TYPE C BLOCKS OF SUPER CATEGORY $\mathcal{O}$

JONATHAN BRUNDAN AND NICHOLAS DAVIDSON

Abstract. We show that the blocks of category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ associated to half-integral weights carry the structure of a tensor product categorification for the infinite rank Kac-Moody algebra of type C_∞. This allows us to prove two conjectures formulated by Cheng, Kwon and Wang. We then focus on the full subcategory consisting of finite-dimensional representations, which we show is a highest weight category with blocks that are Morita equivalent to certain generalized Khovanov arc algebras.

1. Introduction

In this article, we apply some powerful tools from higher representation theory to the study of the BGG category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$, and its subcategory $\mathcal{F}$ of finite-dimensional representations. We restrict our attention throughout to modules with half-integral weights. In fact, by [C], the study of the category $\mathcal{O}$ for $\mathfrak{q}_n(\mathbb{C})$ reduces to studying three types of blocks, known as the type A, type B, and type C blocks. The half-integral weight case studied here constitutes all of the type C blocks. For types A and B blocks, we refer the reader to [CKW, BD2] and [B1, CKW, D], respectively.

The type C blocks are already known to be highest weight categories in the sense of [CPS]. We will prove two conjectures about them formulated by Cheng, Kwon and Wang, namely, [CKW, Conjectures 5.12–5.13]. Roughly speaking, these assert that the combinatorics of type C blocks is controlled by certain canonical bases for the tensor power $V^{\otimes n}$ of the minuscule natural representation V of the quantum group of type C_∞. Actually, in general, one needs to consider Webster's "orthodox basis" from [W1], which is subtley different from Lusztig's canonical basis. Since there is no elementary algorithm to compute Webster's basis, this is still not an entirely satisfactory picture.

Interest in the category $\mathcal{F}$ (again, for half-integral weights) was rekindled by another recent paper of Cheng and Kwon [CK]. We will show here that $\mathcal{F}$ is a highest weight category, answering [CKW, Question 5.1(1)]. When combined with the main result of [BS2], our approach actually allows us to describe $\mathcal{F}$ in purely diagrammatical terms: its blocks are equivalent to finite-dimensional modules over the generalized Khovanov arc algebras denoted $K_r^{+\infty}$ in [BS1].

The remainder of the article is organized as follows.

- In section 2, we set up the underlying combinatorics of the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$. As observed already in [CKW], this may be identified with the Grothendieck group of the category $\mathcal{O}^\Delta$ of Δ-filtered modules of the category $\mathcal{O}$ to be studied later in the paper. We also give a brief review of Lusztig's canonical basis for this module, including an elementary algorithm to compute it in practice, and recall [CKW, Proposition 4.1], which relates this type C canonical basis to some other type A canonical bases.
- In section 3, we introduce the supercategory $s\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ and all half-integral weights. Actually, when n is odd, it is more convenient to

Research supported in part by NSF grants DMS-1161094 and DMS-1700905.

work with supermodules over $\mathfrak{q}_n(\mathbb{C}) \oplus \mathfrak{q}_1(\mathbb{C})$ following the idea of [BD2]. This means that the supercategory $s\mathcal{O}$ considered here in the odd case is the Clifford twist of the one appearing in [CKW]. This trick unifies our treatment of the even and odd cases, and actually makes our results slightly stronger for odd n. Mimicking the approach of [BD2], we then show that $s\mathcal{O}$ splits as $\mathcal{O} \oplus \Pi\mathcal{O}$ for a highest weight category $\mathcal{O}$, and that $\mathcal{O}$ admits the structure of a tensor product categorification of the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$ in the general sense of Losev and Webster [LW]. Our proof depends crucially on a particular instance of the remarkable isomorphisms discovered by Kang, Kashiwara and Tsuchioka [KKT].

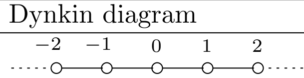
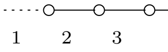
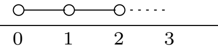
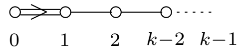
- In section 4, we combine our main result from section 3 with an argument involving truncation from $\mathfrak{sp}_{2\infty}$ to $\mathfrak{sp}_{2k}$ and the uniqueness of $\mathfrak{sp}_{2k}$-tensor product categorifications established in [LW], in order to prove the first Cheng-Kwon-Wang conjecture. This is similar to the proof of the Kazhdan-Lusztig conjecture for the general linear supergroup given in [BLW]. We also give an application to classifying the indecomposable projective-injective supermodules in $s\mathcal{O}$.
- In section 5, we use another form of truncation, this time from $\mathfrak{sp}_{2\infty}$ to $\mathfrak{sl}_{+\infty}$, to establish the second Cheng-Kwon-Wang conjecture. In fact, we show that the category $\mathcal{O}$ admits a filtration whose sections are $\mathfrak{sl}_{+\infty}$-tensor product categorifications, a result which may be viewed as a categorical version of [CKW, Proposition 4.1]. When combined with the uniqueness of $\mathfrak{sl}_{+\infty}$-tensor product categorifications established in [BLW], this also allows us to understand the structure of the subcategory $\mathcal{F}$ of $\mathcal{O}$ consisting of the finite-dimensional supermodules: we show that $\mathcal{F}$ decomposes as
$$\mathcal{F} = \bigoplus_{n_0+n_1=n} \mathcal{F}_{n_0|n_1}$$
with $\mathcal{F}_{n_0|n_1}$ being equivalent to a quotient of the category of rational representations of the general linear supergroup $GL_{n_0|n_1}(\mathbb{C})$. From this, we deduce that $\mathcal{F}$ is a highest weight category, and its blocks are Morita equivalent to certain generalized Khovanov arc algebras like in [BS2].

Acknowledgements. We thank Shunsuke Tsuchioka for allowing us to include his counterexamples to positivity in Example 2.12.

2. Canonical basis

We are going to be interested in categorifications of certain tensor products of minuscule representations of various Kac-Moody algebras. In this section, we define these tensor products and make some elementary combinatorial observations about them. Most of this material also be found in equivalent form in [CKW], but our conventions are somewhat different.

2.1. **Minuscule representations.** We will need the (complex) Kac-Moody algebras of the following types:

Type	Dynkin diagram	Simple roots
$\mathfrak{sl}_\infty$		$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$
$\mathfrak{sl}_{+\infty}$		$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$
$\mathfrak{sp}_{2\infty}$		$\alpha_0 = -2\varepsilon_0,\ \alpha_i = \varepsilon_{i-1} - \varepsilon_i\ (i > 0)$
$\mathfrak{sp}_{2k}$		$\alpha_0 = -2\varepsilon_0,\ \alpha_i = \varepsilon_{i-1} - \varepsilon_i\ (0 < i < k)$

Suppose that $\mathfrak{s}$ is one of these Lie algebras. Letting I denote the set that indexes the vertices of the underlying Dynkin diagram in the above table, $\mathfrak{s}$ is generated by its *Chevalley generators* $\{e_i, f_i \mid i \in I\}$ subject to the usual Serre relations. Let $\mathfrak{t}$ be the Cartan subalgebra spanned by $\{h_i := [e_i, f_i] \mid i \in I\}$. We also introduce the *weight lattice* $P := \bigoplus_{i \in I} \mathbb{Z}\varepsilon_i$, which we identify with an Abelian subgroup of $\mathfrak{t}^*$ so that the *simple roots* $\{\alpha_i \mid i \in I\}$ of $\mathfrak{s}$ are identified with the elements of P indicated in the table. Note then that

$$\langle h_i, \alpha_j \rangle = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \tag{2.1}$$

where $(\cdot, \cdot)$ is the bilinear form on P defined from $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. There is a corresponding *dominance order* $\trianglerighteq$ on P defined from $\lambda \trianglerighteq \mu$ if and only if $\lambda - \mu$ is a sum of simple roots. (The notation $I, P, \dots$ just introduced is potentially ambiguous as it depends on the particular choice of $\mathfrak{s}$, but this should always be clear from the context.)

As is evident from the Dynkin diagrams, there are natural inclusions

$$\mathfrak{sp}_2 \hookrightarrow \mathfrak{sp}_4 \hookrightarrow \mathfrak{sp}_6 \hookrightarrow \cdots \hookrightarrow \mathfrak{sp}_{2\infty} \hookleftarrow \mathfrak{sl}_{+\infty} \hookrightarrow \mathfrak{sl}_\infty$$

sending Chevalley generators to Chevalley generators. These embeddings will play an important role in our applications.

We proceed to introduce various minuscule representations of these Lie algebras.

For $\mathfrak{sl}_\infty$, we will consider both its natural module V^+ and the dual V^-. These have standard bases $\{v_j^+ \mid j \in \mathbb{Z}\}$ and $\{v_j^- \mid j \in \mathbb{Z}\}$, respectively. The weight of the vector $v_j^\pm$ is $\pm\varepsilon_j$, and the Chevalley generators act by

$$f_i v_j^+ = \begin{cases} v_{j+1}^+ & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases} \qquad e_i v_j^+ = \begin{cases} v_{j-1}^+ & \text{if } j = 1 + i \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

$$f_i v_j^- = \begin{cases} v_{j-1}^- & \text{if } j = 1 + i \\ 0 & \text{otherwise,} \end{cases} \qquad e_i v_j^- = \begin{cases} v_{j+1}^- & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Similarly, we have the natural and dual natural modules for $\mathfrak{sl}_{+\infty}$, which will be denoted V_0^+ and V_0^-, respectively. Exploiting the inclusion $\mathfrak{sl}_{+\infty} \hookrightarrow \mathfrak{sl}_\infty$, we identify $V_0^\pm$ with the submodule of the restriction of $V^\pm$ spanned by $\{v_j^\pm \mid j > 0\}$.

For $\mathfrak{sp}_{2\infty}$, we only have its natural module V. This has basis $\{v_j \mid j \in \mathbb{Z}\}$, with v_j of weight ε_{j-1} if $j > 0$ or $-\varepsilon_{-j}$ if $j \leq 0$, and action defined from

$$f_i v_j = \begin{cases} v_{j+1} & \text{if } j = \pm i \\ 0 & \text{otherwise,} \end{cases} \qquad e_i v_j = \begin{cases} v_{j-1} & \text{if } j = 1 \pm i \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Similarly, for any $k \geq 1$, we have the natural module V_k of $\mathfrak{sp}_{2k}$, which is identified with the submodule of the restriction of V spanned by $\{v_j \mid -k < j \leq k\}$.

Lemma 2.1. *As an $\mathfrak{sl}_{+\infty}$-module, V is isomorphic to $V_0^+ \oplus V_0^-$.*

Proof. The map $v_j^+ \mapsto v_j$ defines an isomorphism between V_0^+ and the $\mathfrak{sl}_{+\infty}$-submodule of V spanned by $\{v_j \mid j > 0\}$. Similarly, the map $v_j^- \mapsto v_{1-j}$ defines an isomorphism between V_0^- and the submodule spanned by $\{v_j \mid j \leq 0\}$. □

2.2. **Tensor products.** We are really interested in tensor powers of the minuscule representations defined so far. To introduce these, fix $n \geq 1$ and let $\mathbf{B}$ denote the set $\mathbb{Z}^n$ of n-tuples $\boldsymbol{b} = (b_1, \dots, b_n)$ of integers. Let $\boldsymbol{d}_r \in \mathbf{B}$ be the tuple with 1 in its rth entry and 0 in all other places. Also, for $k \geq 1$ and a tuple of signs $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \{\pm\}^n$, let

$$\mathbf{B}_0 := \{\boldsymbol{b} \in \mathbf{B} \mid b_1, \dots, b_n > 0\}, \tag{2.5}$$

$$\mathbf{B}_k := \{\boldsymbol{b} \in \mathbf{B} \mid -k < b_1, \dots, b_n \leq k\}, \tag{2.6}$$

$$\mathbf{B}_{\boldsymbol{\sigma}} := \{\boldsymbol{b} \in \mathbf{B} \mid b_r > 0 \text{ if } \sigma_r = +, b_r \leq 0 \text{ if } \sigma_r = -\}. \tag{2.7}$$

Let $V^{\otimes\boldsymbol{\sigma}}$ denote the $\mathfrak{sl}_\infty$-module $V^{\sigma_1}\otimes\cdots\otimes V^{\sigma_n}$. It has the natural monomial basis $\left\{v_{\boldsymbol{b}}^{\boldsymbol{\sigma}} := v_{b_1}^{\sigma_1}\otimes\cdots\otimes v_{b_n}^{\sigma_n}\;\middle|\;\boldsymbol{b}\in\mathbf{B}\right\}$. The action of the Chevalley generators of $\mathfrak{sl}_\infty$ on this basis is given explicitly by

$$f_i v_{\boldsymbol{b}}^{\boldsymbol{\sigma}} = \sum_{\substack{1\leq t\leq n\\ i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b})=\mathtt{f}}} v_{\boldsymbol{b}+\sigma_t\boldsymbol{d}_t}^{\boldsymbol{\sigma}}, \qquad e_i v_{\boldsymbol{b}}^{\boldsymbol{\sigma}} = \sum_{\substack{1\leq t\leq n\\ i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b})=\mathtt{e}}} v_{\boldsymbol{b}-\sigma_r\boldsymbol{d}_t}^{\boldsymbol{\sigma}}, \tag{2.8}$$

where $i\text{-}\mathbf{sig}^{\boldsymbol{\sigma}}(\boldsymbol{b}) = (i\text{-sig}_1^{\boldsymbol{\sigma}}(\boldsymbol{b}),\dots,i\text{-sig}_n^{\boldsymbol{\sigma}}(\boldsymbol{b}))$ is the *i-signature* of $\boldsymbol{b}\in\mathbf{B}$ (with respect to $\boldsymbol{\sigma}$) defined from

$$i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b}) := \begin{cases} \mathtt{f} & \text{if } (b_t,\sigma_t)=(i,+) \text{ or } (b_t,\sigma_t)=(1+i,-),\\ \mathtt{e} & \text{if } (b_t,\sigma_t)=(1+i,+) \text{ or } (b_t,\sigma_t)=(i,-),\\ \bullet & \text{otherwise.}\end{cases} \tag{2.9}$$

Similarly, we have the $\mathfrak{sl}_{+\infty}$-module $V_0^{\otimes\boldsymbol{\sigma}} = V_0^{\sigma_1}\otimes\cdots\otimes V_0^{\sigma_n}$, which we identify with the submodule of $V^{\otimes\boldsymbol{\sigma}}$ spanned by $\{v_{\boldsymbol{b}}^{\boldsymbol{\sigma}}\mid\boldsymbol{b}\in\mathbf{B}_0\}$. The projection

$$\mathrm{pr}_0 : V^{\otimes\boldsymbol{\sigma}} \twoheadrightarrow V_0^{\otimes\boldsymbol{\sigma}}, \qquad v_{\boldsymbol{b}}^{\boldsymbol{\sigma}} \mapsto \begin{cases} v_{\boldsymbol{b}}^{\boldsymbol{\sigma}} & \text{if } \boldsymbol{b}\in\mathbf{B}_0,\\ 0 & \text{otherwise}\end{cases} \tag{2.10}$$

is an $\mathfrak{sl}_{+\infty}$-module homomorphism.

We also have the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$, with basis $\{v_{\boldsymbol{b}} := v_{b_1}\otimes\cdots\otimes v_{b_n}\mid\boldsymbol{b}\in\mathbf{B}\}$. The action is given explicitly by the formulae

$$f_i v_{\boldsymbol{b}} = \sum_{\substack{1\leq t\leq n\\ i\text{-sig}_t(\boldsymbol{b})=\mathtt{f}}} v_{\boldsymbol{b}+\boldsymbol{d}_t}, \qquad e_i v_{\boldsymbol{b}} = \sum_{\substack{1\leq t\leq n\\ i\text{-sig}_t(\boldsymbol{b})=\mathtt{e}}} v_{\boldsymbol{b}-\boldsymbol{d}_t}, \tag{2.11}$$

where this time $i\text{-}\mathbf{sig}(\boldsymbol{b}) = (i\text{-sig}_1(\boldsymbol{b}),\dots,i\text{-sig}_n(\boldsymbol{b}))$ is defined from

$$i\text{-sig}_t(\boldsymbol{b}) := \begin{cases} \mathtt{f} & \text{if } b_t=\pm i,\\ \mathtt{e} & \text{if } b_t = 1\pm i,\\ \bullet & \text{otherwise.}\end{cases} \tag{2.12}$$

Similarly, we have the $\mathfrak{sp}_{2k}$-module $V_k^{\otimes n}$, which is identified with the submodule of $V^{\otimes n}$ spanned by $\{v_{\boldsymbol{b}}\mid\boldsymbol{b}\in\mathbf{B}_k\}$. The projection

$$\mathrm{pr}_k : V^{\otimes n} \twoheadrightarrow V_k^{\otimes n}, \qquad v_{\boldsymbol{b}} \mapsto \begin{cases} v_{\boldsymbol{b}} & \text{if } \boldsymbol{b}\in\mathbf{B}_k,\\ 0 & \text{otherwise}\end{cases} \tag{2.13}$$

is an $\mathfrak{sp}_{2k}$-module homomorphism.

From Lemma 2.1, we see that the restriction of $V^{\otimes n}$ to the subalgebra $\mathfrak{sl}_{+\infty}$ is isomorphic to $\bigoplus_{\boldsymbol{\sigma}\in\{\pm\}^n} V_0^{\otimes\boldsymbol{\sigma}}$. To write down an explicit isomorphism, introduce the function

$$\mathbf{B}\to\mathbf{B}_0, \qquad \boldsymbol{b}\mapsto\boldsymbol{b}' \tag{2.14}$$

where $\boldsymbol{b}'$ is the tuple with rth entry b_r if $b_r>0$ or $1-b_r$ if $b_r\leq 0$. This restricts to bijections $\mathbf{B}_{\boldsymbol{\sigma}} \xrightarrow{\sim} \mathbf{B}_0$ for each $\boldsymbol{\sigma}\in\{\pm\}^n$. Define a linear map

$$\mathrm{pr}_{\boldsymbol{\sigma}} : V^{\otimes n} \twoheadrightarrow V_0^{\otimes\boldsymbol{\sigma}}, \qquad v_{\boldsymbol{b}} \mapsto \begin{cases} v_{\boldsymbol{b}'}^{\boldsymbol{\sigma}} & \text{if } \boldsymbol{b}\in\mathbf{B}_{\boldsymbol{\sigma}},\\ 0 & \text{otherwise.}\end{cases} \tag{2.15}$$

Then:

Lemma 2.2. *The map*

$$\sum_{\boldsymbol{\sigma}\in\{\pm\}^n} \mathrm{pr}_{\boldsymbol{\sigma}} : V^{\otimes n} \xrightarrow{\sim} \bigoplus_{\boldsymbol{\sigma}\in\{\pm\}^n} V_0^{\otimes\boldsymbol{\sigma}}$$

is an isomorphism of $\mathfrak{sl}_{+\infty}$-modules.

2.3. **Bruhat order.** Next, we introduce some partial orders on the index set $\mathbf{B}$. These orders arise in Lusztig's construction of canonical bases for the spaces $V^{\otimes\boldsymbol{\sigma}}$ and $V^{\otimes n}$, which we'll review in more detail in the next subsection. To define them, we need the *inverse dominance order* $\preceq$ on P^n from [LW, Definition 3.2]. For $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in P^n$, we write $|\boldsymbol{\beta}|$ for $\beta_1 + \dots + \beta_n \in P$. Then, $\preceq$ is defined by declaring that $\boldsymbol{\beta} \preceq \boldsymbol{\gamma}$ if and only if $|\boldsymbol{\beta}| = |\boldsymbol{\gamma}|$ and $\beta_1 + \dots + \beta_s \trianglerighteq \gamma_1 + \dots + \gamma_s$ for each $s = 1, \dots, n$. (Obviously, $\preceq$ depends on the particular Lie algebra $\mathfrak{s}$ being considered.)

We start with $\mathfrak{sl}_\infty$. So fix $\boldsymbol{\sigma} \in \{\pm\}^n$. Recall that the weight spaces of $V^\pm$ are one-dimensional with $v_j^\pm$ of weight $\pm\varepsilon_j$. There is an injective map

$$\mathbf{wt}^{\boldsymbol{\sigma}} : \mathbf{B} \hookrightarrow P^n, \qquad \boldsymbol{b} \mapsto (\mathrm{wt}_1^{\boldsymbol{\sigma}}(\boldsymbol{b}), \dots, \mathrm{wt}_n^{\boldsymbol{\sigma}}(\boldsymbol{b}))$$

with $\mathrm{wt}_r^{\boldsymbol{\sigma}}(\boldsymbol{b}) := \sigma_r \varepsilon_{b_r}$; in particular, $v_{\boldsymbol{b}}^{\boldsymbol{\sigma}}$ is of weight $|\,\mathbf{wt}^{\boldsymbol{\sigma}}(\boldsymbol{b})|$. The $\mathfrak{sl}_\infty$*-Bruhat order* $\preceq_{\boldsymbol{\sigma}}$ on $\mathbf{B}$ is defined from

$$\boldsymbol{a} \preceq_{\boldsymbol{\sigma}} \boldsymbol{b} \Leftrightarrow \mathbf{wt}^{\boldsymbol{\sigma}}(\boldsymbol{a}) \preceq \mathbf{wt}^{\boldsymbol{\sigma}}(\boldsymbol{b}) \tag{2.16}$$

in the inverse dominance order for $\mathfrak{sl}_\infty$. The induced order on the subset $\mathbf{B}_0$ from (2.5) is the $\mathfrak{sl}_{+\infty}$*-Bruhat order* $\preceq_{\boldsymbol{\sigma}}$. Sometimes the following equivalent description of $\preceq_{\boldsymbol{\sigma}}$ from [BD2, Lemma 4.2] is useful:

Lemma 2.3. *For $i \in I$ (which is either $\mathbb{Z}$ or $\mathbb{Z}_+$ depending on whether we are considering $\mathfrak{sl}_\infty$ or $\mathfrak{sl}_{+\infty}$) and $1 \le s \le n$, we let*

$$N_{[1,s]}^{\boldsymbol{\sigma}}(\boldsymbol{b}, i) := \#\{1 \le r \le s \mid b_r > i, \sigma_r = +\} - \#\{1 \le r \le s \mid b_r > i, \sigma_r = -\}. \tag{2.17}$$

Then, we have that $\boldsymbol{a} \preceq_{\boldsymbol{\sigma}} \boldsymbol{b}$ if and only if
- $N_{[1,s]}^{\boldsymbol{\sigma}}(\boldsymbol{a}, i) \le N_{[1,s]}^{\boldsymbol{\sigma}}(\boldsymbol{b}, i)$ *for all $i \in I$ and $s = 1, \dots, n-1$;*
- $N_{[1,n]}^{\boldsymbol{\sigma}}(\boldsymbol{a}, i) = N_{[1,n]}^{\boldsymbol{\sigma}}(\boldsymbol{b}, i)$ *for all $i \in I$.*

Turning our attention to $\mathfrak{sp}_{2\infty}$, we consider instead the inclusion

$$\mathbf{wt} : \mathbf{B} \hookrightarrow P^n, \qquad \boldsymbol{b} \mapsto (\mathrm{wt}_1(\boldsymbol{b}), \dots, \mathrm{wt}_n(\boldsymbol{b}))$$

defined by setting $\mathrm{wt}_r(\boldsymbol{b}) := \varepsilon_{b_r - 1}$ if $b_r > 0$ or $-\varepsilon_{-b_r}$ if $b_r \le 0$; in particular $v_{\boldsymbol{b}}$ is of weight $|\,\mathbf{wt}(\boldsymbol{b})|$. Then we define the $\mathfrak{sp}_{2\infty}$*-Bruhat order* $\preceq$ on $\mathbf{B}$ as before:

$$\boldsymbol{a} \preceq \boldsymbol{b} \Leftrightarrow \mathbf{wt}(\boldsymbol{a}) \preceq \mathbf{wt}(\boldsymbol{b}) \tag{2.18}$$

in the inverse dominance order for $\mathfrak{sp}_{2\infty}$. The $\mathfrak{sp}_{2k}$*-Bruhat order* $\preceq$ is the induced order on the subset $\mathbf{B}_k$ from (2.6). There is a similar characterization of these orders to Lemma 2.3:

Lemma 2.4. *For $i \in I$ (which is either $\mathbb{N}$ or $\{0, 1, \dots, k-1\}$ for $\mathfrak{sp}_{2\infty}$ or $\mathfrak{sp}_{2k}$) and $1 \le s \le n$, we let*

$$N_{[1,s]}(\boldsymbol{b}, i) := \#\{1 \le r \le s \mid b_r > i\} - \#\{1 \le r \le s \mid b_r \le -i\}. \tag{2.19}$$

Then, we have that $\boldsymbol{a} \preceq \boldsymbol{b}$ if and only if
- $N_{[1,s]}(\boldsymbol{a}, 0) \equiv N_{[1,s]}(\boldsymbol{b}, 0) \pmod 2$ *for each $s = 1, \dots, n-1$;*
- $N_{[1,s]}(\boldsymbol{a}, i) \le N_{[1,s]}(\boldsymbol{b}, i)$ *for all $i \in I$ and $s = 1, \dots, n-1$;*
- $N_{[1,n]}(\boldsymbol{a}, i) = N_{[1,n]}(\boldsymbol{b}, i)$ *for all $i \in I$.*

Recall the set $\mathbf{B}_{\boldsymbol{\sigma}}$ from (2.7) and the bijection $\mathbf{B}_{\boldsymbol{\sigma}} \xrightarrow{\sim} \mathbf{B}_0, \boldsymbol{b} \mapsto \boldsymbol{b}'$ from (2.14).

Lemma 2.5. *The map $\boldsymbol{b} \mapsto \boldsymbol{b}'$ defines a poset isomorphism $(\mathbf{B}_{\boldsymbol{\sigma}}, \preceq) \xrightarrow{\sim} (\mathbf{B}_0, \preceq_{\boldsymbol{\sigma}})$.*

Proof. This follows easily from the characterizations of the two Bruhat orders that we have given, on noting from (2.17)–(2.19) that $N_{[1,s]}(\boldsymbol{b}, 0) = \sigma_1 1 + \dots + \sigma_s 1$ and $N_{[1,s]}(\boldsymbol{b}, i) = N_{[1,s]}^{\boldsymbol{\sigma}}(\boldsymbol{b}', i)$ for $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$ and all $i > 0$. $\square$

The remaining lemmas in this subsection are concerned with the case $\mathfrak{s} = \mathfrak{sp}_{2\infty}$.

Lemma 2.6. *Suppose that* $\boldsymbol{a} \succeq \boldsymbol{b}$ *and* $i\text{-sig}_r(\boldsymbol{a}) = i\text{-sig}_n(\boldsymbol{b}) = \mathtt{f}$ *for some* $i \in I$ *and* $1 \leq r \leq n$. *Then* $\boldsymbol{a} + \boldsymbol{d}_r \succeq \boldsymbol{b} + \boldsymbol{d}_n$, *with equality if and only if* $\boldsymbol{a} = \boldsymbol{b}$ *and* $r = n$.

Proof. This may be checked directly from the characterization of the Bruhat order given by Lemma 2.4. □

Lemma 2.7. *For* $\boldsymbol{b} \in \mathbf{B}$, *there exists* $\boldsymbol{a} \in \mathbf{B}$ *and a monomial* X *in the Chevalley generators* $\{f_i \mid i \in I\}$ *of* $\mathfrak{sp}_{2\infty}$ *such that*

- $a_1 > \cdots > a_n$ *and* $a_r + a_s \neq 1$ *for all* $1 \leq r < s \leq n$;
- $Xv_{\boldsymbol{a}} = v_{\boldsymbol{b}} +$ (*a sum of* $v_{\boldsymbol{c}}$*'s for* $\boldsymbol{c} \succ \boldsymbol{b}$).

Proof. We first explain an explicit construction for $\boldsymbol{a}$ and X. Suppose we are given $\boldsymbol{b} \in \mathbf{B}$. Define $\boldsymbol{a} \in \mathbf{B}$ by setting $a_1 := b_1$, then inductively defining each a_s for $s = 2, \dots, n$ to be the greatest integer such that $a_s \leq b_s$ and $a_s \leq \min(a_r - 1, -b_r)$ for all $1 \leq r < s$. It is clear from the definition of $\boldsymbol{a}$ that $a_1 > \cdots > a_n$. Also for $1 \leq r < s \leq n$, we have that $a_r + a_s \leq b_r - b_r = 0$. Then take $X = X_n \cdots X_2$ where $X_s := f_{|b_s - 1|} \cdots f_{|a_s+1|} f_{|a_s|}$.

To show that $Xv_{\boldsymbol{a}} = v_{\boldsymbol{b}} +$ (a sum of higher $v_{\boldsymbol{c}}$'s), we proceed by induction on n, the result being trivial in case $n = 1$. For $n > 1$, let $\bar{\boldsymbol{a}} := (a_1, \dots, a_{n-1})$, $\bar{\boldsymbol{b}} := (b_1, \dots, b_{n-1})$ and $\bar{X} := X_{n-1} \cdots X_2$. Applying the induction hypothesis in the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes(n-1)}$, we get that $\bar{X} v_{\bar{\boldsymbol{a}}} = v_{\bar{\boldsymbol{b}}} +$(a sum of $v_{\bar{\boldsymbol{c}}}$'s for $\bar{\boldsymbol{c}} \succ \bar{\boldsymbol{b}}$). Now we observe that if f_i is a Chevalley generator appearing in one of the monomials X_r for $r < n$ then $i \neq \pm a_n$, hence, $f_i v_{a_n} = 0$. Letting $\tilde{\boldsymbol{b}} := (b_1, \dots, b_{n-1}, a_n)$, we deduce that $\bar{X} v_{\boldsymbol{a}} = v_{\tilde{\boldsymbol{b}}} +$ (a sum of $v_{\tilde{\boldsymbol{c}}}$'s for $\tilde{\boldsymbol{c}} \succ \tilde{\boldsymbol{b}}$). Finally we act with X_n, which sends v_{a_n} to v_{b_n}, and apply Lemma 2.6. □

2.4. **Canonical basis.** So far, we have introduced the following tensor product modules over various Lie algebras $\mathfrak{s}$:

$\mathfrak{s}$	Tensor space	Monomial basis	Canonical basis
$\mathfrak{sl}_\infty$	$V^{\otimes\boldsymbol{\sigma}}$	$v^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$	$c^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$
$\mathfrak{sl}_{+\infty}$	$V_0^{\otimes\boldsymbol{\sigma}}$	$v^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}_0$	$\mathrm{pr}_0 c^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}_0$
$\mathfrak{sp}_{2\infty}$	$V^{\otimes n}$	$v_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$	$c_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$
$\mathfrak{sp}_{2k}$	$V_k^{\otimes n}$	$v_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}_k$	$\mathrm{pr}_k c_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}_k$

In this subsection, we give meaning to the rightmost column of this table by introducing some *canonical bases*, basically following a construction of Lusztig from [L, §27.3].

In each of the above cases, let $U_q\mathfrak{s}$ be the *quantized enveloping algebra* associated to $\mathfrak{s}$ over the field $\mathbb{Q}(q)$ (q an indeterminate). We denote the standard generators of $U_q\mathfrak{s}$ by $\{\dot{e}_i, \dot{f}_i, \dot{k}_i^{\pm} \mid i \in I\}$. They are subject to the usual q-deformed Serre relations. We view $U_q\mathfrak{s}$ as a Hopf algebra with comultiplication Δ defined from

$$\Delta(\dot{f}_i) = 1 \otimes \dot{f}_i + \dot{f}_i \otimes \dot{k}_i, \quad \Delta(\dot{e}_i) = \dot{k}_i^{-1} \otimes \dot{e}_i + \dot{e}_i \otimes 1, \quad \Delta(\dot{k}_i) = \dot{k}_i \otimes \dot{k}_i.$$

The various minuscule representations introduced in §2.1 all have q-analogs; cf. [J, §5A.1]. We will denote them by decorating our earlier notation with a dot, so we have the $\mathbb{Q}(q)$-vector spaces $\dot{V}^{\pm}, \dot{V}_0^{\pm}, \dot{V}$ and $\dot{V}_k$ with bases $\{\dot{v}_j^{\pm} \mid j \in \mathbb{Z}\}$, $\{\dot{v}_j^{\pm} \mid j > 0\}$, $\{\dot{v}_j \mid j \in \mathbb{Z}\}$ and $\{\dot{v}_j \mid -k < j \leq k\}$, respectively. The Chevalley generators $\dot{f}_i$ and $\dot{e}_i$ act on these bases by the same formulae (2.2)–(2.4) as before, while the diagonal action is given explicitly by

$$\dot{k}_i \dot{v}_j^+ = q^{\delta_{i,j} - \delta_{1+i,j}} \dot{v}_j^+, \qquad \dot{k}_i \dot{v}_j^- = q^{\delta_{1+i,j} - \delta_{i,j}} \dot{v}_j^-,$$

for the $\mathfrak{sl}$ cases, or

$$\dot{k}_i \dot{v}_j = q^{\delta_{i,j} + \delta_{-i,j} - \delta_{1+i,j} - \delta_{1-i,j}} \dot{v}_j$$

for $\mathfrak{sp}$. Taking tensor products, we obtain the modules $\dot{V}^{\otimes\boldsymbol{\sigma}}, \dot{V}_0^{\otimes\boldsymbol{\sigma}}, \dot{V}^{\otimes n}$ and $\dot{V}_k^{\otimes n}$, with their natural monomial bases denoted now by $\{\dot{v}_{\boldsymbol{b}}^{\boldsymbol{\sigma}} \mid \boldsymbol{b} \in \mathbf{B}\}, \{\dot{v}_{\boldsymbol{b}}^{\boldsymbol{\sigma}} \mid \boldsymbol{b} \in \mathbf{B}_0\}, \{\dot{v}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ and $\{\dot{v}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}_k\}$, respectively.

In the infinite rank cases, we need to pass from the q-tensor spaces just defined to completions in which certain infinite sums of the basis vectors also make sense, as follows.

For $\mathfrak{sl}_\infty$, the completed tensor space is denoted $\widehat{V}^{\otimes\boldsymbol{\sigma}}$. It is the $\mathbb{Q}(q)$-vector space consisting of formal linear combinations of the form $\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}^{\boldsymbol{\sigma}}$ for rational functions $p_{\boldsymbol{b}}(q) \in \mathbb{Q}(q)$ such that the *support* $\{\boldsymbol{b} \in \mathbf{B} \mid p_{\boldsymbol{b}}(q) \neq 0\}$ is contained in a finite union of sets of the form $\{\boldsymbol{b} \in \mathbf{B} \mid \mathbf{wt}^{\boldsymbol{\sigma}}(\boldsymbol{b}) \succeq \boldsymbol{\beta}\}$ for $\boldsymbol{\beta} \in P^n$ (working with the inverse dominance order for $\mathfrak{sl}_\infty$). This definition is justified in [BD2, Lemma 8.1]. For $\mathfrak{sl}_{+\infty}$, exactly the same procedure gives a completion $\widehat{V}_0^{\otimes\boldsymbol{\sigma}}$ of $\dot{V}_0^{\otimes\boldsymbol{\sigma}}$, which embeds naturally into $\widehat{V}^{\otimes\boldsymbol{\sigma}}$. Also, as in (2.10), there is a projection

$$\mathrm{pr}_0 : \widehat{V}^{\otimes\boldsymbol{\sigma}} \twoheadrightarrow \widehat{V}_0^{\otimes\boldsymbol{\sigma}}, \qquad \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}^{\boldsymbol{\sigma}} \mapsto \sum_{\boldsymbol{b}\in\mathbf{B}_0} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}^{\boldsymbol{\sigma}},$$

which is left inverse to the inclusion $\mathrm{in}_0 : \widehat{V}_0^{\otimes\boldsymbol{\sigma}} \hookrightarrow \widehat{V}^{\otimes\boldsymbol{\sigma}}$.

For $\mathfrak{sp}_{2\infty}$, we define the completion $\widehat{V}^{\otimes n}$ of $\dot{V}^{\otimes n}$ in an analogous way, replacing the $\mathfrak{sl}_\infty$-Bruhat order by the $\mathfrak{sp}_{2\infty}$-Bruhat order. So it is the $\mathbb{Q}(q)$-vector space consisting of formal linear combinations of the form $\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}$ whose support is contained in a finite union of sets of the form $\{\boldsymbol{b} \in \mathbf{B} \mid \mathbf{wt}(\boldsymbol{b}) \succeq \boldsymbol{\beta}\}$ for $\boldsymbol{\beta} \in P^n$ (working with the inverse dominance order for $\mathfrak{sp}_{2\infty}$). Just like in [BD2, Lemma 8.1], the action of $U_q\mathfrak{sp}_{2\infty}$ on $\dot{V}^{\otimes n}$ extends to an action on $\widehat{V}^{\otimes n}$, and the completion still splits as the direct sum of its weight spaces. The $U_q\mathfrak{sp}_{2k}$-module $\dot{V}_k^{\otimes n}$ embeds naturally into $\dot{V}^{\otimes n}$, hence, its completion $\widehat{V}^{\otimes n}$. As in (2.13), we also have the projection

$$\mathrm{pr}_k : \widehat{V}^{\otimes n} \twoheadrightarrow \dot{V}_k^{\otimes n}, \qquad \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}} \mapsto \sum_{\boldsymbol{b}\in\mathbf{B}_k} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}},$$

which is left inverse to the inclusion $\mathrm{in}_k : \dot{V}_k^{\otimes n} \hookrightarrow \widehat{V}^{\otimes n}$.

The projection (2.15) carries over to the present setting too: there is a $U_q\mathfrak{sl}_{+\infty}$-module homomorphism

$$\mathrm{pr}_{\boldsymbol{\sigma}} : \widehat{V}^{\otimes n} \twoheadrightarrow \widehat{V}_0^{\otimes\boldsymbol{\sigma}}, \qquad \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}} \mapsto \sum_{\boldsymbol{b}\in\mathbf{B}_{\boldsymbol{\sigma}}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}'}^{\boldsymbol{\sigma}}$$

for $\boldsymbol{\sigma} \in \{\pm\}^n$. It is left inverse to $\mathrm{in}_{\boldsymbol{\sigma}} : \widehat{V}_0^{\otimes\boldsymbol{\sigma}} \hookrightarrow \widehat{V}^{\otimes n}, \sum_{\boldsymbol{b}\in\mathbf{B}_{\boldsymbol{\sigma}}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}'}^{\boldsymbol{\sigma}} \mapsto \sum_{\boldsymbol{b}\in\mathbf{B}_{\boldsymbol{\sigma}}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}$.

The key point now is that there are canonical bar involutions on each of the spaces $\widehat{V}^{\otimes\boldsymbol{\sigma}}, \widehat{V}_0^{\otimes\boldsymbol{\sigma}}, \widehat{V}^{\otimes n}$ and $\dot{V}_k^{\otimes n}$, which we'll denote by ψ, ψ_0, ψ and ψ_k, respectively. Each one is antilinear with respect to the field automorphism $\mathbb{Q}(q) \to \mathbb{Q}(q), q \mapsto q^{-1}$, it preserves weight spaces, and it commutes with all $\dot{f}_i$ and $\dot{e}_i$. The construction in finite rank is explained in [L, §27.3.1] using the quasi-R-matrix Θ; note for this due to our different choice of Δ compared to [L] that Lusztig's v is our q^{-1}. The approach in infinite rank is essentially the same; one needs the completion so that the infinite sums that arise still make sense. In the next paragraph, we go through the details of the definition of $\psi : \widehat{V}^{\otimes n} \to \widehat{V}^{\otimes n}$ in the case of $\mathfrak{sp}_{2\infty}$. The constructions for $\mathfrak{sl}_\infty$ and $\mathfrak{sl}_{+\infty}$ are entirely analogous; see also [BD2, Lemma 8.2].

So consider $\widehat{V}^{\otimes n}$. Proceeding by induction on n, we set $\psi(\dot{v}_j) = \dot{v}_j$ for each $j \in \mathbb{Z}$. For $n > 1$, we assume that the analog $\bar{\psi}$ of ψ on the space $\widehat{V}^{\otimes(n-1)}$ has already been defined by induction. Letting $\bar{\boldsymbol{b}}$ denote the $(n-1)$-tuple $(b_1, \dots, b_{n-1})$, we define ψ on

$\widehat{V}^{\otimes n}$ by setting

$$\psi\left(\sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}\right) := \sum_{\boldsymbol{b}\in\mathbf{B}} p_{\boldsymbol{b}}(q^{-1})\,\Theta\big(\bar{\psi}(\dot{v}_{\bar{\boldsymbol{b}}})\otimes \dot{v}_{b_n}\big). \tag{2.20}$$

To better understand this expression, recall that the quasi-R-matrix Θ is a formal sum of terms Θ_β for $\beta \in \bigoplus_{i\in I}\mathbb{N}\,\alpha_i$, with $\Theta_0 = 1$ and $\Theta_\beta \in (U_q^-\mathfrak{sp}_{2\infty})_{-\beta}\otimes(U_q^+\mathfrak{sp}_{2\infty})_\beta$. The only monomials in the generators of $U_q^+\mathfrak{sp}_{2\infty}$ that are non-zero on $\dot{v}_j$ are of the form $\dot{e}_{|i|}\dot{e}_{|i+1|}\cdots\dot{e}_{|j-1|}$ for integers $i\leq j$. Hence, for any $v \in \widehat{V}^{\otimes(n-1)}$ and $j\in\mathbb{Z}$, we have that

$$\Theta\,(v\otimes\dot{v}_j) = v\otimes\dot{v}_j + \sum_{i<j}(\Theta_{i,j}v)\otimes\dot{v}_i \tag{2.21}$$

for $\Theta_{i,j}\in(U_q^-\mathfrak{sp}_{2\infty})_{-(\alpha_{|i|}+\alpha_{|i+1|}+\cdots+\alpha_{|j-1|})}$. Each $\Theta_{i,j}$ lies in Lusztig's $\mathbb{Z}[q,q^{-1}]$-form for $U_q^-\mathfrak{sp}_{2\infty}$ by the integrality of the quasi-R-matrix established in [L, Corollary 24.1.6]. Applying these remarks to (2.20) and using induction, we deduce $\psi(\dot{v}_{\boldsymbol{b}})$ equals $\dot{v}_{\boldsymbol{b}}$ plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of $\dot{v}_{\boldsymbol{a}}$'s for $\boldsymbol{a}\succ\boldsymbol{b}$, which is a well-defined element of $\widehat{V}^{\otimes n}$. The formula (2.20) also makes sense for arbitrary sums $\sum_{\boldsymbol{b}\in\mathbf{B}}p_{\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{b}}$ due to the interval-finiteness of the inverse dominance order on P^n. Finally, to see that ψ commutes with the actions of all $\dot{f}_i$ and $\dot{e}_i$, and that it is an involution, one argues as in [L, §27.3.1].

As the following lemma shows, the various bar involutions we have defined are closely related.

Lemma 2.8. *The following diagrams commute:*

$$\begin{array}{ccc} \widehat{V}_0^{\otimes\boldsymbol{\sigma}} & \xrightarrow{\psi_0} & \widehat{V}_0^{\otimes\boldsymbol{\sigma}} \\ {\scriptstyle\mathrm{in}_0}\big\downarrow & & \big\uparrow{\scriptstyle\mathrm{pr}_0} \\ \widehat{V}^{\otimes\boldsymbol{\sigma}} & \xrightarrow[\psi]{} & \widehat{V}^{\otimes\boldsymbol{\sigma}} \end{array}, \qquad \begin{array}{ccc} \dot{V}_k^{\otimes n} & \xrightarrow{\psi_k} & \dot{V}_k^{\otimes n} \\ {\scriptstyle\mathrm{in}_k}\big\downarrow & & \big\uparrow{\scriptstyle\mathrm{pr}_k} \\ \widehat{V}^{\otimes n} & \xrightarrow[\psi]{} & \widehat{V}^{\otimes n} \end{array}, \qquad \begin{array}{ccc} \widehat{V}_0^{\otimes\boldsymbol{\sigma}} & \xrightarrow{\psi_0} & \widehat{V}_0^{\otimes\boldsymbol{\sigma}} \\ {\scriptstyle\mathrm{in}_{\boldsymbol{\sigma}}}\big\downarrow & & \big\uparrow{\scriptstyle\mathrm{pr}_{\boldsymbol{\sigma}}} \\ \widehat{V}^{\otimes n} & \xrightarrow[\psi]{} & \widehat{V}^{\otimes n} \end{array}.$$

Proof. In each case, this follows because the quasi-R-matrix Θ used to define the bottom map is a sum of the form $\sum_\beta\Theta_\beta$ for β in the positive root lattice of $\mathfrak{sl}_\infty$ or $\mathfrak{sp}_{2\infty}$, while the quasi-R-matrix used to define the top map is a sum of the same Θ_β's for β taken from the positive root lattice of the subalgebra $\mathfrak{sl}_{+\infty}$ or $\mathfrak{sp}_{2k}$. □

Now we can introduce the *canonical basis* for each of our completed tensor spaces. In each case, the bar involution maps the monomial basis vector indexed by $\boldsymbol{b}$ to itself plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of monomial basis vectors indexed by strictly larger $\boldsymbol{a}$'s in the appropriate Bruhat order. Then we apply "Lusztig's Lemma" as in the proof of [L, Theorem 27.3.2]: the canonical basis vector indexed by $\boldsymbol{b}$ is the unique bar-invariant vector that is equal to the monomial basis vector indexed by $\boldsymbol{b}$ modulo a $q\,\mathbb{Z}[q]$-linear combination of other monomial basis vectors. Our notation for the canonical basis in each case is explained in the next two paragraphs.

For $\mathfrak{sl}_\infty$, we denote the canonical basis for $\widehat{V}^{\otimes\boldsymbol{\sigma}}$ as just defined by $\{\dot{c}_{\boldsymbol{b}}^{\boldsymbol{\sigma}}\mid\boldsymbol{b}\in\mathbf{B}\}$. So, $\dot{c}_{\boldsymbol{b}}^{\boldsymbol{\sigma}}$ is the unique vector fixed by ψ such that

$$\dot{c}_{\boldsymbol{b}}^{\boldsymbol{\sigma}} = \sum_{\boldsymbol{a}\in\mathbf{B}} d_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\sigma}}(q)\dot{v}_{\boldsymbol{a}}^{\boldsymbol{\sigma}} \tag{2.22}$$

for polynomials $d_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\sigma}}(q)$ with $d_{\boldsymbol{b},\boldsymbol{b}}^{\boldsymbol{\sigma}}(q)=1$, $d_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\sigma}}(q)=0$ unless $\boldsymbol{a}\succeq\boldsymbol{b}$, and $d_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\sigma}}(q)\in q\,\mathbb{Z}[q]$ if $\boldsymbol{a}\succ\boldsymbol{b}$. These polynomials have a natural representation theoretic interpretation discussed in detail in [BLW, §5.9]. They are some finite type A parabolic Kazhdan-Lusztig polynomials (suitably normalized), hence, all of their coefficients are non-negative. Moreover, each $\dot{c}_{\boldsymbol{b}}^{\boldsymbol{\sigma}}$ is always a *finite* sum of $\dot{v}_{\boldsymbol{a}}^{\boldsymbol{\sigma}}$'s, i.e. $\dot{c}_{\boldsymbol{b}}^{\boldsymbol{\sigma}}\in\dot{V}^{\otimes\boldsymbol{\sigma}}$ before completion. We will not introduce any new notation for the canonical basis of $\widehat{V}_0^{\otimes\boldsymbol{\sigma}}$ in the $\mathfrak{sl}_{+\infty}$-case, because by the first diagram from Lemma 2.8 it is simply the projection $\{\mathrm{pr}_0\dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}_0\}$ of the basis just defined.

Moving on to our notation for $\mathfrak{sp}_{2\infty}$, the canonical basis for $\widehat{V}^{\otimes n}$ is $\{\dot{c}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$. We have that

$$\dot{c}_{\boldsymbol{b}} = \sum_{\boldsymbol{a}\in\mathbf{B}} d_{\boldsymbol{a},\boldsymbol{b}}(q)\dot{v}_{\boldsymbol{a}} \tag{2.23}$$

for polynomials $d_{\boldsymbol{a},\boldsymbol{b}}(q) \in \mathbb{Z}[q]$ with $d_{\boldsymbol{b},\boldsymbol{b}}(q) = 1$, $d_{\boldsymbol{a},\boldsymbol{b}}(q) = 0$ unless $\boldsymbol{a} \succeq \boldsymbol{b}$, and $d_{\boldsymbol{a},\boldsymbol{b}}(q) \in q\,\mathbb{Z}[q]$ if $\boldsymbol{a} \succ \boldsymbol{b}$. Unlike in the previous paragraph, the polynomials $d_{\boldsymbol{a},\boldsymbol{b}}(q)$ may have negative coefficients; see Example 2.12 below. Consequently, it is conceivable that some $\dot{c}_{\boldsymbol{b}}$'s might fail to be finite sums of $\dot{v}_{\boldsymbol{a}}$'s, but this seems unlikely to us. In view of the second diagram from Lemma 2.8, the canonical basis for $\dot{V}_k^{\otimes n}$ is the projection $\{\mathrm{pr}_k\dot{c}_{\boldsymbol{b}} \,|\, \boldsymbol{b} \in \mathbf{B}_k\}$.

The following lemma is an equivalent formulation of [CKW, Proposition 4.1].

Lemma 2.9. *For $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$, we have that $\mathrm{pr}_{\boldsymbol{\sigma}}\dot{c}_{\boldsymbol{b}} = \mathrm{pr}_0\dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}'}$. Hence, $d_{\boldsymbol{a},\boldsymbol{b}}(q) = d^{\boldsymbol{\sigma}}_{\boldsymbol{a}',\boldsymbol{b}'}(q)$ for all $\boldsymbol{a}, \boldsymbol{b} \in B_{\boldsymbol{\sigma}}$.*

Proof. As $\mathrm{pr}_{\boldsymbol{\sigma}}\dot{v}_{\boldsymbol{b}} = \mathrm{pr}_0\dot{v}^{\boldsymbol{\sigma}}_{\boldsymbol{b}'}$, this follows using the third diagram from Lemma 2.8. □

The vectors $c^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ and $c_{\boldsymbol{b}}$ displayed in the table at the beginning of the subsection refer to the specializations of $\dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}}$ and $\dot{c}_{\boldsymbol{b}}$ at $q = 1$.

2.5. **An algorithm.** In [BD2, §8], we described an algorithm to compute the canonical basis $\{\dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ for the $U_q\mathfrak{sl}_\infty$-module $\widehat{V}^{\otimes\boldsymbol{\sigma}}$. In this subsection, we work instead with $U_q\mathfrak{sp}_{2\infty}$, and describe an analogous algorithm to compute the canonical basis $\{\dot{c}_{\boldsymbol{b}} \,|\, \boldsymbol{b} \in \mathbf{B}\}$ for $\widehat{V}^{\otimes n}$. The algorithm goes by induction on n. In case $n = 1$, we have that $\dot{c}_{\boldsymbol{b}} = \dot{v}_{\boldsymbol{b}}$ always. If $n > 1$, we begin by recursively computing $\dot{c}_{\bar{\boldsymbol{b}}} \in \widehat{V}^{\otimes(n-1)}$, where $\bar{\boldsymbol{b}}$ denotes $(b_1, \dots, b_{n-1})$ as usual. It is a linear combination of $\dot{v}_{\bar{\boldsymbol{a}}}$'s for $\bar{\boldsymbol{a}} \succeq \bar{\boldsymbol{b}}$. Then we define j to be the greatest integer such that $j \leq b_n$, and $j \leq -|a_r|$ for all $1 \leq r < n$ and all tuples $\bar{\boldsymbol{a}} = (a_1, \dots, a_{n-1})$ such that $\dot{v}_{\bar{\boldsymbol{a}}}$ occurs with non-zero coefficient in the expansion of $\dot{c}_{\bar{\boldsymbol{b}}}$.

Lemma 2.10. *In the above notation, we have that $\Theta\,(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j) = \dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j$.*

Proof. As in (2.21), we have that $\Theta\,(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j) = \dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j + \sum_{i<j}(\Theta_{i,j}\dot{c}_{\bar{\boldsymbol{b}}}) \otimes \dot{v}_i$, where $\Theta_{i,j}$ is a linear combination of non-trivial monomials in $\dot{f}_{|j-1|}, \dot{f}_{|j-2|}, \dots, \dot{f}_{|i|}$. By the definition of j, all of these generators act as zero on $\dot{c}_{\bar{\boldsymbol{b}}}$. □

Lemma 2.10 shows that the vector $\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j \in \widehat{V}^{\otimes n}$ is fixed by ψ. Hence, so too is $\dot{f}_{|b_n-1|} \cdots \dot{f}_{|j+1|}\dot{f}_{|j|}\,(\dot{c}_{\bar{\boldsymbol{b}}} \otimes \dot{v}_j)$. By Lemma 2.6, this new vector equals $\dot{v}_{\boldsymbol{b}}$ plus a $\mathbb{Z}[q, q^{-1}]$-linear combination of $\dot{v}_{\boldsymbol{a}}$'s for $\boldsymbol{a} \succ \boldsymbol{b}$. If all but its leading coefficient lie in $q\,\mathbb{Z}[q]$, it is already the desired vector $\dot{c}_{\boldsymbol{b}}$. Otherwise, one picks $\boldsymbol{a} \succ \boldsymbol{b}$ minimal so that the $\dot{v}_{\boldsymbol{a}}$-coefficient is not in $q\,\mathbb{Z}[q]$, then subtracts a bar-invariant multiple of the recursively computed vector $\dot{c}_{\boldsymbol{a}}$ to remedy this defficiency. Continuing in this way, we finally obtain a bar-invariant vector with all of the required properties to be $\dot{c}_{\boldsymbol{b}}$.

Example 2.11. The canonical basis of $V^{\otimes 2}$ consists of the following vectors:

$$\begin{aligned}
\dot{v}_i \otimes \dot{v}_j \quad & \text{for } i \geq j \text{ with } i + j \neq 1,\\
\dot{v}_i \otimes \dot{v}_j + q\dot{v}_j \otimes \dot{v}_i \quad & \text{for } i < j \text{ with } i + j \neq 1,\\
\dot{v}_i \otimes \dot{v}_{1-i} + q\dot{v}_{1+i} \otimes \dot{v}_{-i} \quad & \text{for } i > 0,\\
\dot{v}_i \otimes \dot{v}_{1-i} + q\dot{v}_{i+1} \otimes \dot{v}_{-i} + q\dot{v}_{-i} \otimes \dot{v}_{i+1} + q^2\dot{v}_{1-i} \otimes \dot{v}_i \quad & \text{for } i < 0,\\
\dot{v}_0 \otimes \dot{v}_1 + q^2\dot{v}_1 \otimes \dot{v}_0. &
\end{aligned}$$

We refer the reader to `http://pages.uoregon.edu/brundan/papers/C.gap` for some GAP code implementing this algorithm. Using it, we have independently verified the next examples, which were discovered originally by Tsuchioka:

Example 2.12. If $\boldsymbol{a} = (1, 1, 0, 1, 0, 0)$ and $\boldsymbol{b} = (-1, 2, -1, 2, -1, 2)$ then

$$d_{\boldsymbol{a},\boldsymbol{b}}(q) = q^7 + 4q^5 + 3q^3 - q.$$

If $\boldsymbol{a} = (1, -1, 2, -1, 2, 0)$ and $\boldsymbol{b} = (-1, -2, 3, -2, 3, 2)$ then

$$d_{\boldsymbol{a},\boldsymbol{b}}(q) = 8q^3 - q.$$

These examples demonstrate that positivity fails in this situation.

2.6. **Crystals.** To conclude the section, we recall the explicit combinatorial description of the crystal associated to the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$. Later in the article, we will give a representation-theoretic interpretation of this structure; see §4.3. The case of the $\mathfrak{sl}_\infty$-module $V^{\otimes \boldsymbol{\sigma}}$ can be treated entirely similarly on replacing $i\text{-}\mathbf{sig}(\boldsymbol{b})$ with $i\text{-}\mathbf{sig}^{\boldsymbol{\sigma}}(\boldsymbol{b})$; its representation-theoretic significance is discussed e.g. in [BLW, §2.10].

The set underlying the crystal that we need is the set $\mathbf{B}$ that parametrizes our various bases for $V^{\otimes n}$. Its weight decomposition $\mathbf{B} = \bigsqcup_{\gamma \in P} \mathbf{B}_\gamma$ is defined by setting

$$\mathbf{B}_\gamma := \left\{ \boldsymbol{b} \in \mathbf{B} \;\middle|\; |\mathbf{wt}(\boldsymbol{b})| = \gamma \right\}.$$

We need to introduce crystal operators

$$\tilde{f}_i : \mathbf{B}_\gamma \to \mathbf{B}_{\gamma - \alpha_i} \sqcup \{\varnothing\}, \qquad \tilde{e}_i : \mathbf{B}_\gamma \to \mathbf{B}_{\gamma + \alpha_i} \sqcup \{\varnothing\}$$

for each $\gamma \in P$ and $i \in I$. These arise naturally by iterating Kashiwara's tensor product rule, and may be computed as follows. Take $\boldsymbol{b} \in \mathbf{B}_\gamma$. Starting from the *$i$-signature* $i\text{-}\mathbf{sig}(\boldsymbol{b})$ from (2.12), we define the *reduced i-signature* by replacing pairs of entries of the form `ef` (possibly separated by $\bullet$'s) with $\bullet$'s, until all `e` entries appear to the right of the entries `f`. Then define $\tilde{f}_i \boldsymbol{b}$ to be $\boldsymbol{b} + \boldsymbol{d}_r$ if the rightmost `f` in the reduced i-signature appears in position r, or $\varnothing$ if there are no `f`'s remaining in the reduced i-signature. Similarly, define $\tilde{e}_i \boldsymbol{b}$ to be $\boldsymbol{b} - \boldsymbol{d}_s$ if the leftmost `e` in the reduced i-signature appears in position s, or $\varnothing$ if there are no `e`'s present.

Example 2.13. Take $\boldsymbol{b} = (2, -1, -1, 4, -2, -2, 3, 2, -2)$. The 2-signature of $\boldsymbol{b}$ is the tuple $(\mathtt{f}, \mathtt{e}, \mathtt{e}, \bullet, \mathtt{f}, \mathtt{f}, \mathtt{e}, \mathtt{f}, \mathtt{f})$. The reduced 2-signature is $(\mathtt{f}, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \mathtt{f})$. Hence, $\tilde{f}_2 \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{d}_9 = (2, -1, -1, 4, -2, -2, 3, 2, -1)$ and $\tilde{e}_2 \boldsymbol{b} = \varnothing$.

Let $\mathbf{B}^\circ$ denote the set of all elements of $\mathbf{B}$ which can be obtained from $\boldsymbol{z} = (0, \dots, 0)$ by applying a sequence of crystal operators. In other words, $\mathbf{B}^\circ$ is the *connected component* of the crystal $\mathbf{B}$ containing $\boldsymbol{z}$.

Lemma 2.14. *We have that $\boldsymbol{b} \in \mathbf{B}^\circ$ if and only if $\boldsymbol{b}$ is* antidominant *in the sense that $b_1 \leq \cdots \leq b_n$.*

Proof. For the forward implication, we observe that whenever $\boldsymbol{a} \in \mathbf{B}$ is antidominant, then so are $\tilde{f}_i \boldsymbol{a}$ and $\tilde{e}_i \boldsymbol{a}$. For instance, to check that the entries of $\tilde{f}_i \boldsymbol{a}$ are weakly increasing, we have that $\tilde{f}_i \boldsymbol{a} = \boldsymbol{a} + \boldsymbol{d}_r$ where r is the maximal index for which the reduced i-signature of $\boldsymbol{a}$ contains an `f`. We need to see that $a_r < a_{r+1}$. Well, otherwise, we would have that $a_r = a_{r+1}$, in which case $i\text{-sig}_r(\boldsymbol{a}) = i\text{-sig}_{r+1}(\boldsymbol{a}) = \mathtt{f}$. Because we cancel `ef` pairs (and not `fe`!) it would then follow that the reduced i-signature of $\boldsymbol{a}$ contains a `f` in its $(r + 1)$th entry, which contradicts our assumption about r.

Conversely, suppose that $b_1 \leq \cdots \leq b_n$. For every index r, define a monomial

$$\tilde{x}_r := \begin{cases} \tilde{f}_{b_r - 1} \cdots \tilde{f}_1 \tilde{f}_0 & \text{if } b_r \geq 0 \\ \tilde{e}_{-b_r} \cdots \tilde{e}_2 \tilde{e}_1 & \text{if } b_r < 0. \end{cases}$$

Letting t denote the maximal index for which $b_t < 0$, taking $t := 0$ in case $b_r \geq 0$ for all r, one then checks that $\widetilde{x}_t \cdots \widetilde{x}_2 \widetilde{x}_1 \widetilde{x}_{t+1} \widetilde{x}_{t+2} \cdots \widetilde{x}_n \boldsymbol{z} = \boldsymbol{b}$. □

Similarly, one can make the subset $\mathbf{B}_k \subset \mathbf{B}$ into an $\mathfrak{sp}_{2k}$-crystal. The connected component of $\mathbf{B}_k$ containing $\boldsymbol{z}$ is $\mathbf{B}_k^\circ := \mathbf{B}_k \cap \mathbf{B}^\circ$. It is also the connected component containing

$$\boldsymbol{z}_k := (1-k, \dots, 1-k) \in \mathbf{B}_k\,. \tag{2.24}$$

This is significant because the vector $v_{\boldsymbol{z}_k}$ is a highest weight vector in $V_k^{\otimes n}$. Its weight $|\,\mathbf{wt}(\boldsymbol{z}_k)|$ is $-n\varepsilon_{k-1}$.

3. Category $\mathcal{O}$

Next, we introduce the supercategory $s\mathcal{O}$ of representations of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ that is the main object of study of this article. Then, we prove our main categorification theorem, which asserts that $s\mathcal{O}$ splits as $\mathcal{O} \oplus \Pi\mathcal{O}$ with $\mathcal{O}$ being a tensor product categorification of the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$. The proof of this theorem is similar to the proof of a similar assertion for type A blocks from [BD2].

3.1. **Superalgebra.** We will work from now on over the ground field $\mathbb{C}$. A *vector superspace* is a $\mathbb{Z}/2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We denote the parity of a homogeneous vector $v \in V$ by $|v| \in \mathbb{Z}/2$. Any $v \in V$ has a canonical decomposition $v = v_{\bar{0}} + v_{\bar{1}}$ with $|v_p| = p$. Let $\underline{\mathcal{SV}ec}$ be the category of vector superspaces and parity-preserving linear maps. It is symmetric monoidal with braiding $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Then, we make the following definitions following [BE]:

- A *supercategory* is a $\underline{\mathcal{SV}ec}$-enriched category.
- A *superfunctor* is a $\underline{\mathcal{SV}ec}$-enriched functor.
- A *supernatural transformation* $\eta : F \Rightarrow G$ between superfunctors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\eta_M = \eta_{M,\bar{0}} + \eta_{M,\bar{1}} : FM \to GM$ for each $M \in \operatorname{ob} \mathcal{C}$, such that $\eta_{N,p} \circ Ff = (-1)^{|f|p} Gf \circ \eta_{M,p}$ for every homogeneous morphism $f : M \to N$ in $\mathcal{C}$ and each $p \in \mathbb{Z}/2$.

For any supercategory $\mathcal{C}$, there is a supercategory $\mathcal{E}nd(\mathcal{C})$ consisting of superfunctors and supernatural transformations. It is a (strict) *monoidal supercategory* in the sense of [BE, Definition 1.4]. A *superequivalence* between supercategories $\mathcal{C}$ and $\mathcal{D}$ is a superfunctor $F : \mathcal{C} \to \mathcal{D}$ such that there exists another superfunctor $G : \mathcal{D} \to \mathcal{C}$ with $GF : \mathcal{C} \to \mathcal{C}$ and $FG : \mathcal{D} \to \mathcal{D}$ being evenly isomorphic to identity functors.

Given any $\mathbb{C}$-linear category $\mathcal{C}$, one can form the supercategory $\mathcal{C} \oplus \Pi\mathcal{C}$ with objects being pairs (V_1, V_2) of objects from $\mathcal{C}$, and morphisms $(V_1, V_2) \to (W_1, W_2)$ that are 2×2 matrices $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ of morphisms $f_{ij} \in \operatorname{Hom}_{\mathcal{C}}(W_j, V_i)$. The $\mathbb{Z}/2$-grading is defined so $f_{\bar{0}} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$ and $f_{\bar{1}} = \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix}$. We say that a supercategory *splits* if it is superequivalent to a supercategory of this form.

Here is the basic example to keep in mind. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra. There is a supercategory A-$\mathcal{SM}od$ consisting of left A-supermodules. Even morphisms in A-$\mathcal{SM}od$ are parity-preserving linear maps such that $f(av) = af(v)$ for all $a \in A, v \in M$; odd morphisms are parity-reversing linear maps such that $f(av) = (-1)^{|a|} af(v)$ for homogeneous a. If A is purely even, i.e $A = A_{\bar{0}}$, then the category A-$\mathcal{SM}od$ obviously splits as A-$\mathcal{M}od \oplus \Pi(A$-$\mathcal{M}od)$. In general, A-$\mathcal{SM}od$ splits if and only if A is Morita superequivalent to a purely even superalgebra.

3.2. **Supercategory $s\mathcal{O}$.** We assume henceforth that we have fixed $n \geq 1$, and set $m := \lceil n/2 \rceil$. We are interested in a certain supercategory of representations of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$, that is, the subalgebra of the general linear Lie superalgebra $\mathfrak{gl}_{n|n}(\mathbb{C})$ consisting of matrices of the form $\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right)$. In order to unify our treatment of odd versus even n as much as possible, we will adopt the same trick as used in [BD2], setting

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} := \begin{cases} \mathfrak{q}_n(\mathbb{C}) & \text{if } n \text{ is even,} \\ \mathfrak{q}_n(\mathbb{C}) \oplus \mathfrak{q}_1(\mathbb{C}) & \text{if } n \text{ is odd.} \end{cases}$$

The point of the additional $\mathfrak{q}_1(\mathbb{C})$ in case n is odd is that it adjoins an extra odd involution to the supercategory $s\mathcal{O}$ to be defined shortly. In language from the introduction of [BD2], this amounts to working with the *Clifford twist* of the supercategory that one would naturally define without this extra factor.

It will sometimes be helpful to identify $\mathfrak{g}$ with a subalgebra of $\widehat{\mathfrak{g}} := \mathfrak{gl}_{2m|2m}(\mathbb{C})$. Let $x_{r,s}$ be the usual rs-matrix unit in $\widehat{\mathfrak{g}}$, which is even if $1 \leq r, s \leq 2m$ or $2m+1 \leq r, s \leq 4m$, and odd otherwise. Introduce the matrices

$$e_{r,s} := x_{r,s} + x_{2m+r,2m+s}, \qquad e'_{r,s} := x_{r,2m+s} + x_{2m+r,s}, \tag{3.1}$$

$$f_{r,s} := x_{r,s} - x_{2m+r,2m+s}, \qquad f'_{r,s} := x_{r,2m+s} - x_{2m+r,s}, \tag{3.2}$$

$$d_r := e_{r,r}, \qquad d'_r := e'_{r,r}. \tag{3.3}$$

Then $\mathfrak{g}$ is the subalgebra of $\widehat{\mathfrak{g}}$ with basis $\{e_{r,s}, e'_{r,s} \,|\, 1 \leq r, s \leq n\}$ together with $\{d_{2m}, d'_{2m}\}$ if n is odd. The matrices $f_{r,s}$, $f'_{r,s}$ are elements of $\widehat{\mathfrak{g}}$ but not $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ be the Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{1}}$ with basis $\{d_r, d'_r \mid 1 \leq r \leq 2m\}$. Also let $\delta_1, \dots, \delta_{2m}$ be the basis for $\mathfrak{h}_{\bar{0}}^*$ that is dual to the basis $d_1, \dots, d_{2m}$ for $\mathfrak{h}_0$. Finally, let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the matrices $\{e_{r,s}, e'_{r,s} \mid 1 \leq r < s \leq n\}$.

As in the previous section, $\mathbf{B}$ will denote the set $\mathbb{Z}^n$ of n-tuples $\boldsymbol{b} = (b_1, \dots, b_n)$ of integers. For $\boldsymbol{b} \in \mathbf{B}$, let $\lambda_{\boldsymbol{b}} \in \mathfrak{h}_{\bar{0}}^*$ be the weight defined from

$$\lambda_{\boldsymbol{b}} := \begin{cases} \displaystyle\sum_{r=1}^{n} (b_r - \tfrac{1}{2})\delta_r & \text{if } n \text{ is even,} \\ \displaystyle\sum_{r=1}^{n} (b_r - \tfrac{1}{2})\delta_r + \delta_{2m} & \text{if } n \text{ is odd.} \end{cases} \tag{3.4}$$

Then we define $s\mathcal{O}$ to be the supercategory consisting of all $\mathfrak{g}$-supermodules M such that

- M is finitely generated over $\mathfrak{g}$;
- M is locally finite-dimensional over $\mathfrak{b}$;
- M is semisimple over $\mathfrak{h}_{\bar{0}}$ with all weights of the form $\lambda_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$.

We denote the usual *parity switching functor* by $\Pi : s\mathcal{O} \to s\mathcal{O}$. This sends a supermodule M to the same vector space viewed as a superspace with $(\Pi M)_{\bar{0}} := M_{\bar{1}}$ and $(\Pi M)_{\bar{1}} := M_{\bar{0}}$, and new action defined from $x \cdot v := (-1)^{|x|} xv$.

Let $\underline{s\mathcal{O}}$ be the underlying $\mathbb{C}$-linear category consisting of all of the same objects as $s\mathcal{O}$, but only the even morphisms. The category $\underline{s\mathcal{O}}$ is obviously Abelian. In fact, it is *Schurian* in following sense; this follows as in [B2, Lemma 2.3].

Definition 3.1. A $\mathbb{C}$-linear category is *Schurian* if it is Abelian, all of its objects are of finite length, the endomorphism algebras of the irreducible objects are one-dimensional, and there are enough projectives and injectives.

We proceed to introduce the Verma supermodules in $s\mathcal{O}$. We need to do this rather carefully in order to be able to distinguish a Verma supermodule from its parity flip. Since we reserve the letter i for elements of the set I as in the previous section, we'll denote the canonical element of $\mathbb{C}$ by $\sqrt{-1}$. We also need to pick some distinguished square roots for each element of the subset $\mathbb{Z}+\frac{1}{2}$ of $\mathbb{C}$ such that

$$\sqrt{i+\tfrac{1}{2}}\sqrt{i-\tfrac{1}{2}} = \sqrt{-i+\tfrac{1}{2}}\sqrt{-i-\tfrac{1}{2}} \tag{3.5}$$

for each $i \in \mathbb{N}$. For example, this can be done by letting $\sqrt{i+\frac{1}{2}}$ denote the usual positive square root when $i \geq 0$, then setting $\sqrt{i+\frac{1}{2}} := (-1)^{i+1}\sqrt{-1}\sqrt{-i-\frac{1}{2}}$ if $i < 0$.

Lemma 3.2. *For each $\boldsymbol{b} \in \mathbf{B}$, there is a unique (up to even isomorphism) irreducible $\mathfrak{h}$-supermodule $V(\boldsymbol{b})$ of weight $\lambda_{\boldsymbol{b}}$ such that the element $d_1' \cdots d_{2m}' \in U(\mathfrak{g})$ acts on all even (resp. odd) vectors in $V(\boldsymbol{b})$ by multiplication by the scalar $c_{\boldsymbol{b}}$ (resp. $-c_{\boldsymbol{b}}$), where*

$$c_{\boldsymbol{b}} := (\sqrt{-1})^m \sqrt{b_1 - \tfrac{1}{2}} \cdots \sqrt{b_n - \tfrac{1}{2}}. \tag{3.6}$$

Moreover, any $\mathfrak{h}$-supermodule of weight $\lambda_{\boldsymbol{b}}$ splits as a direct sum of copies of $V(\boldsymbol{b})$ and its parity flip $\Pi V(\boldsymbol{b})$.

Proof. This is similar to [BD2, Lemma 2.1]. The supermodule $V(\boldsymbol{b})$ may be constructed explicitly as there as an irreducible supermodule over a Clifford superalgebra of rank $2m$; in particular, $\dim V(\boldsymbol{b}) = 2^m$. □

For each $\boldsymbol{b} \in \mathbf{B}$, we define the *Verma supermodule*

$$M(\boldsymbol{b}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b}), \tag{3.7}$$

viewing $V(\boldsymbol{b})$ as a $\mathfrak{b}$-supermodule via the natural surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$. It is obvious that this belongs to $s\mathcal{O}$. Here we list some more basic facts.

- The Verma supermodule $M(\boldsymbol{b})$ has a unique irreducible quotient $L(\boldsymbol{b})$. The supermodules $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ give a complete set of representatives for the isomorphism classes of irreducible objects in $s\mathcal{O}$. Moreover, $L(\boldsymbol{b})$ is not evenly isomorphic to its parity flip.
- There is a duality $\star$ on $s\mathcal{O}$ such that $L(\boldsymbol{b})$ and $L(\boldsymbol{b})^\star$ are evenly isomorphic for each $\boldsymbol{b} \in \mathbf{B}$; cf. [BD2, Lemma 2.3].
- If $\boldsymbol{b}$ is both *dominant* in the sense that $b_1 \geq \cdots \geq b_n$, and *typical*, meaning that $b_r + b_s \neq 1$ for all $1 \leq r < s \leq n$, then $M(\boldsymbol{b})$ is projective; cf. [BD2, Lemma 2.4].

Let $s\mathcal{O}^\Delta$ be the full subcategory of $s\mathcal{O}$ consisting of all supermodules possessing a Verma flag, i.e. for which there is a filtration $0 = M_0 \subset \cdots \subset M_l = M$ with sections M_k/M_{k-1} that are isomorphic to Verma supermodules. As in [BD2, Lemma 2.5], the multiplicities $(M : M(\boldsymbol{b}))$ and $(M : \Pi M(\boldsymbol{b}))$ of $M(\boldsymbol{b})$ and $\Pi M(\boldsymbol{b})$ in any Verma flag of $M \in \operatorname{ob} s\mathcal{O}^\Delta$ satisfy

$$(M : M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^\star)_{\bar{0}},$$
$$(M : \Pi M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^\star)_{\bar{1}}.$$

Moreover, if M possesses a Verma flag, then so does any direct summand of M.

3.3. **Special projective superfunctors.** Let $\widehat{U}$ be the natural $\widehat{\mathfrak{g}}$-supermodule of column vectors with standard basis $u_1, \dots, u_{2m}, u_1', \dots, u_{2m}'$, so the unprimed vectors are even, the primed ones are odd. Let $\widehat{U}^*$ be its dual, with basis $\phi_1, \dots, \phi_{2m}, \phi_1', \dots, \phi_{2m}'$ that is dual to the basis $u_1, \dots, u_{2m}, u_1', \dots, u_{2m}'$. Then, let $U \subseteq \widehat{U}$ and $U^* \subseteq \widehat{U}^*$ be the $\mathfrak{g}$-supermodules with bases $u_1, \dots, u_n, u_1', \dots, u_n'$ and $\phi_1, \dots, \phi_n, \phi_1', \dots, \phi_n'$, respectively.

It is easy to see that tensoring either with U or with U^* sends supermodules in $s\mathcal{O}$ to supermodules in $s\mathcal{O}$. Hence, we have endofunctors

$$sF := U \otimes - : s\mathcal{O} \to s\mathcal{O}, \qquad sE := U^* \otimes - : s\mathcal{O} \to s\mathcal{O}. \tag{3.8}$$

The superfunctors sF and sE are both left and right adjoint to each other. The canonical adjunction making (sE, sF) into an adjoint pair is induced by the linear maps

$$U^* \otimes U \to \mathbb{C},\ \phi \otimes u \mapsto \phi(u), \qquad \mathbb{C} \to U \otimes U^*,\ 1 \mapsto \sum_{r=1}^{n} (u_r \otimes \phi_r + u'_r \otimes \phi'_r),$$

while the adjunction (sF, sE) is induced by

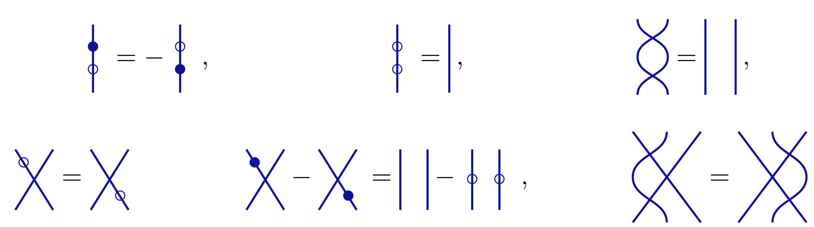
$$U \otimes U^* \to \mathbb{C},\ u \otimes \phi \mapsto (-1)^{|\phi||u|}\phi(u), \quad \mathbb{C} \to U^* \otimes U,\ 1 \mapsto \sum_{r=1}^{n} (\phi_r \otimes u_r - \phi'_r \otimes u'_r).$$

As well as these adjunctions, there are even supernatural transformations $x : sF \Rightarrow sF$ and $t : sF^2 \Rightarrow sF^2$, and an odd supernatural transformation $c : sF \Rightarrow sF$, which are defined on $M \in \operatorname{ob} s\mathcal{O}$ as follows:

- $x_M : U \otimes M \to U \otimes M$ is left multiplication by the tensor
$$\omega := \sum_{r,s=1}^{n} \left(f_{r,s} \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r}\right) \in \widehat{\mathfrak{g}} \otimes \mathfrak{g},$$
which defines a $\mathfrak{g}$-supermodule homomorphism by the proof of [BD2, Lemma 3.1];
- $t_M : U \otimes U \otimes M \to U \otimes U \otimes M$ sends $u \otimes v \otimes m \mapsto (-1)^{|u||v|} v \otimes u \otimes m$;
- $c_M : U \otimes M \to U \otimes M$ is left multiplication by $\sqrt{-1}\, z' \otimes 1$ where
$$z' := \sum_{t=1}^{n} f'_{t,t} \in \widehat{\mathfrak{g}}.$$

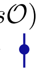
Similarly, there are supernatural transformations $x^* : sE \Rightarrow sE, t^* : sE^2 \Rightarrow sE^2$ and $c^* : sE \Rightarrow sE$: x^* and c^* are defined similarly to x and c but with an additional sign, so they are given by left multiplication by $-\omega$ and by $-\sqrt{-1}\, z' \otimes 1$, respectively; t^* is defined using the braiding on $\mathcal{SV}ec$ in exactly the same way as t. One can check that x^*, t^* and c^* are both the left and right mates of x, t and c, respectively, with respect to the adjunctions fixed in the previous paragraph; cf. [BD2, Lemma 3.6].

Definition 3.3. The (degenerate) *affine Hecke-Clifford supercategory* $\mathcal{AHC}$ is the strict monoidal supercategory with a single generating object 1, even generating morphisms  $: 1 \to 1$ and  $: 1 \otimes 1 \to 1 \otimes 1$, and an odd generating morphism  $: 1 \to 1$, subject to the following relations:



(Here, we are using the string calculus for strict monoidal supercategories as in [BE].)

The following theorem is essentially [HKS, Theorem 7.4.1]; cf. [BD2, Theorem 6.2]. It is proved by explicitly checking the relations.

Theorem 3.4. *There is a strict monoidal superfunctor $\Psi : \mathcal{AHC} \to \mathcal{E}nd(s\mathcal{O})$ sending the generating object 1 to the endofunctor sF, and the generating morphisms*  *,*  *and*  *to the supernatural transformations x, t and c, respectively.*

The superfunctor Ψ from Theorem 3.4 induces superalgebra homomorphisms

$$\Psi_d : AHC_d \to \mathrm{End}(sF^d) \tag{3.9}$$

for each $d \geq 0$, where AHC_d denotes the (degenerate) *affine Hecke-Clifford superalgebra*

$$AHC_d := \mathrm{End}_{\mathcal{AHC}}(1^{\otimes d}). \tag{3.10}$$

These superalgebras were introduced originally in [N], and can be understood algebraically as follows. Numbering the strings of a d-stringed diagram by $1, \dots, d$ from right to left, let x_r (resp. c_r) denote the element of AHC_d defined by a closed dot (resp. an open dot) on the rth string. Let t_r denote the crossing of the rth and $(r+1)$th string. The even elements $x_1, \dots, x_d$ commute, the odd elements $c_1, \dots, c_d$ satisy the relations $c_r^2 = 1$ and $c_r c_s = -c_s c_r$ $(r \neq s)$ of the rank d Clifford superalgebra C_d, and $t_1, \dots, t_{d-1}$ satisfy the same relations as the basic transpositions in the symmetric group S_d. In fact, by the basis theorem for AHC_d from [BK, §2-k], $x_1, \dots, x_d$ generate a copy of the polynomial algebra $A_d := \mathbb{C}[x_1, \dots, x_d]$ inside AHC_d, while $c_1, \dots, c_d, t_1, \dots, t_{d-1}$ generate a copy of the *Sergeev superalgebra* $HC_d := S_d \ltimes C_d$. Moreover, the natural multiplication map $HC_d \otimes A_d \to AHC_d$ is an isomorphism of vector superspaces. We note also that the multiplication in AHC_d satisfies the following:

$$f\, c_r = c_r\, c_r(f), \tag{3.11}$$

$$f\, t_r = t_r\, t_r(f) + \partial_r(f) + c_r c_{r+1} \tilde{\partial}_r(f), \tag{3.12}$$

for each $f \in A_d$. Here, the operators $c_r, t_r, \partial_r, \tilde{\partial}_r : A_d \to A_d$ are defined as follows:

- t_r is the automorphism that interchanges x_r and x_{r+1} and fixes all other generators;
- c_r is the automorphism that sends $x_r \mapsto -x_r$ and fixes all other generators;
- ∂_r is the Demazure operator $\partial_r(f) := \frac{t_r(f)-f}{x_r - x_{r+1}}$;
- $\tilde{\partial}_r$ is the twisted Demazure operator $c_{r+1} \circ \partial_r \circ c_r$.

Next, we are going to decompose sF and sE into generalized eigenspaces with respect to the endomorphisms x and x^*. The key ingredient needed to understand this is the following, whose proof is identical to that of [BD2, Lemma 3.2].

Lemma 3.5. *Suppose that* $\boldsymbol{b} \in \mathbf{B}$ *and let* $M := M(\boldsymbol{b})$.

(1) *There is a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = U \otimes M$$

with $M_t/M_{t-1} \cong M(\boldsymbol{b} + \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} + \boldsymbol{d}_t)$ *for each* $t = 1, \dots, n$. *The endomorphism* x_M *preserves this filtration, and the induced endomorphism of the subquotient* M_t/M_{t-1} *is diagonalizable with exactly two eigenvalues* $\pm\sqrt{b_t + \frac{1}{2}}\sqrt{b_t - \frac{1}{2}}$. *Its* $\sqrt{b_t + \frac{1}{2}}\sqrt{b_t - \frac{1}{2}}$*-eigenspace is evenly isomorphic to* $M(\boldsymbol{b}+\boldsymbol{d}_t)$, *while the other eigenspace is evenly isomorphic to* $\Pi M(\boldsymbol{b} + \boldsymbol{d}_t)$.

(2) *There is a filtration*

$$0 = M^n \subset \cdots \subset M^1 \subset M^0 = U^* \otimes M$$

with $M^{t-1}/M^t \cong M(\boldsymbol{b} - \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} - \boldsymbol{d}_t)$ *for each* $t = 1, \dots, n$. *The endomorphism* x^*_M *preserves this filtration, and the induced endomorphism of the subquotient* M^{t-1}/M^t *is diagonalizable with exactly two eigenvalues* $\pm\sqrt{b_t - \frac{1}{2}}\sqrt{b_t - \frac{3}{2}}$. *Its* $\sqrt{b_t - \frac{1}{2}}\sqrt{b_t - \frac{3}{2}}$*-eigenspace is evenly isomorphic to* $M(\boldsymbol{b}-\boldsymbol{d}_t)$, *while the other eigenspace is evenly isomorphic to* $\Pi M(\boldsymbol{b} - \boldsymbol{d}_t)$.

For the remainder of the section, we let I denote the set $\mathbb{N}$. In the notation from the previous section, this is the index set for the simple roots of the Kac-Moody algebra $\mathfrak{s} = \mathfrak{sp}_{2\infty}$. Let

$$J := \left\{ \pm\sqrt{i+\tfrac{1}{2}}\sqrt{i-\tfrac{1}{2}} \;\middle|\; i \in I \right\}. \tag{3.13}$$

This set is relevant due to the following lemma.

Lemma 3.6. *For any $M \in \operatorname{ob} s\mathcal{O}$, all roots of the minimal polynomials of x_M and x_M^* (computed in the finite dimensional superalgebras $\operatorname{End}_{s\mathcal{O}}(sF\,M)$ and $\operatorname{End}_{s\mathcal{O}}(sE\,M)$) belong to the set J.*

Proof. This reduces to the case that M is a Verma supermodule, when it follows from Lemma 3.5 and (3.5). □

For $j \in J$, let sF_j (resp. sE_j) be the subfunctor of sF (resp. sE) defined by letting $sF_j\,M$ (resp. $sE_j\,M$) be the generalized j-eigenspace of x_M (resp. x_M^*) for each $M \in \operatorname{ob} s\mathcal{O}$. Lemma 3.6 implies that

$$sF = \bigoplus_{j \in J} sF_j, \qquad sE = \bigoplus_{j\in J} sE_j. \tag{3.14}$$

The adjunctions (sE, sF) and (sF, sE) fixed earlier restrict to adjunctions (sE_j, sF_j) and (sF_j, sE_j) for each $j \in J$; this follows because x^* is both the left and right mate of x. Also, by Theorem 3.4, c restricts to an odd isomorphism $sF_j \overset{\sim}{\Rightarrow} sF_{-j}$ for each $j \in J$; similarly, $sE_j \cong sE_{-j}$.

Recalling (2.11), the following theorem reveals the first significant connection between combinatorics in $s\mathcal{O}$ and the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$.

Theorem 3.7. *Given $\boldsymbol{b} \in \mathbf{B}$ and $i \in I$, let $j := \sqrt{i+\frac{1}{2}}\sqrt{i-\frac{1}{2}}$. Then:*

(1) $sF_j\,M(\boldsymbol{b})$ (resp. $sF_{-j}\,M(\boldsymbol{b})$) has a multiplicity-free filtration with sections that are evenly (resp. oddly) isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b}+\boldsymbol{d}_t) \mid \text{for } 1 \le t \le n \text{ such that } i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{f}\},$$

appearing from bottom to top in order of increasing t.

(2) $sE_j\,M(\boldsymbol{b})$ (resp. $sE_{-j}\,M(\boldsymbol{b})$) has a multiplicity-free filtration with sections that are evenly (resp. oddly) isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b}-\boldsymbol{d}_t) \mid \text{for } 1 \le t \le n \text{ such that } i\text{-}\mathrm{sig}_t(\boldsymbol{b}) = \mathtt{e}\},$$

appearing from top to bottom in order of increasing t.

Proof. (1) We just need to check the statement for $sF_j\,M(\boldsymbol{b})$; the one about $sF_{-j}\,M(\boldsymbol{b})$ then follows because it is isomorphic to $sF_j\,M(\boldsymbol{b})$ via an odd isomorphism. Applying Lemma 3.5, we see that $sF_j\,M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the supermodules $M(\boldsymbol{b}+\boldsymbol{d}_t)$ for $t = 1, \dots, n$ such that $\sqrt{b_t+\frac{1}{2}}\sqrt{b_t-\frac{1}{2}} = j = \sqrt{i+\frac{1}{2}}\sqrt{i-\frac{1}{2}}$. Squaring both sides, we deduce that $b_t^2 = i^2$, hence, $b_t = \pm i$. Both cases do indeed give solutions thanks to (3.5). It remains to compare what we have proved with the definition of i-signature from (2.12).

(2) Similar. □

Finally in this subsection, we introduce a completion $\widehat{AHC}_d$ of the affine Hecke-Clifford superalgebra AHC_d from (3.10), following [KKT, Definition 5.3]. As a vector superspace, we have that

$$\widehat{AHC}_d := HC_d \otimes \widehat{A}_d \qquad \text{where} \qquad \widehat{A}_d := \bigoplus_{\boldsymbol{j} \in J^d} \mathbb{C}[[x_1 - j_1, \dots, x_d - j_d]]1_{\boldsymbol{j}}, \tag{3.15}$$

and J^d denotes the set of d-tuples $\boldsymbol{j} = j_d \cdots j_1$ of elements of J. For $h \in HC_d$ and $f \in \mathbb{C}[[x_1 - j_1, \dots, x_d - j_d]]$, we write simply $hf1_{\boldsymbol{j}}$ in place of $h \otimes f1_{\boldsymbol{j}}$. The multiplication in $\widehat{AHC}_d$ is defined so that $\widehat{A}_d$ is a subalgebra, the maps $HC_d \hookrightarrow \widehat{AHC}_d, h \mapsto h1_{\boldsymbol{j}}$ are algebra homomorphisms, and, extending (3.11)–(3.12), we have that:

$$(f1_{\boldsymbol{j}})\,(c_r 1_{\boldsymbol{j}'}) = c_r\, c_r(f) 1_{c_r(\boldsymbol{j})} 1_{\boldsymbol{j}'}, \tag{3.16}$$

$$\begin{aligned}(f1_{\boldsymbol{j}})\,(t_r 1_{\boldsymbol{j}'}) = t_r\, t_r(f) 1_{t_r(\boldsymbol{j})} 1'_{\boldsymbol{j}} + \frac{t_r(f)1_{t_r(\boldsymbol{j})} - f1_{\boldsymbol{j}}}{x_r - x_{r+1}} 1_{\boldsymbol{j}'} \\ + c_r c_{r+1} \frac{t_r(f)1_{t_r(\boldsymbol{j})} - c_{r+1}(c_r(f))1_{c_{r+1}(c_r(\boldsymbol{j}))}}{x_r + x_{r+1}} 1_{\boldsymbol{j}'}.\end{aligned} \tag{3.17}$$

Let $\mathrm{End}(sF^d)$ be the superalgebra of all supernatural transformations $sF^d \Rightarrow sF^d$. Since $(x - j)$ acts *locally nilpotently* on sF_j, i.e. it induces a nilpotent endomorphism of $sF_j\,M$ for each $M \in \mathrm{ob}\, s\mathcal{O}$, we can extend the homomorphism Ψ_d from (3.9) uniquely to a homomorphism

$$\widehat{\Psi}_d : \widehat{AHC}_d \to \mathrm{End}(sF^d) \tag{3.18}$$

such that $\widehat{\Psi}_d(1_{\boldsymbol{j}})$ is the projection of sF^d onto its summand $sF_{j_d} \cdots sF_{j_1}$, and $\widehat{\Psi}_d(a1_{\boldsymbol{j}}) = \Psi_d(a) \circ \widehat{\Psi}_d(1_{\boldsymbol{j}})$ for each $a \in AHC_d$.

3.4. **Indecomposable projectives.** In this subsection, we relate the $\mathfrak{sp}_{2\infty}$-Bruhat order $\preceq$ on $\mathbf{B}$ from §2.3 to the structure of the Verma supermodules in $s\mathcal{O}$. Actually, it is better to work in terms of projectives, so let $P(\boldsymbol{b})$ be a projective cover of $L(\boldsymbol{b})$ in $\underline{s\mathcal{O}}$.

Theorem 3.8. *The indecomposable projective supermodule $P(\boldsymbol{b})$ has a Verma flag with top section evenly isomorphic to $M(\boldsymbol{b})$ and other sections evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \in \mathbf{B}$ with $\boldsymbol{c} \succ \boldsymbol{b}$.*

Proof. By Lemma 2.7, there exists a dominant, typical $\boldsymbol{a} \in \mathbf{B}$ and a monomial X in the Chevalley generators $\{f_i \,|\, i \in I\}$ of $\mathfrak{sp}_{2\infty}$ such that $Xv_{\boldsymbol{a}} = v_{\boldsymbol{b}} +$ (a sum of $v_{\boldsymbol{c}}$'s for $\boldsymbol{c} \succ \boldsymbol{b}$). Suppose that $X = f_{i_l} \cdots f_{i_2} f_{i_1}$ for $i_k \in I$. Let $j_k := \sqrt{i_k + \frac{1}{2}}\sqrt{i_k - \frac{1}{2}}$ and consider the supermodule

$$P := sF_{j_l} \cdots sF_{j_2} sF_{j_1}\, M(\boldsymbol{a}).$$

Since $\boldsymbol{a}$ is dominant and typical, $M(\boldsymbol{a})$ is projective. Since each sF_j sends projectives to projectives (being left adjoint to an exact functor), we deduce that P is projective. Since the combinatorics of (2.11) matches that of Theorem 3.7, we can reinterpret Lemma 2.7 as saying that P has a Verma flag with one section evenly isomorphic to $M(\boldsymbol{b})$ and all other sections evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \succ \boldsymbol{b}$. In fact, the unique section isomorphic to $M(\boldsymbol{b})$ appears at the top of this Verma flag, thanks the order of the sections arising from Theorem 3.7(1). Hence, P has a summand evenly isomorphic to $P(\boldsymbol{b})$, and we are done as $s\mathcal{O}^\Delta$ is closed under passing to summands. □

Corollary 3.9. *For $\boldsymbol{c} \in \mathbf{B}$, we have that $[M(\boldsymbol{c}) : L(\boldsymbol{c})] = 1$. All other composition factors of $M(\boldsymbol{c})$ are evenly isomorphic to $L(\boldsymbol{a})$'s for $\boldsymbol{a} \prec \boldsymbol{c}$.*

Proof. This follows from Theorem 3.8 using *BGG reciprocity*: for $\boldsymbol{a}, \boldsymbol{c} \in \mathbf{B}$ and $p \in \mathbb{Z}/2$, we have that

$$[M(\boldsymbol{c}) : \Pi^p L(\boldsymbol{a})] = [M(\boldsymbol{c})^\star : \Pi^p L(\boldsymbol{a})] = \dim \mathrm{Hom}_{s\mathcal{O}}(P(\boldsymbol{a}), M(\boldsymbol{c})^\star)_p = (P(\boldsymbol{a}) : \Pi^p M(\boldsymbol{c})).$$

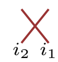
□

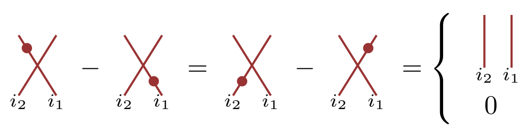
Corollary 3.10. *For any $\boldsymbol{b} \in \mathbf{B}$, every irreducible subquotient of the indecomposable projective $P(\boldsymbol{b})$ is evenly isomorphic to $L(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathbf{B}$ with $|\,\mathbf{wt}(\boldsymbol{a})| = |\,\mathbf{wt}(\boldsymbol{b})|$.*

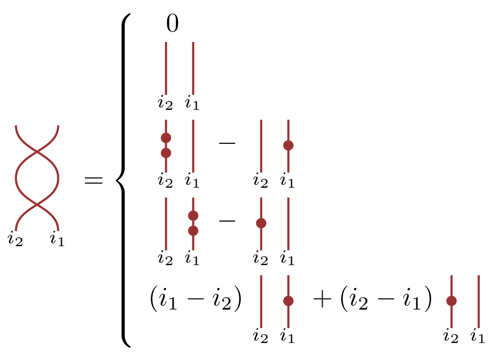
Proof. By Theorem 3.8 and Corollary 3.9, the composition factors of $P(\boldsymbol{b})$ are $L(\boldsymbol{a})$'s for $\boldsymbol{a} \in \mathbf{B}$ such that $\boldsymbol{a} \preceq \boldsymbol{c} \succeq \boldsymbol{b}$ for some $\boldsymbol{c}$. This implies that $|\,\mathbf{wt}(\boldsymbol{a})| = |\,\mathbf{wt}(\boldsymbol{b})|$. ☐

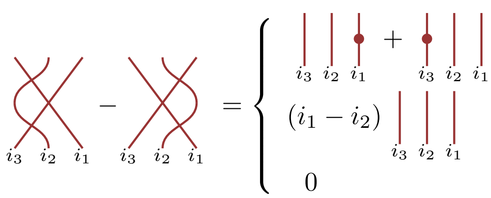
3.5. **The main categorification theorem.** Recall that $I = \mathbb{N}$. The monoidal category in the following definition is one of the categories introduced by Khovanov and Lauda [KL1, KL2] and Rouquier [R], for the graph arising from the Dynkin diagram of $\mathfrak{sp}_{2\infty}$ and the matrix of parameters $(q_{i,j}(u,v))_{i,j\in I}$ defined from

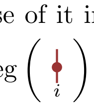
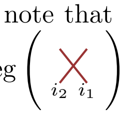
$$q_{i,j}(u,v) := \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } |i-j| > 1, \\ u^2 - v & \text{if } i = 1 \text{ and } j = 0, \\ v^2 - u & \text{if } i = 0 \text{ and } j = 1, \\ (i-j)u + (j-i)v & \text{otherwise.} \end{cases} \tag{3.19}$$

Definition 3.11. The *quiver Hecke category* $\mathcal{QH}$ of type $\mathfrak{sp}_{2\infty}$ is the strict $\mathbb{C}$-linear monoidal category generated by objects I and morphisms  $: i \to i$ and  $: i_2 \otimes i_1 \to i_1 \otimes i_2$ subject to the following relations:







Although we will not make use of it in this section, we note that $\mathcal{QH}$ can be enriched with a $\mathbb{Z}$-grading by setting deg  $:= (\alpha_i, \alpha_i)$ and deg  $:= -(\alpha_{i_1}, \alpha_{i_2})$.

Our final definition is the analog for $\mathfrak{sp}_{2\infty}$ of [BLW, Definition 2.10], which reformulated [LW, Definiton 3.2] for tensor products of minuscule representations of $\mathfrak{sl}_\infty$.

Definition 3.12. A *tensor product categorification* (TPC for short) of $V^{\otimes n}$ is the following data:

- a highest weight category $\mathcal{C}$ with standard objects $\{\Delta(\boldsymbol{b}) \,|\, \boldsymbol{b} \in \mathbf{B}\}$ indexed by the set $\mathbf{B}$ ordered according to the Bruhat order $\preceq$;
- adjoint pairs (F_i, E_i) of endofunctors of $\mathcal{C}$ for each $i \in I$;

- a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{C})$ with $\Phi(i) = F_i$ for each $i \in I$.

We impose the following additional axioms for all $i \in I$ and $\boldsymbol{b} \in \mathbf{B}$:

- E_i is isomorphic to a left adjoint of F_i;
- $F_i\Delta(\boldsymbol{b})$ has a filtration with sections $\{\Delta(\boldsymbol{b} + \boldsymbol{d}_t) \mid 1 \le t \le n, i\text{-sig}_t(\boldsymbol{b}) = \mathtt{f}\}$;
- $E_i\Delta(\boldsymbol{b})$ has a filtration with sections $\{\Delta(\boldsymbol{b} - \boldsymbol{d}_t) \mid 1 \le t \le n, i\text{-sig}_t(\boldsymbol{b}) = \mathtt{e}\}$;
- the natural transformation $\Phi\Big($  $\Big)$ is locally nilpotent.

Now we let $\mathcal{O}$ (resp. $\Pi\mathcal{O}$) be the subcategory of $s\mathcal{O}$ consisting of the supermodules all of whose composition factors are evenly (resp. oddly) isomorphic to $L(\boldsymbol{b})$'s for $\boldsymbol{b} \in \mathbf{B}$. All morphisms between objects of $\mathcal{O}$ are purely even, so we may as well forget the $\mathbb{Z}/2$-grading and view $\mathcal{O}$ simply as a $\mathbb{C}$-linear category.

Our main theorem is as follows.

Theorem 3.13. *We have that $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$, i.e. the supercategory $s\mathcal{O}$ splits. Moreover, the $\mathbb{C}$-linear category $\mathcal{O}$ admits all of the additional structure needed to make it into a TPC of $V^{\otimes n}$.*

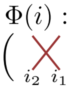
Proof. The fact that $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$ follows from Corollary 3.10; cf. the proof of [BD2, Theorem 5.1]. To make $\mathcal{O}$ into a TPC, we need to introduce the additional data then check the axioms from Definition 3.12.

It is clear that $\mathcal{O}$ is a Schurian category in the sense of Definition 3.1 with irreducible objects $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$. Since $P(\boldsymbol{b})$ belongs to $\mathcal{O}$, it is the projective cover of $L(\boldsymbol{b})$ in $\mathcal{O}$. Theorem 3.8 and Corollary 3.9 then give the necessary technical ingredients needed to check that $\mathcal{O}$ is a highest weight category with the required weight poset; cf. the proof of [BD2, Theorem 5.4]. Its standard objects $\{\Delta(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ are the Verma supermodules $\{M(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$.

To define F_i and E_i, take $i \in I$, and let $j := \sqrt{i + \frac{1}{2}}\sqrt{i - \frac{1}{2}}$. Theorem 3.7 shows that $sF_j\, M(\boldsymbol{b})$ and $sE_j\, M(\boldsymbol{b})$ are objects of $\mathcal{O}$. Hence, by exactness, the functors sF_j and sE_j send arbitrary objects from $\mathcal{O}$ to objects of $\mathcal{O}$. So we obtain the required endofunctors by setting

$$F_i := sF_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}, \qquad E_i := sE_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}.$$

The adjunctions (sF_j, sE_j) and (sE_j, sF_j) discussed earlier give adjunctions (F_i, E_i) and (E_i, F_i) too. Also $F_i\, M(\boldsymbol{b})$ and $E_i\, M(\boldsymbol{b})$ have the required Verma filtrations thanks to Theorem 3.7.

It remains to define $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$. Since $\mathcal{QH}$ is defined by generators and relations, we can do this simply by declaring that $\Phi(i) := F_i$ for each i, then specifying natural transformations $\Phi\Big($  $\Big) : F_i \Rightarrow F_i$ and $\Phi\Big($  $\Big) : F_{i_2}F_{i_1} \Rightarrow F_{i_1}F_{i_2}$ satisfying the quiver Hecke relations from Definition 3.11. The explicit formulae for these natural transformations are recorded in the next two paragraphs. They were derived like in the proof of [BD2, Theorem 6.2] by starting from the supernatural transformations from Theorem 3.4, which satisfy the affine Hecke-Clifford relations of Definition 3.3, then using the remarkable isomorphism from [KKT, Theorem 5.4] to combine these into supernatural transformations satisfying the quiver Hecke-Clifford relations of [KKT, Definition 3.5]. When $i = 0$, the number $j = \sqrt{i + \frac{1}{2}}\sqrt{i - \frac{1}{2}}$ satisfies $j^2 + \frac{1}{4} = 0$. Hence, we are in the situation of [KKT, §5.2(i)(c)] and the appropriate Dynkin diagram is of type $\mathfrak{sp}_{2\infty}$, unlike in [BD2] where it was of type $\mathfrak{sl}_\infty$. This is really the only difference compared to the proof of [BD2, Theorem 6.2], so we omit any further explanations.

Here we give the explicit formula for $\Phi\Big($ 

$\Big)$. Let $j := \sqrt{i+\frac{1}{2}}\sqrt{i-\frac{1}{2}}$, then define $y_1 \in (x_1-j)\,\mathbb{C}[[x_1-j]]$ to be $x_1^2+\frac{1}{4}$ if $i=0$, or the unique power series in $(x_1-j)\,\mathbb{C}[[x_1-j]]$ such that $(y_1+i)^2 = x_1^2+\frac{1}{4}$ if $i \neq 0$. Recalling (3.15), this gives us an element $y_1 1_j \in \widehat{AHC}_1$. Applying the homomorphism $\widehat{\Psi}_1$ from (3.18), we obtain from this an even supernatural transformation $\widehat{\Psi}_1(y_1 1_j) : sF_j \to sF_j$. Since F_i is the restriction of sF_j, this gives us the required natural transformation $\Phi\Big($ 

$\Big)$. It is locally nilpotent because $(x-j)$ acts locally nilpotently on sF_j by the definition of sF_j.

Finally, we give the formula for $\Phi\Big($ 

$\Big)$. For this, we work in $\widehat{AHC}_2$. For $r=1,2$, let $j_r := \sqrt{i_r+\frac{1}{2}}\sqrt{i_r-\frac{1}{2}}$, then define $y_r \in (x_r-j_r)\,\mathbb{C}[[x_r-j_r]$ to be $x_r^2+\frac{1}{4}$ if $i_r=0$, or the unique power series such that $(y_r+i_r)^2 = x_r^2+\frac{1}{4}$ if $i_r \neq 0$ (like in the previous paragraph). Let

$$p := \frac{(x_1^2-x_2^2)^2}{2(x_1^2+x_2^2)-(x_1^2-x_2^2)^2},$$

which is an element of $\mathbb{C}[[x_1-j_1, x_2-j_2]]$ unless $|i_1-i_2|=1$ (when it should be viewed as an element of the fraction field). Then, recalling (3.19), we define $g \in \mathbb{C}[[x_1-j_1,x_2-j_2]]$ from

$$g := \begin{cases} -1 & \text{if } i_1 < i_2,\\ \sqrt{p}/(y_1-y_2) & \text{if } i_1 = i_2,\\ p\,q_{i_2,i_1}(y_2,y_1) & \text{if } i_1 > i_2,\end{cases}$$

choosing the square root when $i_1 = i_2$ so that $g - \frac{x_1-x_2}{y_1-y_2} \in (x_1-x_2)\,\mathbb{C}[[x_1-j_1,x_2-j_2]]$. Using (3.16)–(3.17), one can check that

$$t_1 g 1_{j_2 j_1} + \left(\frac{g}{x_1-x_2} - \frac{\delta_{i_1,i_2}}{y_1-y_2}\right) 1_{j_2 j_1} + c_1 c_2 \frac{g}{x_1+x_2} 1_{j_2 j_1} \in 1_{j_1 j_2}\widehat{AHC}_2 1_{j_2 j_1}.$$

Applying $\widehat{\Psi}_2$, we obtain an even supernatural transformation $sF_{j_2} sF_{j_1} \Rightarrow sF_{j_1} sF_{j_2}$, hence, the desired natural transformation $F_{i_2}F_{i_1} \Rightarrow F_{i_1}F_{i_2}$. □

4. Orthodox basis

In this section, we prove the first Cheng-Kwon-Wang conjecture [CKW, Conjecture 5.12]. Throughout the section, I will denote the set $\mathbb{N}$ that indexes the simple roots of $\mathfrak{sp}_{2\infty}$, and $\mathbf{B} = \mathbb{Z}^n$ as always. For $k \geq 1$, we'll write I_k for the set $\{0,1,\dots,k-1\}$ that indexes the simple roots of the subalgebra $\mathfrak{sp}_{2k} < \mathfrak{sp}_{2\infty}$, and define $\mathbf{B}_k$ as in (2.7).

4.1. **Truncation from $\mathfrak{sp}_{2\infty}$ to $\mathfrak{sp}_{2k}$.** Fix $k \geq 1$. The quiver Hecke category of type $\mathfrak{sp}_{2k}$ is the full subcategory $\mathcal{QH}_k$ of $\mathcal{QH}$ whose objects are monoidally generated by $I_k \subset I$. There is a notion of a *tensor product categorification of* $V_k^{\otimes n}$. This is defined in exactly the same way as Definition 3.12, replacing $\mathfrak{sp}_{2\infty}, V, \mathbf{B}, I$ and $\mathcal{QH}$ with $\mathfrak{sp}_{2k}, V_k, \mathbf{B}_k, I_k$ and $\mathcal{QH}_k$, respectively. In this subsection, we are going to explain how to construct such a structure out of a TPC of $V^{\otimes n}$ by passing to a certain subquotient. The approach is similar to that of [BLW, §2.8].

Recall (2.19). Let $\mathbf{B}_{\leq k}$ denote the set of all $\boldsymbol{b} \in \mathbf{B}$ such that $N_{[1,s]}(\boldsymbol{b},k) \leq 0$ for $s = 1,\dots,n-1$ and $N_{[1,n]}(\boldsymbol{b},k) = 0$. Let $\mathbf{B}_{<k}$ be the set of all $\boldsymbol{b} \in \mathbf{B}_{\leq k}$ such that $N_{[1,s]}(\boldsymbol{b},k) < 0$ for at least one s. Lemma 2.4 implies that these are both ideals (lower sets) in the poset $\mathbf{B}$. Observe moreover that $\mathbf{B}_k$ is the set difference $\mathbf{B}_{\leq k} \setminus \mathbf{B}_{<k}$.

Now let $\mathcal{C}$ be any TPC of $V^{\otimes n}$. Let $\mathcal{C}_{\leq k}$ be the Serre subcategory of $\mathcal{C}$ generated by the irreducible supermodules $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_{\leq k}\}$, and define $\mathcal{C}_{<k}$ similarly using $\mathbf{B}_{<k}$. As

$\mathbf{B}_{\leq k}$ and $\mathbf{B}_{<k}$ are ideals, we are in the same general situation as discussed in [BLW, §2.5]. Hence, $\mathcal{C}_{\leq k}$ and $\mathcal{C}_{<k}$ get induced highest weight structures, as does the Serre quotient $\mathcal{C}_k := \mathcal{C}_{\leq k}/\mathcal{C}_{<k}$. Its weight poset is $(\mathbf{B}_k, \preceq)$.

Theorem 4.1. *The subquotient $\mathcal{C}_k$ of $\mathcal{C}$ admits the structure of a TPC of $V_k^{\otimes n}$.*

Proof. We must check all of the properties from the $\mathfrak{sp}_{2k}$ version of Definition 3.12. We've already explained that $\mathcal{C}_k$ is a highest weight category with the appropriate weight poset. Next, we show that the endofunctors E_i, F_i for $i \in I_k$ leave both $\mathcal{C}_{\leq k}$ and $\mathcal{C}_{<k}$ invariant. As in the proof of [BLW, Lemma 2.18], we just need to verify this on standard objects, when it follows using the observation that

$$N_{[1,s]}(\boldsymbol{b} \pm \boldsymbol{d}_r, k) = N_{[1,s]}(\boldsymbol{b}, k)$$

for all $\boldsymbol{b} \in \mathbf{B}$ and $r, s = 1, \dots, n$ such that $i\text{-sig}_r(\boldsymbol{b}) \in \{\mathtt{e}, \mathtt{f}\}$ for $i \in I_k$. Hence, E_i, F_i induce biadjoint endofunctors of $\mathcal{C}_k$ for each $i \in I_k$. All of the other required structure comes immediately from the definitions. □

4.2. **Proof of the first Cheng-Kwon-Wang conjecture.** Our definition of a TPC of $V_k^{\otimes n}$ is a simplified version of the more general notion of TPC from [LW, Definition 3.2]. The simplification is possible because V_k is a minuscule highest weight representation for $\mathfrak{sp}_{2k}$. The equivalence of our definition with the Losev-Webster definition may be verified by a similar argument to the one explained in [BLW, Remark 2.11]. Hence, we obtain the following as a special case of the uniqueness theorem for TPCs that is the main result of [LW]; we refer to [BD1, Definition 4.7] for the definition of strongly equivariant equivalence being used here.

Theorem 4.2 (Losev–Webster)**.** *All TPCs of $V_k^{\otimes n}$ are strongly equivariantly equivalent via equivalences which preserve the labelling of irreducible objects.*

If we apply the construction from the previous subsection to the category $\mathcal{O}$ of Theorem 3.13, we obtain a subquotient $\mathcal{O}_k := \mathcal{O}_{\leq k}/\mathcal{O}_{<k}$ of $\mathcal{O}$ which is a TPC of $V_k^{\otimes n}$. Let A_k denote Webster's tensor product algebra associated to the n-fold tensor product of the natural representation of $\mathfrak{sp}_{2k}$, that is, the algebra $T^{\boldsymbol{\omega}_k}$ from [W2, §4] associated to the n-tuple of dominant weights $\boldsymbol{\omega}_k := (-\varepsilon_{k-1}, \dots, -\varepsilon_{k-1})$ for $\mathfrak{sp}_{2k}$. Webster's general theory from [W2] shows that the category A_k-*mod* of finite dimensional modules over this algebra also has the structure of a TPC of $V_k^{\otimes n}$; see also [LW, Theorem 3.12]. Hence, applying Theorem 4.2, we obtain the following:

Corollary 4.3. *The category $\mathcal{O}_k$ is equivalent to A_k-mod via an equivalence which preserves the labelling of irreducible objects.*

In particular, this means that the combinatorics of decomposition numbers in the category $\mathcal{O}$ is the same as that of Webster's tensor product algebras. More precisely, given any $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$, we pick k large enough so that $\boldsymbol{a}, \boldsymbol{b}$ both belong to $\mathbf{B}_k$. Then, Corollary 4.3 implies that

$$[M(\boldsymbol{a}) : L(\boldsymbol{b})] = [M_k(\boldsymbol{a}) : L_k(\boldsymbol{b})], \tag{4.1}$$

where $M_k(\boldsymbol{a})$ denotes the standard A_k-module associated to $\boldsymbol{a} \in \mathbf{B}_k$ as constructed in [W2, §5], and $L_k(\boldsymbol{a})$ is its unique irreducible quotient. Indeed, for $\boldsymbol{a} \in \mathbf{B}_k$, the canonical images of the standard objects $M(\boldsymbol{a})$ and their irreducible quotients $L(\boldsymbol{a})$ in the quotient category $\mathcal{O}_k$ map under the equivalence from Corollary 4.3 to copies of $M_k(\boldsymbol{a})$ and $L_k(\boldsymbol{a})$, respectively. Then (4.1) follows just like in the proof of [BLW, Theorem 2.21],

We can reformulate the assertions made in the previous paragraph in terms of Webster's orthodox basis, as follows. Let $P_k(\boldsymbol{a})$ be the projective cover of $L_k(\boldsymbol{a})$ in A_k-*mod*.

As A_k-*mod* is a TPC, there is a vector space isomorphism

$$\iota_k : \mathbb{C} \otimes_{\mathbb{Z}} K_0(A_k\text{-}mod) \xrightarrow{\sim} V_k^{\otimes n}, \qquad [M_k(\boldsymbol{a})] \mapsto v_{\boldsymbol{a}}.$$

By the definition from [W1, §7], Webster's *orthodox basis* of $V_k^{\otimes n}$ (specialized at $q = 1$) is the basis $\left\{\iota_k([P_k(\boldsymbol{b})]) \mid \boldsymbol{b} \in \mathbf{B}_k\right\}$. Analogously, we can consider the isomorphism

$$\iota : \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^{\Delta}) \xrightarrow{\sim} V^{\otimes n}, \qquad [M(\boldsymbol{a})] \mapsto v_{\boldsymbol{a}}.$$

The following defines the *orthodox basis* of $V^{\otimes n}$ (specialized at $q = 1$).

Theorem 4.4. *The space $V^{\otimes n}$ has a unique topological basis $\{o_{\boldsymbol{b}} \mid b \in \mathbf{B}\}$ such that $\mathrm{pr}_k o_{\boldsymbol{b}} = \iota_k([P_k(\boldsymbol{b})])$ for each $k \geq 1$ and $\boldsymbol{b} \in \mathbf{B}_k$. Moreover, we have that $o_{\boldsymbol{b}} = \iota([P(\boldsymbol{b})])$ for any $\boldsymbol{b} \in \mathbf{B}$.*

Proof. Let $o_{\boldsymbol{b}} := \iota([P(\boldsymbol{b})])$. By BGG reciprocity in the highest weight categories $\mathcal{O}$ and A_k-*mod*, respectively, we have that $[P(\boldsymbol{b})] = \sum_{\boldsymbol{a} \in \mathbf{B}} [M(\boldsymbol{a}) : L(\boldsymbol{b})][M(\boldsymbol{a})]$ and $[P_k(\boldsymbol{b})] = \sum_{\boldsymbol{a} \in \mathbf{B}_k} [M_k(\boldsymbol{a}) : L_k(\boldsymbol{b})][M_k(\boldsymbol{a})]$. Hence, for $\boldsymbol{b} \in \mathbf{B}_k$, we have that

$$\mathrm{pr}_k o_{\boldsymbol{b}} = \sum_{\boldsymbol{a} \in \mathbf{B}_k} [M(\boldsymbol{a}) : L(\boldsymbol{b})] v_{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathbf{B}_k} [M_k(\boldsymbol{a}) : L_k(\boldsymbol{b})] v_{\boldsymbol{a}} = \iota_k([P_k(\boldsymbol{b})]),$$

using (4.1) for the middle equality. □

This establishes the truth of [CKW, Conjecture 5.12]. Actually, Cheng, Kwon and Wang formulated their conjecture in terms of tilting modules instead of projective modules, i.e. they work in the Ringel dual setting. The equivalence of our Theorem 4.4 with their conjecture follows by [B2, (7.12)].

Remark 4.5. Webster's algebra A_k admits a natural $\mathbb{Z}$-grading. Hence, one can consider the category A_k-*grmod* of finite-dimensional *graded* A_k-modules. The endofunctors E_i and F_i also admit graded lifts, making A_k-*grmod* into a $U_q\mathfrak{sp}_{2k}$-*tensor product categorification* of $\dot{V}_k^{\otimes n}$. We refer the reader to [BLW, Definition 5.9] for a related definition which is easily adapted to the present situation; this depends on the grading on $\mathcal{QH}_k$ noted at the end of Definition 3.11. The Grothendieck group $K_0(A\text{-}grmod)$ is a $\mathbb{Z}[q, q^{-1}]$-module with q acting as the upward grading shift functor. Also the standard modules $M_k(\boldsymbol{a})$ admit graded lifts $\dot{M}_k(\boldsymbol{a})$, such that there is a $\mathbb{Q}(q)$-vector space isomorphism

$$i_k : \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(A_k\text{-}grmod) \xrightarrow{\sim} \dot{V}_k^{\otimes n}, \qquad [\dot{M}_k(\boldsymbol{a})] \mapsto \dot{v}_{\boldsymbol{a}}.$$

Webster's *orthodox basis* of $\dot{V}_k^{\otimes n}$ is the basis $\left\{i_k([\dot{P}_k(\boldsymbol{b})]) \mid \boldsymbol{b} \in \mathbf{B}_k\right\}$, where $\dot{P}_k(\boldsymbol{b})$ is the projective cover of $\dot{M}_k(\boldsymbol{b})$ in A_k-*grmod*. Using the graded analog of Theorem 4.2, one can show that the coefficients of this basis stabilize as $k \to \infty$, hence, there is a unique topological basis $\{\dot{o}_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ for $\dot{V}^{\otimes n}$ such that $\mathrm{pr}_k \dot{o}_{\boldsymbol{b}} = i_k([\dot{P}_k(\boldsymbol{b})])$ for all $k \geq 1$ and $\boldsymbol{b} \in \mathbf{B}_k$. This is the q-analog of the basis in Theorem 4.4.

Remark 4.6. We expect that the category $\mathcal{O}$ admits a graded lift $\dot{\mathcal{O}}$ which is a $U_q\mathfrak{sp}_{2\infty}$-tensor product categorification of $\dot{V}^{\otimes n}$. Then there should be a $\mathbb{Q}(q)$-vector space isomorphism

$$i : \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\dot{\mathcal{O}}) \xrightarrow{\sim} \dot{V}^{\otimes n}, \quad [\dot{M}(\boldsymbol{a})] \mapsto \dot{v}_{\boldsymbol{a}}, \quad [\dot{P}(\boldsymbol{b})] \mapsto \dot{o}_{\boldsymbol{b}},$$

for suitable graded lifts $\dot{M}(\boldsymbol{a})$ and $\dot{P}(\boldsymbol{b})$ of $M(\boldsymbol{a})$ and $P(\boldsymbol{b})$. It should be possible to prove these statements by mimicking the general approach developed in [BLW]. The argument would also yield an extension of the uniqueness theorem (Theorem 4.2) from $\mathfrak{sp}_{2k}$ to $\mathfrak{sp}_{2\infty}$.

4.3. **Prinjectives and the associated crystal.** The proof of the uniqueness theorem in [LW] gives a great deal of additional information about the structure of TPCs of $V_k^{\otimes n}$. In particular, [LW, Theorem 7.2] gives an explicit combinatorial description of the associated crystal in the general sense of [BD1, §4.4]. Also, [LW, Proposition 5.2] gives a classification of the indecomposable *prinjective* (= projective and injective) objects. Here is a precise statement of these results:

Theorem 4.7 (Losev-Webster)**.** *Let $\mathcal{C}_k$ be a TPC of $V_k^{\otimes n}$. Denote its distinguished irreducible objects by $\{L_k(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_k\}$.*

(1) The associated crystal is the crystal structure on $\mathbf{B}_k$ defined in §2.6. This means that $F_i L_k(\boldsymbol{b}) \neq 0$ (resp. $E_i L_k(\boldsymbol{b}) \neq 0$) if and only if $\tilde{f}_i \boldsymbol{b} \neq \varnothing$ (resp. $\tilde{e}_i \boldsymbol{b} \neq \varnothing$), in which case $F_i L_k(\boldsymbol{b})$ (resp. $E_i L_k(\boldsymbol{b})$) has irreducible head and socle isomorphic to $L_k(\tilde{f}_i \boldsymbol{b})$ (resp. $L_k(\tilde{e}_i \boldsymbol{b})$).
(2) The projective cover of $L_k(\boldsymbol{b})$ is injective if and only if $\boldsymbol{b}$ is antidominant, i.e. it is an element of the connected component $\mathbf{B}_k^\circ$ of the crystal generated by the tuple $\boldsymbol{z}_k$ from (2.24).

Using also Theorem 4.1 and letting $k \to \infty$, we get the following corollary, which extends this result to infinite rank.

Corollary 4.8. *Let $\mathcal{C}$ be a TPC of $V^{\otimes n}$ with irreducible objects $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ (e.g., the category $\mathcal{O}$ from Theorem 3.13).*

(1) The associated crystal is the crystal structure on $\mathbf{B}$ defined in §2.6.
(2) The projective cover of $L(\boldsymbol{b})$ is injective if and only if $\boldsymbol{b}$ is antidominant.

Proof. For (1), choose k so that $i \in I_k$ and all of the composition factors of $F_i L(\boldsymbol{b})$ have label belonging to $\mathbf{B}_k$. Then, $F_i L(\boldsymbol{b}) \in \operatorname{ob} \mathcal{C}_{\leq k}$, and its socle and head can be determined by passing to the quotient category $\mathcal{C}_k$, where the result follows from Theorem 4.7(1). For (2), choose k so that all composition factors of the projective cover of $L(\boldsymbol{b})$ have label belonging to $\mathbf{B}_k$. Then we get done by Theorem 4.7(2), since an object of $\mathcal{C}$ with composition factors labelled by $\mathbf{B}_k$ is projective or injective in $\mathcal{C}$ if and only if its image is projective or injective in $\mathcal{C}_k$. □

5. Category $\mathcal{F}$

To conclude the article, we formulate and prove a generalization of [CKW, Conjecture 5.13], then deduce some consequences for the structure of the category $\mathcal{F}$ of finite-dimensional half-integral weight $\mathfrak{g}$-supermodules. Throughout this section, I denotes $\mathbb{N}$ and $I_0 := \mathbb{Z}_+$, i.e. they are the index sets for the simple roots of $\mathfrak{sp}_{2\infty}$ and $\mathfrak{sl}_{+\infty}$, respectively.

5.1. **Truncation from $\mathfrak{sp}_{2\infty}$ to $\mathfrak{sl}_{+\infty}$.** Recall the $\mathfrak{sl}_{+\infty}$-module $V_0^{\otimes \boldsymbol{\sigma}}$ from §2.2. We gave two different realizations of that, one as a submodule of the $\mathfrak{sl}_\infty$-module $V^{\otimes \boldsymbol{\sigma}}$, the other as a submodule of the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$. In turn, categorifications of $V_0^{\otimes \boldsymbol{\sigma}}$ can be constructed either by truncating from a TPC of the $\mathfrak{sl}_\infty$-module $V^{\otimes \boldsymbol{\sigma}}$ as explained in [BLW, §2.8], or by truncating from a TPC of the $\mathfrak{sp}_{2\infty}$-module $V^{\otimes n}$. In this subsection, we are going to follow the latter route.

We begin with a couple more definitions. The *quiver Hecke category* $\mathcal{QH}_0$ of type $\mathfrak{sl}_{+\infty}$ may be identified with the full subcategory of the quiver Hecke category $\mathcal{QH}$ of type $\mathfrak{sp}_{2\infty}$ from Definition 3.11 whose objects are monoidally generated by $I_0 \subset I$.

Definition 5.1. Fix $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \{\pm\}^n$. A *TPC* of the $\mathfrak{sl}_{+\infty}$-module $V_0^{\otimes \boldsymbol{\sigma}}$ is the following data:

- a highest weight category $\mathcal{C}$ with standard objects $\{\Delta(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_0\}$ indexed by the set $\mathbf{B}_0$ from (2.5) ordered according to the Bruhat order $\preceq_{\boldsymbol{\sigma}}$ from (2.16);
- adjoint pairs (F_i, E_i) of endofunctors of $\mathcal{C}$ for each $i \in I_0$;
- a strict monoidal functor $\Phi : \mathcal{QH}_0 \to \mathcal{E}nd(\mathcal{C})$ with $\Phi(i) = F_i$ for each $i \in I_0$.

We impose the following additional axioms for all $i \in I_0$ and $\boldsymbol{b} \in \mathbf{B}_0$:

- E_i is isomorphic to a left adjoint of F_i;
- $F_i\Delta(\boldsymbol{b})$ has a filtration with sections $\{\Delta(\boldsymbol{b} + \sigma_t \boldsymbol{d}_t) \mid 1 \leq t \leq n, i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b}) = \mathtt{f}\}$;
- $E_i\Delta(\boldsymbol{b})$ has a filtration with sections $\{\Delta(\boldsymbol{b} - \sigma_t \boldsymbol{d}_t) \mid 1 \leq t \leq n, i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b}) = \mathtt{e}\}$;
- the natural transformation $\Phi\Bigg($  $\Bigg)$ is locally nilpotent.

View $\mathbf{B}$ as a poset via the $\mathfrak{sp}_{2\infty}$-Bruhat order from (2.18). Recalling (2.19), let $\mathbf{B}_{\leq\boldsymbol{\sigma}}$ be the set of all $\boldsymbol{b} \in \mathbf{B}$ such that $N_{[1,s]}(\boldsymbol{b}, 0) \leq \sigma_1 + \cdots + \sigma_s$ for $s = 1, \ldots, n-1$ and $N_{[1,n]}(\boldsymbol{b}, 0) = \sigma_1 + \cdots + \sigma_n$. Let $\mathbf{B}_{<\boldsymbol{\sigma}}$ be the set of all $\boldsymbol{b} \in \mathbf{B}_{\leq\boldsymbol{\sigma}}$ such that $N_{[1,s]}(\boldsymbol{b}, 0) < \sigma_1 + \cdots + \sigma_s$ for at least one s. Lemma 2.4 implies that these are both ideals in $\mathbf{B}$. Moreover, the set difference $\mathbf{B}_{\leq\boldsymbol{\sigma}} \setminus \mathbf{B}_{<\boldsymbol{\sigma}}$ is precisely the index set $\mathbf{B}_{\boldsymbol{\sigma}}$.

Now let $\mathcal{C}$ be a TPC of $V^{\otimes n}$ in the sense of Definition 3.12. Let $\mathcal{C}_{\leq\boldsymbol{\sigma}}$ and $\mathcal{C}_{<\boldsymbol{\sigma}}$ be the Serre subcategories of $\mathcal{C}$ corresponding to the ideals $\mathbf{B}_{\leq\boldsymbol{\sigma}}$ and $\mathbf{B}_{<\boldsymbol{\sigma}}$, respectively. Then form the Serre quotient $\mathcal{C}_{\boldsymbol{\sigma}} := \mathcal{C}_{\leq\boldsymbol{\sigma}}/\mathcal{C}_{<\boldsymbol{\sigma}}$. This has a naturally induced structure of highest weight category with weight poset $(\mathbf{B}_{\boldsymbol{\sigma}}, \preceq)$. Its irreducibles $\{L_{\boldsymbol{\sigma}}(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_\sigma\}$ are the canonical images of the $L(\boldsymbol{b})$'s. The following parallels Theorem 4.1.

Theorem 5.2. *The subquotient $\mathcal{C}_{\boldsymbol{\sigma}}$ of $\mathcal{C}$ admits the structure of a TPC of $V_0^{\otimes\boldsymbol{\sigma}}$.*

Proof. Like in [BLW, §2.5], $\mathcal{C}_{\boldsymbol{\sigma}}$ is a highest weight category with weight poset $(\mathbf{B}_\sigma, \preceq)$, which is isomorphic to $(\mathbf{B}_0, \preceq_{\boldsymbol{\sigma}})$ thanks to Lemma 2.5. Also, the endofunctors E_i, F_i for $i \in I_0$ leave both $\mathcal{C}_{\leq\boldsymbol{\sigma}}$ and $\mathcal{C}_{<\boldsymbol{\sigma}}$ invariant, hence, they induce endofunctors of $\mathcal{C}_{\boldsymbol{\sigma}}$. This follows by a similar argument to the proof of Theorem 4.1; the key point this time is that $\boldsymbol{b} \in \mathbf{B}_0$ satisfies

$$N_{[1,s]}(\boldsymbol{b} \pm \boldsymbol{d}_r, 0) = N_{[1,s]}(\boldsymbol{b}, 0)$$

whenever $i\text{-sig}_r(\boldsymbol{b}) \in \{\mathtt{e}, \mathtt{f}\}$ for some $i \in I_0$. We should also note for $i \in I_0$, $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$, and $\boldsymbol{b}' \in \mathbf{B}_0$ defined via (2.2) that:

- $i\text{-sig}_t(\boldsymbol{b}) = \mathtt{e}$ (resp. $\mathtt{f}$) if and only if $i\text{-sig}_t^{\boldsymbol{\sigma}}(\boldsymbol{b}') = \mathtt{e}$ (resp. $\mathtt{f}$);
- $(\boldsymbol{b} \pm \sigma_t \boldsymbol{d}_t)' = \boldsymbol{b}' \pm \boldsymbol{d}_t$.

This follows from Lemma 2.2 using (5.6) and (2.11). □

The extremal choices for $\boldsymbol{\sigma}$ deserve some special mention. For $\boldsymbol{\sigma}$ as in the following lemma, the subquotient $\mathcal{C}_{\boldsymbol{\sigma}}$ of Theorem 5.2 may be identified with a subcategory of $\mathcal{C}$.

Lemma 5.3. *Suppose that $\boldsymbol{\sigma} = (-, \ldots, -, +, \ldots, +)$ with n_1 entries equal to $-$ followed by n_0 entries equal to $+$. Then $\mathbf{B}_{\boldsymbol{\sigma}}$ is an ideal in $\mathbf{B}$.*

Proof. We actually show that $\mathbf{B}_{\boldsymbol{\sigma}} = \mathbf{B}_{\leq\boldsymbol{\sigma}}$, which is an ideal. Take $\boldsymbol{a} \in \mathbf{B}_{\leq\boldsymbol{\sigma}}$. Since $N_{[1,n]}(\boldsymbol{a}, 0) = n_0 - n_1$, exactly n_1 of the entries of $\boldsymbol{a}$ are ≤ 0. Since $N_{[1,n_1]}(\boldsymbol{a}, 0) \leq -n_1$, these must constitute the first n_1 entries of $\boldsymbol{a}$. Hence, $\boldsymbol{a} \in \mathbf{B}_{\boldsymbol{\sigma}}$. □

At the other extreme, for $\boldsymbol{\sigma}$ as in the next lemma, the subquotient $\mathcal{C}_{\boldsymbol{\sigma}}$ may be identified with a quotient of $\mathcal{C}$ itself.

Lemma 5.4. *Suppose that $\boldsymbol{\sigma} = (+, \ldots, +, -, \ldots, -)$ with n_0 entries equal to $+$ followed by n_1 entries equal to $-$. Then $\mathbf{B}_{\boldsymbol{\sigma}}$ is a coideal (upper set) in $\mathbf{B}$.*

Proof. We first observe that

$$\mathbf{B}_{\leq\boldsymbol{\sigma}} = \left\{\boldsymbol{a} \in \mathbf{B} \;\middle|\; N_{[1,n]}(\boldsymbol{a}, 0) = n_0 - n_1\right\}. \tag{5.1}$$

To see this, any $\boldsymbol{a} \in \mathbf{B}_{\leq \boldsymbol{\sigma}}$ satisfies $N_{[1,n]}(\boldsymbol{a}, 0) = \sigma_1 + \cdots + \sigma_n = n_0 - n_1$. Conversely, if $N_{[1,n]}(\boldsymbol{a}, 0) = n_0 - n_1$, then exactly n_0 of the entries of $\boldsymbol{a}$ are > 0 and n_1 entries are ≤ 0. Permuting the positive entries to the beginning makes the numbers $N_{[1,s]}(\boldsymbol{a}, 0)$ bigger, hence, $N_{[1,s]}(\boldsymbol{a}, 0) \leq \sigma_1 + \cdots + \sigma_s$ for all s. This shows $\boldsymbol{a} \in \mathbf{B}_{\leq \boldsymbol{\sigma}}$.

Now we can show that $\mathbf{B}_{\boldsymbol{\sigma}}$ is a coideal. Suppose that $\boldsymbol{a} \in \mathbf{B}_{\boldsymbol{\sigma}}$ and $\boldsymbol{b} \succeq \boldsymbol{a}$. Then $N_{[1,n]}(\boldsymbol{b}, 0) = N_{[1,n]}(\boldsymbol{a}, 0)$, hence, $\boldsymbol{b} \in \mathbf{B}_{\leq \boldsymbol{\sigma}}$. Since $\mathbf{B}_{\boldsymbol{\sigma}}$ is a coideal in $\mathbf{B}_{\leq \boldsymbol{\sigma}}$, this implies that $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$. □

5.2. **Proof of the second Cheng-Kwon-Wang conjecture.** TPCs of $V^{\otimes \boldsymbol{\sigma}}$ and $V_0^{\otimes \boldsymbol{\sigma}}$ are studied in detail in [BLW]. Combining results established there with Theorem 5.2 and our main categorification theorem, recalling the definition of the canonical and orthodox bases from (2.22)–(2.23) and Theorem 4.4, we obtain the following:

Theorem 5.5. *Given $\boldsymbol{b} \in \mathbf{B}$, define $\boldsymbol{\sigma}$ so that $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$, i.e. we take $\sigma_r := +$ if $b_r > 0$ or $\sigma_r := -$ if $b_r \leq 0$. Then, $\mathrm{pr}_{\boldsymbol{\sigma}} o_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} c_{\boldsymbol{b}} = \mathrm{pr}_0 c^{\boldsymbol{\sigma}}_{\boldsymbol{b}'}$.*

Proof. Remembering that $\mathcal{O}$ is a TPC of $V^{\otimes n}$ thanks to Theorem 3.13, let $\mathcal{O}_{\boldsymbol{\sigma}} := \mathcal{O}_{\leq \boldsymbol{\sigma}} / \mathcal{O}_{< \boldsymbol{\sigma}}$ be constructed from $\mathcal{O}$ as in Theorem 5.2. For $\boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$, the canonical image of $P(\boldsymbol{b})$ in the quotient category $\mathcal{O}_{\boldsymbol{\sigma}}$ is the indecomposable projective object of this TPC of $V_0^{\otimes \boldsymbol{\sigma}}$ indexed by $\boldsymbol{b}'$. By [BLW, Corollary 5.30], its isomorphism class is identified with $\mathrm{pr}_0 c_{\boldsymbol{b}'} \in V_0^{\otimes \boldsymbol{\sigma}}$. In view of the definition of $o_{\boldsymbol{b}}$ from Theorem 4.4, this shows that $\mathrm{pr}_{\boldsymbol{\sigma}} o_{\boldsymbol{b}} = \mathrm{pr}_0 c_{\boldsymbol{b}'}$. This equals $\mathrm{pr}_{\boldsymbol{\sigma}} c_{\boldsymbol{b}}$ thanks to Lemma 2.9. □

Corollary 5.6. *Suppose that $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$ have the property that $a_r > 0$ if and only if $b_r > 0$ for each $r = 1, \ldots, n$. Then, $(P(\boldsymbol{b}) : M(\boldsymbol{a})) = [M(\boldsymbol{a}) : L(\boldsymbol{b})] = d_{\boldsymbol{a},\boldsymbol{b}}(1) = d^{\boldsymbol{\sigma}}_{\boldsymbol{a}',\boldsymbol{b}'}(1)$.*

Proof. The first equality is BGG reciprocity in the highest weight category $\mathcal{O}$. Defining $\boldsymbol{\sigma}$ so that $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}_{\boldsymbol{\sigma}}$, we can compute $[M(\boldsymbol{a}) : L(\boldsymbol{b})]$ by passing to the quotient category $\mathcal{O}_{\boldsymbol{\sigma}}$ and computing the corresponding composition multiplicity there. Theorem 5.5 tells us that that is computed by the polynomials (2.22)–(2.23) evaluated at $q = 1$. □

In particular, if all of the strictly positive entries of $\boldsymbol{b} \in \mathbf{B}$ appear after the weakly negative ones, then Corollary 5.6 plus Lemma 5.3 show that all composition multiplicities in the Verma supermodule $M(\boldsymbol{b})$ are determined by computing corresponding coefficients of canonical basis elements (either type A or C). At the other extreme, using Lemma 5.4 instead, if all of the strictly positive entries of $\boldsymbol{b} \in \mathbf{B}$ come before the weakly negative ones, then the same is true for all of the Verma multiplicities in the projective $P(\boldsymbol{b})$. We can state this formally in terms of the orthodox basis as follows:

Corollary 5.7. *If $\boldsymbol{b} \in \mathbf{B}$ has all its strictly positive entries appearing before the weakly negative ones, then $o_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} o_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} c_{\boldsymbol{b}} = \mathrm{pr}_0 c^{\boldsymbol{\sigma}}_{\boldsymbol{b}'} = c_{\boldsymbol{b}}$.*

This is exactly the situation of [CKW, Conjecture 5.13], which follows easily from Corollary 5.7 using also the Ringel duality of [B2, (7.12)].

Remark 5.8. The q-analog of Theorem 5.5 is also true: in the setup of the theorem, we have that $\mathrm{pr}_{\boldsymbol{\sigma}} \dot{o}_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} \dot{c}_{\boldsymbol{b}} = \mathrm{pr}_0 \dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}'}$. If we had proved the assertions in Remark 4.6, this would follow by repeating the proof of Theorem 5.5 in the graded setting. Without this, one needs a slightly more roundabout argument, involving truncating to $\mathfrak{sl}_k \hookrightarrow \mathfrak{sp}_{2k}$. Since we have not introduced notation for this, we omit the detailed argument. This implies also the q-analog of Corollary 5.7: we have that

$$\dot{o}_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} \dot{o}_{\boldsymbol{b}} = \mathrm{pr}_{\boldsymbol{\sigma}} \dot{c}_{\boldsymbol{b}} = \mathrm{pr}_0 \dot{c}^{\boldsymbol{\sigma}}_{\boldsymbol{b}'} = \dot{c}_{\boldsymbol{b}}$$

in case all strictly positive entries of $\boldsymbol{b}$ precede the weakly negative ones.

5.3. **Decomposition of category $\mathcal{F}$.** In this subsection, we view $\mathbf{B}$ as a poset via the $\mathfrak{sp}_{2\infty}$-Bruhat order $\preceq$ from (2.18). Given a decomposition $n = n_0 + n_1$ with $n_0, n_1 \geq 0$, let

$$\mathbf{B}_{n_0|n_1} := \{\boldsymbol{b} \in \mathbf{B} \mid \boldsymbol{b} \text{ has } n_0 \text{ entries that are } > 0 \text{ and } n_1 \text{ entries that are } \leq 0\}, \quad (5.2)$$

$$\mathbf{B}^{\#}_{n_0|n_1} := \{\boldsymbol{b} \in \mathbf{B} \mid b_1, \dots, b_{n_0} > 0, b_{n_0+1}, \dots, b_n \leq 0\}, \quad (5.3)$$

$$\mathbf{B}^{+}_{n_0|n_1} := \{\boldsymbol{b} \in \mathbf{B} \mid b_1 > \cdots > b_{n_0} > 0 \geq b_{n_0+1} > \cdots > b_n\}. \quad (5.4)$$

Lemma 5.9. *Let $\boldsymbol{\sigma} = (+, \dots, +, -, \dots, -)$ with n_0 entries $+$ and n_1 entries $-$. Then $\mathbf{B}_{n_0|n_1} = \mathbf{B}_{\leq \boldsymbol{\sigma}}$ and $\mathbf{B}^{\#}_{n_0|n_1} = \mathbf{B}_{\boldsymbol{\sigma}}$. In particular, $\mathbf{B}^{\#}_{n_0|n_1}$ is a coideal in $\mathbf{B}_{n_0|n_1}$.*

Proof. The first equality follows from (5.1), and the second is clear from (2.7). □

Lemma 5.10. *Any $\boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}$ can be connected to a* typical *$\boldsymbol{a} \in \mathbf{B}^{+}_{n_0|n_1}$ by applying a sequence of the crystal operators $\tilde{e}_i, \tilde{f}_i$ ($i \in I_0$) from §2.6.*

Proof. We proceed by induction on the atypicality of $\boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}$, i.e. the number of pairs $1 \leq r < s \leq n$ such that $b_r + b_s = 1$. If $\boldsymbol{b}$ is typical, the result is trivial. So suppose that $\boldsymbol{b}$ is not typical. Let r be minimal such that $b_r + b_s = 1$ for some $s > r$. Set $i := b_r$, so that $b_s = 1 - i$. Since $b_r > b_s$, we have that $i > 0$. Then let $j \geq i$ be minimal such that $\{j+1, -j\} \cap \{b_1, \dots, b_n\} = \varnothing$.

Now we make a second induction on $j - i$. If $j = i$, then we let $\boldsymbol{c} \in \mathbf{B}^{+}_{n_0|n_1}$ be obtained from $\boldsymbol{b}$ by replacing its entry i with $i+1$. Then $\boldsymbol{c}$ is of smaller atypicality than $\boldsymbol{b}$. Also $\boldsymbol{c} = \tilde{f}_i \boldsymbol{b}$, and we get done by applying the first induction hypothesis to $\boldsymbol{c}$. If $j > i$, we either have that $j \in \{b_1, \dots, b_n\}$ or $1 - j \in \{b_1, \dots, b_n\}$, but not both (by the minimality of r). In the former case, let $\boldsymbol{c} \in \mathbf{B}^{+}_{n_0|n_1}$ be obtained from $\boldsymbol{b}$ by replacing its entry j with $j+1$; then, $\boldsymbol{c} = \tilde{f}_j \boldsymbol{b}$. In the latter case, let $\boldsymbol{c} \in \mathbf{B}^{+}_{n_0|n_1}$ be obtained from $\boldsymbol{b}$ by replacing its entry $1 - j$ with $-j$; then, $\boldsymbol{c} = \tilde{e}_j \boldsymbol{b}$. Either way, $\boldsymbol{c}$ has the same atypicality as $\boldsymbol{b}$, but the analog of the statistic $j - i$ for $\boldsymbol{c}$ is one less than it was for $\boldsymbol{b}$. It remains to apply the second induction hypothesis to $\boldsymbol{c}$ to finish the proof. □

Let $\mathcal{O}_{n_0|n_1}$ be the Serre subcategory of $\mathcal{O}$ generated by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_{n_0|n_1}\}$. One can determine whether $\boldsymbol{b} \in \mathbf{B}$ belongs to $\mathbf{B}_{n_0|n_1}$ just from knowledge of $|\,\mathbf{wt}(\boldsymbol{b})|$ (it does so if and only if $\sum_{i \in I}(|\,\mathbf{wt}(\boldsymbol{b})|, \varepsilon_i) = n_0 - n_1$). So Corollary 3.10 implies that $\mathcal{O}_{n_0|n_1}$ is a sum of blocks of $\mathcal{O}$. Hence:

$$\mathcal{O} = \bigoplus_{n_0+n_1=n} \mathcal{O}_{n_0|n_1}. \quad (5.5)$$

Let $\mathcal{F}$ be the full subcategory of $\mathcal{O}$ consisting of all finite-dimensional supermodules. Setting $\mathcal{F}_{n_0|n_1} := \mathcal{F} \cap \mathcal{O}_{n_0|n_1}$, the decomposition (5.5) induces a decomposition

$$\mathcal{F} = \bigoplus_{n_0+n_1=n} \mathcal{F}_{n_0|n_1}. \quad (5.6)$$

By [P, Theorem 4], the supermodule $L(\boldsymbol{b})$ is finite-dimensional if and only if $\boldsymbol{b}$ is *strictly dominant* in the sense that $b_1 > \cdots > b_n$. Consequently, $\mathcal{F}_{n_0|n_1}$ is the Serre subcategory of $\mathcal{O}$ generated by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}\}$.

The categorical $\mathfrak{sp}_{2\infty}$-action on $\mathcal{O}$ leaves the subcategory $\mathcal{F}$ invariant; this follows because the special projective functors from (3.8) send finite-dimensional supermodules to finite-dimensional supermodules. From this, we get induced categorical $\mathfrak{sl}_{+\infty}$-actions on $\mathcal{F}_{n_0|n_1} \hookrightarrow \mathcal{O}_{n_0|n_1}$ for each $n_0 + n_1 = n$. Recalling Lemma 5.9, let $\overline{\mathcal{O}}_{n_0|n_1}$ be the quotient of $\mathcal{O}_{n_0|n_1}$ by the Serre subcategory generated by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_{n_0|n_1} \setminus \mathbf{B}^{\#}_{n_0|n_1}\}$.

Writing $\overline{L}(\boldsymbol{b})$ for the canonical image of $L(\boldsymbol{b})$ in $\overline{\mathcal{O}}_{n_0|n_1}$, the irreducible objects of $\overline{\mathcal{O}}_{n_0|n_1}$ are represented by $\{\overline{L}(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^{\#}_{n_0|n_1}\}$.

Lemma 5.11. *$\overline{\mathcal{O}}_{n_0|n_1}$ is a TPC of the $\mathfrak{sl}_{+\infty}$-module $(V_0^+)^{\otimes n_0} \otimes (V_0^-)^{\otimes n_1}$.*

Proof. This follows from Lemma 5.9 and Theorem 5.2, since $\overline{\mathcal{O}}_{n_0|n_1}$ is the same as the quotient category $\mathcal{O}_{\boldsymbol{\sigma}}$ for $\boldsymbol{\sigma}$ as in that lemma. □

Let $\overline{\mathcal{F}}_{n_0|n_1}$ be the Serre subcategory of $\overline{\mathcal{O}}_{n_0|n_1}$ generated by $\{\overline{L}(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}\}$. We are going to consider the following commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{F}_{n_0|n_1} & \longrightarrow & \mathcal{O}_{n_0|n_1} \\ {\scriptstyle Q}\downarrow & & \downarrow \\ \overline{\mathcal{F}}_{n_0|n_1} & \longrightarrow & \overline{\mathcal{O}}_{n_0|n_1}. \end{array} \tag{5.7}$$

Here, the horizontal functors are the canonical inclusions, the right hand functor is the quotient functor, and the commutativity of the diagram then determines the left hand functor Q uniquely. The categorical $\mathfrak{sl}_{+\infty}$-action on $\mathcal{O}_{n_0|n_1}$ induces an action on the quotient category $\overline{\mathcal{O}}_{n_0|n_1}$. Then this restricts also to an action on $\overline{\mathcal{F}}_{n_0|n_1}$.

Lemma 5.12. *The functor $Q : \mathcal{F}_{n_0|n_1} \to \overline{\mathcal{F}}_{n_0|n_1}$ is a strongly equivariant equivalence of $\mathfrak{sl}_{+\infty}$-categorifications.*

Proof. It is immediate from the construction that Q is strongly equivariant. Also, since $\mathbf{B}^{+}_{n_0|n_1} \subseteq \mathbf{B}^{\#}_{n_0|n_1}$, the images under Q of all of the irreducible objects of $\mathcal{F}_{n_0|n_1}$ are nonzero. This is enough to show that Q is fully faithful; cf. [BD1, Lemma 2.13]. It just remains to show that Q is dense.

As it is a Serre subcategory of the Schurian category $\overline{\mathcal{O}}_{n_0|n_1}$, the category $\overline{\mathcal{F}}_{n_0|n_1}$ is itself Schurian; in particular, it has enough projectives. For $\boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}$, let $\overline{P}(\boldsymbol{b})$ be the projective cover of $\overline{L}(\boldsymbol{b})$ in $\overline{\mathcal{F}}_{n_0|n_1}$. It suffices to show that each $\overline{P}(\boldsymbol{b})$ is a summand of something in the essential image of Q. Then, to get all other objects of $\overline{\mathcal{F}}_{n_0|n_1}$, one can argue by considering a two-step projective resolution, using the exactness of Q and the Five Lemma.

Suppose in this paragraph that $\boldsymbol{a} \in \mathbf{B}^{+}_{n_0|n_1}$ is typical. Then the Verma supermodule $M(\boldsymbol{a})$ is projective in $\mathcal{O}_{n_0|n_1}$. Hence, the projective object $\overline{P}(\boldsymbol{a})$ may be realized as the largest quotient of the canonical image of $M(\boldsymbol{a})$ in $\overline{\mathcal{O}}_{n_0|n_1}$ which belongs to $\overline{\mathcal{F}}_{n_0|n_1}$. Typicality also implies that there are no strictly dominant $\boldsymbol{b} \in \mathbf{B}$ with $\boldsymbol{b} \prec \boldsymbol{a}$. We deduce that this largest quotient is $\overline{L}(\boldsymbol{a})$. This shows that $\overline{P}(\boldsymbol{a}) = \overline{L}(\boldsymbol{a})$.

Now take any $\boldsymbol{b} \in \mathbf{B}^{+}_{n_0|n_1}$. Applying Lemma 5.10, we can find a typical $\boldsymbol{a} \in \mathbf{B}^{+}_{n_0|n_1}$ connected to $\boldsymbol{b}$ by a sequence of the crystal operators $\tilde{e}_i, \tilde{f}_i$ $(i \in I_0)$. In view of Corollary 4.8, it follows that there is a sequence X of the functors E_i, F_i $(i \in I_0)$ such that $L(\boldsymbol{b})$ appears in the head of $XL(\boldsymbol{a})$. Passing to the quotient category, this shows that

$$\operatorname{Hom}_{\overline{\mathcal{F}}_{n_0|n_1}}(X\overline{L}(\boldsymbol{a}), \overline{L}(\boldsymbol{b})) \neq 0.$$

By the previous paragraph, $\overline{L}(\boldsymbol{a})$ is projective in $\overline{\mathcal{F}}_{n_0|n_1}$. Since X has a biadjoint, it sends projectives to projectives. This means that $X\overline{L}(\boldsymbol{a})$ is projective in $\overline{\mathcal{F}}_{n_0|n_1}$ too. We deduce that $\overline{P}(\boldsymbol{b})$ is a summand of $X\overline{L}(\boldsymbol{a})$. Since $QL(\boldsymbol{a}) = \overline{L}(\boldsymbol{a})$ and Q is strongly equivariant, we have that $Q(XL(\boldsymbol{a})) \cong X\overline{L}(\boldsymbol{a})$. Thus, $\overline{P}(\boldsymbol{b})$ is a summand of something in the essential image of Q. □

5.4. **Realization of $\mathcal{F}_{n_0|n_1}$ via $\mathfrak{gl}_{n_0|n_1}(\mathbb{C})$.** Through the subsection, we fix $n_0, n_1 \geq 0$ with $n_0 + n_1 = n$. The goal is to show that $\mathcal{F}_{n_0|n_1}$ is a highest weight category. To do this, we are going to give a different realization of the categories $\overline{\mathcal{F}}_{n_0|n_1} \hookrightarrow \overline{\mathcal{O}}_{n_0|n_1}$, then appeal to Lemma 5.12. We'll view $\mathbf{B}$ as a poset using the $\mathfrak{sl}_\infty$-Bruhat order $\preceq_{\boldsymbol{\sigma}}$ from (2.16), taking $\boldsymbol{\sigma} := (+, \dots, +, -, \dots, -)$ with n_0 entries $+$ and n_1 entries $-$. Recall also the subset $\mathbf{B}_0$ of $\mathbf{B}$ from (2.5). Let

$$\mathbf{B}^{n_0|n_1} := \{\boldsymbol{b} \in \mathbf{B} \mid b_1 > \cdots > b_{n_0}, b_{n_0+1} < \cdots < b_n\}, \quad \mathbf{B}_0^{n_0|n_1} := \mathbf{B}^{n_0|n_1} \cap \mathbf{B}_0 . \tag{5.8}$$

Recalling the posets (5.3)–(5.4) from the previous subsection, the map $\boldsymbol{b} \mapsto \boldsymbol{b}'$ from (2.14) defines poset isomorphisms $\mathbf{B}^{\#}_{n_0|n_1} \stackrel{\sim}{\rightarrow} \mathbf{B}_0$ and $\mathbf{B}^{+}_{n_0|n_1} \stackrel{\sim}{\rightarrow} \mathbf{B}_0^{n_0|n_1}$.

Lemma 5.13. *The subsets $\mathbf{B}_0$ and $\mathbf{B}_0^{n_0|n_1}$ are coideals in $\mathbf{B}$ and $\mathbf{B}^{n_0|n_1}$, respectively.*

Proof. This follows from Lemma 2.3, on noting that

$$\mathbf{B}_0 = \left\{ \boldsymbol{b} \in \mathbf{B} \;\middle|\; N^{\boldsymbol{\sigma}}_{[1,n_0]}(\boldsymbol{b}, 0) \geq n_0, N^{\boldsymbol{\sigma}}_{[1,n]}(\boldsymbol{b}, 0) = n_0 - n_1 \right\},$$

where $\boldsymbol{\sigma} = (+, \dots, +, -, \dots, -)$ as usual. □

Now we consider the general linear Lie superalgebra $\mathfrak{g}' := \mathfrak{gl}_{n_0|n_1}(\mathbb{C})$. Let $\mathfrak{h}'$ and $\mathfrak{b}'$ be the Cartan subalgebra and Borel subalgebra of $\mathfrak{g}'$ consisting of diagonal and upper triangular matrices, respectively. Let $\delta_1', \dots, \delta_n'$ be the basis for $(\mathfrak{h}')^*$ dual to the diagonal matrix units in $\mathfrak{h}'$. Then define $\mathcal{O}'_{n_0|n_1}$ to be the category of all $\mathfrak{g}'$-supermodules M such that

- M is finitely generated over $\mathfrak{g}'$;
- M is locally finite-dimensional over $\mathfrak{b}'$;
- M is semisimple over $\mathfrak{h}'$ with all weights of the form $\lambda_{\boldsymbol{b}}'$ for $\mathfrak{b} \in \mathbf{B}$, where

$$\lambda_{\boldsymbol{b}}' := \sum_{r=1}^n \lambda_{\boldsymbol{b},r}' \delta_r' \qquad \text{where} \qquad \lambda_{\boldsymbol{b},r}' = \begin{cases} b_r + r - 1 & \text{if } 1 \leq r \leq n_0, \\ -b_r + r - 2n_0 & \text{if } n_0 + 1 \leq r \leq n; \end{cases}$$

- for $\boldsymbol{b} \in \mathbf{B}$, the $\mathbb{Z}/2$-grading on the $\lambda_{\boldsymbol{b}}'$-weight space of M is concentrated in parity $\sum_{r=n_0+1}^n \lambda_{\boldsymbol{b},r}' \pmod 2$.

Note that $\mathcal{O}'_{n_0|n_1}$ is exactly the same as the Abelian category $\mathscr{O}$ defined in [B3, Lemma 2.2]. It is a special case of the category constructed in [BLW, Definition 3.7], taking the type $(\underline{n}, \underline{c})$ there to be $((1^n), (0^{n_0}, 1^{n_1}))$. In particular, [BLW, Theorem 3.10] verifies the following:

Lemma 5.14. *The category $\mathcal{O}'_{n_0|n_1}$ admits additional structure making it into a TPC of the $\mathfrak{sl}_\infty$-module $(V^+)^{\otimes n_0} \otimes (V^-)^{\otimes n_1}$.*

Let us give a little more detail about the highest weight structure here. The irreducible objects of $\mathcal{O}'_{n_0|n_1}$ are parametrized naturally by their highest weights. We denote the one of highest weight $\lambda_{\boldsymbol{b}}'$ by $L'(\boldsymbol{b})$. It can be constructed explicitly as the unique irreducible quotient of the corresponding Verma supermodule $M'(\boldsymbol{b})$. This is the standard object in the highest weight category $\mathcal{O}'_{n_0|n_1}$ indexed by $\boldsymbol{b} \in \mathbf{B}$.

Next, let $\mathcal{F}'_{n_0|n_1}$ be the subcategory of $\mathcal{O}'_{n_0|n_1}$ consisting of all of the finite-dimensional supermodules. Note $\mathcal{F}'_{n_0|n_1}$ may also be described as the Serre subcategory of $\mathcal{O}'_{n_0|n_1}$ generated by the irreducible objects $\left\{ L'(\boldsymbol{b}) \;\middle|\; \boldsymbol{b} \in \mathbf{B}^{n_0|n_1} \right\}$. This follows from Kac' classification of finite dimensional $\mathfrak{g}'$-supermodules in [K]. The argument there realizes each of the finite-dimensional $L'(\boldsymbol{b})$ as a quotient of a corresponding *Kac supermodule* $K'(\boldsymbol{b})$.

The categorical $\mathfrak{sl}_\infty$-action on $\mathcal{O}'_{n_0|n_1}$ restricts to an action on $\mathcal{F}'_{n_0|n_1}$. Taking the type $(\underline{n}, \underline{c})$ of [BLW, Definition 3.7] to be $((n_0, n_1), (0, 1))$, we get the following as another special case of [BLW, Theorem 3.10], recalling also [BLW, Definition 2.10] for this more general sort of TPC.

Lemma 5.15. *The category $\mathcal{F}'_{n_0|n_1}$ is a TPC of the $\mathfrak{sl}_\infty$-module $\bigwedge^{n_0} V^+ \otimes \bigwedge^{n_1} V^-$.*

Part of the content of Lemma 5.15 is that $\mathcal{F}'_{n_0|n_1}$ is a highest weight category. Its standard objects are the Kac supermodules $\{K'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^+\}$ mentioned already.

The idea now is to truncate $\mathcal{F}'_{n_0|n_1} \hookrightarrow \mathcal{O}'_{n_0|n_1}$ from $\mathfrak{sl}_\infty$ to $\mathfrak{sl}_{+\infty}$ to obtain our alternate realization of the categories $\overline{\mathcal{F}}_{n_0|n_1} \hookrightarrow \overline{\mathcal{O}}_{n_0|n_1}$. The construction we need for this has already been developed in [BLW, §2.8] (and is entirely analogous to §§4.1–5.1 above).

Recalling Lemma 5.13, let $\overline{\mathcal{O}}'_{n_0|n_1}$ be the quotient of $\mathcal{O}'_{n_0|n_1}$ by the Serre subcategory generated by $\{L'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B} \setminus \mathbf{B}_0\}$. Denoting the canonical image of $L'(\boldsymbol{b})$ in $\overline{\mathcal{O}}'_{n_0|n_1}$ by $\overline{L}'(\boldsymbol{b})$, the irreducible objects of $\overline{\mathcal{O}}'_{n_0|n_1}$ are represented by $\{\overline{L}'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_0\}$. Let $\overline{\mathcal{F}}'_{n_0|n_1}$ be the Serre subcategory of $\overline{\mathcal{O}}'_{n_0|n_1}$ generated by $\{\overline{L}'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_0^+\}$. Analogously to (5.7), we get a commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{F}'_{n_0|n_1} & \longrightarrow & \mathcal{O}'_{n_0|n_1} \\ {\scriptstyle R}\big\downarrow & & \big\downarrow \\ \overline{\mathcal{F}}'_{n_0|n_1} & \longrightarrow & \overline{\mathcal{O}}'_{n_0|n_1}. \end{array} \tag{5.9}$$

The categorical $\mathfrak{sl}_\infty$-actions on $\mathcal{O}'_{n_0|n_1}$ and $\mathcal{F}'_{n_0|n_1}$ restrict to actions of $\mathfrak{sl}_{+\infty}$. These then induce categorical $\mathfrak{sl}_{+\infty}$-actions on $\overline{\mathcal{O}}'_{n_0|n_1}$ and $\overline{\mathcal{F}}'_{n_0|n_1}$, so that all of the above functors are strongly equivariant.

Lemma 5.16. *Let $\widetilde{\mathcal{F}}'_{n_0|n_1}$ be the quotient of $\mathcal{F}'_{n_0|n_1}$ by the Serre subcategory generated by $\{L'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^{n_0|n_1} \setminus \mathbf{B}_0^{n_0|n_1}\}$. The functor $R : \mathcal{F}'_{n_0|n_1} \to \overline{\mathcal{F}}'_{n_0|n_1}$ induces an equivalence $\widetilde{R} : \widetilde{\mathcal{F}}'_{n_0|n_1} \to \overline{\mathcal{F}}'_{n_0|n_1}$.*

Proof. By the universal property of Serre quotients, R induces $\widetilde{R} : \widetilde{\mathcal{F}}'_{n_0|n_1} \to \overline{\mathcal{F}}'_{n_0|n_1}$. As in the proof of Lemma 5.12, $\widetilde{R}$ is fully faithful. To show that it is dense, we show equivalently that R is dense, again by mimicking the arguments from the proof of Lemma 5.12. This involves replacing the notion of atypicality and the crystal structure used in the proof of that lemma with their counterparts in the category $\mathcal{O}'$. For $\boldsymbol{b} \in \mathbf{B}^{n_0|n_1}$, its atypicality is the number of pairs $1 \leq r < s \leq n$ such that $b_r = b_s$. The appropriate crystal structure, and the required analog of Corollary 4.8, are described in [BLW, Lemma 2.23]. Actually, the bijection $\mathbf{B}^+_{n_0|n_1} \xrightarrow{\sim} \mathbf{B}_0^{n_0|n_1}, \boldsymbol{b} \mapsto \boldsymbol{b}'$ preserves atypicality, and intertwines the crystal operators $\tilde{e}_i, \tilde{f}_i$ $(i \in I_0)$ from §5.10 with the crystal operators $\tilde{e}_i, \tilde{f}_i$ $(i \in I_0)$ defined in [BLW]. Then the argument in the proof of Lemma 5.12 (dependent especially on the combinatorial Lemma 5.10) carries over almost immediately. □

Lemma 5.17. *The categories $\overline{\mathcal{O}}'_{n_0|n_1}$ and $\overline{\mathcal{F}}'_{n_0|n_1}$ are TPCs of $(V_0^+)^{\otimes n_0} \otimes (V_0^-)^{\otimes n_1}$ and $\bigwedge^{n_0} V_0^+ \otimes \bigwedge^{n_1} V_0^-$, respectively.*

Proof. For $\overline{\mathcal{O}}'_{n_0|n_1}$, our statement follows immediately as a special case of [BLW, Theorem 2.19]. The same result shows that $\widetilde{\mathcal{F}}'_{n_0|n_1}$ is a TPC of $\bigwedge^{n_0} V_0^+ \otimes \bigwedge^{n_1} V_0^-$. It remains to appeal to Lemma 5.16 to get the result for $\overline{\mathcal{F}}'_{n_0|n_1}$. □

Theorem 5.18. *The $\mathfrak{sl}_{+\infty}$-categorifications $\overline{\mathcal{O}}_{n_0|n_1}$ and $\overline{\mathcal{O}}'_{n_0|n_1}$ are strongly equivariantly equivalent via an equivalence which sends $\overline{L}(\boldsymbol{b})$ to a copy of $\overline{L}'(\boldsymbol{b}')$ for each $\boldsymbol{b} \in \mathbf{B}^{\#}_{n_0|n_1}$.*

Proof. In Lemmas 5.11 and 5.17, we have shown that both categories are TPCs of $(V_0^+)^{\otimes n_0} \otimes (V_0^-)^{\otimes n_1}$. Now the result follows from the uniqueness theorem for such TPCs, which is a special case of [BLW, Theorem 2.12]. □

Corollary 5.19. *The $\mathfrak{sl}_{+\infty}$-categorifications $\overline{\mathcal{F}}_{n_0|n_1}$ and $\overline{\mathcal{F}}'_{n_0|n_1}$ are strongly equivariantly equivalent via an equivalence which sends $\overline{L}(\boldsymbol{b})$ to a copy of $\overline{L}'(\boldsymbol{b}')$ for each $\boldsymbol{b} \in \mathbf{B}^+_{n_0|n_1}$.*

Proof. Recall $\overline{\mathcal{F}}_{n_0|n_1}$ is the Serre subcategory of $\overline{\mathcal{O}}_{n_0|n_1}$ generated by $\{\overline{L}(\boldsymbol{b}) \,|\, \boldsymbol{b} \in \mathbf{B}^+_{n_0|n_1}\}$, $\overline{\mathcal{F}}'_{n_0|n_1}$ is the Serre subcategory of $\overline{\mathcal{O}}'_{n_0|n_1}$ generated by $\{\overline{L}'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_0^{n_0|n_1}\}$, and the map $\boldsymbol{b} \mapsto \boldsymbol{b}'$ is a bijection between $\mathbf{B}^+_{n_0|n_1}$ and $\mathbf{B}_0^{n_0|n_1}$. Then apply Theorem 5.18. □

Corollary 5.20. *The category $\mathcal{F}_{n_0|n_1}$ is a TPC of $\bigwedge^{n_0} V_0^+ \otimes \bigwedge^{n_1} V_0^-$. In particular, it is a highest weight category with weight poset $(\mathbf{B}^+_{n_0|n_1}, \preceq)$ and irreducible objects represented by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^+_{n_0|n_1}\}$.*

Proof. This follows from Lemma 5.12, Corollary 5.19 and Lemma 5.17. □

5.5. **Realization of $\mathcal{F}$ via arc algebras.** In the final subsection, we are going to briefly explain another realization of the category $\mathcal{F}$ in terms of the generalized Khovanov arc algebras of [BS1]. We will assume the reader is familiar with the language and constructions in [BS1, BS2].

Let Λ be the set of weights in the diagrammatic sense of [BS1, §2] drawn on a number line with vertex set I_0, such that the number of vertices labelled $\times$ plus the number of vertices labelled $\circ$ plus two times the number of vertices labelled $\vee$ is equal to n; all of the (infinitely many) remaining vertices are labelled $\wedge$. The set Λ is in bijection with

$$\mathbf{B}^+ := \{\boldsymbol{b} \in \mathbf{B} \mid b_1 > \cdots > b_n\} = \bigcup_{n_0+n_1=n} \mathbf{B}^+_{n_0|n_1} \tag{5.10}$$

according to the following *weight dictionary*. Given $\boldsymbol{b} \in \mathbf{B}^+$, let

$$I_\vee(\boldsymbol{b}) := \{b_r \mid r = 1, \dots, n, b_r > 0\}$$
$$I_\wedge(\boldsymbol{b}) := I_0 \setminus \{1 - b_r \mid r = 1, \dots, n, b_r \leq 0\}.$$

Then we identify $\boldsymbol{b}$ with the element of Λ whose ith vertex is labelled

$$\begin{cases} \circ & \text{if } i \text{ does not belong to either } I_\vee(\lambda) \text{ or } I_\wedge(\lambda), \\ \vee & \text{if } i \text{ belongs to } I_\vee(\lambda) \text{ but not to } I_\wedge(\lambda), \\ \wedge & \text{if } i \text{ belongs to } I_\wedge(\lambda) \text{ but not to } I_\vee(\lambda), \\ \times & \text{if } i \text{ belongs to both } I_\vee(\lambda) \text{ and } I_\wedge(\lambda). \end{cases} \tag{5.11}$$

Let K_Λ be the generalized Khovanov algebra associated to the set Λ as defined in [BS1]. This is a basic algebra with isomorphism classes of irreducible representations indexed in a canonical way by the set Λ.

Theorem 5.21. *There is an equivalence of categories between $\mathcal{F}$ and the category K_Λ-mod of finite-dimensional left K_Λ-modules. It sends $L(\boldsymbol{b})$ ($\boldsymbol{b} \in \mathbf{B}^+$) to the irreducible K_Λ-module indexed by the element of Λ associated to $\boldsymbol{b}$ according to the above weight dictionary.*

Proof. Corresponding to the decomposition (5.10), we have that $\Lambda = \bigcup_{n_0+n_1=n} \Lambda(n_0|n_1)$ where $\Lambda(n_0|n_1)$ consists of the weights in Λ whose diagrams have n_0 entries equal to $\vee$ or $\times$ and n_1 entries equal to $\vee$ or $\circ$. The algebra K_Λ decomposes as $\bigoplus_{n_0+n_1=n} K_{\Lambda(n_0|n_1)}$. In view of (5.6), to prove the theorem, it suffices to show that $\mathcal{F}_{n_0|n_1}$ is equivalent to $K_{\Lambda(n_0|n_1)}$-*mod*.

By Lemma 5.12, Lemma 5.16 and Corollary 5.19, $\mathcal{F}_{n_0|n_1}$ is equivalent to the quotient $\widetilde{\mathcal{F}}'_{n_0|n_1}$ of $\mathcal{F}'_{n_0|n_1}$ by the Serre subcategory generated by $\{L'(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}^{n_0|n_1} \setminus \mathbf{B}_0^{n_0|n_1}\}$.

By the main theorem of [BS2], $\mathcal{F}'_{n_0|n_1}$ is equivalent to the category K_Δ-*mod* of finite-dimensional modules over another arc algebra K_Δ. The set Δ of weights this time are drawn on a number line with vertex set $\mathbb{Z}$, such that the number of vertices labelled $\vee$ or $\times$ is n_0, and the number labelled $\vee$ or $\circ$ is n_1. Under the weight dictionary from the introduction of [BS2], the set $\mathbf{B}_0^{n_0|n_1}$ is identified with the subset Δ_0 of Δ consisting of weights $\boldsymbol{b}$ whose diagrams have label $\wedge$ on vertex i for all $i \leq 0$.

We conclude that $\widetilde{\mathcal{F}}'_{n_0|n_1}$ is equivalent to the category of finite-dimensional modules over the algebra $\bigoplus_{\boldsymbol{a},\boldsymbol{b}\in\Delta_0} e_{\boldsymbol{a}} K_\Delta e_{\boldsymbol{b}}$, where $e_{\boldsymbol{b}}$ denotes the primitive idempotent in K_Δ indexed by $\boldsymbol{b}$. Noting that Δ_0 is in bijection with $\Lambda(n_0|n_1)$ via the map which deletes all vertices indexed by $\mathbb{Z}_{\leq 0}$, this algebra is obviously isomorphic to $K_{\Lambda(n_0|n_1)}$. □

Theorem 5.21 has a number of consequences for the structure of the category $\mathcal{F}$. We refer to the introduction of [BS2] for a comprehensive list: the present situation is entirely analogous. It shows moreover that any block of $\mathcal{F}$ of atypicality r (which in the diagrammatic setting is the number of vertices labelled $\vee$ in weights belonging to the block) is Morita equivalent to the algebra $K_r^{+\infty}$ from [BS1]. Thus, the category $\mathcal{F}$ gives the first known occurrence "in nature" of the algebras $K_r^{+\infty}$.

References

[B1] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$, *Advances Math.* **182** (2004), 28–77.

[B2] J. Brundan, Tilting modules for Lie superalgebras, *Commun. Alg.* **32** (2004), 2251–2268.

[B3] J. Brundan, Representations of the general linear Lie superalgebra in the BGG category $\mathcal{O}$, in: "Developments and Retrospectives in Lie Theory: Algebraic Methods," eds. G. Mason et al., Developments in Mathematics **38**, Springer, 2014, pp. 71–98.

[BD1] J. Brundan and N. Davidson, Categorical actions and crystals, *Contemp. Math.* **684** (2017), 116–159.

[BD2] J. Brundan and N. Davidson, Type A blocks of super category $\mathcal{O}$, *J. Algebra* **473** (2017), 447–480.

[BE] J. Brundan and A. Ellis, Monoidal supercategories, *Commun. Math. Phys.* **351** (2017), 1045–1089.

[BK] J. Brundan and A. Kleshchev, Hecke-Clifford superalgebras, crystals of type $A_{2\ell}^{(2)}$ and modular branching rules for $\widehat{S}_n$, *Represent. Theory* **5** (2001), 317-403.

[BLW] J. Brundan, I. Losev, and B. Webster, Tensor product categorifications and the super Kazhdan-Lusztig conjecture, to appear in *IMRN*.

[BS1] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov's diagram algebras I: cellularity, *Mosc. Math. J.* **11** (2011), 685–722.

[BS2] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov's diagram algebras IV: the general linear supergroup, *JEMS* **14** (2012), 373–419.

[C] C.-W. Chen, Reduction method for representations of queer Lie superalgebras, *J. Math. Phys.* **57** (2016), no. 5, 051703, 12 pp.

[CK] S.-J. Cheng and J.-H. Kwon, Finite-dimensional half-integer weight modules over queer Lie superalgebras, *Commun. Math. Phys.* **346** (2016) 945–965.

[CKW] S.-J. Cheng, J.-H. Kwon, and W. Wang, Character formulae for queer Lie superalgebras and canonical bases of types A/C, to appear in *Commun. Math. Phys.*.

[CPS] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85–99.

[D] N. Davidson, Type B blocks of super category $\mathcal{O}$, in preparation.

[HKS] D. Hill, J. Kujawa and J. Sussan, Degenerate affine Hecke-Clifford algebras and type Q Lie superalgebras, *Math. Zeit.* **268** (2011), 1091–1158.

[J] J. C. Jantzen, *Lectures on Quantum Groups*, AMS, 1995.

[K] V. Kac, Characters of typical representations of classical Lie superalgebras, *Commun. Algebra* **5** (1977), 889–897.

[KKT] S.-J. Kang, M. Kashiwara and S. Tsuchioka, Quiver Hecke superalgebras, *J. Reine Angew. Math.* **711** (2016), 1–54.

[KL1] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347.

[KL2] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups II, *Trans. Amer. Math. Soc.* **363** (2011), 2685–2700.

[LW] I. Losev and B. Webster, On uniqueness of tensor products of irreducible categorifications, *Selecta Math.* **21** (2015), 345–377.

[L] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, 1993.

[N] M. Nazarov, Young's symmetrizers for projective representations of the symmetric group, *Advances Math.* **127** (1997), 190–257.

[P] I. Penkov, Characters of typical irreducible finite-dimensional $\mathfrak{q}(n)$-modules, *Func. Anal. Appl.* **20** (1986), 30–37.

[R] R. Rouquier, 2-Kac-Moody algebras; `arXiv:0812.5023`.

[W1] B. Webster, Canonical bases and higher representation theory, *Compositio Math.* **151** (2015), 121–166.

[W2] B. Webster, Knot invariants and higher representation theory, *Mem. Amer. Math. Soc.* **1191** (2017), 133pp..

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `brundan@uoregon.edu`

Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
E-mail address: `njd@ou.edu`