TURNER DOUBLES AND GENERALIZED SCHUR ALGEBRAS

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Abstract. Turner's Conjecture describes all blocks of symmetric groups and Hecke algebras up to derived equivalence in terms of certain double algebras. With a view towards a proof of this conjecture, we develop a general theory of Turner doubles. In particular, we describe doubles as explicit maximal symmetric subalgebras of certain generalized Schur algebras and establish a Schur-Weyl duality with wreath product algebras.

1. Introduction

Turner's Conjecture [**Tu**$_1$, Conjecture 165] describes all blocks of symmetric groups and Hecke algebras up to derived equivalence in terms of certain explicitly constructed *double algebras* $D_Q(n, d)$, where Q is a quiver of finite type A. This paper is the first in a series of two papers where we prove Turner's Conjecture. To achieve this goal, in this paper we develop a general theory of Turner doubles, which we believe is of independent interest.

For simplicity, in this introduction we describe the results only over the ground ring $\mathbb{Z}$. We fix a $\mathbb{Z}$-superalgebra $X = X_{\bar{0}} \oplus X_{\bar{1}}$ which is free of finite rank over $\mathbb{Z}$. Consider the invariants $\mathtt{Inv}^d X := (X^{\otimes d})^{\mathfrak{S}_d}$ under the action of the symmetric group $\mathfrak{S}_d$. This action depends crucially on the superstructure on X, as do the structure and the dimension of $\mathtt{Inv}^d X$ and of all algebras defined later in terms of X. There is a natural superbialgebra structure on $\mathtt{Inv}\, X := \bigoplus_{d \geq 0} \mathtt{Inv}^d X$. The *Turner double* is the superalgebra $DX := \mathtt{Inv}\, X \otimes (\mathtt{Inv}\, X)^*$ with product defined in terms of the superbialgebra structures on $\mathtt{Inv}\, X$ and $(\mathtt{Inv}\, X)^*$.

More precisely, $(\mathtt{Inv}\, X)^*$ is naturally a superbimodule over $\mathtt{Inv}\, X$, and the product on DX is described, using Sweedler's notation, as follows:

$$(\xi \otimes x)(\eta \otimes y) = \sum \pm \xi_{(2)}\eta_{(1)} \otimes (x \cdot \eta_{(2)})(\xi_{(1)} \cdot y),$$

for homogeneous $\xi, \eta \in \mathtt{Inv}\, X$ and $x, y \in (\mathtt{Inv}\, X)^*$, with signs determined by superalgebra data. We explain in §4.2 why this agrees with Turner's definition in [**Tu**$_3$]. A key property of DX is that it is always a *symmetric algebra*. Moreover, under some reasonable assumptions on X, the double DX as well as all other algebras defined later in terms of X are *non-negatively graded*. In this case, the theorems below respect the gradings.

The superalgebra $(\mathtt{Inv}\, X)^*$ can be identified with the symmetric superalgebra $\mathtt{Sym}(X^*)$, which is naturally a sublattice in the *divided power* superalgebra $'\mathtt{Sym}(X^*)$. We show that the superalgebra structure on DX extends to that on

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${}'DX := \mathtt{Inv}\, X \otimes {}'\mathtt{Sym}(X^*)$. Thus $DX \subseteq {}'DX$ is a subsuperalgebra. Upon extension of scalars to a filed $\mathbb{K}$ of characteristic 0, the embedding $DX \subseteq {}'DX$ induces an isomorphism $DX_{\mathbb{K}} \stackrel{\sim}{\longrightarrow} {}'DX_{\mathbb{K}}$. But, importantly, if $\mathbb{K}$ has positive characteristic, the induced map is neither injective nor surjective.

Let $T_X = X \oplus X^*$ be the *trivial extension superalgebra* of X, with the product defined by $(\xi, x)(\eta, y) = (\xi\eta, \xi \cdot y + x \cdot \eta)$ for $\xi, \eta \in X$ and $x, y \in X^*$. Let $*$ denote the shuffle product on $\bigoplus_{d\geq 0}(T_X)^{\otimes d}$. We show in Lemma 3.10 that there is a natural isomorphism $\kappa\colon {}'\mathtt{Sym}(X^*) \stackrel{\sim}{\longrightarrow} \mathtt{Inv}(X^*)$. Our first main result is the following theorem, which often allows one to reduce the study of the double over X to that of the invariants over T_X.

Theorem A. *We have:*

(i) *The map* $\varphi\colon {}'DX \to \mathtt{Inv}\, T_X,\ \xi \otimes x \mapsto \xi * \kappa(x)$ *is an isomorphism of superalgebras.*
(ii) *The subalgebra* $\varphi(DX) \subseteq \mathtt{Inv}\, T_X$ *is generated by* $\mathtt{Inv}(X_{\bar 0})$ *and all elements of the form* $t * 1_X^{\otimes d}$ *with* $t \in T_X$ *and* $d \geq 0$.

We have a natural superalgebra decomposition $DX = \bigoplus_{d\geq 0} D^d X$, with

$$D^d X = \bigoplus_{0\leq e\leq d} \mathtt{Inv}^e\, X \otimes (\mathtt{Inv}^{d-e}\, X)^*,$$

where the last direct sum is that of $\mathbb{Z}$-modules, and similarly for ${}'DX$. Then the isomorphism φ of Theorem A restricts to isomorphisms $\varphi\colon {}'D^d X \stackrel{\sim}{\longrightarrow} \mathtt{Inv}^d\, T_X$.

Let A be a $\mathbb{Z}$-superalgebra which is free of finite rank over $\mathbb{Z}$, and consider the case where X is the matrix superalgebra $M_n(A)$ for some fixed n. In this case we use the special notation

$$D^A(n,d) := D^d M_n(A), \quad {}'D^A(n,d) := {}'D^d M_n(A).$$

We refer to the superalgebra $D^A(n,d)$ as a *Schur double*. The following theorem shows that under a natural assumption, the subalgebra $D^A(n,d) \subseteq {}'D^A(n,d)$ is a *maximal* symmetric subalgebra:

Theorem B. *Let* $d \leq n$ *and* C *be a subalgebra of* ${}'D^A(n,d)$ *such that* $D^A(n,d) \subseteq C \subseteq {}'D^A(n,d)$. *Suppose that for every prime* p *the* $\mathbb{F}_p$*-algebra* $C \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is symmetric. Then* $C = D^A(n,d)$.

Let $S^A(n,d) := \mathtt{Inv}^d\, M_n(A)$. If $A = \mathbb{Z}$, then $S^A(n,d)$ is just the (integral version of) the classical Schur algebra. The *generalized Schur algebras* $S^A(n,d)$ bear importance for the doubles, since, by Theorem A and the easy observation that $T_{M_n(A)} \cong M_n(T_A)$, we can identify ${}'D^A(n,d)$ with $S^{T_A}(n,d)$ and $D^A(n,d)$ with an explicit subalgebra of $S^{T_A}(n,d)$.

The superalgebras $S^A(n,d)$ can be studied using a generalized Schur-Weyl duality with the *super wreath product* $W_d^A := A^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d$. The superalgebra $M_n(A)$ can be identified with $\mathrm{End}_A(V)$, where $V := A^{\oplus n}$. The following generalized version of Schur-Weyl duality is crucial for the proof of Turner's Conjecture, but is also of independent interest.

Theorem C. *The natural left* $S^A(n,d)$*-action and the natural right* W_d^A*-action on* $V^{\otimes d}$ *commute and yield an isomorphism* $S^A(n,d) \cong \mathrm{End}_{W_d^A}(V^{\otimes d})$.

As a right W_d^A-supermodule, $V^{\otimes d}$ decomposes explicitly as a direct sum of certain *permutation supermodules* M_λ^A where λ runs over the set $\Lambda(n,d)$ of all compositions of d with n parts. So Theorem C realizes $S^A(n,d)$ as

$$\mathrm{End}_{W_d^A}\Big(\bigoplus_{\lambda\in\Lambda(n,d)} M_\lambda^A\Big).$$

For the purposes of Turner's Conjecture, it is important to 'desuperize' this description of $S^A(n,d)$ in the case where A is a certain *zigzag superalgebra* Z depending on a quiver Q. Let $|X|$ denote the algebra obtained from a superalgebra X by forgetting the superstructure. We construct a (rather delicate) explicit isomorphism σ from the ordinary wreath product $W_d^{|\mathsf{Z}|}$ to $|W_d^{\mathsf{Z}}|$. Twisting with this isomorphism makes the permutation module M_λ^{Z} into an explicit *alternating sign permutation module* $M_\lambda^{|\mathsf{Z}|}$ over $W_d^{|\mathsf{Z}|}$. Then

$$|S^{\mathsf{Z}}(n,d)| \cong \mathrm{End}_{W_d^{|\mathsf{Z}|}}\Big(\bigoplus_{\lambda\in\Lambda(n,d)} M_\lambda^{|\mathsf{Z}|}\Big).$$

Using Theorems A,B,C, we obtain an explicit description of $D_Q(n,d)$ as a maximal symmetric subalgebra of the endomorphism algebra on the right hand side. This description is used in [**EK**] to identify $D_Q(n,d)$ with an algebra Morita equivalent to (a $\mathbb{Z}$-form of) a RoCK block of a Hecke algebra or a more general cyclotomic KLR algebra, thus proving Turner's Conjecture.

Now we describe the contents of the paper in more detail. In Section 2 we set up some basic combinatorial notation. In Section 3 we discuss superspaces and superalgebras, especially symmetric and divided power superalgebras and various products and coproducts on them. In §3.4 we consider trivial extension superalgebras. In Section 4 we begin to study Turner doubles. The properties of invariant algebras $\mathtt{Inv}\,X$ are investigated in §4.1. The definition of DX is given in §4.2, and its divided power version $'DX$ is studied in §4.3. For Theorem A see Theorems 4.26 in §4.4 and 4.30 in §4.4. We discuss gradings on doubles in §4.5 and symmetricity of doubles in §4.6.

Section 5 is on generalized Schur-Weyl duality. In §5.1 we discuss wreath product algebras and permutation modules over them. In §5.2 we study the generalized tensor space, prove Theorem C (see Lemma 5.7) and discuss connections with permutation modules over wreath product algebras. We consider idempotent truncations of generalized Schur algebras in §5.3 and idempotent refinements of permutation modules in §5.4. Desuperization is discussed in §5.5.

Section 6 is on Schur doubles. In §6.1 we identify $D^A(n,d)$ with the subalgebra of $S^{T_A}(n,d)$ generated by certain explicit elements. Theorem B is proved in §6.2, see Theorem 6.6. In §6.3 we discuss bases and product rules of Schur doubles and their divided power versions. Section 7 is on the important special case of the quiver Schur (schiver) doubles. Quivers and zigzag algebras are considered in §7.1. Finally, in §7.2, we discuss the degree zero component of a schiver double and results related to schiver generation and desuperization, which will be needed in [**EK**].

2. Preliminaries

Throughout the paper, $\Bbbk$ is an arbitrary commutative (unital) ring. In some constructions, involving divided powers, we will need to work over a more special ring $\mathcal{O}$, which is assumed to be a (commutative) integral domain with field of fractions $\mathbb{K}$ of *characteristic zero*. We assume that there is a fixed ring homomorphism $\mathcal{O} \to \Bbbk$, which allows us to extend scalars from $\mathcal{O}$ to $\Bbbk$, i.e. to consider

$$V_\Bbbk := V \otimes_{\mathcal{O}} \Bbbk$$

for any $\mathcal{O}$-module V. If U and V are $\Bbbk$-modules, we denote $U \otimes V := U \otimes_\Bbbk V$. Important examples of triples $(\mathbb{K}, \mathcal{O}, \Bbbk)$ are $(\mathbb{Q}, \mathbb{Z}, \mathbb{F}_p)$ and $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$.

2.1. **Weights and sequences.** Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$. We denote by $\Lambda(n)$ the set of compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ with n parts $\lambda_1, \dots, \lambda_n \in \mathbb{Z}_{\geq 0}$. We refer to the elements of $\Lambda(n)$ as *weights*. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n)$, we denote $|\lambda| := \lambda_1 + \cdots + \lambda_n$. We set

$$\Lambda(n, d) := \{\lambda \in \Lambda(n) \mid |\lambda| = d\}.$$

More generally, if S is a finite set, we denote by $\Lambda(S, d)$ the set of tuples $(\lambda_s)_{s \in S}$ of non-negative integers such that $\sum_{s \in S} \lambda_s = d$. For $S = [1, n]$, we identify $\Lambda(S, d)$ with $\Lambda(n, d)$.

For $1 \leq m \leq n$, we have special weights

$$\varepsilon_m := (0, \dots, 0, 1, 0, \dots, 0) \in \Lambda(n, 1),$$

with 1 in the mth position, so that

$$\lambda = (\lambda_1, \dots, \lambda_n) = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n.$$

For $m, n \in \mathbb{Z}$, we consider the (possibly empty) *segments*

$$[m, n] := \{r \in \mathbb{Z} \mid m \leq r \leq n\}, \qquad (m, n] := \{r \in \mathbb{Z} \mid m < r \leq n\},$$
$$[m, n) := \{r \in \mathbb{Z} \mid m \leq r < n\}.$$

The symmetric group $\mathfrak{S}_n$ acts naturally on the left on $[1, n]$.

Let $\mathtt{Seq}(n, d) := [1, n]^d$ be the set of (ordered) d-tuples $\boldsymbol{r} = (r_1, \dots, r_d)$ where $r_1, \dots, r_d \in [1, n]$. The action of the symmetric group $\mathfrak{S}_d$ on $[1, d]$ yields the right action of $\mathfrak{S}_d$ on $\mathtt{Seq}(n, d)$ by place permutations: for $\boldsymbol{r} \in \mathtt{Seq}(n, d)$ and $g \in \mathfrak{S}_d$, we have $\boldsymbol{r}g = \boldsymbol{s}$ where $s_a = r_{ga}$ for all $a \in [1, d]$.

For $\lambda \in \Lambda(n, d)$ we set

$$^{\lambda}\mathtt{Seq} := \{\boldsymbol{r} \in \mathtt{Seq}(n, d) \mid \varepsilon_{r_1} + \cdots + \varepsilon_{r_d} = \lambda\}. \tag{2.1}$$

Then $\mathtt{Seq}(n, d) = \bigsqcup_{\lambda \in \Lambda(n, d)} {}^{\lambda}\mathtt{Seq}$ is the decomposition of $\mathtt{Seq}(n, d)$ into $\mathfrak{S}_d$-orbits.

For $\lambda \in \Lambda(n, d)$ we define

$$\boldsymbol{r}^{\lambda} := (1, \dots, 1, 2, \dots, 2, \dots, n, \dots, n) \in {}^{\lambda}\mathtt{Seq},$$

where each $r \in [1, n]$ is repeated λ_r times.

2.2. **Integer-valued matrices and sequences.** Define $\mathcal{M}(n)$ to be the set of $n \times n$-matrices with non-negative integer coefficients. Let $E_{r,s} \in \mathcal{M}(n)$ denote the *matrix unit* with 1 in the (r,s)th position. For $C = (c_{r,s})_{1\le r,s\le n} \in \mathcal{M}(n)$, we set $|C| := \sum_{r,s=1}^{n} c_{r,s}$, and we define

$$\mathcal{M}(n,d) := \{C \in \mathcal{M}(n) \mid |C| = d\}.$$

Given $C, D \in \mathcal{M}(n)$, define the integers

$$C! = \prod_{r,s\in[1,n]} c_{r,s}!, \qquad \binom{C}{D} := \prod_{r,s\in[1,n]} \binom{c_{r,s}}{d_{r,s}}.$$

For any $C \in \mathcal{M}(n,d)$, we further set

$$\alpha(C) := \big(\textstyle\sum_s c_{1,s}, \sum_s c_{2,s}, \dots, \sum_s c_{n,s}\big) \in \Lambda(n,d),$$
$$\beta(C) := \big(\textstyle\sum_r c_{r,1}, \sum_r c_{r,2}, \dots, \sum_r c_{r,n}\big) \in \Lambda(n,d).$$

Let $\lambda, \mu \in \Lambda(n,d)$. Define

$${}_\mu\mathcal{M}(n,d)_\lambda := \{C \in \mathcal{M}(n,d) \mid \alpha(C) = \mu \text{ and } \beta(C) = \lambda\}.$$

The subsets of $\mathcal{M}(n)$ and $\mathcal{M}(n,d)$ consisting of $\{0,1\}$-matrices are denoted by

$${}'\mathcal{M}(n) := \{C \in \mathcal{M}(n) \mid c_{r,s} \in \{0,1\} \text{ for all } 1 \le r,s \le n\},$$
$${}'\mathcal{M}(n,d) := \mathcal{M}(n,d) \cap {}'\mathcal{M}(n).$$

In §6.3, we will use the following generalization. Let $\mathtt{B} = \mathtt{B}_{\bar 0} \sqcup \mathtt{B}_{\bar 1}$ be a set split as a disjoint union of two subsets $\mathtt{B}_{\bar 0}$ and $\mathtt{B}_{\bar 1}$. Set

$$\mathcal{M}^{\mathtt{B}}(n) := \{\mathbf{C} = (C^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}} \mid C^{\mathtt{b}} \in \mathcal{M}(n) \text{ for } \mathtt{b} \in \mathtt{B}_{\bar 0},\ C^{\mathtt{b}} \in {}'\mathcal{M}(n) \text{ for } \mathtt{b} \in \mathtt{B}_{\bar 1}\}. \tag{2.2}$$

Let $\mathbf{C} = (C^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}} \in \mathcal{M}^{\mathtt{B}}(n)$. For every $\mathtt{b} \in \mathtt{B}$, we write $C^{\mathtt{b}} = (c^{\mathtt{b}}_{r,s})_{1\le r,s\le n}$. Denote $|\mathbf{C}|_{\bar 0} := \sum_{\mathtt{b}\in\mathtt{B}_{\bar 0}} |C^{\mathtt{b}}|$, $|\mathbf{C}|_{\bar 1} := \sum_{\mathtt{b}\in\mathtt{B}_{\bar 1}} |C^{\mathtt{b}}|$,

$$|\mathbf{C}| := |\mathbf{C}|_{\bar 0} + |\mathbf{C}|_{\bar 1} = \sum_{\mathtt{b}\in\mathtt{B}} |C^{\mathtt{b}}| = \sum_{(r,s,\mathtt{b})\in[1,n]^2\times\mathtt{B}} c^{\mathtt{b}}_{r,s}, \tag{2.3}$$

$$\mathcal{M}^{\mathtt{B}}(n,d) := \{\mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n) \mid |\mathbf{C}| = d\}. \tag{2.4}$$

Let $\mathbf{C} = (C^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}}$ and $\mathbf{D} = (D^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}} \in \mathcal{M}^{\mathtt{B}}(n)$. We define $\mathbf{C}+\mathbf{D}$ by $(\mathbf{C}+\mathbf{D})^{\mathtt{b}} = C^{\mathtt{b}} + D^{\mathtt{b}}$ for all $\mathtt{b} \in \mathtt{B}$. Note that $\mathbf{C}+\mathbf{D}$ may or may not be an element of $\mathcal{M}^{\mathtt{B}}(n)$. We set

$$\mathbf{C}! := \prod_{\mathtt{b}\in\mathtt{B}} C^{\mathtt{b}}! = \prod_{\mathtt{b}\in\mathtt{B}_{\bar 0}} C^{\mathtt{b}}!, \quad \binom{\mathbf{C}}{\mathbf{D}} := \prod_{\mathtt{b}\in\mathtt{B}} \binom{C^{\mathtt{b}}}{D^{\mathtt{b}}}.$$

Define $\mathtt{Seq}^{\mathtt{B}}(n,d)^2$ to be the set of tuples

$$(\boldsymbol{r}, \mathbf{b}, \boldsymbol{s}) = ((r_1,\dots,r_d),(\mathtt{b}_1,\dots,\mathtt{b}_d),(s_1,\dots,s_d)) \in \mathtt{Seq}(n,d) \times \mathtt{B}^d \times \mathtt{Seq}(n,d)$$

such that for any distinct $k, l \in [1,d]$ with $(r_k, \mathtt{b}_k, s_k) = (r_l, \mathtt{b}_l, s_l)$ we have $\mathtt{b}_k \in \mathtt{B}_{\bar 0}$. The left action of $\mathfrak{S}_d$ on $[1,d]$ induces a right action on each component of the direct product $\mathtt{Seq}(n,d) \times \mathtt{B}^d \times \mathtt{Seq}(n,d)$ as in §2.1, so we have a right action of $\mathfrak{S}_d$ on $\mathtt{Seq}^{\mathtt{B}}(n,d)^2$. There is a bijection

$$\mathtt{Seq}^{\mathtt{B}}(n,d)^2/\mathfrak{S}_d \xrightarrow{\sim} \mathcal{M}^{\mathtt{B}}(n,d),\ (\boldsymbol{r},\mathbf{b},\boldsymbol{s}) \mapsto \boldsymbol{M}[\boldsymbol{r},\mathbf{b},\boldsymbol{s}] := ((c^{\mathtt{b}}_{r,s})_{r,s\in[1,n]})_{\mathtt{b}\in B}$$

where

$$c^{\mathtt{b}}_{r,s} = \sharp\{k \in [1,d] \mid (r_k, \mathtt{b}_k, s_k) = (r, \mathtt{b}, s)\}.$$

We always identify $\mathtt{Seq}^{\mathtt{B}}(n,d)^2/\mathfrak{S}_d$ with $\mathcal{M}^{\mathtt{B}}(n,d)$ via this bijection. In particular, given $\mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n,d)$, we write $(\boldsymbol{r}, \mathbf{b}, \boldsymbol{s}) \in \mathbf{C}$ if $\boldsymbol{M}[\boldsymbol{r}, \mathbf{b}, \boldsymbol{s}] = \mathbf{C}$.

2.3. **Cosets.** Let $(S,<)$ be a totally ordered finite set. Recall the notation $\Lambda(S,d)$ from §2.1. Let $\lambda = (\lambda_s)_{s\in S} \in \Lambda(S,d)$. The corresponding *standard set partition* Ω^λ is the partition of $[1,d]$ into the segments

$$\Omega^\lambda_s := \big(\textstyle\sum_{t<s} \lambda_t, \sum_{t\leq s} \lambda_t\big] \qquad (s \in S).$$

Note that the segment Ω^λ_s has λ_s elements. Write $S = \{s_1 < \cdots < s_n\}$. The *standard parabolic subgroup*

$$\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_{s_1}} \times \cdots \times \mathfrak{S}_{\lambda_{s_n}} \leq \mathfrak{S}_d \tag{2.5}$$

preserves the set partition Ω^λ. If $\lambda \in \Lambda(n,d)$, we define Ω^λ and $\mathfrak{S}_\lambda$ via the usual total order on $[1,n]$.

Let $\lambda \in \Lambda(S,d)$ and $\mathscr{D}^\lambda$ be the set of shortest coset representatives for $\mathfrak{S}_d/\mathfrak{S}_\lambda$, where the length $\ell(g)$ of an element $g \in \mathfrak{S}_\lambda$ is the smallest integer ℓ such that g can be represented as a product of ℓ transpositions of the form $(r, r+1)$, $1 \leq r < d$. For $\mu \in \Lambda(S,d)$, we also have the set ${}^\mu\mathscr{D}$ of shortest coset representatives for $\mathfrak{S}_\mu\backslash\mathfrak{S}_d$ and the set ${}^\mu\mathscr{D}^\lambda$ of shortest double coset representatives for $\mathfrak{S}_\mu\backslash\mathfrak{S}_d/\mathfrak{S}_\lambda$. Note that we have a bijection

$${}^\mu\mathscr{D} \to {}^\mu\mathtt{Seq},\ \boldsymbol{r}^\mu \mapsto \boldsymbol{r}^\mu g \tag{2.6}$$

and a bijection ${}^\mu\mathscr{D} \to \mathscr{D}^\mu, g \mapsto g^{-1}$.

It is well known and easy to see (cf. e.g. [**JK**, 1.3.10]) that for every $C = (c_{r,s}) \in {}_\mu\mathcal{M}(n,d)_\lambda$ there exists a unique element $g(C) \in {}^\mu\mathscr{D}^\lambda$ such that

$$|g(C)(\Omega^\lambda_s) \cap \Omega^\mu_r| = c_{r,s}$$

for all $r, s \in [1,n]$. Moreover:

Lemma 2.7. *For any $\lambda, \mu \in \Lambda(n,d)$, the map $C \mapsto g(C)$ defines a bijection*

$${}_\mu\mathcal{M}(n,d)_\lambda \xrightarrow{\sim} {}^\mu\mathscr{D}^\lambda.$$

Given $C = (c_{r,s}) \in {}_\mu\mathcal{M}(n,d)_\lambda$ and $1 \leq s \leq n$, we have a composition

$$\boldsymbol{c}_{*,s} := (c_{1,s}, \dots, c_{n,s}) \in \Lambda(n, \lambda_s).$$

Given elements $g_1 \in \mathfrak{S}_{\lambda_1}, \dots, g_n \in \mathfrak{S}_{\lambda_n}$, we consider $(g_1, \dots, g_n) \in \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ as an element of $\mathfrak{S}_d$ via the natural embedding of $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ into $\mathfrak{S}_d$. Another easy and well-known result (see e.g. [**DJ**, Lemma 1.6]) is:

Lemma 2.8. *Let $\lambda, \mu \in \Lambda(n,d)$. There is a bijection*

$$\{(C, g_1, \dots, g_n) \mid C \in {}_\mu\mathcal{M}(n,d)_\lambda,\ g_s \in {}^{\boldsymbol{c}_{*,s}}\mathscr{D} \text{ for } s = 1, \dots, n\} \xrightarrow{\sim} {}^\mu\mathscr{D}$$

defined by $(C, g_1, \dots, g_n) \mapsto g(C)(g_1, \dots, g_n)$.

3. Superspaces and superalgebras

From now on, we write $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a free $\Bbbk$-supermodule of finite rank. We refer to V as a ($\Bbbk$-)*superspace*. The $\Bbbk$-rank of V is denoted by $\dim V$. For parities of elements, we write $\bar{v} = \bar{0}$ if $v \in V_{\bar{0}}$ and $\bar{v} = \bar{1}$ if $v \in V_{\bar{1}}$. Whenever $\bar{v}$ appears in a formula, this means that we assume that v is a homogeneous element. If V is an (associative unital) $\Bbbk$-superalgebra, we denote by $|V|$ the same algebra without the $\mathbb{Z}_2$-grading.

By a $\mathbb{Z}$-*supergrading* on a superspace V we mean a $\mathbb{Z}$-grading $V = \bigoplus_{m\in\mathbb{Z}} V^m$ such that $V^m = (V^m \cap V_{\bar{0}}) \oplus (V^m \cap V_{\bar{1}})$ for all $m \in \mathbb{Z}$.

3.1. **Dual superspaces and tensor products.** The dual $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ is a superspace in a natural way. We have the pairing $\langle \cdot, \cdot \rangle$ between V and V^*:

$$\langle v, \beta \rangle = \langle \beta, v \rangle := \beta(v) \qquad (v \in V,\ \beta \in V^*).$$

Let $d \in \mathbb{Z}_{>0}$, and $V_1, \dots, V_d$ be superspaces. The tensor product $V_1 \otimes \cdots \otimes V_d$ is again a superspace in a natural way. We always identify $(V_1 \otimes \cdots \otimes V_d)^*$ with $V_1^* \otimes \cdots \otimes V_d^*$ via

$$\langle \beta_1 \otimes \cdots \otimes \beta_d, v_1 \otimes \cdots \otimes v_d \rangle := (-1)^{[\beta_1,\dots,\beta_d;v_1,\dots,v_d]} \langle \beta_1, v_1 \rangle \dots \langle \beta_d, v_d \rangle, \tag{3.1}$$

where $\beta_a \in V_a^*, v_a \in V_a$ for $a = 1, \dots, d$, and where

$$[\beta_1, \dots, \beta_d; v_1, \dots, v_d] := \sum_{1\le a<c\le d} \bar{\beta}_c \bar{v}_a \tag{3.2}$$

is defined for (homogeneous) elements $\beta_1, \dots, \beta_d, v_1, \dots, v_d$ of arbitrary superspaces. Note that

$$\begin{aligned}\langle \beta_1 \otimes \cdots \otimes \beta_d, v_1 \otimes \cdots \otimes v_d \rangle &= \langle v_1 \otimes \cdots \otimes v_d, \beta_1 \otimes \cdots \otimes \beta_d \rangle \\ &:= (-1)^{[v_1,\dots,v_d;\beta_1,\dots,\beta_d]} \langle v_1, \beta_1 \rangle \dots \langle v_d, \beta_d \rangle,\end{aligned}$$

since $\langle v_a, \beta_a \rangle = 0$ unless $\bar{v}_a = \bar{\beta}_a$ for any $1 \le a \le d$.

If $V_1, \dots, V_d$ are $\Bbbk$-superalgebras, then $V_1 \otimes \cdots \otimes V_d$ is again a superalgebra with

$$(v_1 \otimes \cdots \otimes v_d)(w_1 \otimes \cdots \otimes w_d) = (-1)^{[v_1,\dots,v_d;w_1,\dots,w_d]} v_1w_1 \otimes \cdots \otimes v_dw_d,$$

for $v_a, w_a \in V_a$, $a = 1, \dots, d$.

The symmetric group $\mathfrak{S}_d$ acts on the superspace $V^{\otimes d}$ on the right by (super) place permutations. More precisely, for $g \in \mathfrak{S}_d$ and $v_1, \dots, v_d \in V$, we define

$$[g; v_1, \dots, v_d] := \sum_{1\le a<c\le d,\ g^{-1}a>g^{-1}c} \bar{v}_a \bar{v}_c, \tag{3.3}$$

and

$$(v_1 \otimes \cdots \otimes v_d)^g := (-1)^{[g;v_1,\dots,v_d]} v_{g1} \otimes \cdots \otimes v_{gd}. \tag{3.4}$$

If V is a superalgebra, then $\mathfrak{S}_d$ acts on $V^{\otimes d}$ with algebra automorphisms.

3.2. **Symmetric and divided power superalgebras.** Recall that $\mathcal{O}$ is a domain of characteristic zero. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be an $\mathcal{O}$-superspace with bases $\mathtt{B}_{\bar{0}} = \{x_1, \dots, x_l\}$ of $V_{\bar{0}}$ and $\mathtt{B}_{\bar{1}} = \{x_{l+1}, \dots, x_{l+m}\}$ of $V_{\bar{1}}$. Then $\mathtt{B} = \mathtt{B}_{\bar{0}} \sqcup \mathtt{B}_{\bar{1}}$ is a homogeneous basis of V. We identify $V_{\Bbbk} := V \otimes_{\mathcal{O}} \Bbbk$ with the free $\Bbbk$-supermodule with basis $\mathtt{B}$, and we identify V with the $\mathcal{O}$-subsupermodule $V \otimes 1 \subseteq V_{\mathbb{K}}$.

For every $d \in \mathbb{Z}_{\geq 0}$, consider the $\mathcal{O}$-superspace

$$\mathtt{Tens}^d V := V^{\otimes d}.$$

Let

$$\mathtt{Tens}\, V := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathtt{Tens}^d V$$

be the tensor superalgebra of V and $\mathtt{Sym}\, V = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathtt{Sym}^d V$ be the *symmetric superalgebra* on V. That is, $\mathtt{Sym}\, V$ is the quotient of $\mathtt{Tens}\, V$ by the ideal generated by all elements of the form $v \otimes u - (v \otimes u)^{(1,2)}$ for $u, v \in V$ and all elements of the form $v \otimes v$ for $v \in V_{\bar{1}}$. Moreover, for every $d \in \mathbb{Z}_{\geq 0}$, the subsuperspace $\mathtt{Sym}^d V \leq \mathtt{Sym}\, V$ is the intersection of $\mathtt{Sym}\, V$ with the subsuperspace $\mathtt{Tens}^d V$ of $\mathtt{Tens}\, V$.

We consider $\mathtt{Sym}\, V$ as an $\mathcal{O}$-form of $\mathtt{Sym}\, V_{\mathbb{K}}$. We will also need another $\mathcal{O}$-form. The *divided powers superalgebra* $'\mathtt{Sym}\, V = \bigoplus {}'\mathtt{Sym}^d V$ is the $\mathcal{O}$-subalgebra of $\mathtt{Sym}\, V_{\mathbb{K}}$ generated by the divided powers $v^{(m)} := v^m/m!$ for all $v \in V_{\bar{0}}$ and $m \in \mathbb{Z}_{\geq 0}$ together with all $v \in V_{\bar{1}}$. We now define $'\mathtt{Sym}\, V_{\Bbbk} := ('\mathtt{Sym}\, V) \otimes_{\mathcal{O}} \Bbbk$ and write $v^{(m)} := v^{(m)} \otimes 1 \in {}'\mathtt{Sym}\, V_{\Bbbk}$.

For every $d \in \mathbb{Z}_{\geq 0}$, we have the fixed points $\mathtt{Inv}^d V := \big(\mathtt{Tens}^d V\big)^{\mathfrak{S}_d}$ of the action (3.4) and set $\mathtt{Inv}\, V := \bigoplus_{d \geq 0} \mathtt{Inv}^d V$. It is a subalgebra of $\mathtt{Tens}\, V$ with respect to a new product, which we now define.

For $d, e \in \mathbb{Z}_{\geq 0}$, recall that ${}^{(d,e)}\mathscr{D}$ stands for the set of the shortest coset representatives for $(\mathfrak{S}_d \times \mathfrak{S}_e)\backslash \mathfrak{S}_{d+e}$. We consider the linear map

$$\mathtt{Tens}^d V \otimes \mathtt{Tens}^e V \to \mathtt{Tens}^{d+e} V,\ t \otimes s \mapsto t * s,$$

defined by

$$(x_1 \otimes \cdots \otimes x_d) * (y_1 \otimes \cdots \otimes y_e) := \sum_{g \in {}^{(d,e)}\mathscr{D}} (x_1 \otimes \cdots \otimes x_d \otimes y_1 \otimes \cdots \otimes y_e)^g \quad (3.5)$$

for all $x_1, \dots, x_d, y_1, \dots, y_e \in V$. This new $*$-*product* (or *shuffle product*) on $\mathtt{Tens}\, V$ makes it an associative supercommutative superalgebra. Moreover, $\mathtt{Inv}\, V$ is a subsuperalgebra of $\mathtt{Tens}\, V$ with respect to the $*$-product.

Let $V = U \oplus W$ be a direct sum decomposition of $\mathcal{O}$-supermodules. For every $e \geq 0$, we identify $\mathtt{Tens}^e U$ and $\mathtt{Tens}^e W$ with subsupermodules of $\mathtt{Tens}^e V$ in the obvious way. The following is easy to see:

Lemma 3.6. *Let $d \in \mathbb{Z}_{\geq 0}$. For every $e \in [0, d]$, the $\mathcal{O}$-supermodule homomorphism*

$$\mathtt{Inv}^e U \otimes \mathtt{Inv}^{d-e} W \to \mathtt{Inv}^d V,\ s \otimes t \mapsto s * t$$

is injective, and we have a direct sum decomposition of $\mathcal{O}$-superspaces:

$$\mathtt{Inv}^d V = \bigoplus_{e=0}^{d} (\mathtt{Inv}^e U) * (\mathtt{Inv}^{d-e} W).$$

To describe bases, set

$$\mathcal{M}^{\mathsf{B}} := \{(c_1, \dots, c_l, c_{l+1}, \dots, c_{l+m}) \mid c_1, \dots, c_l \in \mathbb{Z}_{\geq 0},\ c_{l+1}, \dots, c_{l+m} \in \{0, 1\}\}.$$

For $\boldsymbol{c} = (c_1, \dots, c_{l+m})$, define $|\boldsymbol{c}| := c_1 + \cdots + c_{l+m}$, and denote

$$\mathcal{M}^{\mathsf{B}}_d := \{\boldsymbol{c} \in \mathcal{M}^{\mathsf{B}} \mid |\boldsymbol{c}| = d\}.$$

In terms of (2.2), (2.4), we have $\mathcal{M}^{\mathsf{B}} = \mathcal{M}^{\mathsf{B}}(1)$ and $\mathcal{M}^{\mathsf{B}}_d = \mathcal{M}^{\mathsf{B}}(1, d)$. Then

$$\{x_1^{c_1} \cdots x_{l+m}^{c_{l+m}} \mid (c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathsf{B}}_d\} \tag{3.7}$$

is a basis of $\mathtt{Sym}^d V$,

$$\{x_1^{(c_1)} \cdots x_{l+m}^{(c_{l+m})} \mid (c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathsf{B}}_d\} \tag{3.8}$$

is a basis of $'\mathtt{Sym}^d V$, and

$$\{x_1^{\otimes c_1} * \cdots * x_{l+m}^{\otimes c_{l+m}} \mid (c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathsf{B}}_d\} \tag{3.9}$$

is a basis of $\mathtt{Inv}^d V$.

Define

$$\mathtt{Star}^d V := \underbrace{V * \cdots * V}_{d \text{ times}}, \qquad \mathtt{Star}\, V := \bigoplus_{d \geq 0} \mathtt{Star}^d V,$$

so that $\mathtt{Star}\, V$ is an $\mathcal{O}$-subsupermodule of $\mathtt{Inv}\, V$.

Lemma 3.10. *There is an isomorphism of algebras $\kappa \colon {}'\mathtt{Sym}\, V \xrightarrow{\sim} \mathtt{Inv}\, V$ which maps $x_1^{(c_1)} \cdots x_{l_m}^{(c_{l+m})}$ to $x_1^{\otimes c_1} * \cdots * x_{l+m}^{\otimes c_{l+m}}$ for all $(c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathsf{B}}_d$. Moreover, $\kappa(\mathtt{Sym}(V)) = \mathtt{Star}\, V$.*

Proof. It follows easily from the definitions that there is a homomorphism of superalgebras $\mathtt{Sym}\, V \to \mathtt{Inv}\, V$ which is the identity on V. Under this map, for any $(c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathsf{B}}_d$, the basis element $x_1^{c_1} \dots x_{l_m}^{c_{l+m}}$ is sent to

$$x_1^{*c_1} * \cdots * x_{l+m}^{*c_{l+m}} = c_1! \dots c_{l+m}!\, x_1^{\otimes c_1} * \cdots * x_{l+m}^{\otimes c_{l+m}}.$$

Extending scalars to $\mathbb{K}$ and restricting to $'\mathtt{Sym}\, V$, we obtain the desired isomorphism $'\mathtt{Sym}\, V \xrightarrow{\sim} \mathtt{Inv}\, V$. The final statement of the lemma is clear. ☐

3.3. **Coproducts.** We can also consider $\mathtt{Tens}\, V$ as a *supercoalgebra*, with the coproduct

$$\begin{aligned} \Delta \colon \mathtt{Tens}^d V &\to \bigoplus_{e,f \geq 0,\ e+f=d} \mathtt{Tens}^e V \otimes \mathtt{Tens}^f V, \\ v_1 \otimes \cdots \otimes v_d &\mapsto \sum_{e,f \geq 0,\ e+f=d} (v_1 \otimes \cdots \otimes v_e) \otimes (v_{e+1} \otimes \cdots \otimes v_d). \end{aligned} \tag{3.11}$$

For a supercoalgebra (X, Δ) and $x \in X$, we repeatedly use Sweedler's notation

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$$

where $x_{(1)}$ and $x_{(2)}$ are homogeneous whenever x is.

The following is a superalgebra version of the well-known fact (see e.g. [**Re**, Proposition 1.9]) that $\mathtt{Tens}\, V$ is a bialgebra with respect to $(*, \Delta)$:

Lemma 3.12. *Let $s, t \in \mathtt{Tens}\, V$. Then*

$$\Delta(s * t) = \sum (-1)^{\bar{s}_{(2)}\bar{t}_{(1)}} (s_{(1)} * t_{(1)}) \otimes (s_{(2)} * t_{(2)}).$$

Proof. We may assume that $s = s_1 \otimes \cdots \otimes s_a$ and $t = t_1 \otimes \cdots \otimes t_b$ for some $s_1, \dots, s_a, t_1, \dots, t_b \in V$. Let $\pi^{e,f}$ be the projection from $\mathtt{Tens}\, V \otimes \mathtt{Tens}\, V$ onto the summand $\mathtt{Tens}^e\, V \otimes \mathtt{Tens}^f\, V$. Fix $e \in [0, a+b]$ and denote by $\sum_{(C,g_1,g_2)}$ the sum over all triples (C, g_1, g_2) corresponding to taking $\lambda = (e, a+b-e)$ and $\mu = (a, b)$ in Lemma 2.8. Then using that lemma, we get

$$\begin{aligned}
\pi^{e,a+b-e}\Delta(s * t) =& \pi^{e,a+b-e}\Delta \sum_{h \in {}^{(a,b)}\mathscr{D}} (s \otimes t)^h \\
=& \pi^{e,a+b-e}\Delta \sum_{(C,g_1,g_2)} (s \otimes t)^{g(C)(g_1,g_2)} \\
=& \sum_{(C,g_1,g_2)} (-1)^m (s_1 \otimes \cdots \otimes s_{c_{1,1}} \otimes t_1 \otimes \cdots \otimes t_{c_{2,1}})^{g_1} \\
& \otimes (s_{c_{1,1}+1} \otimes \cdots \otimes s_a \otimes t_{c_{2,1}+1} \otimes \cdots \otimes t_b)^{g_2} \\
=& \sum_C (-1)^m \big((s_1 \otimes \cdots \otimes s_{c_{1,1}}) * (t_1 \otimes \cdots \otimes t_{c_{2,1}})\big) \\
& \otimes \big((s_{c_{1,1}+1} \otimes \cdots \otimes s_a) * (t_{c_{2,1}+1} \otimes \cdots \otimes t_b)\big) \\
=& \pi^{e,a+b-e} \sum (-1)^{\bar{s}_{(2)}\bar{t}_{(1)}} (s_{(1)} * t_{(1)}) \otimes (s_{(2)} * t_{(2)}),
\end{aligned}$$

where $m = (\bar{t}_1 + \cdots + \bar{t}_{c_{2,1}})(\bar{s}_{c_{1,1}+1} + \cdots + \bar{s}_a)$. This completes the proof. ☐

Note that $\mathtt{Inv}\, V$ is a subsupercoalgebra of $\mathtt{Tens}\, V$. The supercoalgebra $\mathtt{Inv}\, V$ is *supercocommutative*, i.e. if $\Delta(\xi) = \sum \xi_{(1)} \otimes \xi_{(2)}$ in Sweedler's notation for a (homogeneous) $\xi \in \mathtt{Inv}\, V$, then

$$\Delta(\xi) = \sum (-1)^{\bar{\xi}_{(1)}\bar{\xi}_{(2)}} \xi_{(2)} \otimes \xi_{(1)}. \tag{3.13}$$

Hence the (restricted) dual

$$(\mathtt{Inv}\, V)^* := \bigoplus_{d \geq 0} (\mathtt{Inv}^d\, V)^*$$

has a superalgebra structure which is dual to the coalgebra structure on $\mathtt{Inv}\, V$. More precisely, the superbialgebra structure on $(\mathtt{Inv}\, V)^*$ is determined by the identity

$$\langle \xi\eta, x \rangle = \langle \xi \otimes \eta, \Delta(x) \rangle \quad (\xi, \eta \in (\mathtt{Inv}\, V)^*, x \in \mathtt{Inv}\, X),$$

where as usual we identify $(\mathtt{Inv}\, V)^* \otimes (\mathtt{Inv}\, V)^*$ with $(\mathtt{Inv}\, V \otimes \mathtt{Inv}\, V)^*$ via (3.1). This makes $(\mathtt{Inv}\, V)^*$ a supercommutative superalgebra. Given $\xi_1, \dots, \xi_d \in V^*$, we have the functional $\xi_1 \otimes \cdots \otimes \xi_d \in (\mathtt{Tens}^d\, V)^*$. Extending by zero to the whole $\mathtt{Tens}\, V$ and restricting to $\mathtt{Inv}\, V$, we can interpret $\xi_1 \otimes \cdots \otimes \xi_d$ as an element of $(\mathtt{Inv}\, V)^*$. The following is now clear:

Lemma 3.14. *The natural map $V^* \to (\mathtt{Inv}\, V)^*$ extends to the isomorphism of superalgebras $\mathtt{Sym}(V^*) \xrightarrow{\sim} (\mathtt{Inv}\, V)^*$, which maps any product $\xi_1 \cdots \xi_d \in \mathtt{Sym}^d(V^*)$ with $\xi_1, \dots, \xi_d \in V^*$ to the functional $\xi_1 \otimes \cdots \otimes \xi_d \in (\mathtt{Inv}\, V)^*$.*

Corollary 3.15. *Identifying the $\mathcal{O}$-submodule $'\mathtt{Sym}(V^*) \subseteq \mathtt{Sym}(V^*_{\mathbb{K}})$ with an $\mathcal{O}$-submodule of $(\mathtt{Inv}\, V_{\mathbb{K}})^*$ via Lemma 3.14, we have*

$$\mathtt{Star}\, V = \{x \in \mathtt{Inv}\, V_{\mathbb{K}} \mid \langle x, \xi\rangle \in \mathcal{O} \text{ for all } \xi \in {}'\mathtt{Sym}(V^*)\}.$$

Proof. Recall the basis $\mathtt{B} = \{x_1, \dots, x_{l+m}\}$ of V, and let $\{\xi_1, \dots, \xi_{l+m}\}$ be the dual basis of V^*. By Lemma 3.10,

$$\{x^{*\boldsymbol{c}} := x_1^{*c_1} * \cdots * x_{l+m}^{*c_{l+m}} \mid \boldsymbol{c} = (c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathtt{B}}\}$$

is an $\mathcal{O}$-basis of $\mathtt{Star}\, V$. On the other hand,

$$\{\xi^{(\boldsymbol{c})} := \xi_1^{(c_1)} \cdots \xi_{l+m}^{(c_{l+m})} \mid \boldsymbol{c} = (c_1, \dots, c_{l+m}) \in \mathcal{M}^{\mathtt{B}}\}$$

is an $\mathcal{O}$-basis of $'\mathtt{Sym}(V^*)$. It remains to note that $\langle x^{*\boldsymbol{c}}, \xi^{(\boldsymbol{d})}\rangle = \pm\delta_{\boldsymbol{c},\boldsymbol{d}}$. ☐

3.4. **Trivial extension algebras.** Let A be a $\Bbbk$-superalgebra. We consider A^* as an A-bimodule with respect to the *dual regular actions* given by

$$\langle \alpha \cdot a, b\rangle = \langle \alpha, ab\rangle,\ \langle b, a \cdot \alpha\rangle = \langle ba, \alpha\rangle \qquad (a, b \in A,\ \alpha \in A^*). \tag{3.16}$$

We refer to this bimodule as the *dual regular superbimodule.*

The *trivial extension superalgebra* T_A of A is $T_A = A \oplus A^*$ as a superspace, with multiplication

$$(a, \alpha)(b, \beta) = (ab, a \cdot \beta + \alpha \cdot b) \qquad (a, b \in A,\ \alpha, \beta \in A^*). \tag{3.17}$$

Let $m\colon A \otimes A \to A$ be the multiplication map on A and

$$m^*\colon A^* \to A^* \otimes A^*$$

be the dual map. For $\alpha \in A^*$, we write $m^*(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)}$ using Sweedler's notation. Then

$$\langle bc, \alpha\rangle = \langle b \otimes c, m^*(\alpha)\rangle = \langle b \otimes c, \sum \alpha_{(1)} \otimes \alpha_{(2)}\rangle = \sum (-1)^{\bar{c}\bar{\alpha}_{(1)}} \langle b, \alpha_{(1)}\rangle\langle c, \alpha_{(2)}\rangle,$$

$$\langle \alpha, bc\rangle = \langle m^*(\alpha), b \otimes c\rangle = \langle \sum \alpha_{(1)} \otimes \alpha_{(2)}, b \otimes c\rangle = \sum (-1)^{\bar{\alpha}_{(2)}\bar{b}} \langle \alpha_{(1)}, b\rangle\langle \alpha_{(2)}, c\rangle.$$

Note that the right hand sides above are equal to each other since $\langle \alpha_{(1)}, b\rangle = 0$ unless $\bar{\alpha}_{(1)} = \bar{b}$ and $\langle \alpha_{(2)}, c\rangle = 0$ unless $\bar{\alpha}_{(2)} = \bar{c}$. The formulas imply that for any $a \in A$ and $\alpha \in A^*$, we have

$$a \cdot \alpha = \sum (-1)^{\bar{a}\bar{\alpha}_{(1)}} \langle a, \alpha_{(2)}\rangle \alpha_{(1)}, \tag{3.18}$$

$$\alpha \cdot a = \sum (-1)^{\bar{\alpha}_{(2)}\bar{a}} \langle a, \alpha_{(1)}\rangle \alpha_{(2)}. \tag{3.19}$$

Let $n \in \mathbb{Z}_{>0}$. The matrix algebra $M_n(A)$ is naturally a superalgebra. For $1 \le r, s \le n$ and $a \in A$, the matrix $aE_{r,s} \in X$ with a in the (r, s)th position and zeros elsewhere will be denoted by $\xi^a_{r,s}$. Then $\xi^a_{r,s}\xi^b_{t,u} = \delta_{s,t}\xi^{ab}_{r,u}$. We have $\overline{\xi^a_{r,s}} = \bar{a}$. For $\alpha \in A^*$ and $1 \le r, s \le n$, we have the element $x^\alpha_{r,s} \in M_n(A)^*$ defined from

$$\langle x^\alpha_{r,s}, \xi^a_{t,u}\rangle = \delta_{r,t}\delta_{s,u}\langle \alpha, a\rangle \qquad (1 \le t, u \le n,\ a \in A). \tag{3.20}$$

Lemma 3.21. *There is an isomorphism of superalgebras*

$$M_n(T_A) \xrightarrow{\sim} T_{M_n(A)},\ \xi^{(a,\alpha)}_{r,s} \mapsto (\xi^a_{r,s}, x^\alpha_{s,r})$$

for all $1 \le r, s \le n$, $a \in A$ and $\alpha \in A^*$.

Proof. Let $1 \le r,s,t,u,v,w \le n$ and $a,b,c \in A$. On the one hand, we have

$$\xi_{r,s}^{(a,\alpha)}\xi_{t,u}^{(b,\beta)} = \delta_{s,t}\xi_{r,u}^{(a,\alpha)(b,\beta)} = \delta_{s,t}\xi_{r,u}^{(ab,a\cdot\beta+\alpha\cdot b)} \mapsto \delta_{s,t}(\xi_{r,u}^{ab}, x_{u,r}^{a\cdot\beta+\alpha\cdot b}).$$

On the other hand,

$$(\xi_{r,s}^a, x_{s,r}^\alpha)(\xi_{t,u}^b, x_{u,t}^\beta) = (\xi_{r,s}^a\xi_{t,u}^b, \xi_{r,s}^a \cdot x_{u,t}^\beta + x_{s,r}^\alpha \cdot \xi_{t,u}^b).$$

Since $\xi_{r,s}^a\xi_{t,u}^b = \delta_{s,t}\xi_{r,u}^{ab}$, we just need to prove that

$$\xi_{r,s}^a \cdot x_{u,t}^\beta + x_{s,r}^\alpha \cdot \xi_{t,u}^b = \delta_{s,t}x_{u,r}^{a\cdot\beta+\alpha\cdot b}. \tag{3.22}$$

But

$$\begin{aligned}(\xi_{r,s}^a \cdot x_{u,t}^\beta + x_{s,r}^\alpha \cdot \xi_{t,u}^b)(\xi_{v,w}^c) &= x_{u,t}^\beta(\xi_{v,w}^c\xi_{r,s}^a) + x_{s,r}^\alpha(\xi_{t,u}^b\xi_{v,w}^c)\\ &= \delta_{w,r}x_{u,t}^\beta(\xi_{v,s}^{ca}) + \delta_{u,v}x_{s,r}^\alpha(\xi_{t,w}^{bc})\\ &= \delta_{w,r}\delta_{u,v}\delta_{t,s}\langle\beta, ca\rangle + \delta_{u,v}\delta_{s,t}\delta_{r,w}\langle\alpha, bc\rangle\\ &= \delta_{s,t}\delta_{u,v}\delta_{r,w}(a\cdot\beta+\alpha\cdot b)(c)\\ &= \delta_{s,t}x_{u,r}^{a\cdot\beta+\alpha\cdot b}(\xi_{v,w}^c),\end{aligned}$$

proving (3.22). □

4. Turner doubles

In this section, we review and develop Turner's theory of doubles [**Tu**$_1$, **Tu**$_2$, **Tu**$_3$]. We will freely use the notation and conventions of Section 3. Let X be an $\mathcal{O}$-superalgebra, free of finite rank as an $\mathcal{O}$-supermodule. We consider $X_{\Bbbk} = X \otimes_{\mathcal{O}} \Bbbk$ as a $\Bbbk$-superalgebra.

4.1. **Invariants.** For $d \in \mathbb{Z}_{\ge 0}$ we have a superalgebra structure on $\mathtt{Tens}^d X := X^{\otimes d}$ induced by that on X. So we have a (locally-unital) superalgebra structure on $\mathtt{Tens}\, X := \bigoplus_{d\ge 0} \mathtt{Tens}^d X$, with the product on each summand $\mathtt{Tens}^d X$ being as above, and $xy = 0$ for $x \in \mathtt{Tens}^d X$ and $y \in \mathtt{Tens}^e X$ with $d \ne e$. Note that this algebra structure is different from the two algebra structures on $\mathtt{Tens}\, X$ considered in §3.2, namely the product $\otimes$ and the product $*$.

In fact, $\mathtt{Tens}\, X$ is now even a *superbialgebra* with the coproduct (3.11). Since $\mathfrak{S}_d$ acts on $\mathtt{Tens}^d X$ with superalgebra automorphisms, the fixed points $\mathtt{Inv}^d X = (\mathtt{Tens}^d X)^{\mathfrak{S}_d}$ is a subsuperalgebra of $\mathtt{Tens}^d X$. By observations made in §3.3, $\mathtt{Inv}\, X = \bigoplus_{d\ge 0} \mathtt{Inv}^d X$ is a supercocommutative subsuperbialgebra of $\mathtt{Tens}\, X$.

Lemma 4.1. *Let $x, y \in \mathtt{Tens}\, X$ and $z \in \mathtt{Inv}\, X$. Then*

$$(x * y)z = \sum (-1)^{\bar{y}\bar{z}_{(1)}}(xz_{(1)}) * (yz_{(2)}),$$
$$z(x * y) = \sum (-1)^{\bar{z}_{(2)}\bar{x}}(z_{(1)}x) * (z_{(2)}y).$$

Proof. We may assume that $z \in \mathtt{Inv}^d X$, $x \in \mathtt{Tens}^e X$ and $y \in \mathtt{Tens}^{d-e} X$ for some non-negative integers $d \ge e$. Write $\sum' z_{(1)} \otimes z_{(2)}$ for the $\mathtt{Inv}^e X \otimes \mathtt{Inv}^{d-e} X$-component of $\Delta(z)$. Then, since z is $\mathfrak{S}_d$-invariant, we have

$$(x * y)z = \sum_{g \in {}^{(e,d-e)}\mathscr{D}} (x \otimes y)^g z$$

$$
\begin{aligned}
&= \sum_{g\in {}^{(e,d-e)}\mathscr{D}} ((x\otimes y)z)^g \\
&= \sum_{g\in {}^{(e,d-e)}\mathscr{D}} \Big(\textstyle\sum' \big((x\otimes y)(z_{(1)}\otimes z_{(2)})\big)\Big)^g \\
&= \sum_{g\in {}^{(e,d-e)}\mathscr{D}} \Big(\textstyle\sum' (-1)^{\bar y \bar z_{(1)}} x z_{(1)} \otimes y z_{(2)}\Big)^g \\
&= \textstyle\sum' (-1)^{\bar y \bar z_{(1)}} (x z_{(1)}) * (y z_{(2)}) \\
&= \textstyle\sum (-1)^{\bar y \bar z_{(1)}} (x z_{(1)}) * (y z_{(2)}),
\end{aligned}
$$

where the last equality holds because a summand on the right hand side is zero unless $z_{(1)} \in \mathtt{Inv}^e X$ and $z_{(2)} \in \mathtt{Inv}^{d-e} X$. The second equality in the lemma is proved similarly. ☐

Lemma 4.2. *Let $x, y, z, u \in \mathtt{Inv}\, X$. Then*

$$(x*y)(z*u) = \sum (-1)^s (x_{(1)}z_{(1)}) * (y_{(1)}z_{(2)}) * (x_{(2)}u_{(1)}) * (y_{(2)}u_{(2)}),$$

where $s = (\bar x_{(2)} + \bar y_{(2)})\bar z + \bar y_{(1)}(\bar x_{(2)} + \bar z_{(1)}) + \bar y_{(2)}\bar u_{(1)}$.

Proof. Writing $\Delta(x*y) = \sum (x*y)_{(1)} \otimes (x*y)_{(2)}$, we have

$$
\begin{aligned}
(x*y)(z*u) &= \sum (-1)^{\overline{(x*y)_{(2)}}\,\bar z} ((x*y)_{(1)}z) * ((x*y)_{(2)}u) \\
&= \sum (-1)^{(\bar x_{(2)}+\bar y_{(2)})\bar z + \bar y_{(1)}\bar x_{(2)}} ((x_{(1)} * y_{(1)})z) * ((x_{(2)} * y_{(2)})u) \\
&= \sum (-1)^s (x_{(1)}z_{(1)}) * (y_{(1)}z_{(2)}) * (x_{(2)}u_{(1)}) * (y_{(2)}u_{(2)}),
\end{aligned}
$$

where $s = (\bar x_{(2)} + \bar y_{(2)})\bar z + \bar y_{(1)}\bar x_{(2)} + \bar y_{(1)}\bar z_{(1)} + \bar y_{(2)}\bar u_{(1)}$ is as in the statement of the lemma, the first and third equalities hold by Lemma 4.1, and the second one is due to Lemma 3.12. ☐

Lemma 4.3. *Let $l \in \mathbb{Z}_{>0}$, $d_1, \dots, d_l, f_1, \dots, f_l \in \mathbb{Z}_{\geq 0}$, and $1_X = e_1 + \dots + e_l$ with $e_i e_j = \delta_{i,j} e_i$ for all i, j. If $x_i \in (\mathtt{Tens}^{d_i} X)e_i^{\otimes d_i}$ and $y_i \in e_i^{\otimes f_i}(\mathtt{Tens}^{d_i} X)$ for $i = 1, \dots, l$, then*

$$(x_1 * \dots * x_l)(y_1 * \dots * y_l) = (-1)^{[x_1,\dots,x_l;y_1,\dots,y_l]} \delta_{d_1,f_1} \dots \delta_{d_l,f_l} (x_1y_1) * \dots * (x_l y_l).$$

Proof. Let $\lambda = (d_1, \dots, d_l)$ and $\mu = (f_1, \dots, f_l)$. Note that $(x_1 \otimes \dots \otimes x_l)^g (y_1 \otimes \dots \otimes y_l)^h = 0$ if $g \in {}^\lambda\mathscr{D}$, $h \in {}^\mu\mathscr{D}$ and either $\lambda \neq \mu$ or $g \neq h$. Since $\mathfrak{S}_d$ acts on $\mathtt{Tens}^d X$ with superalgebra automorphisms for every d, the result follows. ☐

Corollary 4.4. *If $X = X_1 \oplus \dots \oplus X_l$ is a direct sum of superalgebras, then there is an isomorphism of superalgebras*

$$\bigoplus_{(d_1,\dots,d_l)\in\Lambda(l,d)} \mathtt{Inv}^{d_1} X_1 \otimes \dots \otimes \mathtt{Inv}^{d_l} X_l \xrightarrow{\sim} \mathtt{Inv}^d X,\ x_1 \otimes \dots \otimes x_l \mapsto x_1 * \dots * x_l.$$

Proof. This follows from Lemmas 3.6 and 4.3. ☐

Recall from from §3.4 that we consider X^* as a bimodule over X. Note for $d \in \mathbb{Z}_{\geq 0}$ that $\mathtt{Tens}^d(X^*)$ is naturally a bimodule over $\mathtt{Tens}^d X$ with respect to

$$(x_1 \otimes \dots \otimes x_d) \cdot (\xi_1 \otimes \dots \otimes \xi_d) = (-1)^{[x_1,\dots,x_d;\xi_1,\dots,\xi_d]} (x_1 \cdot \xi_1) \otimes \dots \otimes (x_d \cdot \xi_d), \quad (4.5)$$

where $x_1,\dots,x_d \in X$ and $\xi_1,\dots,\xi_d \in X^*$, or $\xi_1,\dots,\xi_d \in X$ and $x_1,\dots,x_d \in X^*$. As usual, if $d \neq e$ we define the action trivially: $\mathtt{Tens}^d X \cdot \mathtt{Tens}^e(X^*) = \mathtt{Tens}^e(X^*)\cdot \mathtt{Tens}^d X = 0$. This yields a $\mathtt{Tens}\, X$-bimodule structure on $\mathtt{Tens}(X^*)$. Upon restriction, we now get an $\mathtt{Inv}\, X$-superbimodule structure on $\mathtt{Inv}(X^*)$. We refer to this superbimodule structure as the *standard superbimodule* structure. On the other hand, we have the dual regular $\mathtt{Inv}\, X$-superbimodule structure on $(\mathtt{Inv}\, X)^*$, see (3.16). By Lemmas 3.14 and 3.10, we have an embedding

$$\iota\colon (\mathtt{Inv}\, X)^* \xrightarrow{\sim} \mathtt{Sym}(X^*) \hookrightarrow {}'\mathtt{Sym}(X^*) \xrightarrow{\sim} \mathtt{Inv}(X^*). \tag{4.6}$$

Lemma 4.7. *The embedding ι is a homomorphism of $\mathtt{Inv}\, X$-bimodules.*

Proof. Every element of $\mathtt{Inv}(X^*)$ is by definition a linear combination of functions of the form $\xi_1 \otimes \cdots \otimes \xi_d$ with $\xi_1,\dots,\xi_d \in X^*$. On the other hand, by Lemma 3.14, $(\mathtt{Inv}\, X)^*$ is spanned by the functions of the form $(\xi_1\otimes\cdots\otimes\xi_d)|_{\mathtt{Inv}\, X}$ with $\xi_1,\dots,\xi_d \in X^*$, and

$$\iota((\xi_1 \otimes\cdots\otimes \xi_d)|_{\mathtt{Inv}\, X}) = \xi_1 * \cdots * \xi_d.$$

Note that

$$(\xi_1 * \cdots * \xi_d)|_{\mathtt{Inv}\, X} = d!(\xi_1 \otimes\cdots\otimes \xi_d)|_{\mathtt{Inv}\, X}.$$

We have proved for any $\xi \in \mathtt{Tens}^d(X^*)$ that

$$\iota(\xi|_{\mathtt{Inv}\, X})|_{\mathtt{Inv}\, X} = d!\xi|_{\mathtt{Inv}\, X}. \tag{4.8}$$

Let $x \in \mathtt{Inv}\, X$. We now prove that $\iota(x\cdot(\xi|_{\mathtt{Inv}\, X})) = x\cdot \iota(\xi|_{\mathtt{Inv}\, X})$, the proof that $\iota((\xi|_{\mathtt{Inv}\, X})\cdot x) = \iota(\xi|_{\mathtt{Inv}\, X})\cdot x$ being similar. Using (4.8), we get

$$\begin{aligned}\iota(x\cdot(\xi|_{\mathtt{Inv}\, X}))|_{\mathtt{Inv}\, X} &= \iota((x\cdot\xi)|_{\mathtt{Inv}\, X})|_{\mathtt{Inv}\, X} = d!(x\cdot\xi)|_{\mathtt{Inv}\, X} = d!\, x\cdot(\xi|_{\mathtt{Inv}\, X})\\ &= x\cdot\big((\iota(\xi|_{\mathtt{Inv}\, X}))|_{\mathtt{Inv}\, X}\big) = (x\cdot\iota(\xi|_{\mathtt{Inv}\, X}))|_{\mathtt{Inv}\, X}.\end{aligned}$$

To prove that $\iota(x\cdot(\xi|_{\mathtt{Inv}\, X})) = x\cdot\iota(\xi|_{\mathtt{Inv}\, X})$ it now suffices to show that the map $\mathtt{Inv}(X^*) \to (\mathtt{Inv}\, X)^*$ given by $\eta \mapsto \eta|_{\mathtt{Inv}\, X}$ is injective. Let $\eta \in \mathtt{Inv}^d(X^*)$ satisfy $\eta|_{\mathtt{Inv}\, X} = 0$. Since $d!({}'\mathtt{Sym}(X^*)) \subseteq \mathtt{Sym}(X^*)$, we can write $d!\eta = \iota(\xi|_{\mathtt{Inv}\, X})$ for some $\xi \in \mathtt{Tens}^d(X^*)$. Then, using (4.8),

$$0 = d!\eta|_{\mathtt{Inv}\, X} = \iota(\xi|_{\mathtt{Inv}\, X})|_{\mathtt{Inv}\, X} = d!\xi|_{\mathtt{Inv}\, X}$$

Hence $\xi|_{\mathtt{Inv}\, X} = 0$. But $\iota(\xi|_{\mathtt{Inv}\, X}) = d!\eta$, whence $\eta = 0$, as desired. ∎

Recall the trivial extension algebra $T_X = X \oplus X^*$ from §3.4. For $d, e \in \mathbb{Z}_{\geq 0}$, we define $\mathtt{Tens}^{d,e}\, T_X$ to be the span in $\mathtt{Tens}^{d+e}\, T_X$ of pure tensors $y_1 \otimes\cdots\otimes y_{d+e}$ such that d of the y's are in X and e of the y's are in X^*. We identify $\mathtt{Tens}^d X$ with $\mathtt{Tens}^{d,0}\, T_X$ and $\mathtt{Tens}^d(X^*)$ with $\mathtt{Tens}^{0,d}\, T_X$ in the obvious way. Then for $\xi \in \mathtt{Tens}^d X$ and $x \in \mathtt{Tens}^e(X^*)$, we have

$$\xi x = \xi\cdot x \quad\text{and}\quad x\xi = x\cdot\xi,$$

where the left hand sides are products in the algebra $\mathtt{Tens}^d\, T_X$ and the right hand sides are the *standard* actions in the sense of (4.5). (Note the change of our notational 'paradigm': from now on we use Greek letters to denote elements of $\mathtt{Tens}\, X$ and Roman letters for elements of $\mathtt{Tens}(X^*)$.)

Lemma 4.9. *Let $a, b, d \in \mathbb{Z}_{\geq 0}$ with $a, b \leq d$. Suppose that $x \in \mathtt{Inv}^a(X^*)$, $y \in \mathtt{Inv}^b(X^*)$, $\xi \in \mathtt{Inv}^{d-a} X$, and $\eta \in \mathtt{Inv}^{d-b} X$. Then in $\mathtt{Inv}^d T_X$ we have*

$$(\xi * x)(\eta * y) = \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} \xi_{(2)}\eta_{(1)} * (x \cdot \eta_{(2)}) * (\xi_{(1)} \cdot y).$$

Proof. Since $X^*X^* = 0$ in T_X, the result follows from (3.13) and Lemma 4.2. □

4.2. **Doubles.** We have a natural pairing $\langle \cdot, \cdot \rangle$ between $\mathtt{Inv}\, X$ and $(\mathtt{Inv}\, X)^*$, with $\langle x, \xi \rangle = \langle \xi, x \rangle = 0$ for $\xi \in \mathtt{Inv}^d X$ and $x \in (\mathtt{Inv}^e X)^*$ with $d \neq e$. Also, for every $d \in \mathbb{Z}_{\geq 0}$ we have the *dual regular* actions (3.16) of $\mathtt{Inv}^d X$ on $(\mathtt{Inv}^d X)^*$. Again, we declare that $\xi \cdot x = x \cdot \xi = 0$ if $\xi \in \mathtt{Inv}^d X$ and $x \in (\mathtt{Inv}^e X)^*$ with $d \neq e$. There is a superbialgebra structure on $(\mathtt{Inv}\, X)^*$ which is dual to that on $\mathtt{Inv}\, X$. We write

$$\nabla \colon (\mathtt{Inv}\, X)^* \to (\mathtt{Inv}\, X)^* \otimes (\mathtt{Inv}\, X)^* \tag{4.10}$$

for the corresponding coproduct. Note that $\nabla((\mathtt{Inv}^d X)^*) \subseteq (\mathtt{Inv}^d X)^* \otimes (\mathtt{Inv}^d X)^*$ for all $d \in \mathbb{Z}_{\geq 0}$.

We now recall Turner's construction [**Tu**$_3$] of a *double superalgebra* DX. As an $\mathcal{O}$-superspace,

$$DX := \mathtt{Inv}\, X \otimes (\mathtt{Inv}\, X)^*.$$

The product is defined, using Sweedler's notation for Δ, as follows:

$$(\xi \otimes x)(\eta \otimes y) = \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} \xi_{(2)}\eta_{(1)} \otimes (x \cdot \eta_{(2)})(\xi_{(1)} \cdot y) \tag{4.11}$$

for $\xi, \eta \in \mathtt{Inv}\, X$ and $x, y \in (\mathtt{Inv}\, X)^*$. The associativity of the product can be checked by a straightforward computation, cf. [**Tu**$_3$, Theorem 1.1]. In view of (3.18) and (3.19), this product formula can be rewritten, using Sweedler's notation for Δ and ∇, to match [**Tu**$_3$, Remark 1.3]:

$$(\xi \otimes x)(\eta \otimes y) = \sum (-1)^s \langle \xi_{(1)}, y_{(2)} \rangle \langle x_{(1)}, \eta_{(2)} \rangle \xi_{(2)}\eta_{(1)} \otimes x_{(2)}y_{(1)}, \tag{4.12}$$

where

$$s = \bar{\xi}_{(1)}\bar{\xi}_{(2)} + \bar{\xi}_{(1)}\bar{\eta}_{(1)} + \bar{x}_{(2)}\bar{y}_{(2)} + \bar{y}_{(1)}\bar{y}_{(2)} + \bar{x}_{(1)}\bar{\eta}_{(1)} + \bar{x}_{(2)}\bar{\eta}_{(2)} + \bar{x}_{(2)}\bar{\eta}_{(1)}.$$

It is easy to see that we can write the superalgebra DX as a direct sum of subsuperalgebras

$$DX = \bigoplus_{d \geq 0} D^d X,$$

where

$$D^d X := \bigoplus_{e,f \geq 0,\ e+f=d} \mathtt{Inv}^e X \otimes (\mathtt{Inv}^f X)^*. \tag{4.13}$$

We use the following notation for the summands on the right hand side above:

$$D^{e,f} X := \mathtt{Inv}^e X \otimes (\mathtt{Inv}^f X)^*. \tag{4.14}$$

Remark 4.15. The definition of the double $D^d X$ makes sense for any $\Bbbk$-algebra X, without any assumption on the ring $\Bbbk$. We also note that Lemmas 4.1, 4.2, and 4.9 do not need the assumption that $\Bbbk = \mathcal{O}$. However, it is crucial to work over $\mathcal{O}$ when we deal with the divided power version ${}'D^d X$ below.

Remark 4.16. The direct sum decomposition in (4.13) is a priori only a decomposition of $\mathcal{O}$-modules. But one can say a little more.

(i) $D^{d,0}X$ is a subalgebra of D^dX naturally isomorphic to the algebra $\mathtt{Inv}^d X$.

(ii) $D^{0,d}X$ is an ideal in D^dX. Moreover,

$$(D^{e,f}X)(D^{0,d}X) = (D^{0,d}X)(D^{e,f}X) = 0$$

unless $e = d$, in which case for $\xi \in \mathtt{Inv}^d X$ and $x \in (\mathtt{Inv}^d X)^*$, we have

$$(\xi \otimes 1)(1 \otimes x) = 1 \otimes (\xi \cdot x), \tag{4.17}$$

$$(1 \otimes x)(\xi \otimes 1) = 1 \otimes (x \cdot \xi). \tag{4.18}$$

In particular, $D^{d,0}X \oplus D^{0,d}X$ is a subalgebra of D^dX, isomorphic to $T_{\mathtt{Inv}^d X}$. As a still more special case, we get $D^1X \cong T_X$.

4.3. **Divided power doubles.** In view of Lemma 3.14, we identify the superalgebras

$$(\mathtt{Inv}\, X)^* = \mathtt{Sym}(X^*). \tag{4.19}$$

Then

$$\mathtt{Sym}(X^*) \subseteq {}'\mathtt{Sym}(X^*) \subseteq \mathtt{Sym}(X^*) \otimes_{\mathcal{O}} \mathbb{K} \cong \mathtt{Sym}(X^*_{\mathbb{K}}) = (\mathtt{Inv}\, X_{\mathbb{K}})^*,$$

where we have used the identification (4.19) over $\mathbb{K}$ for the last equality. We have the left and right dual regular actions of $\mathtt{Inv}\, X_{\mathbb{K}}$ on $(\mathtt{Inv}\, X_{\mathbb{K}})^*$. Since $\mathtt{Inv}\, X \subseteq \mathtt{Inv}\, X_{\mathbb{K}}$ in a natural way, we can also speak of the dual regular actions of $\mathtt{Inv}\, X$ on $(\mathtt{Inv}\, X_{\mathbb{K}})^*$.

Lemma 4.20. *The $\mathcal{O}$-submodule ${}'\mathtt{Sym}(X^*) \subset (\mathtt{Inv}\, X_{\mathbb{K}})^*$ is invariant with respect to the dual regular actions of $\mathtt{Inv}\, X$ on $(\mathtt{Inv}\, X_{\mathbb{K}})^*$. Thus, ${}'\mathtt{Sym}(X^*)$ is an $\mathtt{Inv}\, X$-superbimodule. With respect to this $\mathtt{Inv}\, X$-superbimodule structure on ${}'\mathtt{Sym}(X^*)$ and the standard $\mathtt{Inv}\, X$-superbimodule structure on $\mathtt{Inv}(X^*)$, the map $\kappa\colon {}'\mathtt{Sym}(X^*) \xrightarrow{\sim} \mathtt{Inv}(X^*)$ of Lemma 3.10 is an isomorphism of $\mathtt{Inv}\, X$-superbimodules.*

Proof. By (4.6) and Lemma 4.7, we have an $\mathtt{Inv}\, X$-bimodule homomorphism

$$\iota\colon (\mathtt{Inv}\, X)^* = \mathtt{Sym}(X^*) \hookrightarrow {}'\mathtt{Sym}(X^*) \xrightarrow{\kappa} \mathtt{Inv}(X^*). \tag{4.21}$$

As

$$\mathtt{Sym}(X^*_{\mathbb{K}}) \cong \mathtt{Sym}(X^*) \otimes_{\mathcal{O}} \mathbb{K} \cong {}'\mathtt{Sym}(X^*) \otimes_{\mathcal{O}} \mathbb{K} \cong {}'\mathtt{Sym}(X^*_{\mathbb{K}}),$$

extending scalars in (4.21), we get an $\mathtt{Inv}\, X_{\mathbb{K}}$-superbimodule isomorphism

$$\iota_{\mathbb{K}}\colon (\mathtt{Inv}\, X_{\mathbb{K}})^* = \mathtt{Sym}(X^*_{\mathbb{K}}) = {}'\mathtt{Sym}(X^*_{\mathbb{K}}) \xrightarrow{\sim} \mathtt{Inv}(X^*_{\mathbb{K}}).$$

Considering ${}'\mathtt{Sym}(X^*)$ as the sublattice in $\mathtt{Sym}(X^*_{\mathbb{K}})$, the restriction $\iota_{\mathbb{K}}|_{{}'\mathtt{Sym}(X^*)}$ is the isomorphism $\kappa\colon {}'\mathtt{Sym}(X^*) \xrightarrow{\sim} \mathtt{Inv}(X^*)$. Now the standard left and right actions of $\mathtt{Inv}\, X \subseteq \mathtt{Inv}\, X_{\mathbb{K}}$ on $\mathtt{Inv}(X^*_{\mathbb{K}})$ leave $\mathtt{Inv}(X^*) = \mathtt{Inv}(X^*_{\mathbb{K}}) \cap \mathtt{Tens}(X^*)$ invariant, and we have $\iota_{\mathbb{K}}^{-1}(\mathtt{Inv}(X^*)) = {}'\mathtt{Sym}(X^*)$. This implies the lemma. $\square$

The identification $\mathtt{Sym}(X^*_{\mathbb{K}}) = (\mathtt{Inv}\, X_{\mathbb{K}})^*$ from (4.19) together with the coproduct (4.10) yield a coproduct

$$\nabla_{\mathbb{K}}\colon\ \mathtt{Sym}(X^*_{\mathbb{K}}) \to \mathtt{Sym}(X^*_{\mathbb{K}}) \otimes \mathtt{Sym}(X^*_{\mathbb{K}}).$$

Lemma 4.22. *We have*

$$\nabla_{\mathbb{K}}('\mathtt{Sym}(X^*)) \subseteq \big(\mathtt{Sym}(X^*) \otimes '\mathtt{Sym}(X^*)\big) \cap \big('\mathtt{Sym}(X^*) \otimes \mathtt{Sym}(X^*)\big).$$

Proof. Let $x \in '\mathtt{Sym}^d(X^*)$ for some $d \in \mathbb{Z}_{\geq 0}$. Let $\{\xi_1, \dots, \xi_m\}$ be a homogeneous basis of $\mathtt{Inv}^d X$ and $\{x_1, \dots, x_m\}$ be the dual basis of $(\mathtt{Inv}^d X)^* = \mathtt{Sym}^d(X^*)$, cf. (4.19). We can write $\nabla_{\mathbb{K}}(x) = \sum_{j=1}^m y_j \otimes x_j$, where $y_j \in \mathtt{Sym}^d(X^*_{\mathbb{K}})$ for $j = 1, \dots, m$. By Lemma 4.20, $'\mathtt{Sym}(X^*)$ is invariant under the left dual regular action of $\mathtt{Inv}\, X$, so $\xi_i \cdot x \in '\mathtt{Sym}^d(X^*)$ for any $i \in \{1, \dots, m\}$. On the other hand, by (3.18),

$$\xi_i \cdot x = \sum_{j=1}^m (-1)^{\bar{\xi}_i \bar{y}_j} \langle \xi_i, x_j \rangle y_j = (-1)^{\bar{\xi}_i \bar{y}_i} y_i,$$

whence $y_i \in '\mathtt{Sym}^d(X^*)$. We have proved that $\nabla_{\mathbb{K}}('\mathtt{Sym}(X)) \subseteq \big('\mathtt{Sym}(X^*) \otimes \mathtt{Sym}(X^*)\big)$. The other inclusion is proved similarly. ☐

By Lemma 4.22, we have a coproduct

$$\nabla \colon '\mathtt{Sym}(X^*) \to '\mathtt{Sym}(X^*) \otimes '\mathtt{Sym}(X^*) \tag{4.23}$$

obtained by restricting $\nabla_{\mathbb{K}}$. Recalling (4.19), note that

$$DX = \mathtt{Inv}\, X \otimes (\mathtt{Inv}\, X)^* = \mathtt{Inv}\, X \otimes \mathtt{Sym}(X^*) \tag{4.24}$$

is an $\mathcal{O}$-form of $DX_{\mathbb{K}}$. We define a larger $\mathcal{O}$-form

$$'DX := \mathtt{Inv}\, X \otimes '\mathtt{Sym}(X^*),$$

which is closed under the multiplication (4.11) because $'\mathtt{Sym}(X^*)$ is invariant under the left and right dual regular actions of $\mathtt{Inv}\, X$ by Lemma 4.20. The product in $'DX$ is also given by the formula (4.12) where we use the coproduct (4.23) on x and y. We have $'DX = \bigoplus_{d \geq 0} {'D^d X}$, where

$$'D^d X := \sum_{e,f \geq 0,\; e+f=d} \mathtt{Inv}^e\, X \otimes '\mathtt{Sym}^f(X^*).$$

We use the following notation for the summands on the right hand side above:

$$'D^{e,f} X := \mathtt{Inv}^e\, X \otimes '\mathtt{Sym}^f(X^*). \tag{4.25}$$

The following result often allows one to reduce the study of DX to that of $\mathtt{Inv}\, T_X$. Recall the isomorphism κ from Lemma 3.10.

Theorem 4.26. *There is an isomorphism of $\mathcal{O}$-superalgebras*

$$'DX \xrightarrow{\sim} \mathtt{Inv}\, T_X, \quad \xi \otimes x \mapsto \xi * \kappa(x) \quad (\xi \in \mathtt{Inv}\, X,\ x \in '\mathtt{Sym}(X^*)).$$

Proof. The map φ in the theorem is an isomorphism of $\mathcal{O}$-supermodules by Lemmas 3.10 and 3.6. To see that it is an algebra homomorphism, we compute for $\xi, \eta \in \mathtt{Inv}\, X$ and $x, y \in '\mathtt{Sym}(X^*)$:

$$\begin{aligned}
\varphi\big((\xi \otimes x)(\eta \otimes y)\big) &= \varphi\Big(\sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} \xi_{(2)}\eta_{(1)} \otimes (x \cdot \eta_{(2)})(\xi_{(1)} \cdot y)\Big) \\
&= \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} (\xi_{(2)}\eta_{(1)}) * \kappa\big((x \cdot \eta_{(2)})(\xi_{(1)} \cdot y)\big) \\
&= \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} (\xi_{(2)}\eta_{(1)}) * \kappa(x \cdot \eta_{(2)}) * \kappa(\xi_{(1)} \cdot y) \\
&= \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} (\xi_{(2)}\eta_{(1)}) * (\kappa(x) \cdot \eta_{(2)}) * (\xi_{(1)} \cdot \kappa(y))
\end{aligned}$$

$$= (\xi * \kappa(x))(\eta * \kappa(y))$$
$$= \varphi(\xi \otimes x)\varphi(\eta \otimes y),$$

where we have used (4.11) for the first equality, Lemma 3.10 for the third equality, Lemma 4.20 for the fourth equality and Lemma 4.9 for the fifth equality. □

Example 4.27. Let $\mathcal{O}[z]_d$ be the truncated polynomial algebra $\mathcal{O}[z]/(z^{d+1})$, and ${}'\mathcal{O}[z]_d$ be the divided power truncated polynomial algebra defined as the $\mathcal{O}$-subalgebra of $\mathbb{K}[z]/(z^{d+1})$ spanned by all $z^{(e)}$ with $e = 0, \dots, d$. If X is the trivial algebra $\mathcal{O}$, let $y \in X^*$ be the function which sends 1 to 1. Then $D^d X \cong \mathcal{O}[z]_d$, with $1^{\otimes d-e} \otimes y^e \in \mathtt{Inv}^{d-e} X \otimes \mathtt{Sym}^e(X^*)$ corresponding to z^e, and ${}'D^d X \cong {}'\mathcal{O}[z]_d$, with $1^{\otimes d-e} \otimes y^{(e)} \in \mathtt{Inv}^{d-e} X \otimes {}'\mathtt{Sym}^e(X^*)$ corresponding to $z^{(e)}$.

4.4. **A generating set for a Turner double.** For any $d \in \mathbb{Z}_{\geq 0}$, define $\mathcal{D}^d X \subseteq \mathtt{Inv}^d T_X$ to be the image of $D^d X$ under the isomorphism of Theorem 4.26, and set $\mathcal{D}X := \bigoplus_{d \geq 0} \mathcal{D}^d X$. Of course $\mathcal{D}^d X$ is just an isomorphic copy of $D^d X$, considered as an explicit subalgebra of $\mathtt{Inv}^d T_X$. By (4.24) and Lemma 3.10, we have

$$\mathcal{D}^d X = \bigoplus_{e=0}^{d} \mathtt{Inv}^{d-e}(X) * \mathtt{Star}^e(X^*). \tag{4.28}$$

Let $Y = X_{\bar{1}} \oplus X^*$, so that Y is naturally an $X_{\bar{0}}$-superbimodule and $T_X = X_{\bar{0}} \oplus Y$.

Lemma 4.29. *For any $d \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{D}^d X = \bigoplus_{e=0}^{d} \mathtt{Inv}^{d-e}(X_{\bar{0}}) * \mathtt{Star}^e Y.$$

Proof. By Lemma 3.6,

$$\mathtt{Inv}^{d-e}(X) = \bigoplus_{f=0}^{d-e} \mathtt{Inv}^{d-e-f}(X_{\bar{0}}) * \mathtt{Inv}^f(X_{\bar{1}}).$$

It follows from Lemma 3.10 that $\mathtt{Inv}^f(X_{\bar{1}}) = \mathtt{Star}^f(X_{\bar{1}})$ for all $f \in \mathbb{Z}_{\geq 0}$, so by (4.28) we have

$$\mathcal{D}^d X = \bigoplus_{e=0}^{d} \bigoplus_{f=0}^{e} \mathtt{Inv}^{d-e-f}(X_{\bar{0}}) * \mathtt{Star}^f(X_{\bar{1}}) * \mathtt{Star}^e(X^*)$$
$$= \bigoplus_{e=0}^{d} \mathtt{Inv}^{d-e}(X_{\bar{0}}) * \mathtt{Star}^e Y. \qquad \square$$

In the rest of this subsection, we write 1 for the identity element 1_X of X.

Theorem 4.30. *For any $d \in \mathbb{Z}_{>0}$, the $\mathcal{O}$-superalgebra $\mathcal{D}^d X$ is generated by $\mathtt{Inv}^d X_{\bar{0}}$ and $1^{\otimes(d-1)} * Y$.*

Proof. Let $\mathcal{G}$ be the subalgebra of $\mathcal{D}^d X$ generated by $\mathtt{Inv}^d X_{\bar{0}}$ and $1^{\otimes(d-1)} * Y$. By Lemma 4.29, it suffices to show that $\mathcal{D}^{d-e,e} X := \mathtt{Inv}^{d-e}(X_{\bar{0}}) * \mathtt{Star}^e Y \subseteq \mathcal{G}$ for all $e \in [0, d]$. We will prove this by induction on e, the case $e = 0$ being clear.

Let $0 < e \le d$ and assume that $\mathcal{D}^{d-f,f}X \subseteq \mathcal{G}$ for all $f \in [0, e)$. Let $x \in Y$ and $y \in \mathtt{Star}^{e-1} Y$. It follows from Lemma 4.2 that

$$(1^{\otimes(d-1)} * x)(1^{\otimes(d-e+1)} * y) \in 1^{\otimes(d-e)} * x * y + \bigoplus_{f=0}^{e-1} \mathcal{D}^{d-f,f}X.$$

So $1^{\otimes(d-e)} * \mathtt{Star}^e Y \subseteq \mathcal{G}$.

For every $f \in [0, d-e]$, write $\mathcal{D}^{d-e-f,f,e}X := 1^{\otimes(d-e-f)} * \mathtt{Inv}^f(X_{\bar{0}}) * \mathtt{Star}^e Y$. We claim that $\mathcal{D}^{d-e-f,f,e}X \subseteq \mathcal{G}$ for all such f. If the claim is true, then $\mathcal{D}^{d-e,e}X = \mathcal{D}^{0,d-e,e}X \subseteq \mathcal{G}$, which implies the lemma. We prove the claim by induction on f. The base case $f = 0$ was established in the previous paragraph.

Given $f \in (0, d-e]$ and assuming that our claim is true for smaller f, let $\xi \in \mathtt{Inv}^f(X_{\bar{0}})$ and $z \in \mathtt{Star}^e Y$. By Lemma 4.2, we have

$$\begin{aligned}
&(1^{\otimes(d-f)} * \xi)(1^{\otimes(d-e)} * z) = \\
&= \sum_{a=0}^{\min(d-e,d-f)} \sum \pm(1^{\otimes a}) * (\xi_{(1)}1^{\otimes(d-e-a)}) * (1^{\otimes(d-f-a)}z_{(1)}) * (\xi_{(2)}z_{(2)}) \\
&= \sum_{a=0}^{\min(d-e,d-f)} \sum \pm(1^{\otimes a}) * (\xi_{(1)}1^{\otimes(d-e-a)}) * (\xi_{(2)}z_{(2)}) * (1^{\otimes(d-f-a)}z_{(1)}),
\end{aligned}$$

where supercommutativity of $*$ has been used for the last equality. Note that $\xi_{(1)} \in \mathtt{Inv}^b(X_{\bar{0}})$ for some $b \le f$, so $\xi_{(1)}1^{\otimes(d-e-a)} = 0$ if $a < d-e-f$. Moreover, any term in the sum with $a > d-e-f$ belongs to $\mathcal{D}^{a,d-e-a,e}X$ and hence to $\mathcal{G}$ by the inductive hypothesis. The remaining term is $1^{\otimes(d-e-f)} * \xi * z$, so $1^{\otimes(d-e-f)} * \xi * z \in \mathcal{G}$, and we have proved our claim. □

Let W be an $X_{\bar{0}}$-bimodule. For any $\xi \in X_{\bar{0}}$, define $\mathrm{ad}(\xi) \in \mathrm{End}_{\mathcal{O}}(W)$ by $\mathrm{ad}(\xi)(w) := \xi w - w\xi$ for all $w \in W$. Further, for any $r \in \mathbb{Z}_{\ge 0}$, we define $\mathrm{ad}^r(X_{\bar{0}}) \subseteq \mathrm{End}_{\mathcal{O}}(W)$ as the $\mathcal{O}$-span of all compositions $\mathrm{ad}(\xi_1) \circ \cdots \circ \mathrm{ad}(\xi_r)$ for $\xi_1, \dots, \xi_r \in X_{\bar{0}}$. As usual, if F is a subset of $\mathrm{End}_{\mathcal{O}}(W)$ and U is a subset of W, we denote by $F(U)$ the $\mathcal{O}$-span of the elements $f(u)$ for all $f \in F$ and $u \in U$.

Corollary 4.31. *Let U be a subsuperspace of Y such that $\sum_{r \ge 0} \mathrm{ad}^r(X_{\bar{0}})(U) = Y$, and let $d \in \mathbb{Z}_{>0}$. Then the $\mathcal{O}$-superalgebra $\mathcal{D}^d X$ is generated by $\mathtt{Inv}^d(X_{\bar{0}})$ and $1^{\otimes(d-1)} * U$.*

Proof. If $d = 1$, the result is clear, so we assume that $d \ge 2$. By Lemma 4.2, for any $\xi \in X_{\bar{0}}$ and $x \in Y$, we have

$$\begin{aligned}
(1^{\otimes(d-1)} * \xi)(1^{\otimes(d-1)} * x) &= 1^{\otimes(d-2)} * \xi * x + 1^{\otimes(d-1)} * (\xi x), \\
(1^{\otimes(d-1)} * x)(1^{\otimes(d-1)} * \xi) &= 1^{\otimes(d-2)} * x * \xi + 1^{\otimes(d-1)} * (x\xi).
\end{aligned}$$

Since ξ has degree $\bar{0}$, we have $x * \xi = \xi * x$, so

$$1^{\otimes(d-1)} * (\mathrm{ad}(\xi)(x)) = (1^{\otimes(d-1)} * \xi)(1^{\otimes(d-1)} * x) - (1^{\otimes(d-1)} * x)(1^{\otimes(d-1)} * \xi).$$

We have proved that if $1^{\otimes(d-1)} * x$ belongs to the subalgebra $\mathcal{G} \subseteq \mathcal{D}^d X$ generated by $\mathtt{Inv}^d(X_{\bar{0}})$ and $1^{\otimes(d-1)} * U$, then $1^{\otimes(d-1)} * (\mathrm{ad}(\xi)(x)) \in \mathcal{G}$ for all $\xi \in X_{\bar{0}}$. In view of the hypothesis, this implies that $1^{\otimes(d-1)} * Y \subseteq \mathcal{G}$, and the result now follows by Theorem 4.30. □

4.5. **Gradings.** By (4.11), the algebra D^dX (resp. $'D^dX$) is $\mathbb{Z}_{\geq 0}$-graded with the graded degree e component being $D^{d-e,e}X$ (resp. $'D^{d-e,e}X$) for $e = 0, \dots, d$. We refer to this grading as the *standard grading*. In fact, it is a *supergalgebra grading*, which means that it is an algebra grading and a supergrading in the sense of Section 3. If a superalgebra has a superalgebra grading, we just say that it is graded.

Assume now that the multiplication in X satisfies $X_{\bar 1}X_{\bar 1} = 0$. Then X is a $\mathbb{Z}$-graded algebra with $X^0 = X_{\bar 0}$, $X^1 = X_{\bar 1}$ and $X^m = 0$ for $m \neq 0, 1$. We will always work with the grading on X^* which is the *shift by* 2 of the canonical grading, i.e. $\deg \xi = 2$ if $\xi \in X^*$ satisfies $\xi(X^1) = 0$ and $\deg \xi = 1$ if $\xi \in X^*$ satisfies $\xi(X^0) = 0$. Now $T_X = X \oplus X^*$ is also graded, and it is easy to see that this is a superalgebra grading.

This yields $\mathbb{Z}_{\geq 0}$-gradings on $\mathtt{Inv}\,X$, $\mathtt{Sym}(X^*)$, $'\mathtt{Sym}(X^*)$ and $\mathtt{Inv}\,T_X$. Moreover, we let $(\mathtt{Inv}\,X)^*$ inherit the grading from $\mathtt{Sym}(X^*)$ via the identification (4.19). So we have $\mathbb{Z}_{\geq 0}$-gradings on the $\mathcal{O}$-superspaces $DX = \mathtt{Inv}\,X \otimes (\mathtt{Inv}\,X)^*$ and $'DX = \mathtt{Inv}\,X \otimes {}'\mathtt{Sym}(X^*)$, which we refer to as *Turner's gradings*, cf. [**Tu**$_1$, Remark 156]. If $Y = DX$ or $'DX$ with Turner's grading, then $Y_{\bar 0} = \bigoplus_{m \text{ even}} Y^m$ and $Y_{\bar 1} = \bigoplus_{m \text{ odd}} Y^m$. In particular, Turner's grading is a supergrading.

Lemma 4.32. *Let the superalgebra X have the property that $X_{\bar 1}X_{\bar 1} = 0$. Then, for every $d \in \mathbb{Z}_{\geq 0}$, the superalgebras D^dX and $'D^dX$ are $\mathbb{Z}_{\geq 0}$-graded with respect to Turner's gradings. Moreover, the isomorphism of Theorem 4.26 is an isomorphism of graded superalgebras.*

Proof. It is easy to check that $\mathtt{Inv}\,X$, $\mathtt{Sym}(X^*)$, $'\mathtt{Sym}(X^*)$ are $\mathbb{Z}_{\geq 0}$-graded superalgebras. Moreover, $\mathtt{Inv}(X^*)$ is graded with respect to the $*$-product. Next, one checks that both $\mathtt{Sym}(X^*)$ and $'\mathtt{Sym}(X^*)$ are graded $\mathtt{Inv}\,X$-bimodules. Finally, the homomorphisms $\Delta\colon \mathtt{Inv}\,X \to \mathtt{Inv}\,X \otimes \mathtt{Inv}\,X$ and $\kappa\colon {}'\mathtt{Sym}(X^*) \xrightarrow{\sim} \mathtt{Inv}(X^*)$ are homogeneous of degree zero. So the lemma follows from (4.11). ☐

4.6. **Symmetricity of doubles.** Let X be a $\Bbbk$-superalgebra which is free of finite rank as a $\Bbbk$-module. The Turner double superalgebra D^dX defined in §4.2 is *symmetric*. To see this, we define the bilinear form on D^dX via

$$(\xi \otimes x, \eta \otimes y) := \langle \xi, y \rangle \langle x, \eta \rangle.$$

We give another description of the form $(\cdot,\cdot)$. Recall the standard grading on D^dX from §4.5. Let $F \in (D^dX)^*$ be defined by requiring that F is zero on all standard graded components $D^{d-e,e}X$ for $0 \leq e < d$, and $F(1 \otimes x) = x(1_X^{\otimes d})$ for $x \in (\mathtt{Inv}^d\,X)^*$.

Lemma 4.33. *For any $t, u \in D^dX$, we have $(t, u) = F(tu)$.*

Proof. We may assume that $t = \xi \otimes x$ and $u = \eta \otimes y$, where $\xi \in \mathtt{Inv}^{d-e}\,X$, $x \in (\mathtt{Inv}^e\,X)^*$, $\eta \in \mathrm{Ind}^{d-f}X$ and $y \in (\mathtt{Inv}^f\,X)^*$ for some $0 \leq e, f \leq d$. We may further assume that $e = d - f$, for otherwise both sides of the equation in the lemma are zero. Then, using (4.11), we have

$$\begin{aligned} F\big((\xi \otimes x)(\eta \otimes y)\big) &= \sum (-1)^{\bar\xi_{(1)}(\bar\xi_{(2)}+\bar\eta+\bar x)+\bar\eta_{(1)}\bar x} F\big(\xi_{(2)}\eta_{(1)} \otimes (x \cdot \eta_{(2)})(\xi_{(1)} \cdot y)\big) \\ &= (-1)^{\bar\xi(\bar\eta+\bar x)} F\big(1 \otimes (x \cdot \eta)(\xi \cdot y)\big) \end{aligned}$$

$$= (-1)^{\bar{\xi}(\bar{\eta}+\bar{x})}\big((x\cdot\eta)(\xi\cdot y)\big)(1_X^{\otimes d})$$
$$= (-1)^{\bar{\xi}(\bar{\eta}+\bar{x})}\big((x\cdot\eta)(1_X^{\otimes e})\big)\big((\xi\cdot y)(1_X^{\otimes f})\big)$$
$$= (-1)^{\bar{\xi}(\bar{\eta}+\bar{x})}\langle x,\eta\rangle\langle \xi,y\rangle,$$

where we have used (3.16) for the last equality. It remains to note that we can drop the sign since $\langle x,\eta\rangle = 0$ unless $\bar{x}=\bar{\eta}$. □

Note that over an arbitrary $\Bbbk$, non-degeneracy of a bilinear form $(\cdot,\cdot)$ on a free $\Bbbk$-module V of a finite rank means that for every $\Bbbk$-basis $\{v_1,\dots,v_m\}$ of V there is another basis $\{w_1,\dots,w_m\}$ such that $(v_a,w_b)=\delta_{a,b}$. The following corollary shows that D^dX is a *symmetric algebra*.

Corollary 4.34. [**Tu**$_3$, Theorem 1.1] *The form $(\cdot,\cdot)$ on D^dX is non-degenerate, symmetric and associative.*

Proof. The non-degeneracy and symmetricity are clear, while the associativity follows from Lemma 4.33. □

5. Generalized Schur-Weyl duality

Throughout this section, $A = A_{\bar 0}\oplus A_{\bar 1}$ is a $\Bbbk$-superalgebra with $\Bbbk$-bases $\mathtt{B}_{\bar 0}$ of $A_{\bar 0}$, $\mathtt{B}_{\bar 1}$ of $A_{\bar 1}$, and $\mathtt{B}=\mathtt{B}_{\bar 0}\sqcup\mathtt{B}_{\bar 1}$ of A. Fix $d\in\mathbb{Z}_{\geq 0}$ and $n\in\mathbb{Z}_{>0}$.

5.1. **Wreath product superalgebras.** We will consider *super wreath products*

$$W_d^A := A^{\otimes d}\rtimes \Bbbk\mathfrak{S}_d, \tag{5.1}$$

with $\Bbbk\mathfrak{S}_d$ concentrated in degree $\bar 0$. We identify $A^{\otimes d}$ and $\Bbbk\mathfrak{S}_d$ with the subsuperalgebras $A^{\otimes d}\otimes 1_{\mathfrak{S}_d}$ and $1_A^{\otimes d}\otimes\Bbbk\mathfrak{S}_d$ of W_d^A, respectively. The multiplication in W_d^A is then uniquely determined by the additional requirement that

$$g^{-1}(x_1\otimes\cdots\otimes x_d)g = (x_1\otimes\cdots\otimes x_d)^g \tag{5.2}$$

for $g\in\mathfrak{S}_d$ and $x_1,\dots,x_d\in A$, see (3.4). Given $x\in A$ and $1\leq c\leq d$, we denote

$$x[c] := 1_A\otimes\cdots\otimes 1_A\otimes x\otimes 1_A\otimes\cdots\otimes 1_A\in A^{\otimes d},$$

with x in the cth position. The following lemma is obvious:

Lemma 5.3. *Let A be a superalgebra and $d\in\mathbb{Z}_{\geq 0}$. Then the superalgebra W_d^A is generated by the elements $\{x[c]\mid x\in A,\ 1\leq c\leq d\}\sqcup\mathfrak{S}_d$ subject only to the following relations:*

$$\begin{aligned}
x[c]\cdot y[c] &= xy[c] && (x,y\in A,\ 1\leq c\leq d),\\
x[b]\cdot y[c] &= (-1)^{\bar x\bar y}y[c]\cdot x[b] && (x,y\in A,\ 1\leq b\neq c\leq d),\\
g\cdot h &= gh && (g,h\in\mathfrak{S}_d),\\
g\cdot x[c] &= x[gc]\cdot g && (g\in\mathfrak{S}_d,\ x\in A,\ 1\leq c\leq d).
\end{aligned}$$

Let $\lambda\in\Lambda(n,d)$. We always consider the group algebra $\Bbbk\mathfrak{S}_\lambda$ of the standard parabolic subgroup $\mathfrak{S}_\lambda$ as a subalgebra $\Bbbk\mathfrak{S}_\lambda\subseteq\Bbbk\mathfrak{S}_d\subseteq W_d^A$. In particular, $\Bbbk\mathfrak{S}_\lambda$ acts naturally on the left on W_d^A. This makes W_d^A into a left $\Bbbk\mathfrak{S}_\lambda$-module, which is free with basis

$$\{g(\mathtt{b}_1\otimes\cdots\otimes\mathtt{b}_d)\mid g\in{}^\lambda\mathscr{D},\ \mathtt{b}_1,\dots,\mathtt{b}_d\in\mathtt{B}\}.$$

So, denoting by $\mathtt{triv}_\lambda$ the trivial right $\Bbbk\mathfrak{S}_\lambda$-module $\Bbbk\cdot 1_\lambda$, we have the (right) induced W_d^A-module

$$M_\lambda^A := \mathtt{triv}_\lambda \otimes_{\Bbbk\mathfrak{S}_\lambda} W_d^A \tag{5.4}$$

with generator $m_\lambda := 1_\lambda \otimes 1$. We refer to M_λ^A as a *permutation module.*

5.2. **Tensor space.** The matrix algebra $M_n(A)$ is a superalgebra in its own right. We use the elements

$$\xi_{r,s}^x := xE_{r,s} \in M_n(A) \qquad (x\in A,\ 1\leq r,s\leq n) \tag{5.5}$$

as in §3.4. We also introduce the special notation

$$S^A(n,d) := \mathtt{Inv}^d(M_n(A)) \quad \text{and} \quad S^A(n) := \mathtt{Inv}(M_n(A)) = \bigoplus_{d\geq 0} S^A(n,d).$$

If $A=\Bbbk$, the algebra $S^A(n,d)$ is nothing but the classical Schur algebra $S(n,d)$ as in [**Gr**].

Let $V=A^{\oplus n}$, considered as a right A-supermodule in the natural way. Note that we have a natural isomorphism $M_n(A) \xrightarrow{\sim} \mathrm{End}_A(V)$, where we consider V as column vectors and the isomorphism sends a matrix ξ to the left multiplication by ξ. This implies the isomorphism

$$\mathtt{Tens}^d M_n(A) \xrightarrow{\sim} \mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V). \tag{5.6}$$

Recall from (3.4) that $\mathfrak{S}_d$ acts on $\mathtt{Tens}^d V$ with $\Bbbk$-linear maps, and write $vg := v^g$ for $v\in V$, $g\in\mathfrak{S}_d$. Thus we have right supermodule structures on $\mathtt{Tens}^d V$ over both $k\mathfrak{S}_d$ and $\mathtt{Tens}^d A$. In view of Lemma 5.3, the superspace $\mathtt{Tens}^d V$ becomes a right W_d^A-supermodule. We refer to this right action of W_d^A on $\mathtt{Tens}^d V$ as the *standard permutation action.*

Lemma 5.7. *The natural embedding*

$$S^A(n,d) \hookrightarrow \mathtt{Tens}^d M_n(A) \xrightarrow{\sim} \mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V)$$

defines an isomorphism of superalgebras

$$S^A(n,d) \cong \mathrm{End}_{W_d^A}(\mathtt{Tens}^d V).$$

Proof. The action of $\mathfrak{S}_d$ on $\mathtt{Tens}^d V$ yields the action on $\mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V)$ via $(\varphi\cdot g)(v) = \varphi(vg^{-1})g$ for $\varphi\in \mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V)$, $g\in\mathfrak{S}_d$ and $v\in\mathtt{Tens}^d V$.

Let $\alpha\colon \mathtt{Tens}^d M_n(A) \xrightarrow{\sim} \mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V)$ be the isomorphism (5.6). We have the $\mathfrak{S}_d$-action on $\mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V)$ defined in the previous paragraph, and the $\mathfrak{S}_d$-action on $\mathtt{Tens}^d M_n(A)$ defined by (3.4). It is easy to see that α intertwines the two actions. Taking invariants, we get an isomorphism between $S^A(n,d) = (\mathtt{Tens}^d M_n(A))^{\mathfrak{S}_d}$ and $\mathrm{End}_{W_d^A}(\mathtt{Tens}^d V) = (\mathrm{End}_{\mathtt{Tens}^d A}(\mathtt{Tens}^d V))^{\mathfrak{S}_d}$. □

For $1\leq r\leq n$, we set

$$v_r := (0,\dots,0,1_A,0,\dots,0)\in V, \tag{5.8}$$

where 1_A is in the rth position. For $\boldsymbol{r}=(r_1,\dots,r_d)\in\mathtt{Seq}(n,d)$, we define

$$v_{\boldsymbol{r}} := v_{r_1}\otimes\cdots\otimes v_{r_d} \in \mathtt{Tens}^d V.$$

Since $\{v_1, \dots, v_n\}$ is an A-basis of V, the set $\{v_{\boldsymbol{r}} \mid \boldsymbol{r} \in \mathtt{Seq}(n,d)\}$ is a $\mathtt{Tens}^d A$-basis of $\mathtt{Tens}^d V$. Note that

$$v_{\boldsymbol{r}} g = v_{\boldsymbol{r}g} \qquad (g \in \mathfrak{S}_d,\ \boldsymbol{r} \in \mathtt{Seq}(n,d)). \tag{5.9}$$

Let $\lambda \in \Lambda(n,d)$. We denote by $\mathtt{Tens}^\lambda V$ the $\mathtt{Tens}^d A$-span of all $v_{\boldsymbol{r}}$ such that $\boldsymbol{r} \in {}^\lambda\mathtt{Seq}$, cf. (2.1). By (5.9), $\mathtt{Tens}^\lambda V$ is a W_d^A-submodule of $\mathtt{Tens}^d V$. We have a special vector

$$v_\lambda := v_1^{\otimes \lambda_1} \otimes \dots \otimes v_n^{\otimes \lambda_n} \in \mathtt{Tens}^\lambda V.$$

We have the decomposition of W_d^A-modules:

$$\mathtt{Tens}^d V = \bigoplus_{\lambda \in \Lambda(n,d)} \mathtt{Tens}^\lambda V. \tag{5.10}$$

Lemma 5.11. *Let $\lambda \in \Lambda(n,d)$. There is an isomorphism of right W_d^A-modules $\mathtt{Tens}^\lambda V \xrightarrow{\sim} M_\lambda^A$ which maps v_λ to the standard generator m_λ of M_λ^A.*

Proof. It is immediate that v_λ is $\mathfrak{S}_\lambda$-invariant, which yields a homomorphism $M_\lambda^A \to \mathtt{Tens}^\lambda V,\ m_\lambda \mapsto v_\lambda$. This is an isomorphism, since it maps the $\mathtt{Tens}^d A$-basis $\{m_\lambda g \mid g \in {}^\lambda\mathscr{D}\}$ of M_λ^A to the $\mathtt{Tens}^d A$-basis $\{v_{\boldsymbol{r}} \mid \boldsymbol{r} \in {}^\lambda\mathtt{Seq}\}$ of $\mathtt{Tens}^\lambda V$, cf. the bijection (2.6). ☐

For any $\lambda \in \Lambda(n,d)$, we define

$$\xi_\lambda := E_{1,1}^{\otimes \lambda_1} * \dots * E_{n,n}^{\otimes \lambda_n} \in S^A(n,d). \tag{5.12}$$

Lemma 5.13. *Let $\lambda, \mu \in \Lambda(n,d)$. Then:*

(i) $\xi_\lambda \xi_\mu = \delta_{\lambda,\mu}\xi_\lambda$ *and* $\sum_{\nu \in \Lambda(n,d)} \xi_\nu = 1$.
(ii) $\xi_\lambda \mathtt{Tens}^d V = \mathtt{Tens}^\lambda V$.

Proof. Note that $\xi_\lambda v_\mu = \delta_{\lambda,\mu} v_\lambda$. But v_λ generates $\mathtt{Tens}^\lambda V$ as a right W_d^A-module by Lemma 5.11, and the action of $S^A(n,d)$ on $\mathtt{Tens}^d V$ commutes with that of W_d^A by Lemma 5.7, so ξ_λ acts as the projection onto $\mathtt{Tens}^\lambda V$ along $\bigoplus_{\nu \neq \lambda} \mathtt{Tens}^\nu V$. The lemma follows since $S^A(n,d)$ acts on $\mathtt{Tens}^d V$ faithfully thanks to Lemma 5.7. ☐

5.3. **Idempotent truncation.** Throughout the subsection we assume that $d \leq n$ and set

$$\omega := \varepsilon_1 + \dots + \varepsilon_d \in \Lambda(n,d). \tag{5.14}$$

The main goal of this subsection is to explicitly identify $\xi_\omega S^A(n,d)\xi_\omega$ with W_d^A and $S^A(n,d)\xi_\omega$ with $\mathtt{Tens}^d V$ so that the natural right action of $\xi_\omega S^A(n,d)\xi_\omega$ on $S^A(n,d)\xi_\omega$ becomes the standard permutation action of W_d^A on $\mathtt{Tens}^d V$, cf. [**Gr**, Chapter 6] for the case when $A = \Bbbk$.

Lemma 5.15. *There is a superalgebra isomorphism*

$$\varphi\colon W_d^A \xrightarrow{\sim} \xi_\omega S^A(n,d)\xi_\omega,\ (x_1 \otimes \dots \otimes x_d)g \mapsto \xi_{1,g^{-1}1}^{x_1} * \dots * \xi_{d,g^{-1}d}^{x_d}.$$

Moreover, for any $w \in W_d^A$, its image $\varphi(w)$ is the unique element of $\xi_\omega S^A(n,d)\xi_\omega$ such that $\varphi(w)v_\omega = v_\omega w$.

Proof. Using Lemma 5.7, we have an isomorphism of superalgebras

$$\alpha\colon \xi_\omega S^A(n,d)\xi_\omega \xrightarrow{\sim} \mathrm{End}_{W_d^A}(\xi_\omega \,\mathtt{Tens}^d V)$$

which maps $s \in \xi_\omega S^A(n,d)\xi_\omega$ to the left multiplication by s. On the other hand, by Lemma 5.11, there is an isomorphism of right W_d^A-supermodules $\xi_\omega \,\mathtt{Tens}^d V = \mathtt{Tens}^\omega V \xrightarrow{\sim} M_\omega^A$, $v_\omega \mapsto m_\omega$. But the W_d^A-module M_ω^A is free of rank 1 with generator m_ω. So there is an isomorphism $\beta\colon W_d^A \xrightarrow{\sim} \mathrm{End}_{W_d^A}(\xi_\omega \,\mathtt{Tens}^d V)$ of superalgebras which sends $w \in W_d^A$ to the endomorphism $v_\omega \mapsto v_\omega w$.

Generalizing the notation (5.8), we set

$$v_r^x := (0,\dots,0,x,0,\dots,0) \in V \qquad (1 \le r \le n,\ x \in A), \tag{5.16}$$

where x is in the rth position. Then

$$\begin{aligned}
\alpha(\xi^{x_1}_{1,g^{-1}1} * \dots * \xi^{x_d}_{d,g^{-1}d})(v_\omega) &= (\xi^{x_1}_{1,g^{-1}1} * \dots * \xi^{x_d}_{d,g^{-1}d})(v_1 \otimes \dots \otimes v_d)\\
&= (-1)^{[g;x_1,\dots,x_d]} v_{g1}^{x_{g1}} \otimes \dots \otimes v_{gd}^{x_{gd}}\\
&= (v_1^{x_1} \otimes \dots \otimes v_d^{x_d})g\\
&= v_\omega(x_1 \otimes \dots \otimes x_d)g\\
&= \beta((x_1 \otimes \dots \otimes x_d)g)(v_\omega).
\end{aligned}$$

This proves the lemma. □

Note that $S^A(n,d)\xi_\omega$ is a right $\xi_\omega S^A(n,d)\xi_\omega$-module, so we consider it as a right W_d^A-module via the identification of W_d^A with $\xi_\omega S^A(n,d)\xi_\omega$ coming from the isomorphism φ of Lemma 5.15.

Proposition 5.17. *There is a unique isomorphism $S^A(n,d)\xi_\omega \xrightarrow{\sim} \mathtt{Tens}^d V$ of $(S^A(n,d), W_d^A)$-superbimodules which maps ξ_ω to v_ω.*

Proof. Since $\xi_\omega v_\omega = v_\omega$, there is a unique homomorphism ψ of left $S^A(n,d)$-modules $S^A(n,d)\xi_\omega \to \mathtt{Tens}^d V$ mapping ξ_ω to v_ω. Using Lemma 5.15, we compute for any $s \in S^A(n,d)$ and any $w \in W_d^A$:

$$\begin{aligned}
\psi((s\xi_\omega)w) &= \psi((s\xi_\omega)\varphi(w)) = \psi(s\xi_\omega\varphi(w)) = \psi(s\varphi(w)\xi_\omega)\\
&= s\varphi(w)v_\omega = sv_\omega w = s\xi_\omega v_\omega w = \psi(s\xi_\omega)w,
\end{aligned}$$

so ψ is a homomorphism of $(S^A(n,d), W_d^A)$-superbimodules.

Moreover, ψ is injective since $\psi(s\xi_\omega) = 0$ only if $s\xi_\omega v_\omega = 0$, which implies that $s\xi_\omega \,\mathtt{Tens}^\omega V = 0$ because $v_\omega W_d^A = \mathtt{Tens}^\omega V$. On the other hand, by Lemma 5.13(ii), we have $s\xi_\omega \,\mathtt{Tens}^\mu V = 0$ for all $\mu \neq \omega$, hence $s\xi_\omega \,\mathtt{Tens}^d V = 0$, so $s\xi_\omega = 0$.

Finally, for every $\mu \in \Lambda(n,d)$ there is a homomorphism of right W_d^A-modules $M_\omega^A \to M_\mu^A$, $m_\omega \mapsto m_\mu$, and so there is a homomorphism of right W_d^A-modules $\mathtt{Tens}^\omega V \to \mathtt{Tens}^\mu V$, $v_\omega \mapsto v_\mu$, see Lemma 5.11. By Lemma 5.7, there is $s \in S^A(n,d)$ with $sv_\omega = v_\mu$. As v_μ generates $\mathtt{Tens}^\mu V$ as a W_d^A-module for every $\mu \in \Lambda(n,d)$, we now deduce that v_ω generates $\mathtt{Tens}^d V$ as an $(S^A(n,d), W_d^A)$-bimodule. Hence ψ is surjective. □

Denote the center of an algebra Y by $Z(Y)$. Recall from Section 3 the notation $|X|$ for a superalgebra X. The following technical result, in which we forget the superstructures, will be needed in §6.2:

Lemma 5.18. *Let $d \le n$. If $z \in Z(|S^A(n,d)|)$ and $\xi_\omega \in |S^A(n,d)|z$, then z is invertible.*

Proof. Let $S := |S^A(n,d)|$ and $W := |W_d^A|$. First, note that $\xi_\lambda z \xi_\mu = z\xi_\lambda\xi_\mu = 0$ for any distinct $\lambda, \mu \in \Lambda(n,d)$. So $z = \sum_{\lambda\in\Lambda(n,d)} z_\lambda$, where $z_\lambda := z\xi_\lambda = \xi_\lambda z$.

Let $\lambda \in \Lambda(n,d)$. There is a unique W-module homomorphism sending m_ω to m_λ, so by Lemmas 5.7 and 5.11, there exists a unique element $\xi_{\lambda,\omega} \in \xi_\lambda S \xi_\omega$ such that $\xi_{\lambda,\omega} v_\omega = v_\lambda$.

By the hypothesis, there exists $y_\omega \in S$ such that $y_\omega z = \xi_\omega$. Replacing y_ω with $\xi_\omega y_\omega \xi_\omega$, we may (and do) assume that $y_\omega \in \xi_\omega S \xi_\omega$, and then it is easy to see that $y_\omega \in Z(\xi_\omega S \xi_\omega)$. Let $\tilde{y}_\omega \in W$ be the image of y_ω under the isomorphism $\xi_\omega S \xi_\omega \xrightarrow{\sim} W$ of Lemma 5.15. Then $\tilde{y}_\omega \in Z(W)$ and $y_\omega v_\omega = v_\omega \tilde{y}_\omega$. For any $g \in \mathfrak{S}_\lambda$, we have $m_\lambda \tilde{y}_\omega g = m_\lambda g \tilde{y}_\omega = m_\lambda \tilde{y}_\omega$. Hence, there is a right W-module endomorphism of M_λ sending m_λ to $m_\lambda \tilde{y}_\omega$. By Lemmas 5.7 and 5.11, this implies that there exists $y_\lambda \in \xi_\lambda S \xi_\lambda$ such that $y_\lambda v_\lambda = v_\lambda \tilde{y}_\omega$. Therefore,

$$\begin{aligned} z_\lambda y_\lambda v_\lambda &= z_\lambda v_\lambda \tilde{y}_\omega = z_\lambda \xi_{\lambda,\omega} v_\omega \tilde{y}_\omega = z_\lambda \xi_{\lambda,\omega} y_\omega v_\omega = z \xi_{\lambda,\omega} y_\omega v_\omega = \xi_{\lambda,\omega} z y_\omega v_\omega \\ &= \xi_{\lambda,\omega}\xi_\omega v_\omega = \xi_{\lambda,\omega} v_\omega = v_\lambda. \end{aligned}$$

By Lemma 5.7, it follows that $z_\lambda y_\lambda = \xi_\lambda$. Setting $y := \sum_{\lambda\in\Lambda(n,d)} y_\lambda$, we have $zy = 1$. ☐

5.4. **Idempotent refinements.** In this subsection we suppose that we are given a fixed finite family $\{e_1, \dots, e_l\}$ of non-zero orthogonal idempotents in A with $\sum_{i=1}^l e_i = 1_A$. Moreover, we assume that every $e_i A$ is free as a $\Bbbk$-supermodule with a (homogeneous) finite basis ${}_i\mathtt{B}$, so that $\mathtt{B} = \bigsqcup_{i=1}^l {}_i\mathtt{B}$ is a $\Bbbk$-basis of A.

Set $I := \{1, \dots, l\}$. We order $[1,n] \times I$ lexicographically and, recalling the theory of §2.3, consider the set of compositions $\Lambda([1,n]\times I, d)$. Given $\boldsymbol{\lambda} \in \Lambda([1,n]\times I, d)$, we denote $\lambda_r^{(i)} := \boldsymbol{\lambda}_{(r,i)}$ for $(r,i) \in [1,n] \times I$. We have the map

$$\pi\colon \Lambda([1,n] \times I, d) \to \Lambda(n,d),\ \boldsymbol{\lambda} \mapsto (\textstyle\sum_{i\in I} \lambda_1^{(i)}, \sum_{i\in I} \lambda_2^{(i)}, \dots, \sum_{i\in I} \lambda_n^{(i)}).$$

Let $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$. We have the idempotent

$$e_{\boldsymbol{\lambda}}^A := e_1^{\otimes\lambda_1^{(1)}} \otimes \cdots \otimes e_l^{\otimes\lambda_1^{(l)}} \otimes \cdots \otimes e_1^{\otimes\lambda_n^{(1)}} \otimes \cdots \otimes e_l^{\otimes\lambda_n^{(l)}} \in \mathtt{Tens}^d A. \tag{5.19}$$

Recalling the notation of §2.3, note that

$$\mathtt{B}_{\boldsymbol{\lambda}}^A := \{\mathtt{b}_1 \otimes \cdots \otimes \mathtt{b}_d \mid \mathtt{b}_a \in {}_i\mathtt{B} \text{ if } a \in \Omega_{(r,i)}^{\boldsymbol{\lambda}}\} \tag{5.20}$$

is a $\Bbbk$-basis of $e_{\boldsymbol{\lambda}}^A \mathtt{Tens}^d A$. We define the *parabolic subalgebra*

$$W_{\boldsymbol{\lambda}}^A := e_{\boldsymbol{\lambda}}^A \otimes \Bbbk\mathfrak{S}_{\boldsymbol{\lambda}} \subseteq W_d^A.$$

Note that $e_{\boldsymbol{\lambda}}^A$ is the identity in $W_{\boldsymbol{\lambda}}^A$, and $W_{\boldsymbol{\lambda}}^A$ is a (usually *non-unital*) subsuperalgebra in W_d^A, isomorphic to the group algebra $\Bbbk\mathfrak{S}_{\boldsymbol{\lambda}}$. So we may consider the *trivial* right supermodule $\mathtt{triv}_{\boldsymbol{\lambda}}^A = \Bbbk \cdot 1_{\boldsymbol{\lambda}}^A$ over $W_{\boldsymbol{\lambda}}^A$ with the action on the basis element $1_{\boldsymbol{\lambda}}^A$ given by $1_{\boldsymbol{\lambda}}^A \cdot (e_{\boldsymbol{\lambda}}^A \otimes g) = 1_{\boldsymbol{\lambda}}^A$ for any $g \in \mathfrak{S}_{\boldsymbol{\lambda}}$.

As usual, we view $\mathtt{Tens}^d A$ and $\Bbbk\mathfrak{S}_d$ as subsuperalgebras of W_d^A, so we can also view $e_{\boldsymbol{\lambda}}^A$ as an element of W_d^A. Then $e_{\boldsymbol{\lambda}}^A W_d^A$ is naturally a left $W_{\boldsymbol{\lambda}}^A$-module. We now define the *colored permutation supermodule*

$$M_{\boldsymbol{\lambda}}^A := \mathtt{triv}_{\boldsymbol{\lambda}}^A \otimes_{W_{\boldsymbol{\lambda}}^A} e_{\boldsymbol{\lambda}}^A W_d^A$$

with generator $m^A_{\boldsymbol{\lambda}} := 1^A_{\boldsymbol{\lambda}} \otimes e^A_{\boldsymbol{\lambda}}$.

Lemma 5.21. *The following set is a $\Bbbk$-basis of $M^A_{\boldsymbol{\lambda}}$:*

$$\{m^A_{\boldsymbol{\lambda}}(\mathtt{b}_1 \otimes \cdots \otimes \mathtt{b}_d)g \mid \mathtt{b}_1 \otimes \cdots \otimes \mathtt{b}_d \in \mathtt{B}^A_{\boldsymbol{\lambda}},\ g \in {}^{\boldsymbol{\lambda}}\mathscr{D}\}. \tag{5.22}$$

In particular, $\dim M^A_{\boldsymbol{\lambda}} = |\mathfrak{S}_d : \mathfrak{S}_{\boldsymbol{\lambda}}| \prod_{i\in I}(\dim e_iA)^{\sum_{r=1}^n \lambda^{(i)}_r}$.

Proof. Note that

$$e^A_{\boldsymbol{\lambda}} W^A_d = (e_1A)^{\otimes\lambda^{(1)}_1} \otimes \cdots \otimes (e_lA)^{\otimes\lambda^{(l)}_1} \otimes \cdots \otimes (e_1A)^{\otimes\lambda^{(1)}_n} \otimes \cdots \otimes (e_lA)^{\otimes\lambda^{(l)}_n} \otimes \Bbbk\mathfrak{S}_d,$$

and so

$$\{e^A_{\boldsymbol{\lambda}}(\mathtt{b}_1 \otimes \cdots \otimes \mathtt{b}_d)g \mid \mathtt{b}_1 \otimes \cdots \otimes \mathtt{b}_d \in \mathtt{B}^A_{\boldsymbol{\lambda}},\ g \in {}^{\boldsymbol{\lambda}}\mathscr{D}\}.$$

is a basis of $e^A_{\boldsymbol{\lambda}} W^A_d$ as a left $W^A_{\boldsymbol{\lambda}}$-module. The lemma follows. □

Recalling (5.16), we define

$$v_{r,i} := v^{e_i}_r = (0, \dots, 0, e_i, 0, \dots, 0) \in V \qquad (1 \le r \le n,\ i \in I),$$

where e_i is in the rth position. For $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$, we denote by $\mathtt{Tens}^{\boldsymbol{\lambda}} V \subseteq \mathtt{Tens}^d V$ the (right) $\mathtt{Tens}^d A$-span of all $v_{r_1,i_1} \otimes \cdots \otimes v_{r_d,i_d}$ such that for every $(r,i) \in [1,n] \times I$ we have $\sharp\{a \in [1,d] \mid (r_a, i_a) = (r,i)\} = \lambda^{(i)}_r$. We say that a sequence $((r_1, \mathtt{b}_1), \dots, (r_d, \mathtt{b}_d))$ of elements of $[1,n] \times \mathtt{B}$ is *of type* $\boldsymbol{\lambda}$ if $\sharp\{a \in [1,d] \mid r_a = r \text{ and } \mathtt{b}_a \in {}_i\mathtt{B})\} = \lambda^{(i)}_r$. It is easy to see that

$$\{v^{\mathtt{b}_1}_{r_1} \otimes \cdots \otimes v^{\mathtt{b}_d}_{r_d} \mid ((r_1, \mathtt{b}_1), \dots, (r_d, \mathtt{b}_d)) \text{ is of type } \boldsymbol{\lambda}\} \tag{5.23}$$

is a $\Bbbk$-basis of $\mathtt{Tens}^{\boldsymbol{\lambda}} V$. Hence for any $\lambda \in \Lambda(n,d)$, we have a decomposition

$$\mathtt{Tens}^{\lambda} V = \bigoplus_{\boldsymbol{\lambda} \in \pi^{-1}(\lambda)} \mathtt{Tens}^{\boldsymbol{\lambda}} V \tag{5.24}$$

of $\Bbbk$-modules. We have a special vector

$$v_{\boldsymbol{\lambda}} := v^{\otimes\lambda^{(1)}_1}_{1,1} \otimes \cdots \otimes v^{\otimes\lambda^{(l)}_1}_{1,l} \otimes \cdots \otimes v^{\otimes\lambda^{(1)}_n}_{n,1} \otimes \cdots \otimes v^{\otimes\lambda^{(l)}_n}_{n,l} \in \mathtt{Tens}^{\boldsymbol{\lambda}} V.$$

Lemma 5.25. *We have:*

(i) *For any $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$, we have that $\mathtt{Tens}^{\boldsymbol{\lambda}} V$ is a submodule of the right W^A_d-module $\mathtt{Tens}^d V$. Moreover, there is an isomorphism of right W^A_d-modules $\mathtt{Tens}^{\boldsymbol{\lambda}} V \xrightarrow{\sim} M^A_{\boldsymbol{\lambda}}$ which maps $v_{\boldsymbol{\lambda}}$ to $m^A_{\boldsymbol{\lambda}}$.*

(ii) *For any $\lambda \in \Lambda(n,d)$, we have $\mathtt{Tens}^{\lambda} V = \bigoplus_{\boldsymbol{\lambda}\in\pi^{-1}(\lambda)} \mathtt{Tens}^{\boldsymbol{\lambda}} V$ as right W^A_d-modules. In particular, $\mathtt{Tens}^d V = \bigoplus_{\boldsymbol{\lambda}\in\Lambda([1,n]\times I,d)} \mathtt{Tens}^{\boldsymbol{\lambda}} V$ and $M^A_\lambda \cong \bigoplus_{\boldsymbol{\lambda}\in\pi^{-1}(\lambda)} M^A_{\boldsymbol{\lambda}}$ as right W^A_d-modules.*

Proof. Note that $v_{\boldsymbol{\lambda}} e^A_{\boldsymbol{\lambda}} = v_{\boldsymbol{\lambda}}$ and $v_{\boldsymbol{\lambda}} g = v_{\boldsymbol{\lambda}}$ for any $g \in \mathfrak{S}_{\boldsymbol{\lambda}}$. So, by the adjointness of induction and restriction, there exists a homomorphism of right W^A_d-modules $M^A_{\boldsymbol{\lambda}} \to \mathtt{Tens}^d V$ under which $m^A_{\boldsymbol{\lambda}}$ is mapped to $v_{\boldsymbol{\lambda}}$. It maps the elements of the $\Bbbk$-basis (5.22) of $M^A_{\boldsymbol{\lambda}}$ to the elements of the $\Bbbk$-basis (5.23) of $\mathtt{Tens}^{\boldsymbol{\lambda}} V$ up to signs. This proves (i). Part (ii) follows from (i), (5.24), (5.10) and Lemma 5.11. □

By Lemma 5.7, the superalgebra $S^A(n,d)$ acts naturally on $\mathtt{Tens}^d V$ with W_d^A-homomorphisms. But by Lemma 5.25, we have an explicit identification of right W_d^A-modules

$$\mathtt{Tens}^d V = \bigoplus_{\boldsymbol{\lambda}\in\Lambda([1,n]\times I,d)} \mathtt{Tens}^{\boldsymbol{\lambda}} V = \bigoplus_{\boldsymbol{\lambda}\in\Lambda([1,n]\times I,d)} M_{\boldsymbol{\lambda}}^A.$$

So, for any $y \in S^A(n,d)$, the endomorphism $v \mapsto yv$ of $\mathtt{Tens}^d V$ becomes identified with an endomorphism which we denote by $\varphi(y)$ of $\bigoplus_{\boldsymbol{\lambda}\in\Lambda([1,n]\times I,d)} M_{\boldsymbol{\lambda}}^A$. Recalling Lemma 5.7 again, we deduce:

Corollary 5.26. *Let $M^A(n,d) := \bigoplus_{\boldsymbol{\lambda}\in\Lambda([1,n]\times I,d)} M_{\boldsymbol{\lambda}}^A$. Then $\varphi\colon S^A(n,d) \to \mathrm{End}_{W_d^A}(M^A(n,d))$ is a superalgebra isomorphism.*

5.5. **Desuperization.** Recall from Section 3 that $|X|$ denotes the algebra obtained from a $\Bbbk$-superalgebra X by forgetting the $\mathbb{Z}_2$-grading. In particular, we have the associative algebra $|A|$ and the usual wreath product $W_d^{|A|}$, where the symmetric group acts on $|A|^{\otimes d}$ by place permutations without signs. On the other hand, we can consider the associative algebra $|W_d^A|$. In general, the algebras $W_d^{|A|}$and $|W_d^A|$ are not isomorphic. However, we describe one important situation when they are.

Let $e^{\bar 0}$ and $e^{\bar 1}$ be orthogonal idempotents in A with $1 := 1_A = e^{\bar 0} + e^{\bar 1}$. We call such a pair of idempotents *adapted* if $A_{\bar 0} = e^{\bar 0}Ae^{\bar 0} \oplus e^{\bar 1}Ae^{\bar 1}$ and $A_{\bar 1} = e^{\bar 0}Ae^{\bar 1} \oplus e^{\bar 1}Ae^{\bar 0}$. Let $1 \le r < d$. We denote the elementary transposition $(r, r+1) \in \mathfrak{S}_d$ by τ_r. If in addition $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_2$, we set

$$e^{\varepsilon_1,\varepsilon_2}[r] := e^{\varepsilon_1}[r]e^{\varepsilon_2}[r+1] = 1^{\otimes r-1} \otimes e^{\varepsilon_1} \otimes e^{\varepsilon_2} \otimes 1^{\otimes d-r-1} \in A^{\otimes d}.$$

Lemma 5.27. *Let $(e^{\bar 0}, e^{\bar 1})$ be an adapted pair of idempotents in A. Then there is an isomorphism of associative $\Bbbk$-algebras*

$$\begin{aligned} \sigma\colon W_d^{|A|} &\xrightarrow{\sim} |W_d^A|,\\ x[t] &\mapsto \sum_{\varepsilon_1,\dots,\varepsilon_{t-1}\in\mathbb{Z}_2} (-1)^{(\varepsilon_1+\dots+\varepsilon_{t-1})\bar x} e^{\varepsilon_1} \otimes \dots \otimes e^{\varepsilon_{t-1}} \otimes x \otimes 1^{\otimes d-t},\\ \tau_r &\mapsto \tau_r(e^{\bar 0,\bar 0}[r] + e^{\bar 0,\bar 1}[r] + e^{\bar 1,\bar 0}[r] - e^{\bar 1,\bar 1}[r]). \end{aligned}$$

Proof. It is straightforward to check for all admissible r, t, x, y that the elements $\sigma(\tau_1), \dots, \sigma(\tau_{d-1})$ satisfy the Coxeter relations, that $\sigma(x[t])\sigma(y[t]) = \sigma(xy[t])$, and that $\sigma(\tau_r)\sigma(x[t]) = \sigma(x[t])\sigma(\tau_r)$ if $t \ne r, r+1$.

Let $1 \le s < t \le d$. Then $\sigma(x[t])\sigma(y[s])$ equals

$$\sum_{\varepsilon_1,\dots,\varepsilon_{t-1}\in\mathbb{Z}_2} (-1)^p e^{\varepsilon_1} \otimes \dots \otimes e^{\varepsilon_{s-1}} \otimes e^{\varepsilon_s} y \otimes e^{\varepsilon_{s-1}} \otimes \dots \otimes e^{\varepsilon_{t-1}} \otimes x \otimes 1^{\otimes d-t},$$

where

$$p = (\varepsilon_1 + \dots + \varepsilon_{t-1})\bar x + (\varepsilon_1 + \dots + \varepsilon_{s-1})\bar y + \bar x \bar y,$$

and $\sigma(y[s])\sigma(x[t])$ equals

$$\sum_{\varepsilon_1,\dots,\varepsilon_{t-1}\in\mathbb{Z}_2} (-1)^q e^{\varepsilon_1} \otimes \dots \otimes e^{\varepsilon_{s-1}} \otimes y e^{\varepsilon_s} \otimes e^{\varepsilon_{s-1}} \otimes \dots \otimes e^{\varepsilon_{t-1}} \otimes x \otimes 1^{\otimes d-t},$$

where

$$q = (\varepsilon_1 + \cdots + \varepsilon_{t-1})\bar{x} + (\varepsilon_1 + \cdots + \varepsilon_{s-1})\bar{y}.$$

Considering the $e^{\varepsilon} y e^{\varepsilon'}$ components in the sth tensor position for all $\varepsilon, \varepsilon' \in \mathbb{Z}_2$ in the expressions above, and taking into account that $e^{\varepsilon} y e^{\varepsilon'} = 0$ unless $\bar{y} = \varepsilon + \varepsilon'$ since $(e^{\bar{0}}, e^{\bar{1}})$ is adapted, we see that $\sigma(x[t])\sigma(y[s]) = \sigma(y[s])\sigma(x[t])$.

Let $x \in A$ and $1 \leq r < d$. Then, writing

$$u := \sum_{\varepsilon_1, \dots, \varepsilon_{r-1} \in \mathbb{Z}_2} (-1)^{(\varepsilon_1 + \cdots + \varepsilon_{r-1})\bar{x}} e^{\varepsilon_1} \otimes \cdots \otimes e^{\varepsilon_{r-1}}, \quad v := \underbrace{1 \otimes \cdots \otimes 1}_{d-r-1 \text{ times}},$$

we have

$$\begin{aligned}
\sigma(x[r+1])\sigma(\tau_r) = \\
&= (u \otimes (e^{\bar{0}} + (-1)^{\bar{x}} e^{\bar{1}}) \otimes x \otimes v)(e^{\bar{0},\bar{0}}[r] + e^{\bar{0},\bar{1}}[r] + e^{\bar{1},\bar{0}}[r] - e^{\bar{1},\bar{1}}[r])\tau_r \\
&= (u \otimes (e^{\bar{0}} + e^{\bar{1}}) \otimes e^{\bar{0}} x \otimes v + u \otimes (e^{\bar{0}} - e^{\bar{1}}) \otimes e^{\bar{1}} x \otimes v)\tau_r \\
&= \tau_r (u \otimes e^{\bar{0}} x \otimes 1 \otimes v + u \otimes e^{\bar{1}} x \otimes (e^{\bar{0}} - e^{\bar{1}}) \otimes v) \\
&= \sigma(\tau_r)\sigma(x[r]),
\end{aligned}$$

where the second equality is proved by a case-by-case check using the adaptedness of $(e^{\bar{0}}, e^{\bar{1}})$. Since $\sigma(\tau_r)^2 = 1$, it follows also that $\sigma(\tau_r)\sigma(x[r+1]) = \sigma(x[r])\sigma(\tau_r)$.

In view of Lemma 5.3, we have an algebra homomorphism σ as in the statement of the lemma. Moreover, it is easy to see that for each $g \in \mathfrak{S}_d$, the map σ restricts to an automorphism of the $\Bbbk$-submodule $A^{\otimes d} \otimes g$, whence σ is an isomorphism. $\square$

Let again $(e^{\bar{0}}, e^{\bar{1}})$ be an adapted pair of idempotents in A. Assume in addition that we are given two finite families of non-zero orthogonal idempotents $\{e_i \mid i \in I^{\bar{0}}\}$ and $\{e_i \mid i \in I^{\bar{1}}\}$ such that $e^{\bar{0}} = \sum_{i \in I^{\bar{0}}} e_i$ and $e^{\bar{1}} = \sum_{i \in I^{\bar{1}}} e_i$. Let $I = I^{\bar{0}} \sqcup I^{\bar{1}}$ and recall the theory of §5.4. In particular, I is identified with $\{1, \dots, l\}$ for some l and for any $\boldsymbol{\lambda} \in \Lambda([1, n] \times I, d)$, we have the colored permutation supermodule $M_{\boldsymbol{\lambda}}^A$ over the superalgebra W_d^A. Forgetting the $\mathbb{Z}_2$-gradings, we get the $|W_d^A|$-module $|M_{\boldsymbol{\lambda}}^A|$. On the other hand, by Lemma 5.27, there is an isomorphism of algebras $\sigma\colon W_d^{|A|} \xrightarrow{\sim} |W_d^A|$. Composing with this isomorphism, we get the $W_d^{|A|}$-module $|M_{\boldsymbol{\lambda}}^A|^{\sigma}$. In other words, $|M_{\boldsymbol{\lambda}}^A|^{\sigma} = M_{\boldsymbol{\lambda}}^A$ as a $\Bbbk$-module, but the action is defined by $vh = v\sigma(h)$ for all $v \in M_{\boldsymbol{\lambda}}^A$ and $h \in W_d^{|A|}$.

For every $i \in I$ we define the sign ζ_i as follows:

$$\zeta_i := \begin{cases} +1 & \text{if } i \in I^{\bar{0}}, \\ -1 & \text{if } i \in I^{\bar{1}}. \end{cases}$$

Recall the parabolic subgroup

$$\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda_1^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda_1^{(l)}} \times \cdots \times \mathfrak{S}_{\lambda_n^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda_n^{(l)}} \leq \mathfrak{S}_d.$$

Let ℓ be the usual length function on a symmetric group, cf. §2.3. Define the function $\varepsilon_{\boldsymbol{\lambda}}\colon \mathfrak{S}_{\boldsymbol{\lambda}} \to \{\pm 1\} \subseteq \Bbbk$ by

$$\varepsilon_{\boldsymbol{\lambda}}(g_1^{(1)}, \dots, g_1^{(l)}, \dots, g_n^{(1)}, \dots, g_n^{(l)}) := \zeta_1^{\ell(g_1^{(1)})} \cdots \zeta_l^{\ell(g_1^{(l)})} \dots \zeta_1^{\ell(g_n^{(1)})} \cdots \zeta_l^{\ell(g_n^{(l)})} \tag{5.28}$$

for all $(g_1^{(1)}, \dots, g_1^{(l)}, \dots, g_n^{(1)}, \dots, g_n^{(l)}) \in \mathfrak{S}_{\boldsymbol{\lambda}}$.

The algebra $W_d^{|A|}$ has the parabolic subalgebra $W_{\boldsymbol{\lambda}}^{|A|} := e_{\boldsymbol{\lambda}}^{|A|} \otimes \Bbbk\mathfrak{S}_{\boldsymbol{\lambda}} \cong \Bbbk\mathfrak{S}_{\boldsymbol{\lambda}}$ defined by analogy with the parabolic subsuperalgebra $W_{\boldsymbol{\lambda}}^A \subseteq W_d^A$. We define the *alternating* right module $\mathtt{alt}_{\boldsymbol{\lambda}}^{|A|} = \Bbbk \cdot 1_{\boldsymbol{\lambda}}^{|A|}$ over $W_{\boldsymbol{\lambda}}^{|A|}$ with the action on the basis element $1_{\boldsymbol{\lambda}}^{|A|}$ given by

$$1_{\boldsymbol{\lambda}}^{|A|} \cdot (e_{\boldsymbol{\lambda}}^{|A|} \otimes g) = \varepsilon_{\boldsymbol{\lambda}}(g) 1_{\boldsymbol{\lambda}}^{|A|} \qquad (g \in \mathfrak{S}_{\boldsymbol{\lambda}}).$$

As in the superalgebra situation, $e_{\boldsymbol{\lambda}}^{|A|} W_d^{|A|}$ is naturally a left $W_{\boldsymbol{\lambda}}^{|A|}$-module. We now define the *colored permutation module*

$$M_{\boldsymbol{\lambda}}^{|A|} := \mathtt{alt}_{\boldsymbol{\lambda}}^{|A|} \otimes_{W_{\boldsymbol{\lambda}}^{|A|}} e_{\boldsymbol{\lambda}}^{|A|} W_d^{|A|} \tag{5.29}$$

with generator $m_{\boldsymbol{\lambda}}^{|A|} := 1_{\boldsymbol{\lambda}}^{|A|} \otimes e_{\boldsymbol{\lambda}}^{|A|}$.

Proposition 5.30. *There is an isomorphism of right $W_d^{|A|}$-modules*

$$M_{\boldsymbol{\lambda}}^{|A|} \xrightarrow{\sim} |M_{\boldsymbol{\lambda}}^A|^{\sigma},\ m_{\boldsymbol{\lambda}}^{|A|} \mapsto m_{\boldsymbol{\lambda}}^A.$$

Proof. Let τ_r be an elementary transposition which belongs to $\mathfrak{S}_{\boldsymbol{\lambda}}$. This means that $r, r+1 \in \Omega_{(s,i)}^{\boldsymbol{\lambda}}$ for some $(s,i) \in [1,n] \times I$. By Lemma 5.27, we have $\sigma(e_{\boldsymbol{\lambda}}^{|A|} \otimes \tau_r) = \zeta_i(e_{\boldsymbol{\lambda}}^A \otimes \tau_r)$. This implies that $m_{\boldsymbol{\lambda}}^A \sigma(e_{\boldsymbol{\lambda}}^{|A|} \otimes g) = \varepsilon_{\boldsymbol{\lambda}}(g) m_{\boldsymbol{\lambda}}^A$ for all $g \in \mathfrak{S}_{\boldsymbol{\lambda}}$. By adjointness of induction and restriction, we get a homomorphism of $W_d^{|A|}$-modules as in the statement. Since $|M_{\boldsymbol{\lambda}}^A|^{\sigma}$ is generated by $m_{\boldsymbol{\lambda}}^A$, this homomorphism is surjective. Now, since $M_{\boldsymbol{\lambda}}^{|A|}$ and $|M_{\boldsymbol{\lambda}}^A|^{\sigma}$ are free as $\Bbbk$-modules and have the same rank, the result follows. □

Let $y \in S^A(n,d)$. By Lemmas 5.7 and 5.25, for any $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$, we can write

$$y v_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \in \Lambda([1,n] \times I, d)} v_{\boldsymbol{\mu}} h_{\boldsymbol{\mu},\boldsymbol{\lambda}} \tag{5.31}$$

for some $h_{\boldsymbol{\mu},\boldsymbol{\lambda}} \in W_d^A$. If $\varphi\colon S^A(n,d) \xrightarrow{\sim} \mathrm{End}_{W_d^A}(M^A(n,d))$ is the isomorphism of Corollary 5.26, then

$$\varphi(y)(m_{\boldsymbol{\lambda}}^A) = \sum_{\boldsymbol{\mu} \in \Lambda([1,n] \times I, d)} m_{\boldsymbol{\mu}}^A h_{\boldsymbol{\mu},\boldsymbol{\lambda}}.$$

Consider the right $W_d^{|A|}$-module

$$M^{|A|}(n,d) := \bigoplus_{\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)} M_{\boldsymbol{\lambda}}^{|A|}. \tag{5.32}$$

By Lemma 5.27 and Proposition 5.30, there exists $\psi(y) \in \mathrm{End}_{W_d^{|A|}}(M^{|A|}(n,d))$ such that for any $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$,

$$\psi(y)(m_{\boldsymbol{\lambda}}^{|A|}) = \sum_{\boldsymbol{\mu} \in \Lambda([1,n] \times I, d)} m_{\boldsymbol{\mu}}^{|A|} \sigma^{-1}(h_{\boldsymbol{\mu},\boldsymbol{\lambda}}). \tag{5.33}$$

Corollary 5.34. *The map $\psi\colon |S^A(n,d)| \to \mathrm{End}_{W_d^{|A|}}(M^{|A|}(n,d))$ is an algebra isomorphism.*

Remark 5.35. If the superalgebra A is graded, the theory of this section goes through, yielding gradings on $S^A(n,d)$ and W_d^A, as well as all the modules over them that were considered. To be more precise, $M_n(A)$ inherits a grading from A, and then so do $\mathtt{Tens}\, M_n(A)$ and $\mathtt{Inv}\, M_n(A)$. On the other hand, $W_d^A = A^{\otimes d} \otimes \Bbbk\mathfrak{S}_d$ is graded with $\Bbbk\mathfrak{S}_d$ in degree 0. In particular, the isomorphism ψ from Corollary 5.34 is an isomorphism of graded algebras.

6. Schur doubles

Let $d \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. For an $\mathcal{O}$-superalgebra A which is free of finite rank as an $\mathcal{O}$-supermodule, we denote

$$'D^A(n,d) := {'D}^d M_n(A) \quad \text{and} \quad D^A(n,d) := D^d M_n(A).$$

The main result of this section is Theorem 6.6, which roughly speaking asserts that $D^A(n,d)$ is a maximal symmetric subalgebra of $'D^A(n,d)$. But first, we develop the results of §4.4 on generation in this set-up.

6.1. **Generating $D^A(n,d)$.** Let $T_A = A \oplus A^*$ be the trivial extension superalgebra of A, cf. §3.4. In view of Lemma 3.21, we identify $M_n(T_A)$ with $T_{M_n(A)}$ so that $\xi_{r,s}^{(a,\alpha)} \in M_n(T_A)$ corresponds to $(\xi_{r,s}^a, x_{s,r}^\alpha) \in T_{M_n(A)}$ for all $1 \leq r,s \leq n$, $a \in A$ and $\alpha \in A^*$, where $x_{r,s}^\alpha \in M_n(A)^*$ is the element defined in (3.20).

We also identify $'D^A(n,d) = {'D}^d M_n(A)$ with $\mathtt{Inv}^d\, T_{M_n(A)}$ via the explicit isomorphism of Theorem 4.26. Combining this with the identification $T_{M_n(A)} = M_n(T_A)$ from the previous paragraph, we now can and do identify $'D^A(n,d)$ with $\mathtt{Inv}^d\, M_n(T_A) = S^{T_A}(n,d)$. Since $D^A(n,d)$ is a subsuperalgebra of $'D^A(n,d)$, we now identify it as a subsuperalgebra of $S^{T_A}(n,d)$. As $A_{\bar 0}$ is a subalgebra of T_A, the algebra $S^{A_{\bar 0}}(n,d) = \mathtt{Inv}^d\, M_n(A_{\bar 0})$ is also a subalgebra of the superalgebra $S^{T_A}(n,d) = \mathtt{Inv}^d\, M_n(T_A)$ in the natural way.

Theorem 6.1. *The subsuperalgebra $D^A(n,d) \subseteq S^{T_A}(n,d)$ is precisely the subalgebra generated by $S^{A_{\bar 0}}(n,d)$ and the set $\{\xi_{1,1}^y * 1^{\otimes(d-1)} \mid y \in T_A\} \subseteq S^{T_A}(n,d)$.*

Proof. This follows from Corollary 4.31. Indeed, we consider that corollary with $M_n(A)$ in place of X. Then, taking into account the identifications made in this subsection, $\mathtt{Inv}^d(A_{\bar 0})$ in Corollary 4.31 corresponds to $S^{A_{\bar 0}}(n,d)$, and Y in Corollary 4.31 corresponds to $M_n(A_{\bar 1} \oplus A^*)$. It remains to take $U := \{\xi_{1,1}^y \mid y \in A_{\bar 1} \oplus A^*\}$, which is easily seen to satisfy the assumptions of Corollary 4.31. $\square$

Corollary 6.2. *Let A and A' be $\mathcal{O}$-superalgebras which are free of finite rank as $\mathcal{O}$-supermodules. If we have an isomorphism $\varphi\colon T_A \xrightarrow{\sim} T_{A'}$ which restricts to an isomorphism $A_{\bar 0} \xrightarrow{\sim} A'_{\bar 0}$, then the isomorphism $S^{T_A}(n,d) \xrightarrow{\sim} S^{T_{A'}}(n,d)$ induced by φ restricts to an isomorphism $D^A(n,d) \xrightarrow{\sim} D^{A'}(n,d)$.*

If $A_{\bar 1}A_{\bar 1} = 0$, then $M_n(A)_{\bar 1}M_n(A)_{\bar 1} = 0$, and so we have Turner's grading on $D^A(n,d)$, cf. Lemma 4.32.

Corollary 6.3. *If $A_{\bar 1}A_{\bar 1} = 0$, then $D^A(n,d)$ is generated by the elements of degrees 0, 1 and 2 with respect to Turner's grading.*

Corollary 6.4. *Let $n \geq d$. Then the subsuperalgebra $D^A(n,d) \subseteq S^{T_A}(n,d)$ is precisely the subalgebra generated by $S^{A_{\bar 0}}(n,d)$ and the set*

$$\{\xi^y_{1,1} * E_{2,2}^{\otimes\lambda_2} * \cdots * E_{n,n}^{\otimes\lambda_n} \mid y \in T_A,\ (\lambda_2,\dots,\lambda_n) \in \Lambda(n-1,d-1)\} \subseteq S^{T_A}(n,d).$$

Proof. Let D be the subalgebra generated by the elements in the statement of the corollary. Let $\lambda = (1,\lambda_2,\dots,\lambda_n) \in \Lambda(n,d)$. Recalling the idempotent $\xi_\lambda = E_{1,1} * E_{2,2}^{\otimes\lambda_2} * \cdots * E_{n,n}^{\otimes\lambda_n}$ from (5.12) and using Lemma 4.3, we have

$$\xi_\lambda(\xi^y_{1,1} * 1^{\otimes(d-1)})\xi_\lambda = \xi^y_{1,1} * E_{2,2}^{\otimes\lambda_2} * \cdots * E_{n,n}^{\otimes\lambda_n}.$$

By Theorem 6.1, this shows that $D \subseteq D^A(n,d)$. For the reverse inclusion, it suffices to show that the elements of the form $\xi^y_{1,1} * 1^{\otimes(d-1)}$ with $y \in T_A$ belong to D. For any $\lambda \in \Lambda(n,d-1)$, define

$$x(y,\lambda) := \xi^y_{1,1} * E_{1,1}^{\otimes\lambda_1} * E_{2,2}^{\otimes\lambda_2} * \cdots * E_{n,n}^{\otimes\lambda_n} \in S^{T_A}(n,d).$$

Then $\xi^y_{1,1} * 1^{\otimes(d-1)} = \sum_{\lambda\in\Lambda(n,d-1)} x(y,\lambda)$, so it suffices to prove that each $x(y,\lambda) \in D$. Fix $\lambda \in \Lambda(n,d-1)$. Since $d-1 < n$, there is $k \in [1,n]$ with $\lambda_k = 0$. Let

$$\begin{aligned} b &:= E_{1,1} * E_{1,2}^{\otimes\lambda_1} * \cdots * E_{k-1,k}^{\otimes\lambda_{k-1}} * E_{k+1,k+1}^{\otimes\lambda_{k+1}} * \cdots * E_{n,n}^{\otimes\lambda_n},\\ b' &:= E_{1,1} * E_{2,1}^{\otimes\lambda_1} * \cdots * E_{k,k-1}^{\otimes\lambda_{k-1}} * E_{k+1,k+1}^{\otimes\lambda_{k+1}} * \cdots * E_{n,n}^{\otimes\lambda_n},\\ c &:= \xi^y_{1,1} * E_{2,2}^{\otimes\lambda_1} * \cdots * E_{k,k}^{\otimes\lambda_{k-1}} * E_{k+1,k+1}^{\otimes\lambda_{k+1}} * \cdots * E_{n,n}^{\otimes\lambda_n}.\end{aligned}$$

Then $b, b', c \in D$, and $bcb' = x(y,\lambda)$ by Lemma 4.3, completing the proof. ☐

For every $\lambda \in \Lambda(n,d)$, the idempotent $\xi_\lambda \in S^{T_A}(n,d)$ defined in (5.12) belongs to $S^{A_{\bar 0}}(n,d)$, and so, by Corollary 6.4, to $D^A(n,d) \subseteq S^{T_A}(n,d)$. The following is known, cf. [**Tu**$_2$, Lemma 13]:

Corollary 6.5. *If $d \leq n$, then $\xi_\omega D^A(n,d)\xi_\omega = \xi_\omega S^{T_A}(n,d)\xi_\omega$ and there is a superalgebra isomorphism*

$$\varphi\colon W_d^{T_A} \xrightarrow{\sim} \xi_\omega D^A(n,d)\xi_\omega,\ (x_1 \otimes \cdots \otimes x_d)g \mapsto \xi^{x_1}_{1,g^{-1}1} * \cdots * \xi^{x_d}_{d,g^{-1}d}.$$

Proof. First, we claim that every element of the form $\xi(x_1,\dots,x_d;g) := \xi^{x_1}_{1,g^{-1}1} * \cdots * \xi^{x_d}_{d,g^{-1}d}$ belongs to $D^A(n,d)$. Indeed, the case $g = 1$ is handled using Lemma 4.3 and Corollary 6.4, and the case $x_1 = \cdots = x_d = 1$ is clear since $\xi(1,\dots,1;g) \in S^{A_{\bar 0}}(n,d)$. By Lemma 5.15, the elements $\xi(x_1,\dots,x_d;g)$ span in $S^{T_A}(n,d)$ a copy of $W_d^{T_A}$, with the elements $\xi(x_1,\dots,x_d;1)$ spanning $T_A^{\otimes d}$ and the elements $\xi(1,\dots,1;g)$ spanning $\Bbbk\mathfrak{S}_d$. The claim follows.

Now, using Lemma 5.15 and Corollary 6.4, we conclude that $\xi_\omega D^A(n,d)\xi_\omega = \xi_\omega S^{T_A}(n,d)\xi_\omega$, and another application of Lemma 5.15 completes the proof. ☐

6.2. **Symmetric lattices in $'D^A(n,d)$.** In this subsection, in addition to the hypotheses specified at the beginning of Section 2, we assume that $\mathcal{O}$ is a *principal ideal domain*. Let A be an $\mathcal{O}$-superalgebra which is free of finite rank as an $\mathcal{O}$-supermodule. The following result shows, in particular, that $D^A(n,d)$ is maximal among the symmetric subalgebras of $'D^A(n,d)$. The superstructure on $'D^A(n,d)$ plays no role in the theorem, so the content of the statement does not change if $'D^A(n,d)$ is replaced by $|'D^A(n,d)|$ and $D^A(n,d)$ by $|D^A(n,d)|$.

Theorem 6.6. *Let $d \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$, and assume that $d \leq n$. Let C be an $\mathcal{O}$-subalgebra of $'D^A(n,d)$ such that $D^A(n,d) \subseteq C \subseteq {}'D^A(n,d)$. Suppose that for every maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ the $(\mathcal{O}/\mathfrak{m})$-algebra $C \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$ is symmetric. Then $C = D^A(n,d)$.*

Proof. If the theorem is true in the case where $\mathcal{O}$ is a discrete valuation ring (DVR), then it is true in general. Indeed, for every maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ the localisation $\mathcal{O}_{\mathfrak{m}}$ is a DVR, and so by the DVR case of the theorem, $C \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{m}}$ is equal to the $\mathcal{O}_{\mathfrak{m}}$-span of $D^A(n,d) \otimes 1_{\mathcal{O}_{\mathfrak{m}}}$. Then we have $(C/D^A(n,d)) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, whence the $\mathcal{O}$-module $C/D^A(n,d)$ is 0.

In the rest of the proof, $\mathcal{O}$ is a DVR with the maximal ideal (π) for some $\pi \in \mathcal{O}$, and $\Bbbk := \mathcal{O}/(\pi)$. For any free $\mathcal{O}$-module V of finite rank, we have the $\Bbbk$-vector space $V_{\Bbbk} = V \otimes_{\mathcal{O}} \Bbbk$, which we identify with $V/\pi V$.

Recall the notation (4.14) and (4.25). In this proof, for all $e = 0, \dots, d$, we use the following shorthands:

$$
\begin{aligned}
D &:= D^A(n,d), \quad {}'D := {}'D^A(n,d), \quad S := S^A(n,d), \\
D^{d-e,e} &:= D^{d-e,e} M_n(A) = \mathtt{Inv}^{d-e} M_n(A) \otimes \mathtt{Sym}^e(M_n(A)^*) \subseteq D, \\
{}'D^{d-e,e} &:= {}'D^{d-e,e} M_n(A) = \mathtt{Inv}^{d-e} M_n(A) \otimes {}'\mathtt{Sym}^e(M_n(A)^*) \subseteq {}'D, \\
{}'D^{>e} &:= \bigoplus_{e+1 \leq f \leq d} {}'D^{d-f,f}, \quad C^{d-e,e} := C \cap {}'D^{d-e,e}, \quad C^{>e} := C \cap {}'D^{>e}.
\end{aligned}
$$

Claim 1. The $\mathcal{O}$-submodule $C^{d-e,e}$ is pure in C.

This follows immediately from the fact that $'D^{d-e,e}$ is pure in $'D$.

Claim 2. We have $C^{d,0} = D^{d,0} = {}'D^{d,0}$ and $C = C^{d,0} \oplus C^{>0}$.

Since $'D^{d,0} = D^{d,0}$ by definition, the assumption $D \subseteq C \subseteq {}'D$ implies that $C^{d,0} = D^{d,0}$. The second assertion of the claim follows easily from the first one.

Claim 3. We have $\dim C_{\Bbbk}^{d,0} = \dim C_{\Bbbk}^{0,d}$.

Indeed,

$$
\dim C_{\Bbbk}^{d,0} = \operatorname{rank}_{\mathcal{O}} C^{d,0} = \operatorname{rank}_{\mathcal{O}} D^{d,0} = \operatorname{rank}_{\mathcal{O}} D^{0,d} = \operatorname{rank}_{\mathcal{O}} C^{0,d} = \dim C_{\Bbbk}^{0,d},
$$

where the penultimate equality comes from $D^{0,d} \subseteq C^{0,d} \subseteq {}'D^{0,d}$.

Since $'D^{0,d}$ is an ideal in $'D$ and $C \subseteq {}'D$ is a subalgebra, $C^{0,d}$ is an ideal in C, and so naturally a $C^{d,0}$-bimodule. After extending scalars, $C_{\Bbbk}^{0,d}$ becomes a $C_{\Bbbk}^{d,0}$-bimodule.

Claim 4. The $C_{\Bbbk}^{d,0}$-bimodule $C_{\Bbbk}^{0,d}$ is isomorphic to $(C_{\Bbbk}^{d,0})^*$.

Since $C_{\Bbbk}$ is symmetric by assumption, there is a function $G \in C_{\Bbbk}^*$ such that the bilinear form on $C_{\Bbbk}$ defined by $(x,y) := G(xy)$ is symmetric and non-degenerate. By Claims 1 and 2, we can naturally identify $C_{\Bbbk}^{0,d}$, $C_{\Bbbk}^{d,0}$, and $C_{\Bbbk}^{>0}$ with $\Bbbk$-subspaces of $C_{\Bbbk}$. Using the standard grading on $'D$, we see that the orthogonal complement to $C_{\Bbbk}^{0,d}$ in $C_{\Bbbk}$ contains $C_{\Bbbk}^{>0}$. Comparing dimensions using Claim 3, we deduce that $(\cdot,\cdot)$ restricts to a perfect pairing between $C_{\Bbbk}^{0,d}$ and $C_{\Bbbk}^{d,0}$, which yields the required isomorphism.

In view of Remark 4.16, we identify $D^{d,0}$ with S and $D^{0,d}$ with S^*, so that $S^* \subseteq C^{0,d} \subseteq {}'D^{0,d}$. Extending scalars to the field of fractions $\mathbb{K}$ of $\mathcal{O}$, we identify $S^*_{\mathbb{K}} = C^{0,d}_{\mathbb{K}} = {}'D^{0,d}_{\mathbb{K}}$, and so we can consider $C^{0,d}$ and ${}'D^{0,d}$ as $\mathcal{O}$-submodules of $S^*_{\mathbb{K}}$. Now, define

$$'S = \{x \in S_{\mathbb{K}} \mid \langle x, {}'D^{0,d}\rangle \subseteq \mathcal{O}\} \quad \text{and} \quad N = \{x \in S_{\mathbb{K}} \mid \langle x, C^{0,d}\rangle \subseteq \mathcal{O}\}.$$

Then ${}'S \subseteq N \subseteq S$. The following claim follows easily from the definitions:

Claim 5. We have that ${}'S$ and N are S-subbimodules of S, and there are isomorphisms of S-bimodules ${}'S \cong ({}'D^{0,d})^*$ and $N \cong (C^{0,d})^*$.

Claim 6. We have $\xi_\omega \in {}'S$.

By Corollary 3.15, we have ${}'S = \mathtt{Star}^d\, M_n(A)$, and the claim follows from the definition of ξ_ω.

Claim 7. We have $N = S$.

By Claims 2, 4 and 5, we have isomorphisms $S_{\Bbbk} = C^{d,0}_{\Bbbk} \cong (C^{0,d}_{\Bbbk})^* \cong N_{\Bbbk}$ of $S_{\Bbbk}$-bimodules. Let $z + \pi N \in N/\pi N = N_{\Bbbk}$ be the image of $1 \in S_{\Bbbk}$ under this isomorphism. Then $x(z + \pi N) = (z + \pi N)x$ for all $x \in S_{\Bbbk}$. Since $\pi S \supseteq \pi N$, it follows that $z + \pi S \in Z(S/\pi S) = Z(S_{\Bbbk})$. Since $S_{\Bbbk}$ is generated by 1 as a left $S_{\Bbbk}$-module, $N_{\Bbbk}$ is generated by $z + \pi N$ as a left $S_{\Bbbk}$-module. Moreover, $\xi_\omega \in {}'S$ by Claim 6, so $\xi_\omega \in N$. Hence there exists $y \in S_{\Bbbk}$ such that $y(z + \pi N) = \xi_\omega + \pi N$, whence $y(z + \pi S) = \xi_\omega + \pi S$. By Lemma 5.18, $z + \pi S$ is invertible in $S_{\Bbbk}$. So $N + \pi S = S_{\Bbbk}(z + \pi S)S_{\Bbbk} = S_{\Bbbk}$. By Nakayama's Lemma, this implies that $N = S$.

Now we complete the proof of the theorem. By Claim 7, we have $C^{0,d} = D^{0,d}$. Assume for a contradiction that $C \neq D$. Choose an element $x \in C \setminus D$ such that x lies in ${}'D^{>e-1}$ with e maximal possible. Then we can write $x = x_e + \dots + x_d$, where $x_f \in {}'D^{d-f,f}$ for $f = e, \dots, d$. By the maximality of e, we have $x_e \notin D^{d-e,e}$. Hence $x_e = cy$ for some $c \in \mathbb{K} \setminus \mathcal{O}$ and $y \in D^{d-e,e} \setminus \pi D^{d-e,e}$.

Let $F \in (D_{\Bbbk})^*$ be as in Lemma 4.33. Taking into account Corollary 4.34 and the standard grading on D, we conclude that there exists $u \in D^{e,d-e}$ such that $F(yu+\pi D) \neq 0$ in $\Bbbk$, whence $yu \notin \pi D^{0,d}$. By the standard grading again, $x_f u = 0$ for all $f > e$, and hence $xu = cyu$. Since $c \notin \mathcal{O}$, it follows that $xu \notin D^{0,d} = C^{0,d}$. This is a contradiction, since $x \in C$ and $u \in D^{e,d-e} \subseteq C$. □

Example 6.7. Continuing with Example 4.27, assume that $d = 2e + 1$ for some $e \in \mathbb{Z}_{>0}$, and define the $\mathbb{Z}$-algebra C to be the subalgebra of $\mathbb{Q}[z]_d$ spanned over $\mathbb{Z}$ by the elements $1, z, \dots, z^e, z^{e+1}/2, \dots, z^{2e+1}/2$. We then have $D^d\mathbb{Z} \cong \mathbb{Z}[z]_d \subsetneq C \subsetneq {}'\mathbb{Z}[z]_d \cong {}'D^d\mathbb{Z}$. However, it is easy to see that $C \otimes_{\mathbb{Z}} \mathbb{F}_p$ is symmetric for all primes p. This shows that the assumption $d \leq n$ in Theorem 6.6 is essential.

6.3. **Bases and product rules.** Let $\mathtt{B}_{\bar 0}$ be an $\mathcal{O}$-basis of $A_{\bar 0}$, $\mathtt{B}_{\bar 1}$ be an $\mathcal{O}$-basis of $A_{\bar 1}$, and $\mathtt{B} = \mathtt{B}_{\bar 0} \sqcup \mathtt{B}_{\bar 1}$. The *structure constants* $\kappa^{\mathtt{b}}_{\mathtt{b}'\mathtt{b}''} \in \mathcal{O}$ of A are determined from

$$\mathtt{b}'\mathtt{b}'' = \sum_{\mathtt{b} \in \mathtt{B}} \kappa^{\mathtt{b}}_{\mathtt{b}'\mathtt{b}''}\mathtt{b} \qquad (\mathtt{b}, \mathtt{b}' \in \mathtt{B}).$$

Then

$$\{\xi^{\mathtt{b}}_{r,s} \mid 1 \leq r, s \leq n,\ \mathtt{b} \in \mathtt{B}\} \tag{6.8}$$

is a homogeneous basis of $M_n(A)$ with $\bar{\xi}^{\mathtt{b}}_{r,s} = \bar{\mathtt{b}}$, and

$$\xi^{\mathtt{b}'}_{r,s}\xi^{\mathtt{b}''}_{t,u} = \delta_{s,t}\sum_{\mathtt{b}\in\mathtt{B}}\kappa^{\mathtt{b}}_{\mathtt{b}'\mathtt{b}''}\xi^{\mathtt{b}}_{r,u} \qquad (\mathtt{b}',\mathtt{b}''\in\mathtt{B},\ 1\le r,s,t,u\le n). \tag{6.9}$$

We fix a total order $<$ on the basis (6.8) as follows. First, we fix a total order $<$ on $\mathtt{B}$ so that the elements of $\mathtt{B}_{\bar{0}}$ precede the elements of $\mathtt{B}_{\bar{1}}$. Then for $\mathtt{b}',\mathtt{b}''\in\mathtt{B}$ and $1\le r,s,t,u\le n$, we set $\xi^{\mathtt{b}'}_{r,s} < \xi^{\mathtt{b}''}_{t,u}$ if and only if one of the following happens: (1) $\mathtt{b}'<\mathtt{b}''$, (2) $\mathtt{b}'=\mathtt{b}''$ and $r<t$, (3) $\mathtt{b}'=\mathtt{b}''$, $r=t$ and $s<u$.

Recall the notation of §2.2. For $\mathbf{C} = (C^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}} \in \mathcal{M}^{\mathtt{B}}(n)$, we have the element

$$\xi_{\mathbf{C}} := *_{\mathtt{b},r,s}\left((\xi^{\mathtt{b}}_{r,s})^{\otimes c^{\mathtt{b}}_{r,s}}\right) \in S^A(n),$$

where the $*$-product is taken in the order just defined. This agrees with (3.9), so

$$\{\xi_{\mathbf{C}} \mid \mathbf{C}\in\mathcal{M}^{\mathtt{B}}(n)\} \quad\text{and}\quad \{\xi_{\mathbf{C}} \mid \mathbf{C}\in\mathcal{M}^{\mathtt{B}}(n,d)\}$$

are bases of $S^A(n)$ and $S^A(n,d)$, respectively. The parity of a basis element is $\bar{\xi}_{\mathbf{C}} = \bar{\mathbf{C}} := |\mathbf{C}|_{\bar{1}} \pmod 2$.

Let $\mathbf{C} = (C^{\mathtt{b}})_{\mathtt{b}\in\mathtt{B}} \in \mathcal{M}^{\mathtt{B}}(n,d)$ and $(\boldsymbol{r},\mathbf{b},\boldsymbol{s})\in\mathbf{C}$. Let $(\boldsymbol{r}^0,\mathbf{b}^0,\boldsymbol{s}^0)\in\mathbf{C}$ be the tuple defined by the property that the triples $(r^0_1,\mathtt{b}^0_1,s^0_1),\dots,(r^0_d,\mathtt{b}^0_d,s^0_d)$ appear in the increasing order, i.e. for $1\le k\le l\le d$ we have $\xi^{\mathtt{b}^0_k}_{r^0_k,s^0_k} \le \xi^{\mathtt{b}^0_l}_{r^0_l,s^0_l}$. Let $g\in\mathfrak{S}_d$ be an element such that $(\boldsymbol{r}^0,\mathbf{b}^0,\boldsymbol{s}^0)g = (\boldsymbol{r},\mathbf{b},\boldsymbol{s})$, and define

$$[\boldsymbol{r},\mathbf{b},\boldsymbol{s}] := [g;\mathtt{b}^0_1,\dots,\mathtt{b}^0_d],$$

cf. (3.3). It follows from the definition of $\mathtt{Seq}^{\mathtt{B}}(n,d)^2$ that $[\boldsymbol{r},\mathbf{b},\boldsymbol{s}]$ does not depend on the choice of g. By the definition of the $*$-product, we have

$$\xi_{\mathbf{C}} = \sum_{(\boldsymbol{r},\mathbf{b},\boldsymbol{s})\in\mathbf{C}} (-1)^{[\boldsymbol{r},\mathbf{b},\boldsymbol{s}]}\,\xi^{\mathtt{b}_1}_{r_1,s_1}\otimes\cdots\otimes\xi^{\mathtt{b}_d}_{r_d,s_d}. \tag{6.10}$$

For $\mathbf{C},\mathbf{D}\in\mathcal{M}^{\mathtt{B}}(n)$, we let

$$\varepsilon_{\mathbf{CD}} := \begin{cases} (-1)^{\sum c^{\mathtt{b}'}_{r,s}d^{\mathtt{b}''}_{t,u}} & \text{if } \mathbf{C}+\mathbf{D}\in\mathcal{M}^{\mathtt{B}}(n),\\ 0 & \text{otherwise.}\end{cases}$$

where the summation is over all $1\le r,s,t,u\le n$ and $\mathtt{b}',\mathtt{b}''\in\mathtt{B}_{\bar{1}}$ such that $\xi^{\mathtt{b}'}_{r,s} > \xi^{\mathtt{b}''}_{t,u}$. Using Lemma 3.12, we obtain for all $\mathbf{C}\in\mathcal{M}^{\mathtt{B}}(n)$:

$$\Delta(\xi_{\mathbf{C}}) = \sum_{\mathbf{D},\mathbf{E}\in\mathcal{M}^{\mathtt{B}}(n),\ \mathbf{D}+\mathbf{E}=\mathbf{C}} \varepsilon_{\mathbf{DE}}\,\xi_{\mathbf{D}}\otimes\xi_{\mathbf{E}}. \tag{6.11}$$

Define the structure constants $f^{\mathbf{E}}_{\mathbf{CD}}\in\mathcal{O}$ from

$$\xi_{\mathbf{C}}\xi_{\mathbf{D}} = \sum_{\mathbf{E}\in\mathcal{M}^{\mathtt{B}}(n)} f^{\mathbf{E}}_{\mathbf{CD}}\xi_{\mathbf{E}} \qquad (\mathbf{C},\mathbf{D}\in\mathcal{M}^{\mathtt{B}}(n)). \tag{6.12}$$

In particular, $f^{\mathbf{E}}_{\mathbf{CD}} = 0$ unless $|\mathbf{C}| = |\mathbf{D}| = |\mathbf{E}|$. These structure constants are uniquely determined by the structure constants $\kappa^{\mathtt{b}}_{\mathtt{b}',\mathtt{b}''}$. More precisely, if $(\boldsymbol{r},\mathbf{b},\boldsymbol{s})\in\mathtt{Seq}^{\mathtt{B}}(n,d)^2$ and $\mathbf{E} = \boldsymbol{M}[\boldsymbol{r},\mathbf{b},\boldsymbol{s}]$, then using (6.9) and (6.10) we obtain the formula

$$f^{\mathbf{E}}_{\mathbf{CD}} = \sum_{\mathbf{b}',\boldsymbol{t},\mathbf{b}''} (-1)^{[\boldsymbol{r},\mathbf{b},\boldsymbol{s}]+[\boldsymbol{r},\mathbf{b}',\boldsymbol{t}]+[\boldsymbol{t},\mathbf{b}'',\boldsymbol{s}]+[\mathtt{b}'_1,\dots,\mathtt{b}'_d;\mathtt{b}''_1,\dots,\mathtt{b}''_d]}\,\kappa^{\mathtt{b}_1}_{\mathtt{b}'_1,\mathtt{b}''_1}\cdots\kappa^{\mathtt{b}_d}_{\mathtt{b}'_d,\mathtt{b}''_d}$$

where the sum is over all triples $(\mathbf{b}', \boldsymbol{t}, \mathbf{b}'') \in \mathtt{B}^d \times \mathtt{Seq}(n,d) \times \mathtt{B}^d$ such that $(\boldsymbol{r}, \mathbf{b}', \boldsymbol{t}) \in \mathbf{C}$ and $(\boldsymbol{t}, \mathbf{b}'', \boldsymbol{s}) \in \mathbf{D}$. In the case when $A = \mathcal{O}$ and $\mathtt{B} = \{1\}$, this is Green's formula [**Gr**, (2.3b)] for the structure constants of the Schur algebra.

Let $\{x^{\mathbf{C}} \mid \mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n)\}$ be the basis of $S^A(n)^* = (\mathtt{Inv}\, M_n(A))^*$ dual to the basis $\{\xi_{\mathbf{C}} \mid \mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n)\}$ of $S^A(n)$. As the product and the coproduct on $S^A(n)^*$ are by definition dual to the coproduct and the product on $S^A(n)$, respectively, we have in view of (6.11), (6.12) and (3.1):

$$x^{\mathbf{C}} x^{\mathbf{D}} = (-1)^{\bar{\mathbf{C}}\bar{\mathbf{D}}} \varepsilon_{\mathbf{CD}}\, x^{\mathbf{C}+\mathbf{D}} \qquad (\mathbf{C}, \mathbf{D} \in \mathcal{M}^{\mathtt{B}}(n)), \tag{6.13}$$

$$\nabla(x^{\mathbf{C}}) = \sum_{\mathbf{D},\mathbf{E} \in \mathcal{M}^{\mathtt{B}}(n,d)} (-1)^{\bar{\mathbf{D}}\bar{\mathbf{E}}} f_{\mathbf{DE}}^{\mathbf{C}} x^{\mathbf{D}} \otimes x^{\mathbf{E}} \qquad (\mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n,d)). \tag{6.14}$$

It is easy to see that $S^A(n)^*$ is the free supercommutative superalgebra on the even variables $\{x_{r,s}^{\mathtt{b}} \mid \mathtt{b} \in \mathtt{B}_{\bar{0}},\ 1 \le r, s \le n\}$ and the odd variables $\{x_{r,s}^{\mathtt{b}} \mid \mathtt{b} \in \mathtt{B}_{\bar{1}},\ 1 \le r, s \le n\}$, and

$$x^{\mathbf{C}} = (-1)^{|\mathbf{C}|_{\bar{1}}(|\mathbf{C}|_{\bar{1}}-1)/2} \prod_{\mathtt{b} \in \mathtt{B},\ 1 \le r,s \le n} (x_{r,s}^{\mathtt{b}})^{c_{r,s}^{\mathtt{b}}},$$

with the product taken in the total order on the variables $x_{r,s}^{\mathtt{b}}$ which is the same as the one on the basis $\{\xi_{r,s}^{\mathtt{b}}\}$ fixed above.

Let $x^{(\mathbf{C})} := \frac{x^{\mathbf{C}}}{\mathbf{C}!}$. By (6.13), we have

$$x^{(\mathbf{C})} x^{(\mathbf{D})} = (-1)^{\bar{\mathbf{C}}\bar{\mathbf{D}}} \varepsilon_{\mathbf{CD}} \binom{\mathbf{C}+\mathbf{D}}{\mathbf{D}} x^{(\mathbf{C}+\mathbf{D})} \qquad (\mathbf{C}, \mathbf{D} \in \mathcal{M}^{\mathtt{B}}(n)).$$

Then ${}'\mathtt{Sym}(M_n(A)^*)$ is the $\mathcal{O}$-span in $S^A(n)^*_{\mathbb{K}}$ of all $x^{(\mathbf{C})}$ with $\mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n)$. Let

$$f_{(\mathbf{C})\mathbf{D}}^{(\mathbf{E})} := \frac{f_{\mathbf{CD}}^{\mathbf{E}} \mathbf{C}!}{\mathbf{E}!} \quad \text{and} \quad f_{\mathbf{C}(\mathbf{D})}^{(\mathbf{E})} := \frac{f_{\mathbf{CD}}^{\mathbf{E}} \mathbf{D}!}{\mathbf{E}!} \qquad (\mathbf{C}, \mathbf{D}, \mathbf{E} \in \mathcal{M}^{\mathtt{B}}(n)).$$

A priori, these are elements of $\mathbb{K}$, but by Lemma 4.22, they actually belong to $\mathcal{O}$, and for $\mathbf{C} \in \mathcal{M}^{\mathtt{B}}(n,d)$ we have

$$\nabla(x^{(\mathbf{C})}) = \sum_{\mathbf{D},\mathbf{E} \in \mathcal{M}^{\mathtt{B}}(n,d)} (-1)^{\bar{\mathbf{D}}\bar{\mathbf{E}}} f_{(\mathbf{D})\mathbf{E}}^{(\mathbf{C})} x^{(\mathbf{D})} \otimes x^{\mathbf{E}} = \sum_{\mathbf{D},\mathbf{E} \in \mathcal{M}^{\mathtt{B}}(n,d)} (-1)^{\bar{\mathbf{D}}\bar{\mathbf{E}}} f_{\mathbf{D}(\mathbf{E})}^{(\mathbf{C})} x^{\mathbf{D}} \otimes x^{(\mathbf{E})}.$$

Denoting

$$\mathcal{M}_2^{\mathtt{B}}(n,d) := \{(\mathbf{C}, \mathbf{D}) \mid \mathbf{C}, \mathbf{D} \in \mathcal{M}^{\mathtt{B}}(n),\ |\mathbf{C}| + |\mathbf{D}| = d\},$$

we have bases

$$\{\xi_{\mathbf{C}} \otimes x^{\mathbf{D}} \mid (\mathbf{C}, \mathbf{D}) \in \mathcal{M}_2^{\mathtt{B}}(n,d)\} \quad \text{and} \quad \{\xi_{\mathbf{C}} \otimes x^{(\mathbf{D})} \mid (\mathbf{C}, \mathbf{D}) \in \mathcal{M}_2^{\mathtt{B}}(n,d)\}$$

of $D^A(n,d)$ and ${}'D^A(n,d)$, respectively. If $A_{\bar{1}} A_{\bar{1}} = 0$, then $M_n(A)_{\bar{1}} M_n(A)_{\bar{1}} = 0$, and the Turner gradings on $D^A(n,d)$ and ${}'D^A(n,d)$ satisfy

$$\deg(\xi_{\mathbf{C}} \otimes x^{\mathbf{D}}) = \deg(\xi_{\mathbf{C}} \otimes x^{(\mathbf{D})}) = |\mathbf{C}|_{\bar{1}} + 2|\mathbf{D}|_{\bar{0}} + |\mathbf{D}|_{\bar{1}},$$

for all $(\mathbf{C}, \mathbf{D}) \in \mathcal{M}_2^{\mathtt{B}}(n,d)$, cf. Lemma 4.32.

For $(\mathbf{C}, \mathbf{D}), (\mathbf{E}, \mathbf{F}) \in \mathcal{M}_2^{\mathtt{B}}(n,d)$, we have the following *product rules*, which come from (4.12):

$$\begin{aligned}(\xi_{\mathbf{C}} \otimes x^{\mathbf{D}})(\xi_{\mathbf{E}} \otimes x^{\mathbf{F}}) &= \sum (-1)^s \varepsilon_{\mathbf{C}_1\mathbf{C}_2} \varepsilon_{\mathbf{E}_1\mathbf{E}_2} f_{\mathbf{E}_2\mathbf{D}'}^{\mathbf{D}} f_{\mathbf{F}'\mathbf{C}_1}^{\mathbf{F}} (\xi_{\mathbf{C}_2} \xi_{\mathbf{E}_1} \otimes x^{\mathbf{D}'} x^{\mathbf{F}'}) \\ &= \sum (-1)^t \varepsilon_{\mathbf{C}_1\mathbf{C}_2} \varepsilon_{\mathbf{E}_1\mathbf{E}_2} \varepsilon_{\mathbf{D}'\mathbf{F}'} f_{\mathbf{E}_2\mathbf{D}'}^{\mathbf{D}} f_{\mathbf{F}'\mathbf{C}_1}^{\mathbf{F}} f_{\mathbf{C}_2\mathbf{E}_1}^{\mathbf{G}} (\xi_{\mathbf{G}} \otimes x^{\mathbf{D}'+\mathbf{F}'}),\end{aligned}$$

$$(\xi_{\mathbf{C}}\otimes x^{(\mathbf{D})})(\xi_{\mathbf{E}}\otimes x^{(\mathbf{F})}) = \sum(-1)^{s}\varepsilon_{\mathbf{C}_1\mathbf{C}_2}\varepsilon_{\mathbf{E}_1\mathbf{E}_2}f^{(\mathbf{D})}_{\mathbf{E}_2(\mathbf{D}')}f^{(\mathbf{F})}_{(\mathbf{F}')\mathbf{C}_1}(\xi_{\mathbf{C}_2}\xi_{\mathbf{E}_1}\otimes x^{(\mathbf{D}')}x^{(\mathbf{F}')})$$
$$= \sum(-1)^{t}\varepsilon_{\mathbf{C}_1\mathbf{C}_2}\varepsilon_{\mathbf{E}_1\mathbf{E}_2}\varepsilon_{\mathbf{D}'\mathbf{F}'}f^{(\mathbf{D})}_{\mathbf{E}_2(\mathbf{D}')}f^{(\mathbf{F})}_{(\mathbf{F}')\mathbf{C}_1}$$
$$\times f^{\mathbf{G}}_{\mathbf{C}_2\mathbf{E}_1}\binom{\mathbf{D}'+\mathbf{F}'}{\mathbf{D}'}(\xi_{\mathbf{G}}\otimes x^{(\mathbf{D}'+\mathbf{F}')})$$

where the first sums in both formulas are over all $\mathbf{C}_1,\mathbf{C}_2,\mathbf{D}',\mathbf{E}_1,\mathbf{E}_2,\mathbf{F}'\in\mathcal{M}^{\mathsf{B}}(n)$ such that $\mathbf{C}_1+\mathbf{C}_2=\mathbf{C}$, $\mathbf{E}_1+\mathbf{E}_2=\mathbf{E}$, the second sums have an additional summation parameter $\mathbf{G}\in\mathcal{M}^{\mathsf{B}}(n)$, and

$$s=\bar{\mathbf{C}}_1\bar{\mathbf{C}}_2+\bar{\mathbf{C}}_1\bar{\mathbf{E}}_1+\bar{\mathbf{C}}_1\bar{\mathbf{D}}'+\bar{\mathbf{D}}'\bar{\mathbf{E}}_1+\bar{\mathbf{E}}_1\bar{\mathbf{E}}_2,\quad t=s+\bar{\mathbf{D}}'\bar{\mathbf{F}}'.$$

7. The quiver case

In this section we consider an important class of algebras $D^A(n,d)$ sometimes referred to as *schiver doubles*, from 'schiver=Schur+quiver' [**Tu**$_1$].

7.1. Quivers and quiver algebras. Let Q be a quiver with a finite set of vertices $I=\{1,\dots,l\}$ and a finite set of directed edges E. For an edge $\beta\in E$, we denote by $s(\beta)\in I$ the source of β and by $t(\beta)\in I$ the target of β. We denote by Γ the underlying graph of Q. We assume that Γ is connected and has no loops or multiple edges. If $i,j\in I$, we say that i and j are *neighbors* if they are connected by an edge in Γ.

We define the algebra P_Q to be the quotient of the path algebra $\Bbbk Q$ by all quadratic relations. We consider P_Q as a superalgebra with vertices in parity $\bar 0$ and edges in parity $\bar 1$. The parity $\bar 0$ component $P_{Q,\bar 0}$ has a basis $\{e_i\mid i\in I\}$, and the parity $\bar 1$ component $P_{Q,\bar 1}$ has a basis $\{\beta\mid\beta\in E\}$. Note that $P_{Q,\bar 1}P_{Q,\bar 1}=0$, so P_Q is also $\mathbb{Z}$-graded with the degree 0 component $P_Q^0=P_{Q,\bar 0}$ and degree 1 component $P_Q^1=P_{Q,\bar 1}$.

Let $\{e_i^*,\beta^*\mid i\in I,\beta\in E\}$ be the basis of P_Q^* dual to the basis $\{e_i,\beta\mid i\in I,\beta\in E\}$ of P_Q. According to the agreement made in §4.5, we always work with the $\mathbb{Z}$-grading on P_Q^* which is the *shift by* 2 of the canonical grading, i.e. $\deg e_i^*=2$ and $\deg\beta^*=1$ for all $i\in I$ and $\beta\in E$. Then the trivial extension superalgebra $T_{P_Q}=P_Q\oplus P_Q^*$ is also graded. This superalgebra has an easy description as a zigzag algebra, which we introduce next.

The *zigzag algebra* $\mathsf{Z}=\mathsf{Z}_\Gamma$ *of type* Γ is defined in [**HK**] as follows. First assume that $l>1$. Let $\overline{\Gamma}$ be the quiver obtained by doubling all edges in Γ and then orienting the edges so that if i and j are neighboring vertices in Γ, then there is a directed edge $\mathsf{a}^{i,j}$ from j to i and a directed edge $\mathsf{a}^{j,i}$ from i to j. Then Z is the path algebra $\Bbbk\overline{\Gamma}$, generated by length 0 paths e_i for $i\in I$ and length 1 paths $\mathsf{a}^{i,j}$, subject only to the following relations:

(i) All paths of length three or greater are zero.
(ii) All paths of length two that are not cycles are zero.
(iii) All length-two cycles based at the same vertex are equal.

The algebra Z inherits the path length grading from $\Bbbk\overline{\Gamma}$. If $l=1$, i.e. Γ is of type A_1, we define $\mathsf{Z}_{A_1}:=\Bbbk[\mathsf{c}]/(\mathsf{c}^2)$, where c is an indeterminate in degree 2.

If $l>1$, for every vertex i pick its neighbor j and denote $\mathsf{c}^{(i)}:=\mathsf{a}^{i,j}\mathsf{a}^{j,i}$. The relations in Z imply that $\mathsf{c}^{(i)}=\mathsf{e}_i\mathsf{c}^{(i)}\mathsf{e}_i$ is independent of choice of j. Define

$\mathsf{c} := \sum_{i\in V} \mathsf{c}^{(i)}$. Then in all cases Z has a basis

$$\{\mathsf{a}^{i,j} \mid i,j \in I,\ j \text{ is a neighbor of } i\} \cup \{\mathsf{c}^m \mathsf{e}_i \mid i \in I,\ m \in \{0,1\}\},$$

and the graded $\Bbbk$-rank of Z equals $l(1+q^2) + 2|E|q \in \mathbb{Z}[q]$, where q is an indeterminate. Moreover, we consider Z as a superalgebra with $\mathsf{Z}_{\bar 0} = \mathsf{Z}^0 \oplus \mathsf{Z}^2$ and $\mathsf{Z}_{\bar 1} = \mathsf{Z}^1$.

The following is known [**Tu**$_2$, Lemma 6] and easy to check:

Lemma 7.1. *There is an isomorphism of graded superalgebras $T_{P_Q} \xrightarrow{\sim} \mathsf{Z}$ given by $e_i \mapsto \mathsf{e}_i,\ e_i^* \mapsto \mathsf{c}^{(i)},\ \beta \mapsto \mathsf{a}^{i,j},\ \beta^* \mapsto \mathsf{a}^{j,i}$ if $s(\beta) = j$ and $t(\beta) = i$.*

7.2. **Schiver doubles.** From now on we will work over $\mathcal{O}$. For a quiver Q as in the previous subsection, we define

$$D_Q(n,d) := D^{P_Q}(n,d), \quad {}'D_Q(n,d) := {}'D^{P_Q}(n,d).$$

In view of Lemma 7.1, we identify T_{P_Q} with Z, and so, as in §6.1, we identify ${}'D_Q(n,d)$ with $S^{T_{P_Q}}(n,d) = S^{\mathsf{Z}}(n,d)$. In this way, we identify $D_Q(n,d)$ with a subalgebra of $S^{\mathsf{Z}}(n,d)$. By Corollary 6.2, the superalgebra $D_Q(n,d)$ does not depend on the choice of orientation on Q, cf. [**Tu**$_1$, Theorem 157]. As $P_{Q,\bar 1}P_{Q,\bar 1} = 0$, we have Turner's gradings on $D_Q(n,d)$ and ${}'D_Q(n,d)$, see §4.5. We also have a grading on $S^{\mathsf{Z}}(n,d)$, see Remark 5.35. All our identifications respect gradings.

Note that the degree zero component of P_Q is $P_Q^0 = \sum_{i=1}^l \mathcal{O}e_i \cong \mathcal{O}^{\oplus l}$. Recall that $S(n,d) = S^{\mathcal{O}}(n,d)$ is the classical Schur algebra. By Corollary 4.4,

$$S^{P_Q^0}(n,d) \cong \bigoplus_{(d_1,\dots,d_l)\in\Lambda(l,d)} S(n,d_1)\otimes\cdots\otimes S(n,d_l). \tag{7.2}$$

Lemma 7.3. *The image of the natural embedding $S^{P_Q^0}(n,d) \to S^{\mathsf{Z}}(n,d)$ is exactly the degree zero component $S^{\mathsf{Z}}(n,d)^0$.*

Proof. As $P_Q^0 = \mathsf{Z}^0$, we have $M_n(P_Q^0) = M_n(\mathsf{Z})^0$, which implies the lemma. □

Theorem 7.4. *Let $n \geq d$. Then the subsuperalgebra $D_Q(n,d) \subseteq S^{\mathsf{Z}}(n,d)$ is precisely the subalgebra generated by $S^{\mathsf{Z}}(n,d)^0$ and the set*

$$\{\xi^z_{1,1} * E_{2,2}^{\otimes\lambda_2} * \cdots * E_{n,n}^{\otimes\lambda_n} \mid z \in \mathsf{Z},\ (\lambda_2,\dots,\lambda_n) \in \Lambda(n-1,d-1)\} \subseteq S^{\mathsf{Z}}(n,d).$$

Proof. In view of Lemma 7.3, this is a restatement of Corollary 6.4. □

Note that $D_Q(n,d)$, ${}'D_Q(n,d)$ and $S^{\mathsf{Z}}(n,d)$ are graded *super*algebras, whose constructions depend on the superalgebra structures on P_Q and Z. However, *after* we construct them, we want to forget the superalgebra structures and work with $D_Q(n,d)$ and $S^{\mathsf{Z}}(n,d)$ as usual graded algebras. In order to do that, recall the theory of §5.5. From now on, we assume that Γ has no odd cycles. Then to every vertex $i \in I$, we can assign a sign $\zeta_i \in \{\pm 1\}$ such that $\zeta_i\zeta_j = -1$ whenever i and j are neighbors. Let

$$e^{\bar 0} = \sum_{i\in I,\ \zeta_i = 1} \mathsf{e}_i \quad\text{and}\quad e^{\bar 1} = \sum_{i\in I,\ \zeta_i=-1} \mathsf{e}_i.$$

One can easily check that $(e^{\bar 0}, e^{\bar 1})$ is an adapted pair of idempotents for the superalgebra Z in the sense of §5.5.

By Lemma 5.27, there is an explicit isomorphism of graded algebras $\sigma\colon W_d^{|\mathsf{Z}|} \xrightarrow{\sim} |W_d^{\mathsf{Z}}|$. Moreover, as in (5.29) and (5.32), we have the colored permutation modules $M_{\boldsymbol{\lambda}}^{|\mathsf{Z}|}$ labeled by $\boldsymbol{\lambda} \in \Lambda([1,n] \times I, d)$ and set

$$M^{|\mathsf{Z}|}(n,d) := \bigoplus_{\boldsymbol{\lambda} \in \Lambda([1,n]\times I,d)} M_{\boldsymbol{\lambda}}^{|\mathsf{Z}|}.$$

For $\boldsymbol{\lambda} \in \Lambda([1,n-1] \times I, d-1)$ and $k \in J$, we define $\hat{\boldsymbol{\lambda}}^k \in \Lambda([1,n] \times I, d)$ by

$$\hat{\boldsymbol{\lambda}}^k_{(r,i)} = \begin{cases} \boldsymbol{\lambda}_{(r-1,i)} & \text{if } r > 1, \\ 1 & \text{if } r = 1 \text{ and } i = k, \\ 0 & \text{if } r = 1 \text{ and } i \neq k. \end{cases}$$

Lemma 7.5. *Let $z \in \mathsf{e}_j \mathsf{Z} \mathsf{e}_k$ for some $j,k \in I$ and $\boldsymbol{\lambda} \in \Lambda([1,n-1] \times I, d-1)$. Then there is a unique $\mathtt{i}^{\boldsymbol{\lambda}}(z) \in \mathrm{End}_{W_d^{|\mathsf{Z}|}}(M^{|\mathsf{Z}|}(n,d))$ such that*

$$\mathtt{i}^{\boldsymbol{\lambda}}(z)\colon m_{\boldsymbol{\mu}}^{|\mathsf{Z}|} \mapsto \begin{cases} (m_{\hat{\boldsymbol{\lambda}}^j}^{|\mathsf{Z}|}) z[1] & \text{if } \boldsymbol{\mu} = \hat{\boldsymbol{\lambda}}^k, \\ 0 & \text{otherwise,} \end{cases}$$

where $z[1] = z \otimes 1_{\mathsf{Z}}^{\otimes d-1} \in W_d^{|\mathsf{Z}|}$.

Proof. Recalling (5.19), for any $\boldsymbol{\mu} \in \Lambda(n,d)$ set $e_{\boldsymbol{\mu}} := e_{\boldsymbol{\mu}}^{|A|} \in \mathtt{Tens}^d |A| \subseteq W_d^{|A|}$. Note that for all $i \in I$, we have $e_{\hat{\boldsymbol{\lambda}}^i} = e_i \otimes e_{\boldsymbol{\lambda}}$ and $\mathfrak{S}_{\hat{\boldsymbol{\lambda}}^i} = \mathfrak{S}_1 \times \mathfrak{S}_{\boldsymbol{\lambda}}$. It follows that

$$m_{\hat{\boldsymbol{\lambda}}^j}^{|\mathsf{Z}|} z[1](e_{\hat{\boldsymbol{\lambda}}^k} \otimes g) = \varepsilon_{\hat{\boldsymbol{\lambda}}^k}(g) m_{\hat{\boldsymbol{\lambda}}^j}^{|\mathsf{Z}|} z[1]$$

for all $g \in \mathfrak{S}_{\hat{\boldsymbol{\lambda}}^i}$. By adjointness of induction and restriction, there exists a unique map as in the statement. □

Using the maps of Lemma 7.5, define

$$\mathtt{i}^{\boldsymbol{\lambda}}\colon |\mathsf{Z}| \to \mathrm{End}_{W_d^{|\mathsf{Z}|}}(M^{|\mathsf{Z}|}(n,d)),\ z \mapsto \sum_{j,k \in I} \mathtt{i}^{\boldsymbol{\lambda}}(\mathsf{e}_j z \mathsf{e}_k).$$

The following is easy to see:

Lemma 7.6. *For any $\boldsymbol{\lambda} \in \Lambda([1,n-1] \times I, d-1)$, the map $\mathtt{i}^{\boldsymbol{\lambda}}$ is an injective homomorphism of graded algebras.*

By Corollary 5.34 and Remark 5.35, there is an explicit isomorphism of graded algebras

$$\psi\colon |S^{\mathsf{Z}}(n,d)| \xrightarrow{\sim} \mathrm{End}_{W_d^{|\mathsf{Z}|}}(M^{|\mathsf{Z}|}(n,d)).$$

We use this isomorphism to identify the graded algebra $|'D_Q(n,d)| = |S^{\mathsf{Z}}(n,d)|$ with the graded algebra $\mathrm{End}_{W_d^{|\mathsf{Z}|}}(M^{|\mathsf{Z}|}(n,d))$.

Theorem 7.7. *Let $n \geq d$. The subalgebra $|D_Q(n,d)| \subseteq |S^{\mathsf{Z}}(n,d)|$ is precisely the subalgebra generated by the degree zero component $|S^{\mathsf{Z}}(n,d)|^0$ and the set*

$$\bigcup_{\boldsymbol{\lambda} \in \Lambda([1,n-1]\times I,d-1)} \mathtt{i}^{\boldsymbol{\lambda}}(\mathsf{Z}).$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \Lambda(n-1, d-1)$ and $z \in e_j \mathsf{Z} e_k$ for some $j, k \in I$. We claim that

$$\xi_{1,1}^z * E_{2,2}^{\otimes \lambda_1} * \cdots * E_{n,n}^{\otimes \lambda_{n-1}} = \sum_{\boldsymbol{\lambda} \in \pi^{-1}(\lambda)} \mathtt{i}^{\boldsymbol{\lambda}}(z),$$

which implies the result by Theorem 7.4. To prove the claim, let $\boldsymbol{\nu} \in \Lambda([1,n] \times I, d)$. Note that $(\xi_{1,1}^z * E_{2,2}^{\otimes \lambda_1} * \cdots * E_{n,n}^{\otimes \lambda_{n-1}}) v_{\boldsymbol{\nu}} = 0$ unless $\boldsymbol{\nu}$ is of the form $\hat{\boldsymbol{\mu}}^k$ for some $\boldsymbol{\mu} \in \pi^{-1}(\lambda)$ and $k \in I$. Moreover, for $\boldsymbol{\mu} \in \pi^{-1}(\lambda)$, we have

$$\begin{aligned}(\xi_{1,1}^z * E_{2,2}^{\otimes \lambda_1} * \cdots * E_{n,n}^{\otimes \lambda_{n-1}}) v_{\hat{\boldsymbol{\mu}}^k} &= z v_{1,k} \otimes v_{\boldsymbol{\mu}} = v_{1,j} z \otimes v_{\boldsymbol{\mu}} \\ &= (v_{1,j} \otimes v_{\boldsymbol{\mu}}) z[1] = v_{\hat{\boldsymbol{\mu}}^j} z[1],\end{aligned}$$

where $z[1] = z \otimes 1^{\otimes d-1}$ is viewed as an element of W_d^{Z}. Comparing with (5.31) and (5.33), we deduce that

$$\begin{aligned}(\xi_{1,1}^z * E_{2,2}^{\otimes \lambda_1} * \cdots * E_{n,n}^{\otimes \lambda_{n-1}})(m_{\hat{\boldsymbol{\mu}}^k}^{|\mathsf{Z}|}) &= m_{\hat{\boldsymbol{\mu}}^j}^{|\mathsf{Z}|} \, \sigma^{-1}(z[1]) = m_{\hat{\boldsymbol{\mu}}^j}^{|\mathsf{Z}|} \, z[1] = \mathtt{i}^{\boldsymbol{\mu}}(z)(m_{\hat{\boldsymbol{\mu}}^k}^{|\mathsf{Z}|}) \\ &= \sum_{\boldsymbol{\lambda} \in \pi^{-1}(\lambda)} \mathtt{i}^{\boldsymbol{\lambda}}(z)(m_{\hat{\boldsymbol{\mu}}^k}^{|\mathsf{Z}|}),\end{aligned}$$

where we have used the fact that $\sigma^{-1}(z[1]) = z[1]$, see Lemma 5.27. The claim is proved. □

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