$

where $\mathbf{G}' := \mathcal{R}_{k/l}(\mathrm{SL}_d)$ and $G' := \mathbf{G}'(l) = \mathrm{SL}(d,k)$. Therefore, $\mathbf{N}$ is a smooth group defined over l (see, e.g., [35, Lemma 11.2.4(ii)]). In view of (B-2) above, we have

$$\mathbf{M} \subset \mathbf{N}^\circ \qquad \text{and} \qquad \mathbf{N} \subset N_{\mathbf{G}'}(\mathbf{M}), \tag{6.19}$$

where $\mathbf{N}^\circ$ denotes the connected component of the identity in $\mathbf{N}$.

LEMMA 6.9

We let A_l^{sp} be the group of l-points of the maximal, l-split, torus subgroup of $\mathcal{R}_{k/l}A$. Then there exists some $g_0 \in \mathrm{SL}(d,k)$ so that $A_l^{\mathrm{sp}} \subset g_0\mathbf{N}(l)g_0^{-1}$ and $\overline{Ag_0\Gamma/\Gamma} =$ $\mathrm{supp}(\mu)$.

Proof
Recall that $L\Gamma$ is a closed subset of G and that, for μ-a.e. x and $\mu_x^{\mathcal{P}}$-a.e. z, we have

$$\operatorname{supp}(\nu_z) = gL\Gamma/\Gamma \tag{6.20}$$

for some $g \in G$. We note that, while the element g is not well defined, the set $gL\Gamma$ is well defined. This, in view of (B-2), determines the set $g\mathbf{M}(l)\Gamma$ as the smallest set of the form $\mathbf{R}(l)\Gamma$, where $\mathbf{R}$ is an l-subvariety so that $\nu_z(\mathbf{R}(l)\Gamma/\Gamma) > 0$ (see [28, Theorem 6.9]; see also the original [27, Proposition 3.2]). Now let $g, g' \in G$ be such that $g\mathbf{M}(l)\Gamma = g'\mathbf{M}(l)\Gamma$. Then

$$\mathbf{M}(l) \subset \bigcup_{\gamma} g^{-1}g'\mathbf{M}(l)\gamma.$$

Hence, by Baire's category theorem, there is some γ_0 so that $\mathbf{M}(l) \cap g^{-1}g'\mathbf{M}(l)\gamma_0$ is open in $\mathbf{M}(l)$. Since $\mathbf{M}$ is Zariski-connected, any open (in Hausdorff topology) subset of $\mathbf{M}(l)$ is Zariski-dense in $\mathbf{M}$ (see [26, Chapter 1, Proposition 2.5.3]). This and equality of the dimensions imply that $\mathbf{M}(l) = g^{-1}g'\mathbf{M}(l)\gamma_0$. Therefore, $g^{-1}g'm_0\gamma_0 = 1$ for some $m_0 \in \mathbf{M}(l)$ and we get

$$\mathbf{M}(l) = g^{-1}g'\mathbf{M}(l)\gamma_0 = \gamma_0^{-1}\mathbf{M}(l)\gamma_0.$$

That is, $\gamma_0 \in N_G(\mathbf{M}(l)) \cap \Gamma$ and

$$g^{-1}g' = \gamma_0^{-1}m_0^{-1} \in \big(N_G\big(\mathbf{M}(l)\big) \cap \Gamma\big)\mathbf{M}(l).$$

Hence, by (6.18) and (6.19), we have

$$g^{-1}g' \in \big(N_G\big(\mathbf{M}(l)\big) \cap \Gamma\big)\mathbf{M}(l) \subset \mathbf{N}(l). \tag{6.21}$$

Let $N = \mathbf{N}(l)$ and $G' = \mathbf{G}'(l) = \mathrm{SL}(d,k)$. Then, by (6.21), we get a Borel measurable map f from $\mathfrak{S}(l, \mathbf{M}, M^+(\lambda))$ to $G'/N = \mathrm{SL}(d,k)/N$ defined by $f(x) = g_x N$.

The preceding discussion, in view of (6.17), implies that f is an A-equivariant Borel map, where the action of A on $\mathrm{SL}(d,k)/N$ is induced from the natural action of $\mathcal{R}_{k/l}(\mathbf{A})$ on $\mathbf{G}'/\mathbf{N}$.

Now by Lemma 3.3 there exists some

$$g_0 N \in \operatorname{Fix}_{A_l^{\mathrm{sp}}}\big(\mathrm{SL}(d,k)/N\big)$$

so that $f_*\mu$ is the A-invariant measure on the compact orbit Ag_0N. Using the Birkhoff ergodic theorem for the action of A on X and the compactness of the orbit Ag_0N, we can choose the representative $g_0 \in \mathrm{SL}(d,k)$ so that $\overline{Ag_0\Gamma/\Gamma} = \operatorname{supp}(\mu)$. Let us recall that $\operatorname{Fix}_{A_l^{\mathrm{sp}}}(\mathrm{SL}(d,k)/N) = \{gN : g^{-1}A_l^{\mathrm{sp}}g \subset N\}$. In particular, g_0 satisfies

$$g_0^{-1} A_l^{\mathrm{sp}} g_0 \subset N, \tag{6.22}$$

as we claimed. □

6.4. The algebraic K-groups **F** *and* **H**

While the groups **M** <**N** are still somewhat mysterious at this stage, we can describe their k-Zariski closure quite precisely. Recall that $\Gamma \subset \mathrm{SL}(d,k)$ is a lattice of inner type. Hence, there exists a central simple algebra B over K so that Γ is commensurable with Λ_B (see Section 2.4). We define the shorthand $\Gamma_B := \Gamma \cap \Lambda_B$.

LEMMA 6.10

With notation as in Theorem 1.1, let $\mathbf{F}$ *be a connected, noncommutative algebraic subgroup of* SL_d *so that* $\mathbf{F}(k) \cap \Gamma$ *is Zariski-dense in* $\mathbf{F}$ *and* $A' = A \cap g_0 \mathbf{F}(k) g_0^{-1}$ *is cocompact in* A *for some* $g_0 \in \mathrm{SL}(d,k)$. *Then* $\mathbf{F}$ *is defined over* K *and we have the following:*

(1) $g_0^{-1} \mathbf{A} g_0 \subset \mathbf{F}$;

(2) $\mathbf{F}$ *has no* $\tilde{K}$*-rational character for any purely inseparable algebraic field extension* $\tilde{K}$ *of* K;

(3) $\mathbf{F}$ *is a reductive* K*-group;*

(4) $\mathbf{F}(k) \cap \Gamma$ *is a lattice in* $\mathbf{F}(k)$;

(5) *the commutator group* $[\mathbf{F},\mathbf{F}]$ *is nontrivial, simply connected, and almost* K*-simple;*

(6) *moreover,* $[\mathbf{F},\mathbf{F}](k) \cong \prod_{i=1}^{n} \mathrm{SL}(d_0,k)$ *with* $d = n d_0$; *in fact, apart from the order of the indices, the group* $g_0[\mathbf{F},\mathbf{F}]g_0^{-1}$ *equals the subgroup consisting of* n *block matrices along the diagonal.*

Proof

Since Γ_B is finite index in Γ and $\mathbf{F}$ is connected, we have that $\mathbf{F}(k) \cap \Gamma_B$ is Zariski-dense in $\mathbf{F}$. This and the fact that $\Gamma_B \subset \Lambda_B$ imply that $\mathbf{F}$ is defined over K (see [35, Lemma 11.2.4(ii)]). Since A/A' is compact and since $\mathbf{A}$ is Zariski-connected and k-split, we also have that A' is Zariski-dense in $\mathbf{A}$. Since also $g_0^{-1} A' g_0 \subset \mathbf{F}(k)$, we obtain $g_0^{-1} \mathbf{A} g_0 \subset \mathbf{F}$ as k-groups. Let $\tilde{K}$ be a finite purely inseparable extension of K. For every place w of k, there exists a unique extension $\tilde{w}$ of w to $\tilde{K}$. Recall that $k = K_v$ for a fixed place v of K. Let (see (2.4))

$$\tilde{\Lambda}_B = \{\gamma \in \mathrm{SL}_{1,B}(\tilde{K}) : \gamma \in \mathrm{SL}_{1,B}(\mathfrak{o}_{\tilde{w}}) \text{ for all } \tilde{w} \neq \tilde{v}\}.$$

Let $\tilde{\mathcal{O}}$ be the ring of $\tilde{v}$-integers in $\tilde{K}$. Suppose that χ is an arbitrary $\tilde{K}$-rational character of $\mathbf{F}$. Then there exists some $D \in \tilde{\mathcal{O}}$, depending on χ, so that

$$\mathfrak{B} := \chi(\Gamma_B \cap \mathbf{F}) \subset \chi\big(\tilde{\Lambda}_B \cap \mathbf{F}(\tilde{K})\big) \subset \frac{1}{D}\tilde{\mathcal{O}}.$$

In particular, there exists some $\ell_0 \in \mathbb{Z}$ so that, for any place $\tilde{w} \neq \tilde{v}$ in $\tilde{K}$ and any $r \in \mathfrak{B}$, we have $\tilde{w}(r) \geq \ell_0$. Note further that $\mathfrak{B}$ is a multiplicative group; hence, $\tilde{w}(\mathfrak{B})$ is a subgroup of $(\mathbb{Z}, +)$. In consequence, we have $\tilde{w}(r) = 0$ for any place $\tilde{w} \neq \tilde{v}$ in $\tilde{K}$ and any $r \in \mathfrak{B}$. By the product formula, we also get that $\tilde{v}(r) = 0$ for all $r \in \mathfrak{B}$. Therefore, $\mathfrak{B}$ is a finite group consisting of roots of unity. This implies that there is a finite index subgroup $\Gamma' \subset \Gamma_B \cap \mathbf{F}$ so that $\chi(\Gamma') = 1$. Since $\mathbf{F}$ is connected and $\Gamma_B \cap \mathbf{F}$ is Zariski-dense in $\mathbf{F}$, the group Γ' is also Zariski-dense in $\mathbf{F}$. This implies that χ is trivial on $\mathbf{F}$, as claimed in (2).

We note that part (2) and [5, Theorem 1.3.6] imply part (4) directly. Below we give an argument using (3) which avoids the full force of [5, Theorem 1.3.6]. In particular, the classification of pseudoreductive groups in [6] which is used to resolve the main difficulties in [5] is not used in our proof of (4).

We now prove part (3). Let $\tilde{K}$ be a finite, purely inseparable extension of K so that $R_u(\mathbf{F})$ is defined and splits over $\tilde{K}$ (see [1, Corollaries 15.5, 18.4]). Restricting the adjoint representation of $\mathbf{F}$ to the Lie algebra of $R_u(\mathbf{F})$ and taking the determinant, we obtain a $\tilde{K}$-character. We claim that if $R_u(\mathbf{F})$ is nontrivial, then this character is also nontrivial. In view of this claim, (3) follows from (2).

We now show the claim. Recall that $R_u(\mathbf{F})$ is a $\tilde{K}$-split, unipotent subgroup. Since SL_d is simply connected, we get from [20] (see also [3]) that there exists a $\tilde{K}$-parabolic subgroup $\mathbf{P}$ of SL_d so that $R_u(\mathbf{F}) \subset R_u(\mathbf{P})$ and $N_{\mathrm{SL}_d}(R_u(\mathbf{F})) \subset \mathbf{P}$. The claim now follows; indeed, $g_0^{-1}\mathbf{A}g_0 \subset \mathbf{F} \subset N_{\mathrm{SL}_d}(R_u(\mathbf{F}))$ and $g_0^{-1}\mathbf{A}g_0$ is a maximal torus which is k-split and hence also $\tilde{K}_{\tilde{v}}$-split. Part (4) follows from (2), (3), and [22]. Note that the absence of a unipotent radical (defined over k or not) makes the necessary arguments in our case much simpler. For the rest of the argument, we fix a maximal K-torus $\mathbf{T}$ in $\mathbf{F}$ which is k-split (see [6, Corollary A.2.6]). Note that by [6, Theorem C.2.3], there is some $g \in \mathbf{F}(k)$ so that

$$g\mathbf{T}g^{-1} = g_0^{-1}\mathbf{A}g_0.$$

We now establish part (5). First, note that $\mathbf{F}$ is not commutative, so $[\mathbf{F}, \mathbf{F}]$ is nontrivial. Let K' be a separable field extension of K such that $\mathbf{T}$ splits over K'. Therefore, there exists some $g_1 \in \mathrm{SL}_d(K')$ so that $g_1\mathbf{T}g_1^{-1}$ is the full diagonal subgroup of SL_d. Moreover, let $\mathbf{T}_0 \subset \mathbf{T}$ be the central torus of $\mathbf{F}$. Then

$$g_1[\mathbf{F}, \mathbf{F}]g_1^{-1} \subset g_1\big[\mathbf{Z}_{\mathrm{SL}_d}(\mathbf{T}_0), \mathbf{Z}_{\mathrm{SL}_d}(\mathbf{T}_0)\big]g_1^{-1} = \prod_i \mathrm{SL}_{d_i}$$

for some integers $d_1, d_2, \ldots$ (that depend on the subgroup $g_1\mathbf{T}_0g_1^{-1}$). Since $\mathbf{T} \subset \mathbf{F}$ has absolute rank $d-1$, the rank of $[\mathbf{F}, \mathbf{F}]$ equals $d - 1 - \dim(\mathbf{T}_0)$. Moreover, the torus $\mathbf{T}_0$ is central in $\mathbf{Z}_{\mathrm{SL}_d}(\mathbf{T}_0)$; hence, we have

$$d - 1 - \dim(\mathbf{T}_0) \geq \operatorname{rank}\big(\big[\mathbf{Z}_{\mathrm{SL}_d}(\mathbf{T}_0), \mathbf{Z}_{\mathrm{SL}_d}(\mathbf{T}_0)\big]\big) = \sum_i (d_i - 1).$$

Together with the above inclusion, we thus get that $d - 1 - \dim(\mathbf{T}_0) = \sum_i (d_i - 1)$. Since $[\mathbf{F}, \mathbf{F}]$ is semisimple and $\prod_i \mathrm{SL}_{d_i}$ has no proper semisimple subgroup of the same rank, we get $g_1[\mathbf{F}, \mathbf{F}]g_1^{-1} = \prod_i \mathrm{SL}_{d_i}$. Let $W_1, \ldots$ be the various irreducible subspaces for the action of $[\mathbf{F}, \mathbf{F}]$ on the d-dimensional vector space that are defined over K' and correspond to the various blocks of $g_1[\mathbf{F}, \mathbf{F}]g_1^{-1}$. As $\mathbf{F}$ is nonabelian, at least one of the subspace (say, W_1) has dimension at least 2. Let $\mathbf{W}$ be the sum of W_1 and all its Galois images. Then $\mathbf{W}$ is invariant under $\mathbf{F}$ and is defined over K—recall that K'/K is separable. Since $\mathbf{F}$ has no K-rational characters, we see that $\mathbf{W}$ has full dimension. Otherwise, the determinant of the restriction of $\mathbf{F}$ to $\mathbf{W}$ gives a K-character which is nontrivial since $\mathbf{T}$ is a maximal torus—indeed, over K' we can conjugate $\mathbf{T}$ to $\mathbf{A}$ the diagonal subgroup. Now any subspace of the standard d-dimensional representation of SL_d that is invariant under $\mathbf{A}$ and whose weights sum to zero is trivial. This implies that $[\mathbf{F}, \mathbf{F}]$ is semisimple and almost K-simple. In particular, we obtain $d_i = d_j$ for all i, j, which gives part (6). $\square$

Define

$$\hat{\mathbf{F}} := \text{the Zariski closure of } \mathbf{N}(l) \cap \Gamma \text{ in } \mathrm{SL}_d. \tag{6.23}$$

In particular, $\hat{\mathbf{F}}$ is a smooth group defined over k (see [35, Lemma 11.2.4(ii)]). Put $\mathbf{F} = \hat{\mathbf{F}}^\circ$, the connected component of the identity in $\hat{\mathbf{F}}$. Since $[\Gamma : \Gamma_B] < \infty$, we have that $\mathbf{F}$ coincides with the connected component of the identity in $\hat{\mathbf{F}}_B :=$ the Zariski closure of $\mathbf{N}(l) \cap \Gamma_B$ in SL_d. Now $\hat{\mathbf{F}}_B$ is a smooth group defined over K; therefore, $\mathbf{F}$ is also a smooth group defined over K and hence over k.

LEMMA 6.11

(1) $\mathbf{N}(l) \subset \hat{\mathbf{F}}(k)$ *and hence* $\mathbf{N}(l)$ *is Zariski-dense in* $\hat{\mathbf{F}}$.

(2) $\mathbf{F}$ *satisfies the conditions in Lemma 6.10.*

Proof

For part (1), we note first that the definition (6.23) implies that

$$\mathbf{N}(l) \cap \Gamma \subset \hat{\mathbf{F}}(k) = \mathcal{R}_{k/l}(\hat{\mathbf{F}})(l) \subset \mathbf{G}'(l).$$

Therefore, by (6.18) we have $\mathbf{N} \subset \mathcal{R}_{k/l}(\hat{\mathbf{F}})$. Taking l-points, we get part (1).

We now show that part (1) implies (2). To see this, we first note that $\mathbf{F}$ is connected by definition. Next recall that by (6.19) we have $E_x \subset \mathbf{M}(l) \subset \mathbf{N}(l)$ for μ-a.e. x. In view of the definition of E_x (see (6.12)) and the fact that $\mathbf{F}$ is finite index in $\hat{\mathbf{F}}$, we get that $\mathbf{F}$ is noncommutative. Moreover, note that $\mathbf{F}$ is Zariski-open and closed in $\hat{\mathbf{F}}$. By the definition of $\hat{\mathbf{F}}$ in (6.23), we have that $\hat{\mathbf{F}}(k) \cap \Gamma$ is Zariski-dense in $\hat{\mathbf{F}}$. Together, it follows that $\mathbf{F}(k) \cap \Gamma$ is Zariski-dense in $\mathbf{F}$. Finally, by (6.22) we have $g_0^{-1} A_l^{\mathrm{sp}} g_0 \subset \mathbf{N}(l) \subset \hat{\mathbf{F}}(k)$. Since A_l^{sp} is cocompact in A, we obtain the last assumption—namely, that $A \cap g_0 \mathbf{F}(k) g_0^{-1}$ is cocompact in A. $\square$

Put

$$\hat{\mathbf{H}} := \text{the Zariski closure of } \mathbf{M}(l) \cap \Gamma_B \text{ in } \mathrm{SL}_d. \tag{6.24}$$

Note that $\hat{\mathbf{H}}$ is a smooth group defined over K and hence over k. Put $\mathbf{H} := \hat{\mathbf{H}}^{\circ}$, the connected component of the identity in $\hat{\mathbf{H}}$; then $\mathbf{H}$ is also a smooth group defined over K and hence over k.

LEMMA 6.12
(1) $\mathbf{M}(l) \subset \mathbf{H}(k)$*, and hence* $\mathbf{M}(l)$ *is Zariski-dense in* $\mathbf{H}$.
(2) $[\mathbf{F}, \mathbf{F}] = \mathbf{H}$.
(3) $\mathbf{H}$ *is almost* K*-simple.*
(4) $\mathbf{H}(k) \cong \prod \mathrm{SL}(d_0, k)$ *where* $d = n d_0$.

Proof
Recall from (B-2) that $\mathbf{M} \subset \mathcal{R}_{k/l}(\mathrm{SL}_d)$ is connected and that $\mathbf{M}(l) \cap \Gamma$ is Zariski-dense in $\mathbf{M}$. Since $\mathbf{M}$ is connected and $[\Gamma : \Gamma_B] < \infty$, we get that

$$\mathbf{M}(l) \cap \Gamma_B \text{ is Zariski-dense in } \mathbf{M}. \tag{6.25}$$

Therefore, as in the proof of Lemma 6.11(1), we have

$$\mathbf{M}(l) \cap \Gamma_B \subset \hat{\mathbf{H}}(k) = \mathcal{R}_{k/l}(\hat{\mathbf{H}})(l) \subset \mathbf{G}'(l).$$

This in view of our preceding discussion implies that $\mathbf{M} \subset \mathcal{R}_{k/l}(\hat{\mathbf{H}})$. Since $\mathbf{M}$ is connected and $\mathcal{R}_{k/l}(\mathbf{H})$ is a finite index subgroup of $\mathcal{R}_{k/l}(\hat{\mathbf{H}})$,[1] we get that $\mathbf{M} \subset \mathcal{R}_{k/l}(\mathbf{H})$. Taking l-points, part (1) follows.

By (6.19), we have $g\mathbf{M}(l)g^{-1} = \mathbf{M}(l)$ for all $g \in \mathbf{N}(l)$. Hence, by part (1) and Lemma 6.11(1), we obtain that $\mathbf{H} \subset \mathbf{F}$ is a normal subgroup of $\hat{\mathbf{F}}$ and hence of $\mathbf{F}$. Moreover, since $E_x \subset \mathbf{M}(l)$ for μ-a.e. x, we have that $\mathbf{H}$ is noncommutative. As was mentioned above, $\mathbf{H}$ is a K-subgroup of $\mathbf{F}$. Hence, Lemmas 6.11(2) and 6.10(5) imply that

$$[\mathbf{F}, \mathbf{F}] \subset \mathbf{H}. \tag{6.26}$$

We now show the other inclusion. In view of Lemmas 6.10 and 6.11, we have that $g_0 \mathbf{R}(k)\Gamma/\Gamma$ is a closed orbit with finite $g_0\mathbf{R}(k)g_0^{-1}$-invariant measure for $\mathbf{R} = \mathbf{F}, [\mathbf{F}, \mathbf{F}]$. Moreover, by the choice of g_0 in Lemmas 6.9 and 6.10(1), we have that

[1]Indeed, in view of the smoothness of $\mathbf{H}$, it follows from [6, Proposition A.5.9] that $\mathcal{R}_{k/l}(\mathbf{H})$ is connected.

$$\mu \text{ is supported on } \overline{Ag_0\Gamma/\Gamma} \subset g_0\mathbf{F}(k)\Gamma/\Gamma. \tag{6.27}$$

Since $E_x \subset [\mathbf{F},\mathbf{F}]$ and since any E_x-ergodic component of μ is supported on a homogeneous space $g_0 g[\mathbf{F},\mathbf{F}](k)\Gamma/\Gamma$, for some $g \in \mathbf{F}(k)$ we get that $\mathbf{M}(l) \subset [\mathbf{F},\mathbf{F}](k)$. This completes the proof of part (2) thanks to part (1) and (6.26). The fact that $\mathbf{H}$ satisfies parts (3) and (4) now follow from part (2) and Lemmas 6.11 and 6.10. □

Let us put $A_{\mathbf{H}} = A \cap g_0\mathbf{H}(k)g_0^{-1}$. In view of Lemmas 6.11 and 6.12, we see that $g_0\mathbf{H}(k)g_0^{-1}$ has a block structure. Put $C_{\mathbf{H}} = g_0 Z(\mathbf{F}(k))g_0^{-1}$. Then $A'' := A_{\mathbf{H}}C_{\mathbf{H}}$ is a cocompact subgroup of A. We have the following.

LEMMA 6.13
We can decompose the measure as

$$\mu = \int_{A/\mathrm{Stab}(\eta)} a_*\eta \, \mathrm{d}a,$$

where da *is the Haar measure on the compact group* $A/\mathrm{Stab}(\eta)$*, and* η *is an* $A_{\mathbf{H}}$*-ergodic component of* μ *which is supported on* $g_0\mathbf{H}(k)\Gamma/\Gamma$. *Moreover, we have*

$$\eta = \int \nu_z \, \mathrm{d}\eta(z).$$

Proof
Recall from (6.27) that μ is supported on the closed orbit $g_0\mathbf{F}(k)\Gamma/\Gamma$. Hence, $C_{\mathbf{H}} \cap g_0\Gamma g_0^{-1}$ acts trivially on $\mathrm{supp}(\mu)$. Moreover, by Lemmas 6.11 and 6.10(4), we have that $Z(\mathbf{F}(k))\Gamma/\Gamma$ is compact. This and the fact that A/A'' is compact implies that

$$A/A_{\mathbf{H}}(C_{\mathbf{H}} \cap g_0\Gamma g_0^{-1}) \tag{6.28}$$

is a compact group. Therefore, the $A_{\mathbf{H}}(C_{\mathbf{H}} \cap g_0\Gamma g_0^{-1})$-ergodic decomposition of μ can be written as

$$\int_{A/A_{\mathbf{H}}(C_{\mathbf{H}}\cap g_0\Gamma g_0^{-1})} a_*\eta \, \mathrm{d}a,$$

where η is an $A_{\mathbf{H}}(C_{\mathbf{H}} \cap g_0\Gamma g_0^{-1})$-invariant measure on $g_0\mathbf{H}(k)\Gamma/\Gamma$. This implies the decomposition of μ as in the lemma.

For the final claim, we note that the above discussion also shows that $\mathcal{B}^{A_{\mathbf{H}}} \subset \mathcal{P}$, where $\mathcal{B}^{A_{\mathbf{H}}}$ is the σ-algebra of $A_{\mathbf{H}}$-invariant sets. Hence, the conditional measures $\mu_x^{\mathcal{P}}$ for the Pinsker σ-algebra can be obtained by double conditioning—that is,

$$\mu_y^{\mathcal{P}} = (\mu_x^{\mathcal{B}^{A_{\mathbf{H}}}})_y^{\mathcal{P}}$$

for μ-a.e. x and $\mu_x^{\mathcal{B}^{A_{\mathbf{H}}}}$-a.e. y. Again because of the compactness of (6.28) and the equivariance properties of the conditional measures, it suffices to consider one of the conditional measure $\eta = \mu_x^{\mathcal{B}^{A_{\mathbf{H}}}}$. For the Pinsker conditional measure $\eta_y^{\mathcal{P}}$, we have considered in (6.16) a decomposition into ergodic components for the group E_y. These ergodic components have been completely described in Lemma 6.8. The lemma follows by integration over η. □

The following proposition describes the algebraic structure of the group L in Lemma 6.8. It turns out to be more convenient for us to explicate the structure of the finite index subgroup

$$L_B := \overline{M^+(\lambda)\big(\mathbf{M}(l) \cap \Gamma_B\big)}$$

of L. Note that $L\Gamma/\Gamma = L_B\Gamma/\Gamma$.

PROPOSITION 6.14

Let n be as in Lemma 6.12(4). Then there exist

(1) *a collection $(l_i : 1 \le i \le n)$ of closed (not necessarily distinct) subfields of k,*

(2) *for every $1 \le i \le n$, a connected, simply connected, absolutely almost simple l_i-group $\mathbf{H}_i$ and an isomorphism $\varphi_i : \mathbf{H}_i \times_{l_i} k \to \mathrm{SL}_{d_0}$ (where SL_{d_0} is considered as the ith block subgroup corresponding to the indices $(i-1)d_0 + 1, \dots, id_0$)*

so that $L_B = \prod_{i=1}^n \varphi_i(\mathbf{H}_i(l_i)) \subset \mathbf{H}(k)$.

Proof

In view of (6.25) and parts (3) and (4) of Lemma 6.12, the groups $\mathbf{M}$ and $\mathbf{H}$ satisfy the conditions in [28, Section 7] for the lattice Γ_B. Therefore, [28, Theorem 7.1], which in turn relies heavily on [6], [29], and [23], implies the following. There exist

(a) a collection $(l_i : 1 \le i \le r)$ of closed subfields of k,

(b) for every $1 \le j \le n$, some $1 \le i(j) \le r$ and a continuous field embedding $\tau_j : l_{i(j)} \to k$,

(c) for every $1 \le i \le r$, a connected, simply connected, absolutely almost simple l_i-group $\mathbf{H}_i$ (which is a form of SL_{d_0}),

(d) for every $i \in \{1, \dots, r\}$, some $j \in \{1, \dots, n\}$ with $i(j) = i$,

(e) an isomorphism $\varphi : \coprod_{i=1}^r \mathbf{H}_i \times_{\tau(\bigoplus_{i=1}^r l_i)} \bigoplus_{i=1}^n k \to \coprod \mathrm{SL}_{d_0}$, with $\tau = (\tau_1, \dots, \tau_n)$

so that $L_B = \varphi(\prod_{i=1}^r \mathbf{H}_i(l_i)) \subset \mathbf{H}(k)$.

We now claim that

$$r = n. \tag{6.29}$$

Assuming (6.29), and after possibly renumbering and replacing l_i by $\tau_i(l_i)$ for $1 \le i \le r = n$, we get the proposition. $\square$

We now turn to the proof of (6.29). Put $\Delta := \mathbf{H}(k) \cap \Gamma$, and recall the notation $A_{\mathbf{H}} = A \cap g_0\mathbf{H}(k)g_0^{-1}$. In view of Lemma 6.13, we can reduce the study of the measure μ to the study of the measure η, which is an $A_{\mathbf{H}}$-ergodic invariant measure on $g_0\mathbf{H}(k)/\Delta$. Put

$$\mathbf{H}' := \mathscr{R}_{\bigoplus_{j=1}^n k/\tau(\bigoplus_{i=1}^r l_i)}\Big(\coprod_{j=1}^n \mathrm{SL}_{d_0}\Big).$$

Then $\mathbf{H}'$ is a smooth $\bigoplus_{i=1}^r l_i$-group and $\mathbf{H}'(\bigoplus_{i=1}^r l_i) = \mathbf{H}(k)$ (see [6, Proposition A.5.2]). Moreover, $L_B = \varphi(\prod_{i=1}^r \mathbf{H}_i(l_i))$ is the group of $\bigoplus_{i=1}^r l_i$-points of a $\bigoplus_{i=1}^r l_i$-subgroup of $\mathbf{H}'$ (see [6, Proposition A.5.7]). Define

$$\mathbf{R} := \text{the Zariski closure of } N_{\mathbf{H}(k)}(L_B) \cap \Delta \text{ in } \mathbf{H}'. \tag{6.30}$$

Put

$$R = \mathbf{R}\Big(\bigoplus_{i=1}^r l_i\Big) \subset \mathbf{H}(k).$$

Then $R \subset N_{\mathbf{H}(k)}(L_B)$.

In view of (6.20) and Lemma 6.13, we have the following. For η-a.e. $x \in \mathbf{H}(k)/\Delta$ and $\eta_x^{\mathcal{P}}$-a.e. z, we have

$$\mathrm{supp}(\nu_z) = g_0 g L\Delta/\Delta = g_0 g L_B \Delta/\Delta$$

for some $g \in \mathbf{H}(k)$. Therefore, arguing for each i separately, as in the proof of Lemma 6.9 we get the following. There is a cocompact subgroup $A'_{\mathbf{H}} \subset A_{\mathbf{H}}$ and some $g_1 \in \mathbf{H}(k)$ so that

$$g_1^{-1} g_0^{-1} A'_{\mathbf{H}} g_0 g_1 \subset R;$$

moreover, $\overline{A_{\mathbf{H}} g_0 g_1 \Gamma} = \mathrm{supp}(\eta)$.

In particular, we have that $A'_{\mathbf{H}}$ normalizes the group $g_0 g_1 L_B g_1^{-1} g_0^{-1}$. Recall now that $A_{\mathbf{H}}$ is a maximal torus in the block diagonal group $g_0\mathbf{H}(k)g_0^{-1}$. These and the fact that $A'_{\mathbf{H}}$ is cocompact in $A_{\mathbf{H}}$ imply that the block structure of L_B and $\mathbf{H}$ agree with each other; that is, $r = n$. To see this, assume that $i(j) = 1$ for $j = 1, 2$. Let a be an element in $A'_{\mathbf{H}}$ which equals the identity in all the blocks $j = 2, \dots, n$, and in the first block it is a diagonal element which generates an unbounded group. Then since a normalizes $g_0 g_1 L_B g_1^{-1} g_0^{-1}$, we get a contradiction. $\square$

COROLLARY 6.15
We have that $N_{\mathbf{H}(k)}(L_B)/Z(\mathbf{H}(k))L_B$ is a torsion abelian group.

Proof
In view of Proposition 6.14, it suffices to argue in each SL_{d_0}-block separately. Hence, we fix some $i \in \{1,\dots,n\}$. First, note that $\mathbf{H}_i$ is an l_i-form of SL_{d_0}. Suppose now that $g \in \mathrm{SL}(d_0,k)$ normalizes $\mathbf{H}_i(l_i)$. Since $\mathbf{H}_i(l_i)$ is Zariski-dense in the l_i-group $\mathbf{H}_i$ (see, e.g., [26, Chapter 1, Proposition 2.5.3], we thus get that g induces an l_i-automorphism of $\mathbf{H}_i$. Extending the scalars from l_i to k, we see that the automorphism is inner; that is, this automorphism $\sigma_i(g)$ belongs to $\mathbf{H}_i^{\mathrm{ad}}(k)$. Together it follows that $\sigma_i(g) \in \mathbf{H}_i^{\mathrm{ad}}(l_i)$. This automorphism is, moreover, nontrivial if and only if g is not central in SL_{d_0}. Hence, we get a monomorphism $g \mapsto \sigma(g)$ from

$$N_{\mathrm{SL}(d_0,k)}\big(L_B \cap \mathrm{SL}(d_0,k)\big)/Z\big(\mathrm{SL}(d_0,k)\big)$$

into $\mathbf{H}_i^{\mathrm{ad}}(l_i)$. This map sends $\mathbf{H}_i(l_i)$ to $[\mathbf{H}_i^{\mathrm{ad}}(l_i),\mathbf{H}_i^{\mathrm{ad}}(l_i)]$ by [26, Chapter 1, Theorem 2.3.1], and the claims hold true by [26, Chapter 1, Theorem 2.3.1]. $\square$

Let us now complete the proof of Theorem 1.1.

Proof of Theorem 1.1
In view of Lemma 6.13, we may and will restrict our attention to the measure η appearing in the statement of that lemma. Similar to the proof of (6.29), put $\Delta := \mathbf{H}(k) \cap \Gamma$. Define

$$\mathbf{H}' := \mathcal{R}_{\bigoplus_{j=1}^n k/\bigoplus_{i=1}^n l_i}\Big(\coprod_{i=1}^n \mathrm{SL}_{d_0}\Big).$$

Then $\mathbf{H}'$ is a smooth $\bigoplus_{i=1}^n l_i$-group and $\mathbf{H}'(\bigoplus_{i=1}^n l_i) = \mathbf{H}(k)$ (see [6, Proposition A.5.2]). Moreover, $L_B = \prod_{i=1}^n \mathbf{H}_i(l_i)$ is the group of $\bigoplus_{i=1}^n l_i$-points of a $\bigoplus_{i=1}^n l_i$-subgroup of $\mathbf{H}'$ (see [6, Proposition A.5.7]). Since $\mathbf{H}'(\bigoplus_{i=1}^r l_i) = \mathbf{H}(k)$, we may view $Z(\mathbf{H}(k))$ as a finite subgroup of $\mathbf{H}'(\bigoplus_{i=1}^r l_i)$. Define

$$\mathbf{R} := Z\big(\mathbf{H}(k)\big) \quad \big(\text{the Zariski closure of } N_{\mathbf{H}(k)}(L_B) \cap \Delta \text{ in } \mathbf{H}'\big).$$

Put $R = \mathbf{R}(\bigoplus_{i=1}^n l_i) \subset \mathbf{H}(k)$. Since $\mathbf{H}(k) = \mathbf{H}'(\oplus(l_i))$, we have $Z(\mathbf{H}(k)) \subset R$. Moreover, $R \subset N_{\mathbf{H}(k)}(L_B)$, and by Corollary 6.15, we have

$$[R,R] \subset Z\big(\mathbf{H}(k)\big)L_B. \tag{6.31}$$

In view of (6.20) and Lemma 6.13, for η-a.e. $x \in g_0\mathbf{H}(k)/\Delta$, we have

$$\operatorname{supp}(\nu_x) = g_0 g L\Delta/\Delta = g_0 g L_B \Delta/\Delta \tag{6.32}$$

for some $g \in \mathbf{H}(k)$ depending on x.

Therefore, arguing as in the proof of Lemma 6.9, we get the following. There is a cocompact subgroup $A'_{\mathbf{H}} \subset A_{\mathbf{H}}$ containing $Z(\mathbf{H}(k))$ and there is some $g_1 \in \mathbf{H}(k)$ so that $g_1^{-1} g_0^{-1} A'_{\mathbf{H}} g_0 g_1 \subset R$. We may furthermore require that $\overline{A_{\mathbf{H}} g_0 g_1 \Gamma/\Gamma} = \operatorname{supp}(\eta)$. This gives the decomposition

$$\eta = \int_{A_{\mathbf{H}}/A'_{\mathbf{H}}} a_* \eta' \,\mathrm{d}a, \tag{6.33}$$

where

- $\mathrm{d}a$ is the Haar measure on the compact group $A_{\mathbf{H}}/A'_{\mathbf{H}}$,
- η' is an $A'_{\mathbf{H}}$-invariant and ergodic probability measure on $g'_0 R/\Delta'$, where $\Delta' := R \cap \Delta$ and $g'_0 = g_0 g_1$.

Note that we have implicitly identified here $g'_0 R/\Delta'$ with $g'_0 R\Delta/\Delta$ (which in turn itself has already been implicitly identified with $g'_0 R\Gamma/\Gamma$).

We now further investigate the measure η'. In view of (6.32), we can write

$$\eta' = \int_{g'_0 R/\Delta'} \nu_x \,\mathrm{d}\eta'(x), \tag{6.34}$$

where ν_x is the $g'_0 g L_B g^{-1} {g'_0}^{-1}$-invariant measure on $g'_0 g L_B \Delta'/\Delta'$, where we write x as $x = g'_0 g \Delta'/\Delta'$ for $g \in R$.

Since L_B is normal in R, we get that η' is $g'_0 L_B {g'_0}^{-1}$-invariant. Moreover, since $Z(\mathbf{H}(k)) \subset A'_{\mathbf{H}}$, we also have that η' is $Z(\mathbf{H}(k))$-invariant. Finally, since $L_B\Delta/\Delta$ is closed in $\mathbf{H}(k)/\Delta$, we have that $Z(\mathbf{H}(k))L_B\Delta'$ is a closed subgroup of R. Let $L'_B = Z(\mathbf{H}(k))L_B$. We define η'_1 as the pushforward of η' under the canonical quotient map from $g'_0 R/\Delta'$ into $g'_0 R/L_B\Delta'$, and similarly η'_2 as the pushforward to $g'_0 R/L'_B\Delta'$. With this we obtain from (6.34) that, for $\nu_{L_B} = \nu_{L_B\Delta'/\Delta'}$,

$$\begin{aligned}
\eta' &= \int_{g'_0 R/L_B\Delta'} g_* \nu_{L_B} \,\mathrm{d}\eta'_1(g L_B \Delta') \\
&= \int_{g'_0 R/L'_B\Delta'} g_* \Big(\int_{Z(\mathbf{H}(k))} h_* \nu_{L_B} \,\mathrm{d}h \Big) \mathrm{d}\eta'_2(g L'_B \Delta') \\
&= (g'_0)_* \int_{R/L'_B\Delta'} g_* \Big(\int_{Z(\mathbf{H}(k))} h_* \nu_{L_B} \,\mathrm{d}h \Big) \mathrm{d}\eta_P(g L'_B \Delta'),
\end{aligned} \tag{6.35}$$

for a $(g'_0)^{-1} A'_{\mathbf{H}} g'_0$-invariant and ergodic probability measure η_P on $P = R/L'_B\Delta'$. We note that the measure defined by the inner integral in (6.35) is actually homogeneous. Furthermore, by Corollary 6.15 we know that $P = R/L'_B\Delta'$ is a torsion abelian group.

We claim that

$$\text{the image of } (g_0')^{-1} A_{\mathbf{H}}' g_0' \text{ in } P \text{ is compact and in particular closed.} \tag{6.36}$$

Assuming (6.36), let us finish the proof. Indeed, (6.36) implies that η_P equals the Haar measure on a coset of

$$\big((g_0')^{-1} A_{\mathbf{H}}' g_0'\big) L_B' \Delta' / L_B' \Delta'.$$

This together with (6.33) finishes the proof. □

We now prove (6.36). Let $\{s_1, \ldots, s_r\} \subset (g_0')^{-1} A_{\mathbf{H}}' g_0'$ be a subset which generates a cocompact subgroup of $(g_0')^{-1} A_{\mathbf{H}}' g_0'$. By Corollary 6.15, there exists some $m \in \mathbb{N}$ so that $s_i^m \in Z(\mathbf{H}(k)) L_B = L_B'$ for all $1 \le i \le r$. Let D be the group generated by $\{s_1^m, \ldots, s_r^m\}$. Then D is cocompact in $(g_0')^{-1} A_{\mathbf{H}}' g_0'$ and the natural orbit map from $(g_0')^{-1} A_{\mathbf{H}}' g_0'$ to P factors through the natural map from $(g_0')^{-1} A_{\mathbf{H}}' g_0' / D$ to P. These maps are continuous and $(g_0')^{-1} A_{\mathbf{H}}' g_0' / D$ is compact; thus, (6.36) follows. □

7. Joining classification

7.1. On the group generated by certain commutators

A key to the classification of joinings is the following simple general fact about a rank-2 k-torus. Let $\mathbf{G}$ denote a connected, simply connected, absolutely almost simple group defined over a local field k with $\mathrm{char}(k) > 3$. Let $\lambda : \mathbf{G}_m^2 \to \mathbf{G}$ be an algebraic monomorphism defined over k; let $\mathbf{A} = \lambda(\mathbf{G}_m^2)$. Fix a maximal, k-split, k-torus $\mathbf{S} \subset \mathbf{G}$ so that $\mathbf{A} \subset \mathbf{S}$. Further, let $\mathbf{T} \supset \mathbf{S}$ be a maximal torus of $\mathbf{G}$ which is defined over k. Put $\Phi := \Phi(\mathbf{T}, \mathbf{G})$, ${}_k\Phi := {}_k\Phi(\mathbf{S}, \mathbf{G})$, and $\overline{\Phi} := {}_k\Phi(\mathbf{A}, \mathbf{G})$. For $\Psi \subset \overline{\Phi}$, set

$$\vartheta(\Psi) := \big\{\alpha \in \Phi(\mathbf{T}, \mathbf{G}) : \alpha|_{\mathbf{A}} \in \Psi\big\}. \tag{7.1}$$

PROPOSITION 7.1
The group $\mathbf{G}$ is generated by the commutators $[\mathbf{V}_{[\alpha]}, \mathbf{V}_{[\beta]}]$, where α, β run over all linearly independent *pairs in $\overline{\Phi}$.*

We need the following lemma from [12, Lemma 4.2] (see also [10, Lemma 9.6]).

LEMMA 7.2
Let $\delta \in \overline{\Phi}$ and $\delta' \in \vartheta([\delta])$. Then there exist some $\beta \in \overline{\Phi}$ and some $\beta' \in \vartheta([\beta])$ with the following properties:

(1) *$\{\beta, \delta\}$ is a linearly independent subset of $\overline{\Phi}$,*

(2) $\delta' - \beta' \in \Phi$.

Proof
Let $\bar{k}$ be the algebraic closure of k. Let

$$\Upsilon = \left\{\alpha \in \mathbb{R} \otimes X^*(\mathbf{T}) : \alpha|_{\mathbf{A}} \in \mathbb{R}\delta\right\},$$

where $X^*(\mathbf{T})$ denotes the group of characters of $\mathbf{T}$.

Let $\mathfrak{g}'$ be the $\bar{k}$-span of $\{\mathfrak{g}_{\alpha'}, [\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}] : \alpha', \beta' \in \Phi \setminus \Upsilon\}$. It follows easily from the Jacobi identity (see the proof of [12, Lemma 4.2] for details) that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. Recall that $\mathbf{A} = \lambda(\mathbf{G}_m^2)$. Therefore, $\overline{\Phi}$ has at least two linearly independent roots, and $\mathfrak{g}'$ is not central. Since $\mathfrak{g}$ has no proper noncentral ideals, we have $\mathfrak{g}' = \mathfrak{g}$.

In particular, we get that

$$\mathfrak{g}_{\delta'} \subset \sum_{\alpha' \in \Phi_1 \setminus \Upsilon} \mathfrak{g}_{\alpha'} + \sum_{\alpha', \beta' \in \Phi \setminus \Upsilon} [\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}].$$

Since $\delta' \in \Upsilon$, the above implies that $\mathfrak{g}_{\delta'} \subset \sum_{\alpha', \beta' \in \Phi \setminus \Upsilon} [\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}]$. But for every α', β', we have $[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}] \subseteq \mathfrak{g}_{\alpha'+\beta'}$ and hence $\delta' = \alpha' + \beta'$ for some $\alpha', \beta' \in \Phi \setminus \Upsilon$. In particular, since $\beta' \notin \Upsilon$, it holds that $\beta := \beta'|_{\mathbf{A}}$ is linearly independent from δ. □

Proof of Proposition 7.1
Since the statement of the proposition is on the level of algebraic groups, the validity of the statement over the algebraic closure $\bar{k}$ of k implies that of the statement when the groups are considered as algebraic groups over k. Over $\bar{k}$, we can write for every $\alpha \in \overline{\Phi}$,

$$\mathbf{V}_{[\alpha]} = \prod_{\delta' \in \vartheta([\alpha])} \mathbf{U}_{\delta'}$$

with each $\mathbf{U}_{\delta'}$ a 1-parameter unipotent group over $\bar{k}$.

Since the group $\mathbf{G}$ is absolutely almost simple, and in particular semisimple, the root groups $\mathbf{U}_{\delta'}$ for $\delta' \in \Phi$ generate. Therefore, to prove the proposition, it is enough to show that for every $\delta' \in \Phi$, one can find α and β in $\overline{\Phi}$, linearly independent, so that

$$\mathbf{U}_{\delta'} \subset [\mathbf{V}_{[\alpha]}, \mathbf{V}_{[\beta]}]. \tag{7.2}$$

Let β, β' be as in Lemma 7.2 applied to $\delta := \delta'|_{\mathbf{A}_1}$ and δ', and let $\alpha' = \delta' - \beta'$ and $\alpha = \alpha'|_{\mathbf{A}}$. In particular, α and β are linearly independent.

Recall that $\mathrm{char}(k) \neq 2, 3$. Hence, by [4, Section 4.3], irregular commutation relations do not occur. This means in particular (see also [2, Section 2.5]) that

$$[\mathbf{U}_{\alpha'}, \mathbf{U}_{\beta'}] = \mathbf{U}_{\alpha'+\beta'}.$$

But $\mathbf{U}_{\alpha'} \subset \mathbf{V}_{[\alpha]}$, $\mathbf{U}_{\beta'} \subset \mathbf{V}_{[\beta]}$, and by definition $\alpha' + \beta' = \delta'$. Thus (7.2) is proved, and hence the proposition follows. □

7.2. The main entropy inequality and the invariance group of the leafwise measures
From now on, we use notation from Theorem 1.2. In particular, for $i = 1, 2$, $\mathbf{G}_i$ denotes a connected, simply connected, absolutely almost simple group defined over k. We put $G_i = \mathbf{G}_i(k)$ and $G = G_1 \times G_2$. Recall also that $\mathrm{char}(k) > 3$. Suppose that there are fixed two algebraic monomorphisms $\lambda_i : \mathbf{G}_m^2 \to \mathbf{G}_i$ defined over k; let $\mathbf{A}_i = \lambda_i(\mathbf{G}_m^2)$ and $A_i = \mathbf{A}_i(k)$. For $i = 1, 2$, fix a maximal, k-split, k-torus $\mathbf{S}_i \subset \mathbf{G}_i$ so that $\mathbf{A}_i \subset \mathbf{S}_i$, and set ${}_k\Phi_i := {}_k\Phi(\mathbf{S}_i, \mathbf{G}_i)$ and $\overline{\Phi}_i := {}_k\Phi(\mathbf{A}_i, \mathbf{G}_i)$. Define $\mathbf{A}$ to be the smooth k-group so that

$$\mathbf{A}(R) := \big\{\big(\lambda_1(r), \lambda_2(r)\big) : r \in \mathbf{G}_m(R)^2\big\}$$

for any algebra R/k; let $A := \mathbf{A}(k)$.

Let

$$\overline{\Phi} = {}_k\Phi(\mathbf{A}, \mathbf{G}_1 \times \mathbf{G}_2).$$

Using the natural homomorphisms from $\mathbf{A}$ to $\mathbf{A}_i$, for $i = 1, 2$ we can view ${}_k\Phi(\mathbf{A}_i, \mathbf{G}_i)$ as subsets of $\overline{\Phi}$; moreover, we have

$$\overline{\Phi} = {}_k\Phi(\mathbf{A}_1, \mathbf{G}_1) \cup {}_k\Phi(\mathbf{A}_2, \mathbf{G}_2).$$

For each $\alpha \in \overline{\Phi}$, we can write the coarse Lyapunov group $V_{[\alpha]} \subset G_1 \times G_2$ as a product $V_{[\alpha]}^1 \times V_{[\alpha]}^2$ with $V_{[\alpha]}^i \subset G_i$; by convention, if $\alpha \notin \overline{\Phi}$, then $V_{[\alpha]}^i = \{1\}$. For $i = 1, 2$, fix a maximal, compact, open subgroup $\mathfrak{G}_i \subset G_i$ and put $\mathfrak{G} := \mathfrak{G}_1 \times \mathfrak{G}_2$. Recall that μ denotes an ergodic joining for the action of A_i on (X_i, m_i) for $i = 1, 2$.

PROPOSITION C ([12, Section 3])
Let $a = (a_1, a_2) \in A$, and let $\Psi \subset \overline{\Phi}$ be a positively closed subset. Put

$$W = V_\Psi \subset W^-_{G_1 \times G_2}(a).$$

Then $W = W_1 \times W_2$, where $W_i \subset G_i$ for $i = 1, 2$ and

$$\mathsf{h}_\mu(a, W) \le \mathsf{h}_{m_1}(a_1, W_1) + \mathsf{h}_\mu\big(a, \{\mathrm{id}\} \times W_2\big). \tag{7.3}$$

Furthermore, the following hold.

(1) *If the equality holds in (7.3), then W_1 is the smallest algebraic subgroup of W_1 which contains $\pi_1(\mathrm{supp}(\mu_x^W) \cap \mathfrak{G})$.*
(2) *The equality holds for $W = W^-_{G_1 \times G_2}(a)$.*
(3) *For every $\alpha \in \overline{\Phi}$, the equality holds for $W = V_{[\alpha]}$.*

Proof
The main inequality follows from [12, Proposition 3.1].[2] Parts (2) and (3) follow from

[2]The arguments in [12] generalize to the setting at hand without a change.

[12, Proposition 3.3, Corollary 3.4]. To see part (1), note first that by [13, Proposition 6.2], we have that

$$\pi_1\big(\overline{\operatorname{supp}(\mu_x^W)\cap \mathfrak{G}}^z\big)$$

is a (Zariski-closed) subgroup which is normalized by a and contains $\pi_1(\operatorname{supp}(\mu_x^W))$. Part (1) now follows from [12, Proposition 3.2]. □

COROLLARY 7.3
For any $\alpha \in \overline{\Phi}$, we have that $\pi_i(\mathcal{S}_x^{[\alpha]} \cap \mathfrak{G})$ is Zariski-dense in $\pi_i(V_{[\alpha]})$ for $i = 1, 2$.

Proof
In view of Proposition C(3), this is a direct consequence of Proposition C(1) and the definition of $\mathcal{S}_x^{[\alpha]}$. □

Fix an element $a = (a_1, a_2) \in A$ that is *regular* with respect to $\overline{\Phi}$ (i.e., $\alpha(a) \neq 1$ for any $\alpha \in \overline{\Phi}$). We denote the Pinsker σ-algebra, $\mathcal{P}_a$, simply by $\mathcal{P}$. Disintegrate μ as

$$\mu = \int_X \mu_x^{\mathcal{P}} \, \mathrm{d}\mu(x), \tag{7.4}$$

where $\mu_x^{\mathcal{P}}$ denotes the $\mathcal{P}$-conditional measure for μ-a.e. $x \in X$. Similar to (6.6), define

$$\mathcal{H}_x := \{g \in G_1 \times G_2 : g\mu_x^{\mathcal{P}} = \mu_x^{\mathcal{P}}\}.$$

We have $a\mathcal{H}_x a^{-1} = \mathcal{H}_{ax}$ for all $a \in A$ and μ-a.e. x (see (6.7)).

LEMMA 7.4
For μ-a.e. x and any linearly independent $\alpha, \beta \in \Phi$, the measure $\mu_x^{\mathcal{P}}$ is almost surely invariant under $[\mathcal{S}_x^{[\alpha]}, \mathcal{S}_x^{[\beta]}]$—that is, $[\mathcal{S}_x^{[\alpha]}, \mathcal{S}_x^{[\beta]}] \subset \mathcal{H}_x$.

Proof
By Lemma 3.14, for every $\alpha \in \overline{\Phi}$ and μ-a.e. x, we have that $\mu_x^{\mathcal{P}}$ is invariant under $I_x^{[\alpha]}$, and hence, by Lemma 3.12, is invariant under I_x^{Ψ} for any positively closed $\Psi \subset \overline{\Phi}$. By Lemma 3.11, we have therefore that, for any linearly independent $\alpha, \beta \in \Phi$, the measure $\mu_x^{\mathcal{P}}$ is almost surely invariant under $[\mathcal{S}_x^{[\alpha]}, \mathcal{S}_x^{[\beta]}]$. □

Recall that $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ is a compact, open subgroup of $G = G_1 \times G_2$. Define

$$\mathcal{Q}_x := \big\langle \{g \in \mathcal{H}_x \cap \mathfrak{G} : g \text{ is unipotent}\} \big\rangle.$$

COROLLARY 7.5
For μ-a.e. x, $\pi_i(\mathcal{Q}_x)$ is Zariski-dense in $\mathbf{G}_i$ and $\pi_i(\mathcal{H}_x)$ is unbounded for $i = 1, 2$.

Proof
For any x, let $\mathbf{L}_{i,x}$ denote the Zariski closure of $\pi_i(\mathcal{Q}_x)$ in $\mathbf{G}_i$. Let $\alpha, \beta \in \overline{\Phi}$ be two linearly independent roots. By Corollary 7.3, almost surely $\pi_i(\mathcal{S}_x^{[\alpha]} \cap \mathfrak{G})$ is Zariski-dense in $\pi_i(\mathbf{V}_{[\alpha]})$ and similarly for β, for $i = 1, 2$. By Lemma 7.4, $[\mathcal{S}_x^{[\alpha]} \cap \mathfrak{G}, \mathcal{S}_x^{[\beta]} \cap \mathfrak{G}] \subset \mathcal{Q}_x$. It follows that

$$\pi_i\big([\mathbf{V}_{[\alpha]}, \mathbf{V}_{[\beta]}]\big) \subset \mathbf{L}_{i,x}$$

for any two linearly independent $\alpha, \beta \in \overline{\Phi}$. The first part of the claim follows using Proposition 7.1.

For the second, by Lemmas 7.4 and 3.12 there is an $\alpha \in \overline{\Phi}$ such that $\mathcal{I}_x^{[\alpha]}$ is nontrivial. If $\mathcal{I}_x^{[\alpha]}$ were to be bounded on a set of positive measure, then its diameter would be a monotone, increasing, measurable function under an appropriate subsemigroup of A, in contradiction to Poincaré recurrence. $\square$

7.3. *Proof of Theorem 1.2*
Let $X' \subset X$ be a conull subset so that the conclusions of Lemma 3.5 and Corollary 7.5 hold true on X'. By Corollary 7.5, for all $x \in X'$ the group $\mathcal{Q}_x$ satisfies the conditions of Theorem A.2 in Section 3.1. Therefore, there are two possibilities to consider.

Case 1. There is a subset $X'' \subset X'$ with $\mu(X'') > 0$ so that for all $x \in X''$ and $i = 1, 2$, the following conditions hold. There are
- subfields $k_{i,x} \subset k$,
- $k_{i,x}$-groups $\mathbf{H}_{i,x}$,
- k-isomorphism $\varphi_{i,x} : \mathbf{H}_{i,x} \times_{k_{i,x}} k \to \mathbf{G}_i$, and
- open, compact subgroups $\mathcal{Q}_{i,x} \subset \varphi_{i,x}(\mathbf{H}_{i,x}(k_{i,x}))$

so that $\mathcal{Q}_{1,x} \times \mathcal{Q}_{2,x} \subset \mathcal{Q}_x$.

LEMMA 7.6
For every $x \in X''$ and every $h \in \mathcal{Q}_{1,x}$, define

$$F_x(h) := \big\{v(h,1)v^{-1} : v \in \mathcal{H}_x\big\}.$$

(1) *For every $h \in \mathcal{Q}_{1,x}$, we have $F_x(h) \subset \mathcal{H}_x$.*
(2) *There exists an element $h \in \mathcal{Q}_{1,x}$ such that $F_x^{\alpha}(h)$ is unbounded.*

Proof
Part (1) is immediate since $\mathcal{Q}_{x,1} \times \{1\} \subset \mathcal{Q}_x$.

We now prove part (2). Let $\{v_n\} \subset \mathcal{H}_x$ be a sequence so that $\pi_1(v_n) \to \infty$ (see Corollary 7.5). Let

$$v_n = (v_{n,1}, v_{n,2}) = (r'_{n,1} s_{n,1} r_{n,1}, r'_{n,2} s_{n,2} r_{n,2})$$

be the Cartan decomposition of v_n. Then $s_{n,1} \to \infty$. Passing to a subsequence if necessary, we assume that

- $\{r_{n,i}\}$ and $\{r'_{n,i}\}$ converge for $i = 1, 2$, and moreover,
- $\mathbf{P} := \{g \in \mathbf{G}_1 : \{s_{n,1}^{-1} g s_{n,1}\}$ is bounded$\}$ is a proper parabolic k-subgroup of $\mathbf{G}_1$.

Since $\mathcal{Q}_{1,x}$ is Zariski-dense in the k-group $\mathbf{G}_1$, there exists some $h \in \mathcal{Q}_{1,x}$ which does not lie in $r^{-1}\mathbf{P}r$ where $r_{n,1} \to r$. The claim in part (2) holds for this h. □

Proof of Theorem 1.2: Case 1

Let $x \in X''$, and let h and $F_x(h)$ be as in Lemma 7.6(2). Suppose that $\{(g_n, 1)\} \subset F_x(h)$ is an unbounded sequence. By part (1) of that lemma, we have

$$(g_n, 1) \in \mathcal{H}_x \quad \text{for all } n. \tag{7.5}$$

Recall from Lemma 3.5 that

$$\pi_{i*}(\mu_x^{\mathcal{P}}) = m_i \quad \text{for } i = 1, 2. \tag{7.6}$$

Since $\mathbf{G}_1$ is connected, simply connected, and absolutely almost simple, it follows from the generalized Mautner phenomenon (see [26, Chapter 1, Theorem 2.3.1], [26, Chapter 2, Theorem 7.2]) that (X_1, m_1) is ergodic for the action of the unbounded group $\langle\{g_n\}\rangle$. This, together with (7.5) and (7.6), implies that $\mu_x^{\mathcal{P}} = m_1 \times m_2$ (see, e.g., the argument in Case 1 of the proof of [17, Proposition 4.3]). Since $\mu(X'') > 0$ and μ is A-ergodic, we get that $\mu = m_1 \times m_2$. □

The rest of this section is devoted to the analysis of the following case.

Case 2. Replacing X' by a conull subset, which we continue to denote by X', we have the following. For every $x \in X'$, there are

- a subfield $k_x \subset k$ and a continuous embedding $\tau_x : k_x \to k$,
- a k_x-group $\mathbf{H}_x$,
- a $(k \oplus k)$-isomorphism $\varphi_x : \mathbf{H}_x \times_{\Delta_{\tau_x}(k_x)} (k \oplus k) \to \mathbf{G}_1 \coprod \mathbf{G}_2$ where as in (3.4), $\Delta_{\tau_x}(k_x) = \{(c, \tau_x(c)) : c \in k_x\}$,

so that $\mathcal{Q}_x$ is an open subset of the image under φ_x of $\mathbf{H}_x(k_x)$ with the latter considered as a subset of the $(k \oplus k)$-points of $\mathbf{H}_x \times_{\Delta_{\tau_x}(k_x)} (k \oplus k)$ using the injection of rings $\Delta_{\tau_x} : k_x \to k \oplus k$. Moreover, $\Delta_{\tau_x}(k_x)$ is unique, and $\mathbf{H}_x$ and φ_x are unique up to unique isomorphisms. Let us further recall that

$$k_x = \text{the field of quotients of the ring generated by } \{\mathrm{tr}(\rho(g)) : g \in \mathcal{Q}_x\}, \tag{7.7}$$

where ρ denotes the nonconstant irreducible representation occurring as subquotient of the adjoint representation of $\mathbf{G}_1^{\mathrm{ad}}$.

Put $E_x := \varphi_x(\mathbf{H}_x(k_x)) \subset G_1 \times G_2$.

PROPOSITION 7.7

(1) *There is a subfield $k' \subset k$ and an embedding $\tau : k' \to k$ so that $\Delta_{\tau_x}(k_x) = \Delta_\tau(k')$ on a conull subset of X.*

(2) *The map $x \mapsto E_x$ is an A-equivariant Borel map on a conull subset of X.*

Proof

In view of (7.7) and the fact that $x \mapsto \mathcal{Q}_x$ is a Borel map, we get that $x \mapsto \Delta_{\tau_x}(k_x)$ is a Borel map (see the proof of Lemma 6.5(1)). To see the other claims in part (1), first recall that $a\mathcal{H}_x a^{-1} = \mathcal{H}_{ax}$ for all $a \in A$ and μ-a.e. $x \in X$. Hence, for any $a \in A$ there exists some finite index subgroup $\mathcal{Q}_x(a) \subset \mathcal{Q}_x$ so that

$$a\mathcal{Q}_x(a)a^{-1} \subset \mathcal{Q}_{ax}. \tag{7.8}$$

Therefore, the same arguments as in the proof of Lemma 6.6(1) apply here and finish the proof of part (1) (see (6.13) and (6.14)).

We now turn to the proof of part (2). Put

$$\mathbf{G}' := \mathcal{R}_{k\oplus k/\Delta_\tau(k')}\Big(\mathbf{G}_1 \coprod \mathbf{G}_2\Big).$$

This is a $\Delta_\tau(k')$-group.

Now, part (1), the fact that φ_x is an isomorphism, and the universal property of the restriction of scalars functor (see [6, Section A.5]) imply that

$$E_x = \big(\mathcal{R}_{k\oplus k/\Delta_\tau(k')}(\varphi_x)(\mathbf{H}_x)\big)\big(\Delta_\tau(k')\big).$$

Hence, using [26, Chapter 1, Proposition 2.5.3], we get that E_x is identified with the $\Delta_\tau(k')$-points of the Zariski closure of $\mathcal{Q}_x$ in the $\Delta_\tau(k')$-group $\mathbf{G}'$. Since the map $x \mapsto \mathcal{Q}_x$ is Borel, we thus get that $x \mapsto E_x$ is a Borel map.

To see the A-equivariance, first recall from (7.8) that $a\mathcal{Q}_x(a)a^{-1}$ is an open subgroup of $\mathcal{Q}_{ax}$. Thus, using [26, Chapter 1, Proposition 2.5.3], we get that E_{ax} is the Zariski closure of $a\mathcal{Q}_x(a)a^{-1}$ in $\mathbf{G}'(\Delta_\tau(k'))$. On the other hand, this Zariski closure equals aE_xa^{-1}; the claim follows. $\square$

LEMMA 7.8

For μ-a.e. $x \in X$, we have $E_x \subset \mathcal{H}_x$, and E_x is not compact.

Proof
We first recall from [31, Theorem T] that since $\mathbf{H}_x$ is connected, simply connected, and absolutely almost simple, any open and unbounded subgroup of E_x equals E_x. Thus, since $\mathcal{Q}_x \subset \mathcal{H}_x$ is an open subgroup of E_x, both assertions in the lemma will follow if we show that $\mathcal{H}_x \cap E_x$ is unbounded for μ-a.e. $x \in X$.

However, the proof of Corollary 7.5 shows that for some $\alpha \in \overline{\Phi}$, we have that $\mathcal{Q}_x \cap \mathcal{I}_x^{[\alpha]}$ is nontrivial. Since $x \mapsto E_x$ is an A-equivariant map, using Poincaré recurrence as in Corollary 7.5 it follows that $\mathcal{H}_x \cap E_x$ is unbounded. $\square$

Proof of Theorem 1.2: Case 2
The argument is similar to the proof of Theorem 1.1.

Step 1. Let

$$\mu_x^{\mathcal{P}} = \int_X \nu_z \, \mathrm{d}\mu_x^{\mathcal{P}}(z) \tag{7.9}$$

be the ergodic decomposition of $\mu_x^{\mathcal{P}}$ with respect to E_x.

As before, $k \oplus k$ is a $\Delta_\tau(k')$-algebra. Put

$$\mathbf{G}' := \mathcal{R}_{k\oplus k/\Delta_\tau(k')}\Big(\mathbf{G}_1 \coprod \mathbf{G}_2\Big).$$

This is a connected group defined over $\Delta_\tau(k')$ (see [6, Section A5]). Moreover, $\Gamma_1 \times \Gamma_2$ is a lattice in $\mathbf{G}'(\Delta_\tau(k')) = \mathbf{G}_1(k) \times \mathbf{G}_2(k) = G_1 \times G_2 = G$.

Applying Theorem B in Section 3.2, we conclude that for $\mu_x^{\mathcal{P}}$-a.e. z the measure ν_z is described as follows. There exist

(1) $l_z = (k')^{q_z}$ where $q_z = p^{n_z}$, $p = \mathrm{char}(k)$, and $n_z \geq 1$,

(2) a connected $\Delta_\tau(l_z)$-subgroup $\mathbf{M}_z$ of $\mathcal{R}_{\Delta_\tau(k')/\Delta_\tau(l_z)}(\mathbf{G}')$ so that

$$\mathbf{M}_z\big(\Delta_\tau(l_z)\big) \cap (\Gamma_1 \times \Gamma_2)$$

is Zariski-dense in $\mathbf{M}_z$, and

(3) an element $g_z \in G_1 \times G_2$,

such that ν_z is the $g_z L_z g_z^{-1}$-invariant probability Haar measure on the closed orbit $g_z L_z (\Gamma_1 \times \Gamma_2)/(\Gamma_1 \times \Gamma_2)$ with

$$L_z = \overline{M_z^+(\lambda_z)\big(\mathbf{M}_z\big(\Delta_\tau(l_z)\big) \cap (\Gamma_1 \times \Gamma_2)\big)},$$

where

- the closure is with respect to the Hausdorff topology, and
- $\lambda_z : \mathbf{G}_m \to \mathbf{M}_z$ is a noncentral $\Delta_\tau(l_z)$-homomorphism, $M_z^+(\lambda_z)$ is defined in (3.9), and $E_x \subset M_z^+(\lambda_z)$.

Arguing as in the proof of Lemma 6.8, there exists a triple $(l_0, [\mathbf{M}_0], [M_0^+(\lambda_0)])$ so that

$$\big(l_z, [\mathbf{M}_z], \big[M_z^+(\lambda_z)\big]\big) = \big(l_0, [\mathbf{M}_0], \big[M_0^+(\lambda_0)\big]\big) \quad \text{for } \mu\text{-a.e. } x \text{ and } \mu_x^{\mathcal{P}}\text{-a.e. } z.$$

Put $L_0 := \overline{M_0^+(\lambda_0)(\mathbf{M}_0(\Delta_\tau(l_0)) \cap (\Gamma_1 \times \Gamma_2))}$.

Step 2. One of the following holds:

(a) $L_0 = G_1 \times G_2$, or

(b) $\pi_i(L_0) = G_i$ and $\ker(\pi_i|_{L_0}) \subset C(G_1 \times G_2)$ for $i = 1, 2$.

To see this, first note that, by Lemma 3.5, we have $\pi_{i*}\mu_x^{\mathcal{P}} = m_i$ for μ-a.e. $x \in X$ and $i = 1, 2$. This, together with (7.9), implies that

$$m_i = \pi_{i*}\mu_x^{\mathcal{P}} = \int_X \pi_{i*}\nu_z \, \mathrm{d}\mu_x^{\mathcal{P}}(z) \quad \text{for } \mu\text{-a.e. } x.$$

Since ν_z is invariant under E_x, the projection $\pi_{i*}(\nu_z)$ is invariant under $\pi_i(E_x)$. By Lemma 7.8, the group $\pi_i(E_x)$ is an unbounded subgroup of G_i for $i = 1, 2$. Since $\mathbf{G}_i$ is simply connected, m_i is $\pi_i(E_x)$-ergodic (see [26, Chapter 1, Theorem 2.3.1], [26, Chapter 2, Theorem 7.2]). Therefore,

$$\pi_{i*}\nu_z = m_i \quad \text{for } \mu_x^{\mathcal{P}}\text{-a.e. } z.$$

In particular, we get that $\pi_i(g_z L_0 g_z^{-1}) = G_i$ for $\mu_x^{\mathcal{P}}$-a.e. z and $i = 1, 2$.

Since $\mathbf{G}_i$ is absolutely almost simple, any proper normal subgroup of G_i, as an abstract group, is central (see [26, Chapter 1, Theorem 1.5.6]). This implies that one of the following holds:

- $L_0 = G_1 \times G_2$, or
- $\pi_i(L_0) = G_i$ and $\ker(\pi_i|_{L_0}) \subset C(G_1 \times G_2)$ for $i = 1, 2$,

as we claimed. If $L_0 = G \times G$, then we are done with the proof. Hence, our standing assumption for the rest of the argument is that (b) above holds.

Step 3. The assertion in (b) also holds for M_0 and $M_0^+(\lambda_0)$ in place of L_0. Let us first show this for M_0. Since $L_0 \subset M_0$, we have

$$\pi_i(M_0) = G_i \quad \text{for } i = 1, 2.$$

Therefore, as above, either $M_0 = G_1 \times G_2$ or (b) holds for M_0. Assume to the contrary that $M_0 = G_1 \times G_2$. Recall that $\lambda_0 : \mathbf{G}_m \to \mathbf{M}_0$ is a noncentral homomorphism. Since $\mathbf{G}_i$ is connected, simply connected, and absolutely almost simple for $i = 1, 2$, using [26, Chapter 1, Proposition 1.5.4, Theorem 2.3.1], we have that either

- $M_0^+(\lambda_0) = G_1 \times G_2$, or
- $M_0^+(\lambda_0) \subset G_i$ for some $i = 1, 2$.

However, since $M_0^+(\lambda_0) \subset L_0$, the above contradict our assumption that (b) holds.

We now turn to the proof of the claim for $M_0^+(\lambda_0)$. Since $M_0 \neq G_1 \times G_2$ and $M_0^+(\lambda_0) \subset M_0$, the claim follows if we show that

$$\pi_i\big(M_0^+(\lambda_0)\big) = G_i \quad \text{for } i = 1, 2. \tag{7.10}$$

To see this, note that $\lambda_0(l_0^\times) \subset M_0(\lambda_0)$. Since (b) holds for M_0, we have that $\pi_i(\lambda_0(l_0^\times))$ is unbounded for $i = 1, 2$. Therefore, (7.10) follows from [26, Chapter 1, Proposition 1.5.4, Theorem 2.3.1].

Let us record the following corollaries of the above discussion for later use. Since (b) holds for $M_0^+(\lambda_0)$, L_0, and M_0, we have

$$N_{G_1 \times G_2}(M_0) \subset CM_0, \tag{7.11}$$

where $C := Z(G_1 \times G_2)$. We also have that

$$M_0^+(\lambda_0) \text{ is a finite index subgroup of } L_0 \text{ and of } M_0. \tag{7.12}$$

Step 4. Both

$$M_0^+(\lambda_0)(\Gamma_1 \times \Gamma_2)/(\Gamma_1 \times \Gamma_2) \qquad \text{and} \qquad M_0(\Gamma_1 \times \Gamma_2)/(\Gamma_1 \times \Gamma_2)$$

are closed orbits with probability-invariant Haar measures. In particular, ν_x is the Haar measure on the closed orbit

$$g_x M_0^+(\lambda_0)(\Gamma_1 \times \Gamma_2)/(\Gamma_1 \times \Gamma_2).$$

Indeed, let $\Lambda := M_0 \cap (\Gamma_1 \times \Gamma_2)$. Then by (7.12) and Step 1, Λ is a lattice in M_0, as was claimed for M_0.

Using (7.12) once more, we have that $\Lambda \cap M_0^+(\lambda_0)$ has finite index in Λ. This implies that $\Lambda \cap M_0^+(\lambda_0)$ is a lattice in $M_0^+(\lambda_0)$; hence, the claim for $M^+(\lambda_0)$.

Step 5. We are now in a position to finish the proof. In view of (7.11), (7.12), and Step 4, we can argue as in the proof of Lemma 6.9 (see, in particular, (6.21)) and get the following. Let $C' := C \cap (\Gamma_1 \times \Gamma_2)$. The decomposition

$$\mu = \int \nu_x \, \mathrm{d}\mu$$

yields the Borel map $f(x) = g_x C' M_0$ from a conull subset of X to $G_1 \times G_2 / C' M_0$. Moreover, f is an A-equivariant map. Hence, it follows from Lemma 3.3 that there exists some

$$g_0 \in \mathrm{Fix}_{A_{l_0}^{\mathrm{sp}}}(G_1 \times G_2 / C' M_0)$$

so that $f_* \mu$ is the A-invariant measure on the compact orbit $A g_0$.

By Lemma 3.2 and (7.12), we have that $M_0^+(\lambda_0)$ is a normal and finite index subgroup of M_0; furthermore, C' is a finite group. Therefore, arguing as we did to complete the proof of Theorem 1.1 after (6.34), we get that there is some $g_1 \in M_0$ so that

$$\mu = \int_{A/A\cap g_0 g_1 M_0^+(\lambda_0) g_1^{-1} g_0^{-1}} a_*\nu \, \mathrm{d}a,$$

where da is the probability Haar measure on the compact group

$$A/A \cap g_0 g_1 M_0^+(\lambda_0) g_1^{-1} g_0^{-1},$$

and ν is the probability Haar measure on the closed orbit

$$g_0 g_1 M_0^+(\lambda_0)(\Gamma_1 \times \Gamma_2)/(\Gamma_1 \times \Gamma_2).$$

Hence, Theorem 1.2(2) holds with $\Sigma = g_0 g_1 M_0^+(\lambda_0) g_1^{-1} g_0^{-1}$. □

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