J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2019-0031

Journal für die reine und angewandte Mathematik


# Integral operators, bispectrality and growth of Fourier algebras

By *W. Riley Casper* at Baton Rouge and *Milen T. Yakimov* at Baton Rouge

**Abstract.** In the mid 1980s it was conjectured that every bispectral meromorphic function $\psi(x, y)$ gives rise to an integral operator $K_\psi(x, y)$ which possesses a commuting differential operator. This has been verified by a direct computation for several families of functions $\psi(x, y)$ where the commuting differential operator is of order $\leq 6$. We prove a general version of this conjecture for all self-adjoint bispectral functions of rank 1 and all self-adjoint bispectral Darboux transformations of the rank 2 Bessel and Airy functions. The method is based on a theorem giving an exact estimate of the second- and first-order terms of the growth of the Fourier algebra of each such bispectral function. From it we obtain a sharp upper bound on the order of the commuting differential operator for the integral kernel $K_\psi(x, y)$ leading to a fast algorithmic procedure for constructing the differential operator; unlike the previous examples its order is arbitrarily high. We prove that the above classes of bispectral functions are parametrized by infinite-dimensional Grassmannians which are the Lagrangian loci of the Wilson adelic Grassmannian and its analogs in rank 2.

## 1. Introduction

**1.1. Main result on bispectrality and integral operators.** A meromorphic function $\psi(x, y)$ on an open subset of $\mathbb{C}^2$ is called *bispectral* if it is an eigenfunction of a nonzero differential operator in each of the two variables. This notion was introduced by Duistermaat and Grünbaum in [14] in relation to computer tomography and signal processing and since then has found relations to many other areas: soliton equations [5, 27, 41], Calogero–Moser spaces and systems [9, 10, 42], orthogonal polynomials [18, 25, 26], $W$-algebras and Kac–Moody algebras [6, 8], automorphisms and ideal structure of rings of differential operators [9]. To $\psi(x, y)$ one associates the integral operator

$$T_\psi : f(x) \mapsto \int_{\Gamma_1} K_\psi(x, y) f(y)\, dy \quad \text{with kernel} \quad K_\psi(x, y) = \int_{\Gamma_2} \psi(x, z)\psi(y, z)\, dy.$$

The research of W. Riley Casper was supported by an AMS-Simons Travel Grant and that of Milen T. Yakimov by NSF grants DMS-1601862, DMS-1901830 and Bulgarian Science Fund grant DN02/05.

Here $\Gamma_1$ and $\Gamma_2$ represent paths in $\mathbb{C}$, chosen such that $\psi(x, y) \in L^2(\Gamma_1 \times \Gamma_2)$, along with other analytic convergence conditions. The following property was conjectured in the mid 1980s and was in the heart of the formalization of the notion of bispectrality in [14].

**Conjecture.** Under mild conditions on the bispectral function $\psi(x, y)$ and the paths $\Gamma_1, \Gamma_2$, the integral operator $T_\psi$ posses a commuting differential operator.

The commutativity of integral and differential operators is formalized in Section 3.4. The commutativity property in the conjecture has substantial applications to the *analytic properties* of the integral operator $T_\psi$ and to the *numerical computation* of its spectrum and eigenvalues.

The conjecture was proved in the following special cases:

(1) For the sine and Bessel bispectral functions $\psi(x, y) = e^{xy}$ and $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$, Landau, Pollak, Slepian [29, 36, 37] constructed a second-order differential operator commuting with $T_\psi$ and used it in time-band limiting in signal processing. Mehta proved the fact for $\psi(x, y) = e^{xy}$ independently and applied it to random matrices [30].

(2) Tracy and Widom [38, 39] proved that for the Airy function $\psi(x, y) = \mathrm{Ai}(x + y)$, $T_\psi$ posses a commuting second-order differential operator and applied the facts for the Bessel and Airy functions to asymptotics of Fredholm determinants and scaling limits of random matrix models.

(3) For two one-parameter, one-step Darboux transformations from the Bessel functions $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$ with $\nu = 1$ and $\nu = 2$, Grünbaum constructed a fourth- and sixth-order commuting differential operators [17].

(4) Second-order commuting differential and difference operators were constructed in several discrete-continuous and discrete-discrete situations starting with the Hermite, Laguerre and Jacobi polynomials [16] and expanding to several other situations [34, 35].

(5) Commuting differential and difference operators were also constructed for matrix-valued generalizations of the examples in (4), culminating in the work of Grünbaum, Pacharoni, and Zurrian [19] which constructs second-order commuting differential/difference operators for matrix orthogonal polynomials whose weights satisfy a functional equation.

We obtain the following general solution of the conjecture:

**Theorem A.** (i) *For all self-adjoint bispectral Darboux transformations* $\widetilde{\psi}(x, y)$ *of the exponential function* $e^{xy}$ *and self-adjoint bispectral Darboux transformations of the rank* 2 *Airy* $\mathrm{Ai}(x + y)$ *and Bessel* $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$ *functions, the integral operator with kernel* $K_{\widetilde{\psi}}(x, y)$ *posses a commuting differential operator which is formally symmetric (i.e., equals its formal adjoint).*

(ii) *The class of bispectral functions in part* (i) *of rank* 1 *are parametrized by the points of the infinite-dimensional Grassmannian which is the Lagrangian locus of the Wilson's adelic Grassmannian* [41]. *The class of bispectral functions in part* (i) *of rank* 2 *are parametrized by the points of the Lagrangian loci of the infinite-dimensional Grassmannians of rank* 2 *bispectral functions from* [5].

Additionally we obtain effective upper bounds on minimum order of a nonconstant commuting differential operator in Theorems 5.3, 5.6, 5.9. The arXiv version of the present paper

(arXiv:1807.09314) contains an additional section where we describe an algorithm for fast computation of the operators relying only on the calculation of products of differential operators and the solution of a finite-dimensional system of linear equations. The arXiv version also features a variety of examples illustrating this algorithm, including examples of differential operators of order $6, 8$ and $22$. The construction of differential operators of such high-order commuting with integral operators was not previously feasible.

The first part of the theorem is proved in Section 5. The second part is proved in Section 6. Section 3 contains background on the notion of self-adjoint bispectral Darboux transformations and its role in previous works on the classification of bispectral functions.

Theorem A has the following important special cases:

(1) By the main result of [14], all bispectral meromorphic functions $\widetilde{\psi}(x, y)$ that are eigenfunctions of a second-order differential operator are obtained as iterated self-adjoint bispectral Darboux transformations from the Bessel functions $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$ and are covered as special cases by Theorem A. Even in this special situation the theorem is new and few cases of it were previously known, cf. Section 1.1 (3).

(2) By the main result of [41] (and its interpretation in [5]), all rank 1 bispectral functions $\widetilde{\psi}(x, y)$ are bispectral Darboux transformations from the exponential function $e^{xy}$. Such a function is called self-adjoint if it is an eigenfunction of a formally symmetric differential operator in $x$ and $y$. All rank 1 self-adjoint bispectral functions $\widetilde{\psi}(x, y)$ are also covered by Theorem A.

**1.2. Main result on growth of Fourier algebras.** Our proof of Theorem A is based on the construction of Fourier algebras associated to bispectral functions and a sharp estimate on their growth. For a bispectral meromorphic function $\psi(x, y)$ defined on a connected open subset $U \times V$ of $\mathbb{C}^2$, define the *left and right Fourier algebras of differential operators* for $\psi$ by

$$\mathcal{F}_x(\psi) = \{\mathfrak{d}_x \in \mathfrak{D}(U) : \text{there exists a differential operator } \mathfrak{b}_y \in \mathfrak{D}(V) \\ \text{satisfying } \mathfrak{d}_x \cdot \psi(x, y) = \mathfrak{b}_y \cdot \psi(x, y)\}$$

and

$$\mathcal{F}_y(\psi) = \{\mathfrak{b}_y \in \mathfrak{D}(V) : \text{there exists a differential operator } \mathfrak{d}_x \in \mathfrak{D}(U) \\ \text{satisfying } \mathfrak{d}_x \cdot \psi(x, y) = \mathfrak{b}_y \cdot \psi(x, y)\}.$$

Here and below $\mathfrak{D}(U)$ denotes the algebra of differential operators with meromorphic coefficients on $U$. The algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ are anti-isomorphic, via the map

$$b_\psi : \mathcal{F}_x(\psi) \to \mathcal{F}_y(\psi)$$

defined by

$$\mathfrak{d}_x \cdot \psi(x, y) = b_\psi(\mathfrak{d}_x) \cdot \psi(x, y),$$

see Proposition 2.4. This map will be called the *generalized Fourier map* of $\psi$. The terminology is motivated from the fact that for $\psi(x, y) = e^{xy}$ one recovers the usual Fourier transformation of differential operators. The key idea is that each operator $\mathfrak{d}_x \in \mathcal{F}_x(\psi)$ has well-defined *order* $\operatorname{ord} \mathfrak{d}_x \in \mathbb{N}$ and *co-order* $\operatorname{ord} b_\psi(\mathfrak{d}_x) \in \mathbb{N}$. They give rise to an $\mathbb{N} \times \mathbb{N}$ *filtration* of $\mathcal{F}_x(\psi)$

formed by the subspaces

$$\mathcal{F}_x^{\ell,m}(\psi) = \{\mathfrak{d} \in \mathcal{F}_x(\psi) : \operatorname{ord}(\mathfrak{d}) \leq \ell,\ \operatorname{ord}(b_\psi(\mathfrak{b})) \leq m\} \quad \text{for} \quad \ell, m \in \mathbb{N}.$$

The spaces $\mathcal{F}_x^{\ell,m}(\psi)$ are finite-dimensional and one can study the *growth* of their dimensions as $\ell, m \to \infty$. We will denote by $\mathcal{F}_{x,\mathrm{sym}}^{\ell,m}(\psi)$ the symmetric subspaces of $\mathcal{F}_x^{\ell,m}(\psi)$ consisting of operators $\mathfrak{d}_x \in \mathcal{F}_x^{\ell,m}(\psi)$ such that $\mathfrak{d}_x$ and $b_\psi(\mathfrak{d}_x)$ are *formally symmetric*.

After the breakthrough work of Wilson [41] bispectral functions are classified by rank (see Definition 3.7) via a realization as *bispectral Darboux transformations* in the terminology of [5]. This means that new bispectral functions $\widetilde{\psi}(x, y)$ are constructed from old ones $\psi(x, y)$ via a representation of the form

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)p(x)}\mathfrak{u} \cdot \psi(x, y) \quad \text{and} \quad \psi(x, y) = \widetilde{\mathfrak{u}} \cdot \frac{1}{\widetilde{q}(y)\widetilde{p}(x)}\widetilde{\psi}(x, y)$$

for some differential operators $\mathfrak{u}, \widetilde{\mathfrak{u}} \in \mathcal{F}_x(\psi)$ and polynomials $p(x)$, $\widetilde{p}(x)$ and $q(y)$, $\widetilde{q}(y)$ as in Definition 3.1. We call the pair $(\operatorname{ord}\mathfrak{u}, \operatorname{ord}\widetilde{\mathfrak{u}}) \in \mathbb{N} \times \mathbb{N}$ the *bidegree of the transformation*. We refer the reader to Section 3 for details on the relation of this construction to the classical Darboux process in terms of factorizations of differential operators.

For a fixed bispectral function $\psi(x, y)$, all bispectral Darboux transformations of $\psi(x, y)$ are classified by the points of infinite-dimensional Grassmannians generalizing Wilson's adelic Grassmannian [5, 41]. All rank 1 bispectral functions (classified by the points of the original adelic Grassmannian) are bispectral Darboux transformations from the function $e^{xy}$.

**Theorem B.** (i) *If $\widetilde{\psi}(x, y)$ is a bispectral Darboux transformation from the bispectral meromorphic function $\psi(x, y)$ satisfying natural mild assumptions, then for all $\ell, m \in \mathbb{C}$,*

$$|\dim \mathcal{F}_x^{\ell,m}(\widetilde{\psi}) - \dim \mathcal{F}_x^{\ell,m}(\psi)| \leq \text{const}$$

*for a constant that is independent on $\ell$ and $m$.*

(ii) *If $\widetilde{\psi}(x, y)$ is a self-adjoint bispectral Darboux transformation of bidegree $(d_1, d_2)$ from the exponential function $e^{yx}$, the Airy function $\mathrm{Ai}(x + y)$, or the Bessel functions $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$ with $\nu \in \mathbb{C} \setminus \mathbb{N}$, then*

$$|\dim \mathcal{F}_{x,sym}^{2\ell,2m}(\widetilde{\psi}) - (\ell m + \ell + m + 2)| \leq d_1 d_2$$

*for all $\ell, m \in \mathbb{N}$.*

The precise form of the first part of the theorem is given in Theorem 4.1 (see also Corollary 4.2 and Remark 4.4). The second part of the theorem is a combination of Theorem 4.1 and Lemmas 5.2, 5.5, and 5.8.

**Remark.** (i) Theorem B establishes the fact that the growth of both $\dim \mathcal{F}_x^{\ell,m}(\widetilde{\psi})$ and $\dim \mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\widetilde{\psi})$ as functions of $\ell$ and $m$ is quadratic for huge classes of bispectral functions. The theorem gives exactly the quadratic and the two linear terms of the dimension functions and a sharp upper bound on their constant terms.

(ii) In a typical situation we start with a simple bispectral function $\psi(x, y)$ which is an eigenfunction of low degree differential operators (e.g. $e^{yx}$, $\mathrm{Ai}(x + y)$, or $\sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$) and we built a very complicated bispectral function $\widetilde{\psi}(x, y)$ which is an eigenfunction of differential operators of very high degrees. The remarkable feature of the theorem is that it proves that the Fourier algebras $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_x(\psi)$ have the exactly same growth up to a linear term.

Theorem A is proved using the growth rates from Theorem B in conjunction with dimension estimates for the rank of the linear map taking formally symmetric differential operators to their bilinear concomitant (see Definition 6.5). By comparing these growth rates, we prove for large $\ell$ the algebra $\mathcal{F}^{\ell,\ell}_{x,\mathrm{sym}}(\psi)$ contains an operator whose concomitant is trivial and thus commutes with $T$. Thus growth of Fourier algebras is used as a bridge from bispectrality to commutativity of integral and differential operators.

This paper is a continuation of our collaboration with F. Alberto Grünbaum [20], which aimed at constructing a bridge between bispectrality and commuting integral and differential operators by controlling the sizes of Fourier algebras under Darboux transformation. This strategy was announced in [20]. Among other things, the results in this paper fully justify the statements in [20] which contained no proofs.

When bispectral functions $\widetilde{\psi}(x, y)$ are converted to wave functions $\widetilde{\Psi}(x, y)$ for the KP hierarchy via asymptotic expansions at $\infty$, the elements of the right Fourier algebra $\mathcal{F}_y(\widetilde{\psi})$ correspond to the $W$-constraints for the wave function $\widetilde{\Psi}(x, y)$. The latter generalize the string equation for the Airy wave function [28, 31, 43] which played a key role in quantum gravity and intersection theory on moduli spaces of curves. Consequently, in algebraic geometry the $W$-constraints of Gromov–Witten invariants and the total descendant potential of a simple singularity were extensively studied by Okounkov and Pandharipande [32], Bakalov and Milanov [7] and in many other works.

The $W$-constraints for wave functions of the Toda and multi-component KP hierarchies have played an important role in many situations. In the theory of random matrices, Adler and van Moerbeke [1–3] used Virasoro constraints to derive a system of partial differential equations for the distributions of the spectra of coupled random matrices. Bakalov, Horozov, and Yakimov [6] used $W$-constraints to prove that all tau-functions in the quasifinite representations of the $W_{1+\infty}$-algebra (in the sense of Frenkel, Kac, Radul, and Wang [15]) with highest weight vectors given by Bessel tau-functions are bispectral tau-functions. Our growth estimate theorem for the right Fourier algebra $\mathcal{F}_y(\widetilde{\psi})$ translates directly to a growth estimate theorem for the algebra of $W$-constraints of a wave function. In this way Theorem B has potential applications to the above topics in random matrices and representations of the $W_{1+\infty}$-algebra.

More generally, Virasoro constraints are linked to the isomonodromic deformations approach to random matrices from the works of Palmer [33], Harnad, Tracy, and Widom [22], Its and Harnad [21], and Borodin and Deift [11]. Of particular interest is the relation to the series of results in the literature on random matrices showing that the Fredholm determinants of various kernels are solutions of Painlevé equation. The latter naturally appear in other problems, e.g. representations of $U(\infty)$, see Borodin and Olshanski [12, 13]. We expect that the growth estimates from Theorem B on the algebra of $W$-constraints will also have applications in these respects. The main idea here is to apply the full algebra of $W$-constraints vs concrete Virasoro constraints.

We will use the following conventions for the notation in the paper. Differential operators will always appear in the Gothic font, e.g., $\mathfrak{d} = \partial_x^2 - \frac{2}{x^2}$. When needed, we will write $\mathfrak{d}_x$ in place of $\mathfrak{d}$ to emphasize the action of the operator in the variable $x$. In this case $\mathfrak{d}_y$ will represent the same operator, but acting in the variable $y$. We will denote by $\mathfrak{D}(\mathbb{C}[x])$ and $\mathfrak{D}(\mathbb{C}(x))$ the algebras of differential operators in $x$ with polynomial and rational coefficients. For a noncommutative algebra $R$ and $r, s \in R$, we set

$$\mathrm{ad}_r(s) = [r, s] = rs - sr.$$

**Acknowledgement.** We are grateful to F. Alberto Grünbaum for his insight, suggestions and support throughout the different stages of this project. We are also grateful to the anonymous referee for the very helpful comments, observations, suggestions, and corrections, which greatly improved the quality of the manuscript.

## 2. Bispectrality and double filtrations of Fourier algebras

In this section we collect background material on bispectral functions and Fourier algebras associated to them. We introduce one of the key players in the paper – a double filtration on each such algebra.

**2.1. Bispectral meromorphic functions and associated Fourier algebras.** We first review the definition of a bispectral meromorphic function. For an open subset $U$ of $\mathbb{C}$, denote by $\mathfrak{D}(U)$ the algebra of differential operators on $U$ with meromorphic coefficients.

**Definition 2.1.** A nonconstant meromorphic function $\psi(x, y)$ defined on a connected open subset $U \times V$ of $\mathbb{C}^2$ is said to be *bispectral* if there exist differential operators $\mathfrak{d} \in \mathfrak{D}(U)$ and $\mathfrak{b} \in \mathfrak{D}(V)$ such that

$$\mathfrak{d}_x \cdot \psi(x, y) = g(y)\psi(x, y) \quad \text{and} \quad \mathfrak{b}_y \cdot \psi(x, y) = f(x)\psi(x, y)$$

for some nonconstant functions $f(x)$ and $g(y)$ meromorphic on $U$ and $V$, respectively.

There are three examples of bispectral meromorphic functions that will play a fundamental role in this paper: the exponential, Airy and Bessel functions. We will refer to them as the *elementary bispectral functions*. Due to their central role, we will introduce a notation for each that will be used throughout the paper. The notation and values of the elementary bispectral functions is given in Table 1, where for $\nu \in \mathbb{C}$ the expression $K_\nu(t)$ denotes the Bessel function of the second kind and $\mathrm{Ai}(t)$ denotes Airy function of the first kind.

| name | function | operator |
|---|---|---|
| bispectral exponential | $\psi_{\exp}(x, y) = e^{xy}$ | $\partial_x$ |
| bispectral Airy | $\psi_{\mathrm{Ai}}(x, y) = \mathrm{Ai}(x + y)$ | $\mathfrak{d}_{\mathrm{Ai}} = \partial_x^2 - x$ |
| bispectral Bessel | $\psi_{\mathrm{Be}(\nu)}(x, y) = \sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$ | $\mathfrak{d}_{\mathrm{Be}(\nu)} = \partial_x^2 - \frac{\nu(\nu+1)}{x^2}$ |

Table 1. The elementary bispectral functions.

The bispectral Bessel functions $\psi_{\mathrm{Be}(\nu)}(x, y)$ are in particular a one parameter family of functions indexed by $\nu$. For integer values of $\nu$ the Bessel functions simplify to rational functions multiplied by exponential functions. For example $\psi_{\mathrm{Be}(0)}(x, y)$ is proportional to $e^{xy}$ and $\psi_{\mathrm{Be}(1)}(x, y)$ is proportional to $e^{xy}(1 - (xy)^{-1})$.

Each of the elementary bispectral functions are, as the name suggests, bispectral and thus are families of eigenfunctions in both variables $x$ and $y$. For example the bispectral exponential function satisfies

$$\partial_x \cdot \psi_{\exp}(x, y) = y\psi_{\exp}(x, y) \quad \text{and} \quad \partial_y \cdot \psi_{\exp}(x, y) = x\psi_{\exp}(x, y).$$

The Bessel and Airy bispectral functions are eigenfunctions of the Bessel and Airy operators $\mathfrak{d}_{\mathrm{Be}(\nu)}$ and $\mathfrak{d}_{\mathrm{Ai}}$ from the last column of Table 1. The associated differential equations are

$$\mathfrak{d}_{\mathrm{Be}(\nu),x} \cdot \psi_{\mathrm{Be}(\nu)}(x,y) = y^2 \psi_{\mathrm{Be}(\nu)}(x,y), \tag{2.1}$$
$$\mathfrak{d}_{\mathrm{Be}(\nu),y} \cdot \psi_{\mathrm{Be}(\nu)}(x,y) = x^2 \psi_{\mathrm{Be}(\nu)}(x,y),$$
$$\mathfrak{d}_{\mathrm{Ai},x} \cdot \psi_{\mathrm{Ai}}(x,y) = y \psi_{\mathrm{Ai}}(x,y), \tag{2.2}$$
$$\mathfrak{d}_{\mathrm{Ai},y} \cdot \psi_{\mathrm{Ai}}(x,y) = x \psi_{\mathrm{Ai}}(x,y).$$

Bispectral meromorphic functions in general satisfy a wide collection of differential equations and the associated operators form algebras.

**Definition 2.2.** Let $\psi(x,y)$ be a bispectral meromorphic function defined on the connected open subset $U \times V$ of $\mathbb{C}^2$. We define the *left* and *right Fourier algebras* of differential operators for $\psi$ by

$$\mathcal{F}_x(\psi) = \{\mathfrak{d} \in \mathfrak{D}(U) : \text{there exists a differential operator } \mathfrak{b} \in \mathfrak{D}(V) \text{ satisfying } \mathfrak{d} \cdot \psi(x,y) = \mathfrak{b} \cdot \psi(x,y)\}$$

and

$$\mathcal{F}_y(\psi) = \{\mathfrak{b} \in \mathfrak{D}(V) : \text{there exists a differential operator } \mathfrak{d} \in \mathfrak{D}(U) \text{ satisfying } \mathfrak{d} \cdot \psi(x,y) = \mathfrak{b} \cdot \psi(x,y)\}.$$

The algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ come with distinguished subalgebras

$$\mathcal{B}_x(\psi) = \{\mathfrak{d} \in \mathcal{F}_x(\psi) : \text{there exists a meromorphic function } g(y) \text{ satisfying } \mathfrak{d} \cdot \psi(x,y) = g(y)\psi(x,y)\},$$
$$\mathcal{B}_y(\psi) = \{\mathfrak{b} \in \mathcal{F}_y(\psi) : \text{there exists a meromorphic function } f(x) \text{ satisfying } \mathfrak{b} \cdot \psi(x,y) = f(x)\psi(x,y)\}.$$

The algebras $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are called the algebras of *left* and *right bispectral differential operators* for $\psi(x,y)$.

The function $\psi(x,y)$ is an eigenfunction of all operators in these two algebras. In a neighborhood of a sufficiently nice point $(x_0, y_0)$ of the domain of $\psi(x,y)$, the coefficients of the operators in $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ will be analytic. Then by an appropriate change of variables, we may assume that $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ both contain a nonconstant differential operator whose leading coefficient is constant. Consequently, by [14, equation (1.20)], all functions $f(x)$ and $g(y)$ that appear in the definition of $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are polynomial. Furthermore, the algebras $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are necessarily commutative, so every element in each algebra will have constant leading coefficient. Throughout the rest of the paper, we adopt local coordinates giving us constant leading coefficients.

**Remark 2.3.** The name Fourier algebras is inspired by the case where $\psi$ is the simplest bispectral function $\psi_{\exp}(x,y) = e^{xy}$. In this case $\mathcal{F}_x(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[x])$, the algebra of differential operators with polynomial coefficients. Similarly, $\mathcal{F}_y(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[y])$ and

$$\mathfrak{d} \cdot e^{xy} = \mathfrak{b} \cdot e^{xy} \quad \text{for } \mathfrak{d} = \sum a_{jk} x^k \partial_x^j \text{ and } \mathfrak{b} = \sum a_{jk} y^j \partial_y^k.$$

The operator $\mathfrak{b}$ is the Fourier transform of $\mathfrak{d}$.

**Proposition 2.4.** *Let $\psi(x, y)$ be a bispectral meromorphic function defined on $U \times V$. The algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ are anti-isomorphic via the map $b_\psi : \mathcal{F}_x(\psi) \to \mathcal{F}_y(\psi)$ defined by sending $\mathfrak{d} \in \mathcal{F}_x(\psi)$ to $\mathfrak{b} \in \mathcal{F}_y(\psi)$, where $\mathfrak{d} \cdot \psi(x, y) = \mathfrak{b} \cdot \psi(x, y)$.*

*Proof.* Suppose $\mathfrak{d} \in \mathcal{F}_x(\psi)$ with $\mathfrak{d} \cdot \psi(x, y) = g(y)\psi(x, y)$ for some $g(y) \in \mathcal{F}_y(\psi)$ nonconstant. If there exists a nonzero $\mathfrak{b} \in \mathcal{F}_y(\psi)$ satisfying $\mathfrak{b} \cdot \psi(x, y) = 0$, then

$$\mathrm{ad}^k_{g(y)}(\mathfrak{b}) \cdot \psi(x, y) = 0$$

for all $k$. However, for $k$ equal to the order of $\mathfrak{b}$, the operator $\mathrm{ad}^k_{g(y)}(\mathfrak{b})$ will be equal to a nonzero function $h(y)$. This would imply $h(y)\psi(x, y) = 0$, and that $\psi(x, y)$ is therefore 0, a contradiction. Thus if $\mathfrak{b} \in \mathcal{F}_y(\psi)$ satisfies $\mathfrak{b} \cdot \psi(x, y) = 0$, then $\mathfrak{b} = 0$. Similarly, if $\mathfrak{a} \in \mathcal{F}_x(\psi)$ satisfies $\mathfrak{a} \cdot \psi(x, y) = 0$, then $\mathfrak{a} = 0$.

Using this, we see that for every $\mathfrak{a} \in \mathcal{F}_x(\psi)$ there exists a unique $\mathfrak{b} \in \mathcal{F}_x(\psi)$ such that $\mathfrak{a} \cdot \psi(x, y) = \mathfrak{b} \cdot \psi(x, y)$. This shows that the map $b_\psi$ is well defined and bijective. A simple argument shows that $b_\psi$ is an anti-isomorphism. This completes the proof. □

**Definition 2.5.** We call the map $b_\psi : \mathcal{F}_x(\psi) \to \mathcal{F}_y(\psi)$ from the previous proposition the *generalized Fourier map* or the *bispectral anti-isomorphism* of $\psi$.

**Example 2.6.** (1) By Remark 2.3, for the exponential bispectral function $\psi_{\exp}$, the Fourier algebras are the algebras of differential operators with polynomial coefficients

$$\mathcal{F}_x(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[x]), \quad \mathcal{F}_y(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[y])$$

and the anti-isomorphism

$$b_{\psi_{\exp}} : \mathfrak{D}(\mathbb{C}[x]) \to \mathfrak{D}(\mathbb{C}[y])$$

is the Fourier transform given by $\partial_x \mapsto y$ and $x \mapsto \partial_y$. The algebras of bispectral operators $\mathcal{B}_x(\psi_{\exp})$ and $\mathcal{B}_y(\psi_{\exp})$ are the algebras of differential operators with constant coefficients in $x$ and $y$, respectively.

(2) The Bessel bispectral functions $\psi_{\mathrm{Be}(\nu)}(x, y)$ satisfy

$$x\partial_x \cdot \psi_{\mathrm{Be}(\nu)}(x, y) = y\partial_y \cdot \psi_{\mathrm{Be}(\nu)}(x, y)$$

in addition to equations (2.1). The left and right Fourier algebras are given by

$$\mathcal{F}_x(\psi_{\mathrm{Be}(\nu)}) = \langle \mathfrak{d}_{\mathrm{Be}(\nu),x}, x\partial_x, x^2 \rangle \quad \text{and} \quad \mathcal{F}_y(\psi_{\mathrm{Be}(\nu)}) = \langle \mathfrak{d}_{\mathrm{Be}(\nu),y}, y\partial_y, y^2 \rangle$$

for $\nu \in \mathbb{C} \setminus \mathbb{Z}$. (In the case $\nu \in \mathbb{Z}$ the Fourier algebras are bigger, but this will not a play a role in the paper.) Here $\langle \mathfrak{d}_1, \dots, \mathfrak{d}_k \rangle$ denotes the algebra of differential operators with generators $\mathfrak{d}_1, \dots, \mathfrak{d}_k$. The generalized Fourier map $b_{\psi_{\mathrm{Be}(\nu)}} : \mathcal{F}_x(\psi_{\mathrm{Be}(\nu)}) \to \mathcal{F}_x(\psi_{\mathrm{Be}(\nu)})$ is given by

$$\mathfrak{d}_{\mathrm{Be}(\nu),x} \mapsto y^2, \quad x\partial_x \mapsto y\partial_y, \quad x^2 \mapsto \mathfrak{d}_{\mathrm{Be}(\nu),y}.$$

The algebras of bispectral operators $\mathcal{B}_x(\psi_{\mathrm{Be}(\nu)})$ and $\mathcal{B}_x(\psi_{\mathrm{Be}(\nu)})$ are the polynomial algebras in the Bessel operators $\mathfrak{d}_{\mathrm{Be}(\nu),x}$ and $\mathfrak{d}_{\mathrm{Be}(\nu),y}$, respectively.

(3) The Airy bispectral function $\psi_{\mathrm{Ai}}(x, y)$ satisfies

$$\partial_x \cdot \psi_{\mathrm{Ai}}(x, y) = \partial_y \cdot \psi_{\mathrm{Ai}}(x, y)$$

in addition to the equations (2.2). The left and right Fourier algebras associated to it are the algebras of differential operators with polynomial coefficients

$$\mathcal{F}_x(\psi_{\mathrm{Ai}}) = \mathfrak{D}(\mathbb{C}[x]) \quad \text{and} \quad \mathcal{F}_y(\psi_{\mathrm{Ai}}) = \mathfrak{D}(\mathbb{C}[y]).$$

The generalized Fourier map $b_{\psi_{\mathrm{Ai}}} : \mathcal{F}_x(\psi_{\mathrm{Ai}}) \to \mathcal{F}_x(\psi_{\mathrm{Ai}})$ is given by

$$\partial_x \mapsto \partial_y, \quad x \mapsto \mathfrak{b}_{\mathrm{Ai},y}.$$

It satisfies $\mathfrak{b}_{\mathrm{Ai},x} \mapsto y$. The algebras of bispectral operators $\mathcal{B}_x(\psi_{\mathrm{Ai}})$ and $\mathcal{B}_x(\psi_{\mathrm{Ai}})$ are the polynomial algebras in the Airy operators $\mathfrak{b}_{\mathrm{Ai},x}$ and $\mathfrak{b}_{\mathrm{Ai},y}$, respectively.

**2.2. Double filtrations of Fourier algebras.** Using the generalized Fourier map of a bispectral function $\psi(x, z)$, we define natural filtrations of the left and right Fourier algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ as follows.

**Definition 2.7.** We define the *co-order* of an operator $\mathfrak{d} \in \mathcal{F}_x(\psi)$, denoted $\mathrm{cord}(\mathfrak{d})$, to be the order of $b_\psi(\mathfrak{d})$. Similarly, we define the *co-order* of $\mathfrak{b} \in \mathcal{F}_y(\psi)$, again denoted $\mathrm{cord}(\mathfrak{b})$, to be the order of $b_\psi^{-1}(\mathfrak{b})$. We define $\mathbb{N} \times \mathbb{N}$*-filtrations* of the Fourier algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ by

$$\mathcal{F}_x^{\ell,m}(\psi) = \{\mathfrak{d} \in \mathcal{F}_x(\psi) : \mathrm{ord}(\mathfrak{d}) \le \ell \text{ and } \mathrm{cord}(\mathfrak{d}) \le m\},$$
$$\mathcal{F}_y^{m,\ell}(\psi) = \{\mathfrak{b} \in \mathcal{F}_y(\psi) : \mathrm{ord}(\mathfrak{b}) \le m \text{ and } \mathrm{cord}(\mathfrak{b}) \le \ell\}.$$

**Remark 2.8.** By Proposition 2.4, the generalized Fourier map $b_\psi$ restricts to an isomorphism $\mathcal{F}_x^{\ell,m}(\psi) \to \mathcal{F}_y^{m,\ell}(\psi)$. Moreover, under the above filtration

$$\mathcal{B}_x(\psi) = \mathrm{span} \bigcup_{\ell=0}^{\infty} \mathcal{F}_x^{\ell,0}(\psi) \quad \text{and} \quad \mathcal{B}_y(\psi) = \mathrm{span} \bigcup_{m=0}^{\infty} \mathcal{F}_y^{0,m}(\psi).$$

We pause now to consider the filtrations of the Fourier algebras associated to the bispectral functions $\psi_{\exp}$, $\psi_{\mathrm{Be}(\nu)}$ and $\psi_{\mathrm{Ai}}$, continuing Example 2.6.

**Example 2.9.** (1) For the bispectral exponential function $\psi_{\exp}$ we have

$$\mathcal{F}_x^{\ell,m}(\psi_{\exp}) = \mathrm{span}\{x^k \partial_x^j : 0 \le j \le \ell,\ 0 \le k \le m\},$$
$$\mathcal{F}_y^{m,\ell}(\psi_{\exp}) = \mathrm{span}\{y^j \partial_y^k : 0 \le j \le \ell,\ 0 \le k \le m\}.$$

(2) The filtrations of the Fourier algebras associated to the bispectral Bessel functions $\psi_{\mathrm{Be}(\nu)}$ for $\nu \in \mathbb{C} \setminus \mathbb{Z}$ are given by

$$\begin{aligned}\mathcal{F}_x^{2\ell,2m}(\psi_{\mathrm{Be}(\nu)}) &= \mathrm{span}\{x^{2k} \mathfrak{b}_{\mathrm{Be}(\nu),x}^j : 0 \le j \le \ell,\ 0 \le k \le m\} \\ &\quad \oplus \mathrm{span}\{x^{2k} x\partial_x \mathfrak{b}_{\mathrm{Be}(\nu),x}^j : 0 \le j < \ell,\ 0 \le k < m\}, \\ \mathcal{F}_x^{2m,2\ell}(\psi_{\mathrm{Be}(\nu)}) &= \mathrm{span}\{y^{2j} \mathfrak{b}_{\mathrm{Be}(\nu),y}^k : 0 \le j \le \ell,\ 0 \le k \le m\} \\ &\quad \oplus \mathrm{span}\{y^{2j} y\partial_y \mathfrak{b}_{\mathrm{Be}(\nu),y}^k : 0 \le j < \ell,\ 0 \le k < m\}\end{aligned}$$

for even indices. Similar formulas hold for odd indices, we leave the details to the reader.

(3) For the bispectral Airy function $\psi_{\mathrm{Ai}}$ the filtrations of the left and right Fourier algebras are

$$\begin{aligned}\mathcal{F}_x^{2\ell,2m}(\psi_{\mathrm{Ai}}) &= \operatorname{span}\{x^k \mathfrak{d}_{\mathrm{Ai},x}^j : 0 \le j \le \ell,\ 0 \le k \le m\} \\ &\quad \oplus \operatorname{span}\{x^k \partial_x \mathfrak{d}_{\mathrm{Ai},x}^j : 0 \le j < \ell,\ 0 \le k < m\}, \\ \mathcal{F}_y^{2m,2\ell}(\psi_{\mathrm{Ai}}) &= \operatorname{span}\{y^j \mathfrak{d}_{\mathrm{Ai},y}^k : 0 \le j \le \ell,\ 0 \le k \le m\} \\ &\quad \oplus \operatorname{span}\{y^j \partial_y \mathfrak{d}_{\mathrm{Ai},y}^k : 0 \le j < \ell,\ 0 \le k < m\}\end{aligned}$$

for even indices. Similar formulas hold for odd indices.

All three facts are easily deduced from the facts for the left and right Fourier algebras and generalized Fourier maps in Example 2.6. □

## 3. Self-adjoint bispectral Darboux transformations

This section contains background material on bispectral Darboux transformations and the classification of bispectral functions. We introduce the second main player in the paper – self-adjoint Darboux transformations and prove that they correspond bijectively to self-adjoint bispectral functions.

### 3.1. Bispectral Darboux transformations and classification results.

**Definition 3.1.** Let $\psi(x, y)$ be a bispectral meromorphic function on $U \times V$. A *bispectral Darboux transformation* $\widetilde{\psi}(x, y)$ of $\psi(x, y)$ is a function satisfying

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)p(x)}\mathrm{u} \cdot \psi(x, y) \quad \text{and} \quad \psi(x, y) = \widetilde{\mathrm{u}} \cdot \frac{1}{\widetilde{q}(y)\widetilde{p}(x)}\widetilde{\psi}(x, y)$$

for some pairs of polynomials $p(x)$, $\widetilde{p}(x)$ and $q(y)$, $\widetilde{q}(y)$, and some differential operators $\mathrm{u}, \widetilde{\mathrm{u}} \in \mathcal{F}_x(\psi)$.

We call the pair $(d_1, d_2)$ the *order* of the bispectral Darboux transformation, where $d_1 = \operatorname{ord}(\mathrm{u})$ and $d_2 = \operatorname{cord}(\mathrm{u})$ are the order and co-order of $\mathrm{u}$ (with the latter defined as in Definition 2.7 above).

It follows from the definition that

$$\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathrm{u} \cdot \psi(x, y) = \widetilde{q}(y)q(y)\psi(x, y).$$

Therefore

$$\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathrm{u} \in \mathcal{B}_x(\psi) \quad \text{and} \quad b_\psi\left(\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathrm{u}\right) = \widetilde{q}(y)q(y). \tag{3.1}$$

Similar statements hold when the roles of $\psi(x, y)$ and $\widetilde{\psi}(x, y)$ are interchanged.

**Remark 3.2.** If $\psi(x, y)$ is a bispectral function on $U \times V$, then $f(x)g(y)\psi(x, y)$ is also a bispectral function on $U \times V$ for all meromorphic functions $f(x)$ and $g(y)$ on $U$ and $V$. Because of this, the choice of $p(x)$ and $q(y)$ in Definition 3.1 is merely a choice of normalization. The next paragraph describes the *standard normalization*.

The new bispectral function $\widetilde{\psi}(x,y)$ can be represented in two dual ways as a transformation of $\psi(x,y)$ in the $x$- and $y$-variables:

$$\widetilde{\psi}(x,y) = \frac{1}{q(y)p(x)}\mathrm{u}\cdot\psi(x,y) = \frac{1}{p(x)q(y)}b_\psi(\mathrm{u})\cdot\psi(x,y). \tag{3.2}$$

The polynomials $p(x)$ and $q(y)$ are uniquely determined from the normalization that

$$\text{the differential operators } \frac{1}{p(x)}\mathrm{u} \text{ and } \frac{1}{q(y)}b_\psi(\mathrm{u}) \text{ are monic.} \tag{3.3}$$

**Remark 3.3.** In Theorem 3.6 we prove that if $\widetilde{\psi}(x,y)$ is a bispectral Darboux transformation of $\psi(x,y)$ and $\mathrm{u},\widetilde{\mathrm{u}},p(x),\widetilde{p}(x),q(y),\widetilde{q}(y)$ are as in the definition of a bispectral Darboux transformation, then necessarily $\widetilde{p}(x)\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)}\in\mathcal{F}_x(\widetilde{\psi})$. In fact, the definition of a bispectral Darboux transformation can be replaced with the equivalent condition that there exist differential operators $\mathrm{u}\in\mathcal{F}_x(\psi)$, $\mathrm{v}\in\mathcal{F}_x(\widetilde{\psi})$ and polynomials $p(x),\widetilde{p}(x),q(y),\widetilde{q}(y)$ such that

$$\widetilde{\psi}(x,y) = \frac{1}{q(y)p(x)}\mathrm{u}\cdot\psi(x,y) \quad\text{and}\quad \psi(x,y) = \frac{1}{\widetilde{q}(y)\widetilde{p}(x)}\mathrm{v}\cdot\widetilde{\psi}(x,y).$$

In particular, this latter definition is manifestly symmetric.

The following result of [4] establishes general bispectrality properties of the transformations in Definition 3.1.

**Theorem 3.4** (Bakalov, Horozov, Yakimov [4, Theorem 4.2]). *Let $\psi(x,y)$ be a bispectral function on $U\times V$ and let $\widetilde{\psi}(x,y)$ be a bispectral Darboux transformation from $\psi(x,y)$ with the notation in Definition* 3.1. *Then $\widetilde{\psi}(x,y)$ is also a bispectral function which satisfies the spectral equations*

$$\frac{1}{p(x)}\mathrm{u}\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)}\cdot\widetilde{\psi}(x,y) = q(y)\widetilde{q}(y)\widetilde{\psi}(x,y),$$
$$\frac{1}{q(y)}b_\psi(\mathrm{u})b_\psi(\widetilde{\mathrm{u}})\frac{1}{\widetilde{q}(y)}\cdot\widetilde{\psi}(x,y) = p(x)\widetilde{p}(x)\widetilde{\psi}(x,y).$$

*Dually to* (3.1)*, we have*

$$b_\psi(\widetilde{\mathrm{u}})\frac{1}{\widetilde{q}(y)q(y)}b_\psi(\mathrm{u})\in\mathcal{B}_y(\psi),\quad b_\psi(\widetilde{p}(x)p(x)) = b_\psi(\widetilde{\mathrm{u}})\frac{1}{\widetilde{q}(y)q(y)}b_\psi(\mathrm{u}). \tag{3.4}$$

**Remark 3.5.** (1) It follows from (3.1) and (3.4) that

$$\overline{p}(x) := b_\psi^{-1}\left(b_\psi(\widetilde{\mathrm{u}})\frac{1}{\widetilde{q}(y)q(y)}b_\psi(\mathrm{u})\right)\in b_\psi^{-1}(\mathcal{B}_y(\psi)),$$
$$\overline{q}(y) := b_\psi\left(\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathrm{u}\right)\in b_\psi(\mathcal{B}_x(\psi))$$

and

$$p(x)\widetilde{p}(x) = \overline{p}(x),\qquad q(y)\widetilde{q}(y) = \overline{q}(y).$$

When $\overline{p}(x)\in b_\psi^{-1}(\mathcal{B}_y(\psi))$, $\overline{q}(y)\in b_\psi(\mathcal{B}_x(\psi))$, and $\mathrm{u},\widetilde{\mathrm{u}}\in\mathcal{F}_x(\psi)$ are fixed, there is a freedom in the construction of $\widetilde{\psi}(x,y)$ in that the polynomials $p(x)$ and $q(x)$ can be chosen to be arbitrary divisors of the polynomials $\overline{p}(x)$ and $\overline{q}(x)$. As noted in Remark 3.2, $f(x)g(y)\widetilde{\psi}(x,y)$ is

also a bispectral function on $U \times V$ for all meromorphic functions $f(x)$ and $g(y)$ on $U$ and $V$. Among these, the above bispectral Darboux transformations from $\psi(x, y)$ come from the rational functions $f(x)$, $g(x)$ with denominators $p(x)$, $q(x)$ and numerators given by polynomials that are divisors of $\overline{p}(x)$ and $\overline{q}(x)$.

(2) The normalization (3.3) simply means that $p(x)$ and $q(y)$ are chosen to be the leading terms of $\mathfrak{u}$ and $b_\psi(\mathfrak{u})$. Such a normalization is always possible for the following reasons. Recall from Section 2.1 that we work in local coordinates in which all operators in $\mathcal{B}_y(\psi)$ and $\mathcal{B}_y(\psi)$ have scalar leading term. The ad-condition then implies that all operators in $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ have polynomial leading terms.

Start with a bispectral Darboux transformation $\widetilde{\psi}(x, y)$ of $\psi(x, y)$. Equations (3.1) and (3.4) imply that $\widetilde{p}(x)p(x)$ and $\widetilde{q}(y)q(y)$ are scalar multiples of the leading terms of the differential operators $\widetilde{\mathfrak{u}}\mathfrak{u}$ and $b_\psi(\widetilde{\mathfrak{u}}\mathfrak{u}) = b_\psi(\mathfrak{u})b_\psi(\widetilde{\mathfrak{u}})$. Since $\widetilde{\mathfrak{u}}$ and $b_\psi(\widetilde{\mathfrak{u}})$ have polynomial leading terms, one can use a transformation of the type described in the first part of the remark to change $p(x)$ and $q(y)$ so they equal the leading terms of $\mathfrak{u}$ and $b_\psi(\mathfrak{u})$.

(3) In [4] is was assumed that $p(x), \widetilde{p}(x) \in b_\psi^{-1}(\mathcal{B}_y(\psi))$ and $q(y), \widetilde{q}(y) \in b_\psi(\mathcal{B}_x(\psi))$. The difference here is that we only assume that their products are in the spaces $b_\psi^{-1}(\mathcal{B}_y(\psi))$ and $b_\psi(\mathcal{B}_x(\psi))$. One passes from one setting to the other by the transformations in the first part of the remark.

The next theorem shows that being a bispectral Darboux transformation is a symmetric condition.

**Theorem 3.6.** *Let $\psi(x, y)$ be a bispectral function and suppose that $\widetilde{\psi}(x, y)$ is a bispectral Darboux transformation of $\psi(x, y)$. Then $\psi(x, y)$ is also a bispectral Darboux transformation of $\widetilde{\psi}(x, y)$ and $\widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}, \frac{1}{p(x)}\mathfrak{u}\,p(x) \in \mathcal{F}_x(\widetilde{\psi})$ with*

$$b_{\widetilde{\psi}}\left(\widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}\right) = q(y)b_\psi(\widetilde{\mathfrak{u}})\frac{1}{q(y)} \quad \textit{and} \quad b_{\widetilde{\psi}}\left(\frac{1}{p(x)}\mathfrak{u}\,p(x)\right) = \frac{1}{q(y)}b_\psi(\mathfrak{u})q(y).$$

*Proof.* Let $\mathfrak{u}, \widetilde{\mathfrak{u}}$, $p(x), \widetilde{p}(x), q(y)$, and $\widetilde{q}(y)$ be as in the definition of bispectral Darboux transformations for the bispectral Darboux transformation from $\psi(x, y)$ to $\widetilde{\psi}(x, y)$. Then

$$\begin{aligned}\widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)} \cdot \widetilde{\psi}(x, y) &= \widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)p(x)q(y)}\mathfrak{u} \cdot \psi(x, y)\\ &= \widetilde{p}(x)\widetilde{q}(y) \cdot \psi(x, y).\end{aligned}$$

Therefore by Theorem 3.4 we find

$$\begin{aligned}p(x)\widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)} \cdot \widetilde{\psi}(x, y) &= p(x)\widetilde{p}(x)\widetilde{q}(y)\psi(x, y)\\ &= \widetilde{q}(y)b_\psi(\widetilde{\mathfrak{u}})\frac{1}{\widetilde{q}(y)q(y)}b_\psi(\mathfrak{u}) \cdot \psi,\end{aligned}$$

so that

$$\begin{aligned}\widetilde{p}(x)\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)} \cdot \widetilde{\psi}(x, y) &= \widetilde{q}(y)b_\psi(\widetilde{\mathfrak{u}})\frac{1}{p(x)\widetilde{q}(y)q(y)}b_\psi(\mathfrak{u}) \cdot \psi\\ &= \widetilde{q}(y)b_\psi(\widetilde{\mathfrak{u}})\frac{1}{\widetilde{q}(y)} \cdot \widetilde{\psi}(x, y).\end{aligned}$$

Thus $\widetilde{p}(x)\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)} \in \mathcal{F}_x(\widetilde{\psi})$. Moreover,

$$\begin{aligned}\frac{1}{p(x)}\mathrm{u}\,p(x)\cdot\widetilde{\psi}(x,y) &= \frac{1}{p(x)q(y)}\mathrm{u}^2\cdot\psi(x,y)\\ &= \frac{1}{p(x)q(y)}b_\psi(\mathrm{u})^2\cdot\psi(x,y)\\ &= \frac{1}{q(y)}b_\psi(\mathrm{u})q(y)\frac{1}{p(x)q(y)}b_\psi(\mathrm{u})\cdot\psi(x,y)\\ &= \frac{1}{q(y)}b_\psi(\mathrm{u})q(y)\cdot\widetilde{\psi}(x,y)\end{aligned}$$

so that $\frac{1}{p(x)}\mathrm{u}\,p(x) \in \mathcal{F}_x(\widetilde{\psi})$. Then since

$$\frac{1}{\widetilde{q}(y)\widetilde{p}(x)}\left(\widetilde{p}(x)\widetilde{\mathrm{u}}\frac{1}{\widetilde{p}(x)}\right)\cdot\widetilde{\psi}(x,y) = \psi(x,y)$$

and

$$\left(\frac{1}{p(x)}\mathrm{u}\,p(x)\right)\frac{1}{p(x)q(y)}\cdot\psi(x,y) = \widetilde{\psi}(x,y)$$

we see that $\psi$ is a bispectral Darboux transformation of $\widetilde{\psi}$. □

**Definition 3.7.** We define the *rank* of a bispectral function $\psi(x,y)$ to be the greatest common divisor of the orders of the operators in the left bispectral algebra $\mathcal{B}_x(\psi)$.

In the cases when classification results are available, one can show that the rank of a bispectral function $\psi(x,y)$ also equals the greatest common divisor of the orders of the operators in the right bispectral algebra $\mathcal{B}_y(\psi)$, but there is currently no direct proof of this fact.

In [41] Wilson introduced the powerful idea that the classification of bispectral functions should be performed on a per-rank basis in which case one sees a deep geometric picture of the moduli spaces of such functions. Bispectral Darboux transformations preserve the rank and are especially suited for these purposes.

**Theorem 3.8** (Wilson [41]). *The rank* 1 *bispectral functions are precisely the bispectral Darboux transformations of the exponential function* $\psi_{\exp}(x,y)$.

Wilson's result as stated in [41] is a classification in the case that $\mathcal{B}_x(\psi)$ is rank 1 and *maximally commutative* in the sense that $\mathcal{B}_x(\psi)$ is not contained in any larger commutative subalgebra of the algebra of differential operators. However, in the rank 1 case the common eigenspaces of $\mathcal{B}_x(\psi)$ must be one-dimensional, so $\psi$ is (up to normalization) equal to a wave function of the KP hierarchy [31]. From this, one may show that $\mathcal{B}_x(\psi)$ is necessarily maximally commutative, so the maximality assumption is not explicitly required. For higher rank, it is not known whether maximality follows immediately, so this assumption will required below when necessary.

Bispectral functions $\psi$ for which the bispectral algebra $\mathcal{B}_x(\psi)$ contains an operator of prime order $p$ were classified by Duistermaat and Grünbaum [14] in the case $p = 2$ and by Horozov [23] in the case $p > 2$. These results and the methods of [5, 24, 27] prompted the following conjecture which was has been widely circulated since the mid 1990s.

**Conjecture 3.9.** The maximally commutative rank 2 bispectral meromorphic algebras are precisely the algebras $\mathcal{B}_x(\psi)$ corresponding to all bispectral Darboux transformations $\psi(x, y)$ of the Bessel functions $\psi_{\mathrm{Be}(\nu)}(x, y)$ for $\nu \in \mathbb{C} \setminus \mathbb{Z}$ and the Airy function $\psi_{\mathrm{Ai}}(x, y)$ see Section 2.1.

**Example 3.10.** Continuing Examples 2.6 and 2.9, we describe more explicitly the functions in Theorem 3.8 and Conjecture 3.9.

(1) The bispectral Darboux transformations of the exponential function $\psi_{\exp}$ are precisely the functions in the (infinite-dimensional) Wilson adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$, [41]. They are the functions of the form

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)} \mathfrak{v} \cdot \psi_{\exp}(x, y),$$

where $\mathfrak{v} \in \mathfrak{D}(\mathbb{C}(x))$ is a monic differential operator with rational coefficients satisfying

$$\widetilde{\mathfrak{v}}\mathfrak{v} = f(\partial_x)$$

for a monic polynomial $f(t)$ and some $\widetilde{\mathfrak{v}} \in \mathfrak{D}(\mathbb{C}(x))$. The differential operators $\mathfrak{v}$ that appear in this way are classified in terms of their kernels consisting of quasi-exponential functions (i.e., solutions of homogeneous linear ordinary differential equations with constant coefficients). This classification is recalled in Section 6 below. The polynomial $q(y)$ is uniquely determined from $\mathfrak{v}$ as the polynomial whose roots are the support of the quasi-exponential functions in the kernel of $\mathfrak{v}$.

(2) The bispectral Darboux transformations of the Bessel functions $\psi_{\mathrm{Be}(\nu)}$ (for $\nu \in \mathbb{C} \setminus \mathbb{Z}$) are the functions of the form

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)} \mathfrak{v} \cdot \psi_{\mathrm{Be}(\nu)}(x, y),$$

where $\mathfrak{v} \in \mathfrak{D}(\mathbb{C}(x))$ is a monic differential operator with rational coefficients such that

$$\widetilde{\mathfrak{v}}\mathfrak{v} = f(\mathfrak{d}_{\mathrm{Be}(\nu),x})$$

for a monic polynomial $f(t)$ and some $\widetilde{\mathfrak{v}} \in \mathfrak{D}(\mathbb{C}(x))$. All differential operators $\mathfrak{v}$ satisfying these properties, and as a result, the rank 2 bispectral functions in the Bessel class, form an infinite-dimensional manifold. They are classified by an explicit description of the possible forms of their kernels in terms of Bessel functions and their derivatives, see Section 6 for details.

(3) The bispectral Darboux transformations of the Airy function $\psi_{\mathrm{Ai}}$ are the functions of the form

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)} \mathfrak{v} \cdot \psi_{\mathrm{Ai}}(x, y),$$

where $\mathfrak{v} \in \mathfrak{D}(\mathbb{C}(x))$ is a monic differential operator with rational coefficients satisfying

$$\widetilde{\mathfrak{v}}\mathfrak{v} = f(\mathfrak{d}_{\mathrm{Ai},x})$$

for a monic polynomial $f(t)$ and some $\widetilde{\mathfrak{v}} \in \mathfrak{D}(\mathbb{C}(x))$. The differential operators $\mathfrak{v}$ satisfying these properties and thus, the rank 2 bispectral functions in the Airy class, also form an infinite-dimensional manifold. They are classified by an explicit description of the possible forms of their kernels in terms of derivatives of the Airy functions, see Section 6.

**3.2. Self-adjoint bispectral Darboux transformations and self-adjoint bispectral functions.** We start with a series of definitions.

**Definition 3.11.** Let $\mathfrak{d} = \sum_{j=0}^{n} a_j(x)\partial_x^j$. We define the *formal adjoint* of $\mathfrak{d}$ to be

$$\mathfrak{d}^* = \sum_{j=0}^{n}(-1)^j \partial_x^j a_j(x) = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^j \binom{j}{k} a_j^{(k)}(x)\partial_x^{j-k}.$$

An operator $\mathfrak{d}$ is called *formally symmetric* if $\mathfrak{d}^* = \mathfrak{d}$.

**Definition 3.12.** Let $\Gamma \subseteq \mathbb{C}$ be a smooth path in $\mathbb{C}$. Then we say $\mathfrak{d}$ is *adjointable* with respect to $\Gamma$ if for every $f(x), g(x) \in C_c^\infty(\Gamma)$ the following integral identity holds:

$$\int_\Gamma (\mathfrak{d} \cdot f(x))g(x)\,dx = \int_\Gamma f(x)(\mathfrak{d}^* \cdot g(x))\,dx.$$

If $\mathfrak{d}$ is formally symmetric and adjointable, then we call $\mathfrak{d}$ *symmetric*.

**Definition 3.13.** Let $\psi(x, y)$ be a bispectral meromorphic function. We call an operator $\mathfrak{d} \in \mathcal{F}_x(\psi)$ *formally bisymmetric* if both $\mathfrak{d}$ and $b_\psi(\mathfrak{d})$ are formally symmetric. Fix smooth paths $\Gamma_1, \Gamma_2 \subseteq \mathbb{C}$. We call $\mathfrak{d}$ *bisymmetric* with respect to $(\Gamma_1, \Gamma_2)$ if $\mathfrak{d}$ is symmetric with respect to $\Gamma_1$ and $b_\psi(\mathfrak{d})$ is symmetric with respect to $\Gamma_2$.

**Definition 3.14.** Let $\psi(x, y)$ be a bispectral meromorphic function. We call a bispectral Darboux transformation $\widetilde{\psi}(x, y)$ with the notation of Definition 3.1 a *self-adjoint bispectral Darboux transformation* of $\psi(x, y)$ if

$$\widetilde{p}(x) = p(x), \widetilde{q}(y) = q(y) \quad \text{and} \quad \widetilde{\mathfrak{u}} = (-1)^{d_1}\mathfrak{u}^*, \quad b_\psi(\widetilde{\mathfrak{u}}) = (-1)^{d_2} b_\psi(\mathfrak{u})^*.$$

for $d_1 = \operatorname{ord}(\mathfrak{u})$ and $d_2 = \operatorname{cord}(\mathfrak{u})$.

**Remark 3.15.** In all of the situations that we consider in this paper the essential conditions are

$$\widetilde{\mathfrak{u}} = (-1)^{d_1}\mathfrak{u}^*, \quad b_\psi(\widetilde{\mathfrak{u}}) = (-1)^{d_2} b_\psi(\mathfrak{u})^* \tag{3.5}$$

and the conditions $\widetilde{p}(x) = p(x), \widetilde{q}(y) = q(y)$ follow from them after a normalization. More precisely, if $\psi$ is any of the exponential $\psi_{\exp}$, Bessel $\psi_{\mathrm{Be}(\nu)}$ or the Airy $\psi_{\mathrm{Ai}}$ bispectral functions, then the left and right bispectral algebras $\mathcal{B}_x(\psi)$, $\mathcal{B}_y(\psi)$ consist of differential operators with constant leading terms. The normalization (3.3) implies that $p(x)$ and $q(y)$ equal the leading terms of the differential operators $\mathfrak{u}$ and $b_\psi(\mathfrak{u})$. Therefore, by (3.4), $\widetilde{p}(x)$ and $\widetilde{q}(y)$ are scalar multiples of the leading terms of the differential operators $\widetilde{\mathfrak{u}}$ and $b_\psi(\widetilde{\mathfrak{u}})$

If (3.5) is satisfied, one can rescale $\widetilde{p}(x)$ and $\widetilde{q}(y)$ so that $\widetilde{p}(x) = p(x)$, $\widetilde{q}(y) = q(y)$. This rescaling does not change the function $\widetilde{\psi}(x, y)$.

Theorem 3.4 implies that every self-adjoint bispectral Darboux transformation $\widetilde{\psi}(x, y)$ of a bispectral function $\psi(x, y)$ with the above data satisfies

$$\frac{1}{p(x)}\mathfrak{u}\mathfrak{u}^*\frac{1}{p(x)} \cdot \widetilde{\psi}(x, y) = q(y)^2\widetilde{\psi}(x, y), \tag{3.6}$$

$$\frac{1}{q(y)}b_\psi(\mathfrak{u})b_\psi(\mathfrak{u})^*\frac{1}{q(y)} \cdot \widetilde{\psi}(x, y) = p(x)^2\widetilde{\psi}(x, y). \tag{3.7}$$

Moreover, (3.1) and (3.4) imply that the original wave function $\psi(x,z)$ satisfies

$$\widetilde{\mathfrak{u}}^* \frac{1}{p(x)^2}\mathfrak{u}\cdot\psi(x,y) = q(y)^2\psi(x,y), \tag{3.8}$$

$$b_\psi(\mathfrak{u})^* \frac{1}{q(y)^2} b_\psi(\mathfrak{u})\cdot\psi(x,y) = p(x)^2\psi(x,y). \tag{3.9}$$

**Definition 3.16.** We call a bispectral meromorphic function $\psi(x,y)$ *self-adjoint* if it is an eigenfunction of nonconstant, formally symmetric differential operators in $x$ and $y$.

**Remark 3.17.** A rank 1 bispectral function $\psi(x,y)$ is self-adjoint if and only if $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are preserved by taking formal adjoints $*$. This is because in the rank 1 case $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are maximally commutative and therefore equal to the centralizer of any of its nonconstant elements. The centralizer of the centralizer of a formally symmetric differential operator is closed under taking formal adjoints, so the self-adjointness of $\psi$ follows immediately.

Equations (3.6)–(3.9) imply the following:

**Proposition 3.18.** *Suppose that $\psi(x,y)$ is a bispectral meromorphic function and that $\widetilde{\psi}(x,y)$ is a self-adjoint bispectral Darboux transformation of $\psi(x,y)$. Then both $\psi(x,y)$ and $\widetilde{\psi}(x,y)$ are self-adjoint bispectral meromorphic functions.*

**Lemma 3.19.** *Let $\psi(x,y)$ and $\widetilde{\psi}(x,y)$ be bispectral meromorphic functions and that $\widetilde{\psi}(x,y)$ is a bispectral Darboux transformation of $\psi(x,y)$. If $\mathcal{B}_x(\psi)$ (resp. $\mathcal{B}_y(\psi)$) is maximally commutative, then so too is $\mathcal{B}_x(\widetilde{\psi})$ (resp. $\mathcal{B}_y(\widetilde{\psi})$).*

*Proof.* To be explicit, suppose

$$\widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)}\mathfrak{u}\cdot\psi(x,y) \quad \text{and} \quad \psi(x,y) = \widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)\widetilde{q}(y)}\cdot\widetilde{\psi}(x,y)$$

for some operators $\mathfrak{u}, \widetilde{\mathfrak{u}} \in \mathcal{F}_x(\psi)$ and polynomials $p(x), \widetilde{p}(x), q(y), \widetilde{q}(y)$. Set

$$\mathfrak{d} = \widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathfrak{u} \quad \text{and} \quad \widetilde{\mathfrak{d}} = \frac{1}{p(x)}\mathfrak{u}\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}.$$

Centralizers of nonconstant differential operators are maximally commutative, so to prove that $\mathcal{B}_x(\widetilde{\psi})$ is maximal, it suffices to show that $\mathcal{B}_x(\widetilde{\psi})$ is the centralizer of $\widetilde{\mathfrak{d}}$. Suppose that $\mathfrak{b}$ is a differential operator which commutes with $\widetilde{\mathfrak{d}}$. Then the Darboux conjugate $\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}\mathfrak{b}\widetilde{\mathfrak{d}}\widetilde{p}(x)\widetilde{\mathfrak{u}}^{-1}$ is a differential operator which commutes with $\mathfrak{d}$. By the maximality assumption on $\mathcal{B}_x(\psi)$, we know that $\mathcal{B}_x(\psi)$ is equal to the centralizer of $\mathfrak{d}$ and thus $\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}\mathfrak{b}\widetilde{\mathfrak{d}}\widetilde{p}(x)\widetilde{\mathfrak{u}}^{-1}$ is in $\mathcal{B}_x(\psi)$. Consequently, the Darboux conjugate $\mathfrak{b}\widetilde{\mathfrak{d}}$ is in $\mathcal{B}_x(\widetilde{\psi})$. Since $\widetilde{\mathfrak{d}} \in \mathcal{B}_x(\widetilde{\psi})$, it follows that $\mathfrak{b} \in \mathcal{B}_x(\widetilde{\psi})$. Note that $\mathfrak{b}$ was an arbitrary element in the centralizer of $\widetilde{\mathfrak{d}}$, so this completes the proof. □

A maximally commutative bispectral algebra $\mathcal{B}_x(\psi)$ almost determines the function $\psi(x,y)$, i.e., the family of joint eigenfunctions of a fixed bispectral algebra is almost unique. However, different bispectral meromorphic functions do exist for the same bispectral algebra, such as $\exp(-xy)$ and $\exp(xy)$ for the bispectral algebra $\mathbb{C}[\partial_x]$. Even so, it is not possible to jump between different families via bispectral Darboux transformations, as the next proposition shows.

**Proposition 3.20.** *Suppose that $\psi$ and $\widetilde{\psi}$ are bispectral meromorphic functions with $\mathcal{B}_x(\psi)$ and $\mathcal{B}_x(\psi) = \mathcal{B}_x(\widetilde{\psi})$. If there exists $\mathfrak{u} \in \mathcal{F}_x(\psi)$ satisfying $\widetilde{\psi}(x,y)q(y) = \mathfrak{u} \cdot \psi(x,y)$ for some $q(y)$, then $\mathfrak{u} \in \mathcal{B}_x(\psi)$ and $\widetilde{\psi}(x,y)$ is a constant multiple of $\psi(x,y)$.*

*Proof.* For all $\mathfrak{d} \in \mathcal{B}_x(\psi) = \mathcal{B}_x(\widetilde{\psi})$ we have that

$$\mathfrak{d}\mathfrak{u} \cdot \psi(x,y) = \mathfrak{d} \cdot \widetilde{\psi}(x,y)q(y) = \psi(x,y)q(y)b_{\widetilde{\psi}}(\mathfrak{d}) = \mathfrak{u}b_{\psi}^{-1}(b_{\widetilde{\psi}}(\mathfrak{d})).$$

Hence

$$\mathfrak{d}\mathfrak{u} = \mathfrak{u}b_{\psi}^{-1}(b_{\widetilde{\psi}}(\mathfrak{d})) \quad \text{for all } d \in \mathcal{B}_x(\psi),$$

and in particular $b_{\psi}^{-1} \circ b_{\widetilde{\psi}}$ preserves the order and leading coefficient of operators. Choose a monic differential operator $\mathfrak{d} \in \mathcal{B}_x(\psi)$ with order $\ell$ taken to be positive but as small as possible. Then we see that for some constant $c \in \mathbb{C}$

$$\mathfrak{d}\mathfrak{u} = \mathfrak{u}b_{\psi}^{-1} \circ b_{\widetilde{\psi}}(\mathfrak{d}) = \mathfrak{u}(\mathfrak{d} + c).$$

In particular, $\mathrm{ad}_{\mathfrak{d}}(\mathfrak{u}) = c\mathfrak{u}$. However, since $\mathfrak{d}$ is in the bispectral algebra and $\mathfrak{u}$ is in the Fourier algebra, the ad-condition implies that for some integer $m > 0$ we have $0 = \mathrm{ad}_{\mathfrak{d}}^m(\mathfrak{u}) = c^m\mathfrak{u}$. Hence $c = 0$ and $\mathfrak{d}$ and $\mathfrak{u}$ commute. Since $\mathcal{B}_x(\psi)$ is maximally commutative, this implies $\mathfrak{u} \in \mathcal{B}_x(\psi)$ and $\mathfrak{u} \cdot \psi(x,y) = g(y)\psi(x,y)$ for some $g(y) \in \mathbb{C}[y]$. This implies that

$$\widetilde{\psi}(x,y) = \frac{\psi(x,y)g(y)}{q(y)}$$

and since both bispectral functions are normalized, the ratio $\frac{g(y)}{q(y)}$ must be a constant. □

The next theorem shows that when we restrict our attention to a self-adjoint bispectral meromorphic functions $\psi(x,y)$, the self-adjoint bispectral meromorphic functions arising from bispectral Darboux transformations of $\psi(x,y)$ are precisely those which arise from self-adjoint bispectral Darboux transformations. This statement, which is converse to that in Proposition 3.18, is harder and some easily verifiable conditions on the initial function $\psi(x,y)$ are imposed.

**Theorem 3.21.** *Let $\psi(x,y)$ and $\widetilde{\psi}(x,y)$ be a self-adjoint bispectral meromorphic functions and suppose that $\widetilde{\psi}(x,y)$ is a bispectral Darboux transformation of $\psi(x,y)$ with*

$$\widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)}\mathfrak{u} \cdot \psi(x,y) \quad \textit{and} \quad \psi(x,y) = \widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)\widetilde{q}(y)} \cdot \widetilde{\psi}(x,y)$$

*for some operators $\mathfrak{u}, \widetilde{\mathfrak{u}} \in \mathcal{F}_x(\psi)$ and polynomials $p(x), \widetilde{p}(x), q(y), \widetilde{q}(y)$ such that the bispectral operators*

$$\widetilde{\mathfrak{u}}\frac{1}{p(x)\widetilde{p}(x)}\mathfrak{u} \in \mathcal{B}_x(\psi) \quad \textit{and} \quad b_{\psi}(\widetilde{\mathfrak{u}})\frac{1}{p(x)\widetilde{p}(x)}b_{\psi}(\widetilde{\mathfrak{u}}) \in \mathcal{B}_y(\psi)$$

*are formally symmetric. Assume that $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are maximally commutative, that $p(x) = \widetilde{p}(x)$ and $q(y) = \widetilde{q}(y)$, and that $\mathfrak{u}^*, \widetilde{\mathfrak{u}}^* \in \mathcal{F}_x(\psi)$. Then $\widetilde{\psi}(x,y)$ is a self-adjoint bispectral Darboux transformation of $\psi(x,y)$, and in fact the above bispectral Darboux transformation is self-adjoint.*

**Remark 3.22.** (i) In the case that $\psi(x,y) = \psi_{\mathrm{Ai}}(x,y)$, the assumptions follow trivially since $b_\psi^{-1}(\mathcal{B}_y(\psi)) = \mathbb{C}[x]$, $b_\psi(\mathcal{B}_x(\psi)) = \mathbb{C}[y]$, and $b_\psi(\mathfrak{w}^*) = b_\psi(\mathfrak{w})^*$ for all $\mathfrak{w} \in \mathcal{F}_x(\psi)$. For $\psi(x,y) = \psi_{\mathrm{Be}(\nu)}(x,y)$ (with $\nu \notin \mathbb{Z}$) the conditions follow from the explicit characterization of the bispectral Darboux transformations by Bakalov, Horozov, and Yakimov [5, Theorem 2.7].

(ii) If $\psi(x,y)$ is self-adjoint and rank 2 with maximally commutative $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$, then all the operators in $\mathcal{B}_x(\psi)$ and $\mathcal{B}_y(\psi)$ are formally symmetric.

(iii) The condition $p(x) = \widetilde{p}(x)$ and $q(y) = \widetilde{q}(y)$ in the theorem is equivalent to the seemingly weaker condition that

$$\frac{p(x)}{\widetilde{p}(x)} = \frac{\widetilde{p}_1(x)}{p_1(x)} \quad \text{and} \quad \frac{q(y)}{\widetilde{q}(y)} = \frac{\widetilde{q}_1(y)}{q_1(y)}$$

for some polynomials $p_1(x), \widetilde{p}_1(x) \in \mathcal{F}_x(\psi)$ and $q_1(y), \widetilde{q}_1(y) \in \mathcal{F}_y(\psi)$. This is the condition that is directly verified by [5, Theorem 2.7]. If the second condition holds, then by setting $\mathfrak{d}_1 = b_\psi^{-1}(q_1(y))$ and $\widetilde{\mathfrak{d}_1} = b_\psi^{-1}(\widetilde{q}_1(y))$, and by changing

$$\begin{aligned} &p(x) \mapsto p_1(x)p(x), \quad \mathfrak{u} \mapsto p_1(x)\mathfrak{u}\mathfrak{d}_1, \quad q(y) \mapsto q_1(y)q(y), \\ &\widetilde{p}(x) \mapsto \widetilde{p}_1(x)\widetilde{p}(x), \quad \widetilde{\mathfrak{u}} \mapsto \widetilde{p}_1(x)\widetilde{\mathfrak{u}}\widetilde{\mathfrak{d}_1}, \quad \widetilde{q}(y) \mapsto \widetilde{q}_1(y)\widetilde{q}(y), \end{aligned}$$

we get a new bispectral Darboux transformation from $\psi(x,y)$ to $\widetilde{\psi}(x,y)$ which satisfies the condition (a) of the theorem.

*Proof.* Since $\mathfrak{u}^*$ and $\widetilde{\mathfrak{u}}^*$ belong to $\mathcal{F}_x(\psi)$, the expressions $b_\psi(\mathfrak{u}^*)$ and $b_\psi(\widetilde{\mathfrak{u}}^*)$ make sense. Let $r(y)$ and $\widetilde{r}(y)$ denote the leading coefficients of these two expressions, respectively. Define

$$\varphi(x,y) := \frac{1}{\widetilde{p}(x)\widetilde{r}(y)}\widetilde{\mathfrak{u}}^* \cdot \psi(x,y), \quad \text{and} \quad \widetilde{\mathfrak{d}} = \left(\frac{1}{p(x)}\mathfrak{u}\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)}\right).$$

Note that $\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathfrak{u} \cdot \psi(x,y) = q(y)\widetilde{q}(y)\psi(x,y)$. Furthermore, by assumption

$$\mathfrak{u}^*\frac{1}{p(x)\widetilde{p}(x)}\widetilde{\mathfrak{u}}^* \cdot \psi(x,y) = q(y)\widetilde{q}(y) \cdot \psi(x,y).$$

This expression immediately implies

$$b_\psi(\mathfrak{u}^*)\frac{1}{q(y)\widetilde{q}(y)}b_\psi(\widetilde{\mathfrak{u}}^*) \cdot \psi(x,y) = p(x)\widetilde{p}(x)\psi(x,y).$$

In particular, $b_\psi(\mathfrak{u}^*)\frac{1}{q(y)\widetilde{q}(y)}b_\psi(\widetilde{\mathfrak{u}}^*)$ is in the bispectral algebra and therefore

$$q(y)\widetilde{q}(y) = r(y)\widetilde{r}(y)$$

up to a scalar multiple. This multiple equals 1 by the normalization in (3.3). Thus we have

$$\mathfrak{u}^*\frac{1}{r(y)p(x)} \cdot \varphi(x,y) = \frac{1}{r(y)\widetilde{r}(y)}\mathfrak{u}^*\frac{1}{p(x)\widetilde{p}(x)}\mathfrak{u} \cdot \psi(x,y) = \psi(x,y)$$

and thus $\varphi(x,y)$ is a bispectral Darboux transformation of $\psi(x,y)$.

We also calculate

$$\widetilde{\mathfrak{d}} \cdot \widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)}\mathfrak{u}\left(\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)p(x)}\mathfrak{u}\right) \cdot \psi(x,y) = \widetilde{\psi}(x,y)q(y)\widetilde{q}(y)$$

and

$$\widetilde{\mathfrak{d}}^* \cdot \varphi(x,y) = \frac{1}{\widetilde{p}(x)\widetilde{q}(y)}\widetilde{\mathfrak{u}}^*\left(\mathfrak{u}^*\frac{1}{p(x)\widetilde{p}(x)}\widetilde{\mathfrak{u}}^*\right)\cdot\psi(x,y) = \varphi(x,y)q(y)\widetilde{q}(y).$$

The functions $\widetilde{\psi}(x,y)$ and $\varphi(x,y)$ are both bispectral Darboux transformations of $\psi(x,y)$, so the algebras $\mathcal{B}_x(\widetilde{\psi})$ and $\mathcal{B}_x(\varphi)$ are maximally commutative. Since $\widetilde{\psi}(x,y)$ was self-adjoint and maximally commutative, $\mathcal{B}_x(\psi)$ is closed under $*$. Thus $\widetilde{\mathfrak{d}}^*$ is contained in both bispectral algebras and by maximal commutativity we must have $\mathcal{B}_x(\widetilde{\psi}) = \mathcal{B}_x(\varphi)$. Furthermore, the two bispectral Darboux transformations $\varphi(x,y)$ and $\widetilde{\psi}(x,y)$ are related by the identity

$$\varphi(x,y) = \frac{1}{\widetilde{q}(y)\widetilde{r}(y)}\mathfrak{a}\cdot\widetilde{\psi}(x,y) \quad \text{for } \mathfrak{a} = \frac{1}{\widetilde{p}(x)}\widetilde{\mathfrak{u}}^*\widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)} \in \mathcal{F}_x(\widetilde{\psi}).$$

Then by the previous proposition $\mathfrak{a} \in \mathcal{B}_x(\widetilde{\psi})$ and $\widetilde{\psi}(x,y) = c\varphi(x,y)$ for some constant $c$. In particular,

$$\frac{1}{\widetilde{p}(x)}\widetilde{\mathfrak{u}}^* q(y)\widetilde{q}(y)\cdot\psi = h(y)\frac{1}{p(x)}\mathfrak{u}\cdot\psi.$$

This means

$$\frac{1}{p(x)q(y)}\mathfrak{u}\cdot\psi(x,y) = \frac{c}{\widetilde{p}(x)\widetilde{r}(y)}\widetilde{\mathfrak{u}}^*\cdot\psi(x,y),$$

so that

$$\widetilde{q}(y)r(y)c^{-1}\psi(x,y) = \widetilde{\mathfrak{u}}\frac{1}{\widetilde{p}(x)\widetilde{p}(x)}\widetilde{\mathfrak{u}}^*\cdot\psi(x,y).$$

Consequently,

$$b_\psi(\widetilde{\mathfrak{u}})\frac{c}{\widetilde{q}(y)r(y)}b_\psi(\widetilde{\mathfrak{u}^*})\cdot\psi(x,y) = \widetilde{p}(x)\widetilde{p}(x)\psi(x,y).$$

Since $p(x) = \widetilde{p}(x)$ this implies

$$b_\psi(\widetilde{\mathfrak{u}})\frac{c}{\widetilde{q}(y)r(y)}b_\psi(\widetilde{\mathfrak{u}^*}) = b_\psi(\mathfrak{u}^*)\frac{1}{r(y)\widetilde{r}(y)}b_\psi(\widetilde{\mathfrak{u}^*}),$$

which simplifies to

$$b_\psi(\widetilde{\mathfrak{u}})\frac{1}{\widetilde{q}(y)} = b_\psi(\mathfrak{u}^*)\frac{1}{c\widetilde{r}(y)}.$$

The left side is monic, so the right side must be also and hence $c\widetilde{r}(y) = r(y)$. Thus since $q(y) = \widetilde{q}(y)$, we find

$$r(y)^2 = q(y)^2 c$$

so that $r(y) = \pm\sqrt{c}q(y)$. Inserting this above yields

$$\widetilde{\mathfrak{u}} = \pm\frac{\mathfrak{u}^*}{\sqrt{c}}.$$

Since $\widetilde{\mathfrak{u}}$ and $\mathfrak{u}$ have the same leading coefficient, this simply says that

$$\widetilde{\mathfrak{u}} = (-1)^{d_1}\mathfrak{u}^*$$

for $d_1 = \mathrm{ord}(\mathfrak{u})$. By the same argument applied on the opposite side, we also obtain

$$b_\psi(\widetilde{\mathfrak{u}}) = (-1)^{d_2}b_\psi(\mathfrak{u})^*$$

for $d_2 = \mathrm{cord}(\mathfrak{u})$. □

**Corollary 3.23.** *The self-adjoint bispectral Darboux transformations of the exponential, Airy and Bessel functions are precisely those functions $\widetilde{\psi}(x,y)$ that may be obtained by a bispectral Darboux transformations of $\psi_{\exp}, \psi_{\mathrm{Ai}}$ or $\psi_{\mathrm{Be}(\nu)}$ and are self-adjoint as bispectral functions in the sense of Definition* 3.16*.*

**3.3. Interplay between adjoints and the generalized Fourier map.** In this subsection we discuss the interaction between the formal adjoint $*$ and the generalized Fourier map $b_\psi$. Specifically, we consider in this subsection the value of $b_\psi(\mathfrak{d}^*)$, assuming that both $\mathfrak{d}$ and $\mathfrak{d}^*$ are in $\mathcal{F}_x(\psi)$. It turns out that in the cases we consider, the value of $b_\psi(\mathfrak{d}^*)$ is related to $b_\psi(\mathfrak{d})^*$ by a certain automorphism of the Weyl algebra. As a consequence of this, the elements of $\mathcal{F}_{x,\mathrm{sym}}(\psi)$ are invariant under this automorphism. This invariance property of $\mathcal{F}_{x,\mathrm{sym}}(\psi)$ will play an important role of our proof of Theorem A from the introduction.

For many self-adjoint bispectral meromorphic functions $\psi$ the associated algebras $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ are closed under the formal adjoint operation $*$. This is true in particular of the elementary bispectral meromorphic functions: by Example 2.6, the Fourier algebras for $\psi_{\exp}, \psi_{\mathrm{Ai}}$ and $\psi_{\mathrm{Be}(\nu)}$ are all closed under the operation of taking formal adjoints. The same property turns out to hold for the Fourier algebras of all functions obtained by self-adjoint bispectral Darboux transformations from them.

**Proposition 3.24.** *Let $\widetilde{\psi}$ be a self-adjoint bispectral Darboux transformation of $\psi$. Suppose that $\mathcal{F}_x(\psi)$ is closed under the formal adjoint operation $*$. Then $\mathcal{F}_x(\widetilde{\psi})$ is also closed under the $*$-operation. The same statement holds with $\mathcal{F}_x$ replaced with $\mathcal{F}_y$.*

*Proof.* Let $\mathfrak{u}, \widetilde{\mathfrak{u}} = \mathfrak{u}^*$, $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ be as in the definition of a self-adjoint bispectral Darboux transformation with

$$\widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)}\mathfrak{u}\cdot\psi(x,y).$$

Also define $\widetilde{\mathfrak{a}} = \frac{1}{p(x)}\mathfrak{u}\mathfrak{u}^*\frac{1}{p(x)} \in \mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$ with

$$\widetilde{\mathfrak{a}}\cdot\widetilde{\psi}(x,y) = q(y)^2\widetilde{\psi}(x,y).$$

Next note that for any differential operator $\mathfrak{q}$ with meromorphic coefficients,

$$\mathfrak{q}\widetilde{\mathfrak{a}} \in \mathcal{F}_x(\widetilde{\psi}) \implies \mathfrak{q} \in \mathcal{F}_x(\widetilde{\psi}). \tag{3.10}$$

This follows from the identity

$$\mathfrak{q}\cdot\widetilde{\psi}(x,y)q(y)^2 = \mathfrak{q}\widetilde{\mathfrak{a}}\cdot\widetilde{\psi}(x,y) = b_{\widetilde{\psi}}(\mathfrak{q}\widetilde{\mathfrak{a}})\widetilde{\psi}(x,y),$$

which implies that

$$\mathfrak{q}\cdot\widetilde{\psi}(x,y) = q(y)^{-2}b_{\widetilde{\psi}}(\mathfrak{q}\widetilde{\mathfrak{a}})\cdot\widetilde{\psi}(x,y).$$

For any $\widetilde{\mathfrak{d}} \in \mathcal{F}_x(\widetilde{\psi})$, one easily checks that

$$\mathfrak{d} := \mathfrak{u}^*\frac{1}{p(x)}\widetilde{\mathfrak{d}}\frac{1}{p(x)}\mathfrak{u} \in \mathcal{F}_x(\psi).$$

In fact, $b_\psi(\mathfrak{d}) = q(y)b_{\widetilde{\psi}}(\widetilde{\mathfrak{d}})q(y)$. Since $\mathcal{F}_x(\psi)$ is closed under $*$, it follows that

$$\mathfrak{d}^* = \mathfrak{u}^*\frac{1}{p(x)}\widetilde{\mathfrak{d}}^*\frac{1}{p(x)}\mathfrak{u} \in \mathcal{F}_x(\psi).$$

Consequently,

$$\widetilde{\mathfrak{a}}\widetilde{\mathfrak{d}}^* \cdot \widetilde{\psi}(x,y) = \frac{1}{q(y)} b_\psi(\mathfrak{d}^*) \cdot q(y)\widetilde{\psi}(x,y),$$

so that in particular $\widetilde{\mathfrak{a}}\widetilde{\mathfrak{d}}^* \in \mathcal{F}_x(\widetilde{\psi})$ for all $\widetilde{\mathfrak{d}} \in \mathcal{F}_x(\widetilde{\psi})$. In the remainder of the proof we show that this implies that $\widetilde{\mathfrak{d}}^* \in \mathcal{F}_x(\widetilde{\psi})$.

Let $\ell$ be the co-order of $\widetilde{\mathfrak{d}}$. Then $\mathrm{ad}_{\widetilde{\mathfrak{a}}}^{\ell+1}(\widetilde{\mathfrak{d}})$ is sent under $b_{\widetilde{\psi}}$ to

$$(-1)^{\ell+1} \mathrm{ad}_{q(y)^2}^{\ell+1}(b_\psi(\widetilde{\mathfrak{d}})) = 0.$$

Since the generalized Fourier map is an anti-isomorphism, this implies that $\mathrm{ad}_{\widetilde{\mathfrak{a}}}^{\ell+1}(\widetilde{\mathfrak{d}}) = 0$. Using the fact that $\widetilde{\mathfrak{a}}$ is formally symmetric, we obtain that

$$0 = \mathrm{ad}_{\widetilde{\mathfrak{a}}}^{\ell+1}(\widetilde{\mathfrak{d}}^*) = \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} \widetilde{\mathfrak{a}}^{\ell+1-k} \widetilde{\mathfrak{d}}^* \widetilde{\mathfrak{a}}^k.$$

This identity and the fact that $\widetilde{\mathfrak{a}}\widetilde{\mathfrak{d}}^*, \widetilde{\mathfrak{a}} \in \mathcal{F}_x(\widetilde{\psi})$ give that $\widetilde{\mathfrak{d}}^*\widetilde{\mathfrak{a}}^{\ell+1} \in \mathcal{F}_x(\widetilde{\psi})$. By repeatedly applying (3.10), we obtain that $\widetilde{\mathfrak{d}}^* \in \mathcal{F}_x(\psi)$. This proves that $\mathcal{F}_x(\widetilde{\psi})$ is closed under $*$. In a similar way, one proves that $\mathcal{F}_y(\widetilde{\psi})$ is closed under $*$. □

Assuming that both $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ are closed under the $*$-operation, we can define automorphisms $\iota_{\psi,x}$ and $\iota_{\psi,y}$ of $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ by setting

$$\iota_{\psi,x}(\mathfrak{d}) = b_\psi^{-1}(b_\psi(\mathfrak{d}^*)^*) \quad \text{and} \quad \iota_{\psi,y}(\mathfrak{b}) = b_\psi(b_\psi^{-1}(\mathfrak{d}^*)^*). \tag{3.11}$$

These automorphisms encode the interaction between the generalized Fourier map $b_\psi$ and the formal adjoint. Clearly, $\iota_{\psi,x}$ and $\iota_{\psi,y}$ restrict to the identity on $\mathcal{F}_{x,\mathrm{sym}}(\psi)$ and $\mathcal{F}_{y,\mathrm{sym}}(\psi)$.

In the case $\psi = \psi_{\exp}$ or $\psi = \psi_{\mathrm{Be}(\nu)}$ the automorphisms $\iota_{\psi,x}$ and $\iota_{\psi,y}$ are exactly obtained by restricting the automorphism of the algebra of differential operators with rational coefficients induced by the affine transformation $x \mapsto -x$ and $y \mapsto -y$. In the case $\psi = \psi_{\mathrm{Ai}}$ the automorphisms are just the identity. Moreover, as a consequence of the previous proposition, for any self-adjoint bispectral Darboux transformation $\widetilde{\psi}$ of $\psi = \psi_{\exp}, \psi_{\mathrm{Ai}}$ or $\psi_{\mathrm{Be}(\nu)}$ the involutive automorphisms $\iota_{\widetilde{\psi},x}$ and $\iota_{\widetilde{\psi},y}$ are well defined. The next proposition shows that $\iota_{\widetilde{\psi},x}$ and $\iota_{\widetilde{\psi},y}$ behave identically to $\iota_{\psi,x}$ and $\iota_{\psi,y}$.

**Proposition 3.25.** *Let $\psi$ be a self-adjoint bispectral meromorphic function and let $\widetilde{\psi}$ be a self-adjoint bispectral Darboux transformation of $\psi$. Suppose that $\mathcal{F}_x(\psi)$ and $\mathcal{F}_y(\psi)$ are closed under the formal adjoint operation $*$. Assume moreover that there are involutive automorphisms $\sigma_x$ and $\sigma_y$ of the algebras of differential operators with meromorphic coefficients in $x$ and $y$, respectively, restricting to $\iota_{\psi,x}$ and $\iota_{\psi,y}$. Then $\sigma_x$ and $\sigma_y$ also restrict to $\iota_{\widetilde{\psi},x}$ and $\iota_{\widetilde{\psi},y}$, respectively. In particular, the elements of $\mathcal{F}_{x,sym}(\widetilde{\psi})$ are fixed by $\sigma_x$ and similarly for $\mathcal{F}_{y,sym}(\widetilde{\psi})$.*

*Proof of Proposition* 3.25. Let $\mathfrak{u}, \widetilde{\mathfrak{u}} = \mathfrak{u}^*$, $p(x) = \widetilde{p}(x)$, and $q(x) = \widetilde{q}(x)$ be as in the definition of a self-adjoint bispectral Darboux transformation with

$$\widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)} \mathfrak{u} \cdot \psi(x,y).$$

For sake of brevity set $\mathfrak{n} = \frac{1}{p(x)}\mathfrak{u}$. First note that if $p(x)$ is constant, then $\widetilde{\psi}(x,y)$ is a con-

stant multiple of $\psi(x,y)$ and the proposition is trivial. Therefore we assume otherwise. Suppose that $\widetilde{\mathfrak{d}} \in \mathcal{F}_x(\widetilde{\psi})$. Then $\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n} \in \mathcal{F}_x(\psi)$ with $b_\psi(\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n}) = q(y)b_{\widetilde{\psi}}(\widetilde{\mathfrak{d}})q(y)$. In particular, if $\widetilde{\mathfrak{d}} \in \mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$, then $\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n} \in \mathcal{F}_{x,\mathrm{sym}}(\psi)$. Since $\iota_{\psi,x}$ acts as the identity on $\mathcal{F}_{x,\mathrm{sym}}(\psi)$, for all $\mathfrak{d} \in \mathcal{F}_{x,\mathrm{sym}}(\psi)$ we have

$$\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n} = \iota_{\psi,x}(\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n}) = \sigma_x(\mathfrak{n}^*)\sigma_x(\widetilde{\mathfrak{d}})\sigma_x(\mathfrak{n}).$$

Applying this to the case when $\widetilde{\mathfrak{d}} = 1$, we see that as pseudo-differential operators

$$\sigma_x(\mathfrak{n}^*)^{-1}\mathfrak{n}^* = \sigma_x(\mathfrak{n})\mathfrak{n}^{-1}.$$

Since $\psi(x,y)$ is self-adjoint and $\widetilde{\psi}(x,y)$ is a self-adjoint bispectral Darboux transformation of $\psi(x,y)$, we know that $p(x)^2 \in \mathcal{F}_{x,\mathrm{sym}}(\psi) \cap \mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$. Letting $\widetilde{\mathfrak{d}} = p(x)^2 \in \mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$, we calculate

$$\sigma_x(\mathfrak{n})\mathfrak{n}^{-1}p(x)^2 = p(x)^2\sigma_x(\mathfrak{n})\mathfrak{n}^{-1}.$$

Since $p(x)$ is nonconstant, this implies that $\sigma_x(\mathfrak{n})\mathfrak{n}^{-1} = g(x)$ for some rational function $g(x)$. Thus for all $\widetilde{\mathfrak{d}} \in \mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$ we see

$$g(x)\widetilde{\mathfrak{d}}g(x)^{-1} = \sigma_x(\widetilde{\mathfrak{d}}).$$

Since $\sigma_x$ was assumed to be an involution, this implies that $g(x)^2$ is in the center if $\mathcal{F}_{x,\mathrm{sym}}(\psi)$, and must therefore be constant. Hence $g(x)$ is constant, i.e., $g(x) = c$ for some constant $c \in \mathbb{C}$. In particular, $\sigma_x(\mathfrak{n}) = c\mathfrak{n}$ and since $\sigma_x$ is an involution, $c^2 = 1$ so that $c = \pm 1$. This also implies that $\sigma(\mathfrak{n}^*) = c\mathfrak{n}$.

Suppose that $\widetilde{\mathfrak{d}} \in \mathcal{F}_x(\widetilde{\psi})$. From the proof of the previous proposition, we know that

$$b_\psi(\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n}) = q(y)b_{\widetilde{\psi}}(\widetilde{\mathfrak{d}})q(y).$$

From this, we see

$$\begin{aligned} b_{\widetilde{\psi}}(\iota_{\widetilde{\psi},x}(\widetilde{\mathfrak{d}})) &= b_{\widetilde{\psi}}(\widetilde{\mathfrak{d}}^*)^* \\ &= \frac{1}{q(y)}b_\psi(\mathfrak{n}^*\widetilde{\mathfrak{d}}^*\mathfrak{n})^*\frac{1}{q(y)} \\ &= \frac{1}{q(y)}b_\psi \circ \iota_{\psi,x}(\mathfrak{n}^*\widetilde{\mathfrak{d}}\mathfrak{n})\frac{1}{q(y)} \\ &= \frac{1}{q(y)}b_\psi(\mathfrak{n}^*\sigma_x(\widetilde{\mathfrak{d}})\mathfrak{n})\frac{1}{q(y)} = b_{\widetilde{\psi}}(\sigma_x(\widetilde{\mathfrak{d}})). \end{aligned}$$

It follows that $\iota_{\widetilde{\psi},x}(\widetilde{\mathfrak{d}}) = \sigma_x(\widetilde{\mathfrak{d}})$ for all $\widetilde{\mathfrak{d}} \in \mathcal{F}_x(\psi)$. A similar proof holds for $\mathcal{F}_y(\psi)$. □

The following corollary follows immediately.

**Corollary 3.26.** *If $\psi(x,y)$ is a self-adjoint bispectral Darboux transformation of $\psi_{\exp}$ or $\psi_{\mathrm{Be}(\nu)}$, then the elements of $\mathcal{F}_{x,sym}(\psi)$ and $\mathcal{F}_{y,sym}(\psi)$ are invariant under the affine transformations $x \mapsto -x$ and $y \mapsto -y$, respectively.*

**3.4. Formalization of commutativity of integral and differential operators.** In the final part of this section we formalize commutativity of integral and differential operators.

Let $\Gamma$ be a smooth curve in $\mathbb{C}$ and let $\mathfrak{D}(\Gamma)$ be the space of differential operators on $\Gamma$ with smooth coefficients. Let $C^\infty(\Gamma)$ denote the space of smooth functions on $\Gamma$ and let $C_c^\infty(\Gamma)$ be the subspace of $C^\infty(\Gamma)$ consisting of functions with compact support. Consider an integral operator $T : f(x) \mapsto \int_\Gamma K(x,y) f(y)\, dy$ with kernel $K(x,y)$ smooth on $\Gamma \times \Gamma$ and satisfying the property that the functions $\int_\Gamma |K(x,y)|\, dx$ and $\int_\Gamma |K(x,y)|\, dy$ both lie in $L^\infty(\Gamma)$. Then for any $f(x) \in C_c^\infty(\Gamma)$ the integral $\int_\Gamma K(x,y) f(y)\, dy$ converges uniformly to a smooth function $T(f(x)) \in C^\infty(\Gamma)$. Furthermore, an operator $\mathfrak{d} \in \mathfrak{D}(\Gamma)$ restricts to an endomorphism of $C^\infty(\Gamma)$ and $C_c^\infty(\Gamma)$. Thus for any $f(x) \in C_c^\infty(\Gamma)$, the expressions $T(\mathfrak{d} \cdot f)$ and $\mathfrak{d} \cdot (T(f))$ are both elements of $C^\infty(\Gamma)$.

We say that *the integral operator $T$ and the differential operators $\mathfrak{d}$ commute* if

$$T(\mathfrak{d} \cdot f) = \mathfrak{d} \cdot (T(f)) \quad \text{for all } f \in C_c^\infty(\Gamma).$$

## 4. Darboux transformations and growth of Fourier algebras

In this section we carry out two key steps of our strategy for going from bispectral meromorphic functions to integral operators possessing a commuting differential operator.

Firstly, we prove a theorem that gives a lower bound of the growth of the double filtration of the Fourier algebra $\mathcal{F}_x(\widetilde{\psi})$ of a bispectral Darboux transformation $\widetilde{\psi}$ of $\psi$. Modulo technical details we show that

$$|\dim(\mathcal{F}_x^{\ell,m}(\widetilde{\psi})) - \dim(\mathcal{F}_x^{\ell,m}(\psi))| \leq \text{const}$$

for a constant that is independent on $\ell$ and $m$. We also prove that a similar inequality holds for $\mathcal{F}_x^{\ell,m}$ replaced with $\mathcal{F}_{x,\mathrm{sym}}^{\ell,m}$.

Secondly, we prove that if the dimension of the space $\mathcal{F}_x^{\ell,m}(\widetilde{\psi})$ satisfies a natural lower bound, then it contains a bisymmetric operator with respect to a pair of contours.

### 4.1. Control of the growth of Fourier algebras under Darboux transformations.

**Theorem 4.1.** *Suppose that $\psi(x,y)$ is a bispectral meromorphic function and that $\widetilde{\psi}(x,y)$ is a self-adjoint bispectral Darboux transformation of $\psi(x,y)$ of order $(d_1,d_2)$. Then the following is true for all $\ell > d_2$ and $m > d_1$:*

$$\dim(\mathcal{F}_x^{\ell,m}(\widetilde{\psi})) \geq \dim(\mathcal{F}_x^{\ell,m-2d_2}(\psi)) + \dim(\mathcal{F}_x^{\ell-2d_1,2d_2-1}(\psi)) + 1,$$
$$\dim(\mathcal{F}_{x,sym}^{\ell,m}(\widetilde{\psi})) \geq \dim(\mathcal{F}_{x,sym}^{\ell,m-2d_2}(\psi)) + \dim(\mathcal{F}_{x,sym}^{\ell-2d_1,2d_2-1}(\psi)) + 1.$$

*Proof.* By the definition of a self-adjoint bispectral Darboux transformation, there exists a differential operator $\mathfrak{u} \in \mathcal{F}_x(\psi)$ and polynomials $p(x) \in \mathbb{C}[x]$ and $q(y) \in \mathbb{C}[y]$ such that

$$\widetilde{\psi} = \frac{1}{p(x)q(y)} \mathfrak{u} \cdot \psi,$$

and such that for $\mathfrak{w} = b_\psi(\mathfrak{u})$,

$$b_\psi(p(x)^2) = \mathfrak{w}^* \frac{1}{q(y)^2} \mathfrak{w}$$

and

$$b_\psi^{-1}(q(y)^2) = \mathfrak{u}^* \frac{1}{p(x)^2} \mathfrak{u}.$$

Let $d_1$ be the order of $\mathfrak{u}$ and let $d_2$ be the order of $\mathfrak{w}$. If $\mathfrak{d} \in \mathcal{F}_x^{\ell,m}(\psi)$, then we have

$$\begin{aligned} p(x)\mathfrak{d}\,p(x)\cdot\widetilde{\psi}(x,y) &= \frac{p(x)}{q(y)}\mathfrak{d}\mathfrak{u}\cdot\psi(x,y)\\ &= \frac{1}{q(y)}\mathfrak{w}b_\psi(\mathfrak{d})\cdot p(x)\psi(x,y)\\ &= \frac{1}{q(y)}\mathfrak{w}b_\psi(\mathfrak{d})\cdot p(x)^2\frac{1}{p(x)}\psi(x,y)\\ &= \frac{1}{q(y)}\mathfrak{w}b_\psi(\mathfrak{d})\mathfrak{w}^*\frac{1}{q(y)^2}\mathfrak{w}\cdot\frac{1}{p(x)}\psi(x,y)\\ &= \frac{1}{q(y)}\mathfrak{w}b_\psi(\mathfrak{d})\mathfrak{w}^*\frac{1}{q(y)}\cdot\widetilde{\psi}(x,y). \end{aligned}$$

This shows that $p(x)\mathfrak{d}\,p(x) \in \mathcal{F}_x^{\ell,m+2d_2}(\widetilde{\psi})$ with

$$b_{\widetilde{\psi}}(p(x)\mathfrak{d}\,p(x)) = \frac{1}{q(y)}\mathfrak{w}b_\psi(\mathfrak{d})\mathfrak{w}^*\frac{1}{q(y)}.$$

In particular, we have an inclusion

$$p(x)\mathcal{F}_x^{\ell,m}(\psi)p(x) \subseteq \mathcal{F}_x^{\ell,m+2d_2}(\widetilde{\psi}).$$

Similarly, we can show that if $\mathfrak{b} \in \mathcal{F}_y^{m,\ell}(\psi)$, then $q(y)\mathfrak{b}q(y) \in \mathcal{F}_y^{m,\ell}(\widetilde{\psi})$ with

$$b_{\widetilde{\psi}}^{-1}(q(y)\mathfrak{b}q(y)) = \frac{1}{p(x)}\mathfrak{u}b_\psi^{-1}(\mathfrak{b})\mathfrak{u}^*\frac{1}{p(x)}.$$

Thus we also have an inclusion

$$p(y)\mathcal{F}_y^{m,\ell}(\psi)p(x) \subseteq \mathcal{F}_y^{m,\ell+2d_1}(\widetilde{\psi}).$$

Therefore we have two inclusions

$$p(x)\mathcal{F}_x^{\ell+2d_1,m}(\psi)p(x) \subseteq \mathcal{F}_x^{\ell+2d_1,m+2d_2}(\widetilde{\psi})$$

and

$$b_{\widetilde{\psi}}^{-1}(q(y)\mathcal{F}_y^{m+2d_2,\ell}(\psi)q(y)) \subseteq \mathcal{F}_x^{\ell+2d_1,m+2d_2}(\widetilde{\psi}).$$

Furthermore,

$$p(x)\mathcal{F}_x^{\ell+2d_1,m}(\psi)p(x) \cap b_{\widetilde{\psi}}^{-1}(q(y)\mathcal{F}_y^{2d_2-1,\ell}(\psi)q(y)) = 0,$$

because the nonzero elements in $p(x)\mathcal{F}_x^{\ell+2d_1,m}(\psi)p(x)$ are mapped under $b_{\widetilde{\psi}}$ to elements of order at least $2d_2$. Thus we can write

$$\mathbb{C} \oplus p(x)\mathcal{F}_x^{\ell+2d_1,m}(\psi)p(x) \oplus b_{\widetilde{\psi}}^{-1}(q(y)\mathcal{F}_y^{2d_2-1,\ell}(\psi)q(y)) \subseteq \mathcal{F}_x^{\ell+2d_1,m+2d_2}(\widetilde{\psi}),$$

which in particular gives us the dimension estimate

$$\dim(\mathcal{F}_x^{\ell+2d_1,m+2d_2}(\widetilde{\psi})) \geq \dim(\mathcal{F}_x^{\ell+2d_1,m}(\psi)) + \dim(\mathcal{F}_y^{2d_2-1,\ell}(\psi)) + 1.$$

The above inclusion also sends formally symmetric operators to formally symmetric operators, and therefore

$$\mathbb{C} \oplus p(x)\mathcal{F}_{x,\mathrm{sym}}^{\ell+2d_1,m}(\psi)p(x) \oplus b_{\widetilde{\psi}}^{-1}(q(y)\mathcal{F}_{y,\mathrm{sym}}^{2d_2-1,\ell}(\psi)q(y)) \subseteq \mathcal{F}_{x,\mathrm{sym}}^{\ell+2d_1,m+2d_2}(\widetilde{\psi}),$$

so that

$$\dim(\mathcal{F}_{x,\mathrm{sym}}^{\ell+2d_1,m+2d_2}(\widetilde{\psi})) \geq \dim(\mathcal{F}_{x,\mathrm{sym}}^{\ell+2d_1,m}(\psi)) + \dim(\mathcal{F}_{y,\mathrm{sym}}^{2d_2-1,\ell}) + 1.$$

This completes the proof of the theorem. □

**Corollary 4.2.** *Let $\psi(x,y)$ be a bispectral meromorphic function and let $\widetilde{\psi}(x,y)$ be a self-adjoint bispectral Darboux transformation of $\psi(x,y)$ of order $(d_1,d_2)$. Assume that for some $r>0$ there exist constants $a_{11},a_{10},a_{01},a_{00}$ such that for all $\ell,m\geq 0$ we have*

$$\dim(\mathcal{F}_x^{r\ell,rm}(\psi)) \geq a_{11}\ell m + a_{10}\ell + a_{01}m + a_{00}. \tag{4.1}$$

*Then the growth of the Fourier algebra $\mathcal{F}_x(\widetilde{\psi})$ is controlled by the inequality*

$$\dim(\mathcal{F}_x^{r\ell,rm}(\widetilde{\psi})) \geq a_{11}\ell m + (2a_{10}-a_{11})\ell + a_{01}m + a_{00} - \widetilde{c}, \tag{4.2}$$

*where*

$$\widetilde{c} = a_{11}\left\lfloor \frac{2d_1}{r}\right\rfloor\left\lfloor \frac{2d_2-1}{r}\right\rfloor + a_{10}\left\lfloor \frac{2d_1}{r}\right\rfloor + a_{01}\left(\left\lfloor \frac{2d_2}{r}\right\rfloor - \left\lfloor \frac{2d_2-1}{r}\right\rfloor\right) - a_{00} - 1.$$

*The statement also holds if we replace $\mathcal{F}_x^{r\ell,rm}$ with $\mathcal{F}_{x,sym}^{r\ell,rm}$ in* (4.1) *and* (4.2)*.*

**Remark 4.3.** The precise form of $\widetilde{c}$ in the above inequalities plays an especially important role in practical implementations of the search for differential operators commuting with integral operators, as we shall see below. This is because it provides an explicit bound for orders and co-orders of the operators which we must search through to find an operator commuting with our integral operator, and restricting our search to a finite-dimensional vector space.

**Remark 4.4.** We note that for all the cases of bispectral meromorphic functions $\psi$ which we have computed, the values of $a_{11},a_{10}$, and $a_{01}$ are the same for suitably chosen $r$. Simply put, for all rank 1 or rank 2 bispectral functions $\psi(x,y)$, along with all other cases we have considered, when $\widetilde{\psi}$ is a self-adjoint bispectral Darboux transformation of $\psi$, then

$$|\dim(\mathcal{F}_x^{r\ell,rm}(\widetilde{\psi})) - \dim(\mathcal{F}_x^{r\ell,rm}(\psi))| \leq \widetilde{c}$$

for $\widetilde{c}$ defined as in Corollary 4.2. A similar statement holds for $\mathcal{F}_x^{r\ell,rm}$ replaced with $\mathcal{F}_{x,\mathrm{sym}}^{r\ell,rm}$.

*Proof of Corollary* 4.2. Plugging in the lower bound into the dimension estimate from the previous theorem, we see

$$\begin{aligned}
\dim(\mathcal{F}_x^{r\ell,rm}(\widetilde{\psi})) &\geq \dim(\mathcal{F}_x^{r\ell,rm-2d_2}(\psi) + \dim(\mathcal{F}_x^{r\ell-2d_1,2d_2-1}) \\
&\geq \dim(\mathcal{F}_x^{r\ell,r(m-\lfloor\frac{2d_2}{r}\rfloor)}(\psi)) + \dim(\mathcal{F}_x^{r(\ell-\lfloor\frac{2d_1}{r}\rfloor),r\lfloor\frac{1}{r}(2d_2-1)\rfloor}(\psi)) \\
&\geq a_{11}\ell\left(m-\left\lfloor\frac{2d_2}{r}\right\rfloor\right) + a_{10}\ell + a_{01}\left(m-\left\lfloor\frac{2d_2}{r}\right\rfloor\right) + a_{00} \\
&+ a_{11}\left(\ell-\left\lfloor\frac{2d_1}{r}\right\rfloor\right)\left\lfloor\frac{2d_2-1}{r}\right\rfloor + a_{10}\left(\ell-\left\lfloor\frac{2d_1}{r}\right\rfloor\right) \\
&\qquad + a_{01}\left\lfloor\frac{2d_2-1}{r}\right\rfloor + a_{00} + 1 \\
&\geq a_{11}\ell m + (2a_{10}-a_{11})\ell + a_{01}m + a_{00} - \widetilde{c}.
\end{aligned}$$

□

In the cases considered in this paper, we will find

$$\mathcal{F}^{2\ell,2m}_{x,\mathrm{sym}}(\psi) = \ell m + \ell + m + a_{00},$$

so that

$$\mathcal{F}^{2,2}_{x,\mathrm{sym}}(\psi) = 3 + a_{00}.$$

In this case $\widetilde{c} = d_1 d_2 - a_{00}$, so that if $\widetilde{\psi}$ is a self-adjoint bispectral Darboux transformation of $\psi$ of order $(1,1)$, then

$$\dim(\mathcal{F}^{2,2}_{x,\mathrm{sym}}(\widetilde{\psi})) \geq 2(1 + a_{00}).$$

This means that as long as $a_{00} > 0$, any self-adjoint bispectral Darboux transformation $\widetilde{\psi}$ of order $(1,1)$ will necessarily have a four-dimensional space of formally bisymmetric bispectral differential operators. This will be shown by Proposition 4.7 below to in turn guarantee the existence of a differential operator of order two commuting with an integral operator whose kernel is defined using $\widetilde{\psi}$.

**4.2. Existence of bisymmetric operators.** We next use the growth rate estimates established above to prove that $\mathcal{F}_x(\psi)$ must contain bisymmetric operators.

**Lemma 4.5.** *A differential operator $\mathfrak{d} \in \mathfrak{D}(\mathbb{C}(x))$ is formally symmetric if and only if it has the form*

$$\mathfrak{d} = \sum_{j=0}^{n} \partial_x^j a_j(x) \partial_x^j \tag{4.3}$$

*for some functions $a_0(x), \dots, a_j(x) \in \mathfrak{D}(\mathbb{C}(x))$.*

*Proof.* Clearly every differential operator of the form (4.3) is formally symmetric. In the opposite direction, if $\mathfrak{d}$ is formally symmetric, then necessarily $\mathfrak{d}$ must have even order $2n$. If $n = 0$, then the statement of the lemma is true immediately. As an inductive assumption, suppose that the statement of the lemma is true for formally symmetric operators of order $\leq 2m$. Let $\mathfrak{d}$ be an operator of order $2n$ for $n = m + 1$, and let $a_n(x) \in \mathbb{C}(x)$ be the leading coefficient of $\mathfrak{d}$. Then $\mathfrak{d} - \partial_x^n a_n(x) \partial_x^n$ is a formally symmetric operator of order $\leq 2m$. Therefore by the inductive assumption

$$\mathfrak{d} - \partial_x^n a_n(x) \partial_x^n = \sum_{j=0}^{\ell} \partial_x^j a_j(x) \partial_x^j$$

for some functions $a_0(x), \dots, a_n(x) \in \mathbb{C}(x)$. The lemma now follows by induction. □

**Lemma 4.6.** *Suppose that $\mathfrak{d} \in \mathfrak{D}(\mathbb{C}(x))$ is formally symmetric with*

$$\mathfrak{d} = \sum_{j=0}^{n} \partial_x^j a_j(x) \partial_x^j,$$

*and let $\Gamma$ be a smooth path in $\mathbb{C}$ with endpoints $p_0, p_1$. Assume moreover that the poles of $a_j(x)$ are not on $\Gamma$. If $a_j^{(k)}(p_i) = 0$ for all $0 \leq k < j$ and for all $i$ with $p_i \neq \infty$, then $\mathfrak{d}$ is symmetric with respect to $\Gamma$.*

*Proof.* By linearity, it suffices to show that if $a^{(k)}(p_i) = 0$ for all $0 \le k < n$ then $\mathfrak{d} = \partial_x^n a(x)\partial_x^n$ is symmetric with respect to $\Gamma$. For $f, g \in C_c^\infty(\Gamma)$, integration by parts gives

$$\begin{aligned}\int_\Gamma (a(x)f(x))^{(n)} g(x)\,dx &= (-1)^n \int_\Gamma f(x)a(x)g^{(n)}(x)\,dx \\ &\quad + \sum_{j=0}^{n-1} (-1)^j (a(x)f(x))^{(n-1-j)} g^{(j)}(x)|_{p_0}^{p_1} \\ &= (-1)^n \int_\Gamma f(x)a(x)g^{(n)}(x)\,dx + 0.\end{aligned}$$

Replacing $f(x)$ with $f^{(n)}(x)$ yields

$$\int_\Gamma (a(x)f^{(n)}(x))^{(n)} g(x)\,dx = (-1)^n \int_\Gamma f^{(n)}(x)a(x)g^{(n)}(x)\,dx.$$

Swapping $f$ and $g$ in the last identity leads to

$$(-1)^n \int_\Gamma f^{(n)}(x)a(x)g^{(n)}(x)\,dx = \int_\Gamma f(x)(a(x)g^{(n)}(x))^{(n)}\,dx.$$

Combining the two identities, we obtain

$$\begin{aligned}\int_\Gamma (\mathfrak{d}\cdot f(x))g(x)\,dx &= \int_\Gamma (a(x)f^{(n)}(x))^{(n)} g(x)\,dx \\ &= \int_\Gamma f(x)(a(x)g^{(n)}(x))^{(n)}\,dx \\ &= \int_\Gamma f(x)(\mathfrak{d}\cdot g(x))\,dx.\end{aligned}$$

This shows that $\mathfrak{d}$ is symmetric with respect to $\Gamma$, completing the proof. □

**Proposition 4.7.** *Suppose that $\psi(x, y)$ is a bispectral meromorphic function. Let $\Gamma_1, \Gamma_2$ be two smooth curves in $\mathbb{C}$ with the endpoints of $\Gamma_i$ equal to $p_{i0}$ and $p_{i1}$ (one of which is allowed to be $\infty$ for each $i$). Assume that the coefficients of the bispectral operators of $\psi(x, y)$ are holomorphic in a neighborhood of $\Gamma_1$ and $\Gamma_2$. Assume moreover that for each $i = 0, 1$ either one of the following two conditions holds:*

(i) *$p_{i0} = -p_{i1}$ and every operator of $\mathcal{F}_{x,sym}(\psi)$ and $\mathcal{F}_{y,sym}(\psi)$ is invariant under the transformations $x \mapsto -x$ and $y \mapsto -y$, respectively, or*

(ii) *one of the points $p_{i0}$ or $p_{i1}$ is $\infty$.*

*If for some values of $\ell, m$ we have*

$$\dim(\mathcal{F}_{x,sym}^{2\ell,2m}(\psi)) > \frac{1}{2}\ell(\ell+1) + \frac{1}{2}m(m+1) + 1,$$

*then $\mathcal{F}_x(\psi)$ must contain an operator of positive order which is bisymmetric with respect to $(\Gamma_1, \Gamma_2)$.*

*Proof.* Without loss of generality, assume that $p_{i0}$ is a finite point of $\Gamma_i$. Suppose $\mathfrak{d} \in \mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\psi)$ with $b_\psi(\mathfrak{d}) = \mathfrak{b}$. By Lemma 4.5,

$$\mathfrak{d} = \sum_{j=0}^{\ell} \partial_x^j a_j(x)\partial_x^j \quad \text{and} \quad \mathfrak{b} = \sum_{j=0}^{m} \partial_y^j b_j(y)\partial_y^j$$

for some functions $a_j(x), b_j(y)$ which are holomorphic in neighborhoods of $p_{i0}$, for $i = 1, 2$, respectively. The linear map $\mathcal{F}^{\ell,m}_{x,\mathrm{sym}}(\psi) \to \mathbb{C}^{\frac{1}{2}\ell(\ell+1)} \oplus \mathbb{C}^{\frac{1}{2}m(m+1)}$ defined by

$$\mathfrak{d} \mapsto (a_j^{(r)}(p_{10})_{0\le r<j\le \ell}, b_k^{(s)}(p_{20})_{0\le s<k\le m})$$

has a kernel of dimension at least

$$\dim(\mathcal{F}^{\ell,m}_{x,\mathrm{sym}}(\psi)) - \frac{\ell(\ell+1)}{2} - \frac{m(m+1)}{2}.$$

Moreover, if condition (i) or condition (ii) is satisfied, then the elements in the kernel are bisymmetric with respect to $(\Gamma_1, \Gamma_2)$ by Lemma 4.6. □

## 5. Differential operators commuting with integral operators

In this section, we prove the main theorems in the paper that self-adjoint bispectral meromorphic functions on ranks 1 and 2 give rise to integral operators possessing a commuting differential operator. The kernel of the integral operator corresponding to such a bispectral function $\widetilde{\psi}(x, y)$ is given by

$$\widetilde{K}(x, y) = \int_{\Gamma_2} \widetilde{\psi}(x, z)\widetilde{\psi}(y, z)\, dz \tag{5.1}$$

for an appropriate smooth curve $\Gamma_2 \subseteq \mathbb{C}$. The commuting differential operator will come from $\mathcal{F}_{x,\mathrm{sym}}(\widetilde{\psi})$. In order for the value of $\widetilde{K}(x, y)$ to exist and to define a kernel for an integral operator with the desired domain, we have to make certain assumptions about $\widetilde{\psi}(x, y)$. We specifically assume that $\widetilde{\psi}(x, y)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and for all $j, k, m, n \ge 0$,

$$\begin{aligned} &\int_{\Gamma_1} |x^m y^n \partial_x^j \partial_y^k \cdot \widetilde{\psi}(x, y)|\, dx \in L^\infty(\Gamma_2), \\ &\int_{\Gamma_2} |x^m y^n \partial_x^j \partial_y^k \cdot \widetilde{\psi}(x, y)|\, dy \in L^\infty(\Gamma_1). \end{aligned} \tag{5.2}$$

Note that the inclusion of the $x^m y^n$ multiplier is vacuous unless $\Gamma_1$ or $\Gamma_2$ has an endpoint at infinity, in which case the condition imposes a lower bound on the rate of decay of the partial derivatives of $\widetilde{\psi}(x, y)$. Under the above assumptions, the integral formula for $\widetilde{K}(x, y)$ exists and satisfies the assumptions required for differentiation under the integral for operators of arbitrary order. Therefore $\widetilde{K}(x, y)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_1$ and

$$\partial_x^j \partial_y^k \cdot \widetilde{K}(x, y) = \int_{\Gamma_2} (\partial_x^j \cdot \widetilde{\psi}(x, z))(\partial_y^k \cdot \widetilde{\psi}(y, z))\, dz.$$

Furthermore, we have the norm estimate

$$\|x^m y^n \partial_x^j \partial_y^k \cdot \widetilde{K}(x, y)\|_{1,\Gamma_1} \le \left\| \int |x^m y^n \partial_x^j \cdot \widetilde{\psi}(x, y)|\, dy \right\|_{\infty,\Gamma_1} \left\| \int |\partial_x^k \cdot \widetilde{\psi}(x, y)|\, dy \right\|_{\infty,\Gamma_1},$$

where the $L^1$-norm on the left is taken with respect to either $x$ or $y$. In particular, (5.2) also holds with $\widetilde{\psi}(x, y)$ replaced with $\widetilde{K}(x, y)$ and $\Gamma_2$ replaced with $\Gamma_1$, and so differentiation under the integral may also be applied to the integral operator defined with kernel $\widetilde{K}(x, y)$.

**5.1. Relationship to bisymmetric operators.** The next theorem establishes that every bisymmetric differential operator in the Fourier algebra of a bispectral function $\psi(x, y)$ automatically commutes with the integral operator with kernel (5.1).

**Theorem 5.1.** *Suppose that $\widetilde{\psi}(x, y)$ is a bispectral meromorphic function and that $\Gamma_1, \Gamma_2 \subseteq \mathbb{C}$ are two smooth curves such that $\widetilde{\psi}$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and satisfies* (5.2)*. If $\mathfrak{b} \in \mathcal{F}_{x,sym}(\widetilde{\psi})$ is a bisymmetric differential operator with respect to $(\Gamma_1, \Gamma_2)$ whose coefficients are holomorphic in a neighborhood of $\Gamma_1$, then it commutes with the integral operator*

$$T : f(x) \mapsto \int_{\Gamma_1} \widetilde{K}(x, y) f(y)\, dy \quad \text{with kernel} \quad \widetilde{K}(x, y) = \int_{\Gamma_2} \widetilde{\psi}(x, z)\widetilde{\psi}(y, z)\, dz.$$

*Proof.* Denote $\mathfrak{b} = b_{\widetilde{\psi}}(\mathfrak{b})$. Differentiation under the integral along with the fact that $\mathfrak{b}$ is symmetric with respect to $\Gamma_2$ implies

$$\begin{aligned}
\mathfrak{b}_x \cdot \widetilde{K}(x, y) &= \int_{\Gamma_2} (\mathfrak{b}_x \cdot \widetilde{\psi}(x, z))\widetilde{\psi}(y, z)\, dz \\
&= \int_{\Gamma_2} (\mathfrak{b}_z \cdot \widetilde{\psi}(x, z))\widetilde{\psi}(y, z)\, dz \\
&= \int_{\Gamma_2} \widetilde{\psi}(x, z)(\mathfrak{b}_z \cdot \widetilde{\psi}(y, z))\, dz \\
&= \int_{\Gamma_2} \widetilde{\psi}(x, z)(\mathfrak{b}_y \cdot \widetilde{\psi}(y, z))\, dz = \mathfrak{b}_y \cdot \widetilde{K}(x, y).
\end{aligned}$$

Moreover, for any $f \in C_c^\infty(\mathbb{C})$, Fubini's theorem, differentiation under the integral and the fact that $\mathfrak{b}$ is symmetric with respect to $\Gamma_1$ imply

$$\begin{aligned}
\mathfrak{b} \cdot T(f) &= \int_{\Gamma_1} (\mathfrak{b}_x \cdot \widetilde{K}(x, y)) f(y)\, dy \\
&= \int_{\Gamma_1} (\mathfrak{b}_y \cdot \widetilde{K}(x, y)) f(y)\, dy \\
&= \int_{\Gamma_1} \widetilde{K}(x, y)((\mathfrak{b}_y \cdot f(y))\, dy = T(\mathfrak{b} \cdot f).
\end{aligned}$$

It follows that $\mathfrak{b}$ and $T$ commute. □

**5.2. The rank 1 case.**

**Lemma 5.2.** *The exponential bispectral function $\psi_{\exp}(x, y) = e^{xy}$ satisfies*

$$\mathcal{F}_{x,sym}^{2\ell,2m}(\psi_{\exp}) = \left\{ \sum_{j=0}^{\ell} \partial_x^j a_j(x^2)\partial_x^j : \deg a_j(x^2) \le 2m \right\}.$$

*In particular,*

$$\dim \mathcal{F}_{x,sym}^{2\ell,2m}(\psi_{\exp}) = \ell m + \ell + m + 1.$$

*Proof.* From Example 2.6 (1) we have $\mathcal{F}_x(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[x])$ and $\mathcal{F}_y(\psi_{\exp}) = \mathfrak{D}(\mathbb{C}[y])$, so that both $\mathcal{F}_x(\psi_{\exp})$ and $\mathcal{F}_y(\psi_{\exp})$ are closed under $*$. Recall that $\mathcal{F}_x(\psi_{\exp})$ is closed under

the adjoint $*$ so that the automorphism $\iota = \iota_{\psi_{\exp},x}$ defined by (3.11) is well defined. One may readily check that the automorphism $\iota$ agrees with the endomorphism $\sigma$ of the Weyl algebra induced by the affine transformation $x \mapsto -x$, i.e.,

$$\sigma : \sum_{j=0}^{\ell} a_j(x)\partial_x^j \mapsto \sum_{j=0}^{\ell} a_j(-x)(-1)^j \partial_x^j.$$

Therefore the formally bisymmetric operators in $\mathcal{F}_{x,\mathrm{sym}}(\psi_{\exp})$ are exactly those fixed by $\sigma$, which are those operators of the form

$$\mathfrak{d} = \sum_{j=0}^{\ell} \partial_x^j a_j(x^2)\partial_x^j$$

for some integer $\ell \geq 0$ and some functions $a_j(x^2) \in \mathbb{C}[x]$. The corresponding element of $\mathcal{F}_{y,\mathrm{sym}}(\psi_{\exp})$ is

$$b_{\psi_{\exp}}(\mathfrak{d}) = \sum_{j=0}^{\ell} y^j a_j(\partial_y^2) y^j.$$

From this the statement of the lemma follows immediately. □

**Theorem 5.3.** *Let $\widetilde{\psi}(x, y)$ be a self-adjoint bispectral meromorphic function of rank* 1*, and let $\Gamma_1$ and $\Gamma_2$ be two finite, smooth curves in $\mathbb{C}$ whose endpoints are $\pm p_1$ and $\pm p_2$, respectively. Assume moreover that the coefficients of the operators in $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_y(\widetilde{\psi})$ are holomorphic in a neighborhood of $\Gamma_1$ and $\Gamma_2$, respectively, and that $\widetilde{\psi}(x, y)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and satisfies* (5.2)*. Then there exists a differential operator $\mathfrak{d}$ of positive order commuting with the integral operator*

$$T : f(x) \mapsto \int_{\Gamma_1} \widetilde{K}(x, y) f(y)\, dy \quad \textit{with kernel} \quad \widetilde{K}(x, y) = \int_{\Gamma_2} \widetilde{\psi}(x, z)\widetilde{\psi}(y, z)\, dy.$$

*Moreover, the operator $\mathfrak{d}$ may be taken to be in $\mathcal{F}_{x,sym}^{2d_1d_2,2d_1d_2}(\widetilde{\psi})$, for $(d_1, d_2)$ the bidegree of the self-adjoint bispectral Darboux transformation from $\psi_{\exp}(x, y) = e^{xy}$ to $\widetilde{\psi}(x, y)$.*

**Remark 5.4.** The rank one bispectral functions $\widetilde{\psi}(x, y)$ are all of the form

$$e^{xy} \frac{h(x, y)}{p(x)q(y)}$$

for some polynomials $p(x), q(y), h(x, y)$. The poles of the coefficients of operators in $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_y(\widetilde{\psi})$ occur at the zeros of $p(x)$ and $q(y)$, respectively. In particular, as long as $\Gamma_1$ avoids the zeros of $p(x)$ and $\Gamma_2$ avoids the zeros of $q(y)$, and the endpoints of $\Gamma_1$ and $\Gamma_2$ satisfy the desired symmetry, the assumptions of the above theorem will be satisfied.

*Proof.* Suppose that $\widetilde{\psi}(x, y)$ is a self-adjoint bispectral meromorphic function of rank 1. Then by Lemma 5.2 and Corollary 4.2, we have

$$\dim \mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\widetilde{\psi}) \geq \ell m + \ell + m + 2 - d_1d_2.$$

Thus for $\ell = m$, we have

$$\dim \mathcal{F}_{x,\mathrm{sym}}^{2m,2m}(\widetilde{\psi}) \geq m^2 + 2m + 2 - d_1d_2 > m^2 + m + 1$$

for $m \geq d_1 d_2$. Proposition 4.7 combined with Corollary 3.26 implies that $\mathcal{F}_{x,\mathrm{sym}}^{2d_1d_2,2d_1d_2}(\widetilde{\psi})$ contains a differential operator bisymmetric with respect to $(\Gamma_1, \Gamma_2)$. This differential operator commutes with the integral operator $T$ by Theorem 5.1. □

**5.3. The rank 2 Airy case.** We next deal with the bispectral Darboux transformations in the rank 2 Airy case. The relevant integral operator differs in this case from the integral operators in the rank 1 case and in the rank 2 Bessel case (discussed below) in that the kernel is not compactly supported. For the resultant kernel to satisfy (5.2), the support must be contained in a certain subdomain of the complex plane. For this reason, for all $\epsilon > 0$ we consider the domain

$$\Sigma_\epsilon = \left\{ re^{i\theta} \in \mathbb{C} : r > 0,\ |\theta| < \frac{\pi}{6} - \epsilon \right\}.$$

The Airy function is holomorphic on this domain and has the asymptotic expansion

$$\psi_{\mathrm{Ai}}(x+y) = e^{-\frac{2}{3}(x+y)^{\frac{3}{2}}} \left( \sum_{j=1}^{\infty} c_j (x+y)^{-\frac{j}{4}} \right)$$

for some real constants $c_j \in \mathbb{R}$ where $(x+y)^{\frac{1}{4}}$ is interpreted as the principal fourth root of $(x+y)$. Furthermore, any bispectral Darboux transformation of $\psi_{\mathrm{Ai}}(x+y)$ will be equal to

$$\widetilde{\psi}(x,y) = \frac{1}{p(x)q(y)} \mathfrak{u} \cdot \psi_{\mathrm{Ai}}(x+y)$$

for some rational functions $p(x), q(y)$ and some differential operator $\mathfrak{u}$ with rational coefficients. Thus for any bispectral Darboux transformation of $\psi_{\mathrm{Ai}}(x+y)$ we have the asymptotic estimate

$$\|\partial_x^j \partial_y^k \cdot \psi_{\mathrm{Ai}}(x+y)\| = e^{-\frac{2}{3}(x+y)^{\frac{3}{2}}} \mathcal{O}((|x|+|y|)^{\frac{1}{2}(j+k)+m})$$

for some integer $m$.

Note that $z \mapsto \frac{2}{3} z^{\frac{3}{2}}$ sends $\Gamma$ into the sector $\{re^{i\theta} \in \mathbb{C} : r > 0,\ |\theta| < \frac{\pi}{4} - \frac{3}{2}\epsilon\}$. Therefore if $\Gamma_1, \Gamma_2 \subseteq \Sigma_\epsilon$ are smooth, semi-infinite curves inside this domain with parametrizations $\gamma_i(t) : [0,\infty) \to \mathbb{C}$, then the real part of $-\frac{2}{3}(\gamma_1(t) + \gamma_2(s))^{\frac{3}{2}}$ must go to $-\infty$ as $t \to \infty$ or $s \to \infty$. Therefore the above asymptotic estimate shows that $\widetilde{\psi}(x,y)$ will satisfy (5.2) for any pair of curves $\Gamma_1, \Gamma_2 \subseteq \Sigma_\epsilon$.

**Lemma 5.5.** *The Airy bispectral function* $\psi_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x+y)$ *satisfies*

$$\dim \mathcal{F}_{x,sym}^{2\ell,2m}(\psi_{\mathrm{Ai}}) = \ell m + \ell + m + 1.$$

*Proof.* Recall from Example 2.6 (3) that

$$\mathcal{F}_x(\psi_{\mathrm{Ai}}) = \mathfrak{D}(\mathbb{C}[x]) \quad \text{and} \quad \mathcal{F}_y(\psi_{\mathrm{Ai}}) = \mathfrak{D}(\mathbb{C}[y]).$$

In particular, $\mathcal{F}_x(\psi_{\mathrm{Ai}})$ is closed under $*$. From the description of $b_{\psi_{\mathrm{Ai}}}$ in Example 2.6 (3) one easily sees that $b_{\psi_{\mathrm{Ai}}}$ and $*$ commute. Therefore $\mathfrak{d} \in \mathcal{F}_x(\psi_{\mathrm{Ai}})$ is formally bisymmetric if and only if $\mathfrak{d}^* = \mathfrak{d}$.

Recall that the space $\mathcal{B}_x(\psi_{\mathrm{Ai}})$ contains all polynomials in $x$ as well as the Airy operator $\mathfrak{d}_{\mathrm{Ai},x} = \partial_x^2 - x$. Moreover, for $\mathfrak{d} \in \mathcal{F}_x^{\ell,m}(\psi_{\mathrm{Ai}})$ the Ad-condition $\mathrm{ad}_{\mathfrak{d}_{\mathrm{Ai},x}}^{m+1}(\mathfrak{d}) = 0$ implies that the leading coefficient of $\mathfrak{d}$ must be a polynomial. For each $j,k \geq 0$ consider the formally symmetric differential operator in $\mathcal{F}_x(\psi_{\mathrm{Ai}})$ defined by

$$\mathfrak{a}_{jkx} = \mathfrak{d}_{\mathrm{Ai},x}^j x^k + x^k \mathfrak{d}_{\mathrm{Ai},x}^j,$$

which in particular has order $2j$ and leading coefficient $2x^k$. The anti-isomorphism $b_{\psi_{\mathrm{Ai}}}$ sends $\mathfrak{a}_{jkx}$ to $\mathfrak{a}_{jky}$, so $\mathfrak{a}_{jkx} \in \mathcal{F}_{x,\mathrm{sym}}(\psi_{\mathrm{Ai}})$.

If $\mathfrak{d}$ is self-adjoint, it must have even order with polynomial leading coefficient. Thus by comparing leading coefficients, we see that $\mathcal{F}_{x,\mathrm{sym}}(\psi_{\mathrm{Ai}})$ has basis $\{\mathfrak{a}_{jkx} : j,k \geq 0\}$. Therefore comparing orders we see that $\mathcal{F}^{2\ell,2m}_{x,\mathrm{sym}}(\psi_{\mathrm{Ai}})$ has basis $\{\mathfrak{a}_{jix} : 0 \leq j \leq l,\ 0 \leq k \leq m\}$. In particular, it has dimension $(\ell+1)(m+1)$. □

**Theorem 5.6.** *Let $\widetilde{\psi}(x,y)$ be a self-adjoint bispectral Darboux transformation of the Airy bispectral function $\psi_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x+y)$, and let $\Gamma_1$ and $\Gamma_2$ be two semi-infinite, smooth curves in $\Sigma_\epsilon$ for some $\epsilon > 0$ whose finite endpoints are $p_1$ and $p_2$, respectively. Assume moreover that $\widetilde{\psi}(x,y)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and satisfies* (5.2) *and that the operators $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_y(\widetilde{\psi})$ have holomorphic coefficients in a neighborhood of $\Gamma_1$ and $\Gamma_2$, respectively. Then there exists a differential operator $\mathfrak{d}$ of positive order commuting with the integral operator*

$$T : f(x) \mapsto \int_{\Gamma_1} \widetilde{K}(x,y) f(y)\, dy \quad \textit{with kernel} \quad \widetilde{K}(x,y) = \int_{\Gamma_2} \widetilde{\psi}(x,z)\widetilde{\psi}(y,z)\, dy.$$

*Moreover, the operator $\mathfrak{d}$ may be taken to be in $\mathcal{F}^{2d_1d_2,2d_1d_2}_{x,sym}(\widetilde{\psi})$, for $(d_1,d_2)$ the bidegree the self-adjoint bispectral Darboux transformation from $\psi_{\mathrm{Ai}}(x,y)$ to $\widetilde{\psi}(x,y)$.*

**Remark 5.7.** The bispectral Darboux transformations of the Airy bispectral function will be holomorphic away from the roots of a polynomial $p(x)$ and a polynomial $q(y)$, which are the polynomials entering in the definition of the concrete bispectral Darboux transformation as in (3.2). Furthermore, the poles of the coefficients of the operators in $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_y(\widetilde{\psi})$ occur at the roots of these polynomials also. Therefore the assumptions of the theorem will be automatically satisfied as long as $\Gamma_1$ and $\Gamma_2$ are both semi-infinite paths in $\Sigma_\epsilon$ which avoid the zero sets of $p(x)$ and $q(y)$.

*Proof.* By Lemma 5.5 and Corollary 4.2, we have

$$\dim \mathcal{F}^{2\ell,2m}_{x,\mathrm{sym}}(\widetilde{\psi}) \geq \ell m + \ell + m + 2 - d_1d_2$$

for some constant $c$. Thus for $\ell = m$, we have

$$\dim \mathcal{F}^{2m,2m}_{x,\mathrm{sym}}(\widetilde{\psi}) \geq m^2 + 2m + 2 - d_1d_2 > m^2 + m + 1$$

for $m \geq d_1d_2$. By Proposition 4.7, $\mathcal{F}^{2d_1d_2,2d_1d_2}_{x,\mathrm{sym}}(\widetilde{\psi})$ contains a differential operator which is bisymmetric with respect to $(\Gamma_1,\Gamma_2)$. It follows from Theorem 5.1 that this differential operator commutes with the integral operator $T$. □

### 5.4. The rank 2 Bessel case.

**Lemma 5.8.** *Let $\nu \in \mathbb{R} \setminus \mathbb{Z}$. The Bessel bispectral function*

$$\psi_{\mathrm{Be}(\nu)}(x,y) = \sqrt{xy}K_{\nu+\frac{1}{2}}(xy)$$

*satisfies*

$$\dim \mathcal{F}^{2\ell,2m}_{x,sym}(\psi_{\mathrm{Be}(\nu)}) = \ell m + \ell + m + 1.$$

*Proof.* It follows from Example 2.6 (2) that the algebra $\mathcal{F}_x(\psi_{\mathrm{Be}(\nu)})$ is generated over $\mathbb{C}$ by the Bessel operator $\mathfrak{b}_{\mathrm{Be}(\nu)x} = \partial_x^2 - \frac{\nu(\nu+1)}{x^2}$ the operator $\mathfrak{s}_x = x\partial_x$ and $x^2$. In particular, all of the elements of $\mathcal{F}_x(\psi_{\mathrm{Be}(\nu)})$ are invariant under the change of coordinates $x \mapsto -x$. Furthermore, for all $j, k \geq 0$,

$$\mathfrak{a}_{jk} = \mathfrak{s}_x^k \mathfrak{b}_{\mathrm{Be}(\nu)x}^j (\mathfrak{s}_x^*)^k \quad \text{and} \quad \mathfrak{b}_{jk} = \mathfrak{s}_x^k x^{2j} (\mathfrak{s}_x^*)^k$$

are formally symmetric differential operators with

$$\mathrm{ord}(\mathfrak{a}_{jk}) = 2j + 2k, \quad \mathrm{ord}(\mathfrak{b}_{jk}) = 2k, \quad \mathrm{cord}(\mathfrak{a}_{jk}) = 2k, \quad \mathrm{cord}(\mathfrak{b}_{jk}) = 2j + 2k.$$

In particular, $\mathrm{ord}(\mathfrak{a}_{jk}) \geq \mathrm{cord}(\mathfrak{a}_{jk})$ and $\mathrm{ord}(\mathfrak{b}_{jk}) \leq \mathrm{cord}(\mathfrak{b}_{jk})$ with equality if and only if $j$ is zero. Thus the set

$$\{\mathfrak{a}_{jk} : j + k \leq \ell, k \leq m\} \cup \{\mathfrak{b}_{jk} : j + k \leq m, k \leq \ell,\ j \neq 0\}$$

is a linearly independent collection of $(\ell + 1)(m + 1)$ elements of $\mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\psi_{\mathrm{Be}(\nu)})$.

If $\mathfrak{d} \in \mathcal{F}_{x,\mathrm{sym}}(\psi_{\mathrm{Be}(\nu)}^{2\ell,2m})$ is an arbitrary operator, then the Ad-condition

$$\mathrm{ad}^{2m+1}_{\mathfrak{b}_{\mathrm{Be}(\nu),x}}(\mathfrak{d}) = 0$$

implies that the leading coefficient of $\mathfrak{d}$ must be a polynomial. Furthermore, since $\mathfrak{d}$ is formally symmetric it must have even order. Finally, since $\mathfrak{d}$ must be invariant under the transformation $x \mapsto -x$, the leading coefficient of $\mathfrak{d}$ must be a polynomial in $x^2$. Therefore by comparing leading coefficients, we see that $\mathfrak{d}$ must lie in the span of the $\mathfrak{a}_{jk}$ and $\mathfrak{b}_{jk}$. Thus the set $\{\mathfrak{a}_{jk}, \mathfrak{b}_{jk} : j, k \geq 0\}$ forms a basis for $\mathcal{F}_{x,\mathrm{sym}}(\psi_{\mathrm{Be}(\nu)})$ and by noting the orders and co-orders, we see that the linearly independent collection noted in the previous paragraph is actually a basis for $\mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\psi_{\mathrm{Be}(\nu)})$. □

**Theorem 5.9.** *Let $\widetilde{\psi}(x, y)$ be a self-adjoint bispectral Darboux transformation of the bispectral Bessel function $\psi_{\mathrm{Be}(\nu)}$, and let $\Gamma_1$ and $\Gamma_2$ be two finite, smooth curves in $\mathbb{C}$ whose endpoints are $\pm p_1$ and $\pm p_2$, respectively. Assume moreover that the coefficients of the operators in $\mathcal{F}_x(\widetilde{\psi})$ and $\mathcal{F}_y(\widetilde{\psi})$ are holomorphic in a neighborhood of $\Gamma_1$ and $\Gamma_2$, respectively, and that $\widetilde{\psi}(x, y)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and satisfies* (5.2)*. Then there exists a differential operator $\mathfrak{d}$ of positive order commuting with the integral operator*

$$T : f(x) \mapsto \int_{\Gamma_1} \widetilde{K}(x, y) f(y)\, dy \quad \textit{with kernel} \quad \widetilde{K}(x, y) = \int_{\Gamma_2} \widetilde{\psi}(x, z)\widetilde{\psi}(y, z)\, dy.$$

*Moreover, the operator $\mathfrak{d}$ may be taken to be in $\mathcal{F}_{x,sym}^{2d_1d_2,2d_1d_2}(\widetilde{\psi})$, for $(d_1, d_2)$ the bidegree the self-adjoint bispectral Darboux transformation from $\psi_{\mathrm{Be}(\nu)}$ to $\widetilde{\psi}(x, y)$.*

**Remark 5.10.** As in the rank 1 and the Airy case, the assumptions of Theorem 5.9 will be automatically satisfied as long as the end points of $\Gamma_1$ and $\Gamma_2$ satisfy the symmetry condition stated above and the curves avoid the branching point 0 of the Bessel functions and the roots of the polynomials $p(x)$ and $q(y)$, entering in the definition of the concrete bispectral Darboux transformation as in (3.2).

*Proof.* By Lemma 5.8 and Corollary 4.2, we have

$$\dim \mathcal{F}_{x,\mathrm{sym}}^{2\ell,2m}(\widetilde{\psi}) \geq \ell m + \ell + m + 2 - d_1d_2$$

for some constant $c$. Thus for $\ell = m$, we have

$$\dim \mathcal{F}^{2m,2m}_{x,\mathrm{sym}}(\widetilde{\psi}) \geq m^2 + 2m + 2 - d_1 d_2 > m^2 + m + 1$$

for $m \geq d_1 d_2$. Proposition 4.7 and Corollary 3.26 imply that $\mathcal{F}^{2d_1 d_2, 2d_1 d_2}_{x,\mathrm{sym}}(\widetilde{\psi})$ contains a differential operator bisymmetric with respect to $(\Gamma_1, \Gamma_2)$. This differential operator commutes with the integral operator specified in the statement of the proposition by Theorem 5.1. □

## 6. Classification of self-adjoint bispectral meromorphic functions

In this section we describe a classification of the self-adjoint bispectral meromorphic functions that appear in Theorems 5.3, 5.6, and 5.9. This classification is given in terms of (infinite-dimensional) lagrangian versions of Wilson's adelic Grassmannian [41]. We use the construction of the latter in terms of bispectral Darboux transformations as in [5].

**6.1. The adelic Grassmannian.** By Theorem 3.8, the class of rank 1 bispectral meromorphic functions is precisely the class of bispectral Darboux transformations of the exponential function $\psi_{\exp}(x, y) = e^{xy}$, normalized as in Remark 3.2. A function $\widetilde{\psi}(x, y)$ in this class has the form

$$\widetilde{\psi}(x, y) = \frac{h(x, y)}{p(x)q(y)} e^{xy} \tag{6.1}$$

for some polynomials $p(x) \in \mathbb{C}[x]$, $q(y) \in \mathbb{C}[y]$ and a polynomial $h(x, y) \in \mathbb{C}[x, y]$. (Note however that not all functions of the form (6.1) are bispectral Darboux transformations of $\psi_{\exp}(x, y) = e^{xy}$; the polynomials $p(x), q(y), h(x, y)$ need to satisfy some conditions.) We can recover the bispectral transformation data from this form. Writing

$$h(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j,$$

we define the operators

$$\mathfrak{v}_L = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i \partial_x^j \in \mathcal{F}_{x,\mathrm{sym}}(\psi_{\exp}) \quad \text{and} \quad \mathfrak{v}_R = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} y^j \partial_y^i \in \mathcal{F}_{y,\mathrm{sym}}(\psi_{\exp}).$$

Then $\mathfrak{v}_R$ is the Fourier transform $b_{\psi_{\exp}}$ of $\mathfrak{v}_L$ and by virtue of their definition,

$$\widetilde{\psi}(x, y) = \frac{1}{p(x)q(y)} \mathfrak{v}_L \cdot \psi_{\exp}(x, y) \quad \text{and} \quad \widetilde{\psi}(x, y) = \frac{1}{p(x)q(y)} \mathfrak{v}_R \cdot \psi_{\exp}(x, y).$$

Let $\mathscr{C}$ be the span of all linear functionals of the form $\delta^{(k)}(y - a)$ for $k \in \mathbb{N}$ and $a \in \mathbb{C}$, defined on smooth functions by

$$\langle f(y), \delta^{(k)}(y - a) \rangle = f^{(k)}(a).$$

Note that $\mathscr{C}$ comes with a natural left action of $\mathbb{C}[y]$, defined by

$$f(y) \cdot \delta^{(k)}(y - a) = \sum_{j=0}^{k} \frac{k!}{(k - j)! j!} \delta^{(j)}(y - a) f^{(k-j)}(a).$$

For a given $\widetilde{\psi}(x, y)$, we define the set of linear functionals

$$\mathscr{C}_L(\widetilde{\psi}) = \{\chi \in \mathscr{C} : \langle e^{xy} h(x, y), \chi(y)\rangle = 0\}.$$

The vector space $\mathscr{C}_L(\widetilde{\psi})$ is finite-dimensional and naturally isomorphic to the kernel of the differential operator $\mathfrak{v}_L$. Specifically, for any $\chi \in \mathscr{C}_L(\widetilde{\psi})$ we have

$$0 = \langle e^{xy} h(x, y), \chi(y)\rangle = \langle \mathfrak{v}_L \cdot e^{xy}, \chi(y)\rangle = \mathfrak{v}_L \cdot \langle e^{xy}, \chi(y)\rangle,$$

so that $\chi \mapsto \langle e^{xy}, \chi(y)\rangle$ defines a linear map of $\mathscr{C}_L(\widetilde{\psi})$ into $\ker(\mathfrak{v}_L)$. With a reverse argument one shows that this is an isomorphism.

Conjugation by $\frac{1}{p(x)}\mathfrak{v}_L$ sends $\mathcal{B}_x(\widetilde{\psi})$ into $\mathbb{C}[\partial_x]$. In fact,

$$\mathcal{B}_x(\widetilde{\psi}) = \left\{ \frac{\mathfrak{v}_L f(\partial_x) \mathfrak{v}_L^{-1} p(x)}{p(x)} : f(\partial_x) \in \mathbb{C}[\partial_x], \text{ and } \mathfrak{v}_L f(\partial_x) \mathfrak{v}_L^{-1} \text{ is a differential operator} \right\}.$$

From the study of kernels of differential operators, we know that $\mathfrak{v}_L f(\partial_x)\mathfrak{v}_L^{-1}$ is a differential operator if and only if $f(\partial_x) \cdot \ker(\mathfrak{v}_L) \subseteq \ker(\mathfrak{v}_L)$. Each element of the kernel of $\mathfrak{v}_L$ is of the form $\langle e^{xy}, \chi(y)\rangle$ for some $\chi \in \mathscr{C}_L(\widetilde{\psi})$. Therefore

$$f(\partial_x) \cdot \langle e^{xy}, \chi(y)\rangle = \langle e^{xy}, f(y) \cdot \chi(y)\rangle.$$

Thus, if $f(\partial_x)$ preserves the kernel, then

$$\langle e^{xy}, f(y) \cdot \chi(y)\rangle = \sum_{\lambda \in \mathscr{C}_L(\widetilde{\psi})} \langle e^{xy}, c_\lambda \lambda\rangle,$$

which in turn implies that

$$f(y) \cdot \chi(y) = \sum_{\lambda \in \mathscr{C}_L(\widetilde{\psi})} c_\lambda \lambda.$$

Thus,

$$\mathcal{B}_x(\widetilde{\psi}) = \left\{ \frac{1}{p(x)} \mathfrak{v}_L f(\partial_x) \mathfrak{v}_L^{-1} p(x) : f(\partial_x) \in \mathbb{C}[\partial_x],\ f(y) \cdot \mathscr{C}_L(\widetilde{\psi}) \subseteq \mathscr{C}_L(\widetilde{\psi}) \right\}.$$

For a finite-dimensional subspace $C$ of $\mathscr{C}$, define

$$V_C = \{f(y) \in \mathbb{C}[y] : \langle f(y), \chi(y)\rangle = 0,\ \forall \chi \in C\}.$$

By a direct argument, one shows that $f(y) \cdot C \subseteq C$ if and only if $f(y)V_C \subseteq V_C$. This gives the following characterization of $\mathcal{B}_x(\widetilde{\psi})$:

$$\mathcal{B}_x(\widetilde{\psi}) = \left\{ \frac{1}{p(x)} \mathfrak{v}_L f(\partial_x) \mathfrak{v}_L^{-1} p(x) : f(\partial_x) \in \mathbb{C}[\partial_x],\ f(y) V_{\mathscr{C}_L(\widetilde{\psi})} \subseteq V_{\mathscr{C}_L(\widetilde{\psi})} \right\}.$$

It motivates the following definition of the rational Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$ of Wilson [41].

**Definition 6.1.** We define the *rational Grassmannian* $\mathrm{Gr}^{\mathrm{rat}}$ to be the set of all subspaces $W$ of $\mathbb{C}(y)$ of the form

$$W = \frac{1}{q(y)} V_C$$

for some $C \subseteq \mathscr{C}$ and $q(y) \in \mathbb{C}[y]$ with $\dim(C) = \deg(q(y))$. The subspace $C \subseteq \mathscr{C}$ is called the *space of conditions* of $W$.

To each point $W \in \mathrm{Gr}^{\mathrm{rat}}$, we can associate an algebra

$$A_W = \{f(y) \in \mathbb{C}[y] : f(y)W \subseteq W\}.$$

The pair $(W, A_W)$ is called a Schur pair. From the construction above, we have shown that every bispectral meromorphic function $\widetilde{\psi}(x, y)$ of rank 1 corresponds to a point $W$ in $\mathrm{Gr}^{\mathrm{rat}}$ whose associated algebra $A_W$ is isomorphic to $\mathcal{B}_x(\widetilde{\psi})$.

Not every point of $\mathrm{Gr}^{\mathrm{rat}}$ corresponds to a bispectral meromorphic function. In order for this to be true, the space of conditions $C$ of $W$ must be homogeneous.

**Definition 6.2.** For any point $c \in \mathbb{C}$, let $\mathscr{C}_c$ denote the subspace of $\mathscr{C}$ spanned by linear functionals of the form $\delta^{(k)}(x - c)$ for $k \geq 0$ an integer. Put another way, $\mathscr{C}_c$ is the subspace of linear functionals of $\mathscr{C}$ supported at the point $c$. A linear functional $\chi \in \mathscr{C}$ is called homogeneous if $\chi \in \mathscr{C}_c$ for some value $c \in \mathbb{C}$. A finite-dimensional subspace $C$ of $\mathscr{C}$ is called *homogeneous* if it is spanned by homogeneous elements. Equivalently, $C$ is homogeneous if

$$C = \bigoplus_{c \in \mathbb{C}} C \cap \mathscr{C}_c.$$

Wilson showed that bispectral meromorphic functions give rise to points $W \in \mathrm{Gr}^{\mathrm{rat}}$ with $\mathscr{C}_L(\widetilde{\psi})$ homogeneous. He also proved that the converse is true: a point

$$W = \frac{1}{q(y)} V_C$$

in $\mathrm{Gr}^{\mathrm{rat}}$ with $C$ homogeneous gives rise to a bispectral meromorphic function $\widetilde{\psi}(x, y)$ with $\mathscr{C}_L(\widetilde{\psi}) = C$.

**Definition 6.3.** We define the *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ to be the subset of $\mathrm{Gr}^{\mathrm{rat}}$ consisting of the subspaces $W$ of $\mathbb{C}(y)$ of the form

$$W = \frac{1}{q(y)} V_C,$$

where $C \subseteq \mathscr{C}$ is homogeneous and

$$q(y) = \prod_{c \in \mathbb{C}} (y - c)^{n(c)}, \quad n(c) = \dim(C \cap \mathscr{C}_c). \tag{6.2}$$

The above correspondence between the points of $\mathrm{Gr}^{\mathrm{ad}}$ and the normalized (as in Remark 3.2) bispectral Darboux transformations $\widetilde{\psi}(x, y)$ of $\psi_{\exp}(x, y)$ is a bijection.

**6.2. A classification in rank 1 in terms of fixed points of involutions.** In [41] Wilson defined two involutions of the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$: the adjoint and sign involutions.

The *adjoint involution* on $\mathrm{Gr}^{\mathrm{ad}}$ sends a point $W \in \mathrm{Gr}^{\mathrm{ad}}$ to the point $aW \in \mathrm{Gr}^{\mathrm{ad}}$, given by

$$aW = \left\{ f(y) \in \mathbb{C}[y] : \frac{1}{2\pi i} \oint_{|y|=1} f(y)g(-y)\, dz = 0 \text{ for all } g(y) \in W \right\},$$

see [41, Section 7] for details. Letting $\widetilde{\psi}(x, y)$ and $a\widetilde{\psi}(x, y)$ be the associated bispectral meromorphic functions, the bispectral algebras of $\widetilde{\psi}$ and $a\widetilde{\psi}$ are related by the formal adjoint

$$\mathcal{B}_x(a\widetilde{\psi}) = \{\mathfrak{d}^* : \mathfrak{d} \in \mathcal{B}_x(\widetilde{\psi})\}, \quad \mathcal{B}_y(a\widetilde{\psi}) = \{\mathfrak{d}^* : \mathfrak{d} \in \mathcal{B}_y(\widetilde{\psi})\}, \tag{6.3}$$

see [41, Corollary 7.7]. If $\widetilde{\psi}(x, y)$ is given as a bispectral Darboux transformation of $\psi_{\exp}(x, y)$ as

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)p(x)}\mathfrak{u} \cdot \psi_{\exp}(x, y), \tag{6.4}$$

$$\psi_{\exp}(x, y) = \widetilde{\mathfrak{u}} \cdot \frac{1}{\widetilde{q}(y)\widetilde{p}(x)}\widetilde{\psi}(x, y),$$

then $a\widetilde{\psi}(x, y)$ is given by

$$a\widetilde{\psi}(x, y) = \frac{1}{\widetilde{q}(-y)\widetilde{p}(x)}\widetilde{\mathfrak{u}}^* \cdot \psi_{\exp}(x, y), \tag{6.5}$$

$$\psi_{\exp}(x, y) = \mathfrak{u}^* \frac{1}{q(-y)p(x)} \cdot a\widetilde{\psi}(x, y),$$

see [5, Proposition 1.7 (i)].

The *sign involution s* of $\mathrm{Gr}^{\mathrm{ad}}$, defined in [41, Section 8], is given by

$$W \mapsto sW := \{f(-y) : f(y) \in W\}.$$

On the level of bispectral functions, the sign involution is given by

$$s\widetilde{\psi}(x, y) := \widetilde{\psi}(-x, -y). \tag{6.6}$$

The associated bispectral algebras are related by

$$\mathcal{B}_x(s\widetilde{\psi}) = \sigma\mathcal{B}_x(\widetilde{\psi}), \tag{6.7}$$

where $\sigma$ is the automorphism of the algebra of differential operators with rational coefficients

$$\sigma(x) = -x, \quad \sigma(\partial_x) = -\partial_x. \tag{6.8}$$

**Theorem 6.4.** *Let $\widetilde{\psi}(x, y)$ be a bispectral meromorphic function of rank* 1. *Then the following are equivalent:*

(1) *$\widetilde{\psi}(x, y)$ is self-adjoint.*

(2) *The algebra $\mathcal{B}_x(\widetilde{\psi})$ is closed under the formal adjoint $*$ and the transformation $\sigma$.*

(3) *The plane $W \in \mathrm{Gr}^{\mathrm{ad}}$ corresponding to $\widetilde{\psi}(x, y)$ is invariant under the adjoint a and sign s involutions of* $\mathrm{Gr}^{\mathrm{ad}}$.

*Proof.* (1) $\Rightarrow$ (3) Let $\widetilde{\psi}(x, y)$ be a self-adjoint bispectral Darboux transformation of $\psi_{\exp}(x, y)$ with the notation of Definition 3.1 and the normalization of Remark 3.2. By (6.5),

$$a\widetilde{\psi}(x, y) = \frac{1}{\widetilde{q}(-y)\widetilde{p}(x)}\widetilde{\mathfrak{u}}^* \cdot \psi(x, y) = \frac{1}{q(y)\widetilde{p}(x)}\mathfrak{u} \cdot \psi(x, y) = \widetilde{\psi}(x, y).$$

Since

$$\iota_{\psi_{\exp},x} = \sigma|_{\mathcal{F}_x(\psi_{\exp})}$$

(see Section 3.3), it follows from (6.6) that

$$s\widetilde{\psi}(x, y) = \frac{1}{q(-y)p(-x)}(\sigma(\mathfrak{u})) \cdot \psi(x, y) = \frac{1}{q(y)p(x)}\mathfrak{u} \cdot \psi(x, y) = \widetilde{\psi}(x, y).$$

Here the equalities $q(-y) = q(y)$ and $p(-x) = p(x)$ follow from the facts that $q(y)$ and $p(x)$ are the leading coefficients of the differential operators $\mathfrak{u}$ and $b_{\psi_{\exp}}(\mathfrak{u})$ (see Remark 3.2) and the facts that these operators are fixed under $\iota_{\psi_{\exp},x}$ and $\iota_{\psi_{\exp},y}$, respectively.

(3) $\Rightarrow$ (1) If the plane $W \in \mathrm{Gr}^{\mathrm{ad}}$ corresponding to $\widetilde{\psi}(x, y)$ is invariant under the adjoint and sign involutions of $\mathrm{Gr}^{\mathrm{ad}}$, then it follows from (6.2) that $q(-y) = q(y)$. Applying Wilson's bispectral involution $b$ of $\mathrm{Gr}^{\mathrm{ad}}$ (cf. [41, Section 8]) to $\widetilde{\psi}$, and using that $bs = sb$ and $ab = bsa$ (cf. [41, Section 8]), gives that $p(-x) = p(x)$. The rest of this implication is proved with a reverse argument to the one used for the previous implication.

The equivalence (1) $\Leftrightarrow$ (3) follows from (6.3) and (6.7). □

**6.3. The Lagrangian adelic Grassmannian.** In this subsection we provide a geometric classification of the self-adjoint bispectralfunctions of rank 1. Specifically, we relate the points of $\mathrm{Gr}^{\mathrm{ad}}$ fixed by the adjoint $a$ and sign $s$ involutions to certain Lagrangian subspaces of kernels of formally symmetric differential operators with constant coefficients.

**Definition 6.5.** Let $\mathfrak{d}$ be a differential operator

$$\mathfrak{d} = \sum_{j=0}^{m} d_j(x)\partial_x^j.$$

The *bilinear concomitant* of $\mathfrak{d}$ is the bilinear form $\mathcal{C}_{\mathfrak{d}}(\cdot,\cdot; p)$ defined on pairs of sufficiently smooth functions $f(x), g(x)$ by

$$\begin{aligned}\mathcal{C}_{\mathfrak{d}}(f, g; p) &= \sum_{j=1}^{m}\sum_{k=0}^{j-1}(-1)^k f^{(j-1-k)}(x)(d_j(x)g(x))^{(k)}|_{x=p} \\ &= \sum_{j=1}^{m}\sum_{k=0}^{j-1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^k f^{(j-1-k)}(x)d_j(x)^{(k-\ell)}g(x)^{(\ell)}|_{x=p}.\end{aligned}$$

Equivalently, for $C_{\mathfrak{d}}(x)$ the $m \times m$ matrix whose $n, \ell$-th entry is given by

$$C_{\mathfrak{d}}(x)_{n,\ell} = \sum_{j=n+\ell-1}^{m}\binom{j-n}{\ell-1}(-1)^{j-n}d_j(x)^{(j+1-n-\ell)} \tag{6.9}$$

the bilinear concomitant may be expressed as

$$\mathcal{C}_{\mathfrak{d}}(f, g; p) = [f(x)\ f'(x)\ \ldots\ f^{(m-1)}(x)]C_{\mathfrak{d}}(x)[g(x)\ g'(x)\ \ldots\ g^{(m-1)}(x)]^T|_{x=p}.$$

Note also that via integration by parts, we find that

$$\int_{x_0}^{x_1}[(\mathfrak{d}\cdot f(x))g(x) - f(x)(\mathfrak{d}^*\cdot g(x))]\,dx = \mathcal{C}_{\mathfrak{d}}(f, g; x_1) - \mathcal{C}_{\mathfrak{d}}(f, g; x_0). \tag{6.10}$$

In this way the bilinear concomitant may be seen to act as a means of comparison between the formal adjoint $\mathfrak{d}^*$ of a differential operator $\mathfrak{d}$ and the adjoint of $\mathfrak{d}$ as an unbounded linear operator on a sufficiently nice space of functions on a path connecting $x_0$ to $x_1$.

First we prove a general classification result for the factorizations of a formally symmetric differential operator $\mathfrak{d}$ (i.e., $\mathfrak{d} = \mathfrak{d}^*$) with analytic coefficients into a product $\mathfrak{u}^*\mathfrak{u}$ for differential operators $\mathfrak{u}$ with analytic coefficients.

Start with a (not necessarily formally symmetric) differential operator $\mathfrak{d}$ with analytic coefficients on an open subset of $\mathbb{C}$. Choose a sufficiently small connected open subset $\mathcal{O}$ of $\mathbb{C}$ such that the kernels $\ker\mathfrak{d}$ and $\ker\mathfrak{d}^*$ in the space of holomorphic functions on $\mathcal{O}$ have dimension equal to the order of $\mathfrak{d}$. Note from (6.10) that if $f(x) \in \ker(\mathfrak{d})$ and $g(x) \in \ker(\mathfrak{d}^*)$, then

for all pairs of points $x_0, x_1$ we have $\mathcal{C}_{\mathfrak{b}}(f, g; x_1) = \mathcal{C}_{\mathfrak{b}}(f, g; x_0)$. Thus the bilinear concomitant $\mathcal{C}_{\mathfrak{b}}(\cdot, \cdot; p)$ restricts to a pairing

$$\mathcal{C}_{\mathfrak{b}}(\cdot, \cdot) : \ker(\mathfrak{b}) \times \ker(\mathfrak{b}^*) \to \mathbb{C}$$

which is independent of the value of $p$, where $m = \operatorname{ord} \mathfrak{b}$. Let $f_1(x), \ldots, f_m(x)$ be a basis for the kernel of $\mathfrak{b}$ and let $g_1(x), \ldots, g_m(z)$ be a basis for the kernel of $\mathfrak{b}^*$. Consider the matrix

$$M_{\mathfrak{b}} := \operatorname{Wr}[f_1(x), \ldots, f_m(x)]^T C_{\mathfrak{b}}(x) \operatorname{Wr}[g_1(x), \ldots, g_m(x)]$$

where $\operatorname{Wr}[\cdot]$ denotes the Wronskian matrix and $C_{\mathfrak{b}}(x)$ is the matrix defined in (6.9). Then $M_{\mathfrak{b}}$ is a constant matrix representing the pairing with respect to the chosen bases. If $M_{\mathfrak{b}}$ is singular, then there exists a constant vector $\vec{c}$ with $\vec{c}^T M_{\mathfrak{b}} = 0$. Note that since the Wronskian is nonsingular, the matrix $M_{\mathfrak{b}}$ is singular if and only if there exists an element $f(x) \in \ker(\mathfrak{b})$ with $\mathcal{C}_{\mathfrak{b}}(f, g) = 0$ for all $g$. Using the integral formula in (6.10), this would in fact imply $\int_{x_0}^{x_1} f(x)(\mathfrak{b}^* \cdot g(x))\, dx$ for all $g(x)$. That is, $f(x)$ would have to lie orthogonal to the closure of the image of $\mathfrak{b}^*$ on the Hilbert space of square integrable function of a suitably chosen path from $x_0$ to $x_1$. However, the image of $\mathfrak{b}^*$ will be dense, so this in turn implies that $f(x)$ is zero. Hence $\mathcal{C}_{\mathfrak{b}}(\cdot, \cdot)$ is nonsingular and defines a non-degenerate pairing.

The bilinear concomitant of $\mathfrak{b}^*$ is related to that of $\mathfrak{b}$ by

$$\mathcal{C}_{\mathfrak{b}^*}(f, g; p) = -\mathcal{C}_{\mathfrak{b}}(g, f; p).$$

This in particular, when $\mathfrak{b}$ is formally symmetric the bilinear concomitant defines a symplectic form on $\ker \mathfrak{b}$.

Now suppose that $\mathfrak{b} = \mathfrak{b}^*$. Recall that factorizations of $\mathfrak{b}$ correspond to choices of subspaces of $\ker \mathfrak{b}$. Specifically given a subspace $V \subseteq \ker \mathfrak{b}$, there exists a differential operator $\mathfrak{b}$ (unique up to multiplication by a function on the left) with $\ker \mathfrak{b} = V$ and $\mathfrak{b} = \mathfrak{a}\mathfrak{b}$ for some differential operator $\mathfrak{a}$. Since $\mathfrak{b} = \mathfrak{b}^*$, this also implies that $\mathfrak{b} = \mathfrak{b}^*\mathfrak{a}^*$ so that $\mathfrak{a}^*$ also corresponds to a certain subspace of $\ker \mathfrak{b}$, which turns out to be exactly the orthogonal subspace $V^{\perp}$ of $V$ under the symplectic form defined by the bilinear concomitant. In particular, in the special case of a factorization of the form $\mathfrak{b} = \mathfrak{b}^*\mathfrak{b}$, the subspace $V$ of $\mathfrak{b}$ satisfies $V^{\perp} = V$, i.e., is a Lagrangian subspace of $\ker(\mathfrak{b})$. This is the content of the next theorem.

**Theorem 6.6.** *Let $\mathfrak{b}$ be a monic formally symmetric operator on an open subset $\mathcal{O}$ of $\mathbb{C}$. Choose $\mathcal{O}$ sufficiently small so that the kernel* $\ker \mathfrak{b}$ *in the space of holomorphic functions on $\mathcal{O}$ has dimension equal to the order of $\mathfrak{b}$. Let $a, b \in \mathcal{O}$ with $a \neq b$.*

*If $\mathfrak{b}$ has a factorization $\mathfrak{b} = \mathfrak{a}\mathfrak{b}$ into differential operators $\mathfrak{a}, \mathfrak{b}$ with analytic coefficients on $\mathcal{O}$, then* $\ker \mathfrak{b}$ *and* $\ker \mathfrak{a}^*$ *are complementary subspaces of* $\ker \mathfrak{b}$ *relative to the symplectic form $\mathcal{C}_{\mathfrak{b}}$. Furthermore, given a subspace $V \subseteq \ker \mathfrak{b}$ there exist monic differential operators $\mathfrak{a}, \mathfrak{b}$ satisfying* $\ker \mathfrak{a}^* = V^{\perp}$ *and* $\ker \mathfrak{b} = V$.

*In particular, factorizations of the form $\mathfrak{b} = \mathfrak{b}^*\mathfrak{b}$ for monic differential operators $\mathfrak{b}$ with analytic coefficients on $\mathcal{O}$ are in bijective correspondence with Lagrangian subspaces of* $\ker \mathfrak{b}$.

*Proof.* Suppose that $\mathfrak{b}$ is an operator of order $m$ and has a factorization of the form $\mathfrak{b} = \mathfrak{a}\mathfrak{b}$. If $\operatorname{ord} \mathfrak{b} = r$, then $\operatorname{ord} \mathfrak{a} = m - r$. Since $\mathfrak{b} = \mathfrak{b}^*$, we have $\mathfrak{b}^* = \mathfrak{b}^*\mathfrak{a}^*$. Therefore $\ker \mathfrak{a}^*$ and $\ker \mathfrak{b}$ are subspaces of $\ker \mathfrak{b}$ of dimension $m - r$ and $r$, respectively. Following Wilson in [40], for all $f, g$ we have that

$$\mathcal{C}_{\mathfrak{b}}(f, g; p) = \mathcal{C}_{\mathfrak{a}}(\mathfrak{b} \cdot f, g; p) + \mathcal{C}_{\mathfrak{b}}(f, \mathfrak{a}^* \cdot g; p).$$

This implies that for all $f \in \ker \mathfrak{b}$ and $g \in \ker \mathfrak{a}^*$ we must have $\mathcal{C}_{\mathfrak{d}}(f, g; p) = 0$. Hence

$$\ker \mathfrak{a}^* \subseteq (\ker \mathfrak{b})^{\perp}.$$

Since the dimension of $(\ker \mathfrak{b})^{\perp}$ must be $m - r$, this implies that $\ker \mathfrak{a}^* = (\ker \mathfrak{b})^{\perp}$. This proves the first claim of the theorem.

To prove the second claim, we start with an arbitrary subspace $V \subseteq \ker \mathfrak{d}$. Then the usual construction may be used to produce a differential operator $\mathfrak{b}$ whose kernel is $V$. Then since the kernel of $\mathfrak{b}$ is contained in the kernel of $\mathfrak{d}$, there must exist a differential operator $\mathfrak{a}$ with $\mathfrak{d} = \mathfrak{a}\mathfrak{b}$. By the previous argument then $\ker \mathfrak{a}^* = V^{\perp}$. This proves the second claim of the theorem.

In the special case that $\mathfrak{d} = \mathfrak{b}^*\mathfrak{b}$, we must have that $\ker \mathfrak{b} = (\ker \mathfrak{b})^{\perp}$. Thus in this case, the kernel of $\mathfrak{b}$ defines a Lagrangian subspace of $\ker \mathfrak{d}$. Conversely, a lagrangian subspace $V \subseteq \ker(\mathfrak{d})$ defines a factorization $\mathfrak{d} = \mathfrak{a}\mathfrak{b}$, where $\ker \mathfrak{a}^* = \ker \mathfrak{b}$. This implies that $\mathfrak{a}^* = h(x)\mathfrak{b}$ for some function $h(x)$, and therefore that $\mathfrak{d} = \mathfrak{b}^* h(x)\mathfrak{b}$. Setting $\mathfrak{c} = \pm\sqrt{h(x)}\mathfrak{b}$, this gives us a factorization $\mathfrak{d} = \mathfrak{c}^*\mathfrak{c}$ for a monic differential operator $\mathfrak{c}$. □

By the discussion in Section 6.1, the bispectral functions in the adelic Grassmannian are obtained as follows. Start with a differential operator with constant coefficients

$$\mathfrak{d} = \prod_j (\partial_x - c_j)^{m_j}.$$

Its kernel in the space of entire functions consists of the quasipolynomials

$$\ker \mathfrak{d} = \left\{ \sum_j p_j(x) e^{c_j x} : \deg p_j(x) \leq n_j - 1 \right\}.$$

Let $V$ be a subspace of $\ker \mathfrak{d}$ having a basis consisting of quasiexponential functions each of which contains a single exponent $e^{c_j x}$; this is the adelic condition. Let $\frac{1}{p(x)}\mathfrak{u}$ be the unique monic differential operator such that $\ker \mathfrak{u} = V$ and $\mathfrak{u} \in \mathcal{F}_x(\psi_{\exp})$ (i.e., $\mathfrak{u}$ has polynomial coefficients). Let

$$q(y) = \prod (y - c_j)^{n_j},$$

where $n_j$ equals the number of basis elements of $V$ whose exponent is $e^{c_j x}$. The bispectral functions in the adelic Grassmannian are the functions of the form

$$\widetilde{\psi}(x, y) = \frac{1}{q(y)p(x)} \mathfrak{u} \cdot e^{xy}.$$

The corresponding $W \in \mathrm{Gr}^{\mathrm{ad}}$ (recall Definition 6.3) is obtained as follows: it corresponds to the unique space of conditions $C \subseteq \mathscr{C}$ such that

$$V = C(e^{xy}),$$

keeping in mind that the delta function in $C$ act in the variable $y$.

With this in mind, we define the Lagrangian adelic Grassmannian.

**Definition 6.7.** The *Lagrangian adelic Grassmannian* is the sub-Grassmannian $\mathrm{Gr}^{\mathrm{ad}}_{\mathrm{Lagr}}$ of $\mathrm{Gr}^{\mathrm{ad}}$ consisting of those points for which $\mathfrak{d}^* = \mathfrak{d}$ and the corresponding subspaces $V$ as above are

(1) Lagrangian subspaces of $\ker \mathfrak{d}$ with respect to the symplectic form $\mathcal{C}_{\mathfrak{d}}(\cdot, \cdot)$ and

(2) are preserved under the transformations $x \mapsto -x$.

Combining Theorems 6.4 and 6.6, we obtain the following:

**Theorem 6.8.** *The self-adjoint bispectral functions of rank* 1 *are in bijective correspondence with the points of the Lagrangian adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}_{\mathrm{Lagr}}$.

This theorem gives an explicit algorithmic way to construct all self-adjoint bispectral functions of rank 1.

**6.4. Classification of the self-adjoint transformations in the Airy and Bessel cases.** In this subsection we describe extensions of the results in Sections 6.2–6.3 to the rank 2 Airy and Bessel cases.

The Sato's Grassmannian classifies the solutions of the KP hierarchy. We refer the reader to [31] for details. The adelic Grassmannian is naturally embedded in it. In [5, Sections 2–4] it was proved that the families of (normalized) bispectral Darboux transformations of the Bessel functions $\psi_{\mathrm{Be}(\nu)}$ and Airy functions $\psi_{\mathrm{Ai}}$ are canonically embedded in it, forming sub-Grassmannians which we will denote by $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ and $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$, respectively. (The Grassmannians $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ and $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$ are disjoint from $\mathrm{Gr}^{\mathrm{ad}}$ which is also naturally embedded in the Sato's Grassmannian.)

In [5, Section 1.4] Wilson's adjoint $a$ and sign $s$ involutions were extended to Sato's Grassmannian and it was shown that (6.3)–(6.7) are satisfied whenever $\widetilde{\psi}(x, y)$ and $\psi_{\exp}(x, y)$ are replaced with any pair of wave functions of the KP hierarchy that satisfy the transformation property (6.4). Moreover, in [5, Sections 2 and 4] it was proved that $a$ preserves $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ and $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$, while $s$ preserves $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$. Similarly to Theorem 6.9, using the fact that $\iota_{\psi_{\mathrm{Ai}},x} = \mathrm{id}$ and $\iota_{\psi_{\mathrm{Be}(\nu)},x} = \sigma$ (cf. Section 3.3 and equation (6.8)), one proves the following:

**Theorem 6.9.** (a) *Let* $\widetilde{\psi}(x, y)$ *be a (normalized) bispectral Darboux transformation from the Airy bispectral function* $\psi_{\mathrm{Ai}}(x, y)$. *Then the following are equivalent:*

(1) $\widetilde{\psi}(x, y)$ *is self-adjoint.*

(2) *The algebra* $\mathcal{B}_x(\widetilde{\psi})$ *is closed under the formal adjoint* $*$.

(3) *The plane* $W \in \mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$ *corresponding to* $\widetilde{\psi}(x, y)$ *is invariant under the adjoint involution* $a$ *of* $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$.

(b) *Let* $\widetilde{\psi}(x, y)$ *be a (normalized) bispectral Darboux transformation from the Bessel bispectral function* $\psi_{\mathrm{Be}(\nu)}(x, y)$ *for* $\nu \in \mathbb{C} \setminus \mathbb{Z}$. *Then the following are equivalent:*

(1) $\widetilde{\psi}(x, y)$ *is self-adjoint.*

(2) *The algebra* $\mathcal{B}_x(\widetilde{\psi})$ *is closed under the formal adjoint* $*$ *and the transformation* $\sigma$ *from* (6.8).

(3) *The plane* $W \in \mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ *corresponding to* $\widetilde{\psi}(x, y)$ *is invariant under the adjoint* $a$ *and sign* $s$ *involutions of* $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$.

The (normalized) bispectral Darboux transformations from $\psi_{\mathrm{Ai}}(x, y)$ and $\psi_{\mathrm{Be}(\nu)}(x, y)$ (for $\nu \in \mathbb{C} \setminus \mathbb{Z}$) are constructed as follows, see [5, Sections 2 and 4] for details. Let $\mathfrak{d}$ be a differential operator which is a polynomial with constant coefficients in the differential operators $\mathfrak{d}_{\mathrm{Ai},x}$ (resp. $\mathfrak{d}_{\mathrm{Be}(\nu),x}$) from Table 1. (As a consequence of this, $\mathfrak{d}^* = \mathfrak{d}$.) Let $\mathcal{O}$ be an open

connected subset of $\mathbb{C}$ such that the dimension of the kernel of $\mathfrak{d}$ in the space of holomorphic functions on $\mathcal{O}$ equals the order of $\mathfrak{d}$. The kernel of the operator $\mathfrak{d}$ is given in terms of derivatives of the Airy function (resp. derivatives of the Bessel functions and products of powers of $x$ and logarithmic functions) by [5, Proposition 4.9 and Lemma 2.1].

The planes in $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$ and $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ correspond to bispectral functions of the form

$$\widetilde{\psi}(x,y) = \frac{1}{q(y)p(x)}\mathfrak{u}\cdot\psi(x,y),$$

where

$$\psi(x,y) = \psi_{\mathrm{Ai}}(x,y) \quad \text{resp.} \quad \psi(x,y) = \psi_{\mathrm{Be}(\nu)}(x,y),$$

$\ker\mathfrak{u}$ is a subspace of $\mathfrak{d}$ satisfying certain adelic type conditions [5, Definition 2.5 and Proposition 4.9], and $q(y)$, $p(x)$ are appropriate normalization polynomials.

We define the *Airy Lagrangian adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}_{\mathrm{Lagr}}$ to be the sub-Grassmannian of $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}$ consisting of those points for which $\ker\mathfrak{u}$ is a Lagrangian subspace of $\ker\mathfrak{d}$ with respect to the symplectic form $\mathcal{C}_{\mathfrak{d}}(\cdot,\cdot)$. Similarly, we define the *Bessel Lagrangian adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}_{\mathrm{Lagr}}$ to be the sub-Grassmannian of $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}$ consisting of those points for which $\ker\mathfrak{u}$ is a Lagrangian subspace of $\ker\mathfrak{d}$ with respect to the symplectic form $\mathcal{C}_{\mathfrak{d}}(\cdot,\cdot)$. and is preserved under the transformations $x\mapsto -x$.

From Theorems 6.6 and 6.9, we get:

**Theorem 6.10.** *The self-adjoint bispectral Darboux transformations from the Airy bispectral function $\psi_{\mathrm{Ai}}(x,y)$ (resp. the Bessel bispectral functions $\psi_{\mathrm{Be}(\nu)}(x,y)$ for $\nu\in\mathbb{C}\setminus\mathbb{Z}$) are in bijective correspondence with the points of the Airy Lagrangian adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad},\mathrm{Ai}}_{\mathrm{Lagr}}$ (resp. the Bessel Lagrangian adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad},\mathrm{Be}(\nu)}_{\mathrm{Lagr}}$).*

This theorem gives an explicit algorithmic way to construct all self-adjoint bispectral Darboux transformations from the Airy and Bessel bispectral functions.

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W. Riley Casper, Department of Mathematics, Louisiana State University,
Baton Rouge, LA 70803, USA
https://orcid.org/0000-0002-3783-766X
e-mail: wcasper1@lsu.edu

Milen T. Yakimov, Department of Mathematics, Louisiana State University,
Baton Rouge, LA 70803, USA
e-mail: yakimov@math.lsu.edu

Eingegangen 25. Oktober 2018, in revidierter Fassung 19. Juli 2019