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Quenched invariance principles for orthomartingale-like sequences

Magda Peligrad and Dalibor Volný

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025, USA.
Email: peligrm@ucmail.uc.edu
LMRS, CNRS-Université de Rouen Normandie
e-mail : dalibor.volny@univ-rouen.fr

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Abstract

In this paper we study the central limit theorem and its functional form for random fields which are not started from their equilibrium, but rather under the measure conditioned by the past sigma field. The initial class considered is that of orthomartingales and then the result is extended to a more general class of random fields by approximating them, in some sense, with an orthomartingale. We construct an example which shows that there are orthomartingales which satisfy the CLT but not its quenched form. This example also clarifies the optimality of the moment conditions used for the validity of our results. Finally, by using the so called orthomartingale-coboundary decomposition, we apply our results to linear and nonlinear random fields.

1 Introduction and the quenched CLT

A very interesting type of convergence, with many practical applications, is the almost sure conditional central limit theorem and its functional form. This means that these theorems hold when the process is not started from its equilibrium but it is rather started from a fixed past trajectory. In the Markovian setting such a behavior is called a limit theorem started at a point. In general these results are known under the name of quenched limit theorems, as opposed to the annealed ones. A quenched CLT, for instance, is a stronger form of convergence in distribution and implies the usual CLT. There are examples in the literature showing that the annealed CLT does not necessarily implies the quenched one. See for instance Ouchti and Volný (2008) and Volný and Woodroofe (2010).

The limit theorems started at a point or from a fixed past trajectory are often encountered in evolutions in random media and they are of considerable importance in statistical mechanics. They are also useful for analyzing Markov chain Monte Carlo algorithms.

In the context of random processes, this remarkable property is known for a martingale which is stationary and ergodic, as shown in Ch. 4 in Borodin and Ibragimov (1994) or on page 520 in Derriennic and Lin (2001). By using martingale approximations, this result was extended to larger classes of random variables by Cuny and Peligrad (2012), Volný and Woodroofe (2014), Cuny and Merlevède (2014), among others (for a survey see Peligrad, 2015).

A random field consists of multi-indexed random variables $(X_u)_{u\in Z^d}$. An important class of random fields are orthomartingales which have been introduced by Cairoli (1969) and further developed in Khoshnevisan (1982). They have resurfaced in many recent works. New versions of the central limit theorem for stationary orthomartingales can be found in Wang and Woodroofe (2013), Volný (2015, 2019), which complement the results in Basu and Dorea (1979), where a different definition of multiparameter martingale was used.

In order to exploit the richness of the martingale techniques several authors provided interesting sufficient conditions for orthomartingale approximations, such as Gordin (2009), Volný and Wang (2014), Cuny et al. (2015), El Machkouri and Giraudo (2016), Peligrad and Zhang (2018 a), Giraudo (2018), Volný (2018). Other recent results involve random fields which are functions of independent random variables as in El Machkouri et al. (2013) and Wang and Woodroofe (2013). Peligrad and Zhang (2018 b) obtained necessary and sufficient conditions for an orthomartingale approximation in the mean square. These approximations make possible to obtain the central limit theorem (CLT) for a large class of random fields. As in the case of a stochastic processes, a natural and important question is to get a quenched version of these CLT's. Motivated by this question, we obtain first a quenched CLT for orthomartingales. We show by examples that the situation is different for random fields. An orthomartingale which satisfies the CLT may fail to satisfy the quenched CLT. The example we constructed also throws light on the optimality of the moment conditions we use in our main result. Finally, we extend the quenched CLT to its functional form and to a larger class of random fields which can be decomposed into a orthomartingale and a coboundary. We shall apply our results to linear and nonlinear random fields, often encounters in economics.

For the sake of clarity, due to the complicated notation, we shall explain in detail the case $d = 2$ and the proof of the quenched CLT. Then, in the subsequent sections, we shall discuss the general index set Z^d and other extensions of these results.

Let $(\Omega, \mathcal{K}, P)$ be a probability space, let T and S be two commuting, invertible, bimeasurable, measure preserving transformations from Ω to Ω, and let

$\mathcal{F}_{0,0}$ be a sub-sigma field of $\mathcal{K}$. For all $(i,j) \in Z^2$ define

$$\mathcal{F}_{i,j} = T^{-i}S^{-j}(\mathcal{F}_{0,0}),\, i,j \in Z. \tag{1}$$

Assume the filtration is increasing in i for every j fixed and increasing in j for every i fixed (i.e. $\mathcal{F}_{0,0} \subset \mathcal{F}_{0,1}$ and $\mathcal{F}_{0,0} \subset \mathcal{F}_{1,0}$). For all i and j we also define the following sigma algebras generated by the unions of sigma algebras: $\mathcal{F}_{i,\infty} = \vee_{m\in Z}\mathcal{F}_{i,m}$, $\mathcal{F}_{\infty,j} = \vee_{n\in Z}\mathcal{F}_{n,j}$ and $\mathcal{F}_{\infty,\infty} = \vee_{n,m\in Z}\mathcal{F}_{n,m}$. In addition assume the filtration is commuting, in the sense that for any integrable variable X, with notation $E_{a,b}X = E(X|\mathcal{F}_{a,b})$, we have

$$E_{u,v}E_{a,b}X = E_{a\wedge u, b\wedge v}X. \tag{2}$$

We introduce the stationary sequence as following. Define a function $X_{0,0} : \Omega \to R$, which is $\mathcal{F}_{0,0}-$measurable, and the random field

$$X_{i,j}(\omega) = X_{0,0}(T^iS^j(\omega)). \tag{3}$$

For the filtration $(\mathcal{F}_{i,j})$ defined by (1) we call the random field $(X_{i,j})_{i,j\in Z}$ defined by (3) orthomartingale difference field, if

$$E(X_{i,j}|\mathcal{F}_{u,v}) = 0 \text{ if either } u < i \text{ or } v < j. \tag{4}$$

This definition implies that for any i fixed $(X_{i,j})_{j\in Z}$ is a sequence of martingale differences with respect to the filtration $(\mathcal{F}_{\infty,j})_{j\in Z}$ and also for any j fixed $(X_{i,j})_{i\in Z}$ is a sequence of martingale differences with respect to the filtration $(\mathcal{F}_{i,\infty})_{i\in Z}$. Set

$$S_{n,v} = \sum_{i=0}^{n-1}\sum_{j=0}^{v-1} X_{i,j}.$$

Below, $\Rightarrow$ denotes convergence in distribution.

The results in this paper are motivated by the following annealed CLT in Volný (2015), which was extended to a functional CLT in Cuny et al. (2015).

Theorem A *Assume that $(X_{i,j})_{i,j\in Z}$ is defined by (3) and satisfies (4). Also assume that the filtration $(\mathcal{F}_{i,j})_{i,j\in Z}$ is defined by (1) and satisfies (2). Assume that S (or T) is ergodic and $X_{0,0}$ is square integrable, $E(X_{0,0}^2) = \sigma^2$. Then,*

$$\frac{1}{(nv)^{1/2}} S_{n,v} \Rightarrow N(0,\sigma^2) \text{ when } n\wedge v \to \infty.$$

Let us point out that if S *(or T)* is ergodic, then the Z^2 action generated by S and T is necessarily ergodic. However the ergodicity is not enough for Theorem A to hold. In Example 5.6 in Wang and Woodroofe (2013) and then in more detail by Volný (2015), a simple example of ergodic random field which does not satisfy the central limit theorem is analyzed. Starting with two sequences of i.i.d. random variables, centered with finite second moments, (X_n) and (Y_n), the example is provided by the random field $(Z_{i,j})$, with $Z_{i,j} = X_iY_j$ for all (i,j).

It should be noted that Theorem A has a different area of applications than Theorem 1 in Basu and Dorea (1979). In this latter paper the filtration is not supposed to be commuting. For a random field $(X_{i,j})_{i,j\geq 1}$ their filtration $(\mathcal{K}_{n,m})$ is generated by the variables $\{X_{i,j} : (j \geq 1, 1 \leq i \leq n) \cup (i \geq 1, 1 \leq j \leq m)\}$. Suppose $(\xi_{i,j})$ are i.i.d., standard normal random variables. Then, Theorem A can be applied, for instance, to the random field $(X_{i,j})$, where $X_{i,j}(\omega) = X_{0,0}(T^i S^j(\omega))$ with $X_{0,0} = \xi_{-1,0}\xi_{0,-1}$ and $\mathcal{F}_{0,0} = \sigma(\xi_{i,j}, i \leq 0, j \leq 0)$ but the result in Basu and Dorea (1979) cannot. On the other hand the random field $(Y_{i,j})$, defined by $Y_{i,j} = Y_{0,0}(T^i S^j(\omega))$ with $Y_{0,0} = \sum_{k=1}^{\infty} a_k(\xi_{k,0} + \xi_{0,k})$ and $\sum_{k=1}^{\infty} |a_k| < \infty$, can be treated by the result in Basu and Dorea (1979) but not by Theorem A.

It should also be noted that Theorem A allows to study the central limit theorem for orthomartingales which are not defined by a Bernoulli Z^2-action.

The aim of this paper is to establish a quenched version of Theorem A.

We denote by $P^\omega(\cdot) = P^\omega_{0,0}(\cdot)$ a version of the regular conditional probability $P(\cdot|\mathcal{F}_{0,0})(\omega)$.

One of the results of this paper is the following theorem:

Theorem 1 *Assume that $(X_{i,j})_{i,j\in Z}$ is defined by (3) and satisfies (4). Also assume that the filtration $(\mathcal{F}_{i,j})_{i,j\in Z}$ is defined by (1) and satisfies (2). Assume that S (or T) is ergodic and $X_{0,0}$ is square integrable, $E(X_{0,0}^2) = \sigma^2$. Then, for P-almost all $\omega \in \Omega$,*

$$\frac{1}{n} S_{n,n} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega. \quad (5)$$

In addition, if

$$E(X_{0,0}^2 \log(1 + |X_{0,0}|)) < \infty, \quad (6)$$

then for almost all all $\omega \in \Omega$,

$$\frac{1}{(nv)^{1/2}} S_{n,v} \Rightarrow N(0, \sigma^2) \text{ under } P^\omega \text{ when } n \wedge v \to \infty. \quad (7)$$

We would like to mention that, because by integration the quenched CLT implies the annealed CLT, the conclusion in Theorem 1 implies the CLT in Theorem A. However, when the summation on the rectangles is not restricted, the integrability assumption (6) is stronger than in Theorem A. Later on, in Theorem 5, we shall extend this result to a functional central limit theorem. Let us also notice that the second part of Theorem 1 does not always hold under the assumption $E(X_{0,0}^2) < \infty$. As a matter of fact we are going to provide an example to support this claim.

Theorem 2 *Under the setting used in Theorem 1, there is a stationary sequence $(X_{n,m})_{n,m\in Z}$ satisfying (4), adapted to a commuting filtration $(\mathcal{F}_{i,j})_{i,j\in Z}$, with $E(X_{0,0}^2 \ln(1 + |X_{0,0}|)) = \infty$, for any $0 < \varepsilon < 1$, $E(X_{0,0}^2 \ln^{1-\varepsilon}(1 + |X_{0,0}|)) < \infty$ and such that $(S_{n,m}/\sqrt{nm})_{(n,m)\in Z^2}$ does not satisfy the quenched CLT in (7).*

We mention that, as a matter of fact, in our examples, both transformations constructed for the definition of $(X_{n,m})_{n,m\in Z}$ and for the filtration $(\mathcal{F}_{i,j})_{i,j\in Z}$, are ergodic. Also, this example satisfies the quenched CLT in (5).

The detailed proofs of these two theorems are contained in Section 2. Various extensions of Theorem 1 will be given in subsequent sections.

In Section 3 we formulate the functional form of the quenched CLT and we indicate how to prove it, by adapting the arguments from the proof of Theorem 1 and some other proofs of several known results.

For the sake of applications, in Section 4, we extend the results beyond orthomartingales, to a class of random fields which can be decomposed into an orthomartingale and a generalized coboundary.

In Section 5 we show that Theorem 1 remains valid for random fields indexed by Z^d, $d > 2$. The only difference is that we replace condition (6) by $E(X_{0,0}^2 \log^{d-1}(1 + |X_{0,0}|)) < \infty$.

In Section 6 we apply our results to linear and nonlinear random fields with independent innovations. Several useful results for our proofs are given in Section 7.

2 Proofs of Theorems 1 and 2

Proof of Theorem 1

To fix the ideas, let us suppose that the transformation S is ergodic. Let us denote by $\hat{T}$ and $\hat{S}$ the operators on L_2, defined by $\hat{T}f = f \circ T$ and $\hat{S}f = f \circ S$. Everywhere in the paper, for x real, we shall denote by $[x]$ the integer part of x.

By using a truncation argument, we show first that, without restricting the generality, we can prove the theorem under the additional assumption that the variables are bounded. We shall introduce the following projection operators:

$$\mathcal{P}_{i,j}(X) = E_{i,j}(X) - E_{i,j-1}(X) - E_{i-1,j}(X) + E_{i-1,j-1}(X).$$

Let A be a positive integer. Denote $X_{i,j}^{'} = X_{i,j}I(|X_{i,j}| \le A)$ and $X_{i,j}^{"} = X_{i,j}I(|X_{i,j}| > A)$. Therefore, we can represent $(X_{i,j})$ as a sum of two orthomartingale differences adapted to the same filtration.

$$X_{i,j} = \mathcal{P}_{i,j}(X_{i,j}^{'}) + \mathcal{P}_{i,j}(X_{i,j}^{"}). \tag{8}$$

Note that,

$$|\mathcal{P}_{0,0}(X_{0,0}^{"})| \le |X_{0,0}| + E_{-1,0}|X_{0,0}| + E_{0,-1}|X_{0,0}| + E_{-1,1}|X_{0,0}|.$$

Whence, by the properties of conditional expectation, $E(X_{0,0})^2 < \infty$ implies

$$E(\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 < \infty \tag{9}$$

and $E(X_{0,0}^2 \log(1+|X_{0,0}|)) < \infty$ implies

$$E((\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 \log(1+|(\mathcal{P}_{0,0}(X_{0,0}^{"}))|) < \infty. \tag{10}$$

Set

$$S_{n,v}^{'} = \sum_{i=0}^{n-1}\sum_{j=0}^{v-1}\mathcal{P}_{i,j}(X_{i,j}^{'}) \text{ and } S_{n,v}^{"} = \sum_{i=0}^{n-1}\sum_{j=0}^{v-1}\mathcal{P}_{i,j}(X_{i,j}^{"}).$$

We shall show that, for $P-$almost all ω,

$$\lim_{A\to\infty} \lim \sup_{n\wedge v\to\infty} P^{\omega}(\frac{1}{(nv)^{1/2}}|S_{n,v}^{"}| > \varepsilon) = 0.$$

By conditional Markov inequality, it is enough to show that

$$\lim_{A\to\infty} \lim_{n\wedge v\to\infty} \frac{1}{nv} E_{0,0}(S_{n,v}^{"})^2 = 0 \quad \text{a.s.} \tag{11}$$

By the orthogonality of the orthomartingale differences, we have that

$$\frac{1}{nv} E_{0,0}((S_{n,v}^{"})^2) = \frac{1}{nv}\sum_{i=0}^{n-1}\sum_{j=0}^{v-1} E_{0,0}(\mathcal{P}_{i,j}(X_{i,j}^{"}))^2. \tag{12}$$

Note that the conditional expectation introduces a family of operators defined by

$$Q_1(f) = E_{0,\infty}(\hat{T}f) \ ; \ Q_2(f) = E_{\infty,0}(\hat{S}f).$$

So, using (2), we can write

$$E_{0,0}(\mathcal{P}_{i,j}(X_{i,j}^{"}))^2 = Q_1^i Q_2^j (\mathcal{P}_{0,0}(X_{0,0}^{"}))^2.$$

Since Q_1 and Q_2 are integral preserving Dunford-Schwartz operators, by the ergodic theorem (see Theorem 3.5 in Ch. 6 in Krengel, 1985), if we assume finite second moment, by (9),

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} Q_1^i Q_2^j (\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 = E(\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 \text{ a.s.}$$

If we assume $E(X_{0,0}^2 \log(1+|X_{0,0}|)) < \infty$ then, by (10) and Theorem 1.1 in Ch. 6, Krengel (1985), we obtain

$$\lim_{n\wedge v\to\infty} \frac{1}{nv}\sum_{i=0}^{n-1}\sum_{j=0}^{v-1} Q_1^i Q_2^j (\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 = E(\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 \text{ a.s.} \tag{13}$$

Clearly $\lim_{A\to\infty} \mathcal{P}_{0,0}(X_{0,0}^{"}) = 0$ a.s. So, by the dominated convergence theorem,

$$\lim_{A\to\infty} E(\mathcal{P}_{0,0}(X_{0,0}^{"}))^2 = 0,$$

and (11) is established. By Theorem 3.2 in Billingsley (1999), in order to establish conclusion (7) of Theorem 1, it is enough to show that for A fixed, for almost all $\omega \in \Omega$,

$$\frac{1}{(nv)^{1/2}} S_{n,v}' \Rightarrow N(0,\sigma_A^2) \text{ under } P^{\omega} \text{ as } n\wedge v\to\infty, \text{ and } \sigma_A^2 \to \sigma^2 \text{ as } A\to\infty.$$

Above, $\sigma_A^2 = E(\mathcal{P}_{0,0}(X_{0,0}^{'}))^2$. Clearly, when $A \to \infty$, $\sigma_A^2 \to \sigma^2$. Therefore the result is established if we prove Theorem 1 for orthomartingale differences which are additionally uniformly bounded.

So, in the rest of the proof, without restricting the generality, we shall assume that the variables $(X_{i,j})_{i,j \in Z}$ are bounded by a positive constant C. Also, proving the result for $n > v \to \infty$ is equivalent to proving it for any subsequence (n, v_n) with $v_n \to \infty$ as $n \to \infty$. To ease the notation we shall denote $v = v_n$.

Denote

$$F_{i,v} = \frac{1}{v^{1/2}} \sum\nolimits_{j=0}^{v-1} X_{i,j}. \tag{14}$$

We treat the double summation as a sum of a triangular array of martingale differences $(F_{i,v})_{i \geq 0}$:

$$\frac{1}{(nv)^{1/2}} S_{n,v} = \frac{1}{n^{1/2}} \sum\nolimits_{i=0}^{n-1} F_{i,v}.$$

We shall apply Theorem 1 in Gänssler and Häusler (1979), given for convenience in Theorem 15 from Section 7, to $D_{n,i} = F_{i,v}/\sqrt{n}$. We have to show that for almost all ω, both conditions of this theorem are satisfied. Namely we shall verify that $P-$for almost all $\omega \in \Omega$ and all rationals $q \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{n} E_{0,0} |\sum\nolimits_{i=0}^{[(n-1)q]} (F_{i,v}^2 - \sigma^2)| = 0. \tag{15}$$

and

$$\frac{1}{n} E_{0,0} \max_{0 \leq i \leq n-1} F_{i,v}^2 \text{ is bounded.} \tag{16}$$

We verify first (15). Note that, since the rationals are countable, it is enough to show that for any q rational

$$\lim_{n\to\infty} \frac{1}{n} E_{0,0} |\sum\nolimits_{i=0}^{[(n-1)q]} (F_{i,v}^2 - \sigma^2)| = 0 \quad P-\text{a.s.}$$

We verify it first with $q = 1$ and use a blocking procedure.

Let $m \geq 1$ be a fixed integer and define consecutive blocks of indexes of size m, $I_j(m) = \{(j-1)m, ..., mj-1\}$. In the set of integers from 0 to $n-1$ we have $u = u_n(m) = [n/m]$ such blocks of integers and a last one containing less than m indexes. Practically, by the triangle inequality, we write

$$\frac{1}{n} |\sum\nolimits_{i=0}^{n-1} (F_{i,v}^2 - \sigma^2)| \leq$$
$$\frac{1}{n} \sum\nolimits_{j=1}^{u} |\sum\nolimits_{k \in I_j(m)} (F_{k,v}^2 - \sigma^2)| + \frac{1}{n} |\sum\nolimits_{k=um}^{n-1} (F_{k,v}^2 - \sigma^2)| \leq$$
$$\frac{1}{u} \sum\nolimits_{j=1}^{u} |\frac{1}{m} \sum\nolimits_{k \in I_j(m)} F_{k,v}^2 - \sigma^2| + \frac{1}{n} |\sum\nolimits_{k=um}^{n-1} (F_{k,v}^2 - \sigma^2)| =$$
$$I_{n,m} + II_{n,m}.$$

The task is now to show that

$$\lim_{m\to\infty} \lim_{n \wedge v \to\infty} E_{0,0}(I_{n,m}) = 0 \quad \text{a.s.} \tag{17}$$

and

$$\lim_{m\to\infty}\lim_{n\wedge v\to\infty} E_{0,0}(II_{n,m}) = 0 \text{ a.s.} \tag{18}$$

Let us treat first the limit of $E_{0,0}(I_{n,m})$. Let N_0 be a fixed integer and consider $n \wedge v > N_0$. By using the properties of the conditional expectations and (2) we obtain the following bound for $E_{0,0}(I_{n,m})$:

$$\begin{aligned} E_{0,0}(I_{n,m}) &= \frac{1}{u}E_{0,0}\sum\nolimits_{j=1}^{u}\left|\frac{1}{m}\sum\nolimits_{k\in I_j(m)} F_{k,v}^2 - \sigma^2\right| \\ &= \frac{1}{u}E_{0,0}\sum\nolimits_{j=1}^{u} E_{(j-1)m,0}\left|\frac{1}{m}\sum\nolimits_{k\in I_j(m)} F_{k,v}^2 - \sigma^2\right| \\ &= E_{0,0}\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}E_{0,0}\left|\frac{1}{m}\sum\nolimits_{k=0}^{m-1} F_{k,v}^2 - \sigma^2\right| \\ &\le E_{0,0}\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}(h_{m,N_0}), \end{aligned}$$

where we have used the notation

$$h_{m,N_0} = \sup_{v>N_0} E_{0,0}\left|\frac{1}{m}\sum\nolimits_{k=0}^{m-1} F_{k,v}^2 - \sigma^2\right|.$$

Note that h_{m,N_0} is bounded. Indeed, by the martingale property and the uniform boundedness of the variables by C, it follows that

$$\begin{aligned} h_{m,N_0} &\le \sigma^2 + \frac{1}{m}\sum\nolimits_{k=0}^{m-1}\sup_{v>N_0} E_{0,0}(F_{k,v}^2) \\ &= \sigma^2 + \frac{1}{m}\sum\nolimits_{k=0}^{m-1}\sup_{v>N_0} E_{0,0}\Big(\frac{1}{v}\sum\nolimits_{u=0}^{v-1} X_{k,u}^2\Big) \le \sigma^2 + C^2. \end{aligned}$$

By the ergodic theorem, (see Theorem 11.4 in Eisner et al., 2015 or Corollary 3.8 in Ch. 3, Krengel, 1985) for each m and N_0

$$\lim_{u\to\infty}\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}h_{m,N_0} = E(h_{m,N_0}|I) = E_I(h_{m,N_0}) \text{ a.s.},$$

where I is the invariant sigma field for the operator T. Furthermore, we also have that

$$\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}h_{m,N_0} \le \sigma^2 + C^2.$$

So, by Theorem 34.2 (v) in Billingsley (1995) (see Theorem 16 in Section 7) we derive that

$$\lim_{u\to\infty} E_{0,0}\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}h_{m,N_0} = E_{0,0}E_I(h_{m,N_0}) \text{ a.s.}$$

Since the functions are bounded, by applying twice, consecutively, Theorem 16, we obtain that

$$\lim_{N_0\to\infty}\lim_{u\to\infty} E_{0,0}\frac{1}{u}\sum\nolimits_{i=0}^{u-1}\hat{T}^{im}h_{m,N_0} = E_{0,0}E_I(\lim_{N_0\to\infty} h_{m,N_0}) \text{ a.s.}$$

Clearly, because the variables are bounded, for every m fixed

$$E_{0,0}E_I(\lim_{N_0\to\infty} h_{m,N_0}) = E_{0,0}E_I(\limsup_v E_{0,0}|\frac{1}{m}\sum\nolimits_{k=0}^{m-1} F_{k,v}^2 - \sigma^2|)$$
$$\le E_{0,0}E_I E_{0,0}(\limsup_v E_{\infty,0}|\frac{1}{m}\sum\nolimits_{k=0}^{m-1} F_{k,v}^2 - \sigma^2|).$$

Now, by using again the fact that the variables are bounded and using Theorem 16, in order to show that

$$\lim_{m\to\infty} E_{0,0}E_I(\lim_{N_0\to\infty} h_{m,N_0}) = 0 \ P\text{-a.s.}$$

it is enough to show that

$$\lim_{m\to\infty}\limsup_v E_{\infty,0}|\frac{1}{m}\sum\nolimits_{k=0}^{m-1} F_{k,v}^2 - \sigma^2| = 0 \ \text{ a.s.} \tag{19}$$

With this aim, we note first that by the ergodicity of S and the fact that the variables are bounded, it follows that, for any k,

$$\lim_{v\to\infty} E_{\infty,0}F_{k,v}^2 = \lim_{v\to\infty}\frac{1}{v}E_{\infty,0}(\sum\nolimits_{j=0}^{v-1} X_{k,j}^2) = \sigma^2. \tag{20}$$

Denote $P_{\infty,0}^{\omega}(\cdot) = P(\cdot|\mathcal{F}_{\infty,0})$. We also know that for any k, by the quenched CLT for stationary martingale differences (see, for instance, Ch. 4 in Borodin and Ibragimov (1994) or Derrienic and Lin (2001)), for almost all ω, $F_{k,v} \Rightarrow N_k$ under $P_{\infty,0}^{\omega}$, where N_k is a centered normal random variable with variance σ^2. Therefore, by the sufficiency part of the convergence of moments associated to weak convergence, namely Theorem 3.6 in Billingsley (1999), we have that

$$(F_{k,v}^2)_{v\ge 1} \text{ is uniformly integrable under } P_{\infty,0}^{\omega} \text{ for almost all } \omega. \tag{21}$$

By the functional quenched CLT for martingales (see Ch. 4 in Borodin and Ibragimov (1994)), for almost all ω, we know that

$$(F_{0,v}, F_{1,v}, ..., F_{m-1,v}) \Rightarrow (N_0, N_1, ..., N_{m-1}) \text{ under } P_{\infty,0}^{\omega} \text{ as } v\to\infty,$$

where $(N_0, N_1, ..., N_{m-1})$ is a Gaussian vector of centered normal variables with variance σ^2. But since $(F_{j,v})_{j\in Z}$ are uncorrelated it follows by (21) that the variables in $(N_i)_{i\ge 0}$ are also uncorrelated and therefore $(N_i)_{i\ge 0}$ is an i.i.d. sequence. By the continuous mapping theorem,

$$\frac{1}{m}\sum\nolimits_{k=0}^{m-1}(F_{k,v}^2 - \sigma^2) \Rightarrow \frac{1}{m}\sum\nolimits_{k=0}^{m-1}(N_k^2 - \sigma^2) \text{ under } P_{\infty,0}^{\omega} \text{ for almost all } \omega.$$

By (21) it follows that $(\sum_{k=0}^{m-1}(F_{k,v}^2 - \sigma^2))_{v\ge 1}$ is also uniformly integrable, so we can apply the convergence of moments from Theorem 3.5 in Billingsley (1999). Therefore, denoting by $\mathcal{E}$ the expectation in rapport with the probability on the space where the variables $(N_k)'s$ are defined, we obtain

$$\lim_{v\to\infty} E_{\infty,0}|\frac{1}{m}\sum\nolimits_{k=0}^{m-1}(F_{k,v}^2 - \sigma^2)| = \mathcal{E}|\frac{1}{m}\sum\nolimits_{k=0}^{m-1}(N_k^2 - \sigma^2)| \quad \text{a.s.}$$

By letting $m \to \infty$ and using the law of large numbers for an i.i.d. sequence, we obtain

$$\lim_{m\to\infty} \mathcal{E}(|\frac{1}{m}\sum\nolimits_{k=0}^{m-1}(N_k^2 - \sigma^2)| = 0.$$

Therefore (19) follows. As a consequence, we obtain (17).

In order to treat the term (18), we estimate

$$E_{0,0}(II_{n,m}) = E_{0,0}\frac{1}{n}|\sum\nolimits_{k=um}^{n-1}(F_{k,v}^2 - \sigma^2)| \leq \frac{m}{n}\sigma^2 + E_{0,0}\frac{1}{n}\sum\nolimits_{k=um}^{n-1} F_{k,v}^2$$
$$\leq \frac{m}{n}\sigma^2 + \frac{1}{n}\sum\nolimits_{k=um}^{n-1}\frac{1}{v}\sum\nolimits_{j=0}^{v-1} E_{0,0}X_{k,j}^2 \leq \frac{m}{n}(\sigma^2 + C^2) \text{ a.s.}$$

Whence, (18) follows, by passing to the limit first with $n \to \infty$ followed by $m \to \infty$.

Overall, we have shown that

$$\lim_{n\wedge v\to\infty} \frac{1}{n}E_{0,0}|\sum\nolimits_{u=0}^{n-1}(F_{u,v}^2 - \sigma^2)| = 0 \text{ a.s.}$$

If we replace now $n-1$ by $[(n-1)q]$, with q a rational number, we easily see that we also have convergence to $q\sigma^2$ and (15) follows.

It remains to verify the second condition of Theorem 15, namely to prove (16). To show it, note that, by the martingale property,

$$\frac{1}{n}E_{0,0}(\max_{0\leq i\leq n-1} F_{i,v}^2) \leq \frac{1}{n}E_{0,0}(\sum\nolimits_{i=0}^{n-1} F_{i,v}^2)$$
$$= \frac{1}{nv}(\sum\nolimits_{i=0}^{n-1}\sum\nolimits_{u=0}^{v-1} E_{0,0}(X_{i,u}^2)) \leq C^2 \text{ a.s.}$$

The proof of the theorem is now complete. □

Proof of Theorem 2

We start with an i.i.d. random field $(\xi_{n,m})_{n,m\in Z}$ defined on a probability space $(\Omega, \mathcal{K}, P)$ with the distribution

$$P(\xi_{0,0} = -1) = P(\xi_{0,0} = 1) = 1/2. \tag{22}$$

Without restricting the generality we shall define $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ in a canonical way on the probability space $\Omega = R^{Z^2}$, endowed with the $\sigma-$field $\mathcal{B}$, generated by cylinders. Then, if $\omega = (x_{\mathbf{v}})_{\mathbf{v}\in Z^2}$, we define $\xi'_{\mathbf{u}}(\omega) = x_{\mathbf{u}}$. We construct a probability measure P' on $\mathcal{B}$ such that for all $B \in \mathcal{B}$, any m and $\mathbf{u}_1, ..., \mathbf{u}_m$ we have

$$P'((x_{\mathbf{u}_1}, ..., x_{\mathbf{u}_m}) \in B) = P((\xi_{\mathbf{u}_1}, ..., \xi_{\mathbf{u}_m}) \in B).$$

The new sequence $(\xi'_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ is distributed as $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$ and re-denoted by $(\xi_{\mathbf{u}})_{\mathbf{u}\in Z^2}$. We shall also re-denote P' as P. Now on R^{Z^2} we introduce the operators

$$T_{\mathbf{u}}((x_{\mathbf{v}})_{\mathbf{v}\in Z^2}) = (x_{\mathbf{v}+\mathbf{u}})_{\mathbf{v}\in Z^2}.$$

Two of them will play an important role, namely when $\mathbf{u} = (1, 0)$ and when $\mathbf{u} = (0, 1)$. By interpreting the indexes as notations for the lines and columns of a matrix, we shall call

$$T((x_{u,v})_{(u,v)\in Z^2}) = (x_{u+1,v})_{(u,v)\in Z^2}$$

the vertical shift and

$$S((x_{u,v})_{(u,v)\in Z^2}) = (x_{u,v+1})_{(u,v)\in Z^2}$$

the horizontal shift. Introduce the filtration $\mathcal{F}_{n,m} = \sigma(\xi_{i,j}, i \leq n, j \leq m)$ and notice that this filtration is commuting. We assume $\mathcal{K} = \mathcal{F}_{\infty,\infty}$. The transformations T and S are invertible, measure preserving, commuting and ergodic. Furthermore $T_{i,j} = T^i S^j$.

For a measurable function f defined on R^{Z^2} define

$$X_{j,k} = f(T^j S^k (\xi_{a,b})_{a\leq 0, b\leq 0}). \tag{23}$$

We notice that the variables are adapted to the filtration $(\mathcal{F}_{n,m})_{n,m\in Z}$.

As an important step for constructing our example we shall establish the following lemma:

Lemma 3 *For every n and every $\varepsilon > 0$ we can find a set $F = F(n, \varepsilon)$ which is $\mathcal{F}_{0,0}$ measurable and such that*

$$P(F) \geq \frac{1}{n^2}(1 - \varepsilon).$$

Furthermore, for any $0 \leq i, j \leq n - 1$, $0 \leq k, \ell \leq n - 1$ with $(i, j) \neq (k, \ell)$ we have

$$P(T_{i,j}^{-1} F \cap T_{k,l}^{-1} F) = 0. \tag{24}$$

Proof of Lemma 3.

Let n be an integer and let $\varepsilon > 0$. By using Rokhlin lemma (see Theorem 17 in Section 7), construct $B \in \mathcal{K}$ with

$$P(B) \geq (1 - \frac{\varepsilon}{2})\frac{1}{n^2} \tag{25}$$

and for $0 \leq i, j \leq n - 1$, $T_{i,j}^{-1} B$ are disjoint for distinct pair of indexes. Since $\mathcal{K}$ is generated by the field $\cup_{\mathbf{n}} \mathcal{F}_{\mathbf{n}}$, we can find a set E in $\cup_{\mathbf{n}} \mathcal{F}_{\mathbf{n}}$ such that

$$P(B \Delta E) < \frac{\varepsilon}{8n^4}. \tag{26}$$

Since E belongs to $\cup_{\mathbf{n}} \mathcal{F}_{\mathbf{n}}$, there is a $\mathbf{m}$ such that $E \in \mathcal{F}_{\mathbf{m}}$. So $T_{\mathbf{m}}(E) \in \mathcal{F}_{\mathbf{0}}$. Denote $G = T_{\mathbf{m}}(E)$ and set

$$F = G \setminus \cup_{(i,j)\in D} T_{i,j}^{-1} G,$$

where $D = \{0 \leq i, j \leq n-1, (i,j) \neq (0,0)\}$. Note now that for all $(i,j) \in D$,

$$P(F \cap T_{i,j}^{-1} F) = 0,$$

which implies (24). Also, by stationarity,

$$P(F) = P(E) - P(E \cap (\cup_{(i,j)\in D} T_{i,j}^{-1} E)) \geq P(E) - \sum\nolimits_{(i,j)\in D} P(E \cap T_{i,j}^{-1} E).$$

But for $(i,j) \in D$,

$$P(E \cap T_{i,j}^{-1} E) \leq 2P(E \setminus B) \leq \frac{\varepsilon}{4n^4}.$$

Therefore, by the above considerations, (26) and (25) we obtain

$$P(F) \geq P(E) - \frac{\varepsilon}{4n^2} \geq P(B) - \frac{\varepsilon}{8n^4} - \frac{\varepsilon}{4n^2} \geq \frac{1-\varepsilon}{n^2}.$$

□

Next, we obtain a lemma which is the main step in the construction of the example. In the sequel, we use the notation $a_n \sim b_n$ for $\lim_{n\to\infty} a_n/b_n = 1$.

Lemma 4 *There is a strictly stationary random field of integrable positive random variables $(U_{i,j})_{i,j\in Z}$, coordinatewise ergodic, such that for any $0 < \varepsilon < 1$, $E|U_{0,0}| \ln^{1-\varepsilon}(1 + |U_{0,0}|) < \infty$ and such that for almost all ω, $(U_{n,v}/nv)_{n,v\in Z}$ is not tight under P^ω.*

Proof of Lemma 4.

By Lemma 3, for $n \geq 2$ and $\varepsilon = 1/2$, we can find sets $F_n \in \mathcal{F}_{-n,-n}$ such that $P(F_n) = 1/2n^2$ and such that for any $0 \leq i, j \leq n-1$, $0 \leq k, \ell \leq n-1$ with $(i,j) \neq (k,\ell)$ we have $P(T_{i,j}^{-1} F_n \cap T_{k,l}^{-1} F_n) = 0$.

Now, we consider independent copies of the probability space $(\Omega, \mathcal{K}, P)$, denoted by $(\Omega^{(m)}, \mathcal{K}^{(m)}, P^{(m)})_{m\geq 1}$, and introduce the product space $\mathbf{\Omega} = \prod_{m=1}^{\infty} \Omega^{(m)}$ endowed with the sigma algebra generated by cylinders, $\mathbf{K} = \prod_{m=1}^{\infty} \mathcal{K}^{(m)}$. We also introduce on $\mathbf{K}$ the product probability $\mathbf{P} = \prod_{m=1}^{\infty} P^{(m)}$, $P^{(m)} = P$. In this space consider sets $F_n^{(n)}$ which are products of Ω with the exception of the n-th coordinate which is F_n.

On $\mathbf{\Omega}$, define a random variable f_n by the following formula:

$$f_n = \frac{n}{\ln^2 n} 1_{F_n^{(n)}}. \tag{27}$$

Let A_n be the following event:

$$A_n = \{\text{there are } i, j, \ \ln n \leq i, j \leq n-1, \text{ such that } f_n \circ \mathbf{T}_{i,j}/ij \geq 1\}.$$

where $\mathbf{T}_{i,j} = (T_{i,j}, T_{i,j}, ...)$. Since $f_n \circ \mathbf{T}_{i,j}$ is $\prod_{m=1}^{\infty} \mathcal{F}_{0,0}^{(m)}$ measurable, for $\omega \in A_n$, there are i, j, $\ln n \leq i, j \leq n-1$, such that

$$\mathbf{P}^\omega(f_n \circ \mathbf{T}_{i,j}/ij \geq 1) = 1. \tag{28}$$

Note now that $f_n \circ \mathbf{T}_{i,j}/ij \geq 1$ if and only if $1_{F_n^{(n)}} \circ \mathbf{T}_{i,j} \geq ij(\ln n)^2/n$, if and only if $\omega \in (\mathbf{T}_{i,j})^{-1}(F_n^{(n)})$ and $ij \leq n/(\ln n)^2$.

Then, the probability of A_n can be computed as:

$$\mathbf{P}(A_n) = \mathbf{P}(\bigcup_D \mathbf{T}_{i,j}^{-1}(F_n^{(n)}) = P(\bigcup_D T_{i,j}^{-1}(F_n)),$$

where the union and have indexes in the set $D = \{ij \leq (n-1)/(\ln n)^2; \ln n \leq i, j \leq n-1\}$. By Lemma 3, it follows that

$$\mathbf{P}(A_n) = P(F_n) \sum_{\ln n \leq j \leq n-1} \quad \sum_{\ln n \leq i \leq (n-1)/j(\ln n)^2} 1 \sim \frac{n \ln n}{2n^2 (\ln n)^2} = \frac{1}{2n \ln n}.$$

Therefore

$$\sum_{n \geq 2} \mathbf{P}(A_n) = \sum_{n \geq 2} \frac{1}{2n \ln n} = \infty.$$

By the second Borel-Cantelli lemma, $\mathbf{P}(A_n \text{ i.o.}) = 1$. This means that almost all $\omega \in \mathbf{\Omega}$ belong to an infinite number of A_n. Whence, taking into account (28), for almost all $\omega \in \mathbf{\Omega}$ and every positive B,

$$\lim \sup_{i \wedge j \to \infty} \mathbf{P}^{\omega}(f_m \circ \mathbf{T}_{i,j}/ij \geq B) = 1. \tag{29}$$

Define now

$$U_{0,0} = \sum_{n \geq 2} f_n \quad \text{and} \quad U_{i,j} = \sum_{n \geq 2} f_n \circ \mathbf{T}_{i,j}. \tag{30}$$

Let us estimate the Luxembourg norm of $U_{0,0}$ in the Orlicz space generated by the convex function $g(x) = x \ln^{1-\varepsilon}(1+x)$ for $x > 0$, $0 < \varepsilon < 1$. For each $n \in N$

$$||f_n||_g = \inf_{\lambda} \{\lambda : E(\frac{f_n}{\lambda} \ln^{1-\varepsilon}(1 + \frac{f_n}{\lambda})) \leq 1\} \ .$$

By the definition of f_n, we have

$$\begin{aligned} E(\frac{f_n}{\lambda} \ln^{1-\varepsilon}(1 + \frac{f_n}{\lambda})) &= P(F_n) \frac{n}{\lambda \ln^2 n} \ln^{1-\varepsilon}(1 + \frac{n}{\lambda \ln^2 n}) \\ &= \frac{1}{2\lambda n \ln^2 n} \ln^{1-\varepsilon}(1 + \frac{n}{\lambda \ln^2 n}). \end{aligned}$$

From this identity we see that, after some computations, that for n sufficiently large

$$||f_n||_g \leq \frac{1}{n \ln^{1+\varepsilon/2} n}.$$

Clearly, we have

$$||U_{0,0}||_g \leq \sum_{n \geq 2} ||f_n||_g < \infty. \tag{31}$$

It remains to note that, by definition (30), $U_{i,j} \geq f_n \circ \mathbf{T}_{i,j}$. Therefore, by (29) we also have for almost all $\omega \in \mathbf{\Omega}$ and every positive B,

$$\lim \sup_{i \wedge j \to \infty} \mathbf{P}^{\omega}(U_{i,j}/ij \geq B) = 1$$

and the conclusion of this lemma follows by letting $B \to \infty$. $\square$

End of proof of Theorem 2

On the space constructed in Lemma 4 define the independent random variables $\xi'_{i,j}(\omega_1, \omega_2, ...) = \xi_{i,j}(\omega_1)$ and the random variables $X_{i,j} = \xi'_{i,j} U^{1/2}_{i-1,j-1}$, where $(U_{i,j})_{i,j \in Z}$ and $(\xi_{i,j})_{i,j \in Z}$ are as in Lemma 4. Note that $(X_{i,j})_{i,j \in Z}$ is a sequence of orthomartingale differences with respect to $\prod_{m=1}^{\infty} \mathcal{F}^{(m)}_{i,j}$, where $\mathcal{F}^{(m)}_{i,j}$ are independent copies of $\mathcal{F}_{i,j}$. According to Lemma 4 for $P-$almost all $\omega \in \mathbf{\Omega}$ we have

$$\lim_{B \to \infty} \lim \sup_{i \wedge j \to \infty} P^{\omega}(|X_{i,j}|/\sqrt{ij} \geq B) = 1.$$

If we assume now that $(S_{n,m}/\sqrt{nm})_{n,m \geq 1}$ satisfies the quenched limit theorem (or it is "quenched" tight), because

$$U^{1/2}_{i-1,j-1} = |X_{i,j}| \leq |S_{i,j}| + |S_{i-1,j}| + |S_{i,j-1}| + |S_{i-1,j-1}|,$$

then necessarily the field $(|X_{m,m}|/\sqrt{nm})_{n,m \geq 1}$ should be tight under P^{ω}, for almost all ω, which leads to a contradiction. Note that, by (31), for any $0 < \varepsilon < 1$ we have $EX^2_{0,0} \ln^{1-\varepsilon}(1 + |X_{0,0}|) < \infty$. For this example $EX^2_{0,0} \ln(1 + |X_{0,0}|) = \infty$, since otherwise the quenched result follows by Theorem 1. $\square$

3 Quenched functional CLT

In this section we formulate the functional CLT, which holds under the same conditions as in Theorem 1. For $(s,t) \in [0,1]^2$, we introduce the stochastic process

$$W_{n,v}(t,s) = \frac{1}{\sqrt{nv}} S_{[nt],[vs]}.$$

We shall establish the following result. Denote by $(W(t,s))_{(t,s) \in [0,1]^2}$ the standard 2-dimensional Brownian sheet.

Theorem 5 *Under the setting of Theorem 1, if we assume that $E(X^2_{0,0}) < \infty$ then, for P-almost all ω, the sequence of processes $(W_{n,n}(t,s))_{n \geq 1}$ converges in distribution in $D([0,1]^2)$ endowed with the uniform topology to $\sigma W(t,s)$, under P^{ω}. If we assume now that (6) holds, then for P-almost all ω, the sequence $(W_{n,v}(t,s))_{n,v \geq 1}$ converges in distribution to $\sigma W(t,s)$, as $n \wedge v \to \infty$ under P^{ω}.*

Proof of Theorem 5

Let us first prove the second case, when $n \wedge v \to \infty$. As usual, the proof of this theorem involves two steps, namely the proof of the convergence of the finite dimensional distributions to the corresponding ones of the standard 2-dimensional Brownian sheet and tightness.

For proving tightness we shall verify the moment condition given in relation (3) in Bickel and Wichura (1971) and then the tightness follows from Theorem 3 in the same paper. To verify it is enough to compute the 4−th moment of an increment of the process $W_{n,v}(t,s)$ on the rectangle $A = [t_1, t_2) \times [s_1, s_2)$. That is $E(\Delta^4(A))$ where

$$\Delta(A) = \frac{1}{\sqrt{nv}} \sum\nolimits_{i=[nt_1]}^{[nt_2]-1} \sum\nolimits_{j=[vs_1]}^{[vs_2]-1} X_{i,j}.$$

By applying Burkholder's inequality twice consecutively, and taking into account that the variables are bounded by C, for a positive constant K we obtain

$$E^{\omega}(\Delta^4(A)) \le KC^4(t_2 - t_1)^2(s_2 - s_1)^2 = KC^4\mu^2(A),$$

where μ is the Lebesgue measure on $[0,1]^2$. If B is a neighboring rectangle of A, by the Cauchy-Schwatz inequality we have

$$E^{\omega}(\Delta^2(A)\Delta^2(B)) \le KC^4\mu(A)\mu(B).$$

Therefore the moment condition in relation (3) in Bickel and Wichura (1971) is verified with $\gamma = 4$ and $\beta = 2$.

The proof of the convergence of finite dimensional distribution follows, up to a point, the proof of the corresponding result in Cuny et al. (2015), which will be combined with the method of proof in Theorem 1. As explained in Subsection 3.2 in Cuny et al. (2015), in order to establish the convergence of the finite dimensional distributions, we have to show that for $P-$almost all $\omega \in \Omega$, and for any partitions $0 \le t_1 \le ... \le t_K \le 1$ and $0 \le s_1 \le ... \le s_K \le 1$, we have

$$\frac{1}{\sqrt{nv}} \sum\nolimits_{k=1}^{K} \sum\nolimits_{\ell=1}^{K} a_{k,\ell} \sum\nolimits_{i=[nt_{k-1}]}^{[nt_k]-1} \sum\nolimits_{j=[vs_{\ell-1}]}^{[vs_\ell]-1} X_{i,j} \Rightarrow N(0,\Gamma) \text{ under } P^{\omega}, \tag{32}$$

where $\Gamma = \sigma^2 \sum_{k=1}^{K} \sum_{\ell=1}^{K} a_{k,\ell}^2 (t_k - t_{k-1})(s_\ell - s_{\ell-1})$. Since we have proved tightness in $C([0,1]^2)$, we know that any subsequence contains one which is converges in distribution to a continuous process. Therefore, without restricting the generality we can restrict ourselves to partitions with rational ends which form a countable set.

In order to establish this weak convergence we follow step by step the proof of Theorem 1. We shall just mention the differences. The first step is to decompose $X_{i,j}$ as in formula (8) and to show the negligibility of the term containing $X''_{i,j}$. This is the only step where we need different moment conditions according to whether indexes in the sum are restricted or not. By using simple algebraic manipulations, the triangle inequality along with Theorem 3.2 in Billingsley (1999), we can easily see that this term is negligible P-a.s. for the convergence in $D([0,1]^2)$ endowed with the uniform topology, if, for every $\varepsilon > 0$

$$\lim_{A\to\infty} \ \limsup_{n\wedge v\to\infty} P_{0,0}(\max_{1\leq i\leq n}\max_{1\leq j\leq v}|\sum\nolimits_{k=1}^{i}\sum\nolimits_{\ell=1}^{j}\mathcal{P}_{k,\ell}(X^{"}_{k,\ell})|>\varepsilon\sqrt{nv})=0 \text{ a.s.}$$

But by using Cairoli's maximal inequality for orthomartinagles (see Theorem 2.3.1 in Khoshnevisan, 2002, p. 19) the proof is reduced to showing (11), which was already established in proof of Theorem 1. Without loss of generality we redenote $\mathcal{P}_{i,j}(X'_{i,j})$ by $X_{i,j}$ and assume that it is bounded by a positive constant C. We continue the steps of the proof in Theorem 1 and we shall verify the conditions of Theorem 15 with the exception that we replace $F_{i,v}$ in definition (14) by

$$F_{k,i,v}=\frac{1}{\sqrt{v}}\sum\nolimits_{\ell=1}^{K}a_{k,\ell}\sum\nolimits_{j=[vs_{\ell-1}]}^{[vs_\ell]-1}X_{i,j},$$

where $[nt_{k-1}] \leq i \leq [nt_k]-1$; $1 \leq k \leq K$. We also replace σ^2 by $\eta_k^2 = \sigma^2\sum_{\ell=1}^{K}a_{k,\ell}^2(s_\ell - s_{\ell-1})$ and h_{m,N_0} by

$$h_{k,m,N_0}=\sup_{v>N_0}E_{0,0}|\frac{1}{m}\sum\nolimits_{i=0}^{m-1}F_{k,i,v}^2-\eta_k^2|.$$

For instance, let us convince ourselves that (20) holds. Indeed by the ergodicity of S and the fact that the variables are bounded

$$\lim_{v\to\infty}E_{\infty,0}F_{k,i,v}^2=\lim_{v\to\infty}\frac{1}{v}E_{\infty,0}(\sum\nolimits_{\ell=1}^{K}a_{k,\ell}\sum\nolimits_{j=[vs_{\ell-1}]}^{[vs_\ell]-1}X_{i,j}^2)=\eta_k^2.$$

After we verify the conditions of Theorem 15 for the triangular array of martingale differences $(F_{k,i,v})_{[nt_{k-1}]\leq i\leq[nt_k]-1;\ 1\leq k\leq K}$, we obtain the result in (32) by applying the CLT in Theorem 15. □

4 Quenched functional CLT via coboundary decomposition

Now we indicate a larger class than the orthomartingale, which satisfies a quenched functional CLT. A fruitful approach is to approximate $S_{m,n}$ by an orthomartingale $M_{n,m}$ in a norm that makes possible to transport the quenched functional CLT given in Theorem 5. Such an approximation is of the form: for every $\varepsilon > 0$,

$$\limsup_{n\wedge v\to\infty} P^{\omega}(\max_{1\leq k\leq n,1\leq \ell\leq v}|S_{k,\ell}-M_{k,\ell}|>\varepsilon\sqrt{nv})=0 \text{ a.s.} \tag{33}$$

The random fields we consider can be decomposed into an orthomartingale and a generalized coboundary and therefore satisfy (33). This type of orthomartingale approximation, so called martingale-coboundary decomposition, was introduced for random fields by Gordin (2009) and studied by El Machkouri and Giraudo (2016), Giraudo (2018) and Volný (2018).

Definition 6 *We say that a random field* $(X_{i,j})_{i,j\in Z}$*, defined by (3), adapted to the commuting filtration* $(\mathcal{F}_{i,j})_{i,j\in Z}$*, defined by (1), admits a martingale-coboundary decomposition if*

$$X_{0,0} = m_{0,0} + (1-\hat{T})m'_{0,0} + (1-\hat{S})m''_{0,0} + (1-\hat{T})(1-\hat{S})Y_{0,0}, \qquad (34)$$

with $m_{0,0}$ *an orthomartingale difference (satisfying (4)),* $m'_{0,0}$ *a martingale difference in the second coordinate and* $m''_{0,0}$ *a martingale difference in the first coordinate. All these functions are* $\mathcal{F}_{0,0}-$*measurable.*

We shall obtain the following generalization of Theorem 5:

Theorem 7 *Let us assume that the decomposition (34) holds with all the variables square integrable and* S *(or* T*) is ergodic. Then for almost all* $\omega \in \Omega$,

$$\frac{1}{n}S_{[nt],[ns]} \Rightarrow |c|W(t,s) \text{ under } P^{\omega} \text{ when } n\to\infty, \qquad (35)$$

where $(W(t,s))_{(t,s)\in[0,1]^2}$ *is the standard* 2*-dimensional Brownian sheet and* $c^2 = E(m_{0,0}^2)$*. If we assume that all the variables involved in the decomposition (34) satisfy (6) then, for almost all* $\omega \in \Omega$,

$$\frac{1}{(nv)^{1/2}}S_{[nt],[vs]} \Rightarrow |c|W(t,s) \text{ under } P^{\omega} \text{ when } n\wedge v\to\infty. \qquad (36)$$

It should be noted that Giraudo (2018) have shown that if

$$\sup_{n,v\geq 0} E((E_{0,0}(S_{n,v}))^2) < \infty, \qquad (37)$$

then the decomposition (34) holds and all the variables are in L_2. As a matter of fact this is also a necessary condition for (34). The only condition specific to L_2 needed for his proof is the reflexivity of L_2. Since the Orlicz space L_φ generated by the function

$$\varphi(x) = x^2\log(1+x) : [0,\infty) \to [0,\infty)$$

is reflexive (see Theorem 8 in Milnes (1957)), the proof of Theorem 2.1 in Giraudo is also valid in this context. It follows that if

$$\sup_{n,v\geq 0} E(\varphi(|E_{00}(S_{n,v})|)) < \infty, \qquad (38)$$

then the decomposition in (34) holds all the functions are in L_φ. The reciprocal is also true.

As a matter of fact, by combining Theorem 7 with this result we deduce the following corollary:

Corollary 8 *Let us assume that the random field* $(X_{i,j})_{i,j\in Z}$*, defined by (3), adapted to the commuting filtration* $(\mathcal{F}_{i,j})_{i,j\in Z}$*, defined by (1), satisfies (37). Then* $\lim_{n\wedge v\to\infty}(nv)^{-1}E(S_{n,v}^2) = c^2$*. If in addition we assume that* S *(or* T*) is ergodic, then for* $P-$*almost all* $\omega \in \Omega$*, (35) holds. Also, if condition (38) is satisfied, then for* $P-$*almost all* $\omega \in \Omega$*, (36) holds.*

Proof of Theorem 7

Consider first that the indexes n and m are varying independently. Denote by $m_{i,j} = m_{0,0} \circ T_{i,j}$ and $M_{k,\ell} = \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} m_{i,j}$.

We shall establish (33). A simple computation shows that $(S_{k,\ell} - M_{k,\ell})/\sqrt{nv}$ is the sum of the following three terms:

$$\frac{1}{\sqrt{nv}} \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \hat{T}^i \hat{S}^j (I - \hat{T}) m'_{0,0} = \frac{1}{\sqrt{nv}} \sum_{j=0}^{\ell-1} \hat{S}^j (m'_{0,0} - \hat{T}^k m'_{0,0}) = R_1(k, \ell),$$

$$\frac{1}{\sqrt{nv}} \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \hat{T}^i \hat{S}^j (I - \hat{S}) m''_{0,0} = \frac{1}{\sqrt{nv}} \sum_{i=0}^{k-1} \hat{T}^i (m''_{0,0} - \hat{S}^\ell m''_{0,0}) = R_2(k, \ell),$$

$$\frac{1}{\sqrt{nv}} \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \hat{T}^i \hat{S}^j (I - \hat{T})(I - \hat{S}) Y_{0,0} = \frac{1}{\sqrt{nv}} (I - \hat{S}^\ell)(I - \hat{T}^k) Y_{0,0} = R_3(k, \ell).$$

In order to treat the last term, note that

$$\max_{1 \le k \le n, 1 \le \ell \le v} |R_3(k, \ell)| \le \frac{4}{\sqrt{nv}} \max_{0 \le i \le n} \max_{0 \le j \le v} |Y_{i,j}|.$$

Let A be a positive integer. By truncation at the level A we obtain the following bound

$$\frac{1}{nv} \max_{0 \le i \le n} \max_{0 \le j \le v} |Y_{i,j}|^2 \le \frac{A^2}{nv} + \frac{1}{nv} \sum_{i=0}^{n} \sum_{j=0}^{v} Y_{i,j}^2 I(|Y_{i,j}| > A).$$

Because of the stationarity and the fact that in the second part of Theorem 7 we imposed condition (6), by the ergodic theorem for stationary random fields (see Theorem 1.1 in Ch.6, Krengel (1985)) it follows that for every A,

$$\lim_{n \wedge v \to \infty} \frac{1}{nv} \sum_{i=0}^{n} \sum_{j=0}^{v} Y_{i,j}^2 I(|Y_{i,j}| > A) = E(Y_{0,0}^2 I(|Y_{0,0}| > A)).$$

Therefore $\lim_{A \to \infty} \lim_{n \wedge v \to \infty} |R_3(n, v)| = 0$ $P-$a.s. By Fubini's theorem, it follows that the limit is 0 also under P^ω, for almost all ω.

The terms $R_1(k, \ell)$ and $R_2(k, \ell)$ are treated similarly, with small differences. Let us treat the first one only. It is convenient to truncate at a positive number A. Let

$$m'_{j,k} = m'_{j,k} I(|m'_{j,k}| \le A) - E_{j,k-1} m'_{j,k} I(|m'_{j,k}| \le A) +$$
$$m'_{j,k} I(|m'_{j,k}| > A) - E_{j,k-1} m'_{j,k} I(|m'_{j,k}| > A).$$

We shall use the following bound:

$$E_{0,0}\max_{1\le k\le n,1\le \ell\le v}R_1^2(k,\ell)\le 2E_{0,0}\max_{1\le k\le n,1\le \ell\le v}(\sum_{j=0}^{\ell-1}m'_{j,k})^2\le$$

$$8A^2v+2E_{0,0}\max_{1\le k\le n,1\le \ell\le v}(\sum_{j=0}^{\ell-1}m'_{j,k}I(|m'_{j,k}|>A)-E_{j,k-1}m'_{j,k}I(|m'_{j,k}|>A))^2$$

$$\le 8A^2v+2\sum_{k=1}^{n}E_{0,0}\max_{1\le \ell\le v}(\sum_{j=0}^{\ell-1}m'_{j,k}I(|m'_{j,k}|>A)-E_{j,k-1}m'_{j,k}I(|m'_{j,k}|>A))^2.$$

Now, by the Doob's maximal inequality

$$\frac{1}{nv}E_{0,0}\max_{1\le k\le n,1\le \ell\le v}R_1^2(k,\ell)$$

$$\le \frac{8A^2}{n}+\frac{2}{nv}\sum_{k=1}^{n}\sum_{j=0}^{v-1}E_{0,0}(m'_{j,k}I(|m'_{j,k}|>A)-E_{j,k-1}m'_{j,k}I(|m'_{j,k}|>A))^2$$

$$\le \frac{8A^2}{n}+\frac{4}{nv}\sum_{k=1}^{n}\sum_{j=0}^{v-1}E_{0,0}(m'_{j,k}I(|m'_{j,k}|>A))^2$$

$$=\frac{8A^2}{n}+\frac{4}{nv}\sum_{k=1}^{n}\sum_{j=0}^{v-1}Q_1^jQ_2^k[(m'_{0,0})^2I(|m'_{0,0}|>A)].$$

We let $n\wedge v\to\infty$ and we use Theorem 1.1 in Ch. 6 of Krengel (1985). It follows that, for every A

$$\lim_{n\wedge v\to\infty}\frac{1}{nv}E_{0,0}\max_{1\le k\le n,1\le \ell\le v}R_1^2(k,\ell)=E(m'_{0,0})^2I(|m'_{0,0}|>A).$$

Then, we let $A\to\infty$. This completes the proof of (33). The result follows by using the second part of Theorem 5 along with Theorem 3.2 in Billingsley (1999). Now for the situation $n=m\to\infty$, the proof is similar with the difference that we use Theorem 3.5 in Ch. 6 in Krengel (1985) instead of Theorem 1.1 in the same chapter together with the first part of Theorem 5. □

Remark 9 *If we take $Y_{0,0}$, in the martingale-coboundary decomposition (34), to be the function $U_{0,0}^{1/2}$ found in the proof of Lemma 4, then for almost all ω,*

$$R_3(n,v)=\frac{1}{\sqrt{nv}}\sum_{i=0}^{n-1}\sum_{j=0}^{v-1}\hat{T}^i\hat{S}^j(I-\hat{T})(I-\hat{S})Y_{0,0}$$

does not converge to 0 in probability P^ω when $n\wedge v\to\infty$. Therefore if only the existence of the second moment is assumed or even if $EY_{0,0}^2\ln^{1-\varepsilon}(1+|Y_{0,0}|)<\infty$ for some $0<\varepsilon<1$, this coboundary could spoil the quenched weak convergence.

This is in sharp contrast with the dimension 1. Recall that in dimension 1, when we have a martingale-coboundary decomposition $X_0 = D_0 + G_0 - \hat{T}G_0$ with D_0 a martingale difference and $G_0 \in L_2$, then the coboundary $G_0 - \hat{T}G_0$ does not spoil the quenched invariance principle (see Theorem 8.1 in Borodin and Ibragimov (1994), which is due to Gordin and Lifshits). In higher dimension, in general, we need stronger moment conditions not only for martingale differences but also for the cobounding function $Y_{0,0}$.

5 The case of d-indexed random field

In this section we formulate our results and indicate their proofs for random fields indexed by Z^d with $d > 2$. The proofs are based on induction arguments. When we add on unrestricted d-dimensional rectangles the moment conditions will depend on d. By $\mathbf{u} = (u_1, u_2, ..., u_d)$ we denote elements of Z^d. Let us suppose that $\mathbf{T} = (T_i)_{1 \le i \le d}$ are d commuting, invertible, measure preserving transformations from Ω to Ω and let $\mathcal{F}_{\mathbf{0}}$ be a sub-sigma field of $\mathcal{K}$. For all $\mathbf{u} \in Z^d$ define $\mathcal{F}_{\mathbf{u}} = \mathbf{T}^{-\mathbf{u}}(\mathcal{F}_{\mathbf{0}})$, where $\mathbf{T}^{-\mathbf{u}}$ is the following composition of operators: $\mathbf{T}^{-\mathbf{u}} = \prod_{i=1}^{n} T_i^{-u_i}$. Assume the filtration is coordinatewise increasing and commuting, in the sense that for any integrable variable we have $E_{\mathbf{u}} E_{\mathbf{a}} X = E_{\mathbf{a} \wedge \mathbf{u}} X$, where $\mathbf{a} \wedge \mathbf{u}$ means coordinatewice minimum and we used the notation $E_{\mathbf{u}} X = E(X|\mathcal{F}_{\mathbf{u}})$. We introduce the stationary field by starting with a $\mathcal{F}_{\mathbf{0}}-$measurable function $X_{\mathbf{0}} : \Omega \to R$ and then define the random field $X_{\mathbf{k}}(\omega) = X_{\mathbf{0}}(\mathbf{T}^{\mathbf{k}}(\omega)) = X_{\mathbf{0}}(T_1^{k_1} \circ ... \circ T_d^{k_d})$. The operator $\hat{\mathbf{T}}$ is defined on L_2 as $\hat{\mathbf{T}}(f) = f \circ \mathbf{T}$. For the filtration $(\mathcal{F}_{\mathbf{u}})_{\mathbf{u} \in Z^d}$, defined as above, we call the random field $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbf{Z}^d}$ orthomartingale difference if $E(X_{\mathbf{u}}|\mathcal{F}_{\mathbf{i}}) = 0$ when at least one coordinate of $\mathbf{i}$ is strictly smaller that the corresponding coordinate of $\mathbf{u}$. We also use the notation $\mathbf{i} \le \mathbf{u}$, where the inequality is coordinatewise and $|\mathbf{n}| = n_1 \cdot ... \cdot n_d$. Finally denote $S_{\mathbf{n}} = \sum_{\mathbf{0} \le \mathbf{i} \le \mathbf{n}-\mathbf{1}} X_{\mathbf{i}}$. In this context we have:

Theorem 10 *Assume that there is an integer i, $1 \le i \le d$ such that T_i is ergodic and $X_{\mathbf{0}}$ is square integrable, $E(X_{\mathbf{0}}^2) = \sigma^2$. Then, for $P-$almost all $\omega \in \Omega$,*

$$\frac{1}{n^{d/2}} S_{(n,n,...,n)} \Rightarrow \sigma W(t_1, ..., t_d) \text{ under } P^{\omega} \text{ when } n \to \infty \ .$$

In addition, if $E[X_{\mathbf{0}}^2 \log^{d-1}(1 + |X_{\mathbf{0}}|)] < \infty$, then for almost all $\omega \in \Omega$,

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{(n_1,n_2,...,n_d)} \Rightarrow \sigma W(t_1, ..., t_d) \text{ under } P^{\omega} \text{ when } \min_{1 \le i \le d} n_i \to \infty.$$

Remark 11 *Both Theorems 5 and 7 as well as Corollary 8 also hold for the multi-indexed random field $(X_{\mathbf{u}})_{\mathbf{u} \in \mathbf{Z}^d}$ defined above.*

We shall indicate how to prove these results by induction. We shall follow step by step the proof of Theorem 1 with the following differences. Without restricting the generality, let us assume that the operator T_i is ergodic for an i, $2 \leq i \leq d$. We define now the d-dimensional projection operators. By using the commutative property of the filtrations it is convenient to define:

$$\mathcal{P}_{\mathbf{u}}(X) = \mathcal{P}_{\mathbf{u}_1} \circ \mathcal{P}_{\mathbf{u}_2} \circ ... \circ \mathcal{P}_{\mathbf{u}_d}(X),$$

where

$$\mathcal{P}_{\mathbf{u}_j}(Y) = E(Y|\mathcal{F}_{\mathbf{u}}) - E(Y|\mathcal{F}_{\mathbf{u}_j}).$$

Above we used the notation $\mathbf{u}_j$ for a vector which has the same coordinates as $\mathbf{u}$ with the exception of the j-th coordinate, which is $u_j - 1$. For instance when $d = 3$, $\mathcal{P}_{\mathbf{u}_2}(Y) = E(Y|\mathcal{F}_{u_1,u_2,u_3}) - E(Y|\mathcal{F}_{u_1,u_2-1,u_3})$. We can easily see that, by using the commutativity property of the filtration, this definition is a generalization of the case $d = 2$. We note that, by using this definition of $\mathcal{P}_{\mathbf{u}}(X)$, the truncation argument in Theorem 1 remains unchanged if we replace the index set Z^2 with Z^d. We point out the following two differences in the proof of Theorem 10. One difference is that, for the validity of the limit in (13) when $\min_{1 \leq i \leq d} n_i \to \infty$, in order to apply the ergodic theorem for Dunford-Schwartz operators, conform to Ch. 6 Theorem 2.8 and Theorem 1.1 in Krengel (1985), we have to assume that $E[X_{0,0}^2 \log^{d-1}(1 + |X_{0,0}|)] < \infty$. After we reduce the problem to the case of bounded random variables, we proceed with the proof of the CLT by induction. More precisely, we write the sum in the form

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{(n_1,n_2,...,n_d)} = \frac{1}{n_1^{1/2}} \sum_{k_1=0}^{n_1-1} F_{k_1,(n_2,n_3,...,n_d)} \ ,$$

with

$$F_{k_1,(n_2,n_3,...,n_d)} = \frac{1}{(n_2 \cdot ... \cdot n_d)^{1/2}} \sum_{\mathbf{k} \in B} X_{\mathbf{k}} \ ,$$

where the sum is taken on the set $B = \{(0, ..., 0) \leq (k_2...k_d) \leq (n_2 - 1, ..., n_d - 1)\}$. Because one operator is ergodic, according to the induction hypothesis, $F_{k,(n_2,n_3,...,n_d)} \Rightarrow N(0, \sigma^2)$ under P^ω for almost all ω, and we can replace (20) by

$$\lim \frac{1}{n_2...n_d} E_{\infty,0,...0} \sum_B X_{\mathbf{k}}^2 = \sigma^2 \text{ a.s. when } \min(n_2, ..., n_d) \to \infty.$$

6 Examples

We shall give examples providing new results for linear and Volterra random fields with i.i.d. innovations. Let d be an integer $d > 1$. Denote by $\mathbf{t} = (t_1, t_2, ..., t_d)$ and let $W(\mathbf{t})$ be the standard d-dimensional Brownian sheet.

Example 12 *Let* $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ *be a random field of independent, identically distributed random variables, which are centered and have finite second moment. Let* $(a_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ *be a sequence of real numbers such that* $\sum_{\mathbf{j}\geq\mathbf{0}} a_{\mathbf{j}}^2 < \infty$. *Define*

$$X_{\mathbf{k}} = \sum_{\mathbf{j}\geq\mathbf{0}} a_{\mathbf{j}}\xi_{\mathbf{k}-\mathbf{j}}.$$

Assume that

$$\sup_{\mathbf{n}\geq\mathbf{1}} \sum_{\mathbf{i}\geq\mathbf{0}} b_{\mathbf{n},\mathbf{i}}^2 < \infty, \text{ where } b_{\mathbf{n},\mathbf{i}} = \sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}-\mathbf{1}} a_{\mathbf{k}+\mathbf{i}}. \tag{39}$$

Then, if $\mathbf{n} = (n, n, ..., n)$, *for* $P-$*almost all* ω

$$\frac{1}{n^{d/2}} S_{[(\mathbf{n}-\mathbf{1})\cdot\mathbf{t}]} \Rightarrow |c|W(\mathbf{t}) \text{ under } P^{\omega} \text{ when } n \to \infty. \tag{40}$$

If we assume now that $E(\xi_{\mathbf{0}}^2 \log^{d-1}(1 + |\xi_{\mathbf{0}}|)) < \infty$, *then for* $P-$*almost all* ω

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{[(\mathbf{n}-\mathbf{1})\cdot\mathbf{t}]} \Rightarrow |c|W(\mathbf{t}) \text{ under } P^{\omega} \text{ when } \min(n_1, ..., n_d) \to \infty, \tag{41}$$

where $\mathbf{n} = (n_1, ..., n_d)$.

Proof of Example 12.

For this case we take $\mathcal{F}_{\mathbf{n}} = \sigma(\xi_{\mathbf{u}}, \mathbf{u} \leq \mathbf{n})$. Let us note first that the variables are square integrable and well defined. We also have

$$E(S_{\mathbf{n}}|\mathcal{F}_{\mathbf{0}}) = \sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}-\mathbf{1}} \sum_{\mathbf{j}\leq\mathbf{0}} a_{\mathbf{k}-\mathbf{j}}\xi_{\mathbf{j}}$$

and therefore

$$E(E^2(S_{\mathbf{n}}|\mathcal{F}_{\mathbf{0}})) = \sum_{\mathbf{i}\geq\mathbf{0}} (\sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}-\mathbf{1}} a_{\mathbf{k}+\mathbf{i}})^2 E(\xi_{\mathbf{1}}^2).$$

The result follows for $S_{\mathbf{n}}$ by applying the first part of Corollary 8.

On the other hand, by the Rosenthal inequality for independent random variables (see relation 21.5 in Burkholder (1973)), applied with the function $\varphi(x) = x^2 \log^{d-1}(1 + |x|)$, there is a positive constant C such that

$$E(\varphi(|E(S_{\mathbf{n}}|\mathcal{F}_{\mathbf{0}})|)) \leq C\varphi\left(\left(\sum_{\mathbf{i}\geq\mathbf{0}} b_{\mathbf{n},\mathbf{i}}^2 E(\xi_{\mathbf{1}}^2)\right)^{1/2}\right) + C\sum_{\mathbf{i}\geq\mathbf{0}} E(\varphi(|b_{\mathbf{n},\mathbf{i}}\xi_{\mathbf{0}}|)),$$

which is bounded under condition (39). Indeed, condition (39) implies that $\sup_{\mathbf{n}\geq\mathbf{1}} \sup_{\mathbf{i}\geq\mathbf{0}} |b_{\mathbf{n},\mathbf{i}}| < \infty$, and then, after simple algebraic manipulations we can find a positive constant K such that

$$E(\varphi(|b_{\mathbf{n},\mathbf{i}}\xi_{\mathbf{0}}|)) \leq K b_{\mathbf{n},\mathbf{i}}^2 (E(\varphi(|\xi_{\mathbf{0}}|)) + E(\xi_{\mathbf{0}}^2)).$$

It remains to apply the second part from Corollary 8 and Remark 11 in order to obtain the second part of the example. □

Another class of nonlinear random fields are the Volterra processes, which play an important role in the nonlinear system theory.

Example 13 *Let $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ be a random field of independent random variables, identically distributed, centered and with finite second moment. Define*

$$X_{\mathbf{k}} = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} a_{\mathbf{u},\mathbf{v}}\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}},$$

where $a_{\mathbf{u},\mathbf{v}}$ are real coefficients with $a_{\mathbf{u},\mathbf{u}} = 0$ and $\sum_{\mathbf{u},\mathbf{v}\geq\mathbf{0}} a^2_{\mathbf{u},\mathbf{v}} < \infty$. Denote

$$c_{\mathbf{u},\mathbf{v}}(\mathbf{j}) = \sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{j}-\mathbf{1}} a_{\mathbf{k}+\mathbf{u},\mathbf{k}+\mathbf{v}}.$$

Assume that

$$\sup_{\mathbf{j}\geq\mathbf{1}} \sum_{\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0},\mathbf{u}\neq\mathbf{v}} c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j}) < \infty. \tag{42}$$

Then the quenched functional CLT in (40) holds. If in addition we assume that $E(\xi^2_{\mathbf{0}}\log^{d-1}(1+|\xi_{\mathbf{0}}|)) < \infty$, then the quenched functional CLT in (41) holds for sums of variables in a general d-dimensional rectangle.

Proof of Example 13.

For this case we consider the sigma algebras as in Example 12. We start from the following estimate

$$E(S_{\mathbf{j}}|\mathcal{F}_{\mathbf{0}}) = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})}\sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{j}-\mathbf{1}} a_{\mathbf{k}+\mathbf{u},\mathbf{k}+\mathbf{v}}\xi_{-\mathbf{u}}\xi_{-\mathbf{v}} = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} c_{\mathbf{u},\mathbf{v}}(\mathbf{j})\xi_{-\mathbf{u}}\xi_{-\mathbf{v}}.$$

Since by our conditions $c_{\mathbf{u},\mathbf{u}} = 0$, by Tonelli theorem we obtain

$$\begin{aligned} E(E^2(S_{\mathbf{j}}|\mathcal{F}_{\mathbf{0}})) &= \sum_{\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0},\mathbf{u}\neq\mathbf{v}} (c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j}) + c_{\mathbf{u},\mathbf{v}}(\mathbf{j})c_{\mathbf{v},\mathbf{u}}(\mathbf{j}))E(\xi_{\mathbf{u}}\xi_{\mathbf{v}})^2 \\ &\leq 2\sum_{\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0},\mathbf{u}\neq\mathbf{v}} (c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j}) + c^2_{\mathbf{v},\mathbf{u}}(\mathbf{j}))E(\xi_{\mathbf{u}}\xi_{\mathbf{v}})^2 \\ &\leq 4\sum_{\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0},\mathbf{u}\neq\mathbf{v}} c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j})(E(\xi^2_{\mathbf{0}}))^2. \end{aligned}$$

The first result of this theorem follows by applying the first part of Corollary 8 via Remark 11.

On the other hand, by a moment inequality for U-statistics based on the decoupling procedures, (see Relation 3.1.3. in Giné et al., 2000), we obtain for $\varphi(x) = x^2\log^{d-1}(1+|x|)$,

$$E(\varphi(|E(S_{\mathbf{j}}|\mathcal{F}_{\mathbf{0}})|)) \leq CE\varphi\left(\sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} c_{\mathbf{u},\mathbf{v}}(\mathbf{j})\xi_{-\mathbf{u}}\xi'_{-\mathbf{v}}\right),$$

where $(\xi'_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ in an independent copy of $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ and C is a positive constant. Now, we apply Rosenthal inequality, given in relation 21.5 in Burkholder (1973), and we find a constant $C' > 0$ such that

$$E(\varphi(|E(S_{\mathbf{j}}|\mathcal{F}_{\mathbf{0}})|)) \leq C'\varphi\left(\left(\sum_{\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0},\mathbf{u}\neq\mathbf{v}} c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j})E^2(\xi^2_{\mathbf{1}})\right)^{1/2}\right) + C'\sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} E\varphi\left(|c_{\mathbf{u},\mathbf{v}}(\mathbf{j})\xi_{-\mathbf{u}}\xi'_{-\mathbf{v}}|\right).$$

Note that, by (42), we have $\sup_{\mathbf{u},\mathbf{v}\geq\mathbf{0},\mathbf{j}\geq\mathbf{1}} |c_{\mathbf{u},\mathbf{v}}(\mathbf{j})| < \infty$. Also, because $\xi_{-\mathbf{u}}$ and $\xi'_{-\mathbf{v}}$ are independent and identically distributed, by the properties of φ, we can find positive constants such that

$$E\varphi\left(|c_{\mathbf{u},\mathbf{v}}(\mathbf{j})\xi_{-\mathbf{u}}\xi'_{-\mathbf{v}}|\right) \leq Kc^2_{\mathbf{u},\mathbf{v}}(\mathbf{j})[E(\varphi(\xi_{\mathbf{0}}))E(\xi^2_{\mathbf{0}}) + (E(\xi^2_{\mathbf{0}}))^2] \leq K'c^2_{\mathbf{u},\mathbf{v}}(\mathbf{j}).$$

It remains to note that condition (42) implies condition (38) and then to apply the second part of Corollary 8 and Remark 11.

Remark 14 *In Examples 12 and 13 the innovations are i.i.d. fields. However, the property (2) for the filtration is not restricted to filtrations generated by independent random variables. For example, we can take as innovations the random field $(\xi_{n,m})_{n.m\in Z}$ having as columns independent copies of a stationary and ergodic martingale differences sequence. In this case the filtration generated $(\xi_{n,m})_{n,m\in Z}$ is also commuting. As a matter of fact a commuting filtration could be generated by a stationary random field $(\xi_{n,m})_{n,m\in Z}$ where the columns are independent, i.e. $\bar{\eta}_m = (\xi_{n,m})_{n\in Z}$ are independent.*

7 Auxiliary results

The following is a variant of Theorem 1 in Gänssler and Häusler (1979) (see also Gänssler and Häusler, 1986, pages 315–317).

Theorem 15 *Assume that $(D_{n,k})_{1\leq k\leq n}$ is a triangular array of martingale differences adapted to an increasing filtration $(\mathcal{F}_{n,k})_k$. Assume that for all q rational numbers in $[0,1]$,*

$$\sum_{k=1}^{[nq]} D^2_{n,k} \to^P \sigma^2 q \qquad (43)$$

and $\max_{1\leq k\leq n}|D_{n,k}|$ is uniformly integrable. Then $S_{[nt]} \Rightarrow \sigma W(t)$, where $S_{[nt]} = \sum_{k=1}^{[nt]} D_{n,k}$ and $W(t)$ is a standard Brownian measure. In particular $S_n \Rightarrow N(0,\sigma^2)$.

As a matter of fact, condition (43) in Theorem 1 in Gänssler and Häusler (1979) is formulated for all reals $t \in [0,1]$. We notice however that if (43) holds for any q rational number in $[0,1]$ then it also holds for any $t \in [0,1]$. To see it fix t, $t \in [0,1]$ and let q_1 and q_2 be two rational numbers such that $q_1 \leq t \leq q_2$. Then, by using monotonicity, note that

$$\sum\nolimits_{k=1}^{[nq_1]} D_{n,k}^2 - \sigma^2 q_2 \leq \sum\nolimits_{k=1}^{[nt]} D_{n,k}^2 - \sigma^2 t \leq \sum\nolimits_{k=0}^{[nq_2]} D_{n,k}^2 - \sigma^2 q_1$$

and therefore

$$|\sum\nolimits_{k=1}^{[nt]} D_{n,k}^2 - \sigma^2 t| \leq \max_{i=1,2} |\sum\nolimits_{k=0}^{[nq_i]} D_{n,k}^2 - \sigma^2 q_i| + (q_2 - q_1)\sigma^2.$$

By using the hypothesis (43), and the fact that the rational numbers are dense in R it follows that (43) holds for any $t \in [0,1]$.

We mention now Theorem 34.2 (v) in Billingsley (1995). Further reaching results including comments of the sharpness of the result below can be found in Argiris and Rosenblatt (2006).

Theorem 16 *Assume that the sequence of random variables $(X_n)_{n\geq 0}$ converges a.s. to X and there is an integrable and positive random variable Y such that $|X_n| \leq Y$ a.s. for all $n \geq 0$. Let $\mathcal{F}$ be a sigma algebra. Then the sequence $(E(X_n|\mathcal{F}))_{n\geq 0}$ converges a.s. to $E(X|\mathcal{F})$.*

The following is a result in Katznelson and Weiss (1972) known under the name of Rokhlin lemma for amenable actions.

Theorem 17 *Let $(\Omega, \mathcal{K}, P)$ be a nonatomic probability space and let T be a measure preserving action of Z^2 into Ω*

$$T : Z^2 \times \Omega \to \Omega$$

that is ergodic. Then, for all $\varepsilon > 0$ and $n \in N$, there is a set $B = B(n,\varepsilon) \in \mathcal{K}$ such that for $0 \leq i,j \leq n-1$, $T_{i,j}^{-1}B$ are disjoint for distinct indexes (i,j) and

$$P(B) \geq \frac{1}{n^2}(1-\varepsilon).$$

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