IRREDUCIBLE RESTRICTIONS OF REPRESENTATIONS OF SYMMETRIC AND ALTERNATING GROUPS IN SMALL CHARACTERISTICS

ALEXANDER KLESHCHEV, LUCIA MOROTTI, AND PHAM HUU TIEP

Abstract. Building on reduction theorems and dimension bounds for symmetric groups obtained in our earlier work, we classify the irreducible restrictions of representations of the symmetric and alternating groups to proper subgroups. Such a classification is known when the characteristic of the ground field is greater than 3, but the small characteristics cases require a substantially more delicate analysis and new ideas. Our results fit into the Aschbacher-Scott program on maximal subgroups of finite classical groups.

1. Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$. In this paper we consider the following

Problem 1. *Let H be the symmetric group S_n or the alternating group A_n. Classify the pairs (G, V), where G is a subgroup of H and V is an $\mathbb{F}H$-module of dimension greater than 1 such that the restriction $V{\downarrow}_G$ is irreducible.*

A major application of Problem 1 is to the Aschbacher-Scott program on maximal subgroups of finite classical groups, see [**1**, **7**, **25**, **44**, **51**] for more details on this. We point out that for the purposes of these applications, Problem 1 needs to be solved for all almost quasi-simple groups H and G, but we do not make any additional assumptions on G.

For $p = 0$, Problem 1 has been solved in [**50**]. For $p \geq 5$ and $H = \mathsf{S}_n$ (resp. $H = \mathsf{A}_n$), Problem 1 has been solved in [**8**] (resp. [**36**]). But the small characteristics cases $p = 2$ and 3 require a substantially more delicate analysis as well as new ideas, and remained open for a long time. The first major difficulty is that the submodule structure of certain permutation modules over symmetric groups gets very complicated, making the proof of reduction theorems in [**8**] and [**36**] much harder for $p = 2$ or 3. The task of proving new reduction theorems has now been accomplished in [**32**, **34**], which allows one to mostly reduce the problem to doubly transitive subgroups of S_n. The second major difficulty is that the techniques employed in [**8**] for dealing with doubly transitive subgroups are also inefficient for small p. So in this paper we develop

2010 *Mathematics Subject Classification.* 20C20, 20C30, 20C33, 20D06, 20B35.

The first author was supported by the NSF grant DMS-1700905 and the DFG Mercator program through the University of Stuttgart. The second author was supported by the DFG grant MO 3377/1-1 and the DFG Mercator program through the University of Stuttgart. The third author was supported by the NSF (grants DMS-1839351 and DMS-1840702), and the Joshua Barlaz Chair in Mathematics. This work was also supported by the NSF grant DMS-1440140 and Simons Foundation while all three authors were in residence at the MSRI during the Spring 2018 semester.

The authors are grateful to the referee for careful reading and helpful comments on the paper.

a new approach, which iteratively pitches the dimension bounds against the shape of the labeling partition λ of the $\mathbb{F}\mathsf{S}_n$-module D^λ in question, relying particularly on dimension bounds obtained recently in [**33**] and internal structure of doubly transitive subgroups. This allows us to finally extend the above results to all characteristics.

From now on we assume that $p > 0$. We point out that it is the positive characteristic case that is important for the Aschbacher-Scott program, and that the characteristic 0 case is equivalent to $p > n$. For the reader's convenience, we will formulate our main results for all characteristics, although they are only new for $p = 2, 3$.

Recall that the irreducible $\mathbb{F}\mathsf{S}_n$-modules are labeled by the set $\mathscr{P}_p(n)$ of *p-regular partitions of n*. If $\lambda \in \mathscr{P}_p(n)$, we denote by D^λ the corresponding irreducible $\mathbb{F}\mathsf{S}_n$-module.

The *Mullineux involution*

$$\mathscr{P}_p(n) \to \mathscr{P}_p(n),\ \lambda \mapsto \lambda^{\mathtt{M}}$$

is defined from $D^{\lambda^{\mathtt{M}}} \cong D^\lambda \otimes \mathtt{sgn}$, where $\mathtt{sgn}$ is the 1-dimensional sign representation. Of course the Mullineux involution is trivial when $p = 2$, while for odd p it has several explicit combinatorial descriptions, see [**5**, **13**, **29**, **49**].

We denote by $\mathscr{P}_p^{\mathsf{A}}(n)$ the set of all p-regular partitions of n such that $D^\lambda{\downarrow}_{\mathsf{A}_n}$ is reducible. The set of partitions $\mathscr{P}_p^{\mathsf{A}}(n)$ is well understood—if $p = 2$ it is described explicitly in [**4**] (see Lemma 2.9 below), while for $p > 2$ these are exactly the partitions which are fixed by the Mullineux involution.

If $\lambda \in \mathscr{P}_p^{\mathsf{A}}(n)$ we have

$$D^\lambda{\downarrow}_{\mathsf{A}_n} \cong E^\lambda_+ \oplus E^\lambda_-$$

for irreducible $\mathbb{F}\mathsf{A}_n$-modules $E^\lambda_+ \not\cong E^\lambda_-$. If $\lambda \notin \mathscr{P}_2^{\mathsf{A}}(n)$, we denote

$$E^\lambda := D^\lambda{\downarrow}_{\mathsf{A}_n}.$$

Now,

$$\{E^\lambda \mid \lambda \in \mathscr{P}_p(n) \setminus \mathscr{P}_p^{\mathsf{A}}(n)\} \cup \{E^\lambda_\pm \mid \lambda \in \mathscr{P}_p^{\mathsf{A}}(n)\}$$

is a complete set of irreducible $\mathbb{F}\mathsf{A}_n$-modules, and the only non-trivial isomorphisms among these are $E^\lambda \cong E^{\lambda^{\mathtt{M}}}$ for $p > 2$ and $\lambda \in \mathscr{P}_p(n) \setminus \mathscr{P}_p^{\mathsf{A}}(n)$. For $\lambda \in \mathscr{P}_p(n)$, we will interpret the notation $E^\lambda_{(\pm)}$ as $E^\lambda_\pm$ if $\lambda \in \mathscr{P}_p^{\mathsf{A}}(n)$ and as E^λ otherwise.

We set $I := \mathbb{Z}/p\mathbb{Z}$ identified with $\{0, 1, \dots, p-1\}$. A *node* is an element $(r, s) \in \mathbb{Z}^2_{>0}$ (pictorially, the x-axis goes down and the y-axis goes to the right). We always identify a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ with its Young diagram $\{(r, s) \in \mathbb{Z}^2_{>0} \mid s \leq \lambda_r\}$.

Given a node $A = (r, s)$, we define its *residue* $\operatorname{res} A := s - r \pmod p \in I$. Let $i \in I$ and $\lambda \in \mathscr{P}(n)$. A node $A \in \lambda$ (resp. $B \notin \lambda$) is called *removable* (resp. *addable*) for λ if $\lambda \setminus \{A\}$ (resp. $\lambda \cup \{B\}$) is a Young diagram of a partition. A removable (resp. addable) node is called *i-removable* (resp. *i-addable*) if it has residue i.

Labeling the i-addable nodes of λ by $+$ and the i-removable nodes of λ by $-$, the *i-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs. The *reduced i-signature* of λ is obtained from the i-signature by successively erasing all neighbouring pairs of the form $-+$. The nodes corresponding to $-$'s in the reduced i-signature are called *i-normal* for λ (or *normal nodes of residue i*).

A partition $\lambda \in \mathscr{P}_p(n)$ is called *Jantzen-Seitz* (or *JS*) if its top removable node is its only normal node. Equivalently, writing λ in the form $\lambda = (l_1^{a_1}, \dots, l_m^{a_m})$ with

$l_1 > \cdots > l_m$ and $a_1, \dots, a_m > 0$, λ is JS of and only if p divides $l_k - l_{k+1} + a_k + a_{k+1}$ for all $1 \leq k < m$. It is known that the restriction $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$ is irreducible if and only if λ is *JS*, see [**24**, **27**].

Define the partition

$$\beta_n := \begin{cases} (n/2+1, n/2-1) & \text{if } n \text{ is even,} \\ ((n+1)/2, (n-1)/2) & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

When $p = 2$, the irreducible $\mathbb{F}\mathsf{S}_n$-module D^{β_n} is called the *basic spin* module, cf. [**57**]. The irreducible $\mathbb{F}\mathsf{A}_n$-module $E^{\beta_n}_{(\pm)}$ is also called *basic spin*. Basic spin modules often play a special role, see for example [**35**, Theorem 3.9] and [**32**, Theorem A(vi)]. In particular, in Theorems A,A′,B,B′ below, we exclude the basic spin case, and then consider it separately in Theorems C and C′.

For $m \leq n$ we identify S_m as the subgroup of S_n permuting the first m letters. We also have standard subgroups

$$\mathsf{S}_{m_1,\dots,m_t} \cong \mathsf{S}_{m_1} \times \cdots \times \mathsf{S}_{m_t} \leq \mathsf{S}_{m_1+\cdots+m_t} \leq \mathsf{S}_n \quad \text{and} \quad \mathsf{A}_{m_1,\dots,m_t} := \mathsf{S}_{m_1,\dots,m_t} \cap \mathsf{A}_n.$$

whenever $m_1 + \cdots + m_t \leq n$.

Before stating the main results, in Table I we list the dimensions of the modules which give rise to special cases of irreducible restrictions and indicate when such modules split upon restriction to A_n. This table is obtained using [**21**, Table 1], Lemma 2.6, [**3**, Lemma 2.2], [**4**], [**8**, Lemma 1.21], [**20**, Theorems 24.1, 24.15, Tables] and [**36**, Lemma 1.8]. In the table we will always assume $n \geq 5$.

λ	Assumptions on p and n	$\dim D^\lambda$	$\lambda \in \mathscr{P}^{\mathsf{A}}_p(n)$
β_n	$p=2$	$2^{\lfloor (n-1)/2 \rfloor}$	iff $n \not\equiv 2 \pmod 4$
$(n-1,1)$	$p \nmid n$	$n-1$	no
$(n-1,1)$	$p \mid n$	$n-2$	no
$(n-2,2)$	$p>2,\ n \not\equiv 1,2 \pmod p$ or $p=2,\ n \equiv 3 \pmod 4$	$(n^2-3n)/2$	no
$(n-2,2)$	$p>2,\ n \equiv 1 \pmod p$ or $p=2,\ n \equiv 1 \pmod 4$	$(n^2-3n-2)/2$	iff $p=2$ and $n=5$
$(n-2,2)$	$p>2,\ n \equiv 2 \pmod p$ or $p=2,\ n \equiv 2 \pmod 4$	$(n^2-5n+2)/2$	no
$(n-2,2)$	$p=2,\ n \equiv 0 \pmod 4$	$(n^2-5n+4)/2$	no
$(n-2,1^2)$	$p>2,\ p \nmid n$	$(n^2-3n+2)/2$	iff $n=5$
$(n-2,1^2)$	$p>2,\ p \mid n$	$(n^2-5n+6)/2$	iff $p=3$ and $n=6$
$(5,3)$	$p=5$	21	no
$(6,3)$	$p=5$	21	no
$(3^2,2)$	$p>5$	42	yes
(3^3)	$p>5$	42	yes
$(6,5,1)$	$p=2$	288	yes
$(7,5,1)$	$p=2$	288	yes
$(21,2,1)$	$p \neq 2,3,7,23$	3520	no
$(21,2,1)$	$p=7$	3267	no
$(21,2,1)$	$p=23$	3269	no
$(21,1^3)$	$p>3$	1771	no
$(22,1^3)$	$p=5$	1771	no

TABLE I: CERTAIN SPECIAL MODULES AND THEIR DIMENSIONS

Doubly transitive subgroups G occupy a central place in the solution of Problem 1. Mortimer [**47**] studied the problem for the heart $D^{(n-1,1)}$ of the natural module of S_n and listed the results in [**47**, Table I], although leaving two unsettled instances. These instances can now be completely analyzed, using [**18**, Satz D.2.5] for Ree groups and [**14**] for Co_3. We record the updated version of [**47**, Table I] (with $n \geq 5$) in Table II below, where the last column describes the conditions on p (if needed) for $D^{(n-1,1)}{\downarrow_G}$ to be irreducible.

We point out for the purposes of Theorems B and B′ that, except the third line marked with (†), all listed groups are almost simple. Moreover, not all subgroups G satisfying $\mathsf{C}_r^m \trianglelefteq G \leq AGL_m(r)$ are doubly transitive, but the list of such doubly transitive groups is known by Hering's Theorem, see [**40**]. On the other hand, the subgroups from all other lines are indeed doubly transitive.

G	Degree n	Transitivity	Conditions on p
S_n	n	n	
A_n	n	$n-2$	
(†) $\mathsf{C}_r^m \trianglelefteq G \leq AGL_m(r)$, r prime	r^m	2 or 3	$p \neq r$
$PSL_d(q) \trianglelefteq G \leq P\Gamma L_d(q)$, $d \geq 3$	$\dfrac{q^d-1}{q-1}$	2	$p \nmid q$
$\mathsf{A}_7 \cong G < GL_4(2)$	15	2	$p \neq 2$
$Sp_{2m}(2),\ m \geq 3$	$2^{m-1}(2^m \pm 1)$	2	$p \neq 2$
$SL_2(q) \trianglelefteq G \leq \Sigma L_2(q)$, $2 \mid q$	$q+1$	3	
$PSL_2(q) \trianglelefteq G \leq P\Sigma L_2(q)$, $2 \nmid q$	$q+1$	2	$p \neq 2$
$PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$, $G \not\leq P\Sigma L_2(q),\ 2 \nmid q$	$q+1$	3	
${}^2B_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2B_2(q))$, $q > 2$	q^2+1	2	$p \nmid (q+1+\sqrt{2q})$
$PSU_3(q) \trianglelefteq G \leq P\Gamma U_3(q)$, $q > 2$	q^3+1	2	$p \nmid (q+1)$
${}^2G_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2G_2(q))$	q^3+1	2	$p \nmid (q+1)(q+1+\sqrt{3q})$
M_{24}	24	5	$p \neq 2$
M_{23}	23	4	$p \neq 2$
M_{22}	22	3	$p \neq 2$
M_{12}	12	5	
M_{11}	11	4	
M_{11}	12	3	$p \neq 3$
$PSL_2(11)$	11	2	$p \neq 3$
HS	176	2	$p \neq 2,3$
Co_3	276	2	$p \neq 2,3$

TABLE II: IRREDUCIBILITY OF $D^{(n-1,1)}$ OVER DOUBLY TRANSITIVE SUBGROUPS

Remark 1.2. In [**32**, Theorem B], we have discovered a new exceptional family of imprimitive subgroups G for which $D^{(n-1,1)}{\downarrow_G}$ is irreducible in characteristic 2. Let $p = 2$, n be even, and $G \leq \mathsf{S}_{n/2} \wr \mathsf{S}_2$. Let $B := \mathsf{S}_{n/2} \times \mathsf{S}_{n/2}$ be the base subgroup of $\mathsf{S}_{n/2} \wr \mathsf{S}_2$, and G_1 (resp. G_2) be the projection of $G \cap B$ onto the first (resp.

second) factor $\mathsf{S}_{n/2}$ of B. Then $D^{(n-1,1)}{\downarrow_G}$ is irreducible if and only if $n \equiv 2 \pmod 4$, G is transitive on $\{1, 2 \dots, n\}$, G_1, G_2 are 2-transitive subgroups of $\mathsf{S}_{n/2}$ over which $D^{(n/2-1,1)}$ is irreducible, and $(D^{(n/2-1,1)} \boxtimes D^{(n/2)}){\downarrow_{G\cap B}} \not\cong (D^{(n/2)} \boxtimes D^{(n/2-1,1)})_{G\cap B}$. We refer the reader to [**32**, Section 7], especially [**32**, Example 7.24], for more on this.

For future reference, in Table III, we now list some additional "non-serial" (in the sense that they exist only in a finite number of degrees n) examples of irreducible restrictions of $\mathbb{F}\mathsf{S}_n$-modules D^λ to subgroups $G < \mathsf{S}_n$. In all the cases G acts (at least) 2-transitively on $\{1, 2, \dots, n\}$ or $\{1, \dots, n-1\}$ as indicated in the table, and when $\{1, \dots, n-1\}$ is indicated we have that G fixes n. The fact that the cases listed in Table III do yield irreducible restrictions $D^\lambda{\downarrow_G}$ is part of the statements of Theorems A and C.

Case	λ or $\lambda^{\mathtt{M}}$	G	n	2-transitive on	p
(S1)	$(n-2,2)$	$SL_3(2)$	7	$\{1,\dots,n\}$	$p=5$
		$P\Gamma L_2(8)$	9		$p\neq 2,7$
		M_{11}	11		$p\neq 3,5$
		M_{11}	12		$p=2$
		M_{12}	12		$p\neq 5$
		M_{23}	23		$p\neq 2,3$
		M_{24}	24		$p\neq 2$
(S2)	$(n-2,2)$	M_{11}	12	$\{1,\dots,n-1\}$	$p=2$
		M_{12}	13		$p=11$
		M_{23}	24		$p=11$
		M_{24}	25		$p=23$
(S3)	$(n-2,1^2)$	S_5	6	$\{1,\dots,n\}$	$p=3$
		M_{11}	11		$p\neq 2,11$
		M_{11}	12		$p\neq 2,3$
		M_{12}	12		$p\neq 2$
		$M_{22}, \mathrm{Aut}(M_{22})$	22		$p\neq 2$
		M_{23}	23		$p\neq 2$
		M_{24}	24		$p\neq 2$
(S4)	$(n-2,1^2)$	M_{11}	12	$\{1,\dots,n-1\}$	$p=3$
		M_{11}	13		$p=13$
		M_{12}	13		$p=13$
		$M_{22}, \mathrm{Aut}(M_{22})$	23		$p=23$
		M_{23}	24		$p=3$
		M_{24}	25		$p=5$
(S5)	$(14,1^2)$	$\mathsf{C}_2^4 \rtimes \mathsf{A}_7$	16	$\{1,\dots,16\}$	$p\neq 2$
(S6)	$(15,1^2)$	$\mathsf{C}_2^4 \rtimes \mathsf{A}_7$	17	$\{1,\dots,16\}$	$p=17$
(S7)	$(5,3)$	$AGL_3(2)$	8	$\{1,\dots,8\}$	$p=5$
(S8)	$(6,3)$	$AGL_3(2)$	9	$\{1,\dots,8\}$	$p=5$
(S9)	$(21,2,1)$	M_{24}	24	$\{1,\dots,24\}$	$p\neq 2,3$
(S10)	$(21,1^3)$	M_{24}	24	$\{1,\dots,24\}$	$p\neq 2,3$
(S11)	$(22,1^3)$	M_{24}	25	$\{1,\dots,24\}$	$p=5$
(S12)	$(3,2)$	$\mathsf{C}_5 \rtimes \mathsf{C}_4$	5	$\{1,\dots,5\}$	$p=2$
(S13)	$(4,2)$	S_5	6	$\{1,\dots,6\}$	$p=2$
(S14)	$(6,4)$	$\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)$	10	$\{1,\dots,10\}$	$p=2$

TABLE III: NON-SERIAL EXAMPLES OF IRREDUCIBLE RESTRICTIONS FROM S_n

Note that in the cases (S12)-(S14), we have $(\lambda, p) = (\beta_n, 2)$, i.e. these cases are concerned with restrictions of basic spin modules.

For future reference, in Table IV, we now list some "non-serial" examples of irreducible restrictions of $\mathbb{F}\mathsf{A}_n$-modules $E^\lambda_\pm$ with $\lambda \in \mathscr{P}^{\mathsf{A}}_p(n)$ to subgroups $G < \mathsf{A}_n$. In all but the case (A17), G acts (at least) 2-transitively on $\{1, 2, \dots, m\}$ as indicated in the table (and fixes n if $m = n - 1$). The two additional conditions in Table IV are as follows:

(♠) only one of $E^{(5,4)}_\pm$, namely the one whose Brauer character takes value -1 at elements of order 9 in $SL_2(8)$, is irreducible over G.

(♠♠) $\mathrm{soc}\,(G)$ acts on $\{1, 2, \dots, 6\}$ and $\{7, 8, \dots, 12\}$ via two inequivalent 2-transitive actions.

Case	λ	G	n	2-transitive on	p	Additional conditions
(A1)	$(6,5,1)$	M_{12}	12	$\{1,\dots,12\}$	$p=2$	
(A2)	$(7,5,1)$	M_{12}	13	$\{1,\dots,12\}$	$p=2$	
(A3)	$(4,1^2)$	A_5	6	$\{1,\dots,6\}$	$p=3$	
(A4)	(3^3)	$P\Gamma L_2(8)$	9	$\{1,\dots,9\}$	$p>5$	
(A5)	$(3^2,2)$	$AGL_3(2)$	8	$\{1,\dots,8\}$	$p>5$	
(A6)	(3^3)	$AGL_3(2)$	9	$\{1,\dots,8\}$	$p>5$	
(A7)	$(3,2)$	$\mathsf{C}_5 \rtimes \mathsf{C}_2$	5	$\{1,\dots,5\}$	$p=2$	
(A8)	$(5,4)$	$ASL_2(3), \mathsf{C}_3^2 \rtimes \mathsf{Q}_8$	9	$\{1,\dots,9\}$	$p=2$	
(A9)	$(5,4)$	$SL_2(8), P\Gamma L_2(8)$	9	$\{1,\dots,9\}$	$p=2$	(♠)
(A10)	$(6,4)$	M_{10}	10	$\{1,\dots,10\}$	$p=2$	
(A11)	$(6,5)$	M_{11}	11	$\{1,\dots,11\}$	$p=2$	
(A12)	$(7,5)$	M_{11}, M_{12}	12	$\{1,\dots,12\}$	$p=2$	
(A13)	$(4,3)$	A_5	7	$\{1,\dots,6\}$	$p=2$	
(A14)	$(5,3)$	$\mathsf{A}_5, \mathsf{S}_5$	8	$\{1,\dots,6\}$	$p=2$	
(A15)	$(6,5)$	M_{10}	11	$\{1,\dots,10\}$	$p=2$	
(A16)	$(7,5)$	$\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)$	12	$\{1,\dots,10\}$	$p=2$	
(A17)	$(7,5)$	$\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)$	12		$p=2$	(♠♠)
(A18)	$(7,5)$	M_{11}	12	$\{1,\dots,11\}$	$p=2$	

TABLE IV: NON-SERIAL EXAMPLES OF IRREDUCIBLE RESTRICTIONS FROM A_n

Note that in the cases (A7)-(A18), we have $(\lambda, p) = (\beta_n, 2)$, i.e. these cases are concerned with restrictions of basic spin modules.

We now describe the main results of the paper. In all theorems, the subgroups G are listed up to S_n-conjugation. We note that S_n-conjugate subgroups of A_n need not be A_n-conjugate, and it may happen, as it does in case (A9) listed in Table IV, that one conjugate acts irreducibly while the other does not on an $\mathbb{F}\mathsf{A}_n$-module; such instances are specified explicitly in our results. The case of the basic spin module[1], excluded in Theorems A and A′, will be considered separately in Theorems C and C′.

[1]As pointed out by the anonymous referee, incidentally, the phenomenon of spin modules in characteristic 2 giving rise to long chains of subgroups with irreducible restriction has also been observed in the context of symplectic groups over algebraically closed fields of characteristic 2 in [**9**].

Theorem A. *Let $n \geq 5$, $G < \mathsf{S}_n$, and $\lambda \in \mathscr{P}_p(n)$ be such that $\dim D^\lambda > 1$. Exclude the basic spin case $(p, \lambda) = (2, \beta_n)$. Then $D^\lambda{\downarrow}_G$ is irreducible if and only if one of the following holds:*

(i) $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$ *and* $G = \mathsf{A}_n$.
(ii) λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-1, 1)$, G *is 2-transitive, and* (G, n, p) *is as in Table II.*
(iii) $p = 2$, $n \equiv 2 \pmod 4$, $\lambda = (n-1, 1)$, *and* $G \leq \mathsf{S}_{n/2} \wr \mathsf{S}_2$ *is as in Remark 1.2.*
(iv) $p \neq 2$, λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-2, 1^2)$, $n = 2^m$ *for some* $m \geq 3$ *and* $G = AGL_m(2) < \mathsf{S}_n$ *via its natural action on the points of* $\mathbb{F}_2^m$.
(v) λ *is JS and* $G = \mathsf{S}_{n-1}$.
(vi) λ *is JS,* $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$, *and* $G = \mathsf{A}_{n-1}$.
(vii) $n \equiv 0 \pmod p$, λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-1, 1)$, G *is a 2-transitive subgroup of* S_{n-1}, *and* $(G, n-1, p)$ *is as described in Table II.*
(viii) $p \neq 2$, λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-2, 1^2)$, $n = 2^m + 1 \equiv 0 \pmod p$ *for some* $m \geq 2$, *and* $G = AGL_m(2) < \mathsf{S}_{n-1}$ *embedded via its natural action on the points of* $\mathbb{F}_2^m$.
(ix) (λ, G, n, p) *is as in one of the cases (S1)-(S11) in Table III.*

Theorem A′. *Let $n \geq 5$, $G < \mathsf{A}_n$, and V be a non-trivial irreducible $\mathbb{F}\mathsf{A}_n$-module. If $p = 2$ assume that V is not basic spin. Then $V{\downarrow}_G$ is irreducible if and only if one of the following holds:*

(i) $V \cong E^\lambda$ *with* $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$ *and* (λ, G, n, p) *is as in Theorem A.*
(ii) $V \cong E^\lambda_\pm$ *with* $\lambda \in \mathscr{P}_p^{\mathsf{A}}(n)$ *and one of the following holds:*
  (a) $G = \mathsf{A}_{n-1}$ *and* λ *is JS or it has exactly two normal nodes, both of residue different from* 0.
  (b) $G = \mathsf{A}_{n-2}$ *or* $\mathsf{A}_{n-2,2}$ *and* λ *is JS.*
  (c) (λ, G, n, p) *is as in one of the cases (A1)-(A6) in Table IV.*

A group G is called *almost quasisimple* if $S \trianglelefteq G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for some non-abelian simple group S. In a number of applications, irreducible restrictions to quasisimple subgroups G are of most interest. In the next two theorems we deal just with this important special case.

Theorem B. *Let $n \geq 5$, $G < \mathsf{S}_n$ be an almost quasisimple subgroup, and $\lambda \in \mathscr{P}_p(n)$ be such that $\dim D^\lambda > 1$. Exclude the basic spin case $(p, \lambda) = (2, \beta_n)$. Then $D^\lambda{\downarrow}_G$ is irreducible if and only if one of the following holds:*

(i) $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$ *and* $G = \mathsf{A}_n$.
(ii) λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-1, 1)$, G *is 2-transitive, and* (G, n, p) *is as described in Table II, excluding the third line marked with* (†).
(iii) λ *is JS and* $G = \mathsf{S}_{n-1}$.
(iv) λ *is JS,* $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$ *with* $\lambda \neq \beta_n$ *if* $p = 2$ *and* $n \equiv 2 \pmod 4$, *and* $G = \mathsf{A}_{n-1}$.
(v) $n \equiv 0 \pmod p$, λ *or* $\lambda^{\mathtt{M}}$ *equals* $(n-1, 1)$, G *a 2-transitive subgroup of* S_{n-1}, *and* $(G, n-1, p)$ *is as described in Table II, excluding the third line marked with* (†).
(vi) (λ, G, n, p) *is as in one of the cases (S1)-(S4) or (S9)-(S11) in Table III.*

Theorem B′. *Let $n \geq 5$, $H = \mathsf{A}_n$, $G < H$ be almost quasisimple, and V be a non-trivial irreducible $\mathbb{F}\mathsf{A}_n$-module. If $p = 2$ assume that V is not basic spin. Then $V{\downarrow}_G$ is irreducible if and only if one of the following holds:*

(i) $V \cong E^\lambda$ *with* $\lambda \notin \mathscr{P}_p^{\mathsf{A}}(n)$ *and* (λ, G, n, p) *is as in Theorem B.*

(ii) $V \cong E^\lambda_\pm$ *with* $\lambda \in \mathscr{P}^{\mathsf{A}}_p(n)$ *and one of the following holds:*
 (a) $G = \mathsf{A}_{n-1}$ *and* λ *is JS or it has exactly two normal nodes, both of residue different from 0.*
 (b) $G = \mathsf{A}_{n-2}$ *or* $\mathsf{A}_{n-2,2}$ *and* λ *is JS.*
 (c) (λ, G, n, p) *is as in one of the cases (A1)-(A4) in Table IV.*

For basic spin modules in characteristic 2 we have the following two results.

Theorem C. *Let* $n \geq 5$, $p = 2$, *and* $G < \mathsf{S}_n$ *be a proper subgroup of* S_n *such that* $D^{\beta_n}{\downarrow}_G$ *is irreducible. Then one of the following happens:*

(i) $G \leq \mathsf{S}_{n-k} \times \mathsf{S}_k$ *with* $n-k$ *and* k *odd. In fact,*
$$D^{\beta_n}{\downarrow}_{\mathsf{S}_{n-k}\times\mathsf{S}_k} \cong D^{\beta_{n-k}} \boxtimes D^{\beta_k}$$
is indeed irreducible.

(ii) $G \leq \mathsf{S}_a \wr \mathsf{S}_b$ *with* $n = ab$, $a, b \in \mathbb{Z}_{>1}$ *and* a *is odd. Moreover if* $b > 2$ *then* $G \not\leq \mathsf{S}_a \times \cdots \times \mathsf{S}_a$. *In fact,*
$$D^{\beta_n}{\downarrow}_{\mathsf{S}_a\wr\mathsf{S}_b} \cong D^{\beta_a} \wr D^{\beta_b}$$
is indeed irreducible.

(iii) G *is primitive, in which case* $D^{\beta_n}{\downarrow}_G$ *is irreducible if and only if one of the following happens:*
 (a) $n \equiv 2 \pmod 4$ *and* $G = \mathsf{A}_n$;
 (b) (G, n) *is as in one of the cases (S12)-(S14) in Table III.*

Moreover, if G *is almost quasi-simple then* $D^{\beta_n}{\downarrow}_G$ *is irreducible if and only if one of the following holds:*

(1) n *is even and* $G = \mathsf{S}_{n-1}$.

(2) G *is primitive, and one of the following holds:*
 (a) $n \equiv 2 \pmod 4$ *and* $G = \mathsf{A}_n$;
 (b) (G, n) *is as in one of the cases (S13),(S14) in Table III.*

For restrictions of basic spin modules for A_n we have the following analogous result:

Theorem C′. *Let* $n \geq 5$, $p = 2$ *and* $G < \mathsf{A}_n$. *If* $E^{\beta_n}_{(\pm)}{\downarrow}_G$ *is irreducible then one of the following holds:*

(i) $G \leq \mathsf{A}_{n-k,k}$ *for some* $1 \leq k < n$, *and either* $n \equiv 0 \pmod 4$ *and* k *is odd, or* $n \not\equiv 2 \pmod 4$ *and* $k \equiv 2 \pmod 4$. *Moreover, in all of these cases* $E^{\beta_n}_\pm{\downarrow}_{\mathsf{A}_{n-k,k}}$ *is indeed irreducible.*

(ii) $G \leq (\mathsf{S}_a \wr \mathsf{S}_b) \cap \mathsf{A}_n$ *for* $a, b > 1$ *with* $n = ab$, *and either* a *is odd or* $a \equiv 2 \pmod 4$ *and* $b = 2$. *Moreover, in all of these cases* $E^\lambda_{(\pm)}{\downarrow}_{(\mathsf{S}_a\wr\mathsf{S}_b)\cap\mathsf{A}_n}$ *is indeed irreducible.*

(iii) G *is primitive, in which case* $E^{\beta_n}_{(\pm)}{\downarrow}_G$ *is irreducible if and only if* (G, n) *is as in one of the cases (A7)-(A12) in Table IV.*

Moreover, if G *is almost quasi-simple then* $E^{\beta_n}_{(\pm)}{\downarrow}_G$ *is irreducible if and only if one of the following holds:*

(1) $n \not\equiv 2 \pmod 4$ *and one of the following happens:*
 (a) $4|n$ *and* $G = \mathsf{A}_{n-3,2,1}$ *or* $\mathsf{A}_{n-2,1,1}$.
 (b) $G = \mathsf{A}_{n-2,2}$.
 (c) $n \equiv 0, 3 \pmod 4$ *and* $G = \mathsf{A}_{n-1}$.

(d) (G, n) *is as in one of the cases (A9), (A11)-(A18) in Table IV.*

(2) $(G, n) = (M_{10}, 10)$ *(case (A10) of Table IV).*

Remark 1.3. We point out that [**32**, Theorem C] contains an inaccuracy: since $M_{12} < \mathsf{A}_{12}$ and $\beta_{12} \in \mathscr{P}_2^{\mathsf{A}}(12)$, the restriction $D^{\beta_{12}}{\downarrow}_{M_{12}}$ is reducible, and so this case does not appear in Theorem C above. However, it does appear in Theorem C$'$(iii) and (1)(d) as part of the case (A12).

There is a similar inaccuracy in [**8**, Main Theorem]: let $G = M_{11} \leq \mathsf{S}_{11}$ and $p = 5$. Then $D^{(9,2)}{\downarrow}_G$ is reducible by case (iii) of [**8**, Main Theorem] and so $D^{(10,2)}{\downarrow}_G$ is also reducible.

We point out that the results proved in [**32**, **34**] that reduce the problem mostly to the treatment of doubly transitive groups do not depend on the Classification of Finite Simple Groups (CFSG). However, the main results of this paper depend on CFSG as follows: (i) our treatment of doubly transitive subgroups relies on their explicit list, see [**10**], which is a consequence of CFSG, and (ii) the treatment of "non-generic" situation in Section 3 uses the list of simple subgroups of S_n of large order (Proposition 3.1) which also relies on CFSG.

We now describe the key ingredients of our proof and the organization of the paper. We will exploit various dimension bounds for irreducible representations of symmetric groups, especially new lower bounds obtained in [**33**], see Theorems 2.21 and 2.22. Further dimension bounds and branching results are collected in the preliminary Section 2.

Reduction theorems established in [**32**, **34**] allow us to assume in many situations that the subgroup G is primitive or even doubly transitive. Those subgroups tend to have a relatively large order, and we contrast order bounds with dimension bounds in Section 3, particularly to resolve the "non-generic" situation where the module is either basic spin or not extendible to S_n.

In Sections 4–7 we deal with doubly transitive subgroups $G \leq \mathsf{S}_n$. Given the well-known solution of Problem 1 in the case $(G, H) = (\mathsf{A}_n, \mathsf{S}_n)$, we will assume that $G \not\geq \mathsf{A}_n$. Such subgroups G are subdivided into the following four families, corresponding to the structure of the socle $\mathrm{soc}\,(G)$ and its action on $\{1, 2, \ldots, n\}$:

(A) $\mathrm{soc}\,(G)$ is elementary abelian subgroup;
(B) $\mathrm{soc}\,(G) \cong PSL_m(q)$ (is non-abelian simple) acting on $n = (q^m - 1)/(q - 1)$ 1-dimensional subspaces of $\mathbb{F}_q^m$;
(C) $G \cong Sp_{2m}(2)$, $m \geq 3$, acting on $n = 2^{m-1}(2^m + (-1)^\delta)$ quadratic forms on $\mathbb{F}_2^{2m}$ of the given Witt defect $\delta \in \{0, 1\}$;
(D) all other doubly transitive subgroups; the subgroups from this class will be called *small doubly transitive subgroups*.

The small doubly transitive subgroups of (D) are handled in Section 4, largely relying on the aforementioned results on dimension bounds, branching rules to Young subgroups, and available information about modular representations of H and G.

In Section 5, we handle the family (A) of affine permutation subgroups. Here, the key technical result is Proposition 5.11 that identifies the S_n-modules that have no (nonzero) invariants over $\mathrm{soc}\,(G) \cong \mathsf{C}_r^m$, whose proof in turn relies on representation theory of affine general linear group $AGL_m(r)$ and the new branching recognition result Proposition 2.17.

The families (B) and (C) are handled in Sections 6 and 7, respectively. We note that these large doubly transitive groups are the main reason why the methods of [**8**] and [**36**] break down when one tries to employ them in small characteristics $p = 2, 3$.

The heart of the proof is to show that if the irreducible $\mathbb{F}\mathsf{S}_n$-module D^λ remains irreducible over such a subgroup G from the families (B) and (C) , then the longest part $\lambda_1 = n - \ell$ of λ is very large, in fact, $\ell \leq 3$ most of the time. We will do this in a sequence of steps.

First, using the obvious bound $\dim V \leq |G|^{1/2}$ for any irreducible G-module V and Lemma 2.3, we show in Propositions 6.7 and 7.9 that

$$\dim D^\lambda \leq n^{\frac{1}{2}\log_2 n+1}.$$

Then an application of Proposition 2.23 implies that

$$\ell = O(\log n). \tag{1.4}$$

Next, we choose some L such that $2\ell \leq L < n$. Considering $G \cap \mathsf{S}_{n-L}$ and using Theorem 2.11 and Propositions 6.7(iii) and 7.9(ii), we prove that

$$\dim D^\lambda = n^{O(k)} \tag{1.5}$$

for some $k = O(\log \ell)$. On the other hand, Theorem 2.21 yields a lower bound

$$\dim D^\lambda > O(n^\ell/(\ell!)^2). \tag{1.6}$$

Given (1.4), we can show that (1.6) contradicts (1.5), unless ℓ is small. An iterative application of this argument will allow us to show that $\ell \leq 3$. The remaining possibilities for λ are ruled out using more precise information about D^λ.

Finally, the main theorems are proved in Section 8. First, we use the main results of [**32**, **34**] to reduce to subgroups doubly transitive on $\{1, \dots, n\}$ or doubly transitive on $\{1, \dots, n-1\}$ and fixing n. The results of the previous sections then allow us to complete the proofs of Theorems A, A$'$. The proof of Theorem B requires a delicate argument to rule out the possibility of Theorem A(iii) for almost quasisimple groups. The proof of Theorem C$'$ combines classifications of irreducible restrictions to maximal imprimitive subgroups (from [**34**]) and to primitive subgroups (obtained in Sections 4–7). After that we need to handle the case when $\mathrm{soc}\,(G) \cong \mathsf{A}_m$ has only orbits of length 1 and m on $\{1, 2, \dots, n\}$.

2. Preliminary results

2.1. **Generalities.** Throughout the paper we work over a fixed algebraically closed ground field $\mathbb{F}$ of characteristic $p > 0$. Let G, H be arbitrary finite groups, V, V' be $\mathbb{F}G$-modules, and W be an $\mathbb{F}H$-module. The following notation is used throughout the paper:

$V{\downarrow}_H$ or $V{\downarrow}_H^G$	the *restriction* of V from G to H (if $H \leq G$);
$\mathrm{ind}^G W$ or $\mathrm{ind}_H^G W$	the *induction* of W from H to G (if $H \leq G$);
$V \boxtimes W$	the outer tensor product of V and W (this is a module over $G \times H$);
$V \otimes V'$	the inner tensor product of V and V' (this is a module over G);
V^G	the space of G-*invariant vectors* in V;
$\mathbb{F}_G$	the trivial $\mathbb{F}G$-module;
$\mathrm{Irr}_{\mathbb{F}}(G)$	a complete set of irreducible $\mathbb{F}G$-modules;
$\mathrm{IBr}_p(G)$	the set of irreducible p-Brauer characters of G;
$\mathfrak{b}_p(G)$	the maximal dimension of an irreducible $\mathbb{F}G$-module;
$\mathfrak{b}(G)$	the maximal dimension of an irreducible $\mathbb{C}G$-module;
$d(G)$	the minimal degree of a non-linear irreducible complex character of G (if such exists);
$P(G)$	the smallest index of a (proper) maximal subgroup of G;
$\mathscr{P}_p(n)$	the set of p-regular partitions of n;
$\mathscr{P}_p^{\mathsf{A}}(n)$	the set of $\lambda \in \mathscr{P}_p(n)$ such that $D^\lambda{\downarrow}_{\mathsf{A}_n}$ is reducible;
$h(\lambda)$	the number of nonzero parts in the partition λ.

Let $0 \leq \ell < n$. We denote

$$\mathscr{P}^{(\ell)}(n) := \{\lambda \vdash n \mid \lambda_1 \geq n - \ell\},$$
$$\mathscr{L}^{(\ell)}(n) := \{\lambda \in \mathscr{P}_p(n) \mid \lambda \text{ or } \lambda^{\mathtt{M}} \text{ belongs to } \mathscr{P}^{(\ell)}(n) \cap \mathscr{P}_p(n)\}.$$

Given a partition $\mu = (\mu_1, \mu_2, \dots)$ of ℓ with $\mu_1 \leq n - \ell$, we have a partition

$$(n-\ell, \mu) := (n-\ell, \mu_1, \mu_2, \dots) \tag{2.1}$$

of n. Every partition λ of n can be written in the form $\lambda = (\ell, \mu)$ for a (possibly empty) partition μ of $n - \ell$.

For $\lambda, \lambda^1, \dots, \lambda^s \in \mathscr{P}_p(n)$, we denote

$$[\![D^\lambda]\!] := \{D^\lambda, D^\lambda \otimes \mathtt{sgn}\} \quad \text{and} \quad [\![D^{\lambda^1}, \dots, D^{\lambda^s}]\!] := [\![D^{\lambda^1}]\!] \cup \dots \cup [\![D^{\lambda^s}]\!].$$

Special roles will be played by the sets

$$\mathtt{T}_n := [\![D^{(n)}]\!], \quad \mathtt{N}_n := [\![D^{(n-1,1)}]\!], \quad \mathtt{NT}_n := \mathtt{N}_n \cup \mathtt{T}_n.$$

The following simple observations turn out to be very useful:

Lemma 2.2. *We have:*

(i) *If $G \leq \mathsf{S}_n$, then $n \geq P(G)$. If G is not primitive on $\{1, \dots, n\}$ then $n > P(G)$.*
(ii) *If G is a simple group, then $P(G) > d(G)$.*

Proof. (i) follows by considering point stabilizers. (ii) comes on observing that $\mathrm{ind}_H^G \mathbb{C}_G$ contains some non-trivial irreducible components for any $H < G$. □

Note that $\mathfrak{b}_p(G) \leq \mathfrak{b}(G)$. We will need the following bound:

Lemma 2.3. [**52**, Theorem 2.2] *Let $G = SL_m(q)$ or $Sp_{2m}(q)$ with $m \geq 2$. If B is a Borel subgroup of G, then $\mathfrak{b}(G) \leq [G : B]$.*

2.2. **Representations of symmetric and alternating groups.** Recall the notation and the facts on representation theory of symmetric and alternating groups introduced in Section 1. In addition, we will denote by M^λ the permutation module and by S^λ the Specht module over the symmetric group S_n corresponding to a partition λ of n, see [**20**]. Occasionally, we will need the corresponding Specht module over $\mathbb{C}$, which we denote $S^\lambda_{\mathbb{C}}$. Thus S^λ is a reduction modulo p of $S^\lambda_{\mathbb{C}}$.

Lemma 2.4. *Suppose that $G \leq \mathsf{S}_n$. If $S^\lambda{\downarrow}_G$ is irreducible then so is $S^\lambda_{\mathbb{C}}{\downarrow}_G$.*

Proof. This follows on observing that reduction modulo p and restriction to a subgroup commute. $\square$

Lemma 2.5. *Let $p = 3$ and $n \equiv 0 \pmod 3$. If $G < \mathsf{S}_n$ and $D^{(n-2,2)}{\downarrow}_G$ is irreducible then $n \leq 24$.*

Proof. The assumptions imply that $D^{(n-2,2)} \cong S^{(n-2,2)}$. By Lemma 2.4, if $D^{(n-2,2)}{\downarrow}_G$ is irreducible then so is $S^{(n-2,2)}_{\mathbb{C}}{\downarrow}_G$. The result now follows from [**50**, Theorem 1]. $\square$

We next record some known results on dimensions of special irreducible modules for $p = 2$ and 3.

Lemma 2.6. *We have:*

(i) *If $p = 2$, then*

$$\dim D^{(n-2,2)} = \begin{cases} (n^2-5n+4)/2 & \text{if } n \equiv 0 \pmod 4, \\ (n^2-3n-2)/2 & \text{if } n \equiv 1 \pmod 4, \\ (n^2-5n+2)/2 & \text{if } n \equiv 2 \pmod 4, \\ (n^2-3n)/2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(ii) *If $p = 3$ then*

$$\dim D^{(n-2,2)} = \begin{cases} (n^2-3n)/2 & \text{if } n \equiv 0 \pmod 3, \\ (n^2-3n-2)/2 & \text{if } n \equiv 1 \pmod 3, \\ (n^2-5n+2)/2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

(iii) *If $p = 3$ then*

$$\dim D^{(n-2,1^2)} = \begin{cases} (n^2-5n+6)/2 & \text{if } 3 \mid n, \\ (n^2-3n+2)/2 & \text{if } 3 \nmid n. \end{cases}$$

Proof. This is well known and follows easily from [**20**, 24.15, 24.1]. $\square$

The following results will be needed to study irreducible restrictions to M_{24}:

Lemma 2.7. *Let $n = 24$, $p = 3$, and $\lambda \in \mathscr{P}^{(4)}(24) \setminus \mathscr{P}^{(1)}(24)$. Then the dimension of D^λ and the decomposition of $[S^\lambda]$ in the Grothendieck group are as follows*

(i) $\dim D^{(22,2)} = 252$ *and* $[S^{(22,2)}] = [D^{(22,2)}]$.
(ii) $\dim D^{(22,1^2)} = 231$ *and* $[S^{(22,1^2)}] = [D^{(22,1^2)}] + [D^{(23,1)}]$.
(iii) $\dim D^{(21,3)} = 1726$ *and* $[S^{(21,3)}] = [D^{(21,3)}] + [D^{(23,1)}]$.
(iv) $\dim D^{(21,2,1)} = 1540$ *and* $[S^{(21,2,1)}] = [D^{(21,2,1)}] + [D^{(21,3)}] + [D^{(22,1^2)}] + [D^{(23,1)}] + [D^{(24)}]$.
(v) $\dim D^{(20,4)} = 6854$ *and* $[S^{(20,4)}] = [D^{(20,4)}] + [D^{(21,3)}] + [D^{(23,1)}]$.
(vi) $\dim D^{(20,3,1)} = 26082$ *and* $[S^{(20,3,1)}] = [D^{(20,3,1)}]$.

(vii) $\dim D^{(20,2^2)} = 7315$ *and* $[S^{(20,2^2)}] = [D^{(20,2^2)}]+[D^{(20,4)}]+[D^{(21,3)}]+[D^{(21,2,1)}]+ 2[D^{(23,1)}] + [D^{(24)}]$.

(viii) $\dim D^{(20,2,1^2)} = 26334$ *and* $[S^{(20,2,1^2)}] = [D^{(20,2,1^2)}]$.

Proof. (i), (iii), (v) follow from [**20**, 24.15].
(ii) follows from [**20**, 24.1].
(iv), (vii) follow from [**21**, Appendix].
(vi), (viii) follow from Carter's Criterion, see [**22**]. □

Lemma 2.8. *Let $n = 24$, $p = 2$, and $\lambda \in \mathscr{P}^{(4)}(24) \setminus \mathscr{P}^{(1)}(24)$. Then the dimension of D^λ and the decomposition of $[S^\lambda]$ in the Grothendieck group are as follows*

(i) $\dim D^{(22,2)} = 230$ *and* $[S^{(22,2)}] = [D^{(22,2)}] + [D^{(23,1)}]$.

(ii) $\dim D^{(21,3)} = 1496$ *and* $[S^{(21,3)}] = [D^{(21,3)}] + [D^{(22,2)}] + [D^{(23,1)}]$.

(iii) $\dim D^{(21,2,1)} = 3520$ *and* $[S^{(21,2,1)}] = [D^{(21,2,1)}]$.

(iv) $\dim D^{(20,4)} = 7084$ *and* $[S^{(20,4)}] = [D^{(20,4)}] + [D^{(21,3)}] + [D^{(23,1)}]$.

(v) $\dim D^{(20,3,1)} = 17248$ *and* $[S^{(20,3,1)}] = [D^{(20,3,1)}] + [D^{(20,4)}] + [D^{(21,3)}] + [D^{(22,2)}] + [D^{(23,1)}] + 2[D^{(24)}]$.

Proof. (i), (ii), (iv) follow from [**20**, 24.15].
(iii) follows from Carter's Criterion, see [**22**].
(v) follow from [**19**, Theorem 7.1]. □

For partitions $\mu^1 = (\mu^1_1, \dots, \mu^1_{h_1}), \dots, \mu^k = (\mu^k_1, \dots, \mu^k_{h_k})$, we define the composition

$$(\mu^1, \dots, \mu^k) := (\mu^1_1, \dots, \mu^1_{h_1}, \dots, \mu^k_1, \dots, \mu^k_{h_k}).$$

Recalling (1.1), for a partition $\lambda = (\lambda_1, \dots, \lambda_h)$ of n, we now define its *double* $\mathrm{dbl}(\lambda) := (\beta_{\lambda_1}, \dots, \beta_{\lambda_h})$.

Lemma 2.9. [**4**, Theorem 1.1] *We have*

$$\mathscr{P}_2^{\mathsf{A}}(n) := \mathscr{P}_2(n) \cap \{\mathrm{dbl}(\lambda) \mid \lambda \in \mathscr{P}_2(n),\ \lambda_r \not\equiv 2 \pmod 4 \ \textit{for all } r\}.$$

We record for future reference:

Lemma 2.10. *Let $n \geq 5$ and $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}_p^{\mathsf{A}}(n)$. Then*

$$\lambda_1 \leq \begin{cases} (n+2)/2 & \textit{if } p = 2, \\ (n+p+1)/2 & \textit{if } p \geq 3. \end{cases}$$

Proof. For $p \geq 3$ this is [**37**, Proposition 4.3(i)], and for $p = 2$ this follows from Lemma 2.9. □

To analyze restriction to large doubly transitive subgroups, we will need to know that the trivial submodule $\mathbb{F}_{\mathsf{S}_{n-m}}$ appears in the restriction $D^\lambda{\downarrow}_{\mathsf{S}_{n-m}}$ for some reasonably small m. Recall the notation (2.1).

Theorem 2.11. *Let ℓ, L be integers satisfying $0 \leq 2\ell \leq L < n$, and $\lambda = (n-\ell, \mu) \in \mathscr{P}_p(n)$. Then $D^\lambda{\downarrow}_{\mathsf{S}_{n-L}}$ contains a trivial submodule.*

Proof. We will apply branching rules from [**28**] without further reference. We use induction on $\ell = |\mu|$, the theorem clearly holding if $\ell = 0$ since in that case $D^\lambda = D^{(n)} = \mathbb{F}_{\mathsf{S}_n}$. Let $\ell > 0$.

If λ has a good node below the first row then there exists $\nu \in \mathscr{P}_p(\ell-1)$ such that $(n-\ell,\nu) = (n-1-(\ell-1),\nu)$ is a p-regular partition of $n-1$ and $D^{(n-\ell,\nu)} \subseteq D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. By the inductive assumption, $D^{(n-\ell,\nu)}{\downarrow}_{\mathsf{S}_{n-L}}$ contains a trivial submodule.

Assume now that λ has no good node below the first row. Then $n-\ell = \lambda_1 > \lambda_2 = \mu_1$, $(n-1-\ell,\mu)$ is p-regular and $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}} \cong D^{(n-1-\ell,\mu)}$. If A is the second top removable node of λ then A is normal in $(n-1-\ell,\mu)$. So $(n-1-\ell,\mu)$ has a good node below the first row. In particular there exists $\nu \in \mathscr{P}_p(\ell-1)$ such that $(n-1-\ell,\nu) = (n-2-(\ell-1),\nu)$ is a p-regular partition of $n-2$ and $D^{(n-1-\ell,\nu)} \subseteq D^\lambda{\downarrow}_{\mathsf{S}_{n-2}}$. By the inductive assumption, $D^{(n-1-\ell,\nu)}{\downarrow}_{\mathsf{S}_{n-L}}$ contains a trivial submodule. □

In the following lemma we use functors $e_i : \mathbb{F}\mathsf{S}_n\text{-mod} \to \mathbb{F}\mathsf{S}_{n-1}\text{-mod}$ for which we refer the reader to [**31**]. The integer $\varepsilon_i(\lambda)$ is defined as $\max\{k \mid e_i^k D^\lambda \neq 0\}$.

Lemma 2.12. *Let $\lambda \in \mathscr{P}_p(n)$ with $\varepsilon_i(\lambda) = 2$. Let A and B be the i-normal nodes in λ with A below B. If λ_B is p-regular and the socle of $(e_i D^\lambda)/D^{\lambda_A}$ is isomorphic to D^{λ_B} then $e_i D^\lambda \cong D^{\lambda_A}|D^{\lambda_B}|D^{\lambda_A}$.*

Proof. This follows by self-duality of $e_i D^\lambda$, together with [**30**, Theorem 1.4]. □

We will need the following strengthening of Theorem 2.11 for the partition $(n-2,2)$:

Lemma 2.13. *If $n \geq 5$, then $D^{(n-2,2)}{\downarrow}_{\mathsf{S}_{n-3}}$ contains a trivial submodule, provided $p = 3$ and $n \equiv 0, 1 \pmod 3$, or $p = 2$ and $n \equiv 0, 1, 3 \pmod 4$.*

Proof. We will use branching rules from [**28**] without further reference. Assume first that $p = 3$ and $n \equiv 0, 1 \pmod 3$, or that $p = 2$ and $n \equiv 1, 3 \pmod 4$. Then $D^{(n-2,1)} \subseteq D^{(n-2,2)}{\downarrow}_{\mathsf{S}_{n-3}}$ and so we can conclude using Theorem 2.11. Assume now that $p = 2$ and $n \equiv 0 \pmod 4$, in which case $n \geq 6$. Then $D^{(n-2,2)}{\downarrow}_{\mathsf{S}_{n-1}} \cong D^{(n-3,2)}$. By [**54**], we have in the Grothendieck group

$$[D^{(n-3,2)}{\downarrow}_{\mathsf{S}_{n-2}}] = 2[D^{(n-2)}] + 2[D^{(n-3,1)}] + [D^{(n-4,2)}]$$

(omitting the last summand if $n = 6$). From Lemma 2.12 it follows that there exists $M \subseteq D^{(n-2,2)}{\downarrow}_{\mathsf{S}_{n-2}}$ with $M \sim D^{(n-3,1)}|D^{(n-2)}$. Considering block structure it then follows that $D^{(n-3)} \subseteq M{\downarrow}_{\mathsf{S}_{n-3}} \subseteq D^{(n-2,2)}{\downarrow}_{\mathsf{S}_{n-3}}$. □

To conclude the subsection, we record for future reference the following recognition result for basic spin modules:

Lemma 2.14. *Let $p = 2$, $n \geq 5$, and let $H = \mathsf{A}_n$ or S_n. Suppose that V is an irreducible $\mathbb{F}H$-module in which a 3-cycle t acts with exactly two eigenvalues. Then V is a basic spin module.*

Proof. In the case $H = \mathsf{S}_n$, the statement is [**57**, Theorem 8.1]. Suppose $H = \mathsf{A}_n$. If V extends to S_n, then we are done by the previous case. If V does not extend to S_n, then we can find an irreducible $\mathbb{F}\mathsf{S}_n$-module W such that $W{\downarrow}_G \cong V \oplus V^g$ for any $g \in \mathsf{S}_n \smallsetminus H$. Certainly we can choose such a g to be (a 2-cycle) centralizing t. Thus t has the same eigenvalues on V^g as on V, and so t acts quadratically on W. By the S_n-case, W is basic spin, and so is V. □

2.3. **Branching recognition results.** We begin by recording the following well-known branching recognition result for the modules in $\mathtt{NT}_n$.

Lemma 2.15. [**39**, Proposition 2.3] *Let $n \geq 6$ and D be an irreducible $\mathbb{F}\mathsf{S}_n$-module. Suppose that all composition factors of the restriction $D{\downarrow}_{\mathsf{S}_{n-1}}$ belong to $\mathtt{NT}_{n-1}$. Then $D \in \mathtt{NT}_n$, unless $n = 6$, $p = 3$ and $D \in [\![D^{(4,2)}]\!]$, or $n = 6$, $p = 5$ and $D \in [\![D^{(4,1^2)}]\!]$.*

Define

$$u := \begin{cases} 4 & \text{if } p = 3, \\ 3 & \text{otherwise.} \end{cases}$$

Lemma 2.16. *Let $n \geq 2u$ and D be an irreducible $\mathbb{F}\mathsf{S}_n$-module. If all composition factors of $D{\downarrow}_{\mathsf{S}_{u,u}}$ are of the form $D^\mu \boxtimes D^\nu$ with $D^\mu \in \mathtt{T}_u$ or $D^\nu \in \mathtt{T}_u$, then $D^\lambda \in \mathtt{NT}_n$.*

Proof. If $D \notin \mathtt{NT}_n$ then by Lemma 2.15, the restriction $D{\downarrow}_{\mathsf{S}_{2u}}$ has a composition factor not in $\mathtt{NT}_{2u}$. So it is enough to prove the lemma for $n = 2u$, which is an easy explicit check. □

Proposition 2.17. *Let $s \geq 2$, $m_1, \dots, m_s \geq u$ and $m_1 + \dots + m_s \leq n$, and D be an irreducible $\mathbb{F}\mathsf{S}_n$-module such that all composition factors of $D{\downarrow}_{\mathsf{S}_{m_1,\dots,m_s}}$ are of the form $D^{\mu^1} \boxtimes \dots \boxtimes D^{\mu^s}$ with at most one t such that $D^{\mu^t} \notin \mathtt{T}_{m_t}$. Then $D \in \mathtt{NT}_n$.*

Proof. By assumption, restricting further to the subgroups $\mathsf{S}_u \leq \mathsf{S}_{m_1}$ and $\mathsf{S}_u \leq \mathsf{S}_{m_2}$, we deduce that all composition factors of $D{\downarrow}_{\mathsf{S}_{u,u}}$ are of the form $D^\mu \boxtimes D^\nu$ with $D^\mu \in \mathtt{T}_u$ or $D^\nu \in \mathtt{T}_u$. So the proposition follows from Lemma 2.16. □

We need another special branching recognition result.

Lemma 2.18. *Let $p = 3$, $n \geq 8$ and D^λ be an irreducible $\mathbb{F}\mathsf{S}_n$-module. If all composition factors of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$ belong to $\mathtt{NT}_{n-1} \cup [\![D^{(n-3,1^2)}]\!]$ then $D^\lambda \in \mathtt{NT}_n \cup [\![D^{(n-2,1^2)}]\!]$.*

Proof. Note that

$$\mathtt{NT}_m \cup [\![D^{(m-2,1^2)}]\!] = \{D^{(m)}, D^{(\lceil m/2\rceil, \lfloor m/2\rfloor)}, D^{(m-1,1)}, D^{(\lceil (m-1)/2\rceil, \lfloor (m-1)/2\rfloor, 1)}, D^{(m-2,1^2)}, D^{(\lceil (m-2)/2\rceil, \lfloor (m-2)/2\rfloor, 2)}\}$$

for $m \geq 7$, see for example [**3**, Lemma 2.2]. Throughout the proof we will be using branching rules from [**28**] without further referring to them.

Case 1. $h(\lambda) \geq 4$. Then from [**2**, Lemma 4.7] that $D^\lambda{\downarrow}_{\mathsf{S}_6}$ contains a composition factor $D^{(2,2,1,1)}$. Hence $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$ has a composition factor of the form D^μ with $h(\mu) \geq 4$. In particular $D^\mu \notin \mathtt{NT}_{n-1} \cup [\![D^{(n-3,1^2)}]\!]$.

Case 2. $h(\lambda) = 3$ and $\lambda_3 \geq 3$. Then $n \geq 10$ and by [**2**, Lemma 4.13], $D^\lambda_{\mathsf{S}_{10}}$ contains a composition factor $D^{(4,3^2)}$. So if $n > 10$ then $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$ contains a composition factor D^μ with $\mu_3 \geq 3$, in particular $D^\mu \notin \mathtt{NT}_{n-1} \cup [\![D^{(n-3,1^2)}]\!]$. If $n = 10$ then $\lambda = (4, 3^2)$ and $\lambda^{\mathtt{M}} = (7, 2, 1)$, so $D^{(5,2^2)} \notin \mathtt{NT}_9 \cup [\![D^{(7,1,1)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_9}$ since $(5, 2^2) = (6, 2, 1)^{\mathtt{M}}$.

Case 3. $h(\lambda) = 3$, $\lambda_3 \leq 2$ and $\lambda_1 - \lambda_2 \geq 3$. We may assume that $\lambda_2 \geq 2$, since otherwise $\lambda = (n-2, 1^2)$. But then $D^{(\lambda_1-1,\lambda_2,\lambda_3)} \notin \mathtt{NT}_{n-1} \cup [\![D^{(n-3,1^2)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$.

Case 4. $h(\lambda) = 3$, $\lambda_3 \leq 2$ and $\lambda_1 - \lambda_2 \leq 2$. We may assume that $\lambda_1 - \lambda_2 = 2$, since otherwise $D^\lambda \in \mathtt{NT}_n \cup [\![D^{(n-2,1^2)}]\!]$. If $n > 8$ then $\lambda_2 \geq 3 > \lambda_3$, so $D^{(\lambda_1,\lambda_2-1,\lambda_3)} \notin$

$\mathtt{NT}_{n-1}\cup [\![D^{(n-3,1^2)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. If $n=8$ then $\lambda=(4,2,2)$ and $D^{(4,2,1)}\not\in \mathtt{NT}_7\cup [\![D^{(5,1,1)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_7}$.

Case 5. $h(\lambda)=2$ and $\lambda_1-\lambda_2\geq 3$. We may assume that $\lambda_2\geq 2$, in which case $D^{(\lambda_1-1,\lambda_2)}\not\in \mathtt{NT}_{n-1}\cup [\![D^{(n-3,1^2)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$.

Case 6. $h(\lambda)=2$ and $\lambda_1-\lambda_2\leq 2$. We may assume that $\lambda_1-\lambda_2=2$. Since $n\geq 8$ we have that $\lambda_2\geq 3$ and so $D^{(\lambda_1,\lambda_2-1)}\not\in \mathtt{NT}_{n-1}\cup [\![D^{(n-3,1^2)}]\!]$ is a composition factor of $D^\lambda{\downarrow}_{\mathsf{S}_{n-1}}$. □

Corollary 2.19. *Let $p=3$, $n=2^m$ for $m\geq 4$, and D be an irreducible $\mathbb{F}\mathsf{S}_n$-module. Suppose that all composition factors $D^\mu\boxtimes D^\nu$ of the restriction $D{\downarrow}_{\mathsf{S}_{n/2}\times\mathsf{S}_{n/2}}$ satisfy one of the following three conditions:*

(1) $D^\mu\cong D^\nu\in \mathtt{N}_{n/2}$,
(2) $D^\mu\in \mathtt{T}_{n/2}, D^\nu\in \mathtt{NT}_{n/2}\cup [\![D^{(n/2-2,1,1)}]\!]$,
(3) $D^\nu\in \mathtt{T}_{n/2}, D^\mu\in \mathtt{NT}_{n/2}\cup [\![D^{(n/2-2,1,1)}]\!]$.

Then $D\in \mathtt{NT}_n\cup [\![D^{(n-2,1,1)}]\!]$.

Proof. By assumption, all composition factors of $D{\downarrow}_{\mathsf{S}_{n/2}}$ belong to $\mathtt{NT}_{n/2}\cup[\![D^{(n/2-2,1^2)}]\!]$, and the result follows from Lemma 2.18. □

2.4. **Dimension bounds.** Recall the notation (2.1). We begin by recording James' lower bounds for $\dim D^{(n-\ell,\mu)}$ with $\ell\leq 4$:

Lemma 2.20. [**21**, Appendix] *Let $1\leq\ell\leq 4$, $\mu\in\mathscr{P}_p(\ell)$, and n be such that $(n-\ell,\mu)\in\mathscr{P}_p(n)$ with $\mu\vdash\ell$. Then*

$$\dim D^{(n-\ell,\mu)}\geq\begin{cases} n-2 & \text{if } \ell=1,\\ (n^2-5n+2)/2 & \text{if } \ell=2,\\ (n^3-9n^2+14n)/6 & \text{if } \ell=3.\\ (n^4-14n^3+47n^2-34n)/24 & \text{if } \ell=4.\end{cases}$$

Set

$$\delta_p:=\begin{cases} 0 & \text{if } p\neq 2,\\ 1 & \text{if } p=2.\end{cases}$$

For integers $\ell\geq 0$ and n we define the rational numbers

$$\begin{aligned} C^p_\ell(n) &:= p^\ell\binom{n/p-\delta_p}{\ell}\\ &=\frac{1}{\ell!}\prod_{i=0}^{\ell-1}(n-(\delta_p+i)p)\\ &=\begin{cases}\frac{n(n-p)(n-2p)\cdots(n-(\ell-1)p)}{\ell!} & \text{if } p>2,\\ \frac{(n-p)(n-2p)\cdots(n-\ell p)}{\ell!} & \text{if } p=2.\end{cases}\end{aligned}$$

The following result substantially develops [**21**] (the upper bound $\dim D^\lambda\leq n^\ell$ is trivial, since D^λ is contained in the permutation module M^λ, which has dimension at most $n!/(n-\ell)!\leq n^\ell$).

Theorem 2.21. [**33**, Theorem A] *Let $\ell\geq 4$, $n\geq p(\delta_p+\ell-2)$, and $\lambda=(n-\ell,\mu)\in\mathscr{P}_p(n)$ for some $\mu\in\mathscr{P}_p(\ell)$. Then*

$$n^\ell\geq\dim D^\lambda\geq C^p_\ell(n).$$

While Theorem 2.21 requires that n is relatively large compared to ℓ, we also have the following universal lower bounds which strengthens [**16**, Theorem 5.1]:

Theorem 2.22. [**33**, Theorems B, C] *Let* $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}_p(n)$, *and* $k := \max\{\lambda_1, \lambda_1^{\mathtt{M}}\}$.

(i) *If* $p = 2$, *then* $\dim D^\lambda \geq 2^{n-k}$.

(ii) *If* $p > 2$ *let* $\lambda^{\mathtt{M}} = (\lambda_1^{\mathtt{M}}, \lambda_2^{\mathtt{M}}, \dots)$, *and let* m *be minimal such that* $D^\lambda{\downarrow}_{\mathsf{S}_{n-m}}$ *contains a* 1*-dimensional submodule. Put* $k := \max\{\lambda_1, \lambda_1^{\mathtt{M}}\}$ *and* $t := \max\{n-k, m\}$. *Then*

$$\dim D^\lambda \geq 2 \cdot 3^{(t-2)/3}.$$

In particular, for all p *and* $n \geq 5$, *we have* $\dim D^\lambda \geq 2^{(n-k)/2}$.

The following technical result will be used to study irreducible restrictions to doubly transitive subgroups G with $\mathrm{soc}\,(G) \cong PSL(m,q)$ and $Sp_{2m}(2)$ in Sections 6 and 7.

Proposition 2.23. *Let* $n \geq 324$, $p = 2$ *or* 3, *and define* ℓ *from* $\max(\lambda_1, \lambda_1^{\mathtt{M}}) = n - \ell$. *If*

$$\dim D^\lambda \leq n^{\frac{1}{2}\log_2 n + 1}$$

then $\ell \leq 0.7 \log_2 n + 1.4$.

Proof. Set $L(n) := \frac{1}{2}\log_2 n + 1$. We need to show that $\ell \leq 1.4L(n)$. As $1.4L(324) > 7$, we may assume that $\ell > 7$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda = (n-\ell, \mu)$ for a partition μ of ℓ. By Theorem 2.22 and the assumption, we now have

$$2^{\ell/2} \leq \dim D^\lambda \leq n^{L(n)},$$

and so

$$\ell \leq 2L(n)\log_2 n = (\log_2 n + 2)\log_2 n =: L_1(n). \tag{2.24}$$

As $n \geq 324$, we certainly have that $\ell \leq L_1(n) < \frac{1}{3}n + 2$, whence $n \geq p(\delta_p + \ell - 2)$, and Theorem 2.21 applies to give

$$\dim D^\lambda \geq C_\ell^p(n) > \frac{(n+3-3\ell)^\ell}{\ell!} > \left(\frac{2(n+3)}{\ell} - 6\right)^\ell, \tag{2.25}$$

where we have used $\ell! < (\ell/2)^\ell$ for $\ell \geq 6$ to get the last inequality.

If $\ell > cL(n)$ for some $c > 0$, we get

$$n^{L(n)} > \left(\frac{2(n+3)}{\ell} - 6\right)^{cL(n)},$$

and so

$$\ell > f(n,c) := \frac{2(n+3)}{n^{1/c} + 6}.$$

We have therefore shown that

$$\text{If } f(n,c) \geq \ell \ \text{ for some } c > 0, \text{ then } \ell \leq cL(n). \tag{2.26}$$

We will use this implication repeatedly to prove $\ell \leq 1.4L(n)$.

First we take $c = 16$. By the assumption on n and (2.24), $f(n,16) > L_1(n) \geq \ell$. Hence, (2.26) implies that

$$\ell \leq L_2(n) := 16L(n) = 8\log_2 n + 16.$$

Next we take $c = 9$ and note that $f(n, 9) > L_2(n) \geq \ell$ for $n \geq 324$ (this is the only place where smaller n would not work). Applying (2.26), we deduce that

$$\ell \leq L_3(n) := 9L(n) = 4.5 \log_2 n + 9.$$

Now take $c = 2.8$ and note that $f(n, 2.8) > L_3(n) \geq \ell$ for $n \geq 324$. Applying (2.26), we now obtain

$$\ell \leq L_4(n) := 2.8L(n) = 1.4 \log_2 n + 2.8.$$

Next we take $c = 1.6$ and note that $f(n, 1.6) > L_4(n) \geq \ell$ for $n \geq 324$. Using (2.26), we deduce that

$$\ell \leq L_5(n) := 1.6L(n) = 0.8 \log_2 n + 1.6.$$

Finally, we take $c = 1.4$ and note that $f(n, 1.4) > L_5(n) \geq \ell$ for $n \geq 324$. Again using (2.26), we conclude that $\ell \leq 1.4L(n)$, as stated. □

We now establish some dimension recognition results for modules in $\mathscr{L}^{(\ell)}(n)$ for small ℓ.

Lemma 2.27. *If $n \geq 17$ and $\dim E^{\mu}_{(\pm)} < (n^2 - 5n + 2)/2$, then $\mu \in \mathscr{L}^{(1)}(n)$.*

Proof. The statement follows from [**17**, Lemma 6.1]. □

The following proposition extends [**8**, Lemma 1.20]:

Proposition 2.28. *The following lower bounds hold.*

(i) *Let $n \geq 13$, and assume in addition that $n \geq 23$ if $p = 2$. Then for $\lambda \in \mathscr{P}_p(n)$, we have either $\lambda \in \mathscr{L}^{(2)}(n)$ or*

$$\dim D^\lambda \geq (n^3 - 9n^2 + 14n)/6.$$

(ii) *Suppose that $p \geq 3$ and $n \geq 17$. Then for $\lambda \in \mathscr{P}_p(n)$, we have either $\lambda \in \mathscr{L}^{(3)}(n)$ or*

$$\dim D^\lambda \geq (n^4 - 14n^3 + 47n^2 - 34n)/24.$$

Proof. By Lemma 2.20, if $\lambda \in \mathscr{L}^{(3)}(n) \smallsetminus \mathscr{L}^{(2)}(n)$, then $\dim D^\lambda \geq (n^3 - 9n^2 + 14n)/6$. Now assume that $\lambda \notin \mathscr{L}^{(3)}(n)$, and in addition D^λ is not basic spin if $p = 2$. Then $\dim D^\lambda \geq (n^4 - 14n^3 + 47n^2 - 34n)/24$ by [**48**, (6.2)]. In the case where D^λ is basic spin and $n \geq 23$, one can check directly that $\dim D^\lambda \geq (n^3 - 9n^2 + 14n)/6$. □

Remark 2.29. The statement of Proposition 2.28(i) does not hold for $p = 2$ and $n = 22$, a counterexample given by the basic spin module $D^{\beta_{22}}$. However, a similar argument shows that for $n \geq 17$ we have either $\lambda \in \mathscr{L}^{(2)}(n)$ or $\dim D^\lambda > 2(n^3 - 9n^2 + 14n)/25$.

3. Order bounds and dimension bounds

3.1. **Subgroups of large order.** First we extend Propositions 6.1 and 6.2 of [**26**], following mostly the arguments given therein.

Proposition 3.1. *Let $S < \mathsf{A}_n$ be a non-abelian simple subgroup such that*

$$|\mathrm{Aut}(S)| \geq 2^{n/2-4}. \tag{3.2}$$

Then one of the following happens:

(i) *$S \cong \mathsf{A}_m$ with $m < n$. Moreover, if $m \geq 12$, then S is intransitive, and each of its orbits on $\{1, 2, \ldots, n\}$ has length 1 or m.*

(ii) $S \cong PSL_m(q)$ *with* $(m, q) = (2, \le 37)$, $(3, \le 5)$, $(4, 3)$, $(5, 2)$, *or* $(6, 2)$.
(iii) $S \cong SU_3(3)$ *or* $SU_4(2) \cong PSp_4(3)$.
(iv) $S \cong Sp_6(2)$.
(v) $S \cong M_{11}$, M_{12}, M_{22}, M_{23}, *or* M_{24}.

Proof. (a) First we consider the case $S \cong \mathsf{A}_m$. Note that $m < n$ as $S \neq \mathsf{A}_n$. Assume furthermore that $m \geq 12$ and S has an orbit of length $k \neq 1, m$ on $\{1, 2, \ldots, n\}$. As in [**26**], it follows that $n \geq k \geq m(m-1)/2$. Now one can check that $2^{m(m-1)/4-4} > m!$ for $m \geq 12$, a contradiction.

For the remaining cases, recall from Lemma 2.2 that

$$n \geq P(S) > d(S).$$

Now, if S is one of the 26 sporadic simple groups and not listed in (v), then using the exact value of $P(S)$ given in [**11**] (or of $d(S)$, if $P(S)$ was not listed therein) one can check that (3.2) cannot hold.

(b) Assume now that S is a classical group. Then $|\mathrm{Aut}(S)|$ and $P(S)$ are listed in Tables 5.1.A and 5.2.A of [**25**].

First suppose that $S = PSL_m(q)$. Then $|\mathrm{Aut}(S)| < q^{m^2}$. If $m \geq 4$ (and $S \not\cong \mathsf{A}_8$), then $P(S) = (q^m - 1)/(q - 1)$, and one can check that (3.2) can hold only when $(m, q) = (4, 3)$, $(5, 2)$, or $(6, 2)$. If $(m, q) = (3, \geq 7)$ or $(2, \geq 41)$, then again $P(S) = (q^m - 1)/(q - 1)$ and one checks that (3.2) is violated.

Next suppose that $S = PSU_m(q)$. Then we again have $|\mathrm{Aut}(S)| < q^{m^2}$. If $m \geq 5$, then $P(S) > q^{2m-3}$ and one checks that (3.2) cannot hold. If $(m, q) = (4, \geq 3)$ then $P(S) = (q + 1)(q^3 + 1)$. If $m = 3$, then $P(S) = q^3 + 1$ for $q \geq 7$ or $q = 4$, and 50 if $q = 5$. In all these cases, (3.2) is violated.

Suppose now that $S = PSp_{2m}(q)$ with $m \geq 2$, or $\Omega_{2m+1}(q)$ with $m \geq 3$. If $m \geq 3$, then $|\mathrm{Aut}(S)| < q^{m(2m+1)+1}/2$ whereas $P(S) \geq q^{m-1}(q^m - 1)$, and so (3.2) can possibly hold only when $(m, q) = (3, 2)$. Similarly, if $(m, q) = (2, \geq 4)$, then $P(S) = (q^4 - 1)/(q - 1)$, and (3.2) cannot hold.

Suppose $S = P\Omega^{\pm}_{2m}(q)$ with $m \geq 4$. If $m > 4$ or if $S \not\cong P\Omega^+_8(q)$, then $|\mathrm{Aut}(S)| < q^{m(2m-1)+1}$ whereas $P(S) > q^{2m-2}$, and so (3.2) is impossible. Similarly, (3.2) rules out the remaining case $S = P\Omega^+_8(q)$.

(c) Finally, assume that S is an exceptional group of Lie type. The cases $S = F_4(2)$, ${}^2F_4(2)'$, ${}^3D_4(2)$, $G_2(3)$, $G_2(4)$, or ${}^2B_2(8)$ can be ruled out directly using [**11**]. In all other cases, we can use the Landazuri-Seitz-Zalesskii lower bound on $d(S)$ as recorded in [**25**, Table 5.3.A] to check that (3.2) cannot hold. □

The following known lemma follows from the O'Nan-Scott theorem, see e.g. [**41**]:

Lemma 3.3. *Suppose $G < \mathsf{S}_n$ is a primitive subgroup with an abelian minimal normal subgroup S. Then $n = r^m$ is a power of some prime r, and G is a subgroup of the affine group $AGL(V) = AGL_m(r)$ in its action on the points of $V = \mathbb{F}_r^m$.*

Proposition 3.4. *Let $G < \mathsf{S}_n$ be a primitive subgroup, not containing A_n and such that*

$$|G| \geq 2^{n/2-4}. \tag{3.5}$$

Then one of the following happens:

(i) $\mathrm{soc}\,(G)$ *is elementary abelian of order $n = r^k$, with $(r, k) = (2, \leq 6)$, $(3, \leq 3)$, or $(5, 2)$.*

(ii) *$S \leq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S. Furthermore, either $S \cong \mathsf{A}_m$ with $m \leq 11$, or S satisfies Proposition 3.1(ii)–(v).*
(iii) *$\mathrm{soc}\,(G) = S \times S$ for a non-abelian simple group $S \leq \mathsf{A}_a$, $5 \leq a \leq 9$, and $n = a^2$.*

Proof. We apply the O'Nan-Scott theorem in the version given in [**41**]. First suppose that $\mathrm{soc}\,(G) \cong \mathsf{C}_r^k$, so that $n = r^k$ for a prime r. Then Lemma 3.3 shows that $G \leq AGL_k(r)$ and so $|G| < r^{k^2+k}$. A direct computation shows that (3.5) can hold only in the cases listed in (i).

Next assume that $\mathrm{soc}\,(G) = S$ is non-abelian simple. Then $S \leq G \leq \mathrm{Aut}(S)$ and S is transitive on $\{1, 2, \dots, n\}$. Now we can apply Proposition 3.1 to arrive at (ii).

In the remaining cases, $\mathrm{soc}\,(G) = S^k$ for a non-abelian simple group S and $k \geq 2$, and G is of type III(a), III(b), or III(c) in the notation of [**41**]. Suppose G is of type III(b), so that $n = a^b$ with $a \geq 5$, $b \geq 2$, and $G \leq \mathsf{S}_a \wr \mathsf{S}_b$. In this case, $b \leq \log_5 n$ and

$$(a!)^b \cdot b! \leq (a^a)^b b^b \leq n^a b^b \leq n^{\sqrt{n}} \cdot (\log_5 n)^{\log_5 n}.$$

Now if $n \geq 318$, then

$$2^{n/2-4} > n^{\sqrt{n}} \cdot (\log_5 n)^{\log_5 n},$$

violating (3.5). The cases where $n = a^b \leq 317$ can now be checked directly to show that $b = 2$ and $5 \leq a \leq 9$. This implies that $k = 2$, $S \leq \mathsf{A}_a$, and we arrive at (iii).

Suppose G is of type III(a). Then $n = |S|^{k-1}$ and $G \leq S^k \cdot (\mathsf{S}_k \times \mathrm{Out}(S))$. Since $|S| \geq 60$, we can check that

$$2^{|S|^{k-1}/2-4} > |S|^{k+1} \cdot k!. \tag{3.6}$$

As $|\mathrm{Out}(S)| < |S|$, (3.5) cannot hold.

Finally, assume that G is of type III(c). Then $n = |S|^k$ and

$$G \leq \mathrm{Aut}(S^k) \cong S^k \cdot (\mathrm{Out}(S)^k \rtimes \mathsf{S}_k).$$

Now (3.6) implies that

$$2^{n/2-4} \geq 2^{|S|^k/2-4} > (|S|^{k+1} \cdot k!)^2 > |S|^{2k} \cdot k! > |G|,$$

a contradiction. □

3.2. **Irreducible restrictions for some special modules and groups.** Now we prove main results of this section.

Theorem 3.7. *Let $p = 2$ or 3, $H = \mathsf{A}_n$ or S_n with $n \geq 5$, and V be an irreducible $\mathbb{F}H$-module. Let $G < H$ be a primitive subgroup not containing A_n, with $S := \mathrm{soc}\,(G)$, such that $V{\downarrow}_G$ is irreducible. Assume in addition that either V is a basic spin module in characteristic 2, or $H = \mathsf{A}_n$ and V does not extend to S_n. Then one of the following happens:*

(i) *S is elementary abelian of order $n = r^k$, with $(r, k) = (2, \leq 6)$, $(3, \leq 3)$, or $(5, 2)$.*
(ii) *$S \in \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}\}$, and G is doubly transitive.*
(iii) *$H = \mathsf{A}_9$, $p = 2$, $SL_2(8) \trianglelefteq G \leq SL_2(8) \rtimes \mathsf{C}_3$, V is the basic spin module of dimension 8 whose Brauer character takes value -1 on elements of order 9 of S.*
(iv) *$n = 10$, $p = 2$, $S \cong \mathsf{A}_6$, $G \not\leq PGU_2(9)$, V is basic spin of dimension 16.*
(v) *$H = \mathsf{A}_6$, $p = 3$, $G \cong \mathsf{A}_5$, $\dim V = 3$.*
(vi) *$n = 6$, $p = 2$, $S \cong \mathsf{A}_5$, V is basic spin of dimension 4.*

Moreover, in the cases described in (iii)-(vi) *the restriction $V{\downarrow}_G$ is indeed irreducible.*

Proof. (a) Applying Lemma 2.27 and Proposition 2.28(i) we deduce

$$\dim V \geq \begin{cases} (n^2-5n+2)/2, & \text{if } n \geq 17, \\ (n^3-9n^2+14n)/12, & \text{if } n \geq 23. \end{cases} \tag{3.8}$$

(b) Let λ be a p-regular partition of n such that V is an irreducible constituent of $D^\lambda{\downarrow}_H$. Note that $\lambda = \lambda^{\mathtt{M}}$ if $p=3$. If $p=2$ and V is basic spin then $\dim V \geq 2^{(n-4)/2}$. If V does not extend to S_n, then by Lemma 2.10,

$$\lambda_1 \leq \begin{cases} (n+2)/2 & \text{if } p=2, \\ (n+4)/2 & \text{if } p=3. \end{cases}$$

By Theorems 2.22 and 2.22, we have in all cases

$$\dim V \geq \begin{cases} 2^{(n-8)/4}, & \text{if } p=3, \\ 2^{(n-4)/2}, & \text{if } p=2. \end{cases} \tag{3.9}$$

Since $V{\downarrow}_G$ is irreducible, it follows that $|G| > \dim(V)^2 \geq 2^{n/2-4}$, and so one of the conclusions of Proposition 3.4 must hold. The case (i) of Proposition 3.4 leads to the exception (i) of the theorem.

(c) Suppose we are in the case (iii) of Proposition 3.4. As mentioned in the proof of Proposition 3.4, we have that $G \leq \mathrm{Aut}(S^2) \cong \mathrm{Aut}(S)^2 \rtimes \mathsf{C}_2$, and so

$$\dim V \leq \mathfrak{b}_p(\mathrm{Aut}(S)^2 \rtimes \mathsf{C}_2) \leq 2\mathfrak{b}_p(\mathrm{Aut}(S))^2.$$

Checking $\mathfrak{b}_p(\mathrm{Aut}(S))$ using [**14**], we see that

$$\dim V \leq \begin{cases} 2\cdot 189^2, & a=9, \\ 2\cdot 80^2, & a=8, \\ 2\cdot 20^2, & a=7, \\ 2\cdot 16^2, & a=6, \\ 2\cdot 6^2, & a=5. \end{cases}$$

As $n=a^2$, this contradicts (3.9) when $7 \leq a \leq 9$ and (3.8) when $a=5,6$.

(d) Finally, we consider the case (ii) of Proposition 3.4, so that $S \trianglelefteq G \leq \mathrm{Aut}(S)$, and either $S \cong \mathsf{A}_m$ with $m \leq 11$, or S satisfies Proposition 3.1(ii)–(v). As G is primitive, $S = \mathrm{soc}\,(G)$ is transitive on $\{1,2,\dots,n\}$. Also $n \geq P(S)$ by Lemma 2.2.

(d1) Assume $S = Sp_6(2)$, so that $G=S$ and $n \geq P(S) = 28$ [**11**]. On the other hand, $\dim V \leq \mathfrak{b}(G) = 512$, contradicting (3.8).

(d2) $S = SU_3(3)$. The argument is similar to (d1) but using $P(S)=28$ and $\mathfrak{b}(G) \leq 64$.

(d3) $S = SU_4(2)$. The argument is similar to (d1) but using $P(S)=27$ and $\mathfrak{b}(G) \leq 80$.

(d4) Suppose $S = PSL_2(q)$ with $q = r^f \leq 37$ for a prime r, so that $\mathrm{Aut}(S) \cong PGL_2(q) \rtimes \mathsf{C}_f$.

(d4.1) If $q \geq 16$, then $n \geq P(S) = q+1 \geq 17$, see [**25**, Table 5.2.A], and by [**42**] we have

$$\dim V \leq \mathfrak{b}(G) \leq f\mathfrak{b}(PGL_2(q)) = f(q+1) \leq q(q+1)/4 < (n^2-5n+2)/2,$$

which contradicts (3.8).

(d4.2) If $q=13$ (resp. 11) then $n \geq 14$ (resp. $n \geq 11$), and $\dim V \leq \mathfrak{b}(G) = q+1$. In any of these two cases, we see that $n=14$ (resp. $n \in \{11,12\}$). Furthermore, since V has dimension $\leq q+1$, it extends to S_n (see [**14**]) and is not basic spin, a contradiction.

(d4.3) Suppose $S = PSL_2(9) \cong \mathsf{A}_6$. As $S \neq \mathsf{A}_n$, we have $n \in \{10, 15\}$ or $n \geq 20$, and $\dim V \leq \mathfrak{b}_p(\mathrm{Aut}(S)) \leq 16$, see [**11**] and [**14**]. It follows from (3.8) that $n \in \{10, 15\}$. In this case, any irreducible $\mathbb{F}\mathsf{A}_n$-module of dimension ≤ 16 extends to S_n, a contradiction. On the other hand, the basic spin modules of H are of dimension 16, and their Brauer characters take value -2 at elements $x \in G$ of order 3 and value 1 at elements $y \in G$ of order 5 (see [**14**]), hence they are irreducible over G whenever $G \not\leq S \cdot 2_2 \cong PGU_2(9)$, leading to the exception (iv).

(d4.4) Suppose $S = SL_2(8)$. Here, $n = 9$ or $n \geq 28$, and $\dim V \leq \mathfrak{b}(G) \leq 27$. It follows from (3.8) that $n = 9$. In this case, the only irreducible $\mathbb{F}\mathsf{A}_9$-modules of dimension ≤ 28 that do not extend to S_9 are the two 2-modular basic spin modules of dimension 8, and one can check that exactly one of them is irreducible over G (namely the one whose Brauer character takes value -1 on elements of order 9 in S), leading to the exception (iii).

(d4.5) Suppose $S = PSL_2(7) \cong SL_3(2)$. Here, $n \in \{7, 8, 14\}$ or $n \geq 21$, and $\dim V \leq \mathfrak{b}(G) \leq 8$. It follows from (3.8) that $n \in \{7, 8\}$. In these cases, the only irreducible $\mathbb{F}\mathsf{A}_n$-modules of dimension ≤ 8 that do not extend to S_n are the 2-modular basic spin modules of dimension 4, which restrict reducibly to G. Likewise, the basic spin modules of S_n are reducible over G (as can be seen by checking the value of the Brauer characters at elements of order 3 in G, see [**14**]).

(d4.6) Suppose $S = PSL_2(5) \cong \mathsf{A}_5$. As $S \neq \mathsf{A}_n$, we have $n \in \{6, 10\}$ or $n \geq 15$, and $\dim V \leq \mathfrak{b}(G) \leq 6$, see [**11**]. It follows that $n = 6$. In this case, the irreducible $\mathbb{F}\mathsf{A}_6$-modules of dimension ≤ 6 that do not extend to S_6 are the 3-modular modules of dimension 3, and they restrict irreducibly to G, yielding the exception (v). Next, the basic spin modules of $H = \mathsf{A}_6$ or S_6 are of dimension 4, and their Brauer character takes value 1 at elements of order 3 in G, whence they are irreducible over G, leading to the exception (vi).

(d5) Suppose $S = SL_3(q)$ with $q \leq 5$. The case $q = 2$ is treated in (d4.5).

(d5.1) If $q = 5$, then $n \geq P(S) = 31$ and

$$\dim V \leq \mathfrak{b}(G) \leq 310 < (n^2 - 5n + 2)/2,$$

contradicting (3.8).

(d5.2) If $q = 4$, then $n \geq P(S) = 21$ and

$$\dim V \leq \mathfrak{b}(G) \leq \mathfrak{b}(PGL_3(4) \cdot \mathsf{C}_2^2) \leq 420 < (n^3 - 9n^2 + 14n)/12.$$

This contradicts (3.8) unless $n \leq 22$. As n divides $|S|$, we conclude that $n = 21$, whence $G \leq PGL_3(4) \rtimes \mathsf{C}_2$ and so $\mathfrak{b}(G) \leq 128 < (n^2 - 5n + 2)/2$, again contradicting (3.8).

(d5.3) If $q = 3$, then $n \geq P(S) = 13$, and

$$\dim V \leq \mathfrak{b}(G) \leq 52 < (n^2 - 5n + 2)/2.$$

This contradicts (3.8) unless $n \leq 16$. Inspecting the list of maximal subgroups of S [**11**], we conclude that $n = 13$, whence $G = S$ and $\dim V \leq \mathfrak{b}_p(S) \leq 27$. But then V extends to S_{13} and is not basic spin.

(d6) Suppose $S = PSL_4(3)$. Then $n \geq P(S) = 40$, and

$$\dim V \leq \mathfrak{b}(\mathrm{Aut}(S)) \leq 4\mathfrak{b}(S) = 4160 < (n^3 - 9n^2 + 14n)/12,$$

violating (3.8).

(d7) Suppose $S = SL_5(2)$. Then $n \geq P(S) = 31$, and

$$\dim V \leq \mathfrak{b}_p(\mathrm{Aut}(S)) \leq 1024 < (n^3 - 9n^2 + 14n)/12,$$

violating (3.8).

(d8) Suppose $S = SL_6(2)$. Then $P(S) = 63$ [**25**, Table 5.2.A] and $\mathfrak{b}(G) \leq 66960$ [**14**]. Hence $n \leq 72$ by (3.9). Note that if K is a proper subgroup of S, then either $[S : K] \geq 651$, or K is contained in a maximal subgroup $M \cong \mathsf{C}_2^5 \rtimes SL_5(2)$ of index 63 in S, see [**14**]. Hence, if we take $K = \mathrm{Stab}_S(1)$ in the action of S on $\{1, 2, \ldots, n\}$ (so that $[S : K] = n$), then we must have that $n = [S : M] = 63$, and in fact S acts on n points $\{1, 2, \ldots, n\}$ via its action on 63 lines or 63 hyperplanes of $\mathbb{F}_2^6$. None of these actions can be extended to $\mathrm{Aut}(S)$, so we have that $G = S$. If $p = 2$, then (3.9) implies that $\dim V > 2^{29} > \mathfrak{b}(G)$, a contradiction. Suppose $p = 3$. Then $\lambda_1 \leq (n + 4)/2$ by Lemma 2.10, whence $\lambda_1 \leq 33$. On the other hand, Proposition 2.28(ii) implies that $\lambda \in \mathscr{L}^{(3)}(n)$, and so $\lambda_1 \geq 60$, a contradiction.

(e) Suppose that $S \cong \mathsf{A}_m$ with $m \leq 11$. The case $m = 5$ and 6 are considered in (d4.6) and (d4.3), respectively.

(e1) Let $m = 11$. As $S \neq \mathsf{A}_n$, we have $n \geq 55$ and $\dim V \leq \mathfrak{b}(\mathsf{S}_{11}) = 2310$, see [**11**]. This contradicts (3.8).

(e2) Let $m = 10$. This case is treated similarly to (e1) observing that $n \geq 45$ and $\dim V \leq \mathfrak{b}(\mathsf{S}_{10}) = 768$.

(e3) Let $m = 9$. This case is treated similarly to (e1) observing that $n \geq 36$ and $\dim V \leq \mathfrak{b}(\mathsf{S}_9) = 216$.

(e4) Let $m = 8$. As $S \neq \mathsf{A}_n$, we have $n = 15$ or $n \geq 28$, and $\dim V \leq \mathfrak{b}(\mathsf{S}_8) = 90$, see [**11**]. It follows from (3.8) that $n = 15$ and $G = S$. Now, the only irreducible $\mathbb{F}\mathsf{A}_{15}$-modules of dimension ≤ 90 that does not extend to S_{15} are the two basic spin modules of dimension 64. If φ is the Brauer character of any of these two modules and $g \in S$ is a 3-cycle, then g becomes a disjoint product of five 3-cycles in the doubly transitive embedding $S \hookrightarrow \mathsf{A}_{15}$, and so $\varphi(g) = -2$. However, the unique irreducible 2-Brauer character of S of degree 64 takes value 4 at g, and so $\varphi{\downarrow_G}$ is reducible.

(e4) Let $m = 7$. As $S \neq \mathsf{A}_n$, we have $n = 15$ or $n \geq 21$, and $\dim V \leq \mathfrak{b}(\mathsf{S}_7) = 35$, see [**11**]. It follows from (3.8) that $n = 15$. Now, all irreducible $\mathbb{F}\mathsf{A}_{15}$-modules of dimension ≤ 35 extend to S_{15} and are not basic spin.

(f) Let S be a Mathieu group. Suppose that the conclusion (ii) of the current theorem does *not* hold. Then, if $S = M_{24}$, we have by [**11**] that $n \geq 276$ and $\dim V \leq \mathfrak{b}(G) = 10395$, contradicting (3.8). If $S = M_{23}$, we have by [**11**] that $n \geq 253$ and $\dim V \leq \mathfrak{b}(G) = 2024$, again contradicting (3.8). The same argument applies to $S = M_{22}$, where we have $n \geq 77$ and $\dim V \leq \mathfrak{b}(\mathrm{Aut}(S)) = 560$, to $S = M_{12}$, for which we have $n \geq 66$ and $\dim V \leq \mathfrak{b}(\mathrm{Aut}(S)) = 176$, and to $S = M_{11}$, for which we have $n \geq 55$ and $\dim V \leq \mathfrak{b}(G) = 55$. $\square$

Note that cases (i) and (ii) of Theorem 3.7 will be settled in Theorem 5.13 and Theorem 4.1.

Proposition 3.10. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, $H = \mathsf{A}_n$ or S_n, and V be an irreducible constituent of $D^\lambda{\downarrow_H}$, with $\dim V > 1$. Suppose that $G < H$ is a doubly transitive subgroup such that $S := \mathrm{soc}\,(G) < \mathsf{A}_n$ is one of the following simple groups:*

(a) A_m *with* $5 \leq m \leq 7$*;*

(b) $PSL_2(q)$ *with* $7 \leq q \leq 9$*;*

(c) $PSL_3(q)$ *with* $2 \leq q \leq 4$*;*
(d) $SL_4(2) \cong \mathsf{A}_8$.

Then $V{\downarrow_G}$ *is irreducible if and only if one of the following happens:*

(i) $\lambda \in \mathscr{L}^{(1)}(n)$*; furthermore,* (G,n,p) *fulfills the conditions set in Table II.*
(ii) $S \cong \mathsf{A}_5 \cong SL_2(4) \cong PSL_2(5)$*,* $n=6$*,* $p=3$*,* $\lambda = \lambda^{\mathtt{M}} = (4,1^2)$*, and* $(G,H,\dim V) = (\mathsf{A}_5,\mathsf{A}_6,3)$ *or* $(\mathsf{S}_5,\mathsf{S}_6,6)$.
(iii) $S \cong \mathsf{A}_6 \cong PSL_2(9)$*,* $n=10$*,* $p=2$*,* $V = D^{(6,4)}{\downarrow_H}$ *of dimension* 16*, and* $G \leq \mathrm{Aut}(S)$ *but* $G \not\leq PSU_2(9) = S \cdot 2_2$.
(iv) $S = SL_2(8)$*,* $G=S$ *or* $G = \mathrm{Aut}(S) \cong P\Gamma L_2(8)$*,* $H = \mathsf{A}_9$*,* $p=2$*,* $\dim V = 8$*, and* V *is the only one of* $E^{\beta_9}_{\pm}$ *whose Brauer character takes value* -1 *at elements of order* 9 *in* $SL_2(8)$.
(v) $S = SL_2(8)$*,* $G = \mathrm{Aut}(S) \cong P\Gamma L_2(8)$*,* $n=9$*,* $p=3$*,* $\dim V = 27$*, and* λ *or* $\lambda^{\mathtt{M}}$ *equals* $(7,2)$.

Proof. The 'if-part' in (i) follows [**47**], and in (ii)–(v) from [**14**]. Conversely, suppose $V{\downarrow_G}$ is irreducible. We may assume that $\lambda \notin \mathscr{L}^{(1)}(n)$ again by [**47**].

If $S = \mathsf{A}_5 \cong PSL_2(5)$ then $n=6$, and by [**14**], $\dim V \leq 4$ if $p=2$ and $\dim V \leq 6$ if $p=3$. As $\lambda \notin \mathscr{L}^{(1)}(n)$, we deduce that $p=3$, $\dim D^\lambda = 6$, and arrive at (ii).

If $S = \mathsf{A}_6 \cong PSL_2(9)$ then $n=10$, and by [**14**], $\dim V \leq 16$ if $p=2$ and $\dim V \leq 9$ if $p=3$. As $\lambda \notin \mathscr{L}^{(1)}(n)$, we deduce that $p=2$, $\dim V = 16$, and arrive at (iii).

If $S = \mathsf{A}_7$, then $n=15$ and $G=S$, and by [**14**], $\dim D^\lambda \leq 2(\dim V) \leq 40$, whence $\lambda \in \mathscr{L}^{(1)}(n)$, a contradiction.

If $S = \mathsf{A}_8 \cong SL_4(2)$ then $n=15$, $G=S$, and by [**14**], $\dim V \leq 64$, hence $p=2$ and $\dim V = 64$. Note that $\mathrm{IBr}_2(G)$ contains a unique character of degree 64, which takes value 4 at an element of class $3A$ in A_8, which belongs to class $3D$ in A_{15}, in the notation of [**14**]. However, any character in $\mathrm{IBr}_2(\mathsf{A}_{15})$ of degree 64 takes value -2 at class $3D$, and $\mathrm{IBr}_2(\mathsf{S}_{15})$ contains no character of degree 64, a contradiction.

If $S = SL_3(2) \cong PSL_2(7)$ then $n=7$ or 8, and by [**14**], $\dim D^\lambda = \dim V \leq 8$. If $p=3$, then we conclude using [**14**] that $\lambda \in \mathscr{L}^{(1)}(n)$, a contradiction. Let $p=2$. Since $\dim V \leq 8$ and $\lambda \notin \mathscr{L}^{(1)}(n)$, we deduce that $(H,\dim V)$ is either $(\mathsf{A}_n,4)$ or $(\mathsf{S}_n,8)$. Checking the degrees of characters in $\mathrm{IBr}_2(G)$ we see that $\dim V = 8$. Now, any irreducible 2-Brauer character of degree 8 of H takes value -4 or 2 at elements of order 3, whereas any irreducible 2-Brauer character of degree 8 of G takes value -1 at elements of order 3, a contradiction.

If $S = SL_2(8)$ then $n=9$ or 28, $S \trianglelefteq G \leq \mathrm{Aut}(S) \cong P\Gamma L_2(8) \cong S \rtimes \mathsf{C}_3$, and by [**14**], $\dim V \leq 12$ if $p=2$ and $\dim V \leq 27$ if $p=3$. In particular, if $n=28$, then $\lambda \in \mathscr{L}^{(1)}(n)$ a contradiction. If $n=9$ then, by [**14**], remembering that $\lambda \notin \mathscr{L}^{(1)}(n)$, we can check that (iv) occurs if $p=2$ and (v) occurs if $p=3$.

If $S = SL_3(3)$ then $n=13$, $G=S$, and by [**14**], $\dim V \leq 27$, whence $\lambda \in \mathscr{L}^{(1)}(n)$, a contradiction.

If $S = PSL_3(4)$ then $n=21$, $G \leq PGL_3(4) \rtimes \mathsf{C}_2$, and by [**14**], we have $\dim V \leq 2 \cdot 64 = 128 < (n^2-5n+2)/2$, whence $\lambda \in \mathscr{L}^{(1)}(n)$ by Lemma 2.27, a contradiction. $\square$

We will also need the following extension of Theorem 3.7 to some subgroups of S_n that are not primitive:

Proposition 3.11. *Let $p = 2$, $H = \mathsf{A}_n$ or S_n, and let $G < H$ be an almost simple subgroup with $S = \mathrm{soc}\,(G)$ that is not primitive in H. Assume in addition that (S, n) satisfies one of the following conditions:*

(i) *$S \cong \mathsf{A}_m$ and (m, n) is $(5, \leq 10)$, $(6, \leq 14)$, $(7, \leq 16)$ or $(8, \leq 19)$. Moreover, if $m = 7$ or 8, then some orbit of S on $\Omega := \{1, 2, \ldots, n\}$ has length $> m$.*
(ii) *$S \cong PSL_2(q)$ and (q, n) is $(7, \leq 12)$, $(8, \leq 14)$, $(11, \leq 14)$, $(13, \leq 15)$ or $(16, \leq 17)$.*
(iii) *(S, n) is $(SL_3(3), \leq 17)$ or $(PSL_3(4), 21)$.*
(iv) *$S \cong M_t$ with $t = 11$, 12, 22, 23, or 24.*

Then $V{\downarrow_G}$ is irreducible for a basic spin $\mathbb{F}H$-module V if and only if one of the following holds:

(a) *$H = \mathsf{S}_6$ and $G = \mathsf{S}_5$ fixes one letter.*
(b) *$\mathsf{S}_5 \cong G = \mathsf{A}_{5,2} < H = \mathsf{A}_7$ or $\mathsf{S}_5 \cong G = \mathsf{A}_{5,2,1} < H = \mathsf{A}_8$.*
(c) *$H = \mathsf{A}_7$ or A_8, and $S = \mathsf{A}_6$.*
(d) *$H = \mathsf{A}_7$ or A_8, and $S = \mathsf{A}_5$ acts 2-transitively on $\{1, 2, \ldots, 6\}$.*
(e) *$H = \mathsf{A}_{11}$, and $G = M_{10} < \mathsf{A}_{10}$ acts 2-transitively on $\{1, 2, \ldots, 10\}$.*
(f) *$H = \mathsf{A}_{12}$ and $G \in \{\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)\}$ acts 2-transitively on $\{1, 2, \ldots, 10\}$.*
(g) *$H = \mathsf{A}_{12}$, $G \in \{\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)\}$, and S acts on $\{1, 2, \ldots, 6\}$ and $\{7, 8, \ldots, 12\}$ via two inequivalent 2-transitive actions.*
(h) *$H = \mathsf{A}_{12}$ and $G = M_{11} < \mathsf{A}_{11}$.*

Proof. We will prove the 'only-if-part'. The 'if-part' is then an easy explicit check. Set $\Omega := \{1, 2, \ldots, n\}$. Let U be an irreducible summand of $V{\downarrow_S}$. If U is trivial, then S acts trivially on V by Clifford's theorem, contradicting the faithfulness of the $\mathbb{F}H$-module V. Thus we have

$$\dim U = 2^a, \ \dim V \geq 2^{(n-4)/2} \tag{3.12}$$

for some $a \in \mathbb{Z}_{\geq 1}$. Since G is not primitive, we have by Lemma 2.2(i):

$$n \geq P(S) + 1. \tag{3.13}$$

First, we consider the case (iv). Then (3.12) implies by [**14**] that $\dim U = 16$ and $t = 11$ or 12. If $t = 11$, then $G = S$, $V = U$, $\dim V = 16$, and $n \geq 12$ by (3.13). It follows that $H = \mathsf{A}_{12}$ and $G < \mathsf{A}_{11}$, leading to (h). If $t = 12$, then $\dim V \leq 32$ since $G/S \hookrightarrow \mathrm{Out}(S) \leq \mathsf{C}_2$, whereas $n \geq 13$ by (3.13). It follows that $H = \mathsf{A}_{13}$, $\dim V = 32$, $G = \mathrm{Aut}(M_{12}) \leq \mathsf{A}_{13} \cap \mathsf{S}_{12} = \mathsf{A}_{12}$. The latter implies that $V = E_{\pm}^{\beta_{13}}$ is irreducible over A_{12}, a contradiction.

Next suppose we are in the case (iii). If $S = PSL_3(4)$, then $P(S) = 21$, violating (3.13). If $S = SL_3(3)$, then $n \geq 14$ by (3.13), whereas $\dim U = 16$ by [**14**], and so $\dim V \leq 32$, a contradiction.

Consider now the case (ii). Then $q \neq 16$ because of (3.13), and $q \neq 11, 13$ by (3.12) and [**14**]. If $q = 8$, then $S \leq G \leq \mathrm{Aut}(S)$ and so $\dim V \in \{2, 4, 8\}$ by [**14**]. On the other hand, $n \geq 10$ by (3.13), and this is impossible since $\dim V \geq \dim E^{\beta_{10}} = 16$. Thus $q = 7$, in which case $\dim V = 8$ by [**14**] and $n \geq 8$ by (3.13). The condition $\dim V = 8$ implies that $H \in \{\mathsf{S}_8, \mathsf{A}_9\}$. If $H = \mathsf{S}_8$, then the Brauer character of V can take values -4 or 2 at elements of order 3, whereas any $\psi \in \mathrm{IBr}_2(G)$ of degree 8 takes value -1 at elements of order 3, a contradiction. Thus $H = \mathsf{A}_9$. Note that if we embed $S = PSL_2(7)$ in H via a transitive embedding $S < \mathsf{A}_7$ (so fixing two more points) or a transitive embedding $S < \mathsf{A}_8$, then any element $g \in S$ of order 3 will fix 3 points, and so the Brauer character of V takes value 2 at g, again a contradiction.

Finally, suppose we are in the case (i).

Assume first that $S = \mathsf{A}_8$. Then $\dim V = 4$, 8, or 64 by [**14**]. Since S is not primitive on Ω and has an orbit of length > 8 on Ω, we have by [**11**] that $n \geq 16$, whence $H = \mathsf{A}_{16}$, $\dim V = 64$, and $G = S$ is a 2-transitive subgroup of A_{15}. In this embedding, a 3-cycle $g \in S$ will have 1 fixed point on Ω, so the Brauer character of V takes value -2 at g. But any $\psi \in \mathrm{IBr}_2(S)$ of degree 64 takes value 4 at g, a contradiction.

Next let $S = \mathsf{A}_7$. Then $\dim V = 4$ or 8 by [**14**]. As S is not primitive on Ω and has an orbit of length > 7 on Ω, we have by [**11**] that $n \geq 16$, contradicting (3.12).

Now let $S = \mathsf{A}_6$. Then $\dim V = 4$, 8, or 16 by [**14**], and $n \geq 7$ by (3.13). It follows that $H = \mathsf{A}_n$ with $7 \leq n \leq 12$, or $H = \mathsf{S}_n$ with $7 \leq n \leq 10$.

Assume first that some S-orbit on Ω has length $l > 6$. As S is not primitive, we must then have that $l = 10$, $H = \mathsf{A}_n$ with $n = 11$ or 12, and $\dim V = 16$. In either case, we may assume that S acts 2-transitively on $\{1, 2, \dots, 10\}$ and fixes 11, and also 12 if $n = 12$. In the 2-transitive embedding $\mathsf{A}_6 \hookrightarrow \mathsf{S}_{10}$, elements of S of order 3 acts with one fixed point and elements of order 5 act fixed-point-freely; furthermore a point stabilizer in S is just $\mathbf{N}_S(Q)$ for $Q \in \mathrm{Syl}_3(S)$. The embedding extends to $\mathrm{Aut}(\mathsf{A}_6)$, with the image of $M_{10} = S \cdot 2_3$ (in the notation of [**11**] and [**14**]) contained in A_{10}. Using the class fusion information, it is easy to check in [**14**] that $E^{\beta_{10}}$ is irreducible over $\mathsf{S}_6 \cong S \cdot 2_1$, M_{10}, and $\mathrm{Aut}(S)$, but splits into a direct sum of two simple summands over S and $S \cdot 2_2 \cong PGU_2(9)$. Also,

$$E^{\beta_{12}}_{\pm}{\downarrow}_{\mathsf{A}_{11}} \cong E^{\beta_{11}}_{\pm},\ E^{\beta_{11}}_{\pm}{\downarrow}_{\mathsf{A}_{10}} \cong E^{\beta_{10}}.$$

Hence, if $n = 11$, then G fixes 11 and is contained in the natural A_{10}, and so $G \cong M_{10}$, leading to (e). Note that we can extend the embedding $\mathrm{Aut}(S) \hookrightarrow \mathsf{A}_{10}$ to $\mathrm{Aut}(S) \hookrightarrow \mathsf{A}_{12}$ uniquely by demanding the involution $(1, 2) \in \mathsf{S}_6$ to interchange 11 and 12. This leads to (f) when $n = 12$.

Now we consider the case where all orbits of $S = \mathsf{A}_6$ on Ω have length 1 or 6, and there is more than one orbit of length 6. Then $n = 12$ and $H = \mathsf{A}_{12}$. Let $\pi_1, \pi_2 : S \to \mathsf{A}_6$ be induced by the action of S on its two orbits on Ω, and let ψ_i denote the Brauer characters afforded by $E^{\beta_6}{\downarrow}_{\pi_i(S)}$. Then $\psi_i \in \mathrm{IBr}_2(S)$ and has degree 4; also, ψ_i^2 contains $\mathbb{F}_S$ (with multiplicity 4). As G is irreducible on V and $V{\downarrow}_{\mathsf{A}_6 \times \mathsf{A}_6} \cong E^{\beta_6} \boxtimes E^{\beta_6}$, we see that $\psi_1 \neq \psi_2$. In this case, $\psi_1\psi_2 = \nu_1 + \nu_2$, with $\nu_i \in \mathrm{IBr}_2(S)$ of degree 8, and $\mathrm{Stab}_{\mathrm{Aut}(S)}(\nu_1) = S \cdot 2_2$. So as long as π_1 and π_2 are inequivalent and $G \not\leq S \cdot 2_2$, $V{\downarrow}_G$ is irreducible, leading to (g). (Note that such an action exists: for instance, we can embed S_6 in $\mathsf{A}_{6,6}$, with two inequivalent actions of S_6 on the first and the last six letters.)

Finally, we consider the case where $S = \mathsf{A}_6$ has exactly one orbit $\{1, 2, \dots, 6\}$ of length 6 and fixes each of $7, \dots, n$; in particular, $G \leq \mathsf{S}_{6,n-6}$. If $H = \mathsf{S}_n$ and $n \geq 8$, then it follows that D^{β_n} is irreducible over $\mathsf{S}_{6,n-6}$, contradicting [**32**, Proposition 2.15]. The case $H = \mathsf{S}_7$ is also impossible by dimension consideration. Suppose $H = \mathsf{A}_n$ and let $n_2 \geq n_3 \geq \ldots \geq n_h \geq 1$ denote the lengths of G-orbits on $\{7, \dots, n\}$. Then $G \leq \mathsf{A}_\nu$ for $\nu := (6, n_2, \dots, n_h)$. Since $G/S \leq \mathsf{C}_2^2$, $n_i \in \{1, 2, 4\}$. As V is irreducible over A_ν, we see by [**34**, Proposition 6.3] that $\nu = (6, 1)$, $(6, 1, 1)$, or $(6, 2)$, and arrive at (c).

Finally, let $S = \mathsf{A}_5$. Then $\dim V = 2$ or 4 by [**14**], and $n \geq 6$ by (3.13). It follows that $H = \mathsf{A}_n$ with $6 \leq n \leq 8$, or $H = \mathsf{S}_6$.

If $H = \mathsf{S}_6$, then $G \leq \mathsf{S}_5$ as G is not primitive, and we arrive at (a).

Suppose $H = \mathsf{A}_n$ but some S-orbit has length $l > 5$. As S is not primitive and $n \leq 8$, we have that $l = 6$, $n = 7$ or 8, and S has one orbit $\Omega' := \{1, 2, \dots, 6\}$ and fixes the remaining letters. In this action, elements of order 3 in S act fixed-point-freely on Ω'. Using this class fusion information, we can check in [**14**] that $V{\downarrow_S}$ is irreducible (of dimension 4), giving rise to (d).

Finally, assume $H = \mathsf{A}_n$ and $S = \mathsf{A}_5$ has only orbits of length 5 and 1 on Ω. Then we may assume that S has one orbit $\{1, 2, \dots, 5\}$ and fixes each of $6, \dots, n$. Again let $n_2 \geq n_3 \geq \dots \geq n_h \geq 1$ denote the lengths of G-orbits on $\{6, \dots, n\}$. Then $G \leq \mathsf{A}_\nu$ for $\nu := (5, n_2, \dots, n_h)$. Since $G/S \leq \mathsf{C}_2$, $n_i \in \{1, 2\}$. As V is irreducible over A_ν, we see by [**34**, Proposition 6.3] that $\nu = (5, 2, 1)$, or $(5, 2)$, and arrive at (b). $\square$

4. The small doubly transitive groups

Recall that we call a doubly transitive subgroup $G < \mathsf{S}_n$ small, if $S = \mathrm{soc}\,(G)$ is non-abelian, $S \not\cong \mathsf{A}_n$, $S \not\cong PSL_m(q)$ when $n = (q^m - 1)/(q - 1)$, and $S \not\cong Sp_{2m}(2)$ when $n = 2^{m-1}(2^m \pm 1)$.

All small 2-transitive subgroups are handled in the following theorem:

Theorem 4.1. *Let $p = 2$ or 3, $H = \mathsf{A}_n$ or S_n, and W be an irreducible summand of $D^\lambda{\downarrow_H}$ for some $\lambda \in \mathscr{P}_p(n) \setminus \mathscr{L}^{(1)}(n)$. Let $G < H$ be a small doubly transitive subgroup. Then $W{\downarrow_G}$ is irreducible if and only if one of the following cases occurs.*

- (i) *$G = M_{11}$, $\mathsf{A}_n \leq H \leq \mathsf{S}_n$, and one of the following happens:*
  - (a) *$p = 2$, $n = 11$ or 12, $\lambda = (n-2, 2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 44;*
  - (b) *$p = 3$, $n = 11$, λ or $\lambda^{\mathtt{M}}$ is $(9, 1^2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 45.*
- (ii) *$G = M_{11}$, $p = 2$, $H = \mathsf{A}_n$, $n = 11$ or 12, and $W = E^{\beta_n}_\pm$ has dimension 16.*
- (iii) *$G = M_{12}$, $p = 2$, $n = 12$, and one of the following happens:*
  - (a) *$\mathsf{A}_{12} \leq H \leq \mathsf{S}_{12}$, $\lambda = (10, 2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 44;*
  - (b) *$H = \mathsf{A}_{12}$, $\lambda = \beta_{12}$, and $W = E^{\beta_{12}}_\pm$ has dimension 16;*
  - (c) *$H = \mathsf{A}_{12}$, $\lambda = (6, 5, 1)$, and $W = E^\lambda_\pm$ has dimension 144.*
- (iv) *$G = M_{12}$, $p = 3$, $n = 12$, $\mathsf{A}_{12} \leq H \leq \mathsf{S}_{12}$, and one of the following happens:*
  - (a) *λ or $\lambda^{\mathtt{M}}$ is $(10, 1^2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 45;*
  - (b) *λ or $\lambda^{\mathtt{M}}$ is $(10, 2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 54.*
- (v) *$M_{22} \leq G \leq \mathrm{Aut}(M_{22})$, $p = 3$, $n = 22$, $\mathsf{A}_{22} \leq H \leq \mathsf{S}_{22}$, λ or $\lambda^{\mathtt{M}}$ is $(20, 1^2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 210.*
- (vi) *$G = M_{23}$, $p = 3$, $n = 23$, $\mathsf{A}_{23} \leq H \leq \mathsf{S}_{23}$, λ or $\lambda^{\mathtt{M}}$ is $(21, 1^2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 231.*
- (vii) *$G = M_{24}$, $p = 3$, $n = 24$, $\mathsf{A}_{24} \leq H \leq \mathsf{S}_{24}$, and one of the following happens:*
  - (a) *λ or $\lambda^{\mathtt{M}}$ is $(22, 1^2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 231;*
  - (b) *λ or $\lambda^{\mathtt{M}}$ is $(22, 2)$, and $W = D^\lambda{\downarrow_H}$ has dimension 252.*

Proof. If $S := \mathrm{soc}\,(G)$ is a not a Mathieu group or Co_3, then the arguments in [**8**, Section 5], but using Lemma 2.27 instead of [**8**, Lemma 1.18(i)], show that $D^\lambda{\downarrow_G}$ is reducible. We now consider the remaining possibilities for S. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we will assume that $\lambda_1 \geq \lambda^{\mathtt{M}}_1$.

Case 1: *$G = M_{11}$ in transitive representations of degrees $n = 11$ and 12.*

By comparing the traces in these transitive representations [**14**], one can see that the classes $2a$, $3a$, $4a$, $5a$, $8ab$, $11a$ of G belong to classes $2b$, $3c$, $4b$, $5b$, $8a$, $11a$ in A_{11}, and $2c$, $3d$, $4d$, $5b$, $8b$, $11a$ in A_{12}.

Let $p = 2$. According to [**14**], any $\varphi \in \mathrm{IBr}_2(G)$ has degree 1, 10, 16, or 44; and the degrees of characters in $\mathrm{IBr}_p(H)$ are also known. Hence we need to consider only the cases where $\dim D^\lambda = 32$ or 44. In the latter case, $D^\lambda{\downarrow}_{\mathsf{A}_{11}}$ is irreducible and is obtained by reducing $S_{\mathbb{C}}^{(9,2)}$ modulo 2 (and restricting to A_{11}). Using the above class fusion, we see that the case $\dim D^\lambda = 44$ does give rise to an example (and $\lambda = (n-2,2)$). If $\dim D^\lambda = 32$ (and so $\lambda = (6,5)$, respectively $(7,5)$), then its restriction to A_n is a direct sum of two simple modules of dimension 16, both of which are irreducible over G, giving rise to another example with $(\dim V, H) = (16, \mathsf{A}_n)$.

Let $p = 3$. Using [**14**] as above, when $n = 11$ we see that $\dim V = 45$ and $\lambda = (n-2, 1^2)$ (up to tensoring with $\mathtt{sgn}$), yielding another example. There is no example when $n = 12$, since $\varphi \in \mathrm{IBr}_3(\mathsf{A}_{12})$ of degree 45 takes value 3 at the class $8b$ of A_{12}, whereas $\psi \in \mathrm{IBr}_3(M_{11})$ of degree 45 takes value -1 at the class $8a$ of M_{11}.

Case 2: *$G = M_{12}$ in permutation representations of degree $n = 12$.*

Let $p = 2$. Using the character degrees of G and H as listed in [**14**], we need to consider only the cases where $\dim D^\lambda = 32$, 44 and 288. We can embed M_{11} into G as a 2-transitive subgroup of $G < \mathsf{S}_{12}$. Now the first two cases, with $(\lambda, H) = ((7,5), \mathsf{A}_{12})$ and $((10,2), \mathsf{A}_{12}$ or $\mathsf{S}_{12})$, give rise to examples, since $V{\downarrow}_H$ is irreducible by the results of Case 1. Next, by restricting $\psi \in \mathrm{IBr}_2(\mathsf{A}_{12})$ to G, we see that conjugacy classes $3A$, $3B$, and $5A$ of G as listed in [**14**] correspond to the classes $3D$, $3C$, and $5B$ in A_{12}. It follows that the last case, with $(\lambda, \dim V, H) = ((6,5,1), 144, \mathsf{A}_{12})$, gives rise to another example.

Let $p = 3$. Using [**14**] as above, we see that $\dim V = 45$ or 54, and $\lambda = (10, 1^2)$ or $(10,2)$, respectively (up to tensoring with $\mathtt{sgn}$). In both cases, D^λ is obtained by reducing $S^\lambda_{\mathbb{C}}$ modulo 3. Since $\chi := S^\lambda_{\mathbb{C}}{\downarrow}_G$ is irreducible by [**8**, Main Theorem (iii), (iv)], and $\chi(\mathrm{mod}\ 3)$ is irreducible by [**14**], both cases give rise to examples.

Case 3: *$M_{22} \le G \le \mathrm{Aut}(M_{22})$ in permutation representations of degree $n = 22$.*

Let $p = 2$. According to [**14**], any $\varphi \in \mathrm{IBr}_2(G)$ has degree $\le 140 < (n^2 - 5n + 2)/2$. By Lemma 2.27, this contradicts the assumption $\lambda \notin \mathscr{L}^{(1)}(n)$.

Let $p = 3$. By [**14**], any $\varphi \in \mathrm{IBr}_3(G)$ has degree $\le 231 < (n^3 - 9n^2 + 14n)/12$. Hence $\dim D^\lambda < (n^3 - 9n^2 + 14n)/6$ and so $\lambda \in \mathscr{L}^{(2)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$ by Proposition 2.28(i). By Lemma 2.6 and by checking the possible dimensions of V in [**14**], we see that $\dim D^\lambda = 210$, and that without any loss we may assume that D^λ is obtained by reducing $S_{\mathbb{C}}^{(n-2,1^2)}$ modulo p. As $\chi := S_{\mathbb{C}}^{(n-2,1^2)}{\downarrow}_S$ is irreducible by [**8**, Main Theorem (iv)], and $\chi(\mathrm{mod}\ 3)$ is irreducible by [**14**], both cases $G = S$ and $G = \mathrm{Aut}(S)$ give rise to examples.

Case 4: *$G = M_{23}$ in permutation representations of degree $n = 23$.*

First we consider the case where V does not extend to S_n. As in part (b) of the proof of Theorem 3.7, we have that $\lambda_1 \le (n+2)/2$ if $p = 2$ and $\lambda_1 \le (n+4)/2$ if $p = 3$. Now, if $p = 2$ then $\lambda_1 \le 12$, whence $\dim V = (\dim D^\lambda)/2 \ge 2^{10}$ by Theorem 2.22(i), whereas $\mathfrak{b}_2(G) = 896$ [**14**], a contradiction. If $p = 3$, then $\lambda_1 \le 13$ and so $\dim V = (\dim D^\lambda)/2 \ge 2783$ by Proposition 2.28(ii), contrary to $\mathfrak{b}_3(G) = 1035$ [**14**].

Now we consider the case where $V = D^\lambda{\downarrow}_H$. Then

$$\dim D^\lambda \le \mathfrak{b}_p(G) \le 1035 < (n^3 - 9n^2 + 14n)/6,$$

whence $\lambda \in \mathscr{L}^{(2)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$ by Proposition 2.28(i), and so $\lambda_1 \geq n-2$.

Consider the case $D^\lambda = D^{(n-2,2)}$. If $p=2$, then D^λ is obtained by reducing $S^\lambda_{\mathbb{C}}$ modulo p. Furthermore, $\chi := S^\lambda_{\mathbb{C}}{\downarrow_G}$ is irreducible by [**8**, Main Theorem (iii)], but $\chi(\bmod\ 2)$ is reducible by [**14**], ruling out this case. If $p=3$, then $\dim D^\lambda = 208$. Since $\lambda \notin \mathscr{L}^{(1)}(n)$, D^λ is irreducible over A_n by Lemma 2.27. Since no $\varphi \in \mathrm{IBr}_p(G)$ has degree 208 [**14**], there are no examples in this case either.

Suppose now that $p=3$ and $D^\lambda = D^{(n-2,1^2)}$. Then D^λ is obtained by reducing $S^\lambda_{\mathbb{C}}$ modulo p. Furthermore, $\psi := S^\lambda_{\mathbb{C}}{\downarrow_G}$ is irreducible by [**8**, Main Theorem (iv)], and $\psi(\bmod\ 3)$ irreducible by [**14**], we obtain another example.

Case 5: *$G = M_{24}$ in permutation representations of degree $n=24$.*
According to [**14**], $\mathfrak{b}_p(G) \leq 10395$.

Case 5.1: *$V = D^\lambda{\downarrow_H}$ or $\dim V < 10395/2$.* In this case we have that $\dim D^\lambda \leq 10395$. This implies by [**8**, Lemma 1.23] that either $\lambda \in \mathscr{L}^{(4)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$, or $p=2$ and $\lambda = (13,11)$. In the latter case, D^λ is a basic spin module of dimension 2048. Since no $\varphi \in \mathrm{IBr}_2(G)$ has degree 2048 or 1024, this case is ruled out.

Let $p=2$. Then $\dim D^\mu$ for $\mu \in \mathscr{L}^{(4)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$ is determined by Lemma 2.8, and neither $\dim D^\mu$ nor $(\dim D^\mu)/2$ matches $\varphi(1)$ for any $\varphi \in \mathrm{IBr}_2(G)$ [**14**]. In fact, $D^{(23,1)}$ is also reducible over G. Thus we have no example for $p=2$.

Let $p=3$. Then $\dim D^\mu$ for $\mu \in \mathscr{L}^{(4)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$ is determined by Lemma 2.7, and using [**14**] we see that $\dim D^\mu$ or $(\dim D^\mu)/2$ can match $\varphi(1)$ for some $\varphi \in \mathrm{IBr}_3(G)$ only when $\mu = (22,2)$, $(22,1^2)$, or $(21,2,1)$. If $\lambda = (22,2)$, then D^λ is obtained by reducing $S^\lambda_{\mathbb{C}}$ modulo p. Furthermore, $\psi := S^\lambda_{\mathbb{C}}{\downarrow_G}$ is irreducible by [**8**, Main Theorem (iii)], and $\psi(\bmod\ 3)$ irreducible by [**14**], giving rise to an example. Next, $\alpha := S^{(22,1^2)}_{\mathbb{C}}{\downarrow_G}$ is irreducible by [**8**, Main Theorem (iii)], and $\alpha(\bmod\ 3) = \beta_{22} + \beta_{231}$, where β_i is an irreducible 3-Brauer character of G of degree $i \in \{22, 231\}$. On the other hand, $[S^{(22,1^2)}] = [D^{(22,1^2)}] + [D^{(23,1)}]$ by Lemma 2.7(ii), with $D^{(23,1)}{\downarrow_G} = \beta_{23}$. It follows that $D^{(22,1^2)}{\downarrow_G} = \beta_{231}$, leading to another example. Finally, by Lemma 2.10, $D^{(21,2,1)}{\downarrow_{\mathsf{A}_{24}}}$ is irreducible (of dimension 1540 by Lemma 2.8), ruling out the case $\lambda = (21,2,1)$.

Case 5.2: *V does not extend to S_{24} and $\dim V > 10395/2$.*

Since $\dim V \leq \mathfrak{b}_p(G) \leq 10395$ and $\mathfrak{b}_2(G) = 1792$, we must have that $p=3$, and $10395 < \dim D^\lambda \leq 2 \cdot 10395$. Consider the Young subgroup $\mathsf{S}_{22,2} \cong \mathsf{S}_{22} \times \mathsf{S}_2 < \mathsf{S}_{24}$. Note that the second factor S_2 is generated by a transposition t, which acts semisimply on D^λ and has both 1 and -1 as eigenvalues. The two corresponding t-eigenspaces are invariant under S_{22}. Thus the restriction of D^λ to a natural subgroup S_{22} contains a simple submodule D^μ of dimension at most $(\dim D^\lambda)/2 \leq 10395$. By [**8**, Lemma 1.23] applied to D^μ, we have $\mu \in \mathscr{L}^{(4)}(22)$. By the Frobenius reciprocity, D^λ is a quotient of $\mathrm{ind}^{\mathsf{S}_{24}}_{\mathsf{S}_{22}}(D^\mu)$, and so $\lambda \in \mathscr{L}^{(6)}(24)$, i.e. $\lambda_1 \geq 18$. But this implies by Lemma 2.10 that D^λ is irreducible over A_{24}, contrary to our assumption.

Case 6: *$G = Co_3$ in permutation representations of degree $n = 276$.*

According to [**14**], any $\varphi \in \mathrm{IBr}_2(G)$ has degree $\leq 131584 < (n^3 - 9n^2 + 14n)/12$. Hence $\lambda \in \mathscr{L}^{(2)}(n) \smallsetminus \mathscr{L}^{(1)}(n)$ by Proposition 2.28(i). It follows by Lemma 2.6 that $\dim D^\lambda = 37400$ if $p=2$, 37401 or 37674 if $p=3$. Since no $\varphi \in \mathrm{IBr}_p(G)$ has such degree (or half of it, in case $D^\lambda{\downarrow_{\mathsf{A}_n}}$ is reducible) by [**14**], there are no examples. $\square$

5. Affine permutation groups

In this section we consider restrictions to subgroups G of S_n having regular normal elementary abelian r-subgroups, whose structure is explained in Lemma 3.3. Note that any irreducible r-modular representation of a finite group G with nontrivial normal r-subgroup R must be trivial on R. Therefore, if an $\mathbb{F}_r\mathsf{S}_n$-module V is irreducible over such G and $n \geq 5$, then $\dim V = 1$. Henceforth we may restrict ourselves to S_n-modules in characteristic $p \neq r$.

5.1. **Invariants in modules over wreath products.** Throughout this subsection, we assume that r is a prime different from p. For $m \in \mathbb{Z}_{\geq 1}$, we denote

$$H_m := GL_m(r), \quad V_m := \mathbb{F}_r^m, \quad G_m := AGL_m(r) = V_m \rtimes H_m.$$

We also denote by X_m the set of linear characters $V_m \to \mathbb{F}^\times$, and $X_m^\times \subset X_m$ be the subset of all non-trivial linear characters. Note that X_m is an abelian group via $(\xi+\eta)(v) := \xi(v)\eta(v)$ for $\xi, \eta \in X_m$ and $v \in V_m$. In fact, X_m can be identified with $\mathbb{F}_r^m$. In particular, for any $\xi \in X_m$, we have

$$r\xi = 0. \tag{5.1}$$

Lemma 5.2. *Let $r > 2$ and assume that $m > 1$ if $r = 3$. There exist $\xi_1, \dots, \xi_r \in X_m^\times$ not all equal to each other and such that $\xi_1 + \cdots + \xi_r = 0$.*

Proof. Checked easily identifying X_m with $\mathbb{F}_r^m$. □

A key role in the study of the restriction of irreducible modules D^λ from S_{r^m} to G_m embedded via its natural action on the points of V_m is played by the analysis of the invariant space $(D^\lambda)^{V_m}$. For this, it is convenient to embed V_m into some wreath product subgroup of S_{r^m}.

We will now assume that $m \geq 2$ and denote $n := r^m$. We have $V_m \leq G_m \leq \mathsf{S}_n$ via the natural action of G_m on $n = r^m$ points of V_m. Consider the corresponding embedding $\varphi_m : V_m \hookrightarrow \mathsf{S}_n$—this comes from the regular action of V_m on itself. We consider subgroups of V_m as subgroups of S_n via the embedding φ_m.

Let $e_1, \dots, e_m$ be the standard basis of $V_m = \mathbb{F}_r^m$, and $a \in \mathbb{F}_r$. We denote

$$\begin{aligned} V_m(a) &:= \{b_1e_1 + \cdots + b_{m-1}e_{m-1} + ae_m \mid b_1, \dots, b_{m-1} \in \mathbb{F}_r\} \subseteq V_m, \\ A &:= \{be_m \mid b \in \mathbb{F}_r\} \leq V_m, \\ n' &:= r^{m-1} = n/r. \end{aligned}$$

We identify V_{m-1} with $V_m(0)$ and A with $(\mathbb{F}_r, +)$. Note that $V_m = V_{m-1} \times A$.

For each $a \in A = \mathbb{F}_r$, let $\mathfrak{S}^a \cong \mathsf{S}_{n'}$ be the symmetric group on $V_m(a)$. We have a natural embedding

$$P := \bigtimes_{a \in A} \mathfrak{S}^a \cong \mathsf{S}_{n'}^{\times r} \hookrightarrow \mathsf{S}_n$$

as a parabolic subgroup. Note that V_{m-1} acts on each $V_m(a)$ regularly, so V_{m-1} is embedded into P diagonally via $\bigtimes_{a\in A} \varphi_{m-1}$. The group A acts on the components $\mathfrak{S}^a$ of P via conjugation:

$$b\mathfrak{S}^a b^{-1} = \mathfrak{S}^{a+b} \qquad (a, b \in A).$$

Let

$$W := \langle P, A\rangle = P \rtimes A \cong \mathsf{S}_{n'} \wr A \leq \mathsf{S}_n. \tag{5.3}$$

As V_{m-1} is a subgroup of P, we have that $V_m = V_{m-1} \times A$ is a subgroup of $W = P \rtimes A$.

We now describe irreducible $\mathbb{F}W$-modules. For this we consider the *p-regular A-multipartitions*, i.e. tuples $\boldsymbol{\mu} = (\mu^a)_{a\in A}$ such that $\mu^a \in \mathscr{P}_p(n')$ for all $a \in A$. To any such $\boldsymbol{\mu}$, we associate the $\mathbb{F}P$-module $\boxtimes_{a\in A} D^{\mu^a}$. Elements of $\boxtimes_{a\in A} D^{\mu^a}$ are linear combinations of pure tensors of the form $\otimes_{a\in A} d_a$ with $d_a \in D^{\mu_a}$ for all $a \in A$. Define

$$M(\boldsymbol{\mu}) := \mathrm{ind}_P^W(\boxtimes_{a\in A} D^{\mu^a}).$$

Elements of $M(\boldsymbol{\mu})$ are linear combinations of elements of the form $b\otimes(\otimes_{a\in A} d_a)$, where $b \in A$ and $d_a \in D^{\mu_a}$ for all $a \in A$. Recalling that $V_m = V_{m-1} \times A$ and considering $\mathbb{F}A$ as a left regular $\mathbb{F}A$-module, we have

$$M(\boldsymbol{\mu}){\downarrow}_{V_m} = M(\boldsymbol{\mu}){\downarrow}_{V_{m-1}\times A} \cong (\otimes_{a\in A} D^{\mu^a}{\downarrow}_{V_{m-1}}) \boxtimes \mathbb{F}A. \tag{5.4}$$

We say that a p-regular A-multipartition $\boldsymbol{\mu} = (\mu^a)_{a\in A}$ is *constant* (with value μ) if there exists $\mu \in \mathscr{P}_p(n')$ such that $\mu^a = \mu$ for all $a \in A$. Let $\boldsymbol{\mu} = (\mu)_{a\in A}$ be a constant A-multipartition with value μ. For any linear character $\alpha : A \to \mathbb{F}$, we define a $\mathbb{F}W$-module $M_\alpha(\mu)$ by extending the P-action on $\boxtimes_{a\in A} D^\mu$ to $W = P \rtimes A$-module via

$$b(\otimes_{a\in A} d_a) = \alpha(b)(\otimes_{a\in A} d_{a+b}) \qquad (b \in A).$$

The following result follows from Clifford theory:

Lemma 5.5. *If M is an irreducible $\mathbb{F}W$-module then it is isomorphic to a module of one of the following two types:*

(1) *$M_\alpha(\mu)$ for some $\mu \in \mathscr{P}_p(n')$ and some linear character $\alpha : A \to \mathbb{F}$;*
(2) *$M(\boldsymbol{\mu})$ for some non-constant p-regular A-multipartition $\boldsymbol{\mu}$.*

Conversely, all modules of the forms (1) *and* (2) *are irreducible, and the only isomorphisms between them are $M(\boldsymbol{\mu}) \cong M(\boldsymbol{\nu})$ if there exists $b \in A$ such that $\nu^a = \mu^{a+b}$ for all $a \in A$.*

In the next two lemmas, we study V_m-invariants in irreducible $\mathbb{F}W$-modules.

Lemma 5.6. *Let M be an irreducible $\mathbb{F}W$-module of the form $M = M_\alpha(\mu)$ for some $\mu \in \mathscr{P}_p(n')$. Then $M^{V_m} = 0$ if and only if $\alpha \neq 0$ and one of the following conditions holds:*

(i) *$D^\mu \in \mathtt{T}_{n'}$.*
(ii) *$D^\mu \in \mathtt{N}_{n'}$ and either $r = 2$, or $r = 3$ and $m = 2$.*

Proof. The 'if'-part is an explicit check. For the 'only-if'-part, if $\alpha = 0$, pick a non-zero $d \in D^\mu_\xi$ for some $\xi \in X_{m-1}$. Then, using (5.1), it is easy to see that $\otimes_{a\in A} d \in M^{V_m}\setminus\{0\}$. Now let $\alpha \neq 0$. Suppose we are given the following data:

(a) characters $\{\xi_a \in X_{m-1} \mid a \in A\}$ with $\sum_{a\in A} \xi_a = 0$;
(b) non-zero vectors $\{d_a \in D^\mu_{\xi_a} \mid a \in A\}$, not all proportional to each other.

Then it is easy to see that

$$\sum_{b\in A} \alpha(b)(\otimes_{a\in A} d_{a+b}) \in M^{V_m} \setminus \{0\}.$$

In view of (i), we may assume that $D^\mu \notin \mathtt{T}_{n'}$. So $\dim D^\mu \geq 2$. If $D^\mu_\xi = 0$ for all $\xi \in X^\times_{m-1}$, then $D^\mu_0 = D^\mu$, and we can take $\xi_a = 0$ for all a to satisfy (a) and pick vectors $d_a \in D^\mu_0$, not all proportional to each other, to satisfy (b). Thus we may assume that $D^\mu_\xi \neq 0$ for some $\xi \in X^\times_{m-1}$, in which case we have $D^\mu_\xi \neq 0$ for all $\xi \in X^\times_{m-1}$. Now, we use Lemma 5.2 to find characters $\xi_a \in X^\times_{m-1}$ satisfying (a), and by taking any non-zero $d_a \in D^{\mu}_{\xi_a}$ we also satisfy (b). Lemma 5.2 is applicable unless $r = 2$, or $r = 3$ and $m = 2$, so these cases need to be considered separately. In the case $r = 3$ and $m = 2$ there is actually nothing to check in view of the exception (ii) since for S_3 all irreducible modules are in $\mathtt{NT}_3$.

Let $r = 2$. If there is $\xi \in X_{m-1}$ with $\dim D^{\mu}_{\xi} \geq 2$, we set $\xi_a := \xi$ for all a to satisfy (a), cf. (5.1). Then pick linearly independent vectors $x_1, x_2 \in D^{\mu}_{\xi}$ and set $d_a := x_1$ for $a \in A_1$ and $d_a = x_2$ for $a \in A_2$, where $A = A_1 \sqcup A_2$ for some non-empty sets A_1, A_2. The vectors d_a satisfy (b). Thus we may assume that $\dim D^{\mu}_{\xi} \leq 1$ for all $\chi \in X_{m-1}$, i.e. $\dim D^{\mu} \leq n'$. Using [**21**, Theorem 6(ii)], we deduce that $D^{\mu} \in \mathtt{N}_{n'}$, which is exception (ii). □

Lemma 5.7. *Let M be an irreducible $\mathbb{F}W$-module of the form $M = M(\boldsymbol{\mu})$ for some non-constant p-regular A-multipartition $\boldsymbol{\mu}$. Then $M^{V_m} = 0$ if and only one of the following two conditions holds:*

(i) *there exists $b \in A$ such that $(D^{\mu^b})^{V_{m-1}} = 0$ and $D^{\mu^a} \in \mathtt{T}_{n'}$ for all $a \neq b$.*

(ii) *$r = 2$, $m = 3$, $p > 3$, and there exists $b \in A$ such that $D^{\mu^b} \in \mathtt{N}_4$ and $\mu^a = (2,2)$ for $a \neq b$.*

Proof. We denote by N the $\mathbb{F}V_{m-1}$-module $\otimes_{a \in A} D^{\mu^a}{\downarrow_{V_{m-1}}}$. By (5.4), we have

$$M^{V_m} \cong N^{V_{m-1}} \boxtimes (\mathbb{F}A)^A \cong N^{V_{m-1}}.$$

This gives the 'if'-part, for if there exists $b \in A$ such that $(D^{\mu^b})^{V_{m-1}} = 0$ and $D^{\mu^a} \in \mathtt{T}_{n'}$ for all $a \neq b$, then $N \cong D^{\mu^b}{\downarrow_{V_{m-1}}}$, and so $N^{V_{m-1}} = 0$.

For the 'only-if-part', assume first that $r \neq 2$. If there is at most one $b \in A$ with $D^{\mu_b} \notin \mathtt{T}_n$ the result easily follows. Suppose there are $b \neq c$ in A such that $D^{\mu^b}, D^{\mu^c} \notin \mathtt{T}_{n'}$. Then $D^{\mu^b}_{\xi}$ and $D^{\mu^c}_{\xi}$ are non-zero for all $\xi \in X^{\times}_{m-1}$. For each $a \in A \setminus \{b, c\}$, take $d_a \in D^{\mu^a}_{\xi_a}$ for some $\xi_a \in X_{m-1}$. Now, there exist $\xi_b, \xi_c \in X^{\times}_{m-1}$ such that $\sum_{a \in A} \xi_a = 0$. Pick non-zero $d_b \in D^{\mu^b}_{\xi_b}$ and $d_c \in D^{\mu^c}_{\xi_c}$. Then $\otimes_{a \in A} d_a$ is a non-zero V_{m-1}-invariant vector of N.

Now, let $r = 2$. Note that $N^{V_{m-1}} \neq 0$ if and only if there is a character $\xi \in X_{m-1}$ such that $D^{\mu^0}_{\xi}$ and $D^{\mu^1}_{\xi}$ are non-zero. This is not the case exactly when $(D^{\mu^0})^{V_{m-1}} = D^{\mu^0}$ and $(D^{\mu^1})^{V_{m-1}} = 0$, or $(D^{\mu^0})^{V_{m-1}} = 0$ and $(D^{\mu^1})^{V_{m-1}} = D^{\mu^1}$. But the equality $(D^{\mu})^{V_{m-1}} = D^{\mu}$ holds if and only if $D^{\mu} \in \mathtt{T}_{n'}$, or $m = 3$, $p > 3$, and $D^{\mu} = D^{(2,2)}$, cf. for example [**8**, Lemma 5.5]. □

5.2. **Invariants in modules over symmetric groups.** Recall that we are considering the embeddings $V_m \leq G_m \leq \mathsf{S}_n$ for $n = r^m$ and assuming that $p \neq r$.

Lemma 5.8. *If $D \in \mathtt{N}_n$, then $D^{V_m} = 0$.*

Proof. Since the action of S_n on D is faithful, D affords a non-trivial character of V_m, hence D affords all $n-1$ non-trivial characters of V_m, hence the trivial character does not appear by dimensions. □

Lemma 5.9. *Let $p = 3$ and $r = 2$.*

(i) *Let $m = 3$, i.e. $n = 8$. Then $(D^{\lambda})^{V_3} = 0$ if and only if $D^{\lambda} \in \mathtt{N}_8 \cup [\![D^{(6,1,1)}, D^{(5,3)}]\!]$.*

(ii) *Let $m = 4$, i.e. $n = 16$. Then $(D^{\lambda})^{V_4} = 0$ if and only if $D^{\lambda} \in \mathtt{N}_{16} \cup [\![D^{(14,1,1)}]\!]$.*

Proof. (i) In [**14**], any nontrivial element t in V_3 is of class 2A. If φ is the Brauer character of an irreducible 3-modular module D of S_8, then $D^{V_m}=0$ if and only if $\varphi(t)=-\varphi(1)/7$. Now inspecting the 3-Brauer character table in [**14**] of S_8, we see that there are exactly six possibilities for such φ, two for each dimension 7, 21, and 28. These correspond, respectively, to modules in $\mathtt{N}_8$, $[\![D^{(6,1,1)}]\!]$, and $[\![D^{(5,3)}]\!]$.

(ii) We apply the same argument as in the case $m=3$. Now $t\in V_4\setminus\{1\}$ is of class 2C, and the condition is $\varphi(t)=-\varphi(1)/15$. It follows by checking the 3-Brauer character table of S_{16} in [**14**] that there are exactly four possibilities for such φ, two of dimension 15 and two of dimension 105. These correspond, respectively, to modules in $\mathtt{N}_{16}$ and $[\![D^{(14,1,1)}]\!]$. □

Lemma 5.10. *Let $p=2$, $r=3$, and $m=2$, i.e. $n=9$. Then $(D^\lambda)^{V_2}=0$ if and only if $D^\lambda\cong\mathtt{N}_9\cup D^{(5,4)}$.*

Proof. (i) In [**14**], the elements in $V_2\setminus\{1\}$ are of class 3B. So, arguing as in the proof of Lemma 5.9, we get exactly two 2-modular modules W of S_9 with $W^{V_2}=0$, of dimensions 8 and 16. These correspond, respectively, to modules in N_9 and $D^{(5,4)}$. □

Now we can prove our key technical result which develops [**36**, Proposition 4.6]:

Proposition 5.11. *Let $p=2$ or 3 and $m\geq 2$. Then $(D^\lambda)^{V_m}=0$ if and only if one of the following happens:*

(i) $D^\lambda\in\mathtt{N}_n$;
(ii) $r=2$, $p=3$, $D^\lambda\in[\![D^{(n-2,1,1)},D^{(5,3)}]\!]$;
(iii) $r=3$, $p=2$, $D^\lambda\cong D^{(5,4)}$.

Proof. It follows from [**8**, Lemma 5.6] that in the case $r=2$ and $p=0$, we have $(S^{(n-2,1,1)}_{\mathbb{C}})^{V_m}=0$. Reducing modulo 3, we deduce $(D^{(n-2,1,1)})^{V_m}=0$. The rest of the "if" part follows from Lemmas 5.8, 5.9, 5.10.

For the "only-if" part, recall that $V_m\leq W=P\rtimes A\leq\mathsf{S}_n$, cf. (5.3). By Lemmas 5.6 and 5.7, we have $(D^\lambda)^{V_m}=0$ only if all composition factors of D^λ_P are of the form $\boxtimes_{a\in A}D^{\mu^a}$ satisfying one of the following conditions:

($\mathtt{C}_1$) there is $b\in A$ such that $D^{\mu^a}\in\mathtt{T}_{n'}$ for all $a\neq b$, and either $(D^{\mu^b})^{V_{m-1}}=0$ or $D^{\mu^b}\in\mathtt{T}_{n'}$;
($\mathtt{C}_2$) $\mu^a=\mu^b$ for all $a,b\in A$, $D^{\mu^a}\in\mathtt{N}_{n'}$ for all $a\in A$, and either $r=2$, or $r=3$ and $m=2$;

Assume first that $p=2$. Then $r\neq 2$. If $(r,m)\neq(3,2)$, the restriction D^λ_P only has composition factors $\boxtimes_{a\in A}D^{\mu^a}$ satisfying ($\mathtt{C}_1$). By Proposition 2.17, $D^\lambda\in\mathtt{NT}_n$. In the exceptional case $(r,m)=(3,2)$ use Lemma 5.10.

Now, let $p=3$. Then $r\neq 3$. If $r\neq 2$, the restriction D^λ_P only has composition factors $\boxtimes_{a\in A}D^{\mu^a}$ satisfying ($\mathtt{C}_1$). By Proposition 2.17, $D^\lambda\in\mathtt{NT}_n$. Let $r=2$. By Lemma 5.9, we may assume that $m\geq 5$. Then, by induction on m, we may assume that all composition factors $D^\mu\boxtimes D^\nu$ of $D^\lambda_{\mathsf{S}_{n/2}\times\mathsf{S}_{n/2}}$ satisfy one of the following three conditions:

(1) $D^\mu\cong D^\nu\in\mathtt{N}_{n/2}$,
(2) $D^\mu\in\mathtt{T}_{n/2}, D^\nu\in\mathtt{NT}_{n/2}\cup[\![D^{(n/2-2,1,1)}]\!]$,
(3) $D^\nu\in\mathtt{NT}_{n/2}, D^\mu\in\mathtt{N}_{n/2}\cup[\![D^{(n/2-2,1,1)}]\!]$.

By Corollary 2.19, $D^\lambda \in \mathtt{NT}_n \cup [\![D^{(n-2,1,1)}]\!]$. □

Remark 5.12. One can ask what could be an analogue of Proposition 5.11 in the case $m = 1$, that is, which irreducible modules D^λ of S_r in characteristic $p \neq r$ have no invariants on the cyclic subgroup $\mathsf{C}_r < \mathsf{S}_r$. Until now, this question has been resolved only in the case $p = 0$ (see [**58**]), and in the case $r/2 < p < r$ (see [**55**, Lemma 3.2]).

5.3. **Irreducible restrictions to affine permutation groups.** Let $G \leq \mathsf{S}_n$ be a primitive subgroup with a regular normal abelian subgroup; or more generally, let $G \leq \mathsf{S}_n$ be any subgroup with a regular normal elementary abelian subgroup. Then, up to conjugacy, G is a subgroup of the group $AGL_m(r)$ of all affine transformations of the affine space $V_m = \mathbb{F}_r^m$ for a prime r. The group $AGL_m(r)$ acts naturally on $n = r^m$ points of V_m, which yields an embedding $G \leq \mathsf{S}_n$. Moreover, $AGL_m(r) \cong V_m \rtimes GL_m(r)$, and $G = V_m \rtimes H$ for $H \leq GL_m(r)$.

The following theorem is the main result of the section. It develops [**36**, Corollary 4.7].

Theorem 5.13. *Let $p = 2$ or 3, $n \geq 5$, $H = \mathsf{S}_n$ or A_n, and let M be an irreducible $\mathbb{F}H$-module of dimension greater than 1. Let $G < H$ be a subgroup that contains a regular normal, elementary abelian subgroup. Then $M{\downarrow}_G$ is irreducible if and only if one of the following happens:*

(i) *$M{\downarrow}_{\mathsf{A}_n} \cong E^{(n-1,1)}$ and G is 2-transitive;*
(ii) *$M{\downarrow}_{\mathsf{A}_n} \cong E^{(n-2,1^2)}$, and of the following holds:*
  (a) *$p = 3$, $G = AGL_m(2)$ with $n = 2^m$;*
  (b) *$p = 3$, $G = \mathsf{C}_2^4 \rtimes \mathsf{A}_7$ with $n = 16$;*
(iii) *$p = 2$, $H = \mathsf{A}_9$, $G = ASL_2(3)$ or $\mathsf{C}_3^2 \rtimes \mathsf{Q}_8$, and $M \cong E_\pm^{(5,4)}$.*
(iv) *$p = 2$, $H = \mathsf{A}_5$, $G = \mathsf{C}_5 \rtimes \mathsf{C}_2$, and $M \cong E_\pm^{(3,2)}$.*

Proof. Let $\lambda \in \mathscr{P}_p(n)$ be such that $M = D^\lambda$ if $H = \mathsf{S}_n$, or $M = E^\lambda_{(\pm)}$ if $H = \mathsf{A}_n$.

Recall that $n = r^m$ and $G = V_m \rtimes G_0 \leq AGL_m(r)$. Assume $M{\downarrow}_G$ is irreducible. By Clifford theory, $M^{V_m} = 0$, and so $p \neq r$. In particular, $p \nmid n$, and so $D^{(n-1,1)}$ is a reduction modulo p of the natural $(n-1)$-dimensional representation in characteristic 0, hence $D^{(n-1,1)}{\downarrow}_G$ is irreducible only if G is 2-transitive, in which case it is indeed always irreducible by [**47**]. This gives case (i), and from now on we assume that $E{\downarrow}_{\mathsf{A}_n} \not\cong E^{(n-1,1)}$.

We now exclude the case $m = 1$. In this case $|G| \leq |AGL(1,r)| = r(r-1)$. In particular if $M{\downarrow}_G$ is irreducible then $\dim M < r = n$. If $H = \mathsf{S}_n$, or $H = \mathsf{A}_n$ and M lifts to S_n, then by [**21**, Theorem 6(i)] for $r \geq 7$ we have $D^\lambda \in \mathtt{NT}_n$, which was excluded in the previous paragraph. The special case $r = 5$ is checked using [**23**]. On the other hand, if $H = \mathsf{A}_n$ and $M \cong E^\lambda_\pm$ note first that $G \leq AGL(1,r) \cap \mathsf{A}_r \cong \mathsf{C}_r \rtimes \mathsf{C}_{(r-1)/2}$, and the dimension of an irreducible $\mathbb{F}(\mathsf{C}_r \rtimes \mathsf{C}_{(r-1)/2})$-module is at most $(r-1)/2$. On the other hand, from [**37**, Proposition 4.1], we have that $\dim E^\lambda_\pm \geq 2^{(r-6)/2}$. If $r \geq 11$ we have that $2^{(r-6)/2} > (r-1)/2$. So we only need to consider the cases $r = 5$ and 7. In these cases using modular character tables it can be checked that if $\dim E^\lambda_\pm \leq (r-1)/2$ then $r = 5$, $p = 2$ and $\lambda = (3,2)$, in which case the restriction is indeed irreducible. This corresponds to the special case (iv).

As $M{\downarrow}_G$ is irreducible, so is $M{\downarrow}_{AGL_m(r)\cap H}$. It easily follows that $M{\downarrow}_{V_m}$ is a direct summand of $N{\downarrow}_{V_m}$ for some irreducible $\mathbb{F}AGL_m(r)$-module N. As mentioned above, we have $M^{V_m} = 0$. If $H = \mathsf{S}_n$, or $H = \mathsf{A}_n$ and M lifts to S_n, then $(D^\lambda)^{V_m} = M^{V_m} = 0$. On the other hand, if $H = \mathsf{A}_n$ and $M = E^\lambda_\pm$, note first that any non-trivial element $g \in V_m$ has cycle type $(r^{n/r})$ and $n/r > 1$ since we have already excluded the case $m = 1$. So g^σ is in the same A_n-conjugacy class as g for any $\sigma \in \mathsf{S}_n$. It follows that the Brauer characters of $E^\lambda_+{\downarrow}_{V_m}$ and of $E^\lambda_-{\downarrow}_{V_m}$ coincide and hence $E^\lambda_+{\downarrow}_{V_m} \cong E^\lambda_-{\downarrow}_{V_m}$ as $p \nmid r$. So in this case we can still conclude that $(D^\lambda)^{V_m} = 0$. By Proposition 5.11 we may now assume that one of the following happens:

(1) $r = 2$, $p = 3$, $D^\lambda \in [\![D^{(n-2,1,1)}, D^{(5,3)}]\!]$;
(2) $r = 3$, $p = 2$, $D^\lambda \cong D^{(5,4)}$.

Case 1.1: $r = 2$, $p = 3$, and $D^\lambda \in [\![D^{(n-2,1,1)}]\!]$. By [**20**, Theorem 24.1] that if $2 < p \nmid n$ then $D^{(n-2,1,1)}$ is reduction modulo p of the Specht module $S^{(n-2,1,1)}_{\mathbb{C}}$ in characteristic 0. Here $p = 3 \nmid n = 2^m$, so (by tensoring with $\mathtt{sgn}$ if necessary) we may assume that D^λ is reduction modulo 3 of $S^{(n-2,1,1)}_{\mathbb{C}}$. Moreover, it is easy to see that $D^\lambda{\downarrow}_{\mathsf{A}_n}$ is irreducible. Hence $S^{(n-2,1,1)}_{\mathbb{C}}{\downarrow}_G$ is irreducible. Note that $m \geq 3$ as $n = 2^m \geq 5$. By [**50**], the only proper subgroup of $AGL_m(2)$ that contains V_m and is irreducible on $S^{(n-2,1^2)}$ is $K := V_4 \rtimes \mathsf{A}_7 < \mathsf{A}_{16}$. Furthermore, the only complex irreducible character of degree 7 of $GL_3(2)$ remains irreducible modulo 3, cf. [**23**], so the arguments on pp. 179–180 of [**8**] show that $D^\lambda{\downarrow}_K$ is indeed irreducible. We have shown that either $G = AGL_m(2)$ with $m \geq 3$, or $G = K$ and $m = 4$, as stated in (ii).

Case 1.2. $r = 2$, $p = 3$, $D^\lambda \in [\![D^{(5,3)}]\!]$. By [**20**, Tables], we have $\dim D^{(5,3)} = 28$. Furthermore, $D^{(5,3)}$ is irreducible over A_8, so it suffices to show that $D^\lambda{\downarrow}_{AGL_3(2)}$ is reducible. Since D^λ affords all 7 non-trivial linear characters of V_3, it follows that $D^\lambda{\downarrow}_G = \mathrm{ind}^G_{G_1}(U)$, where U is a 4-dimensional module of $G_1 = V_3 \rtimes \mathsf{S}_4$. Now V_3 acts via scalars on U, and the degree of any irreducible $\mathbb{F}\mathsf{S}_4$-representation is at most 3, whence U, and so $D^\lambda{\downarrow}_G$, is reducible.

Case 2: $r = 3$, $p = 2$, $D^\lambda \cong D^{(5,4)}$. By [**20**, Tables], we have $\dim D^{(5,4)} = 16$. First we consider the case $H = \mathsf{S}_9$. Then it suffices to show that $D^\lambda{\downarrow}_G$ is reducible for $G = AGL_2(3)$. This group G is the 7$^{\text{th}}$ maximal subgroup of S_9 as listed in [**14**]. We can pick two elements $x \in V_2 \smallsetminus \{1\}$ and $y \in G \smallsetminus V_2$, which belong to classes $3A$ and $3C$ in G (in the notation of [**14**]) and which both induce fixed-point-free permutations in S_9. Thus both x and y belong to class $3B$ of S_9. The only irreducible 2-Brauer character of G of degree 16 takes value 1 at y, whereas the character of D^λ takes value -2 at y, cf. [**14**]. Hence we conclude that $D^\lambda{\downarrow}_G$ is reducible. (An alternate way is to note that D^λ is reduction modulo 2 of the basic spin module of a double cover $\hat{\mathsf{S}}_9$, and the latter is reducible over the inverse image of G in $\hat{\mathsf{S}}_9$ by [**26**, Theorem 1.1].)

Now let $H = \mathsf{A}_9$. Then $D^{(5,4)}{\downarrow}_{\mathsf{A}_9}$ splits. Each of $E^{(5,4)}_\pm$ affords 8 non-trivial linear characters of V_2, which are permuted transitively by $AGL_2(3) \cap \mathsf{A}_9 = ASL_2(3)$. Moreover, the only proper subgroup of $SL_2(3)$ that acts transitively on these 8 characters is Q_8. It follows that $G = ASL_2(3)$ or $V_2 \rtimes \mathsf{Q}_8$, in which case the restriction is indeed irreducible, giving the case (iii). ☐

6. Doubly transitive groups with socle $PSL_m(q)$

Throughout the section: $q = r^f$ is a power of a prime r, $m \geq 2$, $W := \mathbb{F}_q^m$ with standard basis $e_1, \ldots, e_m$, and $\mathbb{P}(W)$ is the set of 1-dimensional subspaces of W. Also, unless otherwise stated,

$$n := |\mathbb{P}(W)| = (q^m - 1)/(q-1),$$

and $G < \mathsf{S}_n$ with $S := \mathrm{soc}\,(G) = PSL_m(q)$ acting naturally on $\mathbb{P}(W)$.

6.1. **Bounding the partition λ for groups with socle $PSL_m(q)$.** With the notation as above, we have:

Lemma 6.1. *We have $S \trianglelefteq G \leq PGL_m(q) \rtimes \mathsf{C}_f \leq \mathrm{Aut}(S)$.*

Proof. Note that $N := N_{\mathsf{S}_n}(S)$ is doubly transitive with non-abelian simple normal subgroup S. By [**10**, Proposition 5.2], $\mathrm{soc}\,(N) = S$. Now $C_{\mathsf{S}_n}(S) \cap S = 1$, hence $\mathrm{soc}\,(N) = S$ implies that $C_{\mathsf{S}_n}(S) = 1$. So $S \trianglelefteq G \leq N \leq \mathrm{Aut}(S)$. The group $\mathrm{Aut}(S)$ is described in [**15**, Theorem 2.5.12]. If $m = 2$, we have $\mathrm{Aut}(S) = PGL_m(q) \rtimes \mathsf{C}_f$, and we are done. If $m \geq 3$, the inverse-transpose automorphism of S does not stabilize its action on $\mathbb{P}(W)$, so we have $G \leq PGL_m(q) \rtimes \mathsf{C}_f$. □

By Lemma 6.1, we have $G \leq PGL_m(q) \rtimes \mathsf{C}_f$ where $PGL_m(q) \rtimes \mathsf{C}_f$ acts naturally on $\mathbb{P}(W)$. For $1 \leq k \leq m-1$, let $W_k := \langle e_1, e_2, \dots, e_k \rangle_{\mathbb{F}_q} \subseteq W$, and denote by $\tilde{P}_k$ the subgroup of $PGL_m(q) \rtimes \mathsf{C}_f$ consisting of all elements that fix every point of $\mathbb{P}(W_k)$. (If $k > 1$, then $\tilde{P}_k$ is the image in $PGL_m(q) \rtimes \mathsf{C}_f$ of the subgroup of $GL_m(q) \rtimes C_f$ that acts via scalars on W_k.) Also, let $P_k := \tilde{P}_k \cap G$. By construction, P_k fixes all

$$L_k := (q^k - 1)/(q-1)$$

1-dimensional subspaces of $\langle e_1, e_2, \dots, e_k \rangle_{\mathbb{F}_q}$. Thus:

Lemma 6.2. *The subgroup P_k is contained in a natural subgroup S_{n-L_k} of S_n.*

Lemma 6.3. *Let $\lambda = (n-\ell, \dots) \in \mathscr{P}_p(n)$. For an integer $1 \leq k \leq m-1$ such that $(q^k-1)/(q-1) \geq 2\ell$, we have $(D^\lambda)^{P_k} \neq 0$. In particular, if $D^\lambda{\downarrow}_G$ is irreducible then $\dim D^\lambda \leq [G : P_k]$.*

Proof. The first statement follows from Lemma 6.2 and Theorem 2.11. The second one then follows from the Frobenius Reciprocity. □

Setting

$$P_k \trianglelefteq R_k := \mathrm{Stab}_G(\langle e_1, \dots, e_k \rangle_{\mathbb{F}_q}),$$

we have that

$$PGL_k(q) \cong (R_k \cap S)/(P_k \cap S) \trianglelefteq R_k/P_k \leq PGL_k(q) \rtimes \mathsf{C}_f. \tag{6.4}$$

(Indeed, one can find an element of $SL_n(q)$ that fixes W_k and has any prescribed determinant in its action on W_k.) Since both G and S act transitively on the set $\mathbb{P}_k(W)$ of k-subspaces of W, we have

$$[G : R_k] = [S : S \cap R_k] = |\mathbb{P}_k(W)| = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i - 1}. \tag{6.5}$$

Lemma 6.6. *Let $\lambda = (n-\ell, \dots) \in \mathscr{P}_p(n)$ and $D^\lambda{\downarrow}_G$ be irreducible. For an integer $1 \leq k \leq m-1$ such that $(D^\lambda)^{P_k} \neq 0$, we have*

$$\dim D^\lambda \leq [G : R_k]\mathfrak{b}_p(R_k/P_k) = \frac{[G : P_k]}{[R_k : P_k]}\mathfrak{b}_p(R_k/P_k) = \mathfrak{b}_p(R_k/P_k)\prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}.$$

The assumption $(D^\lambda)^{P_k} \neq 0$ is guaranteed if $(q^k-1)/(q-1) \geq 2\ell$.

Proof. Note that R_k acts on $(D^\lambda)^{P_k}$ and the R_k-module $(D^\lambda)^{P_k}$ contains a simple submodule X of dimension at most $\mathfrak{b}_p(R_k/P_k)$. By the Frobenius reciprocity, we have

$$\dim D^\lambda \le [G:R_k]\dim X \le [G:R_k]\mathfrak{b}_p(R_k/P_k),$$

and it remains to use (6.5) and Lemma 6.3. $\square$

Proposition 6.7. *Let V an irreducible $\mathbb{F}G$-module. Then:*

(i) $\dim V \le |G|^{1/2} \le |\mathrm{Aut}(S)|^{1/2}$.

(ii) $\dim V \le n^{(m+1)/2} < n^{\frac{1}{2}\log_2 n+1}$.

(iii) *Suppose that $V = D^\lambda{\downarrow_G}$ for $\lambda = (n-\ell,\dots) \in \mathscr{P}_p(n)$, and that there exists an integer $1 \le k \le m-1$ such that $(q^k-1)/(q-1) \ge 2\ell$. Then $\dim V < q^{mk}$.*

Proof. (i) Follows from Lemma 6.1.

(ii) Note that $n > 2^{m-1}$, so $m \le 1+\log_2 n$, which implies the second inequality. Let $H := PGL_m(q)$. By Lemma 6.1, $G \le H \rtimes \mathsf{C}_f$. If $m=2$, then $\mathfrak{b}(H) = q+1$. Since $f^2 \le 2^f+1 \le q+1$, we deduce that $\dim V \le f\mathfrak{b}(H) \le (q+1)^{3/2}$, as stated.

Let $m \ge 3$. Using Lemma 2.3, and the estimate $(q^i-1)(q^{m-i}-1) < q^m-1$ for $1 \le i \le m-1$, we get

$$\begin{aligned}\mathfrak{b}(H) &\le \mathfrak{b}(S)\cdot[H:S] \le \mathfrak{b}(SL_m(q))\cdot[H:S]\\ &\le \frac{(q-1)(q^2-1)(q^3-1)\dots(q^m-1)}{(q-1)^m}\cdot\gcd(m,q-1)\\ &\le \frac{(q^m-1)^{(m+1)/2}}{(q-1)^m}\cdot\gcd(m,q-1).\end{aligned}$$

Also, $f \le 2^f-1 \le q-1$. So for $m \ge 5$ we have

$$\dim V \le \mathfrak{b}(G) \le f\mathfrak{b}(H) \le \frac{(q^m-1)^{(m+1)/2}}{(q-1)^m}\cdot f\cdot\gcd(m,q-1) \le \left(\frac{q^m-1}{q-1}\right)^{(m+1)/2},$$

as stated. For $m = 3,4$, using [**42**] one can drop the factor of f in the above estimates for $\mathfrak{b}(H)$ and $\mathfrak{b}(G)$, whence the statement follows again.

(iii) First we consider the case $k=1$. Then note that $R_1 = P_1$ and $S\cap R_1 = S\cap P_1$. It follows from (6.5) and Lemma 6.3 that

$$\dim V \le [G:R_1] = [S:S\cap R_1] = [S:S\cap P_1].$$

Now let $k \ge 2$. As recorded in (6.5), $Y := (S\cap R_k)/(S\cap P_k) \cong PGL_k(q)$. Clearly, $|Y| > q^2 > f^2$, whence $\mathfrak{b}_p(Y) \le |Y|^{1/2} < |Y|/f$. Again using (6.5), we obtain

$$\mathfrak{b}_p(R_k/P_k) \le f\cdot\mathfrak{b}_p(Y) < |Y|.$$

Combining with Lemma 6.3 and (6.5), we get

$$\dim V < [G:R_k]\cdot|Y| = [S:S\cap R_k]\cdot|(S\cap R_k)/(S\cap P_k)| = [S:S\cap P_k].$$

Thus in both cases we have

$$\dim V \le [S:S\cap P_k] \le q^{k(k-1)/2}\prod_{i=m-k+1}^{m}(q^i-1) < q^{mk},$$

which completes the proof. $\square$

Proposition 6.8. *Let $n \geq 324$, $m \geq 4$, $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$ such that $D^\lambda\downarrow_G$ is irreducible, and define ℓ from $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Then $\ell \leq 4$ if $2 \leq q \leq 5$ and $\ell \leq 3$ if $q \geq 7$.*

Proof. We may assume that $\ell \geq 4$, for otherwise there is nothing to prove. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n-\ell$. By Propositions 6.7(ii) and 2.23, we have

$$\ell \leq L(n) := 0.7\log_2 n + 1.4. \tag{6.9}$$

So $n \geq p(\delta_p + \ell - 2)$, and by Theorem 2.21, we have

$$\dim D^\lambda \geq C^p_\ell(n) > \frac{(n+3-3\ell)^\ell}{\ell!}. \tag{6.10}$$

Claim 1: If $1 \leq k \leq m-1$ and $(q^k-1)/(q-1) \geq 2\ell$ then $q^{mk} > \frac{(n+3-3\ell)^\ell}{\ell!}$.
Indeed, by Proposition 6.7(iii), $\dim D^\lambda < q^{mk}$, and the claim follows from (6.10).

Claim 2: If $k := \lceil \log_q(2\ell-1) \rceil + 1$, then $q^{mk} > \frac{(n+3-3\ell)^\ell}{\ell!}$.
To prove Claim 2, it suffices to verify that the given k satisfies the assumptions of Claim 1. Clearly $k \geq 1$, and $(q^k-1)/(q-1) \geq 2\ell$ is easy. Note that $(2L(n)-1)^2 < n$ by our assumption on n, so from (6.9), we have $2\ell - 1 < n^{1/2}$, and hence, using also $m \geq 4$, we get $q^2(2\ell-1) < q^{m/2}n^{1/2} < q^m$. Now $k \leq m-1$ follows from

$$q^k < q^{\log_q(2\ell-1)+2} = q^2(2\ell-1) < q^m.$$

Suppose $\ell \geq 12$. Then for k as in Claim 2, we have

$$\ell/k > \frac{\ell}{\log_2(2\ell-1)+2} > 1.83. \tag{6.11}$$

On the other hand, $n = (q^m-1)/(q-1)$ implies that $m < \log_q n + 1 < \frac{4}{3}\log_q n$, i.e.

$$n > q^{3m/4}. \tag{6.12}$$

Also, for $n \geq 324$ we have $(L(n)/1.87)^{4.27} < n$, and so from (6.9) we get

$$\frac{1.87n}{\ell} > n^{0.765}. \tag{6.13}$$

We also have

$$n + 3 - 3\ell > n + 3 - 3L(n) > 14.8n/15.8 \tag{6.14}$$

for $n \geq 324$. Using Claim 2 and (6.11)–(6.14), and $\ell! < (\ell/2)^\ell$ (which certainly holds for $\ell \geq 12$), we arrive at a contradiction:

$$q^{mk} > \frac{(n+3-3\ell)^\ell}{\ell!} > \left(\frac{14.8n/15.8}{\ell/2}\right)^\ell > \left(\frac{1.87n}{\ell}\right)^\ell > n^{0.765\ell} > n^{1.39k} > q^{mk}.$$

Suppose $8 \leq \ell \leq 11$. If $q \geq 3$, we take k as in Claim 2. If $q = 2$ then $k = 5$ satisfies the assumptions of Claim 1—indeed, $n = (2^m-1) \geq 324$ implies $m \geq 9$, and $2^5 - 1 > 2\ell$. As we have $k \leq 5$ for all q, using Claims 1 and 2 for $q = 2$ and $q \geq 3$, respectively, we get

$$q^{5m} \geq q^{mk} > \frac{(n+3-3\ell)^\ell}{\ell!}. \tag{6.15}$$

If $\ell = 10$ or 11, then

$$\frac{(n+3-3\ell)^\ell}{\ell!} \geq \frac{(n-27)^{10}}{10!} > n^7 > q^{7(m-1)},$$

hence $5m > 7(m-1)$, a contradiction. If $\ell = 9$, then

$$\frac{(n+3-3\ell)^\ell}{\ell!} = \frac{(n-24)^9}{9!} > n^{20/3} > q^{20(m-1)/3},$$

hence $5m > 20(m-1)/3$, a contradiction since $m \geq 4$. Let $\ell = 8$. Then for $q \geq 3$ we have $k = \lceil \log_q(15) \rceil + 1 \leq 4$. So by Claim 2, we have

$$q^{4m} \geq q^{mk} > \frac{(n-21)^8}{8!} > n^6 > q^{6(m-1)},$$

a contradiction. For $q = 2$, we have $m \geq 9$, $k = 5$, and we again get a contradiction:

$$q^{5m} = q^{mk} > \frac{(n-21)^8}{8!} > n^6 > q^{6(m-1)}.$$

Suppose $\ell = 7$. If $q \geq 3$, choose k as in Claim 2. If $q = 2$, then choose $k = 4$. In both cases we have $k \leq 4$. Now we get a contradiction using Claims 1 and 2:

$$q^{4m} \geq q^{mk} > \frac{(n-18)^7}{7!} > n^{16/3} > q^{16(m-1)/3}.$$

Suppose $5 \leq \ell \leq 6$. If $q \geq 3$ take $k = 3$ and apply Claim 1 to get a contradiction:

$$q^{3m} \geq q^{mk} > \frac{(n-12)^5}{5!} > n^4 > q^{4(m-1)}.$$

If $q = 2$ take $k = 4$ and apply Claim 1 to get a contradiction:

$$2^{4m} = q^{mk} > \frac{(n-12)^5}{5!} > (n+1)^4 > 2^{4m}.$$

If $q \geq 7$ and $\ell = 4$ take $k = 2$ and apply Claim 1 to get a contradiction:

$$q^{2m} = q^{mk} > \frac{(n-9)^4}{4!} > n^{8/3} > q^{8(m-1)/3}.$$

□

6.2. **Ruling out the remaining D^λ for groups with socle $PSL_m(q)$.** Proposition 6.8 rules out irreducible restrictions $D^\lambda{\downarrow}_G$ in the generic case where $n \geq 324$, $m \geq 4$, and ℓ not too small. In this subsection we deal with the remaining cases.

Lemma 6.16. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, $m \geq 5$, and $q = 2$. If $D^\lambda{\downarrow}_G$ is irreducible then $\lambda \in \mathscr{L}^{(1)}(n)$.*

Proof. By Lemma 6.1, we have $G = S = SL_m(2)$. Set $V := D^\lambda$ and suppose $V{\downarrow}_G$ is irreducible. Write $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n - \ell$. We need to prove that $\ell \leq 1$.

Claim 1: $\ell \leq 5$.

If $m \geq 9$, then $n = 2^m - 1 \geq 511$ and so $\ell \leq 4$ by Proposition 6.8. Let $m = 8$. By [**14**], $\mathfrak{b}(G) = 361416600 < 2^{28.5}$. As $\dim V \geq 2^{\ell/2}$ by Theorem 2.22, we have that $\ell \leq 56 < n/4$. By Theorem 2.21,

$$\dim V > \left(\frac{2(n+3)}{\ell} - 6\right)^\ell = \left(\frac{516}{\ell} - 6\right)^\ell > \mathfrak{b}(G),$$

for $6 \leq \ell \leq 56$, which contradicts the irreducibility of $V{\downarrow_G}$. Let $m = 7$. By [**14**], $\mathfrak{b}(G) = 2731008 < 2^{21.5}$, whence $\ell \leq 42 < n/3$ by Theorem 2.22. By Theorem 2.21,

$$\dim V > \left(\frac{2(n+3)}{\ell} - 6\right)^{\ell} = \left(\frac{262}{\ell} - 6\right)^{\ell} > \mathfrak{b}(G),$$

for $6 \leq \ell \leq 42$, which contradicts the irreducibility of $V{\downarrow_G}$. Let $m = 6$. By [**14**] for $p = 3$ and [**43**] for $p = 2$,

$$\max(\mathfrak{b}_2(G), \mathfrak{b}_3(G)) = \max(32768, 29295) = 32768 < (n^3 - 9n^2 + 14n)/6,$$

whence $\ell \leq 2$ by Proposition 2.28(i). The case $m = 5$ is treated similarly, using the bound $\dim V \leq 1024$ coming from [**14**].

Claim 1: $\ell \leq 2$.
By Claim 1, we may assume that $\ell \leq 5$, so we can take $k = 4$ in Lemma 6.6 to get

$$\dim V \leq \frac{[G:P_4]}{[R_4:P_4]}\mathfrak{b}_p(R_4/P_4) = \frac{[G:P_4]}{[R_4:P_4]}\mathfrak{b}_p(SL_4(2)) < \frac{(2^m-1)^4}{20160} \cdot 64 = \frac{n^4}{315}.$$

If $\ell = 4$ or 5, then $\dim V \geq \min\{(n-9)^4/24, (n-12)^5/120\}$ by Theorem 2.21, a contradiction since $n \geq 31$. So $\ell \leq 3$. Taking $k = 3$ in Lemma 6.6, we get

$$\dim V \leq \frac{[G:P_3]}{[R_3:P_3]}\mathfrak{b}_p(SL_3(2)) < \frac{(2^m-1)^3}{21} = \frac{n^3}{21} < \frac{n^3 - 9n^2 + 14n}{6},$$

since $n \geq 31$. By Proposition 2.28(i), this implies that $\ell \leq 2$.

Now we consider the case $\ell = 2$. If $\lambda = (n-2, 1^2)$ then $p = 3$. Using Lemma 2.6(iii) one can show that $D^\lambda = \wedge^2(D^{(n-1,1)})$. Thus $SL_m(2)$ admits a non-trivial (irreducible) module $V{\downarrow_G}$ whose exterior square is irreducible. This is impossible by [**45**, Proposition 3.4]. (An alternative argument is to note that G is not 3-homogeneous and then apply [**32**, Theorem A].)

Finally, let $\lambda = (n-2, 2)$. By Lemma 6.2, $P_2 \leq S_{n-3}$, and by Lemma 2.13 we have $V^{P_2} \neq 0$, so by Lemma 6.6, we obtain

$$\dim V \leq \frac{[G:P_2]}{[R_2:P_2]}\mathfrak{b}(SL_2(2)) < \frac{2(2^m-1)^2}{6} = \frac{n^2}{3} < \frac{(n^2 - 5n + 2)}{2},$$

since $n \geq 31$. This contradicts Lemma 2.27. □

Lemma 6.17. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, $m \geq 4$, and $q = 3$. If $D^\lambda{\downarrow_G}$ is irreducible then $\lambda \in \mathscr{L}^{(1)}(n)$.*

Proof. By Lemma 6.1, we have $PSL_m(3) \trianglelefteq G \leq PGL_m(3)$. Set $V := D^\lambda$ and suppose $V{\downarrow_G}$ is irreducible. Write $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n - \ell$. We need to prove that $\ell \leq 1$.

If $m \geq 6$, then $n \geq 364$ and so $\ell \leq 4$ by Proposition 6.8. If $m = 5$, by [**14**], we have $\mathfrak{b}(G) \leq 98010 < (n^3 - 9n^2 + 14n)/6$, hence $\ell \leq 2$ by Proposition 2.28(i). The same argument applies in the case $m = 4$ where $\mathfrak{b}(G) \leq 2080$. Thus, we have $\ell \leq 4$ in all cases.

By [**14**], $\mathfrak{b}_p(SL_3(3)) \leq 27$. So by Lemma 6.6, we obtain

$$\dim V \leq \frac{27(3^m-1)(3^{m-1}-1)(3^{m-2}-1)}{(3^3-1)(3^2-1)(3-1)} < \frac{(3^m-1)^3}{416} = \frac{n^3}{52} < \frac{n^3 - 9n^2 + 14n}{6},$$

since $n \geq 40$. This in turn implies by Proposition 2.28(i) that $\ell \leq 2$. We can take $k = 2$ in Lemma 6.6, and, using $\mathfrak{b}_p(R_2/P_2) = \mathfrak{b}_p(PGL_2(3)) = \mathfrak{b}_p(\mathsf{S}_4) \leq 3$, to get

$$\dim V \leq \frac{3(3^m-1)(3^{m-1}-1)}{(3^2-1)(3-1)} < \frac{n^2}{4} < \frac{(n^2-5n+2)}{2},$$

and so $\ell = 1$ by Lemma 2.27. □

Lemma 6.18. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, $m \geq 4$, and $q = 4$. If $D^\lambda{\downarrow}_G$ is irreducible then $\lambda \in \mathscr{L}^{(1)}(n)$.*

Proof. By Lemma 6.1, we have $PSL_m(4) \trianglelefteq G \leq PGL_m(4) \rtimes \mathsf{C}_2$. Set $V := D^\lambda$ and suppose $V{\downarrow}_G$ is irreducible. Write $n-\ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n - \ell$. We need to prove that $\ell \leq 1$.

If $m \geq 5$, then $n \geq 341$ and so $\ell \leq 4$ by Proposition 6.8. If $m = 4$ then by [**14**], we have $\dim V \leq 2 \cdot 7140 < (n^3 - 9n^2 + 14n)/6$, whence $\ell \leq 2$ by Proposition 2.28(i). Thus we always have $\ell \leq 4$, and we can take $k = 3$ in Lemma 6.6. Note using (6.4) that $R_3/P_3 \leq PGL_3(4) \rtimes \mathsf{C}_2$, so $\mathfrak{b}_p(R_3/P_3) \leq 2\mathfrak{b}_p(PGL_3(4)) = 128$ by [**14**], and by Lemma 6.6,

$$\dim V \leq \frac{128(4^m-1)(4^{m-1}-1)(4^{m-2}-1)}{(4^3-1)(4^2-1)(4-1)} < \frac{2(4^m-1)^3}{2835} = \frac{2n^3}{105} < \frac{n^3-9n^2+14n}{6}$$

since $n \geq 85$. By Proposition 2.28(i), we have $\ell \leq 2$. So we can take $k = 2$ in Lemma 6.6. Note that $R_3/P_3 \leq \mathsf{S}_5$ as $\mathbb{F}_4^2$ contains 5 lines, whence $\mathfrak{b}_p(R_3/P_3) \leq \mathfrak{b}_p(\mathsf{S}_5) \leq 6$, and by Lemma 6.6,

$$\dim V \leq \frac{6(4^m-1)(4^{m-1}-1)}{(4^2-1)(4-1)} < \frac{3n^2}{10} < \frac{(n^2-5n+2)}{2}.$$

Now $\ell = 1$ by Lemma 2.27. □

Lemma 6.19. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, $m \geq 4$, and $q \geq 5$. If $D^\lambda{\downarrow}_G$ is irreducible then $\lambda \in \mathscr{L}^{(1)}(n)$.*

Proof. By Lemma 6.1, we have $PSL_m(q) \trianglelefteq G \leq PGL_m(q) \rtimes \mathsf{C}_f$. Note that $f < q/2.6$ as $q \geq 5$. Set $V := D^\lambda$ and suppose $V{\downarrow}_G$ is irreducible. Write $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n - \ell$. We need to prove that $\ell \leq 1$.

We claim that $\ell \leq 3$. If $m = 4$ then $|G| \leq f \cdot |PGL_4(q)| < q^{15} f < q^{16}/2.6$, and

$$\dim V \leq |G|^{1/2} < \frac{q^8}{\sqrt{2.6}} < \frac{q^9}{8} < \frac{n^3}{8} < \frac{n^3-9n^2+14n}{6},$$

hence $\ell \leq 2$ by Proposition 2.28(i). Let $m \geq 5$. Then $n \geq 781$, and we have $\ell \leq 4$ by Proposition 6.8, so we may assume that $\ell = 4$. We show that $\dim V < \frac{(n-9)^4}{24}$, which contradicts Theorem 2.21. If $m = 5$ then

$$\dim V \leq \sqrt{|G|} \leq \sqrt{f \cdot |PGL_5(q)|} < \sqrt{q^{24} f} < \sqrt{q^{25}/2.6} < \frac{q^{12.5}}{\sqrt{2.6}} < \frac{n^{25/8}}{\sqrt{2.6}} < \frac{(n-9)^4}{24}.$$

If $m \geq 6$, Proposition 6.7(iii) with $k = 3$ yields

$$\dim V < q^{3m} \leq q^{3.6(m-1)} < n^{3.6} < \frac{(n-9)^4}{24}.$$

As $\ell \le 3$, we can take $k = 2$ in Lemma 6.1. Note using (6.4) that $R_2/P_2 \le PGL_2(q) \rtimes \mathsf{C}_f$, so

$$\mathfrak{b}_p(R_2/P_2) \le f\mathfrak{b}_p(PGL_2(q)) \le f(q+1),$$

and by Lemma 6.6,

$$\begin{aligned}\dim V &\le f(q+1)\frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)} = \frac{f(q^m-1)(q^{m-1}-1)}{(q-1)^2} \\ &< \frac{n^2 f}{q} < \frac{n^2}{2.6} < \frac{n^2-5n+2}{2}\end{aligned}$$

since $n \ge 156$. We conclude that $\ell = 1$ by by Lemma 2.27. $\square$

Lemma 6.20. *Let $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$, and $(m, q) = (3, q \ge 5)$ or $(2, q \ge 11)$. If $D^\lambda{\downarrow}_G$ is irreducible then $\lambda \in \mathscr{L}^{(1)}(n)$.*

Proof. By Lemma 6.1, we have $PSL_m(q) \trianglelefteq G \le PGL_m(q) \rtimes \mathsf{C}_f$. Note that $f \le 3q/8$ as $q \ge 5$. Set $V := D^\lambda$ and suppose $V{\downarrow}_G$ is irreducible. Write $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda_1 = n - \ell$. We need to prove that $\ell \le 1$.

If $m = 3$ then $n = q^2 + q + 1$, and

$$\dim V \le \mathfrak{b}(G) \le f\mathfrak{b}(PGL_3(q)) \le f(q+1)(q^2+q+1) < 3n^2/8 < (n^2-5n+2)/2,$$

hence $\ell = 1$ by Lemma 2.27.

Let $m = 2$, so $n = q + 1$. If $q \ge 16$, we have that

$$\dim V \le \mathfrak{b}(G) \le f\mathfrak{b}(PGL_2(q)) \le f(q+1) < 3n(n-1)/8 < (n^2-5n+2)/2,$$

hence $\ell = 1$ by Lemma 2.27. If $q = 13$ or 11, then $\dim V \le \mathfrak{b}(G) = q + 1 = n$, and we conclude that $\ell = 1$ using [**14**]. $\square$

Now we can prove the main result of this section:

Theorem 6.21. *Let $p = 2$ or 3, and $\lambda \in \mathscr{P}_p(n)$ such that $\dim D^\lambda > 1$. Suppose that $G < \mathsf{S}_n$ is a doubly transitive subgroup with $S = \mathrm{soc}\,(G) = PSL_m(q)$ acting on $n = (q^m-1)/(q-1)$ 1-subspaces of $\mathbb{F}_q^m$, and either $m \ge 3$, or $m = 2$ and $q \ge 4$. Then $D^\lambda{\downarrow}_G$ is irreducible if and only if one of the following holds:*

(i) *$\lambda \in \mathscr{L}^{(1)}(n)$. Furthermore, $p \nmid q$ if $m \ge 3$, and $G \not\le P\Sigma L_2(q)$ if $m = p = 2 \nmid q$.*
(ii) *$m = 2$, and one of* (ii), (iii), (v) *of Proposition 3.10 occurs.*

Proof. Define ℓ from $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Then $\ell \ge 1$. Recall the notation (2.1). Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda = (n - \ell, \mu)$ for a partition μ of ℓ. If (m, q) are as listed in Proposition 3.10, then we are done. Otherwise we apply Lemmas 6.16–6.19 when $m \ge 4$ and Lemma 6.20 when $2 \le m \le 3$ to conclude that $\ell = 1$, in which case the theorem follows from the main result of [**47**]. $\square$

7. Doubly transitive groups $Sp_{2m}(2)$

Throughout the section: $\delta \in \{0, 1\}$, $m \ge 3$, W is a $2m$-dimensional vector space over $\mathbb{F}_2$ with symplectic form $(\cdot,\cdot)$ and symplectic basis $(e_1, \ldots, e_m, f_1, \ldots, f_m)$, Ω^δ is the set of the quadratic forms of Witt defect δ on W associated with $(\cdot,\cdot)$,

$$n = n(\delta) := |\Omega^\delta| = 2^{m-1}(2^m + (-1)^\delta),$$

and $G = Sp(W) \cong Sp_{2m}(2)$ is embedded into S_n via its doubly transitive action on Ω^δ. For $1 \le k \le m$ we put $W_k := \langle e_1, \ldots, e_k \rangle_{\mathbb{F}_2}$.

7.1. **Bounding D^λ for $Sp_{2m}(2)$.** We follow [**12**, §7.7] and [**8**, §5]. Let Ω be the set of all quadratic forms on W which satisfy $Q(v+w)-Q(v)-Q(w)=(v,w)$ for all $v,w\in W$. The group G acts on Ω via $g\cdot Q(w)=Q(g^{-1}w)$ for $g\in G, Q\in\Omega, w\in W$. Let $Q_0\in\Omega$ be the quadratic form defined by $Q_0\big(\sum_{i=1}^m(a_ie_i+b_if_i)\big)=\sum_{i=1}^m a_ib_i$. Then $\Omega=\{Q^v\mid v\in W\}$, where $Q^v(-):=Q_0(-)+(v,-)$. For $\delta\in\{0,1\}$, we set

$$\Omega^\delta=\{Q^v\mid Q_0(v)=\delta\}. \tag{7.1}$$

By [**12**, Theorem 7.7A], Ω^0 and Ω^1 are the G-orbits on Ω, and the G-action on both of them is doubly transitive. Note that $Q_0=Q^0\in\Omega^0$, also fix $Q_1:=Q^{e_m+f_m}\in\Omega^1$.

Let $1\le k\le m-1$. We define certain subgroups $P_k^\delta\trianglelefteq R_k^\delta\le G$. First, let

$$P_k^\delta:=\mathrm{Stab}_G(Q_\delta,e_1,\dots,e_k)=\mathrm{Stab}_{O(Q_\delta)}(e_1,\dots,e_k) \tag{7.2}$$

be the subgroup of G that fixes Q_δ and each of k vectors $e_1,\dots,e_k$. Also set

$$R_k^\delta:=\mathrm{Stab}_{O(Q_\delta)}(W_k). \tag{7.3}$$

Note that $P_k^\delta\trianglelefteq R_k^\delta$ and

$$R_k^\delta/P_k^\delta\cong SL_k(2). \tag{7.4}$$

Lemma 7.5. *Let $1\le k\le m-1$. Then P_k^δ fixes 2^k quadratic forms in Ω^δ, so P_k^δ is contained in a natural subgroup S_{n-2^k} in S_n.*

Proof. Note that P_k^δ fixes each of the 2^k forms $\{Q^{v+\delta(e_m+f_m)}\mid v\in W_k\}$ in Ω^δ. □

Lemma 7.6. *Let $\lambda=(n-\ell,\mu)\in\mathscr{P}_p(n)$ for a partition μ of ℓ. For an integer $1\le k\le m-1$ such that $2^{k-1}\ge\ell$, we have $(D^\lambda)^{P_k^\delta}\ne0$. In particular, if $D^\lambda{\downarrow}_G$ is irreducible then $\dim D^\lambda\le[G:P_k^\delta]$.*

Proof. The first statement follows from Lemma 7.5 and Theorem 2.11. The second one then follows from the Frobenius Reciprocity. □

Lemma 7.7. *Let $\lambda=(n-\ell,\mu)\in\mathscr{P}_p(n)$ with $\mu\vdash\ell$ and $D^\lambda{\downarrow}_G$ be irreducible. For an integer $1\le k\le m-1$ such that $(D^\lambda)^{P_k^\delta}\ne0$, we have*

$$\dim D^\lambda\le[G:R_k^\delta]\mathfrak{b}_p(SL_k(2))=\frac{[G:P_k^\delta]}{|SL_k(2)|}\mathfrak{b}_p(SL_k(2)).$$

The assumption $(D^\lambda)^{P_k^\delta}\ne0$ is guaranteed if $2^{k-1}\ge\ell$.

Proof. Note that R_k^δ acts on $(D^\lambda)^{P_k^\delta}$ and the R_k^δ-module $(D^\lambda)^{P_k^\delta}$ contains a simple submodule X of dimension at most $\mathfrak{b}_p(R_k/P_k)$. By the Frobenius reciprocity, we have

$$\dim D^\lambda\le[G:R_k^\delta]\dim X\le[G:R_k^\delta]\mathfrak{b}_p(R_k/P_k)=\frac{[G:P_k^\delta]}{[R_k^\delta:P_k^\delta]}\mathfrak{b}_p(R_k^\delta/P_k^\delta),$$

and it remains to use (7.4) and Lemma 7.6. □

Lemma 7.8. *For $1\le k\le m-1$, we have*

$$[G:P_k^\delta]\;=\;2^{m-1+k(k-1)/2}(2^{m-k}+(-1)^\delta)\prod_{i=m-k+1}^{m}(2^{2i}-1)\;<\;2^{(4m-k)(k+1)/2}.$$

Proof. Let $\varepsilon := +$ if $\delta = 0$ and $\varepsilon := -$ if $\delta = 1$. We have

$$\begin{aligned}
\dim V &\le [G : P_k^\delta] = [G : O(Q_\delta)]\,[O(Q_\delta) : P_k^\delta] \\
&= 2^{m-1}(2^m + (-1)^\delta)\,[O^\varepsilon_{2m}(2) : (GL_k(2) \times O^\varepsilon_{2m-2k}(2))]_{2'}\,|GL_k(2)| \\
&= 2^{m-1}(2^m + (-1)^\delta)(2^m - (-1)^\delta)(2^{m-k} + (-1)^\delta) \prod_{i=m-k+1}^{m-1} (2^{2i} - 1) \cdot 2^{k(k-1)/2} \\
&= 2^{m-1+k(k-1)/2}(2^{m-k} + (-1)^\delta) \prod_{i=m-k+1}^{m} (2^{2i} - 1) < 2^{(4m-k)(k+1)/2},
\end{aligned}$$

as required. $\square$

Proposition 7.9. *Let V be an irreducible $\mathbb{F}G$-module.*

(i) *If $m \ge 4$ then $\dim V < (\sqrt{3}/2) \cdot 2^{m(m+1/2)} < n^{\frac14 \log_2 n + 1.52}$.*
(ii) *Suppose that $V = D^\lambda{\downarrow}_G$ for $\lambda = (n-\ell, \dots) \in \mathscr{P}_p(n)$, and that there exists an integer $1 \le k \le m-1$ such that $2^{k-1} \ge \ell$. Then $\dim V < 2^{(4m-k)(k+1)/2}$.*

Proof. (i) Note that $n > 2^{2m-2}$, so $m < \frac12 \log_2 n + 1$. As $m \ge 4$, we have $n \ge 28$, and

$$\begin{aligned}
\dim V &\le |G|^{1/2} < (3 \cdot 2^{m(2m+1)-2})^{1/2} = (\sqrt{3}/2) \cdot 2^{m(m+1/2)} \\
&< (\sqrt{3}/2) \cdot 2^{(1+\frac12 \log_2 n)(\frac32 + \frac12 \log_2 n)} = \sqrt{6} \cdot 2^{(\log_2 n)\cdot(\frac14 \log_2 n + \frac54)} \\
&= \sqrt{6} \cdot n^{\frac14 \log_2 n + \frac54} < n^{\frac14 \log_2 n + 1.52}.
\end{aligned}$$

(ii) Follows from Lemmas 7.6 and 7.8. $\square$

Proposition 7.10. *Let $m \ge 3$, $p = 2$ or 3, $\lambda \in \mathscr{P}_p(n)$ such that $D^\lambda{\downarrow}_G$ is irreducible. Determine ℓ from $n - \ell = \max(\lambda_1, \lambda_1^{\mathtt{M}})$. Then $\ell \le 2$ if $m = 3, 4$, and $\ell \le 3$ if $m \ge 5$.*

Proof. Replacing λ by $\lambda^{\mathtt{M}}$ if necessary, we may assume that $\lambda = (n-\ell, \mu)$ for a partition μ of ℓ. Let $m = 4$. By [**14**], $\mathfrak{b}_p(Sp_8(2)) \le 2^{16}$. So

$$\dim D^\lambda \le 2^{16} < (n^3 - 9n^2 + 14)/6,$$

hence $\ell \le 2$ by Proposition 2.28(i). The same argument applies to the case $m = 3$. So we may assume that $m \ge 5$, hence $n \ge 496$, and $\ell \ge 4$.

By Proposition 7.9(i), we have that $\dim V < n^{L(n)}$ with

$$L(n) := \frac14 \log_2 n + 1.52 < \frac12 \log_2 n + 1.$$

So by Proposition 2.23, we get

$$\ell \le L'(n) := 0.7 \log_2 n + 1.4 < n/p + 2 - \delta_p. \tag{7.11}$$

Now Theorem 2.21 applies to give $\dim D^\lambda \ge C_\ell^p(n)$, hence

$$n^{L(n)} > \dim D^\lambda \ge C_\ell^p(n) = \frac{1}{\ell!} \prod_{i=0}^{\ell-1} (n - (\delta_p + i)p). \tag{7.12}$$

Assume that $\ell = 4$. By Lemmas 7.7 and 7.8 with $k = 3$, we have

$$\dim V \le \frac{[G : P_3^\delta]}{|SL_3(2)|} \mathfrak{b}_p(SL_3(2)) < \frac{2^{8m-6}}{168} \cdot 8 = \frac{2^{8m-6}}{21}.$$

On the other hand, (7.12) implies

$$\dim V \geq \frac{(n-9)^4}{24} \geq \frac{(2^{m-1}(2^m-1)-9)^4}{24} > \frac{2^{8m-6}}{21}$$

as $m \geq 5$, a contradiction. So we may assume that $\ell \geq 5$.

Assume that $\ell = 5$. By Lemma 7.7 with $k = 4$, we have

$$\dim V \leq \frac{[G : P_4^\delta]}{|SL_4(2)|}\mathfrak{b}_p(SL_4(2)) < \frac{n^5}{20160} \cdot 64 = \frac{n^5}{315},$$

where we have used Lemma 7.8 to get $[G : P_4^\delta] < 2^{10m-10} < n^5$. On the other hand, (7.12) implies

$$\dim V \geq \frac{(n-12)^5}{120} > \frac{n^5}{315}$$

as $n \geq 496$, a contradiction.

Now we may assume that $\ell \geq 6$. In particular, $\ell! < (\ell/2)^\ell$, and by (7.12), we get

$$n^{L(n)} > \dim D^\lambda > \frac{(n+3-3\ell)^\ell}{\ell!} > \left(\frac{2(n+3)}{\ell} - 6\right)^\ell. \tag{7.13}$$

If $\ell \geq 1.3L(n)$, then

$$n^{L(n)} > \left(\frac{2(n+3)}{\ell} - 6\right)^{1.3L(n)}$$

and so, since $n \geq 496$,

$$\ell \geq \frac{2(n+3)}{n^{1/1.3}+6} > L'(n),$$

contradicting (7.11). So

$$\ell < 1.3L(n) < 0.33\log_2 n + 2. \tag{7.14}$$

Now $\ell \geq 6$ implies $m \geq 7$ and $n \geq 8128$. In this case, (7.14) implies that

$$\ell < \sqrt{n}/16. \tag{7.15}$$

As $n < 2^{2m}$, we get $\ell < 2^{m-4}$ and so for

$$k := \lceil \log_2 \ell \rceil + 1 \tag{7.16}$$

we have $1 < k < m-2$ and $2^{k-1} \geq \ell$. By Proposition 7.9(ii), we now conclude that

$$\dim D^\lambda < 2^{(4m-k)(k+1)/2}. \tag{7.17}$$

If $\ell \geq 14$ then (7.16) implies that $k+1 < \log_2 \ell + 3 < \ell/2$. As $n > 2^{2m-2}$ and $k \geq 4$, we then have from (7.17) that

$$\dim D^\lambda < 2^{(2m-2)(k+1)} < n^{k+1} < n^{\ell/2}.$$

On the other hand, using (7.15) and (7.13), we obtain

$$\dim D^\lambda > \left(\frac{2(n+3)}{\ell} - 6\right)^\ell > (32\sqrt{n} - 6)^\ell > n^{\ell/2},$$

a contradiction.

If $9 \leq \ell \leq 13$ then $k = 5$ by (7.16). Using (7.13) and (7.17) we get

$$\frac{(n-36)^9}{13!} < \dim D^\lambda < 2^{12m-15} < n^6,$$

which is a contradiction since $n \geq 8128$.

If $6 \le \ell \le 8$ then $k = 4$ by (7.16). Again using (7.13) and (7.17), we obtain

$$\min\left\{\frac{(n-15)^6}{6!}, \frac{(n-21)^7}{8!}\right\} < \dim V < 2^{10m-10} < n^5,$$

a contradiction. The proof of the claim is complete. ☐

7.2. **Ruling out the remaining D^λ for $Sp_{2m}(2)$.**

Proposition 7.18. *Let $p = 3$, $n \ge 28$, and $\lambda = (n-a, a)$ with $a = 2$ or 3. Then $D^\lambda{\downarrow}_G$ is reducible.*

Proof. Note that $n = 2^{m-1}(2^m + (-1)^\delta) \not\equiv 2 \pmod 3$. In particular, by Lemma 2.5, if $\lambda = (n-2, 2)$ we may assume that $n \equiv 1 \pmod 3$. We will use the following notation from [**32**]:

$$S_k := S^{(n-k,k)},\ M_k := M^{(n-k,k)},\ i_k(G) := \dim M_k^G,$$
$$\mathcal{E}(\lambda) := \mathrm{End}_{\mathbb{F}}(D^\lambda),\ \mathcal{I}(G) := \mathrm{ind}_G^{\mathsf{S}_n} \mathbb{F}_G.$$

If $\lambda = (n-2, 2)$ and $n \equiv 1 \pmod 3$, or $\lambda = (n-3, 3)$ and $n \equiv 0 \pmod 3$, then by [**46**, Lemma 6.8], there exists a homomorphism $\zeta : M_3 \to \mathcal{E}(\lambda)$ with $[\mathrm{im}\,\zeta : D_3] \ne 0$. If $\lambda = (n-3, 3)$ and $n \equiv 1 \pmod 3$, then by [**46**, Lemma 6.12], there exists a homomorphism $\zeta : M_3 \to \mathcal{E}(\lambda)$ with $[\mathrm{im}\,\zeta : D_3] \ne 0$ or there exists a homomorphism $\zeta' : M_4 \to \mathcal{E}(\lambda)$ with $[\mathrm{im}\,\zeta' : D_4] \ne 0$.

From [**32**, Corollary 2.31], $(S_1^*)^G = 0$. Further if $n \equiv 1 \pmod 3$ then by [**32**, Corollary 7.5], we have $(S_2^*)^G = 0$. By [**8**, Lemmas 5.11, 5.12], we have $i_2(G) = 1$, $i_3(G) = 2$ and $i_4(G) > 2$. It then follows by [**32**, Lemmas 3.3, 3.4] that there exists a homomorphism $\psi : \mathcal{I}(G) \to M_3$ with $[\mathrm{im}\,\psi : D_3] \ne 0$. If $n \equiv 1 \pmod 3$ then by [**46**, Lemma 3.5] there exists a homomorphism $\psi' : \mathcal{I}(G) \to M_4$ with $[\mathrm{im}\,\psi' : D_4] \ne 0$.

Therefore, $[\mathrm{im}\,(\zeta \circ \psi) : D_3] \ne 0$ or $[\mathrm{im}\,(\zeta' \circ \psi') : D_4] \ne 0$. The proposition then follows from [**32**, Lemma 2.18]. ☐

Lemma 7.19. *Let $p = 2$, $n \ge 28$, and $\lambda = (n-a, a)$ with $a = 2$ or 3. Then $D^\lambda{\downarrow}_G$ is reducible.*

Proof. Assume the contrary. Note that D^λ is a subquotient of the $\mathbb{F}\mathsf{S}_n$-module $\mathtt{Sym}^a(U)$, where U denotes the $\mathbb{F}\mathsf{S}_n$-permutation module on the set Ω^δ of cardinality $n = 2^{m-1}(2^m + (-1)^\delta$. It is shown on [**47**, p. 10] that the $\mathbb{F}G$-module U contains a subquotient B of dimension $2m + 1 \ge 7$. Thus, in the Grothendieck group of $\mathbb{F}G$-modules we can write $U = A + B$ for a $\mathbb{F}G$-module A of dimension $n - (2m+1)$. Note that $\dim A \ge 4(\dim B)$ since $m \ge 3$. This implies that, in the following decomposition in the Grothendieck group

$$\mathtt{Sym}^a(U) = \sum_{i=0}^{a} \mathtt{Sym}^{a-i}(A) \otimes \mathtt{Sym}^i(B), \tag{7.20}$$

the summand $\mathtt{Sym}^a(A)$ has the largest dimension. Now, by Proposition 2.28(i) we have

$$\dim \mathtt{Sym}^3(A) \le (n-5)(n-6)(n-7)/6 < (n^3 - 9n^2 + 14n)/6 \le \dim D^{(n-3,3)}$$

as $n \ge 28$. Thus, when $a = 3$ every summand in (7.20) has dimension less than $\dim V$, and so $V{\downarrow}_G$ cannot be irreducible. Likewise,

$$\dim \mathtt{Sym}^2(A) \le (n-6)(n-7)/2 < (n^2 - 5n + 2)/2 \le \dim D^{(n-2,2)}$$

by Lemma 2.27. Thus, when $a = 2$ every summand in (7.20) has dimension less than $\dim V$, and so $V{\downarrow_G}$ cannot be irreducible. $\square$

We now prove the main result of the section:

Theorem 7.21. *Let $m \geq 3$, $\delta = 0$ or 1, and let $G = Sp_{2m}(2) < \mathsf{S}_n$ with $G = Sp(W)$ acting on the $n = 2^{m-1}(2^m + (-1)^\delta)$ quadratic forms of Witt defect δ on the symplectic space $W := \mathbb{F}_2^{2m}$. Let $p = 2$ or 3, and let $\lambda \in \mathscr{P}_p(n)$ be such that $\dim D^\lambda > 1$. Then $D^\lambda{\downarrow_G}$ is irreducible if and only if $p = 3$ and $\lambda = (n-1, 1)$ or $(n-1, 1)^{\mathtt{M}}$.*

Proof. Assume that $D^\lambda{\downarrow_G}$ is irreducible. By Proposition 7.10, we may assume that $\lambda = (n - \ell, \mu)$ with $\ell \leq 3$ and $\mu \vdash \ell$. Next, by Proposition 7.18 and Lemma 7.19, $\lambda \neq (n-2, 2)$, $(n-3, 3)$.

The cases where $\lambda = (n-3, 2, 1)$, or $p = 3$ and $\lambda = (n-2, 1^2)$, when $G = Sp_{2m}(2) < \mathsf{S}_n$ with $m \geq 3$, are ruled out by [**32**, Theorem A]. Indeed, it was shown in [**8**, Lemma 5.11] that G has (exactly) two orbits on the set of 3-element subsets of Ω^ε, and so G is not 3-homogeneous. Also by [**3**, Lemma 2.2] we have that if $p = 3$ and $n \geq 10$ then $(n-3, 2, 1)^{\mathtt{M}} = ((n-3)^{\mathtt{M}}, 3)$ and $(n-2, 1^2)^{\mathtt{M}} = ((n-2)^{\mathtt{M}}, 2)$, so that in either case $h(\lambda^{\mathtt{M}}) = 3$.

This leaves only one possibility $\lambda = (n-1, 1)$. Now we apply the main result of [**47**] to see that $D^{(n-1,1)}$ is irreducible over G if and only if $p = 3$. $\square$

8. Proofs of Main Theorems

8.1. **Proof of Theorem A.** For $p \geq 5$ this is [**8**, Main Theorem] (and Remark 1.3). So we may assume that $p = 2$ or 3. Since the case $(p, \lambda) = (2, \beta_n)$ is excluded, by [**32**, Theorems A, B], we may assume that one of the following happens:

(1) $p = 2$, $n \equiv 2 \pmod 4$, $\lambda = (n-1, 1)$, and $G \leq \mathsf{S}_{n/2} \wr \mathsf{S}_2$ is as in [**32**, Theorem B];
(2) G is 2-transitive on $\{1, \dots, n\}$;
(3) $G \leq \mathsf{S}_{n-1}$ and λ is JS.

Since (1) is Theorem A(iii), we assume from now on that this case does not occur.

Suppose we are in the case (2). If $\lambda \in \mathscr{L}^{(1)}(n)$ then by [**47**] and the remarks preceding Table II, we arrive at Theorem A(ii). If $G = \mathsf{A}_n$, then, by definition of $\mathscr{P}_p^{\mathsf{A}}(n)$, we arrive at Theorem A(i). So we may assume that $G \neq \mathsf{A}_n$ and $\lambda \notin \mathscr{L}^{(1)}(n)$. By the classification of 2-transitive groups [**10**], we are in one of the following situations:

(A) $\mathrm{soc}\,(G)$ is an elementary abelian subgroup;
(B) $\mathrm{soc}\,(G) \cong PSL_m(q)$ (is non-abelian simple) acting on $n = (q^m - 1)/(q-1)$ 1-dimensional subspaces of $\mathbb{F}_q^m$;
(C) $G \cong Sp_{2m}(2)$, $m \geq 3$, acting on $n = 2^{m-1}(2^m + (-1)^\delta)$ quadratic forms on $\mathbb{F}_2^{2m}$ of the given Witt defect $\delta \in \{0, 1\}$;
(D) G is any of the other doubly transitive subgroups (which we call small).

We now apply Theorems 5.13, 6.21, 7.21, and 4.1 for the cases (A), (B), (C) and (D), respectively.

Suppose we are in the case (3). If $n = 5$ then λ is JS only if $\lambda = (5)$ and $p = 2$, or $\lambda \in \{(5), (3, 2)\}$ and $p = 3$, and in either case we have $\dim D^\lambda = 1$. So we may assume that $n \geq 6$. By [**27**], we have $D^\lambda{\downarrow_{\mathsf{S}_{n-1}}} \cong D^\mu$, where μ is obtained from λ by removing the top removable node of λ. If $G = \mathsf{S}_{n-1}$ we arrive at Theorem A(v). Now we may assume that $G < \mathsf{S}_{n-1}$.

We now apply [**32**, Theorems A, B] again with $n-1$ in place of n and μ in place of λ to arrive to the cases (1'),(2'),(3') parallel to the cases (1),(2),(3) above. For example, by [**35**, Theorems 3.3, 3.6], μ is not JS, so (3') is excluded. The case (1') is also excluded, since $\mu = (n-2,1)$ implies $\lambda = (n-1,1)$, but $n-1 \equiv 2 \pmod 4$ implies that n is odd, and so λ is not JS. Thus, we are in the case (2'), i.e. G is 2-transitive on $\{1, 2, \dots, n-1\}$.

Suppose $G = \mathsf{A}_{n-1}$. Then $D^\lambda{\downarrow}_{\mathsf{A}_{n-1}} \cong D^\mu{\downarrow}_{\mathsf{A}_{n-1}}$ is irreducible if and only if $\mu \notin \mathscr{P}^{\mathsf{A}}_p(n-1)$. If $p = 2$, since $\lambda \neq \beta_n$ is JS, it can be easily seen from Lemma 2.9 that $\mu \notin \mathscr{P}^{\mathsf{A}}_2(n-1)$ if and only if $\lambda \notin \mathscr{P}^{\mathsf{A}}_2(n)$. If $p \neq 2$, since λ is JS, we have from [**6**, Theorem 5.10] that $\mu \notin \mathscr{P}^{\mathsf{A}}_p(n-1)$ if and only if $\lambda \notin \mathscr{P}^{\mathsf{A}}_p(n)$. So $D^\lambda{\downarrow}_{\mathsf{A}_{n-1}}$ is irreducible if and only if λ is JS and $\lambda \notin \mathscr{P}^{\mathsf{A}}_p(n)$. We have arrived at Theorem A(vi).

Assume finally that $\mathsf{A}_{n-1} \neq G < \mathsf{S}_{n-1}$. As $n \geq 6$, passing from λ to $\lambda^{\mathtt{M}}$ if necessary, we may assume by Theorems 5.13, 6.21, 7.21, and 4.1 that $\mu = (n-2,1)$, $(n-3,2)$ or $(n-3,1^2)$ (the last partition only for $p = 3$). Since μ is obtained from λ by removing the top removable node it follows that $\lambda = (n-1,1)$, $(n-2,2)$ or $(n-2,1^2)$ respectively. Note that $(n-1,1)$ and $(n-2,1^2)$ are JS if and only if $n \equiv 0 \pmod p$, while $(n-2,2)$ is JS if and only if $n \equiv 2 \pmod p$. The result then easily follows in this case by checking when the required congruences modulo p hold and when $D^\mu{\downarrow}_G$ is irreducible using Theorems 5.13, 6.21, 7.21, and 4.1.

8.2. **Proof of Theorem A$'$.** For $p \geq 5$ this is [**36**, Main Theorem]. So we may assume that $p = 2$ or 3. If V lifts to S_n, we arrive at Theorem A$'$(i). Otherwise $\lambda \in \mathscr{P}^{\mathsf{A}}_p(n)$. From Lemma 2.10 it then follows that $\lambda_1 \leq (n+4)/2$. By [**34**, Theorem A], we are in one of the following situations:

(1) G is primitive on $\{1, 2, \dots, n\}$;
(2) $G \leq \mathsf{A}_{n-1}$ and either λ is JS or λ has exactly two normal nodes, both of residue different from 0.
(3) $G \leq \mathsf{A}_{n-2,2} \cong \mathsf{S}_{n-2}$ and λ is JS.

Suppose we are in the case (1). By Theorem 3.7, we see that either G is an affine group, which is subsequently ruled out by Theorem 5.13, or G is a Mathieu group, in which case one can apply Theorem 4.1 to arrive at the case (A1) from Table IV, or else the case (A3) from Table IV occurs.

Consider the case (2). Suppose first that λ is JS. Then $E^\lambda_\pm{\downarrow}_{\mathsf{A}_{n-1}} \cong E^\pi_\pm$ for some $\pi \in \mathscr{P}^{\mathsf{A}}_p(n-1)$. As π can not be JS, applying [**34**, Theorem A] to $n-1$ instead of n, we deduce that either G is a subgroup of A_{n-1} primitive on $\{1, 2, \dots, n-1\}$, or $G \leq \mathsf{A}_{n-2}$. The former case is considered as in the case (1) using Theorems 3.7, 4.1 and 5.13. The case $G \leq \mathsf{A}_{n-2}$ is subsumed by the case (3) to be considered below.

Suppose now that λ has exactly two normal nodes both of residue different from 0. From [**36**, Proposition 3.8] or the proofs of [**34**, Theorems B, 5.3] we have that $D^\lambda{\downarrow}_{\mathsf{A}_{n-1}} \cong E^\nu$ with $\nu \in \mathscr{P}_p(n-1) \setminus \mathscr{P}^{\mathsf{A}}_p(n-1)$ obtained by removing a good node from λ. If $p = 3$ we also have that $\nu^{\mathtt{M}}$ is obtained from λ by removing a good node. In particular $\nu_1, \nu^{\mathtt{M}}_1 \leq (n+4)/2 < (n-1)-2$ if $n \geq 11$, so by Theorem A we have that $G \in \{\mathsf{A}_{n-2}, \mathsf{A}_{n-1}\}$. Using [**34**, Theorem A], we arrive at Theorem A$'$(ii)(a). If $n \leq 10$ and $p = 2$, $\lambda = (4,3,1)$ and $\nu = (4,2,1)$, in which case we can conclude as above. If $n \leq 10$ and $p = 3$, $\lambda = (3,1^2)$ and $\nu^{(\mathtt{M})} = (3,1)$. In this case $E^\lambda_\pm \cong E^{(4,1^2)}_\pm{\downarrow}_{\mathsf{A}_5}$, which will be considered below when covering case (3).

Consider the case (3). Using the isomorphism $\mathsf{A}_{n-2,2} \cong \mathsf{S}_{n-2}$, by [**34**, Theorem 5.4] and [**36**, Theorem 3.6], we can write $E^\lambda_\pm{\downarrow}_{\mathsf{A}_{n-2,2}} \cong D^\mu$ where $\mu \in \mathscr{P}_p(n-2)\backslash\mathscr{P}^{\mathsf{A}}_p(n-2)$ is obtained from λ by removing two good nodes. If $p = 3$ we also have that $\mu^{\mathtt{M}}$ is obtained from λ by removing two good nodes. In particular $\mu_1, \mu^{\mathtt{M}}_1 \leq (n+4)/2 < (n-2)-2$ if $n \geq 13$. In this case it follows from Theorem A that $G \in \{\mathsf{A}_{n-2}, \mathsf{A}_{n-3}\}$. The case $G = \mathsf{A}_{n-3}$ can be excluded, since $E^\lambda_\pm{\downarrow}_{\mathsf{A}_{n-3,3}}$ is not irreducible by [**34**, Theorem A]. So $G \in \{\mathsf{A}_{n-2}, \mathsf{A}_{n-2,2}\}$, in which case $E^\lambda_\pm{\downarrow}_G$ is irreducible by [**34**, Theorem C], and we arrive at Theorem A′(ii)(b). If $n \leq 12$ then $p = 2$, $\lambda = (5,3,1)$ and $\mu = (4,2,1)$ or $p = 3$ and $(\lambda, \mu^{(\mathtt{M})}) \in \{((4,1^2),(3,1)),((7,3,2),(5,3,2))\}$. If $p = 2$ and $\lambda = (5,3,1)$ or $p = 3$ and $\lambda = (7,3,2)$ we can conclude as above. If $p = 3$ and $\lambda = (4,1^2)$ then $E^\lambda_\pm{\downarrow}_{\mathsf{A}_{4,2}} \cong D^{(3,1)^{(\mathtt{M})}}$. Since $\dim E^\lambda_\pm = 3$, we have that $E^\lambda_\pm{\downarrow}_G$ is reducible if G is abelian or a 2-group. Further $E^\lambda_\pm{\downarrow}_{\mathsf{A}_{3,2}} \cong D^{(3,1)^{(\mathtt{M})}}{\downarrow}_{\mathsf{S}_3}$ is reducible, under the identification of $\mathsf{A}_{3,2} \cong \mathsf{S}_3$. Considering the submodule structure of S_4 it then follows that $G \in \{\mathsf{A}_{4,2}, \mathsf{A}_4\}$ and so we can again conclude by [**34**, Theorem C].

8.3. **Proof of Theorem B.** For the ‘if’ direction, by Theorem A, the cases listed in Theorem B do give rise to irreducible restrictions $D^\lambda{\downarrow}_G$.

For the ‘only-if’ direction, assume that $D^\lambda{\downarrow}_G$ is irreducible. By Schur’s Lemma, $\mathbf{Z}(G)$ acts on D^λ via scalars, and so $\mathbf{Z}(G) \leq \mathbf{Z}(\mathsf{S}_n) = 1$ as S_n acts faithfully on D^λ. Thus G is in fact almost simple, i.e. $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S. Inspecting the list of exceptions in Theorem A for almost simple groups, we conclude that it is enough to show that such a group cannot occur in the case (iii) of Theorem A.

So assume for a contradiction that G is almost simple with socle S and satisfies the conditions described in Remark 1.2. Recall that $B = \mathsf{S}_{n/2,n/2}$ is the base subgroup of $\mathsf{S}_{n/2} \wr \mathsf{S}_2$. As G is almost simple, we have $S \trianglelefteq G \cap B$, and

$$2 \text{ divides } |\mathrm{Out}(S)|. \tag{8.1}$$

Let π_1 (resp. π_2) denote the permutation representations of *odd* degree $n/2$ of $G \cap B$, induced by the projection of B onto the first (resp. second) factor $\mathsf{S}_{n/2}$ of B. By assumption, $\pi_i(G \cap B)$ is 2-transitive, but the homomorphisms

$$G \cap B \xrightarrow{\pi_i} \mathsf{S}_{n/2} \to GL(D^{(n/2-1,1)})$$

for $i = 1, 2$ give rise to non-isomorphic irreducible representations (of degree $n/2 - 1$). This implies that

$$\pi_1 \text{ and } \pi_2 \text{ induce two distinct 2-transitive permutation characters of } G \cap B. \tag{8.2}$$

We also note that both π_1 and π_2 are faithful. Indeed, if $\mathrm{Ker}(\pi_i) \neq 1$ for some i, then $\mathrm{Ker}(\pi_i) \geq \mathrm{soc}\,(G) = S$. Since G interchanges π_1 and π_2, it follows that $S \leq \mathrm{Ker}(\pi_{3-i})$, whence S acts trivially on $\{1, 2, \ldots, n\}$, a contradiction.

Now we can go over the list of 2-transitive permutation groups of odd degree $n/2$ with socle S, e.g. in [**47**, Table I]. Then (8.1) rules out the cases $S = M_{11}$, M_{23}, and ${}^2B_2(q)$. If $(S, n/2) = (\mathsf{A}_m, m \geq 5)$, then, since $|\mathrm{Out}(S)| = 2$, we must have that $G \cong \mathsf{S}_m$ and $G \cap B = \mathsf{A}_m$, which has a unique 2-transitive permutation character of degree m, violating (8.2). Likewise, if $(S, n/2) = (PSL_2(11), 11)$ or $(\mathsf{A}_7, 15)$, then again $|\mathrm{Out}(S)| = 2$, and $G \cap B = S$ has a unique 2-transitive permutation character of degree $n/2$, a contradiction.

Consider the cases $(S, n/2) = (PSL_2(q), q+1)$ or $(PSU_3(q), q^3+1)$. In these cases, $2|q$ as $n/2$ is odd. If S_1 and G_1 denote the stabilizer of 1 in S, respectively in $G \cap B$, then it is easy to see that $S_1 = \mathbf{N}_S(Q)$ and $Q = \mathbf{O}_2(S_1) \trianglelefteq G_1$. As

$$[G \cap B : G_1] = n/2 = [S : S_1],$$

by Frattini argument we have $G_1 = \mathbf{N}_{G_1}(Q)$. The same argument also applies to the stabilizer of $n/2+1$ in $G \cap B$. Thus the 2-transitive representations of $G \cap B$ induced by π_1 and π_2 are in fact $G \cap B$-conjugate and so have the same character, contradicting (8.2).

Finally, consider the case $(S, n/2) = (PSL_d(q), (q^d-1)/(q-1))$ with $d \geq 3$. In this case, the 2-transitive permutation action $\pi_i(S)$ extends to $P\Gamma L_d(q)$, but not to the entire $\mathrm{Aut}(S) \cong P\Gamma L_d(q) \rtimes \mathsf{C}_2$ (where C_2 is generated by the inverse-transpose automorphism τ). As $\pi_i{\downarrow_S}$ extends to $G \cap B$, $G \cap B \leq P\Gamma L_d(q)$. We may assume that the stabilizer S_1 of 1 in S is the stabilizer of a fixed one-dimensional subspace in the natural module $\mathbb{F}_q^d$ for $SL_d(q)$. Then $Q := \mathbf{O}_r(S_1)$ is an elementary abelian r-subgroup of order q^{d-1}, if r is the prime dividing q. Note that $P\Gamma L_d(q)$ preserves the S-conjugacy classes of Q, and so

$$[G \cap B : \mathbf{N}_{G \cap B}(Q)] = [S : \mathbf{N}_S(Q)].$$

Arguing as in the previous case, we obtain that the stabilizer G_1 of 1 in $G \cap B$ is precisely $\mathbf{N}_{G \cap B}(Q)$. Thus the representation π_1 of $G \cap B$ is uniquely determined once we fix (the S-conjugacy class of) Q, whence it must be the restriction to $G \cap B$ of the usual action of $P\Gamma L_d(q)$ on 1-spaces of $\mathbb{F}_q^d$, with character say ψ. Clearly, $\psi(g)$ is the number of g-invariant 1-spaces on $\mathbb{F}_q^d$ for all $g \in P\Gamma L_d(q)$. Note that S has only one more 2-transitive representation that is not equivalent to $\pi_1{\downarrow_S}$, namely the one on hyperplanes of $\mathbb{F}_q^d$, which extends to the usual action of $P\Gamma L_d(q)$ on hyperplanes of $\mathbb{F}_q^d$, with character say ψ'. Again, $\psi'(g)$ is the number of g-invariant hyperplanes on $\mathbb{F}_q^d$ for all $g \in P\Gamma L_d(q)$. Now, $\psi' = \psi^\tau$, and ψ is τ-invariant by the proof of [**56**, Lemma 6.2]. It follows that $\psi' = \psi$. As the 2-transitive permutation character of $G \cap B$ induced by π_2 is either $\psi{\downarrow_{G \cap B}}$ or $\psi'_{G \cap B}$, we see that π_1 and π_2 induce the same permutation character, again violating (8.2).

8.4. **Proof of Theorem B′.** Inspect the list of exceptions in Theorem A′ for almost simple groups.

8.5. **Proof of Theorem C.** The first statement of the theorem and the 'if' part of the second statement is [**32**, Theorem C], but see Remark 1.3. For the 'only-if' part of the second statement, in view of part (iii) of the first statement, we may assume that G is not primitive. By [**57**, Table III], D^{β_n} is obtained by reducing modulo 2 a basic spin representation $B_\mathbb{C}$ of $\hat{\mathsf{S}}_n$. So $B_\mathbb{C}{\downarrow_G}$ is irreducible. By [**38**, Theorem C] we have that

$$G \in \{\mathsf{S}_{n-1}, \mathsf{A}_{n-1}, \mathsf{S}_{n-2}, \mathsf{A}_{n-2,2}\}.$$

If $D^{\beta_n}{\downarrow_{\mathsf{A}_{n-1}}}$ is irreducible then $D^{\beta_n}{\downarrow_{\mathsf{A}_n}}$ and $D^{\beta_{n-1}}{\downarrow_{\mathsf{A}_{n-1}}}$ must be irreducible, which is impossible. The cases $G = \mathsf{S}_{n-2}$ and $\mathsf{A}_{n-2,2}$ can be also ruled out, since by part (i) of the first statement of the theorem, we have that $D^\lambda{\downarrow_{\mathsf{S}_{n-2,2}}}$ is reducible. Finally, if $G = \mathsf{S}_{n-1}$ we apply part (i) of the first statement of the theorem to arrive to part (1) of the second statement.

8.6. **Proof of Theorem C′.** For the first statement of the theorem, taking into account [**34**, Propositions 6.3, 6.6, 6.7], which deal with irreducible restrictions of basic spin modules to the subgroups of the from $\mathsf{A}_n \cap (\mathsf{S}_{n-k} \times \mathsf{S}_k)$ and $\mathsf{A}_n \cap (\mathsf{S}_a \wr \mathsf{S}_b)$, we may assume that G is primitive. If $n \equiv 2 \pmod 4$ then $\beta_n \notin \mathscr{P}_2^{\mathsf{A}}(n)$, so in this case the first statement follows from Theorem C. So we may also assume that $n \not\equiv 2 \pmod 4$, in which case $\beta_n \in \mathscr{P}_2^{\mathsf{A}}(n)$. By Theorems 3.7, 4.1 and 5.13, if $E_{\pm}^{\beta_n}{\downarrow}_G$ is irreducible then we are in one of the exceptional cases (A7)-(A12) listed in Theorem C′(iii). Conversely, the cases (A11),(A12) give rise to examples by Theorem 4.1; the cases (A7),(A8) occur by Theorem 5.13; the case (A9) occurs by Theorem 3.7 (and the case (A10) is covered by Theorem C since in this case $n \equiv 2 \pmod 4$).

For the second statement, the 'if' part follows from the first statement, and [**34**, Proposition 6.3].

We finally prove the 'only-if' part of the second statement. In view of Theorem C, we may assume that $\beta_n \in \mathscr{P}_2^{\mathsf{A}}(n)$, i.e. $n \not\equiv 2 \pmod 4$. As in the proof of Theorem B, we have that $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S. By the case (iii) of the first statement, we may also assume that G is not primitive.

Since $\dim V = 2^{\lfloor (n-1)/2 \rfloor - 1} \geq 2^{(n-4)/2}$, we have that $|\mathrm{Aut}(S)| \geq |G| \geq 2^{n-4}$. Now we apply [**26**, Proposition 6.1] and consider the possible cases for G listed there. If we are in one of the cases listed in Proposition 3.11, then we arrive at the exceptional cases covered in part (1). So we may assume that $S = \mathsf{A}_m$, with $m \geq 7$ and each orbit of S on $\Omega := \{1, 2, \dots, n\}$ having length 1 or m.

Let $\Omega_1, \dots, \Omega_a$ be the S-orbits of length m so that S fixes $b := n - am$ points in

$$\Omega' := \Omega \setminus (\Omega_1 \cup \cdots \cup \Omega_a).$$

Let π_i denote the permutation action of S on Ω_i, and also of G on Ω_i in the case G stabilizes Ω_i. Let $\mathsf{S}(\Omega_i) \cong \mathsf{S}_m$ and $\mathsf{A}(\Omega_i) \cong \mathsf{A}_m$ denote the natural subgroups of S_n that act only on Ω_i.

Restricting V to $\prod_{i=1}^{a} \mathsf{A}(\Omega_i)$, we see that $V{\downarrow}_S$ contains a submodule

$$U := V_1 \otimes V_2 \otimes \ldots \otimes V_a \otimes X,$$

where $V_1, \dots, V_a$ are basic spin modules of S. If $a \geq 3$, then, as S has at most two non-isomorphic basic spin modules, we may assume $V_1 \cong V_2$ and note that $\dim V_i \geq 2$. The same holds if $a = 2$ and S has a unique basic spin module, i.e. $m \equiv 2 \pmod 4$. Thus in either case U has a proper submodule $\mathtt{Sym}^2(V_1) \otimes V_3 \otimes \ldots \otimes V_a \otimes X$ of dimension greater than $(\dim U)/2$. Hence $V{\downarrow}_S$ has a nonzero subquotient of dimension less than $(\dim V)/2$, contradicting to the irreducibility of G on V. We deduce that $a \leq 2$, and if $a = 2$ then S has two basic spin modules, i.e. $a = 2$ implies $m \not\equiv 2 \pmod 4$.

Let c denote the number of G-orbits on Ω. Since $|G/S| \leq 2$ we have that

$$c \geq \lfloor (a+1)/2 \rfloor + \lfloor (b+1)/2 \rfloor.$$

By [**34**, Proposition 6.3], $c \leq 3$, which implies that $b \leq 4$. If $b = 4$, then G must have two orbits of length 2 on the set Ω' of S-fixed points, contradicting [**34**, Proposition 6.3]. Suppose $b = 3$. Then G must have two orbits of length 2 and 1 on Ω', so [**34**, Proposition 6.3(1)] implies that $4 \mid n$, $c = 3$, $m = n - 3$, and also $G = \langle \mathsf{A}_m, h \rangle \cong \mathsf{S}_m$. Since h does not centralize S, h must act non-trivially, in fact as an odd permutation on Ω_1. As G has three orbits of length $n-3$, 2, and 1 on Ω, we see that $G = \mathsf{A}_{n-3,2,1}$.

Assume now that $b = 2$ and G fixes the two points of Ω'. Then $c = 3$, and so $4|n$ by [**34**, Proposition 6.3(1)]. If $m = n - 2$, we have arrived at the second case of

Theorem C′(1)(a). As $4|n$, the restriction of V to $\mathsf{A}(\Omega_1 \cup \Omega_2) \cong \mathsf{A}_{n-2}$ is $E^{\beta_{n-2}}$, which extends to $D^{\beta_{n-2}}$. Hence $V{\downarrow_S}$ contains a subquotient

$$D^{\beta_{n-2}}{\downarrow_{\pi_1(S)\times\pi_2(S)}} \cong D^{\beta_m}{\downarrow_{\pi_1(S)}} \otimes D^{\beta_m}{\downarrow_{\pi_2(S)}}.$$

Note that all embedding $\mathsf{A}_m \to \mathsf{S}_m$ are S_m-conjugate. It follows that

$$X := D^{\beta_m}{\downarrow_{\pi_1(S)}} \cong D^{\beta_m}{\downarrow_{\pi_2(S)}}$$

as $\mathbb{F}S$-modules, of dimension $e \geq 4$. Hence $V{\downarrow_S}$ contains subquotients $\mathtt{Sym}^2(X)$ and $\wedge^2(X)$ of distinct dimensions, contradicting the irreducibility of G on V.

Consider the case $b = 2$ and Ω' forms a G-orbit of length 2. Recall that $a \leq 2$. Now if $c = 3$, then $4|n$ by [**34**, Proposition 6.3(1)], $n = 2m+2$, $G \cong \mathsf{S}_m$, and we can repeat the above argument with $\mathtt{Sym}^2 / \wedge^2 (X)$ to reach a contradiction. Suppose $c = a = 2$. Then G has orbits of length 2 and $n-2 = 2m$ on Ω, whence $4|n$ as $n \not\equiv 2 \pmod 4$ by assumption. Then we can again repeat the above argument with $\mathtt{Sym}^2 / \wedge^2 (X)$. So we must have $a = 1$, $n = m+2$, $G = \langle S, h\rangle \mathsf{S}_m$. Again, since $[h, S] \neq 1$, we must have that h acts non-trivially on Ω_1 (and on Ω'), and so $G = \mathsf{A}_{n-2,2}$, and we have arrived at the case (1)(b) of Theorem C′.

Now assume that $b = 1$. As $G \leq \mathsf{A}_{n-1}$ is irreducible on V, by [**34**, Proposition 6.3] we have $n \equiv 0, 3 \pmod 4$. If $c = 3$, then, since $a \leq 2$, we have that G has three orbits of length m, m, and 1 on Ω, but this contradicts [**34**, Proposition 6.3(1)]. Suppose $c = a = 2$, so that $n = 2m+1 \equiv 3 \pmod 4$. In this case, the restriction of V to $\mathsf{A}(\Omega_1 \cup \Omega_2) \cong \mathsf{A}_{n-1}$ is $E^{\beta_{n-1}}$, which extends to $D^{\beta_{n-1}}$. Now we can repeat the argument with $\mathtt{Sym}^2 / \wedge^2 (X)$ to reach a contradiction. Thus $a = 1$, $n = m+1$, $G = \mathsf{A}_m$, and we and we have arrived at the case (1)(c) of Theorem C′.

Finally, we consider the case $b = 0$. As $n > m$ and $a \leq 2$, we must have that $a = 2$, $n = 2m \equiv 0 \pmod 4$. By [**34**, Proposition 6.3] for $c = 2$ (where $G \leq \mathsf{A}_{m,m}$) and [**34**, Proposition 6.6] for $c = 1$ (where $G \leq \mathsf{S}_m \wr \mathsf{S}_2$), we have $m \equiv 2 \pmod 4$, which contradicts what was proved above.

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), 469–514.

[2] A. Baranov and A. Kleshchev, Maximal ideals in modular group algebras of the finitary symmetric and alternating groups, *Trans. Amer. Math. Soc.* **351** (1999), 595–617.

[3] A.A. Baranov, A.S. Kleshchev, A.E. Zalesskii, Asymptotic results on modular representations of symmetric groups and almost simple modular group algebras, *J. Algebra* **219** (1999), 506–530.

[4] D.J. Benson, Spin modules for symmetric groups, *J. London Math. Soc.* **38** (1988), 250–262.

[5] C. Bessenrodt and J.B. Olsson, On residue symbols and the Mullineux conjecture, *J. Algebraic Combin.* **7** (1998), 227–251.

[6] C. Bessenrodt and J.B. Olsson, Branching of modular representations of the alternating groups, *J. Algebra* **209** (1998), 143–174.

[7] J.N. Bray, D.F. Holt, and C.M. Roney-Dougal, '*The Maximal Subgroups of the Low-dimensional Finite Classical Groups*', London Math. Soc. Lecture Note Series no. **407**. Camb. Univ. Press, 2013.

[8] J. Brundan and A.S. Kleshchev, Representations of the symmetric group which are irreducible over subgroups, *J. Reine Angew. Math.* **530** (2001), 145–190.

[9] T. Burness, C. Marion and D. Testerman, On irreducible subgroups of simple algebraic groups, *Math. Ann.* **367** (2017), 1259–1309.

[10] P.J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[11] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, '*An ATLAS of Finite Groups*', Clarendon Press, Oxford, 1985.
[12] J. D. Dixon and B. Mortimer, *Permutation Groups*, Grad. Texts Math. 163, Springer-Verlag, Berlin- Heidelberg-New York 1996.
[13] B. Ford and A.S. Kleshchev, A proof of the Mullineux conjecture, *Math. Z.* **226** (1997), 267–308.
[14] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4; 2004, `http://www.gap-system.org.`
[15] D. Gorenstein, R. Lyons, and R.M. Solomon, '*The Classification of the Finite Simple Groups*', Number 3. Part I. Chapter A, vol. **40**, *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1998.
[16] R.M. Guralnick, M. Larsen, P.H. Tiep, Representation growth in positive characteristic and conjugacy classes of maximal subgroups, *Duke Math. J.* **161** (2012), 107–137.
[17] R.M. Guralnick and P.H. Tiep, The non-coprime $k(GV)$-problem, *J. Algebra* **293** (2005), 185–242.
[18] G. Hiss, Zerlegungszahlen endlicher Gruppen vom Lie-Typ in nicht-definierender Charakteristik, Habilitationschirft, RWTH Aachen, 1990.
[19] G.D. James, Representations of the symmetric groups over the field of order 2, *J. Algebra* **38** (1976), 280–308.
[20] G.D. James, '*The Representation Theory of the Symmetric Groups*', Lecture Notes in Mathematics, vol. **682**, Springer, NewYork/Heidelberg/Berlin, 1978.
[21] G.D. James, On the minimal dimensions of irreducible representations of symmetric groups, *Math. Proc. Camb. Phil. Soc.* **94** (1983), 417–424.
[22] G.D. James and G.E. Murphy, The determinant of the Gram matrix for a Specht module, *J. Algebra* **59** (1979), 222–235.
[23] C. Jansen, K. Lux, R. Parker, R. Wilson, '*An Atlas of Brauer Characters*', Clarendon Press, Oxford 1995.
[24] J.C. Jantzen and G.M. Seitz, On the representation theory of the symmetric groups, *Proc. London Math. Soc.* **65** (1992), 475–504.
[25] P.B. Kleidman and M.W. Liebeck, '*The Subgroup Structure of the Finite Classical Groups*', London Math. Soc. Lecture Note Ser. no. **129**, Cambridge University Press, 1990.
[26] P.B. Kleidman and D.B. Wales, The projective characters of the symmetric groups that remain irreducible on subgroups, *J. Algebra* **138** (1991), 440–478.
[27] A.S. Kleshchev. On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I. *Proc. London Math. Soc.* **69** (1994), 515–540.
[28] A.S. Kleshchev, Branching rules for modular representations of symmetric groups. II, *J. Reine Angew. Math.* **459** (1995), 163–212.
[29] A.S. Kleshchev, Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux. *J. Lond. Math. Soc.* **54** (1996), 25–38.
[30] A.S. Kleshchev, On decomposition numbers and branching coefficients for symmetric and special linear groups, *Proc. Lond. Math. Soc.* 75 (1997), 497-558.
[31] A.S. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge University Press, Cambridge, 2005.
[32] A.S. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of symmetric groups in small characteristics: reduction theorems, *Math. Z.* **293** (2019), 677–723.
[33] A.S. Kleshchev, L. Morotti and P.H. Tiep, Lower bounds for dimensions of irreducible representations of symmetric groups, *Proc. Amer. Math. Soc.* **148** (2020), 1439–1445.
[34] A.S. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of alternating groups in small characteristics: reduction theorems, *Rep. Theory* **24** (2020), 115–150.
[35] A.S. Kleshchev and J. Sheth, Representations of the symmetric group are reducible over simply transitive subgroups, *Math. Z.* **235** (2000), 99–109.
[36] A.S. Kleshchev and J. Sheth, Representations of the alternating group which are irreducible over subgroups, *Proc. London Math. Soc.* **84** (2002), 194–212.
[37] A.S. Kleshchev, P. Sin, and P.H. Tiep, Representations of the alternating group which are irreducible over subgroups. II, *Amer. J. Math.* **138** (2016), 1383–1423.

[38] A.S. Kleshchev and P.H. Tiep, On restrictions of modular spin representations of symmetric and alternating groups, *Trans. Amer. Math. Soc.* **356** (2004), 1971–1999.
[39] A.S. Kleshchev and A.E. Zalesski, Minimal polynomials of elements of order p in p-modular projective representations of alternating groups, *Proc. Amer. Math. Soc.* **132** (2004), 1605–1612.
[40] M.W. Liebeck, The affine permutation groups of rank three, *Proc. London Math. Soc.* **54**(1987), 477–516.
[41] M.W. Liebeck, C.E. Praeger, and J. Saxl, On the O'Nan-Scott reduction theorem for finite primitive permutation groups, *J. Austral. Math. Soc.* **44** (1988), 389–396.
[42] F. Lübeck, Character Degrees and their Multiplicities for some Groups of Lie Type of Rank < 9, http://www.math.rwth-aachen.de/ Frank.Luebeck/chev/DegMult/index.html
[43] F. Lübeck, Weight Multiplicities: Groups of type A5, all 2-restricted highest weights in characteristic 2, http://www.math.rwth-aachen.de/ Frank.Luebeck/chev/WMSmall/A5-mod2.html
[44] K. Magaard, Some remarks on maximal subgroups of finite classical groups; in: '*Finite Simple Groups: Thirty Years of the Atlas and Beyond*', Contemp. Math. **694**, Amer. Math. Soc., Providence, RI, 2017, pp. 123–137.
[45] K. Magaard and G. Malle, Irreducibility of alternating and symmetric squares, *Manuscripta Math.* **95** (1998), 169–180.
[46] L. Morotti, Irreducible tensor products for alternating groups in characteristics 2 and 3, arXiv:1901.06997.
[47] B. Mortimer, The modular permutation representations of the known doubly transitive groups, *Proc. Lond. Math. Soc.* **41** (1980), 1–20.
[48] J. Müller, On low-degree representations of the symmetric group, *J. Algebra* **465** (2016), 356–398.
[49] G. Mullineux, Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups, *J. London Math. Soc.* **20** (1979), 60–66.
[50] J. Saxl, The complex characters of the symmetric groups that remain irreducible in subgroups, *J. Algebra* **111** (1987), 210–219.
[51] L. L. Scott, Representations in characteristic p, in: '*The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*', Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 319–331.
[52] G.M. Seitz, Cross-characteristic embeddings of finite groups of Lie type, *Proc. London Math. Soc.* **60** (1990), 166–200.
[53] J.-P. Serre, '*Linear Representations of Finite Groups*', Gradute Texts in Mathematics **42**, Springer-Verlag, Berlin, 1977.
[54] J. Sheth, Branching rules for two row partitions and applications to the inductive systems for symmetric groups, *Comm. Alg.* **27** (1999), 3303–3316.
[55] J.G. Thompson, Composition factors of rational finite groups, *J. Algebra* **319** (2008), 558–594.
[56] P.H. Tiep, Real ordinary characters and real Brauer characters, *Trans. Amer. Math. Soc.* **367** (2015), 1273–1312.
[57] D.B. Wales, Some projective representations of S_n, *J. Algebra* **61** (1979), 37–57.
[58] A.E. Zalesskii, Minimal polynomials and eigenvalues of p-elements in representations of quasi-simple groups with a cyclic Sylow p-subgroup, *J. London Math. Soc.* **59** (1999), 845–866.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: klesh@uoregon.edu

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, 30167 Hannover, Germany
E-mail address: morotti@math.uni-hannover.de

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
E-mail address: tiep@math.rutgers.edu