LOWER BOUNDS FOR DIMENSIONS OF IRREDUCIBLE REPRESENTATIONS OF SYMMETRIC GROUPS

ALEXANDER KLESHCHEV, LUCIA MOROTTI, AND PHAM HUU TIEP

Abstract. We give new, explicit and asymptotically sharp, lower bounds for dimensions of irreducible modular representations of finite symmetric groups.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic $p > 0$. We denote by $\mathscr{P}(n)$ the set of all partitions of n and by $\mathscr{P}_p(n)$ the set of all p-regular partitions of n, see [**4**]. Given a partition $\mu = (\mu_1, \mu_2, \dots) \in \mathscr{P}(m)$ and $n \in \mathbb{Z}_{\geq m+\mu_1}$, we denote

$$(n-m, \mu) := (n-m, \mu_1, \mu_2, \dots) \in \mathscr{P}(n).$$

Let S_n be the symmetric group on n letters, and denote by D^λ the irreducible $\mathbb{F}\mathsf{S}_n$-module corresponding to a p-regular partition λ of n, see [**4**]. In [**5**], James gave sharp lower bounds for $\dim D^{(n-m,\mu)}$ for $m \leq 4$, and here we obtain asymptotically sharp lower bounds for all m.

Set

$$\delta_p := \begin{cases} 0 & \text{if } p \neq 2, \\ 1 & \text{if } p = 2. \end{cases}$$

For integers $m \geq 0$ and n we define the rational numbers

$$\begin{aligned} C_m^p(n) &:= p^m \binom{n/p - \delta_p}{m} \\ &= \frac{1}{m!} \prod_{i=0}^{m-1} (n - (\delta_p + i)p) \\ &= \begin{cases} \frac{n(n-p)(n-2p)\cdots(n-(m-1)p)}{m!} & \text{if } p > 2, \\ \frac{(n-p)(n-2p)\cdots(n-mp)}{m!} & \text{if } p = 2. \end{cases} \end{aligned}$$

Our first main result develops [**5**] as follows:

Theorem A. *Let $m \geq 4$, p a prime, $n \geq p(\delta_p + m - 2)$, and let $\mu \in \mathscr{P}_p(m)$. Then for $\lambda := (n-m, \mu) \in \mathscr{P}_p(n)$ we have*

$$\dim D^\lambda \geq C_m^p(n).$$

2010 *Mathematics Subject Classification.* 20C30, 20C20.

The first author was supported by the NSF grant DMS-1700905 and the DFG Mercator program through the University of Stuttgart. The second author was supported by the DFG grant MO 3377/1-1 and the DFG Mercator program through the University of Stuttgart. The third author was supported by the NSF (grants DMS-1839351 and DMS-1840702), and the Joshua Barlaz Chair in Mathematics. This work was also supported by the NSF grant DMS-1440140 and Simons Foundation while all three authors were in residence at the MSRI during the Spring 2018 semester.

The authors are grateful to the referee for careful reading and helpful comments on the paper.

Note that $C^p_m(n) \approx n^m/m!$ when p, m are fixed and $n \to \infty$. Hence, in view of [**5**, Theorem 1], the lower bound of Theorem A is asymptotically sharp. Theorem A will be crucially used in [**11**].

While Theorem A requires that n is relatively large compared to m, we also prove the following universal lower bound which improves [**3**, Theorem 5.1].

Theorem B. *Let $p \geq 3$ and $\lambda \in \mathscr{P}_p(n)$. Let $\lambda^{\mathtt{M}} = (\lambda^{\mathtt{M}}_1, \lambda^{\mathtt{M}}_2, \dots)$ be the p-regular partition determined from $D^\lambda \otimes \mathtt{sgn} \cong D^{\lambda^{\mathtt{M}}}$. Let a be minimal such that $D^\lambda{\downarrow}_{\mathsf{S}_{n-a}}$ contains a submodule of dimension 1, and let*

$$k := \max\{\lambda_1, \lambda^{\mathtt{M}}_1\},\ t := \max\{n-k, a\}.$$

Then

$$\dim D^\lambda \geq 2 \cdot 3^{(t-2)/3}.$$

For $p = 2$ we have the following result, which is a special case of Lemma 2.7:

Theorem C. *Let $p = 2$ and $\lambda \in \mathscr{P}_2(n)$. Then $\dim D^\lambda \geq 2^{n-\lambda_1}$.*

2. Main results

2.1. **Preliminaries on modular branching rules.** In this subsection, we review modular branching rules for symmetric groups, which will be used below without further comment. The reader is referred to [**8–10**] for more details.

We identify $\lambda \in \mathscr{P}(n)$ and its Young diagram, which consists of nodes, i.e. elements of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. Given any node $A = (r, s)$, its *residue* $\operatorname{res} A := s - r \pmod p \in \mathbb{Z}/p\mathbb{Z}$. For $i \in \mathbb{Z}/p\mathbb{Z}$ a node $A \in \lambda$ (resp. $B \notin \lambda$) is called *i-removable* (resp. *i-addable*) for λ if $\operatorname{res} A = i$ and $\lambda_A := \lambda \setminus \{A\}$ (resp. $\lambda^B := \lambda \cup \{B\}$) is a Young diagram of a partition.

Let $\lambda \in \mathscr{P}_p(n)$. Labeling the i-addable nodes of λ by $+$ and the i-removable nodes of λ by $-$, the *i-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs. The *reduced i-signature* of λ is obtained from the i-signature by successively erasing all neighbouring pairs of the form $-+$. The nodes corresponding to $-$'s in the reduced i-signature are called *i-normal* for λ. The leftmost i-normal node is called *i-good* . A node is called *removable* (resp. *normal*, *good*) if it is i-removable (resp. i-normal, i-good) for some i. We denote

$$\varepsilon_i(\lambda) := \sharp\{i\text{-normal nodes of } \lambda\}.$$

If $\varepsilon_i(\lambda) > 0$, let A be the i-good node of λ and set $\tilde{e}_i\lambda := \lambda_A$. Let e_i be the *i-restriction functor* so that $V{\downarrow}_{\mathsf{S}_{n-1}} = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} e_i V$ for any $\mathbb{F}\mathsf{S}_n$-module V.

Lemma 2.1. *Let $\lambda \in \mathscr{P}_p(n)$ and $i \in \mathbb{Z}/p\mathbb{Z}$. Then:*

- (i) *$e_i D^\lambda \neq 0$ if and only if $\varepsilon_i(\lambda) > 0$, in which case $e_i D^\lambda$ is a self-dual indecomposable module with socle and head both isomorphic to $D^{\tilde{e}_i\lambda}$.*
- (ii) *Let A be a removable node of λ such that λ_A is p-regular. Then D^{λ_A} is a composition factor of $e_i D^\lambda$ if and only if A is i-normal, in which case $[e_i D^\lambda : D^{\lambda_A}]$ is one more than the number of i-normal nodes for λ above A.*

It follows easily from Lemma 2.1 that $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$ is irreducible if and only if the top removable node of λ is its only normal node, in which case λ is called a *Jantzen-Seitz* (or *JS*) partition, cf. [**6, 7**].

2.2. **Properties of $C^p_m(n)$.**

Lemma 2.2. *For any $q \in \mathbb{R}_{\geq 1}$, $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{R}_{\geq k}$ we have*

$$\prod_{i=0}^{k}(a-i) \leq \left(a-k+\frac{k}{q}\right)\prod_{i=0}^{k-1}\left(a-i-\frac{1}{q}\right).$$

Proof. Induction on k. For inductive step, it suffices to check that

$$a-(k+1) \leq \left(a-k-\frac{1}{q}\right)\left(a-k-1+\frac{k+1}{q}\right)\left(a-k+\frac{k}{q}\right)^{-1},$$

which is elementary. □

Lemma 2.3. *Let $m \geq 1$. Then:*

(i) $C^p_m(n) = C^p_m(n-p) + pC^p_{m-1}(n-p)$.

(ii) *If $n \geq p(\delta_p + m - 1)$ then $C^p_m(n) \leq C^p_m(n-1) + C^p_{m-1}(n-1)$.*

Proof. (i) follows from

$$\begin{aligned}\frac{C^p_m(n)}{p^m} = \binom{n/p-\delta_p}{m} &= \binom{n/p-\delta_p-1}{m} + \binom{n/p-\delta_p-1}{m-1}\\ &= \binom{(n-p)/p-\delta_p}{m} + \binom{(n-p)/p-\delta_p}{m-1}\\ &= \frac{C^p_m(n-p)}{p^m} + \frac{C^p_{m-1}(n-p)}{p^{m-1}}.\end{aligned}$$

(ii) Note that

$$\begin{aligned}&C^p_m(n-1) + C^p_{m-1}(n-1)\\ &= \frac{1}{m!}\prod_{i=0}^{m-1}(n-1-(\delta_p+i)p) + \frac{1}{(m-1)!}\prod_{i=0}^{m-2}(n-1-(\delta_p+i)p)\\ &= \frac{1}{m!}((n-1-(\delta_p+m-1)p)+m)\prod_{i=0}^{m-2}(n-1-(\delta_p+i)p).\end{aligned}$$

Multiplying by $m!$ and dividing by p^m, it suffices to prove that

$$\prod_{i=0}^{m-1}\left(\frac{n}{p}-\delta_p-i\right) \leq \left(\frac{n}{p}-\delta_p-m+1+\frac{m-1}{p}\right)\prod_{i=0}^{m-2}\left(\frac{n}{p}-\delta_p-i-\frac{1}{p}\right)$$

This holds by Lemma 2.2 with $a = \frac{n}{p} - \delta_p$, $k = m-1$ and $q = p$. □

2.3. **Proof of Theorem A.**

Lemma 2.4. [**5**] *Let $1 \leq m \leq 4$, $\mu \in \mathscr{P}_p(m)$, and n be such that $(n-m,\mu) \in \mathscr{P}_p(n)$. Then*

$$\dim D^{(n-m,\mu)} \geq \begin{cases} n-2 & \text{if } m=1,\\ (n^2-5n+2)/2 & \text{if } m=2,\\ (n^3-9n^2+14n)/6 & \text{if } m=3.\\ (n^4-14n^3+47n^2-34n)/24 & \text{if } m=4.\end{cases}$$

Theorem 2.5. *Let $m \geq 4$, $n \geq p(\delta_p + m - 2)$, $\mu \in \mathscr{P}_p(m)$, and suppose that $\lambda := (n-m, \mu) \in \mathscr{P}_p(n)$. Then $\dim D^\lambda \geq C^p_m(n)$.*

Proof. If $p(\delta_p + m - 2) \leq n \leq p(\delta_p + m - 1)$, we have $C^p_m(n) \leq 0$ and there is nothing to prove. So we assume that $n > p(\delta_p + m - 1)$.

Let $m = 4$ and set $f(n) := (n^4 - 14n^3 + 47n^2 - 34n)/24$, see Lemma 2.4. If $p \geq 3$ then $n > p(\delta_p + m - 1) \geq 9$ and $f(n) \geq C^p_m(n)$, and so we are done in this case. If $p = 2$, then $n > p(\delta_p + m - 1) \geq 8$, while $f(n) \geq C^p_m(n)$ for $n > 10$. For $n = 9$ and 10, the claimed dimension bound holds by inspection of [**4**, Tables].

So, in addition to $n > p(\delta_p + m - 1)$ we now assume that $m \geq 5$. We apply induction on n. Note that $n > p(\delta_p + m - 1)$ implies $n - 2m > 1$, unless $p = 2$, in which case we have $n - 2m \geq 1$. Hence $\lambda_1 - \lambda_2 \geq 2$, unless $p = 2$ and $\lambda = (m+1, m)$. In the exceptional case, D^λ is the basic spin module of dimension 2^m, and the bound boils down to $2^m \geq \frac{(2m-1)!!}{m!}$, which is easily checked. Thus we may assume that $\lambda_1 - \lambda_2 \geq 2$. Let $A = (1, \lambda_1)$ be the top removable node of λ.

Suppose first that λ is not JS. Then A is not the only normal node of λ, so there exists a good node B of λ with $B \neq A$. Then D^{λ_A} and D^{λ_B} are composition factors of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. The inductive assumption applies to D^{λ_A} to give $\dim D^{\lambda_A} \geq C^p_m(n-1)$. Since $m \geq 5$, the inductive assumption applies to D^{λ_B} to give $\dim D^{\lambda_B} \geq C^p_{m-1}(n-1)$. Now the result follows from Lemma 2.3(ii).

Next, let λ be JS, and let B be the second removable node from the top. Suppose first that $\lambda_1 - \lambda_2 > p$ and for $t = 0, 1, 2, \dots, p$, set $A_t := (1, \lambda_1 + 1 - t)$. We denote

$$\lambda^{(t)} := (\lambda_1 - t, \lambda_2, \lambda_3, \dots) = (\dots(\lambda_{A_1})_{A_2}\dots)_{A_t} \qquad (1 \leq t \leq p).$$

As λ is JS, we have $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}} \cong D^{\lambda^{(1)}}$. As λ is JS, we have $\operatorname{res} B = \operatorname{res} A_0 = \operatorname{res} A_p$. So successive application of the branching rules implies that $D^\lambda{\downarrow}_{\mathsf{S}_{n-p+1}}$ contains composition factors $D^{\lambda^{(p-1)}}$ and $D^{(\lambda_B)^{(p-2)}}$, the second one with multiplicity at least $p-2$. Modular branching rules now imply that $[D^{\lambda^{(p-1)}}{\downarrow}_{\mathsf{S}_{n-p}} : D^{(\lambda_B)^{(p-1)}}] = 2$, and so we deduce that $D^\lambda{\downarrow}_{\mathsf{S}_{n-p}}$ contains composition factors $D^{\lambda^{(p)}}$ and $D^{(\lambda_B)^{(p-1)}}$, the second one with multiplicity at least p. Now result follows from the inductive assumption and Lemma 2.3(i).

Thus we may assume that λ is JS, and $\lambda_1 - \lambda_2 \leq p$. If $p \geq 3$, we deduce

$$p \geq \lambda_1 - \lambda_2 \geq n - 2m > p(\delta_p + m - 1) - 2m = p(m-1) - 2m = (p-2)m - p,$$

implying $p = 3$, $m = 5$ and $n = 13$, hence $\lambda = (8, 5)$, which is not JS.

Finally, let $p = 2$. Then $\lambda_1 - \lambda_2 = 2$ since λ is JS. The assumption $n > p(\delta_p + m - 1) = 2m$ now implies that $\lambda = (m+2, m)$ or $\lambda = (m+1, m-1, 1)$. In the first case, λ is a basic spin module of dimension 2^m, and the required bound boils down to $2^m \geq \frac{(2m)!!}{m!}$, which is actually an equality! In the second case we have $\lambda = (m+1, m-1, 1)$. By the modular branching rules, $D^{(m, m-2, 1)}$ appears in $D^\lambda{\downarrow}_{\mathsf{S}_{n-2}}$ with multiplicity at least 2, and the result follows from

$$2C^2_{m-1}(n-2) = 2\frac{(2m-3)!!}{(m-1)!} > \frac{(2m-1)!!}{m!} = C^2_m(n).$$

The theorem is proved. $\square$

Remark 2.6. Some other lower bounds on the dimensions of irreducible modular representations of S_n were obtained in [**12**], based on an improved version [**12**, Theorems (5.2), (5.6)] of James' [**5**, Lemma 4].

2.4. **Proof of Theorems B and C.**

Lemma 2.7. *Let $\lambda \in \mathscr{P}_p(n)$. Then*

$$\dim D^\lambda \geq \prod_{i\geq p} \lceil i/(p-1)\rceil^{\lambda_i}.$$

In particular,

$$\dim D^\lambda \geq 2^{n-\lambda_1-\ldots-\lambda_{p-1}}.$$

Proof. Let $A_1, A_2, \ldots$ be the removable nodes counting from the top and let $A = A_j$ be minimal such that λ_{A_j} is p-regular. If A_j is on row i then $(j-1)(p-1) < i \leq j(p-1)$ and nodes $A_1, \ldots, A_j$ are all normal of the same residue. So

$$[D^\lambda{\downarrow}_{\mathsf{S}_{n-1}} : D^{\lambda_A}] = j = \lceil i/(p-1)\rceil,$$

from which the lemma follows by induction. □

Lemma 2.8. *Let $a, b \geq 0$ with $a-b \geq p-1$. Then $\dim D^{(a,b)} \geq 2^b$.*

Proof. If $a-b > p-1$ then $D^{(a-1,b)}$ is a composition factor of $D^{(a,b)}{\downarrow}_{\mathsf{S}_{a+b-1}}$, while if $a-b = p-1$ then $D^{(a,b-1)}$ is a composition factor with multiplicity 2 of $D^{(a,b)}{\downarrow}_{\mathsf{S}_{a+b-1}}$. The lemma then follows.

Alternatively, the lemma follows from Lemma 2.7 and [**1**, Lemma 2.3]. □

Lemma 2.9. *Let $\lambda \in \mathscr{P}_p(n)$. If $\lambda_1 \geq p-1$ and $((\lambda_1)^{\mathtt{M}}, (\lambda_2, \lambda_3, \ldots)^{\mathtt{M}}) \in \mathscr{P}_p(n)$ then $\dim D^\lambda \geq 2^{n-\lambda_1}$.*

Proof. The lemma follows from Lemma 2.7 and [**1**, Lemma 2.2]. □

The following result improves [**3**, Theorem 5.1].

Theorem 2.10. *Let $p \geq 3$ and $\lambda \in \mathscr{P}_p(n)$. Further let $k := \max\{\lambda_1, \lambda_1^{\mathtt{M}}\}$ and $a \in \mathbb{Z}_{>0}$ be minimal such that $D^\lambda{\downarrow}_{\mathsf{S}_{n-a}}$ contains a submodule of dimension 1. Then*

$$\dim D^\lambda \geq 2 \cdot 3^{(\max\{n-k,a\}-2)/3}.$$

Proof. If $\lambda \in \{(n), (n)^{\mathtt{M}}\}$ then the statement clearly holds. So we will assume that this is not the case. If μ is obtained from λ by removing a sequence of b good nodes, then $\mu^{\mathtt{M}}$ can also be obtained from $\lambda^{\mathtt{M}}$ by removing a sequence of b good nodes. In particular $\max\{\mu_1, \mu_1^{\mathtt{M}}\} \leq k$. Also if $D^\mu{\downarrow}_{\mathsf{S}_{n-b-c}}$ contains a submodule of dimension 1 then $c \geq a-b$ by minimality of a. By induction we can assume that $\dim D^\mu \geq 2\cdot 3^{(\max\{n-k,a\}-2-b)/3}$.

Case 1. *λ is not JS.* If $\varepsilon_i(\lambda) \geq 2$ for some i then $[D^\lambda{\downarrow}_{\mathsf{S}_{n-1}} : D^{\tilde{e}_i\lambda}] \geq 2$ and $D^{\tilde{e}_i\lambda} \subseteq D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. Otherwise there exist $i \neq j$ with $\varepsilon_i(\lambda), \varepsilon_j(\lambda) = 1$ and then $D^{\tilde{e}_i\lambda} \oplus D^{\tilde{e}_j\lambda} \subseteq D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. In either case

$$\dim D^\lambda \geq 4 \cdot 3^{(\max\{n-k,a\}-3)/3} > 2 \cdot 3^{(\max\{n-k,a\}-2)/3}.$$

Case 2. *λ is JS.* Let A be the top normal node of λ. Then A is good in λ and $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}} \cong D^{\lambda_A}$. From [**5**, Lemma 3] we have that λ_A has at least 2 normal nodes. If λ_A has at least 3 normal nodes we can conclude similarly to the previous case that

$$\dim D^\lambda \geq 6 \cdot 3^{(\max\{n-k,a\}-4)/3} > 2 \cdot 3^{(\max\{n-k,a\}-2)/3}.$$

So we may assume that λ_A has exactly 2 normal nodes. Further notice that $D^{(2)}$ and $D^{(1^2)}$ are both composition factors of $D^\lambda{\downarrow}_{\mathsf{S}_2}$ since $\lambda \notin \{(n),(n)^{\mathtt{M}}\}$. Since $p \geq 3$, it follows that

$$D^\lambda{\downarrow}_{\mathsf{S}_{n-2,2}} \cong (D^\mu \boxtimes D^{(2)}) \oplus (D^\nu \boxtimes D^{(1^2)}),$$

where $\mu,\nu \in \mathscr{P}_p(n-2)$ can each be obtained from λ_A by removing a good node. In particular if $D^\pi \subseteq D^\lambda{\downarrow}_{\mathsf{S}_{n-3}}$ then π can be obtained from λ by removing a sequence of 3 good nodes. Also $\mu = \tilde{e}_i\lambda_A$ and $\nu = \tilde{e}_j\lambda_A$ with $i \neq j$.

If μ and ν are not both JS then similar to before

$$\dim D^\lambda \geq 6 \cdot 3^{(\max\{n-k,a\}-5)/3} = 2 \cdot 3^{(\max\{n-k,a\}-2)/3}.$$

If $p \geq 5$ and μ and ν are both JS, then $D^\lambda{\downarrow}_{\mathsf{S}_{n-3}}$ has only 2 composition factors. From

$$D^\lambda{\downarrow}_{\mathsf{S}_{n-2,2}} \cong (D^\mu \boxtimes D^{(2)}) \oplus (D^\nu \boxtimes D^{(1^2)})$$

it follows that either $D^\lambda{\downarrow}_{\mathsf{S}_{n-3,3}} \cong (D^\pi \boxtimes D^{(2,1)})$ or

$$D^\lambda{\downarrow}_{\mathsf{S}_{n-3,3}} \cong (D^\psi \boxtimes D^{(3)}) \oplus (D^\xi \boxtimes D^{(1^3)})$$

for certain partitions π,ψ,ξ. So from [**2**, Corollary 3.9] with $k = 3$ or from [**2**, Corollary 4.3] we have that $n \leq 5$ or $p \mid n$ and $\lambda \in \{(n-1,1),(n-1,1)^{\mathtt{M}}\}$. The cases $n \leq 5$ can be checked separately. If $p \mid n$ and $\lambda \in \{(n-1,1),(n-1,1)^{\mathtt{M}}\}$ then $n-k = 1$, $a = 2$ and $\dim D^\lambda = n-2 \geq 3 > 2$.

So we can now assume that $p = 3$. We will show that in this case μ and ν are not both JS, from which the lemma follows. From the previous part all normal node of λ_A are good. So it is enough to show that that for a certain normal node B of λ_A we have that $(\lambda_A)_B$ is not JS.

Case 2.1. $\lambda_1 \geq \lambda_2 + 3$. If $B := (1,\lambda_1 - 1)$ then B is normal in λ_A and $(1,\lambda_1-2)$ and the second top removable node of λ are normal in $(\lambda_A)_B$.

Case 2.2. $\lambda_1 = \lambda_2 + 2$. Then λ is not JS.

Case 2.3. $\lambda_1 = \lambda_2 + 1$. Then $\lambda = (\lambda_1, \lambda_1 - 1, \lambda_3, \dots)$ with $1 \leq \lambda_3 \leq \lambda_1 - 2$. If $B = (2,\lambda_1 - 1)$ then B is normal in λ_A and $(1,\lambda_1-1)$ and the third top removable node of λ are normal in $(\lambda_A)_B$.

Case 2.4. $\lambda_1 = \lambda_2$. Then $\lambda = (\lambda_1^2, \lambda_3, \dots)$ with $1 \leq \lambda_3 \leq \lambda_1 - 2$. If $B = (1,\lambda_1)$ then B is normal in λ_A and $(2,\lambda_1-1)$ and the second top removable node of λ are normal in $(\lambda_A)_B$. ∎

References

[1] A.A. Baranov, A.S. Kleshchev, and A.E. Zalesskii, Asymptotic results on modular representations of symmetric groups and almost simple modular group algebras, *J. Algebra* **219** (1999), 506–530.

[2] J. Brundan and A.S. Kleshchev, Representations of the symmetric group which are irreducible over subgroups, *J. Reine Angew. Math.* **530** (2001), 145–190.

[3] R.M. Guralnick, M. Larsen, and P.H. Tiep, Representation growth in positive characteristic and conjugacy classes of maximal subgroups, *Duke Math. J.* **161** (2012), 107–137.

[4] G.D. James, '*The Representation Theory of the Symmetric Groups*', Lecture Notes in Mathematics, vol. **682**, Springer, NewYork/Heidelberg/Berlin, 1978.

[5] G.D. James, On the minimal dimensions of irreducible representations of symmetric groups, *Math. Proc. Camb. Phil. Soc.* **94** (1983), 417–424.

[6] J.C. Jantzen and G.M. Seitz, On the representation theory of the symmetric groups, *Proc. London Math. Soc.* **65** (1992), 475–504.

[7] A.S. Kleshchev. On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I. *Proc. London Math. Soc.* **69** (1994), 515–540.

[8] A.S. Kleshchev, Branching rules for modular representations of symmetric groups. II, *J. Reine Angew. Math.* **459** (1995), 163–212.

[9] A.S. Kleshchev, On decomposition numbers and branching coefficients for symmetric and special linear groups, *Proc. London Math. Soc.* **75** (1997), 497–558.

[10] A. Kleshchev, '*Linear and Projective Representations of Symmetric Groups*', Cambridge University Press, Cambridge, 2005.

[11] A.S. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of symmetric and alternating groups in small characteristics, preprint `arXiv:1903.09854`.

[12] J. Müller, On low-degree representations of the symmetric group, *J. Algebra* **465** (2016), 356–398.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `klesh@uoregon.edu`

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, 30167 Hannover, Germany
E-mail address: `morotti@math.uni-hannover.de`

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
E-mail address: `tiep@math.rutgers.edu`