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# Topological crackle of heavy-tailed moving average processes☆

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**Abstract**

The main focus of this paper is topological crackle, the layered structure of annuli formed by heavy-tailed random points in $\mathbb{R}^d$. In view of extreme value theory, we study the topological crackle generated by a heavy-tailed discrete-time moving average process. Because of the clustering effect of a moving average process, various topological cycles are produced consecutively in time in the layers of the crackle. We establish the limit theorems for the Betti numbers, a basic quantifier of topological cycles. The Betti number converges to the sum of stochastic integrals, some of which induce multiple cycles because of the clustering effect.



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*Keywords:* Extreme value theory; Random topology; Topological crackle; Moving average process; Betti number

## 1. Introduction

### *1.1. Topological crackle*

The main theme of this paper is topological crackle — the annuli structure of a continued presence of topological cycles in a manifold. Topological crackle is a phenomenon that has been recognized, at least empirically, in the field of manifold learning. Among many relevant studies, those presented in [30] and [31] showed that if a sample is taken from a nice manifold $\mathcal{M}$ with a small, e.g., Gaussian, error, then one can recover the topology of $\mathcal{M}$ with high probability. However, if the error becomes large, i.e., has a heavy-tailed distribution, then one is no longer

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able to recover the underlying topology. The layers of extraneous homology elements, caused by heavy-tailed errors, that damage the estimation of an original manifold is called the *crackle phenomenon*, in an analogy to audio crackle in temporal signal analysis; see [2].

Although not necessarily related to the manifold learning problem, this paper addresses the crackle phenomenon in view of extreme value theory (EVT), in light of the fact that it is in general caused by data with a heavy tail. Typically, we suppose that independently and identically distributed (i.i.d) random points on $\mathbb{R}^d$, $d \geq 2$ are taken from a (spherically symmetric) distribution with a heavy tail. Let $\text{Ann}(K, L)$ denote a closed annulus with inner radius $K$ and outer radius $L$. We then divide $\mathbb{R}^d$ into the layers of annuli at different radii, all of which grow to infinity as the sample size $n$ increases:

$$\mathbb{R}^d = \bigcup_{i=1}^{d} \text{Ann}(R_{i,n}, R_{i-1,n}),$$

where

$$0 = R_{d,n} < R_{d-1,n} \ll R_{d-2,n} \ll \cdots \ll R_{1,n} < R_{0,n} = \infty, \quad n \to \infty.$$

This layered structure provides basic modeling of the topological crackle, with each annulus containing an extreme sample cloud generated by a heavy-tailed distribution.

The study of the topological features of the layered structure above belongs to EVT. EVT addresses, as its name implies, the extremal behavior of stochastic processes at the intersection of probability theory and statistics; an excellent treatment of the field is provided in [35] and a more recent exposition in [21], with other key publications over the years including [13,18,19,27], and [36]. Over the last decade or so, many articles have provided geometric descriptions of multivariate extremes, among them [3,4], and [5]. In particular, Poisson limits of point processes with a U-statistic structure were discussed in [14,37], and [17], the last two of which also include a number of stochastic geometry examples.

As for topological crackle, its study from the viewpoints of EVT has just begun, and hence, there exist only a limited number of relevant publications, e.g., [2,33], and [32]. More importantly, these papers intrinsically assume i.i.d (heavy-tailed) observations or the similar variant, such as a Poisson point process with spatial independence. The main contribution of this paper is to remove the independence assumption and investigate the crackle phenomenon of a discrete-time moving average process of finite order $q$,

$$Y_n = \sum_{j=0}^{q} A_j X_{n-j}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $A_j$ represents $d \times d$ non-random matrices and $(X_j,\ j \in \mathbb{Z})$ denotes a noise sequence of i.i.d $\mathbb{R}^d$-valued heavy-tailed random variables. Of course, many other stochastic processes can give non-trivial dependency to heavy-tailed observations; see, e.g., [6,15]. However, a moving average process is the most basic in time series analysis. For example, at least in a classical setup, every autoregressive moving average (ARMA) process and many of the stationary Gaussian sequences have moving average representation [12]. Moreover, the tail behavior of heavy-tailed moving average processes has been studied in depth by many authors; see, e.g., [16,26,28], and [23], to mention just a few. Therefore, from a more practical viewpoint, some of the techniques developed previously are also applicable to our analyses. The main characteristic of a heavy-tailed moving average process is "clustering of extremes". Namely, the process forms a significant amount of clusters at a distance from the origin, so that "cluster Poisson limits" may arise; see [16,35]. In this paper, the clustering effect of extremes plays a crucial role in characterizing the crackle phenomenon.

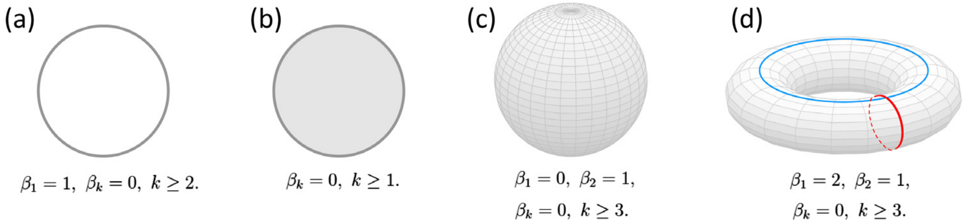


**Fig. 1.** (a) One-dimensional sphere. (b) One-dimensional disk. (c) Two-dimensional sphere. (d) Two-dimensional torus.

### *1.2. Basic notions in algebraic topology*

We now collect the basic notions in algebraic topology necessary for this paper. Although not necessarily dealing with extremes, there has been increasing interest in random topology beyond classical issues of connectivity; see [24,25,40,41], and [11], and recent progress is nicely summarized in [10]. Among these articles, [40] is somewhat relevant to our study; the authors discussed the impact of clustering on random topology for stationary point processes. It seems that in most of the studies on random topology, *Betti numbers* were chosen as a good quantifier of topological complexity. Following this convention, we also employ Betti numbers to measure the topological complexity of various layers in the crackle. Given a topological space $X$ and an integer $k \geq 1$, the $k$th homology group $H_k(X)$ is the quotient group $\ker \delta_k / \mathrm{im} \delta_{k+1}$, where $\delta_k, \delta_{k+1}$ are boundary maps for $X$. Equivalently, $H_k(X)$ represents a topological invariant generated by elements representing (non-trivial) $k$-dimensional "cycles" as the boundary of a $(k+1)$-dimensional body. (Hereinafter, we write "$k$-cycle" for short.) The $k$th Betti number, denoted by $\beta_k(X)$, is the rank of $H_k(X)$, representing the number of $k$-cycles in $X$. More intuitively, $\beta_1(X)$ counts the number of "closed loops" in $X$, whereas $\beta_2(X)$ counts the number of "voids" within $X$. More formal coverage of Betti numbers in view of homology theory can be found in, e.g., [22,39], and [29]. An excellent review [1] contains a gentle introduction of the topological concepts used in the current paper.

Since it is impossible to define Betti numbers formally in a few paragraphs, we would like instead to discuss a few actual examples of objects in the Euclidean space; see Fig. 1. First, a one-dimensional sphere, i.e., a circle, shown in Fig. 1(a) has $\beta_1 = 1$ and $\beta_k = 0$ for all $k \geq 2$. A two-dimensional sphere as shown in Fig. 1(c) has $\beta_1 = 0$, because even if we wind a closed loop around the sphere, the loop ultimately vanishes as it moves upward (or downward) along the sphere until the pole. In addition, it has $\beta_2 = 1$ because of the "void" consisting of the interior of the sphere. In general, a $d$-dimensional sphere has $\beta_d = 1$, and $\beta_k = 0$ for all $k \geq 1$ with $k \neq d$. Finally, a two-dimensional torus, as shown in Fig. 1(d), has $\beta_1 = 2$ (i.e., there are two independent closed loops), and $\beta_2 = 1$ (i.e., the inside of the torus is void).

In order to extract topological information from extreme sample clouds, we consider the Betti numbers of some *geometric complexes*. These complexes will be constructed from a set of random points generated by the process (1). Among many candidates of geometric complexes (see, e.g., [20]), we chose the two most studied ones known as the *Vietoris–Rips complex* and the *Čech complex*.

**Definition 1.1.** Let $\mathcal{X}$ be a collection of points in $\mathbb{R}^d$ and $t$ be a positive number. The Vietoris–Rips complex $\mathcal{R}_t(\mathcal{X})$ is defined as follows.

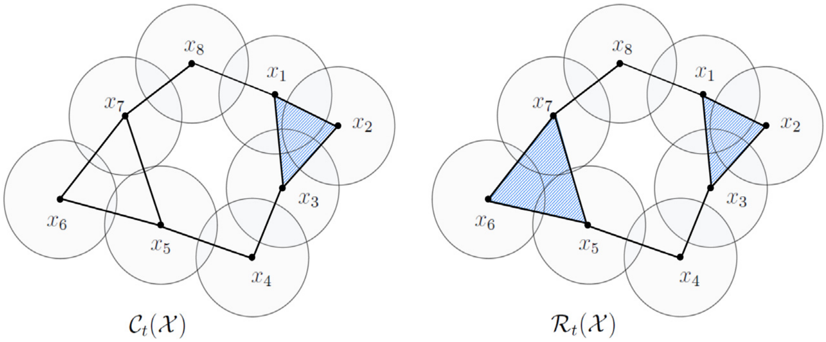


**Fig. 2.** Čech complex $\mathcal{C}_t(\mathcal{X})$ and Vietoris–Rips complex $\mathcal{R}_t(\mathcal{X})$ with $\mathcal{X} = \{x_1, \dots, x_8\} \subset \mathbb{R}^2$. The first order Betti numbers are $\beta_1\big(\mathcal{C}_t(\mathcal{X})\big) = 2$ and $\beta_1\big(\mathcal{R}_t(\mathcal{X})\big) = 1$.

1. The 0-simplices are the points in $\mathcal{X}$.
2. A $p$-simplex $\sigma = [x_{i_0}, \dots, x_{i_p}]$ belongs to $\mathcal{R}_t(\mathcal{X})$ if $B(x_{i_k}, t/2) \cap B(x_{i_m}, t/2) \neq \emptyset$ for every $0 \le k < m \le p$,

where $B(x, r)$ is a closed ball of radius $r$ centered at $x$.

**Definition 1.2.** The Čech complex $\mathcal{C}_t(\mathcal{X})$ is defined as follows.

1. The 0-simplices are the points in $\mathcal{X}$.
2. A $p$-simplex $\sigma = [x_{i_0}, \dots, x_{i_p}]$ is in $\mathcal{C}_t(\mathcal{X})$ if a family of closed balls $\big\{B(x_{i_j}, t/2),\ j = 0, \dots, p\big\}$ has a non-empty intersection.

The difference between these complexes is that, in order for a $p$-simplex to be its element, the Čech complex requires a common intersection of all $p + 1$ balls, whereas the Vietoris–Rips complex merely requires pairwise intersections between the balls. The main importance of the Čech complex is its equivalence to the union of balls; the Nerve theorem, e.g., Theorem 10.7 in [9], claims that $\mathcal{C}_t(\mathcal{X})$ is homotopy equivalent to the union of balls of radius $t/2$ around $\mathcal{X}$. Although the Vietoris–Rips complex does not have such a nice link to the union of balls, it is much more efficient in computational applications. Since $\mathcal{R}_{t'}(\mathcal{X}) \subset \mathcal{C}_t(\mathcal{X}) \subset \mathcal{R}_t(\mathcal{X})$ for $t/t' > \sqrt{2d/(d+1)}$ (see [38]), the Vietoris–Rips complex can be used to approximate the Čech complex.

To obtain a clearer picture of the two geometric complexes, a simple example is presented in Fig. 2. Take $\mathcal{X} = \{x_1, \dots, x_8\} \subset \mathbb{R}^2$. Note that the 2-simplex $[x_1, x_2, x_3]$ is in $\mathcal{C}_t(\mathcal{X})$, since the three balls with radius $t$ centered at $x_1, x_2, x_3$ have a common intersection. Then, the Betti numbers are $\beta_1\big(\mathcal{C}_t(\mathcal{X})\big) = 2$ (two closed loops) and $\beta_k = 0$ for $k \ge 2$. For the Vietoris–Rips complex $\mathcal{R}_t(\mathcal{X})$, there exists another 2-simplex $[x_5, x_6, x_7]$, because the balls around $x_5, x_6, x_7$ have pairwise intersections. As a result, the first order Betti number is $\beta_1\big(\mathcal{R}_t(\mathcal{X})\big) = 1$, since one of the closed loops in $\mathcal{C}_t(\mathcal{X})$ has been filled in.

### *1.3. Poissonian regime*

Now that all the necessary notions in algebraic topology have been covered, we return to topological crackle and add a few comments. In the spirit of EVT, throughout the paper, we focus on the "Poissonian" regime, under which the appearance of topological $k$-cycles becomes a rare event. In this case, if the geometric complex is generated by a set of i.i.d heavy-tailed points in $\mathbb{R}^d$, the $k$th Betti number is known to converge to some non-trivial limit without any centering and scaling. See Theorem 2.2 in [32] and Theorem 5.1 in [33]. In particular, the former represents the limit process as the difference of two dependent Poisson processes, with one representing the birth and the other representing the death of $k$-cycles. The objective of this study is to establish the limit theorems for the Betti numbers, under the same Poissonian regime, when the geometric complex is generated by a heavy-tailed moving average process. For the description of the limit process, the notion of a *Poisson random measure* is important. Let $(E, \mathcal{E}, \mu)$ be a $\sigma$-finite measure space. The Poisson random measure $N$ on $E$, with mean measure $\mu$, is defined by the finite-dimensional distributions

$$\mathbb{P}\big\{N(A) = m\big\} = \frac{e^{-\mu(A)}\big(\mu(A)\big)^m}{m!}, \quad m = 0, 1, 2, \dots$$

for all measurable sets $A \subset E$ with $\mu(A) < \infty$. Furthermore, if $A_1, \dots, A_m$ are disjoint, $N(A_1), \dots, N(A_m)$ are independent.

The organization of the paper is as follows. In Section 2, we establish the required limit theorems for the Betti numbers associated to a Vietoris–Rips complex. In Section 3, the same is done for the Čech complex. We discuss some future research topics in Section 4. All the necessary arguments for the proof are collected in Section 5.

Finally let us add a few comments on our assumptions. First we assume a spherically symmetric distribution of the noise sequence in (1). Although this assumption is never crucial, we adopt it to avoid unnecessary technicalities. Second, this paper addresses only the case in which the density of an i.i.d noise sequence has a regularly varying tail. As is well-known in EVT, in the one-dimensional case, regular variation of the tail in the density suffices for the distribution to be in the max-domain of attraction of the Fréchet law. Lastly, this paper assumes the process (1) is of finite order. In the infinite order case (i.e. $q = \infty$), the moving average will exhibit longer range dependence due to the emergence of larger clusters. We conjecture that even in the infinite order case, the Poissonian limit theorem holds for our Betti numbers. However, the discussion should involve much more technicalities, which may blur the message of this paper. So we have decided to assume the finiteness of $q$.

## 2. Vietoris–Rips complex

### 2.1. Setup and assumptions

Let $(X_j,\ j \in \mathbb{Z})$ be a sequence of i.i.d $\mathbb{R}^d$-valued random variables with spherically symmetric probability density $f$. Let $S_{d-1}$ be the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and assume that $f$ has a regularly varying tail with parameter $-\alpha$. That is, for some $\alpha > d$ and some $\theta \in S_{d-1}$ (equivalently, for every $\theta \in S_{d-1}$),

$$\lim_{r\to\infty} \frac{f(rt\theta)}{f(r\theta)} = t^{-\alpha} \ \text{ for all } t > 0. \tag{2}$$

Taking $(X_j,\ j \in \mathbb{Z})$ as a heavy-tailed noise sequence, we consider a discrete-time moving average process of order $1 \le q < \infty$.

$$Y_n = \sum_{j=0}^{q} A_j X_{n-j}, \quad n \ge 1, \tag{3}$$

where $A_j$ represents $d \times d$ non-singular matrices with non-random real entries.

The main focus of the current paper is the topological crackle occurring in the tail of $(Y_n,\ n \ge 1)$. We employ the $k$th Betti numbers, for $k \ge 1$, to quantify the topological complexity of the crackle phenomenon. We attempt to follow the dynamical evolution of the topology outside a growing ball, through the filtration associated to Vietoris–Rips complexes:

$$\Big\{\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(1)})^c\big),\ t \ge 0\Big\}, \tag{4}$$

where $\mathcal{Y}_n = (Y_1, \dots, Y_n)$, and $R_{k,n}^{(1)} \to \infty$ is determined by

$$n^{2k+2}(R_{k,n}^{(1)})^d f(R_{k,n}^{(1)} e_1)^{2k+2} \to 1 \ \text{ as } n \to \infty \tag{5}$$

with $e_1 = (1, 0, \dots, 0)' \in \mathbb{R}^d$.

Note that $\big(R_{k,n}^{(1)},\ k = 1, 2, \dots, d-1\big)$ grow at regularly varying rates as a function of $n$, such that

$$R_{d-1,n}^{(1)} \ll R_{d-2,n}^{(1)} \ll \cdots \ll R_{2,n}^{(1)} \ll R_{1,n}^{(1)}, \quad n \to \infty.$$

A set of radii above divides $\mathbb{R}^d$ as

$$\mathbb{R}^d = \bigcup_{i=1}^{d} \mathrm{Ann}(R_{i,n}^{(1)}, R_{i-1,n}^{(1)}), \tag{6}$$

where $\mathrm{Ann}(K, L)$ is a closed annulus of inner radius $K$ and outer radius $L$, and $R_{0,n}^{(1)} \equiv \infty$, $R_{d,n}^{(1)} \equiv 0$.

For $k = 1, \dots, d-1$, the $k$th Betti number studied in the section is

$$\beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(1)})^c\big)\Big), \quad t \ge 0. \tag{7}$$

Clearly, this can be viewed as a stochastic process, in the parameter $t$, possessing right continuous sample paths with left limits. By analyzing (7) as a stochastic process, we expect to capture the birth and death of topological cycles as $t$ varies.

The main point for (5) is that the Vietoris–Rips complex (4) defined outside $B(0, R_{k,n}^{(1)})$ is so sparse that, as $n \to \infty$, we observe only "finitely many" $k$-cycles. Consequently, the $k$th Betti number should be governed by a Poissonian type limit theorem. On the contrary, the Vietoris–Rips complex inside $B(0, R_{k,n}^{(1)})$ becomes denser, and there should be "infinitely many" $k$-cycles there. To be more specific on this point, let us return to the layered structure (6). Then, we have, as $n \to \infty$,

- Outside $B(0, R_{1,n}^{(1)})$, there are finitely many 1-cycles, but no $i$-cycles for all $i \ge 2$.
- Outside $B(0, R_{2,n}^{(1)})$, equivalently inside $\mathrm{Ann}(R_{2,n}^{(1)}, R_{1,n}^{(1)})$, there are infinitely many 1-cycles and finitely many 2-cycles, but no $i$-cycles for all $i \ge 3$.

In general,

- Outside $B(0, R_{k,n}^{(1)})$, equivalently inside $\mathrm{Ann}(R_{k,n}^{(1)}, R_{k-1,n}^{(1)})$, there are infinitely many $i$-cycles for all $i = 1, \dots, k-1$, and finitely many $k$-cycles, but no $i$-cycles for all $i \ge k+1$.

In the following, for a fixed $1 \le k \le d-1$, we wish to give a complete description of the topological crackle of a moving average process in terms of how the dependence structure of the process affects the spatial distribution of finitely many $k$-cycles occurring in $\mathrm{Ann}(R_{k,n}^{(1)}, R_{k-1,n}^{(1)})$. Before moving to the next section, we shall add a few conditions on a sequence of matrices $(A_j)$ in (3).

- For all distinct $i, j \in \{0, \dots, q\}$,

$$\inf_{\theta \in S_{d-1}} \| (A_i - A_j)\theta \| > 0. \tag{8}$$

- There exists a constant $c > 0$ such that

$$c \, \|A_j^{-1}\| \, \|x\| \le \|A_j^{-1} x\| \tag{9}$$

for all $j = 0, \dots, q$ and $x \in \mathbb{R}^d$. $\|A\|$ denotes the usual matrix norm of a $d \times d$ matrix $A$.
- For all off-diagonal elements $(\ell_1, \dots, \ell_{2k+2}) \in \{0, \dots, q-1\}^{2k+2}$ (i.e., $\ell_i \ne \ell_j$ for some $i \ne j$) and $m = 1, \dots, q - \max_{1 \le i \le 2k+2} \ell_i$, assume that for some $i \ne j$,

$$A_{\ell_i + m} A_{\ell_i}^{-1} \ne A_{\ell_j + m} A_{\ell_j}^{-1}. \tag{10}$$

These three conditions do not appear in the classical study of extremes of moving averages. In particular, the first two, (8) and (9), are needed in our proof when using standard techniques in EVT, such as the Potter bound for regularly varying functions (e.g., Theorem 1.5.6 in [8]), and the approximation argument by point processes. In contrast, the last one plays a more decisive role. It actually restricts multiple occurrence of $k$-cycles in a certain situation. More details on this point are given in Example 2.1 and Theorem 2.5.

### 2.2. *Limit process*

The objective of this section is to formalize the limit process of the $k$th Betti number (7). First, let

$$h(x_1, \dots, x_{2k+2}) = \mathbf{1}\Big\{ \beta_k\big(\mathcal{R}_1(x_1, \dots, x_{2k+2})\big) = 1 \Big\}, \quad x_i \in \mathbb{R}^d. \tag{11}$$

It is obvious that $h$ is translation invariant; namely,

$$h(x_1, \dots, x_{2k+2}) = h(x_1 + y, \dots, x_{2k+2} + y) \ \text{ for all } x_i, y \in \mathbb{R}^d. \tag{12}$$

Since the Vietoris–Rips complex appearing in (11) is necessarily connected (with connectivity radius $1/2$), there exists a finite $M > 0$, such that

$$h(0, x_1, \dots, x_{2k+1}) = 0 \ \text{ if } \|x_i\| > M \text{ for some } i \in \{1, \dots, 2k+1\}. \tag{13}$$

In addition, for every $t > 0$, we define a scaled version of $h$ by setting

$$\begin{aligned} h_t(x_1, \dots, x_{2k+2}) &:= h(x_1/t, \dots, x_{2k+2}/t) \\ &= \mathbf{1}\Big\{ \beta_k\big(\mathcal{R}_t(x_1, \dots, x_{2k+2})\big) = 1 \Big\}. \end{aligned} \tag{14}$$

As mentioned earlier, the occurrence of $k$-cycles is rare in the sense that, outside $B(0, R_{k,n}^{(1)})$, there asymptotically appear only finitely many $k$-cycles. Because of their rareness, all the $k$-cycles in the limit must be supported on $2k+2$ vertices; this is actually the minimum number of vertices to form a single $k$-cycle. Furthermore, any of the components built on more than $2k+2$ vertices would not contribute to the limit. From this viewpoint, at least in the limit, it suffices only to count the $k$-cycles on exactly $2k+2$ vertices.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a generic probability space on which our limit process is constructed. Define $\big(X_j^{(i)}, \ j \in \mathbb{Z}, \ i = 1, \dots, 2k+2\big)$ as a collection of $\mathbb{R}^d$-valued i.i.d random variables on a different probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, with common density $f$. Then, we define

$$Y_n^{(i)}(\omega') = \sum_{j=0}^{q} A_j X_{n-j}^{(i)}(\omega'), \quad i = 1, \dots, 2k+2, \ \ n \ge 0, \ \ \omega' \in \Omega'. \tag{15}$$

Observe that, for every $n \geq 0$, $Y_n^{(1)}, \dots, Y_n^{(2k+2)}$ are independent random variables, but, for each $i \in \{1, \dots, 2k+2\}$, $Y_n^{(i)}$ and $Y_m^{(i)}$ are not independent whenever $|n-m| \leq q$.

Subsequently, we need to introduce a family of Poisson random measures on a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. First, for $\ell_1, \dots, \ell_{2k+2} \in \{0, \dots, q\}$, define a positive constant

$$C_{(\ell_1,\dots,\ell_{2k+2})} := \frac{\left(\prod_{j=1}^{2k+2} \det A_{\ell_j}\right)^{-1}}{\alpha(2k+2)-d} \int_{S_{d-1}} \prod_{j=1}^{2k+2} \|A_{\ell_j}^{-1}\theta\|^{-\alpha} J(\theta) d\theta, \tag{16}$$

where $J(\theta) = |\partial x / \partial \theta|$ is the Jacobian given by $J(\theta) = \sin^{d-2}(\theta_1) \sin^{d-3}(\theta_2) \cdots \sin(\theta_{d-2})$. Writing $\lambda_m$ for the Lebesgue measure on $(\mathbb{R}^d)^m$, denote by $M(\cdot) = M(\cdot\,; \omega)$, $\omega \in \Omega$, a Poisson random measure on $(0,\infty) \times S_{d-1} \times \Omega' \times (\mathbb{R}^d)^{2k+1}$ with mean measure

$$\frac{1}{(2k+2)!}\, \rho^{d-1-2\alpha(k+1)} d\rho \times J(\theta) d\theta \times \mathbb{P}' \times \lambda_{2k+1}.$$

Moreover, for $(\ell_1, \dots, \ell_{2k+2}) \in \{0, \dots, q\}^{2k+2}$, denote by $M_{(\ell_1,\dots,\ell_{2k+2})}(\cdot) = M_{(\ell_1,\dots,\ell_{2k+2})}(\cdot\,; \omega)$, $\omega \in \Omega$, independent Poisson random measures on $\Omega' \times (\mathbb{R}^d)^{2k+1}$, each of which is independent of $M$ and has a mean measure

$$\frac{C_{(\ell_1,\dots,\ell_{2k+2})}}{(2k+2)!}\, \mathbb{P}' \times \lambda_{2k+1}.$$

Finally, for $m \geq 1$, let

$$\mathcal{L}_m = \Big\{ (\ell_1, \dots, \ell_m) \in \{0, \dots, q\}^m \Big\}, \tag{17}$$

$$\widetilde{\mathcal{L}}_m = \Big\{ (\ell_1, \dots, \ell_m) \in \mathcal{L}_m : \ell_i \neq \ell_j \text{ for some } i \neq j \Big\}. \tag{18}$$

The latter denotes a collection of off-diagonal elements in $\{0, \dots, q\}$. Now, we are ready to define the limit process for the Betti number (7) by

$$V_k(t\,; \omega) := V_k^{(1)}(t\,; \omega) + V_k^{(2)}(t\,; \omega), \tag{19}$$

where

$$V_k^{(1)}(t\,;\omega) = \sum_{\ell=0}^{q} \int_{(\|A_\ell \theta\|^{-1},\infty)\times S_{d-1}\times\Omega'\times(\mathbb{R}^d)^{2k+1}} h_t\big(Y_\ell^{(1)}(\omega'), \dots, Y_\ell^{(2k+2)}(\omega')\big)\Big|_{(X_0^{(1)}(\omega'),\dots,X_0^{(2k+2)}(\omega'))=(0,\mathbf{y})} M(d\rho\, d\theta\, d\omega'\, d\mathbf{y}\,; \omega),$$

and

$$V_k^{(2)}(t\,;\omega) = \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \int_{\Omega'\times(\mathbb{R}^d)^{2k+1}} h_t\big(Y_{\ell_1}^{(1)}(\omega'), \dots, Y_{\ell_{2k+2}}^{(2k+2)}(\omega')\big)\Big|_{X_0^{(i)}(\omega')=A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,2k+2} M_{\boldsymbol{\ell}}(d\omega'\, d\mathbf{y}\,; \omega)$$

with $\mathbf{y} = (y_1, \dots, y_{2k+1}) \in (\mathbb{R}^d)^{2k+1}$, $y_0 \equiv 0$, and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{2k+2}) \in \widetilde{\mathcal{L}}_{2k+2}$. For the process $V_k^{(1)}$, the notation $h_t(\dots)\big|_{(X_0^{(1)}(\omega'),\dots,X_0^{(2k+2)}(\omega'))=(0,\mathbf{y})}$ requires to substitute

$$(X_0^{(1)}(\omega'), \dots, X_0^{(2k+2)}(\omega')) = (0, \mathbf{y})$$

into $(Y_\ell^{(1)}(\omega'), \dots, Y_\ell^{(2k+2)}(\omega'))$. This type of notation will frequently appear throughout the paper. From onward however, for ease of description we omit the dependence on $\omega$ and $\omega'$. For example, we simply write $V_k^{(1)}(t) = V_k^{(1)}(t\,;\omega)$, $Y_\ell^{(i)} = Y_\ell^{(i)}(\omega')$ etc.

The process $V_k(t)$ looks quite complicated, but it clarifies how topological $k$-cycles are formed by the moving average (3). First, the Poisson random measure $M$ in the process $V_k^{(1)}(t)$ is driving multiple, i.e., $q+1$, indicator functions. More intuitively $M$ possibly produces multiple $k$-cycles over $q+1$ periods of time. To be more specific, suppose that $M$ has driven $(Y_0^{(1)}, \dots, Y_0^{(2k+2)})$ and forms a $k$-cycle at time 0. Then, at time 1, $(Y_1^{(1)}, \dots, Y_1^{(2k+2)})$ can also form another $k$-cycle. This is because, for every $i = 1, \dots, 2k+2$, $Y_0^{(i)}$ and $Y_1^{(i)}$ have the same $X_j^{(i)}$s in common, that is, they are both functions of $X_0^{(i)}, \dots, X_{-(q-1)}^{(i)}$. Since our moving average process is of order $q$, we may eventually obtain multiple (up to $q+1$) $k$-cycles, all of which are driven by a "single" Poisson random measure $M$. Note that as $q$ gets larger, more and more $k$-cycles are induced by $M$. In other words the "clustering effect" of a moving average process continues for longer periods of time. As a result, the $k$th Betti number asymptotically increases. For the process $V_k^{(2)}(t)$, however, each of the Poisson random measures, denoted by $M_\ell$, triggers at most a single $k$-cycle.

Before presenting the main weak limit theorem, it is beneficial to display a simple example to understand how the moving average (3) forms non-trivial cycles, and the Betti number converges to the limit (19).

**Example 2.1.** We consider a moving average process

$$Y_n = X_n + AX_{n-1} + BX_{n-2},$$

where $A, B$ are $d \times d$ non-singular matrices with $A \neq BA^{-1}$; see (10). Assume, for simplicity, that $A$ and $B$ are orthogonal matrices with $0 < \|B\| < \|A\| < 1$. Taking $R_{1,n}^{(1)}$ as in (5), we are interested in the topological crackle outside $B(0, R_{1,n}^{(1)})$. Condition (5) imposes a restriction that one can take at most four extremal points (i.e. points at a large distance from the origin) from $(X_j,\ j \in \mathbb{Z})$. This is a crucial requirement throughout this example.

We first consider the quadruple $(Y_{i_1}, \dots, Y_{i_4})$, for which we set $X_{i_j-1} = X_{i_j-2} = 0$ for all $j = 1, \dots, 4$, and assume that $(Y_{i_1}, \dots, Y_{i_4}) = (X_{i_1}, \dots, X_{i_4})$ forms a 1-cycle outside $B(0, R_{1,n}^{(1)})$. See Fig. 3(i). Since $\|A\| < 1$, the four points $(AX_{i_1}, \dots, AX_{i_4})$ approach closer to the origin and asymptotically are equal to each other. Nevertheless, these four points may still lie outside $B(0, R_{1,n}^{(1)})$. Then,

$$(Y_{i_1+1}, \dots, Y_{i_4+1}) = (X_{i_1+1} + AX_{i_1}, \dots, X_{i_4+1} + AX_{i_4})$$

may, once again, create a 1-cycle outside $B(0, R_{1,n}^{(1)})$. Since $(AX_{i_1}, \dots, AX_{i_4})$ are already extremal at a large distance from the origin, any of the points in $(X_{i_1+1}, \dots, X_{i_4+1})$ cannot be a fifth extremal point, but their small perturbations may create a 1-cycle. Similarly, $(Y_{i_1+2}, \dots, Y_{i_4+2})$ can form another 1-cycle as well, and hence, there could eventually occur multiple (up to three) 1-cycles in the extreme. The sum of all 1-cycles thus created converges to $V_1^{(1)}(t)$.

Returning to the quadruple $(Y_{i_1}, \dots, Y_{i_4})$, we suppose alternatively that $X_{i_1-1} = X_{i_1-2} = X_{i_2-1} = X_{i_2-2} = X_{i_3} = X_{i_3-2} = X_{i_4} = X_{i_4-2} = 0$, and

$$(Y_{i_1}, \dots, Y_{i_4}) = (X_{i_1}, X_{i_2}, AX_{i_3-1}, AX_{i_4-1}) \tag{20}$$

forms a 1-cycle in the outside of $B(0, R_{1,n}^{(1)})$. See Fig. 3(ii). In this case, since $0 < \|B\| < \|A\| < 1$, we have that $AX_{i_1} \approx AX_{i_2}$ and $BX_{i_3-1} \approx BX_{i_4-1}$. At the same $A \neq BA^{-1}$ implies that $AX_{i_1}$ and $BX_{i_3-1}$ are necessarily far apart from each other. Accordingly,

$$(Y_{i_1+1}, \dots, Y_{i_4+1}) = (X_{i_1+1} + AX_{i_1}, X_{i_2+1} + AX_{i_2}, X_{i_3+1} + BX_{i_3-1}, X_{i_4+1} + BX_{i_4-1})$$

cannot create a 1-cycle. More precisely since $(AX_{i_1}, AX_{i_2}, BX_{i_3-1}, BX_{i_4-1})$ are all extremal, there are no other extremal points in $(X_{i_1+1}, \dots, X_{i_4+1})$. Since $AX_{i_1}$ and $BX_{i_3-1}$ are far apart

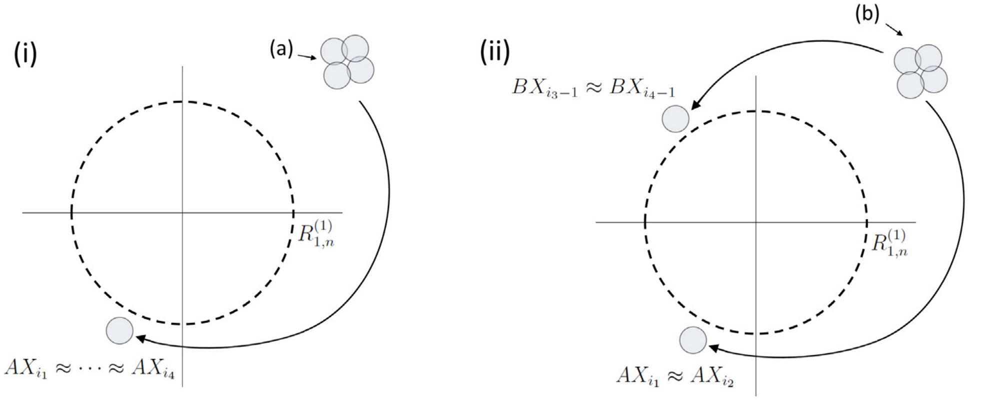


**Fig. 3.** Topological crackle of a moving average process. (i): (a) denotes a 1-cycle formed by $(Y_{i_1}, \dots, Y_{i_4}) = (X_{i_1}, \dots, X_{i_4})$. Since $A$ is an orthogonal matrix with $\|A\| < 1$, we have that $AX_{i_1} \approx AX_{i_2} \approx AX_{i_3} \approx AX_{i_4}$; however, $(Y_{i_1+1}, \dots, Y_{i_4+1})$ can create a 1-cycle because of the perturbation of $(X_{i_1+1}, \dots, X_{i_4+1})$. (ii): (b) denotes a 1-cycle formed by $(Y_{i_1}, \dots, Y_{i_4}) = (X_{i_1}, X_{i_2}, AX_{i_3-1}, AX_{i_4-1})$. Since $0 < \|B\| < \|A\| < 1$, we have that $AX_{i_1} \approx AX_{i_2}$ and $BX_{i_3-1} \approx BX_{i_4-1}$, while $A \neq BA^{-1}$ implies $AX_{i_1}$ and $BX_{i_3-1}$ are at a distance from each other. In this case, $(Y_{i_1+1}, \dots, Y_{i_4+1})$ does not create a 1-cycle.

from one another, it is impossible to make $Y_{i_j+1}$'s close together to form a 1-cycle. By the same reasoning $(Y_{i_1+2}, \dots, Y_{i_4+2})$ does not form a 1-cycle. Then, the sum of all 1-cycles occurring "individually", as that induced by (20), converges to $V_1^{(2)}(t)$.

### 2.3. Weak limit theorem

We now formally describe the main result of this paper. The proof is provided in Section 5. Hereinafter, $\Rightarrow$ denotes the weak convergence and $D[0, \infty)$ is the space of right continuous functions on $[0, \infty)$ with left limits.

**Theorem 2.2.** *Under the assumptions (2), (5), (8), (9), and (10), the Betti number (7) satisfies*

$$\beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R^{(1)}_{k,n})^c\big)\Big) \Rightarrow V_k(t) \ \ \text{in } D[0, \infty), \quad n \to \infty. \tag{21}$$

**Remark 2.3.** If $q = 0$ and $A_0 = I$, i.e., $d \times d$ identity matrix, the moving average in (3) reduces to an i.i.d heavy-tailed sequence, i.e., $Y_n = X_n$ for all $n \geq 1$. In such a case, an elementary calculation shows that

$$V_k(t) = \int_{(\mathbb{R}^d)^{2k+1}} h_t(0, \mathbf{y}) M'(d\mathbf{y}), \tag{22}$$

where $M'$ is a Poisson random measure on $(\mathbb{R}^d)^{2k+1}$ with mean measure

$$C_k := \frac{s_{d-1}}{(2k+2)!\big(2\alpha(k+1) - d\big)} \lambda_{2k+1}$$

($s_{d-1}$ is the surface area of a $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$). It is clear that the process (22) has Poisson marginals, i.e., $V_k(t)$ is a Poisson random variable with mean parameter $C_k \int_{(\mathbb{R}^d)^{2k+1}} h_t(0, \mathbf{y}) d\mathbf{y} t^{d(2k+1)}$; see [32]. The general limit (19), however, does not possess

Poisson marginals, unless $q = 0$. More precisely, each of the integrals in $V_k^{(2)}(t)$ has Poisson marginals, because their integrands are all indicator functions, as in (22). However, the process $V_k^{(1)}(t)$ drives many indicator functions, and therefore, it does not possess Poisson marginals.

**Remark 2.4.** One of the main techniques used in the proof are those from EVT via convergence of appropriate point processes. Indeed our argument extends the discussion in Section 2 of [16] to a multidimensional setting. Another key step is to transform point process convergence into the convergence of some $k$-cycle counts. Since we are dealing with the Poissonian regime, these $k$-cycle counts can be approximated by "isolated" $k$-cycle counts, and the latter can be further approximated by the Betti number (7). In the case of i.i.d heavy-tailed points, a similar treatment was made in [2] and [33].

Before concluding this section, we want to add one more result to clarify the role of condition (10). As implied in the latter case of Example 2.1, it prevents multiple occurrence of $k$-cycles. Although generalizing Theorem 2.2 by removing (10) seems feasible, it makes the description of the limit process extremely complicated, because of the involvement of more general coefficient matrices $(A_j)$. Therefore, in the following, we set $A_j \equiv I$ and consider a simple moving average process

$$Y_n = \sum_{j=0}^{q} X_j. \tag{23}$$

Clearly, this process does not fulfill (10) (and also (8)). We now state the limit theorem for the Betti number (7). Since the proof is very similar to that of Theorem 2.2, we omit it.

**Theorem 2.5.** *For the moving average process (23), assume all the conditions in Theorem 2.2 except (10) and (8). Then, we have*

$$\beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(1)})^c\big)\Big) \Rightarrow W_k(t) \ \text{ in } D[0,\infty), \quad n \to \infty,$$

*where* $W_k(t) := W_k^{(1)}(t) + W_k^{(2)}(t)$ *such that*

$$W_k^{(1)}(t) = \sum_{\ell=0}^{q} \int_{\Omega' \times (\mathbb{R}^d)^{2k+1}} h_t\big(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)}\big)\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} M(d\omega', d\mathbf{y}),$$

$$W_k^{(2)}(t) = \sum_{p=1}^{q} \sum_{\substack{\boldsymbol{\ell} \in \tilde{\mathcal{L}}_{2k+2} \\ \min_{1\le i\le 2k+2} \ell_i = 0 \\ \max_{1\le i\le 2k+2} \ell_i = p}} \int_{\Omega' \times (\mathbb{R}^d)^{2k+1}} \sum_{j=0}^{q-p} h_t\big(Y_{\ell_1+j}^{(1)}, \dots, Y_{\ell_{2k+2}+j}^{(2k+2)}\big)\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \times M_{\boldsymbol{\ell}}(d\omega', d\mathbf{y}).$$

*Here, $M$ and $M_{\boldsymbol{\ell}}$'s are independent Poisson random measures on $\Omega' \times (\mathbb{R}^d)^{2k+1}$ with common mean measure*

$$\frac{s_{d-1}}{(2k+2)!\,\big(2\alpha(k+1)-d\big)} \mathbb{P}' \times \lambda_{2k+1}.$$

In the corollary, the process $W_k^{(1)}(t)$ is essentially the same as $V_k^{(1)}(t)$ in the sense that both represent multiple occurrence of $k$-cycles over $q+1$ periods of time. The process $W_k^{(2)}(t)$, however, consists of many more integrals than $V_k^{(2)}(t)$ does, some of which relate to multiple occurrences of $k$-cycles, whereas all the integrals in $V_k^{(2)}(t)$ induce at most a single $k$-cycle.

Specifically, for every $\boldsymbol{\ell} \in \widetilde{\mathcal{L}}_{2k+2}$ with $p = \max_{1\le i\le 2k+2} \ell_i$, the corresponding Poisson random measure $M_{\boldsymbol{\ell}}$ drives multiple (up to $q - p + 1$) $k$-cycles, unless $p = q$.

## 3. Čech complex

In this section, we consider the filtration relating to the Čech complex $\mathcal{C}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(2)})^c\big)$, where $\mathcal{Y}_n = (Y_1, \dots, Y_n)$ is the same moving average process (3), and $R_{k,n}^{(2)}$ is defined as a solution to the asymptotic equation

$$n^{k+2}(R_{k,n}^{(2)})^d f(R_{k,n}^{(2)}e_1)^{k+2} \to 1 \text{ as } n \to \infty. \tag{24}$$

Then one can obtain the same layered structure as in (6), with radii of different regularly varying rates. The corresponding $k$th Betti number is defined as

$$\beta_k\Big(\mathcal{C}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(2)})^c\big)\Big) = \beta_k\Bigg(\bigcup_{Y\in\mathcal{Y}_n\cap B(0,R_{k,n}^{(2)})^c} B(Y, t/2)\Bigg), \ \ t \ge 0, \tag{25}$$

where the equality holds because of homotopy equivalence between the Čech complex and the union of balls.

As for the coefficient matrices $A_j$, we again assume (8) and (9). Additionally we need to slightly modify (10) by changing the dimensions. Namely, for all off-diagonal elements $(\ell_1, \dots, \ell_{k+2}) \in \{0, \dots, q-1\}^{k+2}$ (i.e., $\ell_i \ne \ell_j$ for some $i \ne j$) and $m = 1, \dots, q - \max_{1\le i\le k+2} \ell_i$, assume that for some $i \ne j$,

$$A_{\ell_i+m}A_{\ell_i}^{-1} \ne A_{\ell_j+m}A_{\ell_j}^{-1}. \tag{26}$$

Subsequently, we define the limiting process in a way analogous to that in the previous section. By slightly abusing notations in (11) and (14), define

$$h(x_1, \dots, x_{k+2}) := \mathbf{1}\Big\{\beta_k\big(\mathcal{C}_1(x_1, \dots, x_{k+2})\big) = 1\Big\}, \ \ x_i \in \mathbb{R}^d,$$
$$h_t(x_1, \dots, x_{k+2}) := h(x_1/t, \dots, x_{k+2}/t).$$

Moreover, $N$ and $N_{\boldsymbol{\ell}}$ with $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{k+2}) \in \widetilde{\mathcal{L}}_{k+2}$ denote independent Poisson random measures with mean measures

$$\frac{1}{(k+2)!}\,\rho^{d-1-\alpha(k+2)}d\rho \times J(\theta)d\theta \times \mathbb{P}' \times \lambda_{k+1}$$

and

$$\frac{C_{(\ell_1,\dots,\ell_{k+2})}}{(k+2)!}\,\mathbb{P}' \times \lambda_{k+1}, \quad (\ell_1, \dots, \ell_{k+2}) \in \widetilde{\mathcal{L}}_{k+2}, \tag{27}$$

respectively.

Then, the following process becomes the weak limit for (25).

$$\begin{aligned} Z_k(t) &= \sum_{\ell=0}^{q} \int_{(\|A_\ell\theta\|^{-1},\infty)\times S_{d-1}\times\Omega'\times(\mathbb{R}^d)^{k+1}} h_t\big(Y_\ell^{(1)}, \dots, Y_\ell^{(k+2)}\big)\big|_{(X_0^{(1)},\dots,X_0^{(k+2)})=(0,\mathbf{y})} \\ &\qquad N(d\rho\, d\theta\, d\omega'\, d\mathbf{y}) \\ &\quad + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{k+2}} \int_{\Omega'\times(\mathbb{R}^d)^{k+1}} h_t\big(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{k+2}}^{(k+2)}\big)\big|_{X_0^{(i)}=A_{\ell_i}^{-1}y_{i-1}, i=1,\dots,k+2} N_{\boldsymbol{\ell}}(d\omega'\, d\mathbf{y}) \\ &:= Z_k^{(1)}(t) + Z_k^{(2)}(t), \end{aligned} \tag{28}$$

where $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{k+2}) \in \widetilde{\mathcal{L}}_{k+2}$, $\mathbf{y} = (y_1, \dots, y_{k+1}) \in (\mathbb{R}^d)^{k+1}$ with $y_0 \equiv 0$.

The structure of (28) is essentially the same as that of (19) except for the dimensions of the domain of functions and measures, as well as a multiplicative constant in (27). The reason for the difference in dimensions is that the minimum number of points to form a single $k$-cycle differs between the two complexes. In fact, in order to make a $k$-cycle, there need to be at least $2k + 2$ points for the Vietoris–Rips complex, while there need to be only $k + 2$ points for the Čech complex.

**Theorem 3.1.** *Under the assumptions (2), (8), (9), (24), and (26), the kth Betti number (25) satisfies*

$$\beta_k\Big(\mathcal{C}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n}^{(2)})^c\big)\Big) \Rightarrow Z_k(t), \quad n \to \infty, \tag{29}$$

*in a finite-dimensional sense.*

**Remark 3.2.** In contrast to the Vietoris–Rips complex, the definition of a $k$-simplex in the Čech complex requires a non-empty intersection of "multiple" closed balls. This difference makes it much harder to establish the required tightness. Although the weak limit theorem above seems to hold in the space $D[0, \infty)$, we have proven it only in a finite-dimensional sense.

## 4. Future work

The current paper presented a novel result for the topological crackle when the underlying moving average process exhibits non-trivial linear dependence. However, this study is just a first step toward developing the theory of the crackle phenomenon for more general stochastic processes. With this goal in mind, we discuss a few possible future research directions.

1. The crackle phenomenon can be observed not only in heavy-tailed distributions with a regularly varying tail, but also in the distributions of lighter tails, such as subexponential or exponential distributions; see [33]. From this viewpoint, we conjecture that, if the moving average process is constructed from i.i.d subexponential (or exponential) random variables, then the process once again exhibits the crackle phenomenon. The resulting limit process will be expressed in terms of the sum of stochastic integrals as in (19), all of which are driven by Poisson random measures. However, the mean measure of these Poisson random measures should be a different one, reflecting the light tail of the moving average, as well as its clustering property.
2. For the current work, Eq. (5) plays a crucial role in determining $R_{k,n}^{(1)}$, so that there appear only finitely many $k$-cycles outside $B(0, R_{k,n}^{(1)})$. Suppose, on the contrary, that one takes a different radius $R_n$ growing more slowly than $R_{k,n}^{(1)}$, such that

$$n^{2k+2} R_n^d f(R_n e_1)^{2k+2} \to \infty \ \text{ as } n \to \infty. \tag{30}$$

   Then, the Vietoris–Rips complexes in the exterior of $B(0, R_n)$ are more dense than those outside $B(0, R_{k,n}^{(1)})$. Consequently, as $n \to \infty$, we would observe "infinitely many" $k$-cycles outside $B(0, R_n)$. This implies that by proper centering and scaling, the normalized $k$th Betti number follows a functional central limit theorem. The weak limits of the normalized Betti numbers are represented as the sum of many integrals as in (19), but all the integrals should be driven by certain Gaussian random measures. In the case of i.i.d heavy-tailed points, the normal convergence under the condition similar to that in (30) was proven in [32].

## 5. Proof of the main theorem

The proofs of (21) and (29) are basically the same. In fact, the argument is completely parallel up to finite-dimensional convergence, regardless of the type of complex. Thus, we prove only (21) together with the required tightness. Since the proof of (21) is rather long and heavy on notations, we would like to introduce shorthand notations to save space. Let $\mathbb{N}$ be the non-negative integers. For $m \geq 1$, let

$$\mathcal{I}_m = \big\{(i_1, \dots, i_m) \in \mathbb{N}_+^m : 1 \leq i_1 < \cdots < i_m \leq n\big\}$$

and

$$\widetilde{\mathcal{I}}_m = \big\{(i_1, \dots, i_m) \in \mathcal{I}_m : i_j - i_{j-1} > 2q,\ j = 2, \dots, m\big\}.$$

We need to recall notations (17) and (18). Recall also $R_{k,n}^{(1)}$ given in (5), but hereafter, we drop the superscript "(1)" for typographical ease.

Given a moving average process (3), define, for $i \in \mathbb{Z}$,

$$\mathcal{X}_i = (X_i, X_{i-1}, \dots, X_{i-q}) \in (\mathbb{R}^d)^{q+1}$$

and for $\mathbf{i} = (i_1, \dots, i_{2k+2}) \in \mathcal{I}_{2k+2}$,

$$\mathcal{Y}_{\mathbf{i}} = (Y_{i_1}, \dots, Y_{i_{2k+2}}) \in (\mathbb{R}^d)^{2k+2}.$$

Finally, we provide a basic fact: for every $m \geq 1$,

$$\binom{n}{m} \sim \frac{n^m}{m!} \quad \text{as } n \to \infty, \tag{31}$$

where $\sim$ means that the ratio of the two sides tends to 1 as $n \to \infty$. Moreover, for $x \in \mathbb{R}^d$ and $\mathbf{y} = (y_1, \dots, y_m) \in (\mathbb{R}^d)^m$, we write

$$x + \mathbf{y} := (x + y_1, \dots, x + y_m).$$

In the following, $C^*$ denotes a generic positive constant, which does not depend on $n$ and may vary between lines.

An entire proof is divided into four parts. Our proof employs, at least partially, some arguments in EVT via point process convergence [16] and [35]. We first propose some claims about point process convergence. Assuming these claims, we transform the point process into some $k$-cycle counts. Part I shows that such $k$-cycle counts weakly converges to $V_k(t)$ in a finite-dimensional sense. The required tightness is proven in Part II. Subsequently, Part III verifies that the $k$-cycle counts can be approximated by the $k$th Betti number. Finally, Part IV proves all the claims regarding point process convergence and finishes the entire proof.

Now we state three claims (32)–(34) below about point process convergence. Among them, the major statement is (33), asserting the multidimensional version of the result in Section 2 of [16]. Writing $\epsilon$ for the usual Dirac measure, we define a point process

$$\sum_{\mathbf{i} \in \mathcal{I}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \epsilon_{R_{k,n}^{-1}\left(\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_{2k+2}}\right)}(\cdot)$$

on the space $E_k := \big([-\infty, \infty]^{d(q+1)} \setminus \{\mathbf{0}\}\big)^{2k+2}$ ($\mathbf{0}$ is the vector of zeros in $\mathbb{R}^{d(q+1)}$). Based on this point process, the first claim is that

$$\sum_{\mathbf{i} \in \mathcal{I}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \epsilon_{R_{k,n}^{-1}\left(\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_{2k+2}}\right)} - \sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \epsilon_{R_{k,n}^{-1}\left(\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_{2k+2}}\right)} \xrightarrow{p} 0 \tag{32}$$

in the space $M_p(E_k)$ of point measures on $E_k$.

For $i \in \{0, \dots, q\}$, let $\delta_i = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0)'$ be an $\mathbb{R}^{d(q+1)}$-valued column vector with ones from the $(di+1)$th position to the $d(i+1)$th position, and all other entries 0. The second claim is that in the space $M_p(E_k)$,

$$\sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, \epsilon_{R_{k,n}^{-1}\left(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}}\right)} - \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, \epsilon_{R_{k,n}^{-1}\left(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\right)} \xrightarrow{p} 0. \tag{33}$$

Here, $\mathrm{diag}(A) := (a_{11}, a_{22}, \dots, a_{mm})$ for an $m \times m$ matrix $A = (a_{ij})_{i,j=1}^m$ and so,

$$\mathrm{diag}(\delta_{\ell_j}\mathcal{X}_{i_j}) = (0, \dots, 0, X_{i_j-\ell_j}, 0, \dots, 0) \in \mathbb{R}^{d(q+1)},$$

where the components in $X_{i_j-\ell_j}$ are distributed from the $(d\ell_j + 1)$th position to the $d(\ell_j + 1)$th position. A key insight for this claim is that for each $j = 1, \dots, 2k+2$, exactly one component in $\mathcal{X}_{i_j}$ is likely to be so large that it cannot be driven to zero by the normalization $R_{k,n}$, while all the other components in $\mathcal{X}_{i_j}$ tend to zero by the $R_{k,n}$ (remember that $\mathbf{0} \notin [-\infty, \infty]^{d(q+1)}$).

In addition, we also claim that

$$\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, \epsilon_{R_{k,n}^{-1}\left(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\right)} - \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\, \epsilon_{R_{k,n}^{-1}\left(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1+\ell_1}),\dots,\mathrm{diag}(\delta_{\ell_{k+2}}\mathcal{X}_{i_{2k+2}+\ell_{2k+2}})\right)} \xrightarrow{p} 0, \tag{34}$$

where, by definition,

$$\mathrm{diag}(\delta_{\ell_j}\mathcal{X}_{i_j+\ell_j}) = (0, \dots, 0, X_{i_j}, 0, \dots, 0) \in \mathbb{R}^{d(q+1)}.$$

When (32), (33), and (34) have been established, we can immediately derive

$$\sum_{\mathbf{i}\in\mathcal{I}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\} - \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|A_{\ell_j}X_{i_j}\| \ge R_{k,n}\Big\} \xrightarrow{p} 0. \tag{35}$$

In what follows, we temporarily assume (32), (33), and (34), together with (35), and start with the discussion in Part I.

*Part I*: The goal of Part I is to establish the convergence of the finite-dimensional distributions of

$$G_n(t) := \sum_{\mathbf{i}\in\mathcal{I}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\} \Rightarrow V_k(t).$$

By virtue of (35) and Slutsky's lemma, it suffices to show that

$$\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|A_{\ell_j}X_{i_j}\| \ge R_{k,n}\Big\} \Rightarrow V_k(t) \tag{36}$$

in a finite-dimensional sense. For typographical ease, we consider only one-dimensional distributions. In the case of more than one-dimension, the argument is similar because of the Cramér–Wold device, but the notation becomes more cumbersome.

We begin with defining a point process on $\mathbb{N}$,

$$\xi_n(\cdot) := \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} \mathbf{1}\Big\{ \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\,\mathbf{1}\big\{\min_{j=1,\dots,2k+2} \|A_{\ell_j} X_{i_j}\| \ge R_{k,n}\big\} \neq 0 \Big\}$$
$$\times\, \epsilon_{\left(\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\mathbf{1}\left\{\min_{j=1,\dots,2k+2}\|A_{\ell_j}X_{i_j}\|\ge R_{k,n}\right\}\right)}(\cdot).$$

Additionally let $\zeta$ be a Poisson random measure on $\mathbb{N}$ with *finite* mean measure $\mu + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \nu_{\boldsymbol{\ell}}$, where

$$\mu(\cdot) = \frac{1}{(2k+2)!}\int_0^\infty d\rho \int_{S_{d-1}} J(\theta)d\theta \int_{(\mathbb{R}^d)^{2k+1}} d\mathbf{y}\,\rho^{d-1-2\alpha(k+1)}$$
$$\times\, \mathbb{P}'\Big\{ \sum_{\ell=0}^{q} h_t(Y_\ell^{(1)},\dots,Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \mathbf{1}\big\{\rho \ge \|A_\ell\theta\|^{-1}\big\} \in \cdot \setminus \{0\}\Big\},$$

and for $\boldsymbol{\ell} = (\ell_1,\dots,\ell_{2k+2}) \in \widetilde{\mathcal{L}}_{2k+2}$,

$$\nu_{\boldsymbol{\ell}}(\cdot) = \frac{C_{\boldsymbol{\ell}}}{(2k+2)!}\int_{(\mathbb{R}^d)^{2k+1}} \mathbb{P}'\Big\{ h_t(Y_{\ell_1}^{(1)},\dots,Y_{\ell_{2k+2}}^{(2k+2)})\big|_{X_0^{(i)}=A_{\ell_i}^{-1}y_{i-1},\, i=1,\dots,2k+2} \in \cdot\setminus\{0\}\Big\} d\mathbf{y}.$$

For the latter we take $y_0 \equiv 0$ and $C_{\boldsymbol{\ell}}$ is given in (16).

Under this setup we show that

$$\xi_n \Rightarrow \zeta \ \text{ in } M_p(\mathbb{N}). \tag{37}$$

Before proving (37) we would like to demonstrate that (37) implies (36). To see this let $\widehat{T} : M_p(\mathbb{N}) \to \mathbb{N}$ be a functional defined by $\widehat{T}\big(\sum_m \epsilon_{z_m}\big) = \sum_m z_m$. This functional is continuous on a set of finite point measures. Combining (37) and the continuous mapping theorem yields

$$\widehat{T}(\xi_n) \Rightarrow \widehat{T}(\zeta).$$

Clearly $\widehat{T}(\xi_n)$ is equal to the left hand side of (36). Furthermore, we claim that

$$\widehat{T}(\zeta) \stackrel{d}{=} V_k(t). \tag{38}$$

To show (38) let us represent $\zeta$ as

$$\zeta \stackrel{d}{=} \sum_{i=1}^{M_n} \epsilon_{Z_i},$$

where $Z_1, Z_2, \dots$ are i.i.d with common distribution

$$\Big(\mu(\mathbb{N}) + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \nu_{\boldsymbol{\ell}}(\mathbb{N})\Big)^{-1}\Big(\mu + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \nu_{\boldsymbol{\ell}}\Big),$$

and $M_n$ is a Poisson random variable with parameter $\mu(\mathbb{N}) + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \nu_{\boldsymbol{\ell}}(\mathbb{N})$. Moreover, $(Z_i)$ and $M_n$ are independent. It then follows from the Laplace functional of a Poisson random measure (see Theorem 5.1 in [36]) that for every $\lambda > 0$,

$$\mathbb{E}\Big\{\exp\big(-\lambda V_k(t)\big)\Big\} = \mathbb{E}\Big\{\exp\big(-\lambda V_k^{(1)}(t)\big)\Big\}\mathbb{E}\Big\{\exp\big(-\lambda V_k^{(2)}(t)\big)\Big\} \tag{39}$$
$$= \exp\Big(-\sum_{m=1}^\infty (1-e^{-\lambda m})\mu(m)\Big)\exp\Big(-\sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}}\sum_{m=1}^\infty (1-e^{-\lambda m})\nu_{\boldsymbol{\ell}}(m)\Big).$$

On the other hand it is elementary to see that

$$\mathbb{E}\Big\{\exp\big(-\lambda\widehat{T}(\zeta)\big)\Big\} = \mathbb{E}\Big\{\exp\Big(-\lambda\sum_{i=1}^{M_n} Z_i\Big)\Big\} = \exp\Big(-\sum_{m=1}^{\infty}(1-e^{-\lambda m})\Big(\mu(m)+\sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}}\nu_{\boldsymbol{\ell}}(m)\Big)\Big) \tag{40}$$

Since (39) and (40) are the same, we get (38) and hence (36) follows as desired.

In order to finish the argument in Part I, it now remains to verify (37). According to Kallenberg's theorem (e.g., Proposition 3.22 in [35]), it is enough to show that for every $m = 1, 2, \dots$

$$\mathbb{E}\{\xi_n(m)\} \to \mathbb{E}\{\zeta(m)\} = \mu(m) + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} \nu_{\boldsymbol{\ell}}(m) \tag{41}$$

and

$$\mathbb{P}\big\{\xi_n(m)=0\big\} \to \mathbb{P}\big\{\zeta(m)=0\big\}. \tag{42}$$

Since $\xi_n(m)$ denotes a sum of indicators over $\mathbf{i} = (i_1, \dots, i_{2k+2}) \in \widetilde{\mathcal{I}}_{2k+2}$ with $i_j - i_{j-1} > 2q$, $j = 2, \dots, 2k+2$, we see that for every $\boldsymbol{\ell} \in \mathcal{L}_{2k+2}$, the components in $\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}} = (Y_{i_1+\ell_1}, \dots, Y_{i_{k+2}+\ell_{k+2}})$ do not share the same $X_j$s. Therefore, by (31) and (15), we have, as $n \to \infty$,

$$\begin{aligned}\mathbb{E}\{\xi_n(m)\} &\sim \frac{n^{2k+2}}{(2k+2)!}\mathbb{P}'\Big\{\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)}) \\ &\qquad\qquad \times \mathbf{1}\big\{\min_{j=1,\dots,2k+2}\|A_{\ell_j}X_0^{(j)}\| \ge R_{k,n}\big\} = m\Big\}. \\ &= \frac{n^{2k+2}}{(2k+2)!}\int_{(\mathbb{R}^d)^{2k+2}}\mathbb{P}'\Big\{\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(z_1,\dots,z_{2k+2})} \\ &\qquad\qquad \times \mathbf{1}\big\{\min_{j=1,\dots,2k+2}\|A_{\ell_j}z_j\| \ge R_{k,n}\big\} = m\Big\}\prod_{i=1}^{2k+2} f(z_i)d\mathbf{z} \\ &\equiv J_n.\end{aligned}$$

For $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{2k+2}) \in \mathcal{L}_{2k+2}$ and $\mathbf{z} = (z_1, \dots, z_{2k+2}) \in (\mathbb{R}^d)^{2k+2}$, we set

$$I_{\boldsymbol{\ell}}(\mathbf{z}) := h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(z_1,\dots,z_{2k+2})}\mathbf{1}\big\{\min_{j=1,\dots,2k+2}\|A_{\ell_j}z_j\| \ge R_{k,n}\big\}. \tag{43}$$

Using (43), we can write $J_n = J_n^{(1)} + \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} J_{n,\boldsymbol{\ell}}^{(2)} + J_n^{(3)}$, where

$$J_n^{(1)} := \frac{n^{2k+2}}{(2k+2)!}\int_{(\mathbb{R}^d)^{2k+2}}\mathbb{P}'\Big\{\sum_{\ell=0}^{q} I_{(\ell,\dots,\ell)}(\mathbf{z}) \ne 0, \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}}(\mathbf{z}) = m\Big\}\prod_{i=1}^{2k+2} f(z_i)d\mathbf{z},$$

$$J_{n,\boldsymbol{\ell}}^{(2)} := \frac{n^{2k+2}}{(2k+2)!}\int_{(\mathbb{R}^d)^{2k+2}}\mathbb{P}'\Big\{I_{\boldsymbol{\ell}}(\mathbf{z}) \ne 0, \sum_{\boldsymbol{\ell}'\in\mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}'}(\mathbf{z}) = m\Big\}\prod_{i=1}^{2k+2} f(z_i)d\mathbf{z},$$

and $J_n^{(3)} := J_n - J_n^{(1)} - \sum_{\boldsymbol{\ell}\in\widetilde{\mathcal{L}}_{2k+2}} J_{n,\boldsymbol{\ell}}^{(2)}$. Then, we prove the following: as $n \to \infty$,

$$J_n^{(1)} \to \mu(A), \tag{44}$$

$$J_{n,\boldsymbol{\ell}}^{(2)} \to \nu_{\boldsymbol{\ell}}(A) \ \text{ for every } \boldsymbol{\ell} \in \widetilde{\mathcal{L}}_{2k+2}, \tag{45}$$

$$J_n^{(3)} \to 0. \tag{46}$$

To handle $J_n^{(1)}$, changing the variables $z_1 \leftrightarrow x$, $z_i \leftrightarrow x + y_{i-1}$, $i = 2, \dots, 2k+2$ yields

$$\begin{aligned} J_n^{(1)} = \frac{n^{2k+2}}{(2k+2)!} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{2k+1}} \mathbb{P}'\Big\{ \sum_{\ell=0}^{q} h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \\ \times \mathbf{1}\Big\{ \min_{j=1,\dots,2k+2} \|A_\ell(x + y_{j-1})\| \ge R_{k,n} \Big\} \neq 0, \\ \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}}(x, x+\mathbf{y}) = m \Big\} \\ \times f(x) \prod_{i=1}^{2k+1} f(x+y_i) d\mathbf{y} dx, \end{aligned}$$

where $y_0 \equiv 0$, and we have applied

$$\begin{aligned} &\sum_{\ell=0}^{q} I_{(\ell,\dots,\ell)}(x, x+\mathbf{y}) \\ &= \sum_{\ell=0}^{q} h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \mathbf{1}\Big\{ \min_{j=1,\dots,2k+2} \|A_\ell(x+y_{j-1})\| \ge R_{k,n} \Big\}, \end{aligned} \tag{47}$$

which is derived from the translation invariance (12).

Subsequently, the polar coordinate transform $x \leftrightarrow (r, \theta)$ with $J(\theta) = |\partial x/\partial\theta|$ and another change of variable $\rho = r/R_{k,n}$ yield

$$\begin{aligned} J_n^{(1)} = \frac{n^{2k+2}}{(2k+2)!} R_{k,n}^d f(R_{k,n}e_1)^{2k+2} \int_0^\infty d\rho \int_{S_{d-1}} J(\theta) d\theta \int_{(\mathbb{R}^d)^{2k+1}} d\mathbf{y} \rho^{d-1} \\ \times \mathbb{P}'\Big\{ \sum_{\ell=0}^{q} h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \\ \times \mathbf{1}\Big\{ \min_{j=1,\dots,2k+2} \|A_\ell(\rho\theta + y_{j-1}/R_{k,n})\| \ge 1 \Big\} \neq 0, \\ \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}}(R_{k,n}\rho\theta, R_{k,n}\rho\theta + \mathbf{y}) = m \Big\} \\ \times \frac{f(R_{k,n}\rho e_1)}{f(R_{k,n}e_1)} \prod_{i=1}^{2k+1} \frac{f\big(R_{k,n}\|\rho\theta + y_i/R_{k,n}\|e_1\big)}{f(R_{k,n}e_1)}. \end{aligned} \tag{48}$$

By (5), we see that

$$\frac{n^{2k+2}}{(2k+2)!} R_{k,n}^d f(R_{k,n}e_1)^{2k+2} \to \frac{1}{(2k+2)!}, \quad n \to \infty. \tag{49}$$

By the regular variation of $f$, we have, as $n \to \infty$,

$$\frac{f(R_{k,n}\rho e_1)}{f(R_{k,n}e_1)} \prod_{i=1}^{2k+1} \frac{f\big(R_{k,n}\|\rho\theta + y_i/R_{k,n}\|e_1\big)}{f(R_{k,n}e_1)} \to \rho^{-2\alpha(k+1)}$$

for all $\rho > 0$, $\theta \in S_{d-1}$, and $\mathbf{y} = (y_1, \dots, y_{2k+1}) \in (\mathbb{R}^d)^{2k+1}$.

In addition, it follows from (13) and (14) that for every $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{2k+2}) \in \widetilde{\mathcal{L}}_{2k+2}$ (assume, without loss of generality, $\ell_1 \neq \ell_2$),

$$\begin{aligned}
&h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(R_{k,n}\rho\theta, R_{k,n}\rho\theta+\mathbf{y})} \\
&\le \mathbf{1}\big\{\|Y_{\ell_1}^{(1)} - Y_{\ell_2}^{(2)}\| \le Mt\big\}\big|_{(X_0^{(1)},\, X_0^{(2)})=(R_{k,n}\rho\theta, R_{k,n}\rho\theta+y_1)} \\
&= \mathbf{1}\big\{\|R_{k,n}\rho(A_{\ell_1} - A_{\ell_2})\theta - A_{\ell_2}y_1 + \widetilde{Y}_{\ell_1}^{(1)} - \widetilde{Y}_{\ell_2}^{(2)}\| \le Mt\big\},
\end{aligned}$$

where $\widetilde{Y}_{\ell_1}^{(1)} = Y_{\ell_1}^{(1)} - A_{\ell_1}X_0^{(1)}$ and $\widetilde{Y}_{\ell_2}^{(2)} = Y_{\ell_2}^{(2)} - A_{\ell_2}X_0^{(2)}$. From (8) we have that $(A_{\ell_1} - A_{\ell_2})\theta \neq 0$ and hence, the last expression converges to 0 as $n \to \infty$ a.s. Thus, the probability $\mathbb{P}'$ on the right hand side of (48) converges to

$$\mathbb{P}'\Big\{\sum_{\ell=0}^{q} h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \mathbf{1}\{\rho \ge \|A_\ell\theta\|^{-1}\} = m\Big\}.$$

Therefore, the proof of (44) will be complete if one can find an integrable upper bound for the application of the dominated convergence theorem. First, the probability $\mathbb{P}'$ in (48) can be bounded by

$$\sum_{\ell=0}^{q} \mathbb{P}'\Big\{h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \neq 0\Big\},$$

and, by (13), we have

$$\int_{(\mathbb{R}^d)^{2k+1}} \mathbb{P}'\Big\{h_t(Y_\ell^{(1)}, \dots, Y_\ell^{(2k+2)})\big|_{(X_0^{(1)},\dots,X_0^{(2k+2)})=(0,\mathbf{y})} \neq 0\Big\} d\mathbf{y} < \infty$$

for all $\ell = 0, \dots, q$.

Next, we need to provide another bound for the probability $\mathbb{P}'$ in (48); namely, it can be bounded above by

$$\begin{aligned}
&\mathbf{1}\Big\{\max_{\ell=0,\dots,q}\ \min_{j=1,\dots,2k+2} \big\|A_\ell(\rho\theta + y_{j-1}/R_{k,n})\big\| \ge 1\Big\} \\
&\quad \le \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|\rho\theta + y_{j-1}/R_{k,n}\| \ge \big(\max_{\ell=0,\dots,q}\|A_\ell\|\big)^{-1}\Big\}.
\end{aligned}$$

Using this bound together with the Potter bound for regularly varying functions (e.g., Theorem 1.5.6 in [8]), we derive that, for every $\eta \in (0, \alpha - d)$, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&\mathbf{1}\Big\{\rho \ge \big(\max_{\ell=0,\dots,q}\|A_\ell\|\big)^{-1}\Big\} \frac{f(R_{k,n}\rho e_1)}{f(R_{k,n}e_1)} \\
&\le C_1\mathbf{1}\Big\{\rho \ge \big(\max_{\ell=0,\dots,q}\|A_\ell\|\big)^{-1}\Big\}(\rho^{-(\alpha+\eta)} + \rho^{-(\alpha-\eta)})
\end{aligned} \tag{50}$$

and

$$\prod_{i=1}^{2k+1} \mathbf{1}\Big\{\|\rho\theta + y_i/R_{k,n}\| \ge \big(\max_{\ell=0,\dots,q}\|A_\ell\|\big)^{-1}\Big\} \frac{f\big(R_{k,n}\|\rho\theta + y_i/R_{k,n}\|e_1\big)}{f(R_{k,n}e_1)} \le C_2. \tag{51}$$

Here, we have introduced specific constants $C_i$ (not a generic one) for later use. Since

$$\int_{\left(\max_{\ell=0,\dots,q}\|A_\ell\|\right)^{-1}}^{\infty} (\rho^{d-1-(\alpha+\eta)} + \rho^{d-1-(\alpha-\eta)})d\rho < \infty,$$

the dominated convergence theorem justifies the convergence under the integral sign and (44) has been proven.

We now turn to proving (45). The argument basically proceeds in the same manner as in (44). Fix $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{2k+2}) \in \widetilde{\mathcal{L}}_{2k+2}$. Changing the variables $z_1 \leftrightarrow A_{\ell_1}^{-1}x$, $z_i \leftrightarrow A_{\ell_i}^{-1}(x + y_{i-1})$, $i = 2, \dots, 2k+2$ and using the translation invariance (12) as in (47), we obtain

$$\begin{aligned} J_{n,\boldsymbol{\ell}}^{(2)} &= \Big(\prod_{j=1}^{2k+2} \det A_{\ell_j}\Big)^{-1} \frac{n^{2k+2}}{(2k+2)!} \\ &\times \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{2k+1}} \mathbb{P}'\Big\{ h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)})\big|_{X_0^{(i)}=A_{\ell_i}^{-1}y_{i-1},\, i=1,\dots,2k+2} \\ &\qquad \times \mathbf{1}\Big\{ \min_{j=1,\dots,2k+2} \|x + y_{j-1}\| \geq R_{k,n} \Big\} \neq 0, \\ &\qquad \sum_{\boldsymbol{\ell}' \in \mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}'}\big(A_{\ell_1}^{-1}x, A_{\ell_2}^{-1}(x+y_1), \dots, A_{\ell_{2k+2}}^{-1}(x+y_{2k+1})\big) = m \Big\} \\ &\qquad \times f(A_{\ell_1}^{-1}x) \prod_{i=2}^{2k+2} f\big(A_{\ell_i}^{-1}(x+y_{i-1})\big) d\mathbf{y}dx \end{aligned}$$

(with $y_0 \equiv 0$).

Next, the polar coordinate transform $x \leftrightarrow (r, \theta)$ followed by the change of variable $\rho = r/R_{k,n}$ gives

$$\begin{aligned} J_{n,\boldsymbol{\ell}}^{(2)} &= \Big(\prod_{j=1}^{2k+2} \det A_{\ell_j}\Big)^{-1} \frac{n^{2k+2}}{(2k+2)!} R_{k,n}^d f(R_{k,n}e_1)^{2k+2} \qquad (52) \\ &\times \int_0^\infty d\rho \int_{S_{d-1}} J(\theta)d\theta \int_{(\mathbb{R}^d)^{2k+1}} d\mathbf{y}\rho^{d-1} \\ &\times \mathbb{P}'\Big\{ h_t(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{2k+2}}^{(2k+2)})\big|_{X_0^{(i)}=A_{\ell_i}^{-1}y_{i-1},\, i=1,\dots,2k+2} \neq 0, \\ &\quad \sum_{\boldsymbol{\ell}' \in \mathcal{L}_{2k+2}} I_{\boldsymbol{\ell}'}\big(A_{\ell_1}^{-1}R_{k,n}\rho\theta, A_{\ell_2}^{-1}(R_{k,n}\rho\theta + y_1), \dots, A_{\ell_{2k+2}}^{-1}(R_{k,n}\rho\theta + y_{2k+1})\big) = m \Big\} \\ &\qquad \times \mathbf{1}\Big\{ \min_{j=1,\dots,2k+2} \|\rho\theta + y_{j-1}/R_{k,n}\| \geq 1 \Big\} \\ &\qquad \times \frac{f\big(R_{k,n}\|A_{\ell_1}^{-1}\rho\theta\|e_1\big)}{f(R_{k,n}e_1)} \prod_{i=2}^{2k+2} \frac{f\big(R_{k,n}\|A_{\ell_i}^{-1}(\rho\theta + y_{i-1}/R_{k,n})\|e_1\big)}{f(R_{k,n}e_1)}. \end{aligned}$$

By the regular variation of $f$, as $n \to \infty$,

$$\begin{aligned} &\frac{f\big(R_{k,n}\|A_{\ell_1}^{-1}\rho\theta\|e_1\big)}{f(R_{k,n}e_1)} \prod_{i=2}^{2k+2} \frac{f\big(R_{k,n}\|A_{\ell_i}^{-1}(\rho\theta + y_{i-1}/R_{k,n})\|e_1\big)}{f(R_{k,n}e_1)} \\ &\quad \to \rho^{-2\alpha(k+1)} \prod_{j=1}^{2k+2} \|A_{\ell_j}^{-1}\theta\|^{-\alpha}. \end{aligned}$$

Observe that for every $\boldsymbol{\ell}' = (\ell'_1, \dots, \ell'_{2k+2}) \in \mathcal{L}_{2k+2}$ with $\boldsymbol{\ell}' \neq \boldsymbol{\ell}$,

$$h_t(Y^{(1)}_{\ell'_1}, \dots, Y^{(2k+2)}_{\ell'_{2k+2}})\big|_{X_0^{(i)} = A_{\ell_i}^{-1}(R_{k,n}\rho\theta + y_{i-1}),\, i=1,\dots,2k+2} \to 0 \ \text{ as } n \to \infty, \ \text{a.s.}$$

Hence, the probability $\mathbb{P}'$ in (52) converges to

$$\mathbb{P}'\big\{ h_t(Y^{(1)}_{\ell_1}, \dots, Y^{(2k+2)}_{\ell_{2k+2}})\big|_{X_0^{(i)} = A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,2k+2} = m \big\}.$$

Putting these results together, while using (49) and assuming that all convergences take place under the integral sign, we can obtain (45).

It now remains to find an integrable upper bound. Note first that (13) implies

$$\int_{(\mathbb{R}^d)^{2k+1}} \mathbb{P}'\big\{ h_t(Y^{(1)}_{\ell_1}, \dots, Y^{(2k+2)}_{\ell_{2k+2}})\big|_{X_0^{(i)} = A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,2k+2} \neq 0 \big\} d\mathbf{y} < \infty.$$

By virtue of (9), on the set $\big\{\min_{j=1,\dots,2k+2} \|\rho\theta + y_{j-1}/R_{k,n}\| \ge 1 \big\}$, we have

$$\begin{aligned} \|A_{\ell_1}^{-1}\rho\theta\| &\ge c\|A_{\ell_1}^{-1}\|(>0), \\ \big\|A_{\ell_j}^{-1}(\rho\theta + y_{j-1}/R_{k,n})\big\| &\ge c\|A_{\ell_j}^{-1}\|(>0), \quad j = 2, \dots, 2k+2, \end{aligned}$$

for all $\rho > 0$, $\theta \in S_{d-1}$ and $\mathbf{y} = (y_1, \dots, y_{2k+1}) \in (\mathbb{R}^d)^{2k+1}$, from which Potter's bounds are applicable once again, and the required integrable bound can be established as before.

Finally, the proof of (46) is mostly parallel to those of (44) and (45), and therefore, we omit it. Now, we can conclude (41).

Subsequently we turn our attention to verifying (42). First we write

$$\xi_n(m) = \sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} \eta_{\mathbf{i},n}, \quad m \ge 1,$$

with

$$\eta_{\mathbf{i},n} := \mathbf{1}\Big\{ \sum_{\boldsymbol{\ell} \in \mathcal{L}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}+\boldsymbol{\ell}})\, \mathbf{1}\big\{ \min_{j=1,\dots,2k+2} \|A_{\ell_j} X_{i_j}\| \ge R_{k,n} \big\} = m \Big\}.$$

Our argument needs the *total variation distance*, which is defined for real-valued random variables $Y_1, Y_2$ on the same probability space:

$$d_{\mathrm{TV}}(Y_1, Y_2) := \sup_{A \subset \mathbb{R}} \big| \mathbb{P}\{Y_1 \in A\} - \mathbb{P}\{Y_2 \in A\} \big|.$$

Then, we see that

$$\begin{aligned} &\big| \mathbb{P}\{\xi_n(m) = 0\} - \mathbb{P}\{\zeta(m) = 0\} \big| \\ &\le d_{\mathrm{TV}}\Big(\xi_n(m),\ \mathrm{Poi}\big(\mathbb{E}\{\xi_n(m)\}\big)\Big) + \Big| \mathbb{P}\big\{ \mathrm{Poi}\big(\mathbb{E}\{\xi_n(m)\}\big) = 0 \big\} - \mathbb{P}\big\{\zeta(m) = 0\big\} \Big|, \end{aligned} \tag{53}$$

where $\mathrm{Poi}(a)$ is a Poisson random variable with mean parameter $a$. Since $\mathrm{Poi}\big(\mathbb{E}\{\xi_n(m)\}\big)$ and $\zeta(m)$ are both Poisson random variables, an elementary calculation shows that

$$\Big| \mathbb{P}\big\{ \mathrm{Poi}\big(\mathbb{E}\{\xi_n(m)\}\big) = 0 \big\} - \mathbb{P}\big\{\zeta(m) = 0\big\} \Big| \le \big| \mathbb{E}\{\xi_n(m)\} - \mathbb{E}\{\zeta(m)\} \big| \to 0,$$

where the last convergence is due to (41). To handle the first term in (53), we use the so-called *Stein's method for Poisson approximation* (see Theorem 2.1 in [34]). To fulfill this aim, note first that $\eta_{\mathbf{i},n}$ is a Bernoulli random variable. For $\mathbf{i} = (i_1, \dots, i_{2k+2}) \in \widetilde{\mathcal{I}}_{2k+2}$ and

$\mathbf{j} = (j_1, \dots, j_{2k+2}) \in \widetilde{\mathcal{I}}_{2k+2}$, write $\mathbf{i} \sim \mathbf{j}$ if and only if they are "close" to one another in the sense of

$$\big\{(i_1 + r_1, \dots, i_{2k+2} + r_{2k+2}) : |r_p| \le q,\ p = 1, \dots, 2k+2\big\} \\ \cap \big\{(j_1 + r_1, \dots, j_{2k+2} + r_{2k+2}) : |r_p| \le q,\ p = 1, \dots, 2k+2\big\} \ne \emptyset.$$

Recalling that all elements in $\widetilde{\mathcal{I}}_{2k+2}$ are separate from one another by at least $2q$, we find that $(\widetilde{\mathcal{I}}_{2k+2}, \sim)$ becomes a *dependency graph* with respect to $(\eta_{\mathbf{i},n},\ \mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2})$. That is, for all $I_1, I_2 \subset \widetilde{\mathcal{I}}_{2k+2}$ with no edges connecting $I_1$ and $I_2$, we have that $(\eta_{\mathbf{i},n},\ \mathbf{i} \in I_1)$ and $(\eta_{\mathbf{i},n},\ \mathbf{i} \in I_2)$ are independent. Therefore, Stein's method for Poisson approximation yields

$$d_{\mathrm{TV}}\Big(\xi_n(m),\ \mathrm{Poi}\big(\mathbb{E}\{\xi_n(m)\}\big)\Big) \\ \le 3\Big(\sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} \sum_{j \in N_{\mathbf{i}}} \mathbb{E}\{\eta_{\mathbf{i},n}\}\mathbb{E}\{\eta_{\mathbf{j},n}\} + \sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} \sum_{\mathbf{j} \in N_{\mathbf{i}} \setminus \{\mathbf{i}\}} \mathbb{E}\{\eta_{\mathbf{i},n}\eta_{\mathbf{j},n}\}\Big), \tag{54}$$

where $N_{\mathbf{i}} = \{j \in \widetilde{\mathcal{I}}_{2k+2} : \mathbf{i} \sim \mathbf{j}\} \cup \{\mathbf{i}\}$.

For the first term on the right hand side of (54), we see that (41) implies, for each $\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}$,

$$\mathbb{E}\{\eta_{\mathbf{i},n}\} \sim \binom{n}{2k+2}^{-1} \mathbb{E}\big\{\zeta(m)\big\}, \quad n \to \infty.$$

Furthermore,

$$\sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} \sum_{j \in N_{\mathbf{i}}} 1 = o\bigg(\binom{n}{2k+2}^2\bigg),$$

and thus, the first term in (54) goes to 0 as $n \to \infty$. Proceeding as in the derivation of (41), we have, for all $\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}$ and $\mathbf{j} \in N_{\mathbf{i}} \setminus \{\mathbf{i}\}$ with $p := |\mathbf{i} \cap \mathbf{j}| \in \{0, \dots, 2k+1\}$,

$$\mathbb{E}\{\eta_{\mathbf{i},n}\eta_{\mathbf{j},n}\} \sim C^* R_{k,n}^d f(R_{k,n}e_1)^{4k+4-p} \ \text{ as } n \to \infty.$$

It follows from (31) and (5) that

$$\sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}} \sum_{\mathbf{j} \in N_{\mathbf{i}} \setminus \{\mathbf{i}\}} \mathbb{E}\{\eta_{\mathbf{i},n}\eta_{\mathbf{j},n}\} \\ \sim C^* \sum_{p=0}^{2k+1} \binom{n}{2k+2}\binom{2k+2}{p}\binom{n-(2k+2)}{2k+2-p} \mathbb{E}\{\eta_{\mathbf{i},n}\eta_{\mathbf{j},n}\}\mathbf{1}\{|\mathbf{i} \cap \mathbf{j}| = p\} \\ \sim C^* \sum_{p=0}^{2k+1} n^{4k+4-p} R_{k,n}^d f(R_{k,n}e_1)^{4k+4-p} \\ \sim C^* \sum_{p=0}^{2k+1} \big(nf(R_{k,n}e_1)\big)^{2k+2-p} \to 0.$$

We now obtain (42), and accordingly, we also obtain (37) as desired.

*Part II*: For the tightness of

$$G_n(t) = \sum_{\mathbf{i} \in \mathcal{I}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\},$$

we need to check a sufficient condition given in Theorem 13.5 of [7]. For a fixed $L > 0$, we show that there exists $B > 0$, for which

$$\mathbb{P}\Big\{ \min\big\{ \big|G_n(t) - G_n(s)\big|, \ \big|G_n(s) - G_n(r)\big| \big\} \geq \lambda \Big\} \leq \frac{B}{\lambda^2} (t-r)^2$$

for all $0 \leq r < s < t \leq L$ and $\lambda > 0$.

By Markov's inequality, we have to show only that

$$\mathbb{E}\Big\{ \big|G_n(t) - G_n(s)\big| \big|G_n(s) - G_n(r)\big| \Big\} \leq B(t-r)^2 \tag{55}$$

for all $0 \leq r < s < t \leq L$. In the below, the following functions are used. Define

$$h_{t,s}(\mathbf{x}) := h_t(\mathbf{x}) - h_s(\mathbf{x}), \ 0 \leq s \leq t, \ \mathbf{x} = (x_1, \dots, x_{2k+2}) \in (\mathbb{R}^d)^{2k+2}$$

and define, for $0 \leq m \leq 2k+2$ and $\mathbf{x} = (x_1, \dots, x_{4k+4-m}) \in (\mathbb{R}^d)^{4k+4-m}$,

$$h^{(m)}_{t,s,r}(\mathbf{x}) := h_{t,s}(x_1, \dots, x_{2k+2}) h_{s,r}(x_1, \dots, x_m, x_{2k+3}, \dots, x_{4k+4-m}) \tag{56}$$

for $0 \leq r < s < t$. In particular, we set

$$h_{s,r}(x_1, \dots, x_m, x_{2k+3}, \dots, x_{4k+4-m}) := \begin{cases} h_{s,r}(x_{2k+3}, \dots, x_{4k+4}) & \text{if } m = 0 \\ h_{s,r}(x_1, \dots, x_{2k+2}) & \text{if } m = 2k+2. \end{cases}$$

Using these functions, the left hand side in (55) is bounded by

$$\sum_{m=0}^{2k+2} \mathbb{E}\Big\{ \sum_{\mathbf{i} \in \mathcal{I}_{4k+4-m}} \big| h^{(m)}_{t,s,r}(\mathcal{Y}_{\mathbf{i}}) \big| \mathbf{1}\big\{ \min_{j=1,\dots,4k+4-m} \|Y_{i_j}\| \geq R_{k,n} \big\} \Big\}, \tag{57}$$

where $\mathcal{Y}_{\mathbf{i}} = (Y_{i_1}, \dots, Y_{i_{4k+4-m}})$ for $\mathbf{i} = (i_1, \dots, i_{4k+4-m}) \in \mathcal{I}_{4k+4-m}$. Observe that

$$\begin{aligned} &\mathbf{1}\big\{ \min_{j=1,\dots,4k+4-m} \|Y_{i_j}\| \geq R_{k,n} \big\} \\ &\quad \leq \sum_{\boldsymbol{\ell} \in \mathcal{L}_{4k+4-m}} \mathbf{1}\big\{ \min_{j=1,\dots,4k+4-m} \|A_{\ell_j} X_{i_j - \ell_j}\| \geq R_{k,n}/(q+1) \big\}, \end{aligned} \tag{58}$$

and, because of (32), $\mathcal{I}_{4k+4-m}$ in (57) can be asymptotically replaced with $\widetilde{\mathcal{I}}_{4k+4-m}$. Therefore, (57) is further bounded by a constant multiple of

$$\begin{aligned} &\sum_{m=0}^{2k+2} \sum_{\boldsymbol{\ell} \in \mathcal{L}_{4k+4-m}} \mathbb{E}\Big\{ \sum_{\mathbf{i} \in \widetilde{\mathcal{I}}_{4k+4-m}} \big| h^{(m)}_{t,s,r}(\mathcal{Y}_{\mathbf{i}}) \big| \mathbf{1}\big\{ \min_{j=1,\dots,4k+4-m} \|A_{\ell_j} X_{i_j - \ell_j}\| \geq R_{k,n}/(q+1) \big\} \Big\} \\ &:= \sum_{m=0}^{2k+2} \sum_{\boldsymbol{\ell} \in \mathcal{L}_{4k+4-m}} K_{\boldsymbol{\ell},m}. \end{aligned}$$

We now bound $K_{\boldsymbol{\ell},m}$ for every $1 \leq m \leq 2k+2$ and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{4k+4-m}) \in \mathcal{L}_{4k+4-m}$. In the case of $m = 0$, the task of bounding $K_{\boldsymbol{\ell},0}$ is even easier, because $h_{t,s}$ and $h_{s,r}$ in (56) do not share the same elements. Thus, in the following, we restrict our consideration to the case $1 \leq m \leq 2k+2$ and $\boldsymbol{\ell} \in \mathcal{L}_{4k+4-m}$. Then, the same calculation as that for (45), i.e., the same change of variables as well as applications of the Potter bound, gives

$$\begin{aligned} K_{\boldsymbol{\ell},m} &\leq C^* n^{4k+4-m} R^d_{k,n} f(R_{k,n} e_1)^{4k+4-m} \\ &\quad \times \int_{(\mathbb{R}^d)^{4k+3-m}} \mathbb{E}' \Big\{ \big| h^{(m)}_{t,s,r}(Y^{(1)}_{\ell_1}, \dots, Y^{(4k+4-m)}_{\ell_{4k+4-m}}) \big| \Big|_{X^{(i)}_0 = A^{-1}_{\ell_i} y_{i-1}, \, i=1,\dots,4k+4-m} \Big\} d\mathbf{y} \end{aligned} \tag{59}$$

$$\leq C^* \int_{\Omega'} \int_{(\mathbb{R}^d)^{4k+3-m}} \big| h_{t,s,r}^{(m)}(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{4k+4-m}}^{(4k+4-m)}) \big| \Big|_{X_0^{(i)}=A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,4k+4-m} d\mathbf{y} d\mathbb{P}'.$$

In order to bound the rightmost term, one needs some properties on $h_t$. First, for $\mathbf{x} = (x_1, \dots, x_{2k+2}) \in (\mathbb{R}^d)^{2k+2}$,

$$h_t(\mathbf{x}) \leq a_t(\mathbf{x}) := \mathbf{1}\big\{\mathcal{R}_t(\mathbf{x}) \text{ is connected}\big\}.$$

In particular, $a_t$ is non-decreasing in the sense that $a_s(\mathbf{x}) \leq a_t(\mathbf{x})$ for all $0 \leq s < t$ and $\mathbf{x} \in (\mathbb{R}^d)^{2k+2}$. Thus, we have, for $\mathbf{x} \in (\mathbb{R}^d)^{2k+2}$,

$$\begin{aligned}\big|h_{t,s}(\mathbf{x})\big| &= \mathbf{1}\Big\{\big\{h_t(\mathbf{x}) = 1,\ h_s(\mathbf{x}) = 0\big\} \cup \big\{h_t(\mathbf{x}) = 0,\ h_s(\mathbf{x}) = 1\big\}\Big\} \\ &\leq a_L(\mathbf{x})\, \mathbf{1}\big\{s < \|x_i - x_j\| \leq t \text{ for some } i, j \in \{1, \dots, 2k+2\}\big\}.\end{aligned}$$

This is because, whenever the values of $h_s(\mathbf{x})$ and $h_t(\mathbf{x})$ are different, there always exist two points $(x_i, x_j)$ such that the Euclidean distance between them is greater than $s$ and less than $t$.

Finally, similarly to (56), we define an augmented version of $a_t$ by setting, for $1 \leq m \leq 2k+2$ and $\mathbf{x} = (x_1, \dots, x_{4k+4-m}) \in (\mathbb{R}^d)^{4k+4-m}$,

$$a_t^{(m)}(\mathbf{x}) := a_t(x_1, \dots, x_{2k+2})\, a_t(x_1, \dots, x_m, x_{2k+3}, \dots, x_{4k+4-m}).$$

Returning to the rightmost term in (59), we now have that

$$\begin{aligned}&\big|h_{t,s,r}^{(m)}(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{4k+4-m}}^{(4k+4-m)})\big| \\ &\quad\leq a_L^{(m)}(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{4k+4-m}}^{(4k+4-m)}) \sum_{p_1>p_2} \sum_{q_1>q_2} \mathbf{1}\big\{s < \|Y_{\ell_{p_1}}^{(p_1)} - Y_{\ell_{p_2}}^{(p_2)}\| \leq t\big\} \\ &\qquad\qquad \times \mathbf{1}\big\{r < \|Y_{\ell_{q_1}}^{(q_1)} - Y_{\ell_{q_2}}^{(q_2)}\| \leq s\big\},\end{aligned}$$

where $p_i, i = 1, 2$ ranges over $\{1, \dots, 2k+2\}$ and $q_i, i = 1, 2$ ranges over $\{1, \dots, m\} \cup \{2k+3, \dots, 4k+4-m\}$. It therefore remains to show that

$$\begin{aligned}&\int_{\Omega'} \int_{(\mathbb{R}^d)^{4k+3-m}} a_L^{(m)}(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{4k+4-m}}^{(4k+4-m)})\, \mathbf{1}\big\{s < \|Y_{\ell_{p_1}}^{(p_1)} - Y_{\ell_{p_2}}^{(p_2)}\| \leq t\big\} \\ &\qquad \times \mathbf{1}\big\{r < \|Y_{\ell_{q_1}}^{(q_1)} - Y_{\ell_{q_2}}^{(q_2)}\| \leq s\big\}\Big|_{X_0^{(i)}=A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,4k+4-m} d\mathbf{y} d\mathbb{P}' \\ &\qquad \leq B(t-r)^2\end{aligned}$$

for all $1 \leq m \leq 2k+2$, $p_i \in \{1, \dots, 2k+2\}$, and $q_i \in \{1, \dots, m\} \cup \{2k+3, \dots, 4k+4-m\}$ with $p_1 > p_2, q_1 > q_2$.

If $p_1 = q_1$ and $p_2 = q_2$, the product of the two indicator functions is identically zero. Thus, we may consider only the case $(p_1, p_2) \neq (q_1, q_2)$. Assuming, without loss of generality, $p_1 \neq q_1$, we consider the integral

$$\begin{aligned}&\int_{(\mathbb{R}^d)^2} a_L^{(m)}(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_{4k+4-m}}^{(4k+4-m)})\, \mathbf{1}\big\{s < \|Y_{\ell_{p_1}}^{(p_1)} - Y_{\ell_{p_2}}^{(p_2)}\| \leq t\big\} \\ &\qquad \times \mathbf{1}\big\{r < \|Y_{\ell_{q_1}}^{(q_1)} - Y_{\ell_{q_2}}^{(q_2)}\| \leq s\big\}\Big|_{X_0^{(i)}=A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,4k+4-m} dy_{p_1-1} dy_{q_1-1}.\end{aligned} \tag{60}$$

Note that

$$\begin{aligned}\|Y_{\ell_{p_1}}^{(p_1)} - Y_{\ell_{p_2}}^{(p_2)}\|\big|_{X_0^{(i)}=A_{\ell_i} y_{i-1},\, i=p_1,p_2} &= \|y_{p_1-1} - y_{p_2-1} + \widetilde{Y}_{\ell_{p_1}}^{(p_1)} - \widetilde{Y}_{\ell_{p_2}}^{(p_2)}\| \\ \|Y_{\ell_{q_1}}^{(q_1)} - Y_{\ell_{q_2}}^{(q_2)}\|\big|_{X_0^{(i)}=A_{\ell_i} y_{i-1},\, i=q_1,q_2} &= \|y_{q_1-1} - y_{q_2-1} + \widetilde{Y}_{\ell_{q_1}}^{(q_1)} - \widetilde{Y}_{\ell_{q_2}}^{(q_2)}\|\end{aligned}$$

($y_{p_2-1}$ and $y_{q_2-1}$ can be zero), where $\widetilde{Y}^{(p_i)}_{\ell_{p_i}} = Y^{(p_i)}_{\ell_{p_i}} - A_{\ell_{p_i}} X^{(p_i)}_0$ and $\widetilde{Y}^{(q_i)}_{\ell_{q_i}} = Y^{(q_i)}_{\ell_{q_i}} - A_{\ell_{q_i}} X^{(q_i)}_0$ for $i = 1, 2$.

By changing the variables $z_1 = y_{p_1-1} - y_{p_2-1} + \widetilde{Y}^{(p_1)}_{\ell_{p_1}} - \widetilde{Y}^{(p_2)}_{\ell_{p_2}}$ and $z_2 = y_{q_1-1} - y_{q_2-1} + \widetilde{Y}^{(q_1)}_{\ell_{q_1}} - \widetilde{Y}^{(q_2)}_{\ell_{q_2}}$, it turns out that the integral (60) has an upper bound

$$\begin{aligned}
&\sup_{\|z_i\|\le L, i=1,2} a^{(m)}_L\big(Y^{(1)}_{\ell_1}, \dots, Y^{(4k+4-m)}_{\ell_{4k+4-m}}\big) \int_{(\mathbb{R}^d)^2} \mathbf{1}\big\{s < \|z_1\| \le t,\ r < \|z_2\| \le s\big\} dz_1 dz_2 \\
&\quad \le C^* \sup_{\|z_i\|\le L, i=1,2} a^{(m)}_L\big(Y^{(1)}_{\ell_1}, \dots, Y^{(4k+4-m)}_{\ell_{4k+4-m}}\big)(t-r)^2,
\end{aligned}$$

for which we have substituted

$$X^{(i)}_0 = A^{-1}_{\ell_i} y_{i-1},\ i = 1, \dots, 4k+4-m,\ i \ne p_1, q_1, \tag{61}$$

$$X^{(p_1)}_0 = A^{-1}_{\ell_{p_1}}\big(z_1 + y_{p_2-1} - \widetilde{Y}^{(p_1)}_{\ell_{p_1}} + \widetilde{Y}^{(p_2)}_{\ell_{p_2}}\big), \tag{62}$$

$$X^{(q_1)}_0 = A^{-1}_{\ell_{q_1}}\big(z_2 + y_{q_2-1} - \widetilde{Y}^{(q_1)}_{\ell_{q_1}} + \widetilde{Y}^{(q_2)}_{\ell_{q_2}}\big). \tag{63}$$

Writing

$$g(\mathbf{y} \setminus \{y_{p_1-1}, y_{q_1-1}\}, z_1, z_2; \omega') := a^{(m)}_L\big(Y^{(1)}_{\ell_1}, \dots, Y^{(4k+4-m)}_{\ell_{4k+4-m}}\big), \quad \mathbf{y} \in (\mathbb{R}^d)^{4k+3-m},$$

with (61), (62), and (63) all substituted on the right hand side, we need to show that

$$\begin{aligned}
&\int_{\Omega'} \int_{(\mathbb{R}^d)^{4k+1-m}} \sup_{\|z_i\|\le L, i=1,2} g(\mathbf{y} \setminus \{y_{p_1-1}, y_{q_1-1}\}, z_1, z_2; \omega') \\
&\quad \times d\big(\mathbf{y} \setminus \{y_{p_1-1}, y_{q_1-1}\}\big) < \infty.
\end{aligned} \tag{64}$$

Because of (13) and (14), we can see that

$$\begin{aligned}
&\sup_{\|z_i\|\le L, i=1,2} g(\mathbf{y} \setminus \{y_{p_1-1}, y_{q_1-1}\}, z_1, z_2; \omega') \\
&\quad \le \prod_{i=2\ i\ne p_1,q_1}^{4k+4-m} \mathbf{1}\big\{\, \|Y^{(i)}_{\ell_i} - Y^{(1)}_{\ell_1}\| \le ML\big\}\Big|_{X^{(i)}_0 = A^{-1}_{\ell_i} y_{i-1},\, X^{(1)}_0 = 0} \\
&\quad = \prod_{i=2,\, i\ne p_1,q_1}^{4k+4-m} \mathbf{1}\big\{\, \|y_{i-1} + \widetilde{Y}^{(i)}_{\ell_i} - \widetilde{Y}^{(1)}_{\ell_1}\| \le ML\big\}.
\end{aligned}$$

This in turn implies that the integral (64) is bounded by $\lambda_d\big(B(0, ML)\big)^{4k+1-m} < \infty$, and thus, the proof for the tightness is complete.

*Part III*: The arguments in Part I and Part II have proven

$$G_n(t) = \sum_{\mathbf{i}\in\mathcal{I}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\} \Rightarrow V_k(t) \ \text{ in } D[0,\infty).$$

Denote $\mathcal{Y}_n = (Y_1, \dots, Y_n)$ and, for $\mathbf{i} = (i_1, \dots, i_{2k+2}) \in \mathcal{I}_{2k+2}$, define

$$g_t(\mathcal{Y}_{\mathbf{i}}, \mathcal{Y}_n) := h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\big\{\mathcal{R}_t(\mathcal{Y}_{\mathbf{i}}) \text{ is an isolated component of } \mathcal{R}_t(\mathcal{Y}_n)\big\}.$$

We claim that

$$\widetilde{G}_n(t) := \sum_{\mathbf{i}\in\mathcal{I}_{2k+2}} g_t(\mathcal{Y}_{\mathbf{i}}, \mathcal{Y}_n)\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\}$$

has the same weak limit as $G_n(t)$ does. Equivalently, we claim that

$$G_n(t) - \widetilde{G}_n(t) \stackrel{p}{\to} 0 \ \text{ in } D[0,\infty). \tag{65}$$

Since $a_t$ is non-decreasing in $t$, we have, for every $L > 0$,

$$\begin{aligned}
&\sup_{0\le t\le L} \big(G_n(t) - \widetilde{G}_n(t)\big) \\
&\le \sum_{\mathbf{i}\in\mathcal{I}_{2k+2}} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected but is not an isolated component of } \mathcal{R}_L(\mathcal{Y}_n)\big\} \\
&\qquad\qquad \times \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|Y_{i_j}\| \ge R_{k,n}\Big\}.
\end{aligned}$$

Since an isolated component on $p$ ($\ge 2k+3$) points may have at most $\binom{p}{2k+2}$ non-trivial $k$-cycles, it follows from (58) that

$$\begin{aligned}
&\sup_{0\le t\le L} \big(G_n(t) - \widetilde{G}_n(t)\big) \\
&\le \sum_{p=2k+3}^{\infty} \binom{p}{2k+2} \sum_{\mathbf{i}\in\mathcal{I}_p} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\}\, \mathbf{1}\Big\{\min_{j=1,\dots,p} \|Y_{i_j}\| \ge R_{k,n}\Big\} \\
&\le \sum_{p=2k+3}^{\infty} \binom{p}{2k+2} \sum_{\boldsymbol{\ell}\in\mathcal{L}_p} \sum_{\mathbf{i}\in\mathcal{I}_p} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\} \\
&\qquad\qquad \times \mathbf{1}\Big\{\min_{j=1,\dots,p} \|A_{\ell_j} X_{i_j-\ell_j}\| \ge R_{k,n}/(q+1)\Big\}.
\end{aligned}$$

From (32), we show only that, as $n \to \infty$,

$$\begin{aligned}
A_n := \sum_{p=2k+3}^{\infty} \binom{p}{2k+2} (q+1)^p \sup_{\boldsymbol{\ell}\in\mathcal{L}_p} \mathbb{E}\Bigg\{ &\sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_p} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\} \\
&\times \mathbf{1}\Big\{\min_{j=1,\dots,p} \|A_{\ell_j} X_{i_j-\ell_j}\| \ge R_{k,n}/(q+1)\Big\}\Bigg\} \to 0.
\end{aligned}$$

Repeating the same calculation as that for (45),

$$\begin{aligned}
&\mathbb{E}\Bigg\{ \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_p} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\}\, \mathbf{1}\Big\{\min_{j=1,\dots,p} \|A_{\ell_j} X_{i_j-\ell_j}\| \ge R_{k,n}/(q+1)\Big\}\Bigg\} \qquad (66) \\
&\le C^* C^p \frac{n^p}{p!} R_{k,n}^d f(R_{k,n} e_1)^p \\
&\quad \times \int_{\Omega'} \int_{(\mathbb{R}^d)^{p-1}} \mathbf{1}\big\{\mathcal{R}_L(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_p}^{(p)}) \text{ is connected}\big\}\Big|_{X_0^{(i)} = A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,p} d\mathbf{y}\, d\mathbb{P}',
\end{aligned}$$

where $C = \max\{C_1, C_2\}$ (see (50) and (51)). Exploiting the well-known fact that there exist $p^{p-2}$ spanning trees on a set of $p$ vertices, we have

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{p-1}} \mathbf{1}\big\{\mathcal{R}_L(Y_{\ell_1}^{(1)}, \dots, Y_{\ell_p}^{(p)}) \text{ is connected}\big\}\Big|_{X_0^{(i)} = A_{\ell_i}^{-1} y_{i-1},\, i=1,\dots,p} d\mathbf{y} \\
&\qquad \le p^{p-2} \lambda_d\big(B(0,L)\big)^{p-1}, \quad \mathbb{P}'\text{-a.s.}
\end{aligned}$$

Substituting this result back into (66), while applying (5) and Stirling's formula, i.e., $p! \geq (p/e)^p$ for sufficiently large $p$, we get

$$
\begin{aligned}
A_n &\leq C^* \sum_{p=2k+3}^{\infty} \binom{p}{2k+2} (q+1)^p \, C^p \frac{n^p}{p!} \, R_{k,n}^d f(R_{k,n}e_1)^p p^{p-2} \lambda_d\big(B(0,L)\big)^{p-1} \\
&\leq C^* \sum_{p=2k+3}^{\infty} p^{2k} (q+1)^p C^p \big(nf(R_{k,n}e_1)\big)^{p-(2k+2)} e^p \lambda_d\big(B(0,L)\big)^{p-1} \\
&\leq C^* \sum_{p=1}^{\infty} (p+2k+2)^{2k} \Big(Ce(q+1)\lambda_d\big(B(0,L)\big) nf(R_{k,n}e_1)\Big)^p .
\end{aligned}
$$

The rightmost term goes to 0 as $n \to \infty$, since $nf(R_{k,n}e_1) \to 0$, $n \to \infty$, and thus, (65) is obtained.

Finally we wish to conclude

$$
\beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n})^c\big)\Big) \Rightarrow V_k(t) \ \text{ in } D[0,\infty). \tag{67}
$$

To that aim we observe that

$$
\widetilde{G}_n(t) \leq \beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n})^c\big)\Big) \leq \widetilde{G}_n(t) + L_n(t), \tag{68}
$$

where

$$
L_n(t) = \sum_{\mathbf{i} \in \mathcal{I}_{2k+3}} \mathbf{1}\big\{\mathcal{R}_t(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\} \, \mathbf{1}\big\{\min_{j=1,\dots,2k+3} \|Y_{i_j}\| \geq R_{k,n}\big\}.
$$

The inequality at the left side of (68) comes from the fact that $k$-cycles can be built not only on $2k+2$ points but also on more than $2k+2$ points. At the right side of (68), $L_n(t)$ counts $2k+3$ tuples constituting a connected graph. Then, adding the total count to $\widetilde{G}_n(t)$ exceeds the $k$th Betti number because of an overcounting by $L_n(t)$.

For every $L > 0$,

$$
\begin{aligned}
&\mathbb{E}\Big\{ \sup_{0 \leq t \leq L} \Big[ \beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n})^c\big)\Big) - \widetilde{G}_n(t) \Big] \Big\} \\
&\leq \mathbb{E}\big\{ \sup_{0 \leq t \leq L} L_n(t) \big\} \\
&\leq \mathbb{E}\Big\{ \sum_{\mathbf{i} \in \mathcal{I}_{2k+3}} \mathbf{1}\big\{\mathcal{R}_L(\mathcal{Y}_{\mathbf{i}}) \text{ is connected}\big\} \, \mathbf{1}\big\{\min_{j=1,\dots,2k+3} \|Y_{i_j}\| \geq R_{k,n}\big\} \Big\} \\
&= \mathcal{O}\Big(n^{2k+3} R_{k,n}^d f(R_{k,n}e_1)^{2k+3}\Big) \\
&= \mathcal{O}\big(nf(R_{k,n}e_1)\big) \to 0, \quad n \to \infty.
\end{aligned}
$$

This implies that

$$
\beta_k\Big(\mathcal{R}_t\big(\mathcal{Y}_n \cap B(0, R_{k,n})^c\big)\Big) - \widetilde{G}_n(t) \xrightarrow{p} 0 \ \text{ in } D[0,\infty),
$$

and thus, (67) follows.

*Part IV*: Finally, we prove (32), (33), and (34). For the proof of (32), note that the difference of the two point processes becomes another point process represented as the sum over $\mathbf{i} \in \mathcal{I}_{2k+2}$ with $i_j - i_{j-1} \leq 2q$ for at least one $j \in \{2, \dots, 2k+2\}$. We here only show that, for every

$1 \le r \le 2q$,

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \epsilon_{R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})} \xrightarrow{p} 0$$

in the space $M_p(E_k)$. Equivalently, for every non-negative continuous function $f : E_k \to \mathbb{R}_+$ with compact support,

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, f\Big(R_{k,n}^{-1}\big(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}}\big)\Big) \xrightarrow{p} 0. \tag{69}$$

More specifically we shall show that the expectation of (69) decays at the rate of

$$\mathcal{O}\big(n^{2k+1} R_{k,n}^d f(R_{k,n}e_1)^{2k+2}\big) = \mathcal{O}(n^{-1}) \to 0, \quad n \to \infty.$$

Before handling (69) we want to make a quick comment on what happens if multiple $i_j'$s are not separate enough. For example, if $i_2 = i_1 + r_1$, $i_3 = i_2 + r_2$ for some $r_1, r_2 \in \{0,\dots,2q\}$, and $i_j - i_{j-1} > 2q$ for $j = 4,\dots,2k+2$, then one can show that the expectation of

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r_1,\, i_3=i_2+r_2,\\ i_j-i_{j-1}>2q,\, j=4,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, f\Big(R_{k,n}^{-1}\big(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}}\big)\Big) \tag{70}$$

vanishes at a faster rate

$$\mathcal{O}\big(n^{2k} R_{k,n}^d f(R_{k,n}e_1)^{2k+2}\big) = \mathcal{O}(n^{-2}) \to 0, \quad n \to \infty.$$

Similarly if more and more $i_j'$s get closer to one another, the expectation of the quantities as those in (69) and (70) will decay to 0 at even faster rates. Since the necessary arguments are very similar, we only show (69).

Since $f$ has compact support on $E_k = \big([-\infty,\infty]^{d(q+1)} \setminus \{\mathbf{0}\}\big)^{2k+2}$ ($\mathbf{0}$ is the vector of zeros in $\mathbb{R}^{d(q+1)}$), there exists $\delta > 0$ such that the support of $f$, denoted by $\operatorname{supp} f$, satisfies

$$\operatorname{supp} f \subset \Big\{\mathbf{x} = (x_1^{(0)},\dots,x_1^{(q)},\dots,x_{2k+2}^{(0)},\dots,x_{2k+2}^{(q)}) \in E_k : \min_{j=1,\dots,2k+2}\ \max_{i=0,\dots,q} \|x_j^{(i)}\| \ge \delta\Big\}. \tag{71}$$

Hence, we have that

$$\begin{aligned}
&\sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, f\Big(R_{k,n}^{-1}\big(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}}\big)\Big)\\
&\le \|f\|_\infty \sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2}\ \max_{\ell_j=0,\dots,q} \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}\Big\}\\
&\le \|f\|_\infty \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}}\ \sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}\Big\},
\end{aligned}$$

where $\|f\|_\infty = \sup_{x\in E_k} |f(x)|$ is a finite and positive constant.

Now, what needs to be proven is that, for every $\boldsymbol{\ell} = (\ell_1,\dots,\ell_{2k+2}) \in \mathcal{L}_{2k+2}$,

$$\mathbb{E}\Bigg\{\sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\Big\{\min_{j=1,\dots,2k+2} \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}\Big\}\Bigg\} \to 0.$$

It then suffices to consider the case $r + \ell_1 = \ell_2$, because it implies $X_{i_1-\ell_1} = X_{i_2-\ell_2}$ and reduces one of the constraints in the indicators. Taking, without loss of generality, $\ell_1 = 0$, $\ell_2 = r$, as well as $\ell_3 = \cdots = \ell_{2k+2} = 0$, we have to show only that

$$\mathbb{E}\Bigg\{ \sum_{\substack{\mathbf{i}\in\mathcal{I}_{2k+2},\, i_2=i_1+r,\\ i_j-i_{j-1}>2q,\, j=3,\dots,2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, \mathbf{1}\big\{ \|X_{i_2-r}\| \ge \delta R_{k,n},\ \min_{j=3,\dots,2k+2} \|X_{i_j}\| \ge \delta R_{k,n}\big\} \Bigg\} \to 0.$$

Because of the constraints that $i_1 = i_2 - r$ and all the indices in $(i_2 - r, i_3, \dots, i_{2k+2})$ are separate from each other by at least $2q$, the entire expression is asymptotically equal to

$$\frac{n^{2k+1}}{(2k+1)!}\, \mathbb{E}\Big\{ h_t(Y_0^{(2)}, Y_r^{(2)}, Y_0^{(3)}, \dots, Y_0^{(2k+2)})\, \mathbf{1}\big\{ \min_{j=2,\dots,2k+2} \|X_0^{(j)}\|\big\} \ge \delta R_{k,n} \Big\},$$

where $Y_j^{(i)}$'s are given in (15). Note that if $\|X_0^{(2)}\| \ge \delta R_{k,n}$,

$$\begin{aligned}
&h_t(Y_0^{(2)}, Y_r^{(2)}, Y_0^{(3)}, \dots, Y_0^{(2k+2)})\\
&\le \mathbf{1}\big\{ \|Y_0^{(2)} - Y_r^{(2)}\| \le Mt \big\}\\
&\le \mathbf{1}\Big\{ \big\|(A_0 - A_r)X_0^{(2)}\big\| \le Mt + \Big\| \sum_{j=1}^{q} A_j X_{-j}^{(2)} - \sum_{j=0,\, j\neq r}^{q} A_j X_{r-j}^{(2)} \Big\| \Big\}\\
&\le \mathbf{1}\Big\{ \delta R_{k,n} \inf_{\theta\in S_{d-1}} \big\|(A_0 - A_r)\theta\big\| \le Mt + \sum_{j=1}^{q} \|A_j\|\|X_{-j}^{(2)}\| + \sum_{j=0,\, j\neq r}^{q} \|A_j\|\|X_{r-j}^{(2)}\| \Big\}.
\end{aligned}$$

The first inequality above follows from (13) and (14), and the second is due to the triangle inequality. The third is a result of the triangle inequality together with the constraint $\|X_0^{(2)}\| \ge \delta R_{k,n}$. Note that (8) ensures $\inf_{\theta\in S_{d-1}} \big\|(A_0 - A_r)\theta\big\| > 0$; thus, the condition in the last indicator function implies that there is an $\eta > 0$ such that $\|X_s^{(2)}\| \ge \eta R_{k,n}$ for some $s \in \{-q, \dots, r\} \setminus \{0\}$.

Thus, we now need to verify

$$\frac{n^{2k+1}}{(2k+1)!}\, \mathbb{E}\Big\{ h_t(Y_0^{(2)}, Y_r^{(2)}, Y_0^{(3)}, \dots, Y_0^{(2k+2)}) \times \mathbf{1}\big\{ \|X_s^{(2)}\| \ge \eta R_{k,n},\ \min_{j=2,\dots,2k+2} \|X_0^{(j)}\| \ge \delta R_{k,n}\big\} \Big\} \to 0$$

for every $s \in \{-q, \dots, r\} \setminus \{0\}$. The change of variables, as in the derivation of (41), together with proper applications of the Potter bound, shows that the left hand side above is equal to

$$\mathcal{O}\big(n^{2k+1} R_{k,n}^d f(R_{k,n} e_1)^{2k+2}\big) = \mathcal{O}(n^{-1}) \to 0, \quad n \to \infty.$$

Now, the proof of (32) is complete.

To prove (33), it is sufficient to show that for every non-negative continuous function $f : E_k \to \mathbb{R}_+$ with compact support,

$$\sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, f\Big( R_{k,n}^{-1}(\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_{2k+2}})\Big) - \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_{\mathbf{i}})\, f\Big( R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}), \dots, \mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big) \xrightarrow{p} 0. \tag{72}$$

Assume, without loss of generality, that $f$ satisfies (71). Then, the first term in (72) can be written as

$$\sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})\Big) \tag{73}$$
$$= \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})\Big) \prod_{j=1}^{2k+2} \mathbf{1}\big\{\max_{\ell=0,\dots,q} \|X_{i_j-\ell}\| \ge \delta R_{k,n}\big\}.$$

We here claim that, for each $\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}$, exactly one component in each of the $\mathcal{X}_{i_j}$s can asymptotically be at distance at least $\delta R_{k,n}$ from the origin. More specifically, we claim that the last term in (73) is equal to

$$\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})\Big) \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\neq\ell_j} \|X_{i_j-\ell}\| < \delta R_{k,n} \Big\} + o_p(1). \tag{74}$$
$$:= B_n + o_p(1).$$

To see this, for every $\boldsymbol{\ell} = (\ell_1,\dots,\ell_{2k+2}) \in \mathcal{L}_{2k+2}$, $1 \le m \le 2k+2$, and $\ell \neq \ell_m$,

$$\mathbb{E}\Bigg\{ \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \prod_{j=1}^{2k+2} \mathbf{1}\big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n} \big\} \times \mathbf{1}\big\{ \|X_{i_m-\ell}\| \ge \delta R_{k,n}\big\}\Bigg\}$$
$$= \mathcal{O}\big(n^{2k+2} R_{k,n}^d f(R_{k,n}e_1)^{2k+3}\big)$$
$$= \mathcal{O}\big(f(R_{k,n}e_1)\big) \to 0, \quad n\to\infty.$$

Turning to the second term in (72), we find that (71) ensures

$$\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big)$$
$$= \sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big) \times \prod_{j=1}^{2k+2} \mathbf{1}\big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}\big\}$$

From the same reasoning as in (74), the above is equal to

$$\sum_{\boldsymbol{\ell}\in\mathcal{L}_{2k+2}} \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i})\, f\Big(R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big) \times \prod_{j=1}^{2k+2} \mathbf{1}\big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\neq\ell_j} \|X_{i_j-\ell}\| < \delta R_{k,n}\big\} + o_p(1)$$
$$:= C_n + o_p(1).$$

Hence, it now remains to demonstrate that $B_n - C_n \xrightarrow{p} 0$. This can be established, provided that for every $\boldsymbol{\ell} = (\ell_1,\dots,\ell_{2k+2}) \in \mathcal{L}_{2k+2}$,

$$D_n := \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \Big| f\Big(R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})\Big) - f\Big(R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big)\Big| \times \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\ne\ell_j} \|X_{i_j-\ell}\| < \delta R_{k,n}\Big\} \xrightarrow{p} 0.$$

For every $0 < \eta < \delta$, the same approximation argument as that in (74) yields

$$\begin{aligned} D_n &= \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \Big| f\Big(R_{k,n}^{-1}(\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_{2k+2}})\Big) - f\Big(R_{k,n}^{-1}\big(\mathrm{diag}(\delta_{\ell_1}\mathcal{X}_{i_1}),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}\mathcal{X}_{i_{2k+2}})\big)\Big)\Big| \\ &\quad \times \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\ne\ell_j} \|X_{i_j-\ell}\| < \eta R_{k,n}\Big\} + o_p(1) \\ &:= E_n + o_p(1). \end{aligned}$$

Then, we have that

$$\begin{aligned} E_n \le &\sup_{(y_1,\dots,y_{2k+2})\in K} \Big| f(y_1,\dots,y_{2k+2}) - f\big(\mathrm{diag}(\delta_{\ell_1}y_1),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}y_{2k+2})\big)\Big| \\ &\times \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\ne\ell_j} \|X_{i_j-\ell}\| \le \eta R_{k,n}\Big\}, \end{aligned}$$

where

$$K = \Big\{(y_1,\dots,y_{2k+2}) = (y_1^{(0)},\dots,y_1^{(q)},\dots,y_{2k+2}^{(0)},\dots,y_{2k+2}^{(q)}) \in E_k : \max_{j=1,\dots,2k+2}\ \max_{\ell=0,\dots,q,\,\ell\ne\ell_j} \|y_j^{(\ell)}\| \le \eta\Big\}$$

(note that $y_j \in \mathbb{R}^{d(q+1)}$ and $y_j^{(\ell)} \in \mathbb{R}^d$).

Since $f$ is uniformly continuous on its compact support,

$$\sup_{(y_1,\dots,y_{2k+2})\in K} \Big| f(y_1,\dots,y_{2k+2}) - f\big(\mathrm{diag}(\delta_{\ell_1}y_1),\dots,\mathrm{diag}(\delta_{\ell_{2k+2}}y_{2k+2})\big)\Big| \to 0$$

as $\eta \to 0$. Furthermore, as $n \to \infty$,

$$\begin{aligned} &\mathbb{E}\Big\{ \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}, \max_{\ell=0,\dots,q,\,\ell\ne\ell_j} \|X_{i_j-\ell}\| \le \eta R_{k,n}\Big\}\Big\} \\ &\sim \mathbb{E}\Big\{ \sum_{\mathbf{i}\in\widetilde{\mathcal{I}}_{2k+2}} h_t(\mathcal{Y}_\mathbf{i}) \prod_{j=1}^{2k+2} \mathbf{1}\Big\{ \|X_{i_j-\ell_j}\| \ge \delta R_{k,n}\Big\}\Big\} \\ &= \mathcal{O}\big(n^{2k+2} R_{k,n}^d f(R_{k,n}e_1)^{2k+2}\big) = \mathcal{O}(1). \end{aligned}$$

Since $\eta > 0$ is arbitrary, we now get $E_n \xrightarrow{p} 0$ as required.

Finally, the claim (34) is obvious, since translating indices from $\mathbf{i} \in \widetilde{\mathcal{I}}_{2k+2}$ to $\mathbf{i}+\boldsymbol{\ell}$ for $\boldsymbol{\ell} \in \mathcal{L}_{2k+2}$ does not change the distributions.

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