A Programming Framework for Differential Privacy with Accuracy Concentration Bounds

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***Abstract*—Differential privacy offers a formal framework for reasoning about privacy and accuracy of computations on private data. It also offers a rich set of building blocks for constructing private data analyses. When carefully calibrated, these analyses simultaneously guarantee the privacy of the individuals contributing their data, and the accuracy of the data analyses results, inferring useful properties about the population. The compositional nature of differential privacy has motivated the design and implementation of several programming languages aimed at helping a data analyst in programming differentially private analyses. However, most of the programming languages for differential privacy proposed so far provide support for reasoning about privacy but not for reasoning about the accuracy of data analyses. To overcome this limitation, in this work we present DPella, a programming framework providing data analysts with support for reasoning about privacy, accuracy and their trade-offs. The distinguishing feature of DPella is a novel component which statically tracks the accuracy of different data analyses. In order to make tighter accuracy estimations, this component leverages taint analysis for automatically inferring *statistical independence* of the different noise quantities added for guaranteeing privacy. We evaluate our approach by implementing several classical queries from the literature and showing how data analysts can figure out the best manner to calibrate privacy to meet the accuracy requirements.**

***Keywords*-accuracy; concentration bounds; differential privacy; functional programming; databases; haskell**

I. Introduction

Differential privacy (DP) [1] is emerging as a viable solution to release statistical information about the population without compromising data subjects' privacy. A standard way to achieve DP is adding some statistical noise to the result of a data analysis. If the noise is carefully calibrated, it provides a *privacy* protection for the individuals contributing their data, and at the same time it enables the inference of *accurate* information about the population from which the data are drawn. Thanks to its quantitative formulation quantifying privacy by means of the parameters ϵ and δ, DP provides a mathematical framework for rigorously reasoning about the *privacy-accuracy trade-offs*. The accuracy requirement is not baked in the definition of DP, rather it is a constraint that is made explicit for a specific task at hand when a differentially private data analysis is designed.

An important property of DP is *composeability*: multiple differentially private data analyses can be composed with a graceful degradation of the privacy parameters ϵ and δ. This property allows to reason about privacy as a *budget*: a data analyst can decide how much privacy budget (the ϵ parameter) to assign to each of her analyses. The compositionality aspects of DP motivated the design of several programming frameworks [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and tools [14, 15, 16, 17] with built-in basic data analyses to help analysts to design their own differentially private consults. At a high level, most of these programming frameworks and tools are based on a similar idea for reasoning about privacy: use some primitives for basic tasks in DP as building blocks, and use composition properties to combine these building blocks making sure that the privacy cost of each data analysis sum up and that the total cost does not exceed the privacy budget. Programming frameworks such as [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] also provide general support to further combine, through programming techniques, the different building blocks and the results of the different data analyses. Differently, tools such as [14, 15, 16, 17] are optimized for specific tasks at the price of restricting the kinds of data analyses they can support.

Unfortunately, this simple approach for privacy cannot be directly applied to accuracy. Reasoning about accuracy is *less compositional* than reasoning about privacy, and it depends both on the specific task at hand and on the specific accuracy measure that one is interested in offering to data analysts. Despite this, when restricted to specific mechanisms and specific forms of data analyses, one can measure accuracy through estimates given as *confidence intervals*, or error bounds. As an example, most of the standard mechanisms from the differential privacy literature come with theoretical confidence intervals or error bounds that can be exposed to data analysts in order to allow them to take informed decisions about the consults they want to run. This approach has been integrated in tools such as GUPT [15], PSI [17], and Apex [18]. Users of these tools, can specify the target confidence interval they want to achieve, and the tools adjust accordingly the privacy parameters, when sufficient budget is available[1].

In contrast, all the programming frameworks proposed so far [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] do not offer any support to programmers or data analysts for tracking, and reasoning about, the accuracy of their data analyses. This phenomenon is in large part due to the complex nature of accuracy reasoning, with respect to privacy analyses, when designing arbitrary data analyses that users of these frameworks may want to program

[1] Apex actually goes beyond this by also helping user by selecting the right differentially private mechanism to achieve the required accuracy.

and run. In this paper, we address this limitation by building a programming framework for designing differentially private analysis which also supports a compositional form of reasoning about accuracy.

Our Contribution

Our main contribution is showing how programming frameworks can internalize the use of *probabilistic bounds* [19] for composing different confidence intervals or error bounds, in an automated way. Probabilistic bounds are part of the classical toolbox for the analysis of randomized algorithms, and are the tools that differential privacy algorithms designers usually employ for the accuracy analysis of classical mechanisms [20, 21]. Two important probabilistic bounds are the *union bound*, that can be used to compose errors with no assumption on the way the random noise is generated, and *Chernoff bound*, which applies to the sum of random noise when the different random variables characterizing noise generation are statistically independent (see Section IV). When applicable, and when the number of random variables grows, Chernoff bound usually gives a much "tighter" error estimation than the union bound.

Barthe et. al [22] have shown how the union bound can be internalized in a Hoare-style logic for reasoning about probabilistic imperative programs, and how this logic can be used to reason in a mechanized way about the accuracy of probabilistic programs, and in particular of programs implementing differentially private primitives.

Building on this idea, we propose a programming framework where this kind of reasoning is automated, and can be combined with reasoning about privacy. The aim of our framework is to offer programmers the tools that they need for implementing differentially private data analyses and explore their privacy-accuracy trade-offs, in a *compositional way*. Our framework supports not only the use of union bound as a reasoning principle, but also the use of the Chernoff bound. Our insight is that probabilistic bounds relying on probabilistic independence of random variables can be smoothly integrated in a programming framework by using techniques from information-flow control [23] (in the form of taint analysis [24]). While these probabilistic bounds are not enough to express every accuracy guarantee one wants to express for arbitrary data analyses, they allow the analysis of a large class of user-designed programs. Our approach allow programmers to exploit the compositional nature of both privacy and utility, complementing in this way the support provided by tools such as GUPT [15], PSI [17], which provide confidence intervals estimate only at the level of individual queries, and by Apex [18], which provide confidence intervals estimate only at the level of a query workload for queries of the same type.

We implement our ideas into a programming framework called DPella—an acronym for Differential Privacy in Haskell with accuracy—where data analysts can explore the privacy-accuracy trade-off while writing their differentially private data analyses. DPella provides several basic differentially private building blocks and composition techniques, which can be used by a programmer to design complex differentially private data analyses. The analyses that can be expressed in DPella are *data independent* and can be built using primitives for counting, average, max as well as any aggregation of their results. DPella supports both pure-DP, with parameter ϵ, and approximate-DP, with parameters ϵ and δ. Accordingly, it supports the use of both Laplace and Gaussian noise, and the use of sequential or advanced [20] composition, respectively, together with parallel composition for both notions. For simplicity, in the main part of the paper we focus only on ϵ-DP and we discuss the use of the Laplace mechanism. DPella is implemented as a library in the general purpose language Haskell; a programming language that is well-known to easily support information-flow analyses [25, 26]. Furthermore, DPella is designed to be *extensible* through the addition of new primitives (see Section VI).

To reason about privacy and accuracy, DPella provides two primitives responsible to symbolically interpret programs (which implement data analyses). DPella's symbolic interpretation for privacy consists on decreasing the privacy budget of a query by deducing the required budget of its sub-parts. On the other hand, the accuracy interpretation uses as abstraction the *inverse Cumulative Distribution Function* (iCDF) representing an upper bound on the (theoretical) error that the program incurs when guaranteeing DP. The iCDF of a query is build out of the iCDFs of the different components, by using as a basic composition principle the *union bound*. These interpretations provide overestimates of the corresponding quantities that they track. In order to make these estimates as precise as possible, DPella uses *taint analysis* to track the use of noise to identify which variables are *statistically independent*. This information is used by DPella to *soundly* replace, when needed, the union bound with the *Chernoff bound*, something that to the best of our knowledge other program logics or program analyses also focusing on accuracy, such as [22] and [27], do not consider. We envision DPella's accuracy estimations to be used in scenarios which align with those considered by tools like GUPT, PSI, and Apex.

In summary, our contributions are:

► We present DPella, a programming framework that allows data analysts to reason compositionaly about privacy-accuracy trade-off.

► We show how to use taint analysis to detect statistical independence of the noise that different primitives add, and how to use this information to achieve better error estimates.

► We inspect DPella's expressiveness and error estimations by implementing PINQ-like queries from previous work [28, 2, 29] and workloads from the matrix mechanism [30, 31, 32].

II. DPella by example

We start by providing a brief background on the notions of privacy and accuracy DPella considers.

A. Background

Differential privacy [1] is a quantitative notion of privacy that bounds how much a single individual's private data can

affect the result of a data analysis. More formally, we can define differential privacy as a property of a randomized query $\tilde{Q}(\cdot)$ representing the data analysis, as follow.

Definition II.1 (Differential Privacy (DP)[1])**.** *A randomized query $\tilde{Q}(\cdot) : \mathrm{db} \to \mathbb{R}$ satisfies* ϵ-differential privacy *if and only if for any two datasets D_1 and D_2 in* db*, which differ in one row, and for every output set $S \subseteq \mathbb{R}$ we have*

$$\Pr[\tilde{Q}(D_1) \in S] \leqslant e^{\epsilon} \Pr[\tilde{Q}(D_2) \in S] \tag{1}$$

In the definition above, the parameter ϵ determines a bound on the distance between the distributions induced by $\tilde{Q}(\cdot)$ when adding or removing an individual from the dataset—the farther away they are, the more at risk the privacy of an individual is, and vice versa. In other words, ϵ imposes a limit on the *privacy loss* that an individual can incur in, as a result of running a data analysis.

A standard way to achieve ϵ-differential privacy is adding some carefully calibrated noise to the result of a query. To protect all the different ways in which an individual's data can affect the result of a query, the noise needs to be calibrated to the maximal change that the result of the query can have when changing an individual's data. This is formalized through the notion of *sensitivity*.

Definition II.2 ([1])**.** *The* (global) sensitivity *of a query $Q(\cdot) : \mathrm{db} \to \mathbb{R}$ is the quantity $\Delta_Q = \max\{|Q(D_1) - Q(D_2)|$ for D_1, D_2 differing in one row*

The sensitivity gives a measure of the amount of noise needed to protect one individual's data. Besides, in order to achieve differential privacy, it is also important the choice of the kind of noise that one adds. A standard approach is based on the addition of noise sampled from the Laplace distribution.

Theorem II.1 (Laplace Mechanism [1])**.** *Let $Q(\cdot) : \mathrm{db} \to \mathbb{R}$ be a deterministic query with sensitivity Δ_Q. Let $\tilde{Q}(\cdot) : \mathrm{db} \to \mathbb{R}$ be a randomized query defined as $\tilde{Q}(D) = Q(D) + \eta$, where η is sample from the Laplace distribution with mean $\mu = 0$ and scale $b = \Delta_Q/\epsilon$. Then $\tilde{Q}$ is ϵ-differentially private.*

Notice that in the theorem above, for a given query, the smaller the ϵ is, the more noise $\tilde{Q}(\cdot)$ needs to inject in order to hide the contribution of one individual's data to the result—this protects privacy but degrades how meaningful the result of the query is—and vice versa. In general, the notion of *accuracy* can be defined more formally as follows.

Definition II.3 (Accuracy, see e.g.[20])**.** *Given an ϵ-differentiallly private query $\tilde{Q}(\cdot)$, a target query $Q(\cdot)$, a distance function $d(\cdot)$, a bound α, and the probability β, we say that $\tilde{Q}(\cdot)$ is* $(d(\cdot), \alpha, \beta)$-accurate *with respect to $Q(\cdot)$ if and only if for all dataset D:*

$$\Pr[d(\tilde{Q}(D) - Q(D)) > \alpha] \leqslant \beta \tag{2}$$

This definition allows one to express data independent error statements such as: with probability at least $1 - \beta$ the query $\tilde{Q}(D)$ diverge from $Q(D)$, in terms of the distance $d(\cdot)$, for less than α. Then, we will refer to α as the *error* and $1 - \beta$ as the *confidence probability* or simply *confidence*. In general, the lower the β is, i.e., the higher the confidence probability is, the higher the error α is.

As previously discussed, an important property of differential privacy is composeability.

Theorem II.2 (Sequential Composition [1])**.** *Let $\tilde{Q}_1(\cdot)$ and $\tilde{Q}_2(\cdot)$ be two queries which are ϵ_1- and ϵ_2-differentially private, respectively. Then, their sequential composition $\tilde{Q}(\cdot) = (\tilde{Q}_1(\cdot), \tilde{Q}_2(\cdot))$ is $(\epsilon_1 + \epsilon_2)$-differentially private.*

Theorem II.3 (Parallel Composition [2])**.** *Let $\tilde{Q}(\cdot)$ be a ϵ-differentially private query. and $\mathrm{data}_1, \mathrm{data}_2$ be a partition of the set of data. Then, the query $\tilde{Q}_1(D) = (\tilde{Q}(D \cap \mathrm{data}_1), \tilde{Q}(D \cap \mathrm{data}_2))$ is ϵ-differentially private.*

Thanks to the composition properties of differential privacy, we can think about ϵ as a *privacy budget* that one can spend on a given data before compromising the privacy of individuals' contributions to that data. The *global* ϵ for a given program can be seen as the privacy budget for the entire data. This budget can be consumed by selecting the *local* ϵ to "spend" in each intermediate query. Thanks to the composition properties, by tracking the local ϵ that are consumed, one can guarantee that a data analysis will not consume more than the allocated privacy budget.

Given an ϵ, DPella gives data analysts the possibility to explore how to spend it on different queries and analyze the impact on accuracy. For instance, data analysts might decide to spend "more" epsilon on sub-queries which results are required to be more accurate, while spending "less" on the others. The next examples (inspired by the use of DP in network trace analyses [28]) show how DPella helps to quantify what "more" and "less" means.

B. Example: CDF

Suppose we have a *tcpdump* trace of packets which yields a table where each row is represented as list of `String` values containing the following information:

```
[<id>, <timestamp>, <src>, <dest>, <protocol>,
 <length>, <payload>]
```

From this table, we would like to inspect—in a differentially private manner—the packet's length distribution by computing its Cumulative Distribution function (CDF), defined as $\mathrm{CDF}(x) =$ number of records with value $\leqslant x$. Hence, we are just interested in the values of the attribute `<length>`.

McSherry and Mahajan [28] proposed three different ways to approximate (due to the injected noise) CDFs with DP, and they argued for their different levels of accuracy. We revise two of these approximations (the third one can be found in the extended version of the paper) to show how DPella can assist in showing the accuracy of these analyses.

1) Sequential `CDF`*:* A simple approach to compute the CDF consists in splitting the range of lengths into `bins` and, for each `bin`, count the number of records that are $\leqslant$ `bin`. A natural way to make this computation differentially private is to add independent Laplace noise to each count.

```
1 cdf₁ bins eps dataset = do
2   sizes  ← dpSelect getPktLen dataset
3   counts ← sequence [ do elems ← dpWhere (⩽ bin)
4                                          sizes
5                          dpCount localEps elems
6                       | bin ← bins]
7   return (norm∞ counts)
8     where localEps = eps / (length bins)
```

(a) Sequential approach

```
9  cdf₂ bins eps dataset = do
10   sizes ← dpSelect ((⩽ max bins) ∘ getPktLen) dataset
11     -- parts :: Map Integer (Value Double)
12   parts ← dpPartRepeat (dpCount eps) bins assignBin
13                        sizes
14   let counts      = Map.elems parts
15       cumulCounts = [add (take i counts)
16                      | i ← [1..length counts]]
17   return (norm∞ cumulCounts)
```

(b) Parallel approach

Fig. 1: CDF's implementations

We show how to do this using DPella in Figure 1a. We define a function $\texttt{cdf}_1$ which takes as input the list of `bins` describing length ranges, the amount of budget `eps` to be spent by the entire query, and the `dataset` where it will be computed. For now, we assume that we have a fixed list of bins for packets' length. $\texttt{cdf}_1$ uses the primitive transformation[2] `dpSelect` to obtain from the dataset the length of each packet via a selector function `getPktLen`::`String` → `Integer` (where :: is used to describe the type of a term in Haskell). This computation results in a new dataset `sizes`. Then, we create a counting query for each bin using the primitive `dpWhere`. This filters all records that are less than the `bin` under consideration (⩽ `bin`). Finally, we perform a noisy count using the DPella primitive `dpCount`. The noise injected by the primitive `dpCount` is calibrated so that the execution of `dpCount` is `localEps`-DP (line 8 [3]). The function `sequence` then takes the list of queries and compute them sequentially collecting their results in a list—to create a list of noisy counts. We then return this list. The combinator $\texttt{norm}_\infty$ in line 7 is used to mark where we want the accuracy information to be collected, but it does not have any impact on the actual result of the cdf.

To ensure that $\texttt{cdf}_1$ is `eps`-differential privacy, we distributed the given budget `eps` evenly among the sub-queries (this is done in lines 5 and 8). However, a data analyst may forget to do so, e.g., she can define `localEps` = `eps`, and in this case the final query is (`length bins`)*`eps`-DP, which is a significant change in the query's privacy price. To prevent such budget miscalculations or unintended expenditure of privacy budget, DPella provides the analyst with the function `budget` (see Section III) that, given a query, statically computes an upper bound on how much budget it will spend. To see how to use this function, consider the function $\texttt{cdf}_1$ and a its modified version $\texttt{cdf}_1'$ with `localEps` = `eps`. Suppose that we want to compute how much budget will be consumed by running it on a list of bins of size 10 (identified as $\texttt{bins}_{10}$) and on a dataset `networkTraffic`. Then, the data analyst can ask this as follow:

```
>budget (cdf₁ bins₁₀ 1 networkTraffic)
ϵ = 1
>budget (cdf₁' bins₁₀ 1 networkTraffic)
ϵ = 10
```

The function `budget` *will not* execute the query, it simply performs an static analysis on the code of the query by symbolically interpreting it. The static analysis uses information encoded by the *type* of the database `networkTraffic` (explained in Section III).

DPella also provides primitives to statically explore the accuracy of a query. The function `accuracy` takes a query $Q(\cdot)$ and a probability β and returns an estimate of the (theoretical) error that can be achieved with confidence probability $1-\beta$. Suppose that we want to estimate the error we will incur in by running $\texttt{cdf}_1$ with a budget of $\epsilon = 1$ on with the same list of bins and dataset as above, and we want to have this estimate for $\beta = 0.05$ and $\beta = 0.2$, respectively. Then, the data analyst can ask this as follow:

```
>accuracy (cdf₁ bins₁₀ 1 networkTraffic) 0.05
α = 53
>accuracy (cdf₁ bins₁₀ 1 networkTraffic) 0.2
α = 40
```

Since the result of the query is a vector of counts, we measure the error α in terms of ℓ_∞ distance with respect to the CDF without noise. This is the max difference that we can have in a bin due to the noise. The way to read the information provided by DPella is that with confidence 95% and 80%, we have errors 53 and 40, respectively. These error bounds can be used by a data analyst to figure out the exact set of parameters that would be useful for her task.

2) Parallel `CDF`*:* Another way to compute a CDF is by first generating an histogram of the data according to the bins, and then building a cumulative sum for each bin. To make this function private, an approach could be to add noise at the different bins of the histogram, rather than to the cumulative sums themself, so that we could use the parallel composition, rather than the sequential one [28], which we show how to implement in DPella in Figure 1b. —where double-dashes are used to introduce single-line comments.

In $\texttt{cdf}_2$, we first select all the packages whose length is smaller than the maximum bin, and then we partition the data accordingly to the given list of bins. To do this, we use `dpPartRepeat` operator to create as many (disjoint) datasets as given `bins`, where each record in each partition belongs to the range determined by an specific `bin`—where the record belongs is determined by the function

[2]Anticipating on Section III, in our code we will usually use the red color for transformations, the blue color for aggregate operations, and the green color for combinators for privacy and accuracy.

[3]The casting operation `fromIntegral` is omitted for clarity

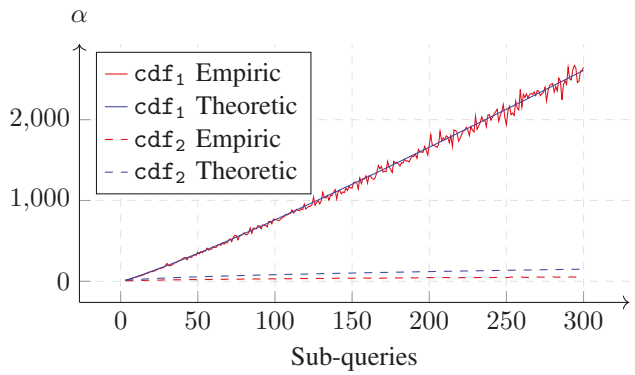

Fig. 2: Error comparison (95% confidence)

`assignBin :: Integer → Integer`. After creating all partitions, the primitive `dpPartRepeat` computes the given query `dpCount eps` in *each partition*—the name `dpPartRepeat` comes from repetitively calling `dpCount eps` as many times as partitions we have. As a result, `dpPartRepeat` returns a finite map where the keys are the `bins` and the elements are the noisy count of the records per partition—i.e., the histogram. In what follows (lines 15–17), we compute the cumulative sums of the noisy counts using the DPella primitive `add`, and finally we build and return the list of values denoting the CDF.

The privacy analysis of $\texttt{cdf}_2$ is similar to the one of $\texttt{cdf}_1$. The accuracy analysis, however, is more interesting: first it gets error bounds for each cumulative sum, then these are used to give an error bound on the maximum error of the vector. For the error bounds on the cumulative sums DPella uses either the union bound or the Chernoff bound, depending on which one gives the lowest error. For the maximum error of the vector, DPella uses the union bound, similarly to what happens in $\texttt{cdf}_1$. A data analyst can explore the accuracy of $\texttt{cdf}_2$.

```
>accuracy (cdf2 bins10 1 networkTraffic) 0.05
α = 22
>accuracy (cdf2 bins10 1 networkTraffic) 0.2
α = 20
```

3) Exploring the privacy-accuracy trade-off: Let us assume that a data analyst is interested in running a CDF with an error bounded with 90% confidence, i.e., with $\beta = 0.1$, having three bins (named $\texttt{bins}_3$), and $\epsilon = 1$. With those assumptions in mind, which implementation should she use? To answer that question, the data analyst can ask DPella:

```
>accuracy (cdf1 bins3 1 networkTraffic) 0.1
 α = 11
>accuracy (cdf2 bins3 1 networkTraffic) 0.1
 α = 12
```

So, the analyst would know that using $\texttt{cdf}_1$ in this case would give, likely, a lower error. Suppose further that the data analyst realize that she prefers to have a finer granularity and have 10 bins, instead of only 3. Which implementation should she use? Again, she can compute:

```
>accuracy (cdf1 bins10 1 networkTraffic) 0.1
 α = 46
>accuracy (cdf2 bins10 1 networkTraffic) 0.1
 α = 20
```

So, the data analyst would know that using $\texttt{cdf}_2$ in this case would give, likely, a lower error. One can also use DPella to show a comparison between $\texttt{cdf}_1$ and $\texttt{cdf}_2$ in terms of error when we keep the privacy parameter fixed and we change the number of bins, where $\texttt{cdf}_2$ gives a better error when the number of bins is large [28] as illustrated in Figure 2. In the figure, we also show the empirical error to confirm that our estimate is tight—the oscillations on the empirical $\texttt{cdf}_1$ are given by the relative small (300) number of experimental runs we consider.

Now, what if the data analyst choose to use $\texttt{cdf}_2$ because of what we discussed before but she realizes that she can afford an error $\alpha \leqslant 50$; what would be then the epsilon that gives such α? One of the feature of DPella is that the analyst *can write a simple program that finds it by repetitively calling* `accuracy` *with different epsilons*—this is one of the advantages of providing a programming framework. These different use cases shows the flexibility of DPella for different tasks in private data analysis.

Synthetic data: When compared with (non-compositional) approaches for estimating accuracy based on synthetic or public data, such as [33], the static analysis of DPella can be used in a complimentary manner to quickly (and precisely) estimate privacy and accuracy for a wide range of simple queries. There are also certain kind of queries (e.g., k-way marginal) where it is more convenient to use our static analysis than synthetic data for high-dimensional datasets—see Appendix G for details.

The following sections will introduce the theoretical and technical aspects of DPella.

III. Privacy

DPella have two kind of actors: *data curators*, owners of the private dataset that decide the global privacy budget and split it among the *data analysts*, the ones who write queries to mine useful information from the data and spend the budget they received. DPella is designed to help data analysts to have an informed decision about how to spend their budget based on exploring the trade-offs between privacy and accuracy.

A. Components of the API

Figure 3 shows part of DPella API. DPella introduces two abstract data types to respectively denote datasets and queries:

```
data Data s r   -- datasets
data Query a    -- queries
```

The attentive reader might have observed that the API also introduces the data type `Value a`. This type is used to capture values resulting from data aggregations. However, we defer its explanation for Section IV since it is only used for accuracy calculations—for this section, readers can consider the type `Value a` as isomorphic to the type `a`. It is also worth noticing that the API enforces an invariant by construction: *it is not possible to branch on results produced by aggregations*—observe that there is no primitive capable to destruct a value

```
  -- Transformations (data analyst)
dpWhere      :: (r → Bool) → Data s r → Query (Data s r)
dpSelect     :: (r → r′) → Data s r → Query (Data s r′)
dpGroupBy    :: Eq k ⇒ (r → k) → Data s r
             → Query (Data (2*s) (k, [r]))
dpIntersect  :: Eq r ⇒ Data s₁ r → Data s₂ r
             → Query (Data (s₁+s₂) r)
dpUnion      :: Data s₁ r → Data s₂ r
             → Query (Data (s₁+s₂) r)
dpPart       :: Ord k ⇒ (r → k) → Data s r
             → Map k (Data s r) → Query (Value a))
             → Query (Map k (Value a))
  -- Aggregations (data analyst)
dpCount :: Stb s ⇒ ϵ → Data s r → Query (Value Double)
dpSum   :: Stb s ⇒ ϵ → (r → Double) → Data s r
        → Query (Value Double)
dpAvg   :: Stb s ⇒ ϵ → (r → Double) → Data s r
        → Query (Value Double)
dpMax   :: Eq a ⇒ ϵ → Responses a → (r → a)
        → Data 1 r → Query (Value a)
  -- Budget
budget  :: Query a → ϵ
  -- Execution (data curator)
dpEval :: (Data 1 r → Query (Value a)) → [r] → ϵ → IO a
```

Fig. 3: DPella API: Part I

of type `Value a`. While it might seem restrictive, it enables to write counting queries, which are the bread and butter of statistical analysis and have been the focus of the majority of the work in DP. Section VI discusses, however, how to lift this limitation for specific analyses.

Values of type `Data s r` represent sensitive datasets with *accumulated stability* `s`, where each row is of type `r`. Accumulated stability, on the other hand, is instantiated to type-level positive natural numbers, i.e., `1`, `2`, etc. Stability is a measure that captures the number of rows in the dataset that could have been affected by transformations like selection or grouping of rows. In DP research, stability is associated with dataset transformations rather than with datasets themselves. In order to simplify type signatures, DPella uses the type parameter s in datasets to represent the accumulated stability of the transformations for which datasets have gone through—as done in [34]. Different than, e.g., PINQ [2], one novelty of DPella is that it computes stability *statically* using Haskell's type-system.

Values of type `Query a` represent *computations*, or queries, that yield values of type `a`. Type `Query a` is a monad [35], and because of this, computations of type `Query a` are built by two fundamental operations:

```
return :: a → Query a
(>>=)  :: Query a → (a → Query b) → Query b
```

The operation `return x` outputs a query that just produces the value `x` without causing side-effects, i.e., without touching any dataset. The function `(>>=)`—called *bind*—is used to sequence queries and their associated side-effects. Specifically, `qp >>= f` executes the query `qp`, takes its *result*, and passes it to the function `f`, which then returns a second query to run. Some languages, like Haskell, provide syntactic sugar for monadic computations known as **do**-notation. For instance, the program $\texttt{qp}_1 \gg\!= (\lambda \texttt{x}_1 \rightarrow \texttt{qp}_2 \gg\!= (\lambda \texttt{x}_2 \rightarrow \texttt{return}\ (\texttt{x}_1, \texttt{x}_2)))$, which performs queries $\texttt{qp}_1$ and $\texttt{qp}_2$ and returns their results in a pair, can be written as $\textbf{do}\ \texttt{x}_1 \leftarrow \texttt{qp}_1; \texttt{x}_2 \leftarrow \texttt{qp}_2; \texttt{return}\ (\texttt{x}_1, \texttt{x}_2)$ which gives a more "imperative" feeling to programs. We split the API in four parts: transformations, aggregations, budget prediction, and execution of queries—see next section for the description of API's accuracy components. The first three parts are intended to be used by data analysts, while the last one is intended to be *only* used by data curators[4].

1) Transformations: The primitive `dpWhere` filters rows in datasets based on a predicate functions (`r → Bool`). The created query (of type `Query (Data s r)`) produces a dataset with the same row type `r` and accumulated stability `s` as the dataset given as argument (`Data s r`). Observe that if we consider two datasets which differ in `s` rows in two given executions, and we apply `dpWhere` to both of them, we will obtain datasets that will still differ in `s` rows—thus, the accumulated stability remains the same. The primitive `dpGroupBy` returns a dataset where rows with the same key are grouped together. The functional argument (of type `r → k`) maps rows to keys of type `k`. The rows in the return dataset (`Data (2*s) (k, [r])`) consist of key-rows pairs of type `(k, [r])`—syntax `[r]` denotes the type of lists of elements of type `r`. What appears on the left-hand side of the symbol ⇒ are type constraints. They can be seen as static demands for the types appearing on the right-hand side of ⇒. Type constraint `Eq k` demands type `k`, denoting keys, to support equality; otherwise grouping rows with the same keys is not possible. The accumulated stability of the new dataset is multiplied by `2` in accordance with stability calculations for transformations [2, 34]—observe that `2*s` is a type-level multiplication done by a type-level function (or type family [37]) `*`. Our API also considers transformations similar to those found in SQL like intersection (`dpIntersect`), union (`dpUnion`), and selection (`dpSelect`) of datasets, where the accumulated stability is updated accordingly. Providing a general join transformation is known to be challenging [2, 38, 39, 40]. The output of a join may contain duplicates of sensitive rows, which makes difficult to bound the accumulated stability of datasets. However, and similar to PINQ, DPella supports a limited form of joins, where a limit gets imposed on the number of output records mapped under each key in order to obtain stability. For brevity, we skip its presentation and assume that all the considered information is contained by the rows of given datasets.

2) Partition: Primitive `dpPart` deserves special attention. This primitive is a mixture of a transformation and aggregations since it partitions the data (transformation) to subsequently apply aggregations on each of them. More specifically, it splits the given dataset (`Data s r`) based on a row-to-key mapping (`r → k`). Then, it takes each partition for a given key `k` and applies it to the corresponding function `Data s r → Query (Value a)`, which is given as an element of a key-

[4] A separation that can be enforced via Haskell modules [36]

query mapping (`Map k ((Data s r) → Query (Value a))`). Subsequently, it returns the values produced at every partition as a key-value mapping (`Query (Map k (Value a))`). The primitive `dpPartRepeat`, used by the examples in Section II, is implemented as a special case of `dpPart` and thus we do not discuss it further.

Partition is one of the most important operators to save privacy budget. It allows to run the same query on a dataset's partitions but only paying for one of them—recall Theorem II.3. The essential assumption that makes this possible is that every query runs on *disjoint* datasets. Unfortunately, data analysts could ignore this assumption when writing queries—see Appendix A for an example. To catch such possible coding errors, DPella deploys an static information-flow control (IFC) analysis similar to that provided by MAC [41]. IFC ensures that queries run by `dpPart` do not perform queries on shared datasets by attaching provenance labels to datasets `Data s r` indicating to which part of the query they are associated with and propagates that information accordingly. The implemented IFC mechanism is *transparent* to data analysts and curators, i.e., they do not need to understand how it works. Analysts and curators only need to know that, when the IFC analysis raises an alarm, is due to a possibly access to non-disjoint datasets when using `dpPart`.

3) Aggregations: DPella presents primitives to count (`dpCount`), sum (`dpSum`), and average (`dpAvg`) rows in datasets. These primitives take an argument `eps` :: ϵ, a dataset, and build a Laplace mechanism which is `eps`-differentially private from which a noisy result gets return as a term of type `Value Double`. The purpose of data type `Value a` is two fold: to encapsulate noisy values of type `a` originating from aggregations of data, and to store information about its accuracy—intuitively, how "noisy" the value is (explained in Section IV). The injected noise of these queries gets adjusted depending on three parameters: the value of type ϵ, the accumulated stability of the dataset `s`, and the sensitivity of the query (recall Definition II.2). More specifically, the Laplace mechanism used by DPella uses accumulated stability s to scale the noise, i.e., it consider b from Theorem II.1 as $b = s \cdot \frac{\Delta_Q}{\epsilon}$. The sensitivity of DPella's aggregations are hard-coded into the implementation—similar to what PINQ does. The sensitivities of `dpSum` and `dpAvg` are set to `1` and `2`, respectively, by applying a clipping function (`r → Double`). This function maps the values under scrutiny into the interval $[-1, 1]$ before executing the query. The sensitivity of `dpCount` and `dpMax` is set to `1`. To implement the Laplace mechanism, the type constrain `Stb s` in `dpCount`, `dpSum`, and `dpAvg` demands the accumulated stability parameter `s` to be a type-level natural number in order to obtain a term-level representation when injecting noise. Finally, primitive `dpMax` implements report-noisy-max [20]. This query takes a list of possible responses (`Responses a` is a type synonym for `[a]`) and a function of type `r → a` to be applied to every row. The implementation of `dpMax` adds uniform noise to every score—in this case, the amount of rows *voting* for a response—and returns the *response* with the highest noisy score. This primitive becomes relevant to obtain the winner option in elections without singling out any voter. However, it requires that the accumulated stability of the dataset to be `1` in order to be sound [22]. DPella guarantees such requirement by typing: the type of the given dataset as argument is `Data 1 r`.

4) Privacy budget and execution of queries: The primitive `budget` statically computes how much privacy budget is required to run a query. It is worth notice that DPella returns an upper bound of the required privacy budget rather than the exact one—an expected consequence of using a type-system to compute it and provide early feedback to data analysts. Finally, the primitive `dpEval` is used by data curators to run queries (`Query a`) under given privacy budgets (ϵ), where datasets are just lists of rows (`[r]`). It assumes that the initial accumulated stability as `1` (`Data 1 r`) since the dataset has not yet gone through any transformation, and DPella will automatically calculate the accumulated stability for datasets affected by subsequent transformations via the Haskell's type system. This primitive returns a computation of type `IO a`, which in Haskell are computations responsible to perform side-effects—in this case, obtaining randomness from the system in order to implement the Laplace mechanism.

5) Implementation: DPella is implemented as a *deep embedded domain-specific language* (EDSL) in Haskell. Due to such design choice, data analysts can piggyback on Haskell's infrastructure to build queries in a creative way. For instance, it is possible to leverage on any of Haskell's pure functions. The following one-liner (of type `Query [Value Double]`) shows how to write a query that generates possibly non-disjoint datasets from `ds` :: `Data` s r based on different criteria for then performing a counting.

```
mapM (flip dpSelect ds>=>dpCount eps) fs
```

Variable `eps` is the epsilon to spend in each counting while `fs` :: $[r \to$ `Bool`$]$ is the criteria list. The high-order functions `flip`, `mapM`, and `(>=>)` are standard in Haskell and represent a function who switches arguments, the monadic versions of `map`, and the Kleisli arrow, respectively. Despite DPella being a first-order interface, data analysts can use Haskell's high-order functions to compactly describe queries.

IV. Accuracy

DPella uses the data type `Value a` responsible to store a result of type `a` as well as information about its accuracy. For instance, a term of type `Value Double` stores a noisy number (e.g., coming from executing `dpCount`) together with its accuracy in terms of *a bound on the noise introduced to protect privacy*.

DPella provides an static analysis capable to compute the accuracy of queries via the following function

```
accuracy :: Query (Value a) → β → α
```

which takes as an argument a query and returns a function, called *inverse Cumulative Distribution Function* (iCDF), capturing the theoretical error α for a given confidence `1-`β. Function `accuracy` does not execute queries but rather symbolically

```
  -- Accuracy analysis (data analyst)
accuracy :: Query (Value a) → β → α
  -- Norms (data analyst)
norm∞ :: [Value Double] → Value [Double]
norm2 :: [Value Double] → Value [Double]
norm1 :: [Value Double] → Value [Double]
rmsd  :: [Value Double] → Value [Double]
  -- Accuracy combinators (data analyst)
add   :: [Value Double] → Value Double
neg   :: Value Double → Value Double
```

Fig. 4: DPella API: Part II

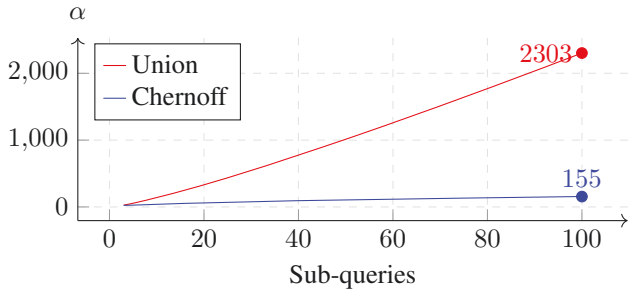

Fig. 5: Union vs. Chernoff bounds

interpret all of its components in order to compute the accuracy of the result based on the sub-queries and how data gets aggregated. DPella follows the principle of improving accuracy calculations by detecting statistical independence. For that, it implements taint analysis [24] in order to track if values were drawn from statistically independent distributions.

A. Accuracy calculations

DPella starts by generating iCDFs at the time of running aggregations based on the following known result of the Laplace mechanism:

Definition IV.1 (Accuracy for the Laplace mechanism)**.** *Given a randomized query $\tilde{Q}(\cdot) : \mathrm{db} \to \mathbb{R}$ implemented with the Laplace mechanism as in Theorem II.1, we have that*

$$\Pr\left[|\tilde{Q}(D) - Q(D)| > \log(1/\beta) \cdot \frac{\Delta_Q}{\epsilon}\right] \leqslant \beta \tag{3}$$

Recall that the Laplace mechanism used by DPella utilizes accumulated stability s to scale the noise, i.e., it consider b from Theorem II.1 as $b = s \cdot \frac{\Delta_Q}{\epsilon}$. Consequently, DPella stores the iCDF $\lambda\beta \to \log(1/\beta) \cdot s \cdot \frac{\Delta_Q}{\epsilon}$ for the values of type `Value Double` returned by aggregation primitives like `dpCount`, `dpSum`, and `dpAvg`. However, queries are often more complex than just calling aggregation primitives—as shown by $\mathtt{CDF}_2$ in Figure 1b. In this light, DPella provides combinators responsible to aggregate noisy values, while computing its iCDFs based on the iCDFs of the arguments. Figure 4 shows DPella API when dealing with accuracy.

1) Norms: DPella exposes primitives to aggregate the magnitudes of several errors predictions into a *single* measure—a useful tool when dealing with vectors. Primitives $\mathtt{norm}_\infty$, $\mathtt{norm}_2$, and $\mathtt{norm}_1$ take a list of values of type `Value Double`, where each of them carries accuracy information, and produces a *single value* (or vector) that contains a list of elements (`Value [Double]`) whose accuracy is set to be the well-known ℓ_∞-, ℓ_2-, ℓ_1-norms, respectively. Finally, primitive `rmsd` implements *root-mean-square deviation* among the elements given as arguments. In our examples, we focus on using $\mathtt{norm}_\infty$, but other norms are available for the taste, and preference, of data analysts.

2) Adding values: The primitive `add` aggregates values and, in order to compute accuracy of the addition, it tries to apply the Chernoff bound if all the values are statistically independent; otherwise, it applies the union bound. More precisely, for the next definitions we assume that primitive `add` receives n terms $\mathtt{v}_1$`::Value Double`, $\mathtt{v}_2$`::Value Double`, ... , $\mathtt{v}_n$`::Value Double`. Importantly, since we are calculating the theoretical error, we should consider random variables rather than specific numbers. The next definition specifies how `add` behaves when applying union bound.

Definition IV.2 (`add` using union bound)**.** *Given $n \geqslant 2$ random variables V_j with their respective $iCDF_j$, where $j \in 1 \ldots n$, and $\alpha_j = iCDF_j(\frac{\beta}{n})$, then the addition $Z = \sum_{j=1}^{n} V_j$ has the following accuracy:*

$$\Pr[|Z| > \textstyle\sum_{j=1}^{n} \alpha_j] \leqslant \beta \tag{4}$$

Observe that to compute the $iCDF$ of Z, the formula uses the $iCDFs$ from the operands applied to $\frac{\beta}{n}$. Union bound makes no assumption about the distribution of the random variables V_j.

In contrast, the Chernoff bound *often* provides a tighter error estimation than the commonly used union bound when adding several *statistically independent* queries *sampled from a Laplace distribution*. To illustrate this point, Figure 5 shows that difference for the $\mathtt{cdf}_2$ function we presented in Section II with $\epsilon = 0.5$ (for each DP sub-query) and $\beta = 0.1$. Clearly, the Chernoff bound is asymptotically much better when estimating accuracy, while the union bound works best with a reduced number of sub-queries—observe how lines get crossed in Figure 5. In this light, and when possible, DPella computes both union bound and Chernoff bound and selects the tighter error estimation. However, to apply Chernoff bound, DPella needs to be certain that the events are independent. Before explaining how DPella detects that, we give an specification of the formula we use for Chernoff.

Definition IV.3 (`add` using Chernoff bound [42])**.** *Given $n \geqslant 2$ independent random variables $V_j \sim Lap(0, b_j)$, where $j \in 1 \ldots n$, $b_M = max\ \{b_j\}_{j=1\ldots n}$, and $\nu > max\{\sqrt{\sum_{j=1}^{n} b_j^2}, b_M\sqrt{\ln \frac{2}{\beta}}\}$, then the addition $Z = \sum_{j=1}^{n} V_j$ has the following accuracy:*

$$\Pr[|Z| > \nu \cdot \sqrt{8 \ln \tfrac{2}{\beta}}] \leqslant \beta \tag{5}$$

DPella uses the value $\nu = max\{\sqrt{\sum_{j=1}^{n} b_j^2}, b_M\sqrt{\ln \frac{2}{\beta}}\} + 0.00001$ to satisfy the conditions of the definition above when applying the Chernoff bound—any other positive increment to

the computed maximum works as well[5].

Lastly, to support subtraction, DPella provides primitive `neg` responsible to change the sign of a given value. We next explain how DPella checks that values come from statistically independent sampled variables.

3) Detecting statistical independence: To detect statistical independence, we apply taint analysis when considering terms of type `Value a`. Specifically, every time a result of type `Value Double` gets generated by an aggregation query in DPella's API (i.e., `dpCount`, `dpSum`, etc.), it gets assigned a label indicating that it is *untainted* and thus statistically independent. The label also carries information about the scale of the Laplace distribution from which it was sampled—a useful information when applying Definition IV.3. When the primitive `add` receives all untainted values as arguments, the accuracy of the aggregation is determined by the best estimation provided by either the union bound (Definition IV.2) or the Chernoff bound (Definition IV.3). Importantly, values produced by `add` are considered *tainted* since they depend on other results. When `add` receives any tainted argument, it proceeds to estimate the error of the addition by just using union bound—we refer readers to Appendix B for a piece of DPella code which intituively illustrates how our taint analysis works. In the next Section, we proceed to formally define our accuracy analysis.

B. Implementation

The accuracy analysis consists on *symbolically* interpreting a given query, calculating the accuracy of individual parts, and then combining them using our taint analysis. We introduce two polymorphic symbolic values: $\mathcal{D} :: \texttt{Data s r}$ and $\mathcal{S}[\texttt{iCDF}, s, \texttt{ts}] :: \texttt{Value a}$. Symbolic dataset $\mathcal{D}$ represents concrete datasets arising from data transformations. A symbolic value $\mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]$ represents concrete values with tags `ts` and a `iCDF` which is computed assuming a noise scale s. Tags are used to detect the provenance of symbolic values and when they arise from different *noisy sources*.

Function `accuracy` takes queries that produce a result of type `Value a`. Such queries are essentially built by performing data aggregation queries (e.g., `dpCount`) preceded by a (possibly empty) sequence of other primitives like data transformations[6]. Figure 6 and 7 show the interesting parts of our analysis—Appendix C shows the calculation of norms and thus we skip them here for brevity. Given a *well-typed* query `q :: Query (Value a)`, `accuracy q = iCDF` where $\texttt{q} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]$ for some s and `ts`. The rules in Figure 6 are mainly split into two cases: considering data aggregation queries and sequences of primitives glued together with $(\gg\!=)$.

The symbolic interpretation of `dpCount` is captured by rule DPCount—see Figure 6a. This rule populates the `iCDF` of the return symbolic value with the corresponding error

[5] There are perhaps other ways to compute the Chernoff bound for the sum of independent Laplace distributions, changing this equation in DPella does not require major work.

[6] We ignore the case of `return val :: Query (Value a)` since the definition of `accuracy` is trivial for such case.

$$\textsc{DPCount}\quad \frac{\texttt{dataset} :: \texttt{Data s r} \qquad \texttt{iCDF} = \lambda\beta \to \mathit{log}(\frac{1}{\beta}) \cdot \texttt{s} \cdot \frac{1}{\epsilon} \qquad \texttt{t}\ \mathit{fresh}}{\texttt{dpCount}\ \epsilon\ \texttt{dataset} \triangleright \mathcal{S}[\texttt{iCDF}, \texttt{s} \cdot \frac{1}{\epsilon}, \{\texttt{t}\}]}$$

(a) DP-queries

$$\textsc{Seq-Trans}\quad \frac{\texttt{k}\ \mathcal{D} \rightsquigarrow^* \texttt{next} \qquad \texttt{next} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]}{\texttt{transform} \gg\!= \texttt{k} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]}$$

$$\textsc{Seq-Query}\quad \frac{\texttt{query} \triangleright \mathcal{S}[\texttt{iCDF}_\texttt{q}, s_\texttt{q}, \texttt{ts}_\texttt{q}] \qquad \texttt{k}\ (\mathcal{S}[\texttt{iCDF}_\texttt{q}, s_\texttt{q}, \texttt{ts}_\texttt{q}]) \rightsquigarrow^* \texttt{next} \qquad \texttt{next} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]}{\texttt{query} \gg\!= \texttt{k} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]}$$

(b) Sequential traversal

$$\textsc{Seq-Part}\quad \frac{\begin{array}{c}(\texttt{m j}\ \mathcal{D} \rightsquigarrow^* \texttt{next}_j)_{j \in dom(\texttt{m})} \\ (\texttt{next}_j \triangleright \mathcal{S}[\texttt{iCDF}_j, s_j, \texttt{ts}_j])_{j \in dom(\texttt{m})} \\ \texttt{m'} = (j \mapsto \mathcal{S}[\texttt{iCDF}_j, s_j, \texttt{ts}_j])_{j \in dom(\texttt{m})} \\ \texttt{k m'} \rightsquigarrow^* \texttt{next} \qquad \texttt{next} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]\end{array}}{\texttt{dpPart sel dataset m} \gg\!= \texttt{k} \triangleright \mathcal{S}[\texttt{iCDF}, s, \texttt{ts}]}$$

(c) Accuracy calculation when partitioning data

Fig. 6: Accuracy analysis implemented by `accuracy`

calculations for Laplace as presented in Definition A.1 (with the scale adjusted with the accumulated stability). Observe that it extracts the stability information from the type of the considered dataset (`ds :: Data s r`) and attaches a fresh tag indicating an independently generated noisy value. The symbolic interpretation of `dpSum` and `dpAvg` proceeds similarly to `dpCount` and we thus omit them for brevity. We also omit the symbolic interpretation of `dpMax` for brevity—readers can refer to Appendix D for details.

To symbolically interpret a sequence of primitives, the analysis gets further split into two cases depending if the first operation to interpret is a transformation or an aggregation, respectively—see Figure 6b. Rule Seq-Trans considers the former, where `transform` can be any of the transformation operations in Figure 3. It simply uses the symbolic value $\mathcal{D}$ to pass it to the continuation `k`. It can happen that $\texttt{k}\ \mathcal{D}$ does not match (yet) any part of DPella's API required for our analysis to continue[7]. However, the EDSL nature of DPella makes Haskell's to reduce $\texttt{k}\ \mathcal{D}$ to the `next` primitive to be considered, which we capture as $\texttt{k}\ \mathcal{D} \rightsquigarrow^* \texttt{next}$—and we know that it will occur thanks to type preservation. We represent $\rightsquigarrow$ ($\rightsquigarrow^*$) to pure reduction(s) in the host language like function application, pair projections, list comprehension, etc. The analysis then continues symbolically interpreting the `next` yield instruction. Rule Seq-Query computes the corresponding symbolic value for the aggregation `query`. The symbolic value is then passed to the continuation, and the analysis continues with the `next`

[7] For instance, $\texttt{k}\ \mathcal{D} = (\lambda\texttt{x} \to \texttt{dpCount 1 x})\ \mathcal{D}$, and thus $((\lambda\texttt{x} \to \texttt{dpCount 1 x})\ \mathcal{D}) \rightsquigarrow^* \texttt{dpCount 1}\ \mathcal{D}$.

UNION-BOUND

$$\frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \quad \alpha_j = \mathtt{iCDF}_j(\frac{b}{n}) \quad \mathtt{iCDF} = \lambda\beta \rightarrow \sum_{j=1}^{n} \alpha_j}{\mathtt{ub}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathtt{iCDF}}$$

CHERNOFF-BOUND

$$\frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \quad \mathtt{v_M} = max\ \{s_j\}_{j=1\ldots n} \quad \nu = max\{\sqrt{\textstyle\sum_{j=1}^{n} s_j^2}, \mathtt{v_M}\sqrt{\ln \tfrac{2}{\beta}}\} + 0.0001 \quad \mathtt{iCDF} = \lambda\beta \rightarrow \nu \cdot \sqrt{8 \ln \tfrac{2}{\beta}}}{\mathtt{cb}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathtt{iCDF}}$$

ADD-UNION

$$\frac{(\exists j \cdot \mathtt{ts}_j = \emptyset) \vee \bigcap_{j=1\ldots n} \mathtt{ts}_j \neq \emptyset}{\mathtt{add}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathcal{S}[\mathtt{ub}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}], 0, \emptyset]}$$

ADD-CHERNOFF-UNION

$$\frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \quad (\forall j \cdot \mathtt{ts}_j \neq \emptyset) \quad \bigcap_{j=1\ldots n} \mathtt{ts}_j = \emptyset \quad \mathtt{iCDF} = \lambda\beta \rightarrow \mathtt{min}\ (\mathtt{ub}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}]\ \beta)\ (\mathtt{cb}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}]\ \beta)}{\mathtt{add}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathcal{S}[\mathtt{iCDF}, 0, \emptyset]}$$

Fig. 7: Calculation of concentration bounds

yield instruction.

Rule SEQ-PART shows the symbolic interpretation of `dpPart`. The argument `m :: Map k (Data s r → Query (Value a))` describes the queries to execute once given the corresponding bins. Since these queries produce values, we need to symbolically interpret each of them to obtain their accuracy estimations. The rule applies each of those queries to a symbolic dataset ($\mathtt{m\ j}\ \mathcal{D}$)[8]. The symbolic values yield by each bin are collected into the mapping `m'`, which is then passed to continuation `k` in order to continue the analysis on the `next` yield instruction.

Figure 7 shows the part of our analysis responsible to apply concentration bounds. Rules UNION-BOUND and CHERNOFF-BOUND define pure functions (reduction $\rightsquigarrow$) which produce the concentration bounds as described in Definitions IV.2 and IV.3, respectively. We define the function `add` based on two cases. Rule ADD-UNION produces a symbolic value with a `iCDF` generated by the union bound ($\mathtt{ub}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}]$). The symbolic value is tainted, which is denoted by the empty tags ($\emptyset$). The scale `0` denotes that the scale of the noise and its distribution is unknown—adding Laplace distributions do not yield a Laplace distribution. (However, the situation is different with Gaussians where the analysis keeps the scale of the noise and taint tags—see Appendix E for details.) This rule gets exercised when either the list of symbolic values contains a tainted one ($\exists j \cdot \mathtt{ts}_j = \emptyset$) or have not been independently generated ($\bigcap_{k=1\ldots n} \mathtt{ts}_j \neq \emptyset$). Differently, ADD-CHERNOFF-UNION produces a symbolic value with a `iCDF` which chooses the minimum error estimation between union and Chernoff bound for a given β—sometimes union bound provides tighter estimations when aggregating few noisy-values (recall Figure 5). This rule triggers when all the values are untainted ($\forall j \cdot \mathtt{ts}_j \neq \emptyset$) and independently generated ($\bigcap_{j=1\ldots n} \mathtt{ts}_j = \emptyset$). At a first glance, one could believe that it would be enough to use the scale of the noise to track when values are untainted, i.e., if the scale is different from 0, then the value is untainted. Unfortunately, this design choice is unsound: it will classify adding a variable twice as an independent sum: **do** `x` ← `dpCount` ϵ `ds; return (add [x, x])`. It is also possible to consider various ways to add symbolic values to boost accuracy. We could easily write a pre-processing function which, for instance, firstly partitions the arguments into subset of independently generated values, applies `add` to them (thus triggering ADD-CHERNOFF-UNION), and finally applies `add` to the obtained results (thus triggering ADD-UNION). The implementation of DPella enables to write such functions in a few lines of code.

[8]For simplicity, we assume that maps are implemented as functions

V. CASE STUDIES

Category	Application	Programs
PINQ-like	CDFs [28]	`cdf1`, `cdf2`, `cdfSmart`
	Term frequency [2]	`queryFreq`, `queriesFreq`
	Network analysis [28]	`packetSize`, `portSize`
	Cumulative sums [29]	`cumulSum1` `cumulSum2` `cumulSumSmart`
Counting queries	Range queries via Identity, Histograms [31], and Wavelet [32]	`i_n` `h_n` `y_n`

TABLE I: Implemented literature examples

In this section, we will discuss the advantages and limitations of our programming framework. Moreover, we will go in-depth into using DPella to analyze the interplay of privacy and accuracy parameters in hierarchical histograms.

A. DPella expressiveness

First, we start by exploring the expressiveness of DPella. For this, we have built several analyses found in the DP literature—see Table I—which we classify into two categories, *PINQ-like queries* and *counting queries*. The former class allows us to compare DPella expressivity with the one of PINQ, while the latter with APEx.

PINQ-like queries: We have implemented most of PINQ's examples [2, 28], such as, different versions of CDFs (sequential, parallel, and hybrid) and network tracing-like analyses (such as determining the frequency a term or several terms have been searched by the users, and computing port's and packets' size distribution); additionally, we considered analyses of *cumulative sums* [29]—which are queries that share some commonalities with CDFs. The interest over differentially private CDFs and cumulative partial sums applications rely on the existing several approaches to inject noise, such choices will directly impact the accuracy of our results, and therefore,

are ideal to be tested and analyzed in DPella. The structures of these examples follow closely the ones of the CDFs presented in previous sections, which are straightforward implementations. DPella supports these queries naturally since its expressiveness relies on its primitives and, by construction, they follow PINQ's ones very closely. However, as stated in previous sections, our framework goes a step further and exposes to data analysts the accuracy bound achieved by the specific implementation. This feature allows data analyst to reason about accuracy of the results—without actually executing the query—by varying i) the strategy of the implementation ii) the parameters of the query. For instance, in Section II, we have shown how an analyst can inspect the error of a sequential and parallel strategy to compute the CDF of packet lengths. Furthermore, the data analyst can take advantage of DPella being an embedded DSL and write a Haskell function that takes any of the approaches (`cdf1` or `cdf2`) and varies epsilon aiming to certain error tolerance (for a fixed confidence interval), or vice versa. Such a function can be as simple as a brute force analysis or as complex as an heuristic algorithm.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 8: $\mathbf{W_{R_4}}$

Counting queries: To compare our approach with the tool APEx [18], we consider range queries analyses—an specific subclass of counting queries. APEx uses the *matrix mechanism* [30] to compute counting queries. This algorithm answers a set of linear queries (called the *workload*) by calibrating the noise to specific properties of the workload while preserving differential privacy. More in detail, the matrix mechanism uses some *query strategies* as an intermediate device to answer a workload; returning a DP version of the query strategies (obtained using the Laplace or Gaussian mechanism), from which noisy answers of the workload are derived. The matrix mechanism achieves an almost optimal error on counting queries. To achieve such error, the algorithm uses several non-trivial transformations which cannot be implemented easily in terms of other components. APEx implements it as a black-box and we could do the same in DPella (see Section VI). Instead, in this section we show how DPella can be directly used to answer sets of counting queries using some of the ideas behind the design of the matrix mechanism, and how these answers improve with respect to answering the queries naively, thanks to the use of partition and the Chernoff bound.

To do this, we have implemented several strategies to answer an specific workload $\mathbf{W_R}$: the set of all range queries over a domain. Figure 8 illustrates the workload that would be answer for a frequency count of four ranges. Having the identity $\mathbf{I_4}$, hierarchical $\mathbf{H_4}$ and wavelet $\mathbf{Y_4}$ strategies to compute the noisy count of each range, binary hierarchy of sums, and the Haar wavelet, respectively. Our implementation generates noisy counts and any possible combination of them will yield (at least) the same error as using strategy $\mathbf{I_4}$. In other words, the more accurate answer for $\mathbf{W_R}$ will be yield by the identity strategy. This is not unexpected, since in order to use the other queries strategies more efficiently we would need transformations similar to the ones used in the matrix mechanism.

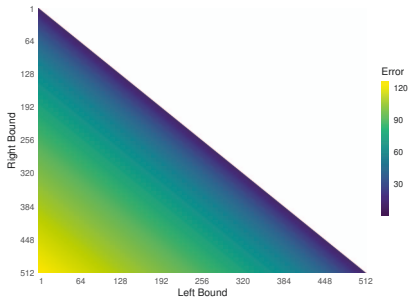

Fig. 9: Error of each range query in $\mathbf{W_R}$ using strategy $\mathbf{I_n}$ with $n = 512, \epsilon = 1$, and $\beta = 0.05$

Figure 9 exposes the error of answering each range query (i.e., each row) in $\mathbf{W_R}$ with strategy $\mathbf{I_n}$ and $n = 512$. While we use the same kind of plot, this error cannot be directly compared with the one shown in Figure 7 of [30], since we use a different error metrics: (α,β)-accuracy vs MSE. Nonetheless, we share the tendency of having lower error on small ranges and significant error on large ranges. Now, since the noisy values that will be added (using the function `add`) are statistically independent, we can use the Chernoff bound to show that the error is approximately $\mathcal{O}(\sqrt{n})$ for each range query, and a maximum error of $\mathcal{O}(\sqrt{n \log n})$ for answering any query in $\mathbf{W_R}$. If we compare our maximum error $\mathcal{O}(\sqrt{n \log n})$ with the one of the matrix mechanism based on the identity strategy $\mathcal{O}(n/\epsilon^2)$, it becomes evident how Chernoff bound is useful to provide tighter accuracy bounds. Unfortunately, as previously stated, the error of strategies $\mathbf{H_n}$ and $\mathbf{Y_n}$ in DPella is not better than the one of the strategy $\mathbf{I_n}$, so we cannot reach the same accuracy the matrix mechanism achieves with these strategies (see Figure 7 of [30]). This limitation can be addressed by leveraging the fact that DPella is a programming framerwork that could be *extended* by adding the matrix mechanism—and some other features—as black-box primitives.

VI. Limitations & Extensions

We have discussed so far the use of DPella as an API allowing a programmer to implement her own data analyses. However, we foreseen DPella to also serve as a "glue" which enables a programmer to integrate arbitrary DP-algorithms, as (black-box) building blocks while reasoning about accuracy. In this light, our design supports the introduction of new primitives when some analyses cannot be directly implemented because either (i) the static analysis for accuracy provided by DPella is too conservative, or (ii) DPella's API building blocks are not enough to express the desired analysis. Below, we describe several possible such extensions.

The matrix mechanism (MM): As we discussed in the previous section, in some situations DPella allows to answer in an accurate way multiple counting queries in a way that is similar to the MM. As an example, DPella estimates accuracy better then MM for the strategy $\mathbf{I}$—recall Section V. However,

for other workloads and other strategies the accuracy provided by DPella is too conservative. To consider other workloads and strategies, the MM can be incorporated into DPella as a primitive for answering counting queries. The requirements for this are that the return values are tainted, and that we have an iCDFs for it—this can be calculated as in [18]. In general, it is sound to add new primitives which permit a more precise accuracy analysis as long as the return values are tainted, and an accuracy information is provided—thus effectively allowing to further compose the primitive with other analyses by means of the union bound.

Primitives with non-compositional privacy analyses: Several DP-algorithms have a privacy analysis which does not follow directly by composition. Some well-known examples are report-noisy-max, the exponential mechanism, and the sparse-vector technique—see [20] and [43] for more details. In their natural implementations, these algorithms branch on the result of some noised query's result, and the privacy analyses use some properties of the noise distributions that are not directly expressible in terms of composition of differentially private components. Because DPella's API does not allow to branch on the results of noised queries, and because the privacy analyses that DPella support are based on composition, we cannot implement these analyses directly using the DPella API. However, we can provide them as (black-box) primitives. We already discussed how to integrate report-noisy-max through a primitive `dpMax` (Figure 3). The exponential mechanism (EM) can be incorporated into DPella in a similar way. One subtleties that one has to consider is the fact that the privacy guarantee of EM depends on a bound of the sensitivity of the score function. We handle this by requiring the score function's output to be bound between 0 and 1, bounding the sensitivity to be at most 1. As with `dpMax`, the output of EM is tainted. The EM is an important mechanism which allows to implement many other techniques. In particular, we can use EM to implement the offline version of the sparse vector technique, as discussed in [20]. These components allow DPella to support automated reasoning about accuracy for complex algorithms such as the offline version of the MWEM algorithm [44] following an analysis similar to the one discussed in [22].

Online adaptive algorithms: Several DP-algorithms have different implementations depending if they work offline—where all the decision are taken upfront before running the program—or online—where some of the decision are taken while running the program. Online algorithms usually have a more involved control flow which depend on information that are available at runtime. As an example, the online version of the sparse vector technique uses the result of a DP query to decide whether to stop or not the computation (or whether to stop or not giving meaningful answers). These kind of algorithms usually are based on some re-use of a noised result which correspond to a taint value in DPella. So, the current design of DPella cannot support them. We plan to explore as future how to integrate these algorithms in DPella.

Improving accuracy through post-processing: Several works have explored the use of post-processing techniques to improve on accuracy, e.g. [31, 45, 46]. Most of these works use accuracy measure that differ from the one we consider here, and use some specific properties of the particular problem at hand. As an example, the work by Hay et al. [31] describes how to boost accuracy in terms of *Mean Squared Error* (MSE) for DP hierarchical queries by post-processing the DP results by means of some relatively simple optimization. This improvement in accuracy relies among other things on the impact that the optimization has on the MSE, which does not directly apply to the α-β notion of accuracy we use here. We expect that, also for the notion of α-β accuracy we use, it is possible to use post-processing for improve accuracy. However, we leave this for future works. Moreover, the reason for us to chose α-β accuracy as the principal notion of accuracy in DPella is because of its compositional nature expressible through the use of probability bounds. It is an interesting future direction to design a similar compositional theory also for other accuracy notions such as MSE. We expect DPella to be extensible to incorporate such a theory, once it is available.

VII. Related work

Programming frameworks for DP: PINQ [2] uses dynamic tracking and sensitivity information to guarantee privacy of computations. Among the frameworks and tools sharing features with PINQ we highlight: Airavat [3] ; wPINQ [47]; DJoin [38]; Ektelo [12]; Flex [40]; and PrivateSQL [48]. In contrast to DPella, none of these works keeps track of accuracy, nor static analysis for privacy or accuracy. As discussed in Section III, DPella supports a limited form of joins, and it is still able to provide accuracy estimates. We leave as future work to support more general join operations through techniques similar to the ones proposed in Flex and PrivateSQL. While several of the components from the frameworks discussed above are not supported in the current implementation of DPella, these can be added as black-box primitives, as we discussed in Section VI. All the programming frameworks discussed above support reasoning about privacy for complex data analyses while neglecting accuracy, whereas DPella supports accuracy, but restricts the programming framework to rule out certain analysis (e.g., adaptive ones) for which we do not have a general compositional theory, yet.

Tools for DP: In a way similar to DPella, there exist tools which support reasoning about accuracy and restrict the kind of data analyses they support. GUPT [15] is a tool based on the sample-and-aggregate framework for differential privacy [49]. GUPT allows analysts to specify the target accuracy of the output, and compute privacy from it—or vice versa. This approach has inspired several of the subsequent works and also our design. The limitations of GUPT are that it supports only analyses that fit in the sample-and-aggregate framework, and it supports only confidence intervals estimates expressed at the level of individual queries. In contrast, DPella supports analyses of a more general class, such as the ones we discussed in Section II and Section V, and it also allows to reason about the accuracy of combined queries, rather that just about the individual ones. PSI [17] offers to the data analyst an interface

for selecting either the level of accuracy that she wants to reach, or the level of privacy she wants to impose. The error estimates that PSI provides are similar to the ones that are supported in DPella. However, similarly to GUPT, PSI supports only a limited set of transformations and primitives, it supports only confidence intervals at the level of individual queries, and in its current form it does not allow analysts to submit their own (programmed) queries.

APEx [18] has similar goals as DPella and it allows data analysts to write queries as SQL-like statements. However, the model that APEx uses is different from DPella's. It supports three kind of queries: WCQ (counting queries), ICQ (iceberg counting queries), and TCQ (top-k counting queries). To answer WCQ queries, APEx uses the matrix mechanism (recall Section V) and applies a Monte Carlo simulations to achieve accuracy bounds in terms of α and β, and to determine the least privacy parameter (ϵ) that fits those bounds. We have shown how DPella can be used to answer queries based on the identity strategies using partition and concentration bounds. To answer effectively different workloads and strategies as well as ICQ and TCQ queries, we would need to extend DPella with the matrix mechanism as a black-box (recall Section VI). While APEx supports advanced query strategies, it does not provide means to reason about combinations of analyses, e.g., it does not support reasoning about the accuracy of a query using results from WCQs queries to perform TCQs ones. DPella instead has been designed specifically to support the combination of different queries. As we discussed in Section VI, DPella can be seen as a programming environment that could be combined with some of the analyses supported by tools similar to PSI, GUPT or APEx in order to reason about the accuracy of the combined queries.

Formal Calculi for DP: There are several works on enforcing differential privacy relying on different models and techniques. Within this group are Fuzz [4]—a programming language which enforces (pure) differential privacy of computations using a linear type system which keeps track of program sensitivity—and its derivatives DFuzz [6], Adaptive Fuzz [10], Fuzzi [13], and Duet [50]. Hoare2 [7], a programming language which enforces (pure or approximate) differential privacy using program verification, together with its extension PrivInfer [8] supporting differentially private Bayesian programming; and other systems using similar ideas [43, 51, 9, 52].

Barthe et al. [29] devise a method for proving differential privacy using Hoare logic. Their method uses accuracy bounds for the Laplace Mechanism for proving privacy bounds of the Propose-Test-Release Mechanism, but cannot be used to prove accuracy bounds of arbitrary computations. Later, Barthe et al. [22] develop a Hoare-style logic, named aHL, internalizing the use of the union bound for reasoning about probabilistic imperative programs. The authors show how to use aHL for reasoning in a mechanized way about accuracy bounds of several basic techniques such as report-noisy-max, sparse vector and MWEM. This work has largely inspired our design of DPella but with several differences. First, aHL mixes the reasoning about accuracy with the more classical Hoare-style reasoning. This choice makes aHL very expressive but difficult to automate. DPella instead favors automation over expressivity. As discussed before, the use of DPella to derive accuracy bound is transparent to a programmer thanks to its automation. On the other hand, there are mechanisms that can be analyzed using aHL and cannot be analyzed using DPella, e.g. adaptive online algorithms. Second, aHL supports only reasoning about accuracy but it does not support reasoning about privacy. This makes it difficult to use aHL for reasoning about the privacy-accuracy trade-offs. Finally, aHL supports only reasoning using the union bound and it does not support reasoning based on the Chernoff bound. This makes DPella more precise on the algorithms that can be analyzed using the Chernoff Bound. Barthe et al [53] use aHL, in combination with a logic supporting reasoning by coupling, to verify differentially private algorithms whose privacy guarantee depends on the accuracy guarantee of some sub-component. We leave exploring this direction for future works. More recently, Smith et al. [27] propose an automated approach for computing accuracy bounds of probabilistic imperative programs. This work shares some similarities with our. However, it does not support reasoning about privacy, and it only uses the Union Bound and do not attempt to reason about probabilistic independence to obtain tighter bounds.

Other works: In a recent work, Ligett et al. [54] propose a framework for developing differentially private algorithms under accuracy constraints. This allows one to chose a given level of accuracy first, and then finding the private algorithm meeting this accuracy. This framework is so far limited to empirical risk minimization problems and it is not supported by a system, yet.

VIII. Conclusions

DPella is a programming framework for reasoning about privacy, accuracy, and their trade-offs. DPella uses taint analysis to detect probabilistic independence and derive tighter accuracy bounds using Chernoff bounds. We believe the principles behind DPella, i.e., the use of concentration bounds guided by taint analysis, could generalize for more notions of privacy such as Renyi-DP [55], concentrated differential privacy [56], zero concentrated differential privacy [57], or truncated concentrated differential privacy [58] (as done with (ϵ, δ)-DP). As future work, we envision lifting the restriction that programs should not branch on query outputs.

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APPENDIX

A. Primitive `dpPart` and disjoint datasets

```
1 q :: ϵ → [Color] → Data 1 Double
2   → Query (Map Color Double)
3 q eps bins dataset = dpPart id dataset dps
4   where dps = fromList [(c, λds → dpCount eps dataset)
5       -- dps = fromList [(c, λds → dpCount eps ds
6                           | c ← bins]
```

Fig. 10: DP-histograms by using `dpPart`

We present the code in Figure 10. Query q produces a ϵ-DP histogram of the colors found in the argument `dataset`, which rows are of type `Color` and variable `bins` enumerates all the possible values of such type. The code partitions the dataset by using the function `id::Color → Color` (line 2) and executes the aggregation counting query (`dpCount`) in each partition (line 3)—function `fromList` creates a map from a list of pairs. The attentive reader could notice that `dpCount` is applied to the original `dataset` rather than the partitions. This type of errors could lead to break privacy as well as inconsistencies when estimating the required privacy budget. A correct implementation consists on executing `dpCount` on the corresponding partition as shown in the commented line 4.

The IFC analysis assigns the provenance of `dataset` in q to the top-level fragment of the query rather than to sub-queries executed in each partition—and DPella will raise an error at compile time when `ds` is accessed by the sub-queries! Instead, if we comment line 3 and uncomment line 4, the query `q` is successfully run by DPella (when there is enough privacy budget) since every partition is only accessing their own partitioned data (denoted by variable `ds`).

B. Taint analysis example

Figure 11 presents the query plan `totalCount` which adds the results of hundred `dpCount` queries over different datasets, namely ds_1, ds_2, ..., ds_{100}. (The `...` denotes code intentionally left unspecified.) The code calls the primitive `add` with the results of calling `dpCount`. (We use $[x_1, x_2, x_3]$ to denote the list with elements x_1, x_2, and x_3.) What would it be then the theoretical error of `totalCount`? The accuracy calculation depends on whether all the values are untainted in line 7. When no dependencies are detected between $\mathtt{v}_1$, $\mathtt{v}_2$, ..., v_{100}, namely all the values are untainted, DPella applies Chernoff bound in order to give a tighter error estimation. Instead, for instance, if $\mathtt{v}_3$ were computed as an aggregation

```
1 totalCount :: Query (Value Double)
2 totalCount = do
3   v1 ← dpCount 0.3  ds1
4   v2 ← dpCount 0.25 ds2
5   ...
6   v100 ← dpCount 0.5  ds100
7   return (add [v1, v2, .., v100])
```

Fig. 11: Combination of sub-queries results

$$\textsc{Norm-Inf}\quad \frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \qquad \mathtt{iCDF} = \lambda\beta \to max\ \{|\mathtt{iCDF}_j(\frac{\beta}{n})|\}_{j=1\ldots n}}{\mathtt{norm}_\infty\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathcal{S}[\mathtt{iCDF_M}, 0, \emptyset]}$$

$$\textsc{Norm-1}\quad \frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \qquad \mathtt{iCDF} = \lambda\beta \to \sum_{j=1}^{n} |\mathtt{iCDF}_j(\frac{\beta}{n})|}{\mathtt{norm}_1\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathcal{S}[\mathtt{iCDF}, 0, \emptyset]}$$

Fig. 12: Calculation of norms

of $\mathtt{v_1}$ and $\mathtt{v_2}$, e.g., **let** $\mathtt{v_3} = \mathtt{add}\ [\mathtt{v_1}, \mathtt{v_2}]$, then line 7 applies union bound since $\mathtt{v_3}$ is a tainted value. With taint analysis, DPella is capable to detect dependencies among terms of type `Value Double`, and leverages that information to apply different concentrations bounds.

C. Norms calculation

Figure 12 shows our static analysis when computing $\mathtt{norm}_\infty$ and $\mathtt{norm}_1$, respectively. There is nothing special about the rules except to note that the results are symbolic values which are tainted. The reason for that is that norms are designed to condense (in one measure) the error of the list of the arguments. By doing so, it is hard to assign an specific Laplace distribution with sensitivity s to the overall given vector. We simply say that the return symbolic values are tainted—thus they can only be aggregated by ADD-UNION in Figure 7.

D. Accuracy for dpMax

Figure 13 shows the `iCDF` computed by `dpMax`, which aligns with the one appearing in [22]. Observe that the return value is tainted. The reason for that relies in the fact that the result, which is one of the responses in `res`, contains no noise—it is rather the process that lead to determining the winning response which has been "noisy." In this light, no scale of noise nor distribution can be associated to the response—as we did, for instance, with `dpCount`.

$$\textsc{DPCount}\quad \frac{ds :: \mathtt{Data\ 1\ r} \qquad \mathtt{iCDF} = \lambda\beta \to \frac{4}{\epsilon} \cdot log(\frac{\mathtt{length\ res}}{\beta})}{\mathtt{dpMax}\ \epsilon\ \mathtt{res\ vote\ ds} \rhd \mathcal{S}[\mathtt{iCDF}, 0, \emptyset]}$$

Fig. 13: `iCDF` implemented by `dpMax`

$$\textsc{DPCount}\quad \frac{ds :: \mathtt{Data\ s\ r} \qquad \sigma = \mathtt{s} \cdot 1 \cdot \sqrt{2 \cdot log(1.25/\delta)}/\epsilon \qquad \mathtt{iCDF} = \lambda\beta \to \sigma \cdot \sqrt{2 \cdot log(2/\beta)} \qquad \mathtt{t}\ fresh}{\mathtt{dpCount}\ \epsilon\ ds \rhd \mathcal{S}[\mathtt{iCDF}, \sigma^2, \{\mathtt{t}\}]}$$

Fig. 14: Accuracy analysis for aggregations

$$\textsc{Chernoff-Bound}\quad \frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \qquad \mathtt{iCDF} = \lambda\beta \to \sqrt{2 \cdot \sum_{j=1}^{n} s_j \cdot \log(1/\beta)}}{\mathtt{cb}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathtt{iCDF}}$$

$$\textsc{Add-Chernoff-Union}\quad \frac{\mathtt{v_j} = \mathcal{S}[\mathtt{iCDF_j}, \mathtt{s_j}, \mathtt{ts_j}] \qquad (\forall j \cdot \mathtt{ts}_j \neq \emptyset) \qquad \bigcap_{j=1\ldots n} \mathtt{ts}_j = \emptyset \qquad \mathtt{iCDF} = \lambda\beta \to \mathtt{min}\ (\mathtt{ub}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}]\ \beta)\ (\mathtt{cb}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}]\ \beta)}{\mathtt{add}\ [\mathtt{v_1}, \mathtt{v_2}, \ldots, \mathtt{v_n}] \rightsquigarrow \mathcal{S}[\mathtt{iCDF}, \sum_{j=1}^{n} \mathtt{s_j}, \bigcup_{j=1\ldots n} \mathtt{ts_j}]}$$

Fig. 15: Calculation of concentration bounds

E. Accuracy of Gaussian mechanism

For $Q : \mathrm{db} \to \mathbb{R}$ an arbitrary function with sensitivity Δ_Q as defined in II.2, the Gaussian mechanism with parameter σ add noise scaled to $\mathcal{N}(0, \sigma^2)$ to its output.

Theorem A.1 (Gaussian Mechanism [59])**.** *For any $\epsilon, \delta \in (0, 1)$, the Gaussian output perturbation mechanism with standard deviation $\sigma = \Delta_Q \sqrt{2 \log(\frac{1.25}{\delta})}/\epsilon$ is (ϵ, δ)-differentially private*

Definition A.1 (Accuracy for the Gaussian mechanism)**.** *Given a randomized query $\tilde{Q}(\cdot) : \mathrm{db} \to \mathbb{R}$ implemented with the Gaussian mechanism as previously described, we have that*

$$\Pr\left[|\tilde{Q}(D) - Q(D)| > \sigma\sqrt{2\log(2/\beta)}\right] \leqslant \beta \tag{6}$$

Figure 15 shows how concentration bounds are applied for the case of the Gaussian mechanism—UNION-BOUND and ADD-UNION are omitted since they are the same as the ones in Figure 7. In general, the accuracy analysis for addition of aggregations follows the one presented in Section IV. The main difference is seen when adding independent values. In this case, we use the well-known fact the addition of independent normally distributed random variables is also normally distributed. This means that after executing the ADD-CHERNOFF-UNION we do not lose information about the distribution of our result as we used to do under the Laplacian setting.

F. Privacy and accuracy trade-off analysis in DPella

We study histograms with certain hierarchical structure (commonly seen in Census Bureaus analyses) where different accuracy requirements are imposed per level and where varying one privacy or accuracy parameter can have a *cascade impact* on the privacy or accuracy of others. We consider the scenario where we would like to generate histograms from the

Adult database[9] to perform studies on gender balance. The information that we need to mine is not only an histogram of the genders (for simplicity, just male and female) but also how the gender distributes over age, and within that, how age distributes over nationality—thus exposing a hierarchical structure of three levels.

```
1 hierarchical1 [e1, e2, e3] dat = do
2   -- h1 :: Map Gen (Value Double)
3   -- h2 :: Map (Gen, Age) (Value Double)
4   -- h3 :: Map (Gen, Age, Nationality) (Value Double)
5   h1 ← byGen         e1 dat
6   h2 ← byGenAge      e2 dat
7   h3 ← byGenAgeNat   e3 dat
8   return (h1, h2, h3)
```

(a) Hierarchical histogram I: distribute budget among the levels

```
9  hierarchical2 e dat = do
10   h3 ← byGenAgeNat e dat
11   h2 ← level2 h3
12   h1 ← level1 h3
13   return (h1, h2, h3)
```

(b) Hierarchical histogram II: spend budget only on the most detailed histogram

Fig. 16: Implementation of hierarchical histograms

Our first approach is depicted in Figure 16a, where query $\texttt{hierarchical}_1$ generates three histograms with different levels of details. This query puts together the results produced by queries byGen, byGenAge, and byGenAgeNationality where each query generates an histogram of the specified set of attributes. Observe that these sub-queries are called with potentially different epsilons, namely $\texttt{e}_1$, $\texttt{e}_2$, and $\texttt{e}_3$, then under sequential composition, we expect $\texttt{hierarchical}_1$ to be $\texttt{e}_1$+$\texttt{e}_2$+$\texttt{e}_3$-differentially private.

We proceed to explore the possibilities to tune the privacy and accuracy parameters to our needs. In this case, we want a confidence of 95% for accuracy, i.e., $\beta = 0.05$, with a total budget of 3 ($\epsilon = 3$). We could manually try to take the budget $\epsilon = 3$ and distribute it to the different histograms in many different ways and analyze the implication for accuracy by calling accuracy on each sub-query. Instead, we write a small (simple, brute force) optimizer in Haskell that splits the budget uniformly among the queries, i.e., $\texttt{e}_1 = 1$, $\texttt{e}_2 = 1$, and $\texttt{e}_3 = 1$, and tries to find the minimum epsilon that meets the accuracy demands per histogram. In other words, we are interested in minimizing the *privacy loss* at each level bounding the maximum accepted error. The optimizer essentially adjusts the different epsilons and calls accuracy during the minimization process. To ensure termination, the optimizer aborts after a fixed number of calls to accuracy, or when the local budget $\texttt{e}_\texttt{i}$ is exhausted.

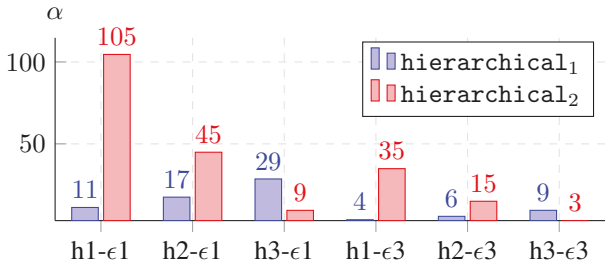
Table II shows some of our findings. The first row shows what happens when we impose an error of 100 at every level of detail, i.e., each bar in all the histograms could be at most $+/-100$ off. Then, we only need to spend a little part of our budget—the optimizer finds the minimum epsilons that adheres to the accuracy constrains. Instead, the second row shows that if we ask to be *gradually* more accurate on more detailed histograms, then the optimizer could fulfill the first two demands and aborted on the most detailed histogram (byGenAgeNat) since it could not find an epsilon that fulfills that requirement—the best we can do is spending all the budget and obtain and error bound of 9.43. Finally, the last row shows what happens if we want *gradually* tighter error bounds on the less detailed histograms. In this case, the middle layer can be "almost" fulfilled by expending all the budget and obtaining an error bound of 5.76 instead of 5. While the results from Table II could be acceptable for some data analysts, they might not be for others.

Histogram	α tolerance	Status	ϵ	α
byGen	100	✓	0.06	61.48
byGenAge	100	✓	0.06	96.13
byGenAgeNat	100	✓	0.11	85.74
byGen	10	✓	0.41	8.99
byGenAge	50	✓	0.16	36.05
byGenAgeNat	5	× MaxBud	1	9.43
byGen	5	✓	0.76	4.85
byGenAge	5	× MaxBud	1	5.76
byGenAgeNat	10	✓	0.96	9.82

TABLE II: Budgeting with α tolerances, $\beta = 0.05$, & total $\epsilon = 3$

We propose an alternative manner to implement the same query which consists on spending privacy budget *only* for the most detailed histogram. As shown in Figure 16b, this new approach spends all the budget e on calling $\texttt{h}_3 \leftarrow$ byGenAgeNat e dat. Subsequently, the algorithm builds the other histograms based on the information extracted from the most detailed one. For that, we add the noisy values of $\texttt{h}_3$ (using helper functions $\texttt{level}_2$ and $\texttt{level}_1$) creating the rest of the histograms representing the Cartesian products of gender and age, and gender, respectively. These methodology will use add and $\texttt{norm}_\infty$ to compute the derived histograms, and therefore will not consume more privacy budget. Observe that the query proceeds in a bottom-up fashion, i.e., it starts with the most detailed histogram and finishes with the less detailed one. Now that we have two implementations, which one is better? Which one yields the better trade-offs between privacy and accuracy? Figure 17 shows the accuracy of the different level of histograms, i.e., $\texttt{h}_1$, $\texttt{h}_2$, and $\texttt{h}_3$, when fixing $\beta = 0.05$ and a global budget of $\epsilon = 1$ (h1-ϵ1, h2-ϵ2, and h3-ϵ3) and $\epsilon = 3$ (h1-ϵ3, h2-ϵ3, and h3-ϵ3)—we obtained all this information by running repetitively the function accuracy. Form the graphics, we can infer that the splitting of the privacy budget per level often gives rise to more accurate histograms. However, observe the exception when $\epsilon = 3$ for $\texttt{hierarchical}_2$: in this case, $\texttt{hierarchical}_1$ will use an $\epsilon = 1$ in that histogram so it will receive a more noisy count than using $\epsilon = 3$.

[9]https://archive.ics.uci.edu/ml/datasets/adult



$\mathtt{h}_1 = \mathtt{byGen}, \mathtt{h}_2 = \mathtt{byGenAge}, \mathtt{h}_3 = \mathtt{byGenAgeNat}$

Fig. 17: $\mathtt{hierarchical}_1$ vs. $\mathtt{hierarchical}_2$

G. K-way marginal queries on synthetic data

We focus on the problem of releasing, in a differentially private manner, the k-way marginals of a binary dataset $D \in (0,1^d)^n$. This is a classical learning problem that has been extensively studied in the DP literature, see [60, 61, 62] among others. A k-marginal query, also called a k-conjunction, returns the count of how many individual records in D have $k < d$ attributes set to certain values. For simplicity, we focus on 3-way marginal queries to compare performance between DPella and using synthetic data. The goal of our analysis is to release all the 3-way marginals of a dataset.

This is implemented through the following code:

```
1   -- Perform all 3-way combinations up to attribute dim
2  allChecks :: ϵ → Int → Data s {0,1}^d
3             → [Query (Value Double)]
4  allChecks localEps dim db = do
5    (i,j,k) ← combinatory (dim-1) 3
6    let allOne r = (r !! i) ≡ (r !! j) ≡ (r !! k) ≡ 1
7    return (do tab ← dpWhere allOne db
8               dpCount localEps tab
9           )

10   -- Compute k-way marginals
11 threeMarginal :: ϵ → Int → Data s {0,1}^d
12    → Query (Value [Double])
13 threeMarginal localEps dim db = do
14   checks ← sequence (allChecks localEps dim db)
15   return (norm∞ checks)
```

Function `allChecks` counts how many records have 3-attributes set to 1. Auxiliary function `combinatory d k` generates k-tuples arising from the combination of indexes $0, 1, \ldots, d$ taken k at the time. In our example, the number of generated tuples is $\binom{dim}{3}$. For each tuple, `allChecks` filters the rows which have attributes `i`, `j`, and `k` set to 1 (`dpWhere allOne db`) for then making a noisy count (`dpCount localEps tab`). Function `threeMarginal` collects the counts for the different considered attributes and places them into a vector ($\mathtt{norm}_\infty$ `checks`).

We run `threeMarginal` considering a synthetic dataset (`db`) which has only *1 row with all the attributes set to zeros*. Setting all the attributes to zero produces that all the counts are 0, thus we are able to measure the noisy on each run and accuracy accordingly. We run `threeMarginal` approx. 1000 times for each dimension to measure the noisy magnitude, where we took the `1-`β percentile with $\beta = 0.05$ (as we did in many of our case studies). Notice that we have $\binom{dim}{3}$ queries and so $\binom{dim}{3}$ independent sources of noise, which need an high number of runs to be well-represented. In general, for this kind of task one is interested in bounding the max error that can occur in one of the queries (the ℓ_∞ norm over the output). For this task, the empirical error is well aligned with the theoretical one provided by DPella by calling the function `accuracy`. The latter is computed by taking a union bound over the error of each individual query. For each query we have a tight bound and the union bound gives us a tight bound over the max. However, we observe a significant different in performance.

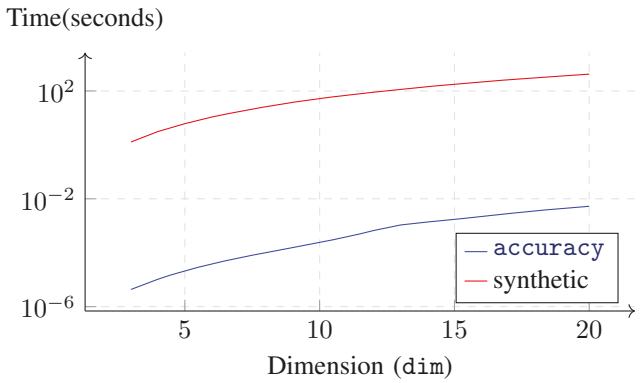


Fig. 18: Performance comparison between `accuracy` (DPella) and estimating errors using synthetic analysis

Figure 18 shows (in log scale) the time difference when calculating accuracy by DPella and on synthetic data when the dimension of the dataset increases. Already in low dimension, the difference in performance is many orders of magnitude in favor of DPella—a tendency which does not change when the dimension goes above 20. The main reason for that comes down to that DPella, as an static analysis, do not execute the filtering `dpWhere allOne db` (as well as any other transformation, recall Section IV-B) which an approach based on synthetic data should do and many times—in our case 1000 iterations for each dimension. We expect that for more complex tasks this difference is even more evident.