SOME EXTENSION ALGEBRAS FOR STANDARD MODULES OVER KLR ALGEBRAS OF TYPE A

DOEKE BUURSMA, ALEXANDER KLESHCHEV, AND DAVID J. STEINBERG

Abstract. Khovanov-Lauda-Rouquier algebras R_θ of finite Lie type are affine quasihereditary with standard modules $\Delta(\pi)$ labeled by Kostant partitions of θ. Let Δ be the direct sum of all standard modules. It is known that the Yoneda algebra $\mathcal{E}_\theta := \mathrm{Ext}^*_{R_\theta}(\Delta,\Delta)$ carries a structure of an A_∞-algebra which can be used to reconstruct the category of standardly filtered R_θ-modules. In this paper, we explicitly describe $\mathcal{E}_\theta$ in two special cases: (1) when θ is a positive root in type $\mathtt{A}$, and (2) when θ is of Lie type $\mathtt{A}_2$. In these cases, $\mathcal{E}_\theta$ turns out to be torsion free and intrinsically formal. We provide an example to show that the A_∞-algebra $\mathcal{E}_\theta$ is non-formal in general.

1. Introduction

Let $R_{\theta,\mathbb{F}}$ be a Khovanov-Lauda-Rouquier (KLR) algebra of finite Lie type over a field $\mathbb{F}$ corresponding to $\theta \in Q_+$ [7, 16]. It is known that $R_{\theta,\mathbb{F}}$ is affine quasihereditary [1, 4, 9, 10], and in particular it comes with a family of *standard modules* $\{\Delta(\pi)_\mathbb{F} \mid \pi \in \mathrm{KP}(\theta)\}$, where $\mathrm{KP}(\theta)$ is the set of Kostant partitions of θ. KLR algebras are defined over $\mathbb{Z}$, so we have a $\mathbb{Z}$-algebra R_θ with $R_{\theta,\mathbb{F}} \cong R_\theta \otimes_\mathbb{Z} \mathbb{F}$. The standard modules have natural integral forms $\Delta(\pi)$ with $\Delta(\pi)_\mathbb{F} \cong \Delta(\pi) \otimes_\mathbb{Z} \mathbb{F}$. All modules and algebras are explicitly graded, and we refer to these gradings as KLR gradings.

Let $\Delta := \bigoplus_{\pi\in\mathrm{KP}(\theta)} \Delta(\pi)$. The Yoneda algebra $\mathcal{E}_\theta := \mathrm{Ext}^*_{R_\theta}(\Delta,\Delta)$ carries a structure of an A_∞-algebra [3], which can be used to reconstruct the category $\mathcal{F}(\Delta)$ of modules which admit a finite filtration by standard modules [5, 6, 12]. In view of [1, Corollary 3.14] and [11, Theorem 4.28], understanding $\mathcal{F}(\Delta)$ is relevant for computing formal characters of the simple modules $L(\pi)_\mathbb{F}$ of $R_{\theta,\mathbb{F}}$. The smallest known example where the formal characters depend on the characteristic of $\mathbb{F}$ occurs in Lie type $\mathtt{A}_5$, see [18] (cf. [1, §2.6]).

We now assume that the Lie type is $\mathtt{A}_\infty$ with simple roots $\{\alpha_i \mid i \in \mathbb{Z}\}$ so that the set of positive roots Φ_+ is $\{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid i \le j\}$. There is a natural lexicographic total order $>$ on Φ_+. Let Q_+ be the positive root lattice, and fix $\theta \in Q_+$. If $\theta = \sum k_i\alpha_i$, we define the *height* of θ as $\mathrm{ht}(\theta) := \sum k_i$. A *Kostant partition* of θ is a sequence $\pi = (\beta_1^{m_1}, \ldots, \beta_t^{m_t})$ where $m_1, \ldots, m_t \in \mathbb{Z}_{>0}$, $\beta_1 > \cdots > \beta_t$ are positive roots, and $m_1\beta_1 + \cdots + m_t\beta_t = \theta$.

2010 *Mathematics Subject Classification.* 16G99, 16E05, 17B37.

The second author was supported by the NSF grants DMS-1161094, DMS-1700905, the Max-Planck-Institut, the Fulbright Foundation, and the DFG Mercator program through the University of Stuttgart.

We consider the Yoneda algebra $\mathcal{E}_\theta$ as the $\mathbb{Z}$-linear category whose objects are $\mathrm{KP}(\theta)$, and the set of morphisms from $\rho \in \mathrm{KP}(\theta)$ to $\sigma \in \mathrm{KP}(\theta)$ is

$$\mathcal{E}_\theta(\rho,\sigma) := \mathrm{Ext}^*_{R_\theta}(\Delta(\rho),\Delta(\sigma)).$$

The composition gf of $g \in \mathcal{E}_\theta(\sigma,\tau)$ and $f \in \mathcal{E}_\theta(\rho,\sigma)$ is obtained using the composition of lifts of g in $\mathrm{Hom}_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet)$ and f in $\mathrm{Hom}_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$, where $P^\pi_\bullet$ is a projective resolution of $\Delta(\pi)$ for $\pi \in \mathrm{KP}(\theta)$. The category $\mathcal{E}_\theta$ has a *homological grading* for which the homogeneous components are $\mathcal{E}^m_\theta(\rho,\sigma) := \mathrm{Ext}^m_{R_\theta}(\Delta(\rho),\Delta(\sigma))$, and a *KLR grading* which is inherited from the KLR grading on the standard modules. We use q to denote the KLR degree shift functor. Theorems A and B describe the category $\mathcal{E}_\theta$ (as a bigraded category) in two special cases: (1) when θ is an arbitrary positive root, and (2) when θ is of type $\mathtt{A}_2$, i.e. θ is of the form $c_1\alpha_1 + c_2\alpha_2$.

1.1. **The case where θ is a positive root.** Let $\theta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b+1} \in \Phi_+$. Set $l := b+2-a = \mathrm{ht}(\theta)$ and consider the polynomial algebra $\mathcal{X} := \mathbb{Z}[x_1,\dots,x_l]$. We consider $\mathcal{X}$ to be graded with $\deg x_r = 2$. Note that $\mathrm{KP}(\theta)$ is in bijection with the set of subsets of $[1, l-1]$: the subset associated to $\rho = (\beta_1,\dots,\beta_u) \in \mathrm{KP}(\theta)$ is $D_\rho := \{d_1,\dots,d_{u-1}\}$ where $d_t := \mathrm{ht}(\beta_1) + \cdots + \mathrm{ht}(\beta_t)$. For such D_ρ, set $d_0 := 0$ and $d_u := l$, and let J^ρ be the ideal of $\mathcal{X}$ generated by all $x_r - x_s$ such that there is $1 \le t \le u$ with $d_{t-1} < r, s \le d_t$. Define $\mathcal{X}^\rho := \mathcal{X}/J^\rho$. If $D_\rho \subseteq D_\sigma$, then $J^\sigma \subseteq J^\rho$ so we have a natural projection $\mathtt{p}^\sigma_\rho : \mathcal{X}^\sigma \twoheadrightarrow \mathcal{X}^\rho$. We use the notation $C \subseteq_m D$ to indicate that $C \subseteq D$ with $|D \setminus C| = m$.

Theorem A. *Let $\theta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b+1} \in \Phi_+$ be a positive root. We have*

$$\mathcal{E}^m_\theta(\rho,\sigma) \cong \begin{cases} q^{-m}\mathcal{X}^\rho & \text{if } D_\rho \subseteq_m D_\sigma,\\ 0 & \text{otherwise.}\end{cases}$$

If $D_\rho \subseteq_m D_\sigma \subseteq_n D_\tau$ with $f \in q^{-m}\mathcal{X}^\rho \cong \mathcal{E}_\theta(\rho,\sigma)$ and $g \in q^{-n}\mathcal{X}^\sigma \cong \mathcal{E}_\theta(\sigma,\tau)$, then the composition of g with f is given by $\mathtt{p}^\sigma_\rho(g)f \in q^{-(m+n)}\mathcal{X}^\rho \cong \mathcal{E}_\theta(\rho,\tau)$.

1.2. **The $\mathtt{A}_2$ case.** For a nonnegative integer k, let Λ_k be the algebra of symmetric polynomials in k variables. We impose a grading on Λ_k where linear symmetric polynomials have degree 2. The space Λ_k is a free $\mathbb{Z}$-module with basis $\{s_\lambda \mid \lambda \in \mathscr{P}(k)\}$, where $\mathscr{P}(k)$ is the set of partitions with at most k parts, and s_λ is the *Schur polynomial* corresponding to λ [14, §I.3]. Letting V be the free graded $\mathbb{Z}$-module with basis $\{v_0, v_1, v_2, \dots\}$ such that $\deg v_i := 2i$, there is an isomorphism of graded $\mathbb{Z}$-modules

$$\gamma_k : \Lambda_k \xrightarrow{\sim} q^{-k(k-1)}\textstyle\bigwedge^k V,\ \ s_{(\lambda_1,\dots,\lambda_k)} \mapsto v_{\lambda_k} \wedge v_{\lambda_{k-1}+1} \wedge \cdots \wedge v_{\lambda_1+k-1} \qquad (1.1)$$

where $\bigwedge^k V$ is the kth exterior power of V. Define

$$- \star - : \Lambda_a \otimes \Lambda_b \to q^{2ab}\Lambda_{a+b},\ \ f \otimes g \mapsto \gamma^{-1}_{a+b}(\gamma_a(f) \wedge \gamma_b(g)). \qquad (1.2)$$

Considering Λ_{a+b} to be a subalgebra of $\Lambda_a \otimes \Lambda_b$ in the obvious way, we have that $\Lambda_a \otimes \Lambda_b$ is free as a Λ_{a+b}-module with basis $\{s_\lambda \otimes 1 \mid \lambda \in \mathscr{P}(a,b)\}$, where $\mathscr{P}(a,b)$ is the set of partitions with at most a nonzero parts, the first part being at most b, see [13, PARTL.1.5] and [15, Proposition 2.6.8]. Moreover, [13, SCHUB.1.7] provides an explicit algorithm for writing any element of $\Lambda_a \otimes \Lambda_b$ as a Λ_{a+b}-linear combination of the basis elements $s_\lambda \otimes 1$.

Let $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $\theta = c_1\alpha_1 + c_2\alpha_2$. Note that there is a bijection

$$[0, \min\{c_1, c_2\}] \xleftrightarrow{\sim} \mathrm{KP}(\theta),\ r \mapsto (\alpha_2^{c_2-r}, (\alpha_1+\alpha_2)^r, \alpha_1^{c_1-r}),\ r_\rho \leftarrow\!\shortmid \rho.$$

For $\rho, \sigma \in \mathrm{KP}(\theta)$ with $r_\rho \geq r_\sigma$, let

$$\begin{aligned} \omega(\rho,\sigma) &:= -(r_\rho - r_\sigma)(1 + (c_1 - r_\rho) + (c_2 - r_\rho)),\\ \Lambda(\rho,\sigma) &:= q^{\omega(\rho,\sigma)}\Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho - r_\sigma} \otimes \Lambda_{r_\sigma} \otimes \Lambda_{c_1-r_\rho},\\ \mathscr{P}_{\rho,\sigma} &:= \mathscr{P}(r_\rho - r_\sigma, r_\sigma). \end{aligned}$$

If $f \in \Lambda_{r_\rho - r_\sigma}$, we write

$$f^{\rho,\sigma} := 1_{\Lambda_{c_2-r_\rho}} \otimes f \otimes 1_{\Lambda_{r_\sigma}} \otimes 1_{\Lambda_{c_1-r_\rho}} \in \Lambda(\rho,\sigma).$$

Then note that $\Lambda(\rho,\sigma)$ is a free right $\Lambda(\rho,\rho)$-module with basis $\{s_\lambda^{\rho,\sigma} \mid \lambda \in \mathscr{P}_{\rho,\sigma}\}$. We make $\Lambda(\rho,\sigma)$ into a left $\Lambda(\sigma,\sigma)$-module via the composition of algebra homomorphisms:

$$\xi : \Lambda(\sigma,\sigma) \hookrightarrow \Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{r_\sigma} \otimes \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{c_1-r_\rho} \twoheadrightarrow q^{-\omega(\rho,\sigma)}\Lambda(\rho,\sigma);$$

the first map uses the embeddings $\Lambda_{c_2-r_\sigma} \hookrightarrow \Lambda_{c_2-r_\rho} \otimes \Lambda_{r_\rho-r_\sigma}$ and $\Lambda_{c_1-r_\sigma} \hookrightarrow \Lambda_{r_\rho-r_\sigma} \otimes \Lambda_{c_1-r_\rho}$, and the second map is $a \otimes b \otimes c \otimes d \otimes e \mapsto a \otimes bd \otimes c \otimes e$ (which we think of as identifying the two factors of $\Lambda_{r_\rho-r_\sigma}$).

If $\rho, \sigma, \tau \in \mathrm{KP}(\theta)$ with $r_\rho \geq r_\sigma \geq r_\tau$, the tensor product $\Lambda(\sigma,\tau) \otimes_{\Lambda(\sigma,\sigma)} \Lambda(\rho,\sigma)$ is now a free right $\Lambda(\rho,\rho)$-module with basis

$$\{s_\mu^{\sigma,\tau} \otimes s_\lambda^{\rho,\sigma} \mid \mu \in \mathscr{P}_{\sigma,\tau}, \lambda \in \mathscr{P}_{\rho,\sigma}\}$$

and we define a map of right $\Lambda(\rho,\rho)$-modules

$$\Theta : \Lambda(\sigma,\tau) \otimes_{\Lambda(\sigma,\sigma)} \Lambda(\rho,\sigma) \to \Lambda(\rho,\tau),\ s_\mu^{\sigma,\tau} \otimes s_\lambda^{\rho,\sigma} \mapsto (s_\mu \star s_\lambda)^{\rho,\tau}.$$

Let

$$- \diamond - : \Lambda(\sigma,\tau) \otimes_{\mathbb{Z}} \Lambda(\rho,\sigma) \to \Lambda(\rho,\tau),\ g \otimes f \mapsto \Theta(g \otimes f).$$

Thus, to compute $g \diamond f$ for some $g \in \Lambda(\sigma,\tau)$ and $f \in \Lambda(\rho,\sigma)$, the following steps must be performed: (1) write $g = \sum_{\mu \in \mathscr{P}_{\sigma,\tau}} s_\mu^{\sigma,\tau} g_\mu$ with $g_\mu \in \Lambda(\sigma,\sigma)$, (2) for each $\mu \in \mathscr{P}_{\sigma,\tau}$, write $\xi(g_\mu)f = \sum_{\lambda \in \mathscr{P}_{\rho,\sigma}} s_\lambda^{\rho,\sigma} h_{\mu,\lambda}$ with $h_{\mu,\lambda} \in \Lambda(\rho,\rho)$, (3) we have

$$g \diamond f = \sum_{\mu \in \mathscr{P}_{\sigma,\tau},\, \lambda \in \mathscr{P}_{\rho,\sigma}} (s_\mu \star s_\lambda)^{\rho,\tau} h_{\mu,\lambda}.$$

Theorem B. *Let $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $\theta = c_1\alpha_1 + c_2\alpha_2$. We have*

$$\mathscr{E}_\theta^m(\rho,\sigma) = \begin{cases} \Lambda(\rho,\sigma) & \text{if } m = r_\rho - r_\sigma \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $r_\rho \geq r_\sigma \geq r_\tau$ with $f \in \Lambda(\rho,\sigma) \cong \mathcal{E}_\theta(\rho,\sigma)$ and $g \in \Lambda(\sigma,\tau) \cong \mathcal{E}_\theta(\sigma,\tau)$, then the composition of g with f is given by $g \diamond f \in \Lambda(\rho,\tau) \cong \mathcal{E}_\theta(\rho,\tau)$.

1.3. **Formality.** For θ as in Theorem A or B, note that $\mathrm{Ext}_{R_\theta}(\Delta(\rho),\Delta(\sigma))$ is torsion-free as a $\mathbb{Z}$-module. We do not know if this is true in general.

Also note that because $\mathrm{Ext}_{R_\theta}(\Delta(\rho),\Delta(\sigma))$ is concentrated in homological degree $|X_\sigma \setminus X_\rho|$ (in the case of Theorem A) or $r_\rho - r_\sigma$ (in the case of Theorem B), the A_∞-category structure of $\mathcal{E}_\theta$ must have $m_n = 0$ unless $n = 2$, so that $\mathcal{E}_\theta$ is *intrinsically formal*, see [5, §3.3]. In Section 5, we show that intrinsic formality and even formality does not occur in general:

Example C. *If $\theta = \alpha_1 + 2\alpha_2 + \alpha_3$, then the A_∞-category $\mathcal{E}_\theta$ is non-formal.*

1.4. **The structure of the paper.** The proofs of Theorems A and B occupy Sections 3 and 4, respectively. In the preliminary Section 2, we review the definition of the KLR algebra R_θ and the standard modules $\Delta(\pi)$.

In §3.1 and §4.1, we record the relevant special cases of the projective resolution $P^\rho_\bullet$ of $\Delta(\rho)$ constructed in [2]. This resolution is finite and has the form $P^\rho_\bullet = \cdots \stackrel{d_1}{\to} P^\rho_1 \stackrel{d_0}{\to} P^\rho_0 \stackrel{\varepsilon_\rho}{\to} \Delta(\rho)$ with $P^\rho_n = \bigoplus_{x\in X_n} q^{s_x} R_\theta 1_x$ for some explicit index set X_n, integers s_x, and idempotents 1_x. The map $d_n : P^\rho_{n+1} \to P^\rho_n$ can be described as right multiplication by an $X_{n+1}\times X_n$ matrix $(d_n^{y,x})$ for some $d_n^{y,x} \in 1_y R_\theta 1_x$.

In §3.2 and §4.3, we use the isomorphism $\mathrm{Hom}_{R_\theta}(R_\theta 1_x, \Delta(\sigma)) \cong 1_x\Delta(\sigma)$ to describe the complex $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ in terms of familiar objects from commutative algebra; in the case of Theorem A, these objects are polynomial rings and in the case of Theorem B, they are rings of symmetric polynomials. It turns out that in both cases, the complex $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ is isomorphic to a Koszul complex corresponding to an explicit regular sequence, and we can therefore compute its homology $H(\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))) =: \mathcal{E}_\theta(\rho,\sigma)$ as a bigraded $\mathbb{Z}$-module.

It remains to describe the composition in the category $\mathcal{E}_\theta$. This is done in §3.3 and §4.4, where we explicitly lift elements of $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma))$ to $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$. The function composition map $\mathrm{Hom}^\bullet_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet)\otimes \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet) \to \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\tau_\bullet)$ induces a map on homology $\mathcal{E}_\theta(\sigma,\tau)\otimes\mathcal{E}_\theta(\rho,\sigma) \to \mathcal{E}_\theta(\rho,\tau)$ which is the composition in the category $\mathcal{E}_\theta$.

In Section 5, we review the A_∞-category structure on $\mathcal{E}_\theta$ following [3, 5, 6] and provide details for Example C.

2. Preliminaries

2.1. **Basic notation.** Throughout, we work over an arbitrary principal ideal domain $\Bbbk$ (since everything is defined over $\mathbb{Z}$, one could just consider the case $\Bbbk = \mathbb{Z}$).

For $r, s \in \mathbb{Z}$, we use the segment notation $[r, s] := \{t \in \mathbb{Z} \mid r \le t \le s\}$, $[r, s) := \{t \in \mathbb{Z} \mid r \le t < s\}$, etc.

Let q be a variable, and $\mathbb{Z}((q))$ be the ring of Laurent series. For $n \in \mathbb{Z}_{\ge 0}$, we define

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_\pm := q^{\pm(n-1)}[n], \quad [n]^!_{(\pm)} := [1]_{(\pm)}[2]_{(\pm)}\cdots[n]_{(\pm)},$$

and if $0 \le m \le n$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_{(\pm)} := \frac{[n]^!_{(\pm)}}{[m]^!_{(\pm)}[n-m]^!_{(\pm)}}.$$

We denote by $\mathfrak{S}_d$ the symmetric group on d letters considered as a Coxeter group with generators $\{s_r := (r, r+1) \mid 1 \le r < d\}$ and the corresponding length function

ℓ. The longest element of $\mathfrak{S}_d$ is denoted w_0 or $w_{0,d}$. By definition, $\mathfrak{S}_d$ acts on $[1,d]$ on the left. For a set I the d-tuples from I^d are written as words $\boldsymbol{i} = i_1 \cdots i_d$. The group $\mathfrak{S}_d$ acts on I^d via place permutations: $w \cdot \boldsymbol{i} = i_{w^{-1}(1)} \cdots i_{w^{-1}(d)}$.

Given a composition $\mu = (\mu_1, \dots, \mu_k)$ of d, we have the corresponding standard parabolic subgroup $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k} \leq \mathfrak{S}_d$. We denote by $\mathscr{D}^\mu$ the set of the shortest coset representatives for $\mathfrak{S}_d/\mathfrak{S}_\mu$.

2.2. **Symmetric polynomials and the nil-Hecke algebra.** We impose a grading on the polynomial algebra $\mathcal{X}_d := \Bbbk[x_1, \dots, x_d]$ such that $\deg(x_r) = 2$. The symmetric group $\mathfrak{S}_d$ acts by automorphisms on $\mathcal{X}_d$ via $(w \cdot f)(x_1, \dots, x_d) := f(x_{w(1)}, \dots, x_{w(d)})$. The symmetric polynomial algebra $\Lambda_d := \mathcal{X}_d^{\mathfrak{S}_d}$ has a basis consisting of Schur polynomials

$$\{s_\lambda \mid \lambda \in \mathscr{P}(d)\},$$

where $\mathscr{P}(d)$ is the set of partitions with at most d nonzero parts, see [14, §I.3].

For a composition $\mu = (\mu_1, \dots, \mu_k)$ of d, the algebra of μ-partially symmetric polynomials is $\Lambda_\mu := \mathcal{X}_d^{\mathfrak{S}_\mu}$. We often write $\Lambda_{\mu_1,\dots,\mu_k}$ for Λ_μ and identify it with $\Lambda_{\mu_1} \otimes \cdots \otimes \Lambda_{\mu_k}$. For $a, b \in \mathbb{Z}_{\geq 0}$, let $\mathscr{P}(a,b)$ be the set of partitions with at most a nonzero parts, each part being at most b. The following is known:

Proposition 2.1. *The algebra Λ_μ is free as a Λ_d-module, and in the case where $\mu = (a,b)$ with $a + b = d$, a basis is given by*

$$\{s_\lambda \otimes 1 \mid \lambda \in \mathscr{P}(a,b)\}.$$

Proof. The freeness assertion is [13, PARTL.1.5], which, when combined with [15, Proposition 2.6.8], gives the basis assertion. □

For an integer r with $1 \leq r < d$ and a reduced decomposition $w = s_{r_1} \cdots s_{r_k} \in \mathfrak{S}_d$, the Demazure operators on $\mathcal{X}_d$ are defined as follows:

$$\partial_r := \frac{\mathrm{id}_{\mathcal{X}_d} - s_r}{x_{r+1} - x_r} \quad \text{and} \quad \partial_w := \partial_{r_1} \cdots \partial_{r_k}.$$

Note that ∂_w does not depend on the choice of reduced decomposition and is a degree $-2\ell(w)$ element of $\mathrm{End}_\Bbbk \mathcal{X}_d$.

For integers $r, i, j \geq 0$ with $r + i + j \leq d$, define

$$U_{r;i,j} \in \mathfrak{S}_d \tag{2.2}$$

to be the permutation which maps the interval $[r+1, r+i]$ increasingly onto $[r+1+j, r+i+j]$, and the interval $[r+i+1, r+i+j]$ increasingly onto $[r+1, r+j]$, and fixes all other elements of $[1,d]$. For example, we have $U_{r;1,1} = s_{r+1} = (r+1\ r+2)$. Recalling (1.2), we have:

Proposition 2.3. *[8, Proposition 2.9] Let $a, b \in \mathbb{Z}_{\geq 0}$ with $f \in \Lambda_a$ and $g \in \Lambda_b$. Then $\partial_{U_{0;a,b}}(f \otimes g) = f \star g$.*

Proof. Since we use different conventions from [8], we provide a translation for the reader's convenience. For $w \in \mathfrak{S}_{a+b}$, let $\partial'_w := (-1)^{\ell(w)} \partial_w$. If $k \in \mathbb{Z}_{\geq 0}$, recalling (1.1), let $\gamma'_k := (-1)^{\binom{k}{2}} \gamma_k$ and note that $\gamma'_k(s_\lambda) = v_{\lambda_1+k-1} \wedge v_{\lambda_2+k-2} \wedge \cdots \wedge v_{\lambda_k} \in \bigwedge^k V$ for any $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathscr{P}(k)$. By [8, Proposition 2.9], we have $\partial'_{U_{0;a,b}}(f \otimes g) = (\gamma'_{a+b})^{-1} m(\gamma'_a \otimes \gamma'_b)(f \otimes g)$, so

$$\partial_{U_{0;a,b}}(f \otimes g) = (-1)^{\ell(w)} \partial'_{U_{0;a,b}}(f \otimes g)$$

$$= (-1)^{ab}(\gamma'_{a+b})^{-1}m(\gamma'_a \otimes \gamma'_b)(f \otimes g)$$
$$= (-1)^{ab+\binom{a+b}{2}+\binom{a}{2}+\binom{b}{2}}(\gamma_{a+b})^{-1}m(\gamma_a \otimes \gamma_b)(f \otimes g)$$
$$= f \star g. \qquad \square$$

Define

$$x_0 = x_{0,d} := \prod_{r=1}^{d} x_r^{r-1} \in \mathcal{X}_d.$$

The following is well-known and easy to check:

Lemma 2.4. *We have $\partial_{w_0}(fx_0) = f$ for all $f \in \Lambda_d$.*

The *nil-Hecke algebra* $\mathcal{NH}_d$ is given by generators $\tau_1, \dots, \tau_{d-1}$, $x_1, \dots, x_d$ subject only to the relations

$$x_r x_t = x_t x_r,\ \tau_r^2 = 0,\ \tau_r\tau_{r+1}\tau_r = \tau_{r+1}\tau_r\tau_{r+1},\ \tau_r\tau_s = \tau_s\tau_r\ (|r-s| > 1),$$
$$\tau_r x_r = x_{r+1}\tau_r - 1,\ \tau_r x_{r+1} = x_r\tau_r + 1,\ \tau_r x_s = x_s\tau_r\ (s \neq r, r+1).$$

For a reduced decomposition $w = s_{r_1} \cdots s_{r_k} \in \mathfrak{S}_d$, we have a well-defined element $\tau_w := \tau_{r_1} \cdots \tau_{r_k}$. It is well-known that $\{\tau_w x_1^{k_1} \dots x_d^{k_d} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\geq 0}\}$ is a basis of $\mathcal{NH}_d$. In particular, we identify $\mathcal{X}_d$ as a subalgebra of $\mathcal{NH}_d$.

Theorem 2.5. *[7, Theorem 2.9] The center of $\mathcal{NH}_d$ is equal to Λ_d.*

We consider $\mathcal{X}_d$ as an $\mathcal{NH}_d$-module with τ_w acting by ∂_w and x_r acting by multiplication with x_r. Then we have the following easy to check and well-known properties:

Lemma 2.6. *Let $d \in \mathbb{Z}_{\geq 0}$, $f \in \mathcal{X}_d$, and $w \in \mathfrak{S}_d$. Then in $\mathcal{NH}_d$, we have*

$$\tau_{w_0} f \tau_w = \tau_{w_0}\partial_{w^{-1}}(f) \quad \textit{and} \quad \tau_w f \tau_{w_0} = \partial_w(f)\tau_{w_0}.$$

Define the following elements of $\mathcal{NH}_d$:

$$e_d := x_{0,d}\tau_{w_{0,d}} \quad \text{and} \quad e'_d := \tau_{w_{0,d}} x_{0,d}. \tag{2.7}$$

Lemmas 2.4 and 2.6 yield:

Lemma 2.8. *In $\mathcal{NH}_d$, the elements e_d and e'_d are idempotents. Moreover,*

(i) *$\tau_{w_0} f \tau_{w_0} = 0$ for any $f \in \mathcal{X}_d$ with $\deg f < d(d-1)$, and*

(ii) *$\tau_{w_0} x_0 \tau_{w_0} = \tau_{w_0}$.*

2.3. **KLR Algebras.** From now on, we set $I := \mathbb{Z}$. If $i, j \in I$ with $|i-j| = 1$ we set $\varepsilon_{i,j} := j - i \in \{1, -1\}$. We identify I with the set of vertices of the Dynkin diagram of type $\mathtt{A}_\infty$ and denote by $(\mathtt{c}_{i,j})_{i,j \in I}$ the corresponding Cartan matrix so that $\mathtt{c}_{i,j} = 2$ if $i = j$, $\mathtt{c}_{i,j} = -1$ if $|i-j| = 1$, and $\mathtt{c}_{i,j} = 0$ otherwise. We use the notation Q_+, Φ_+, ht, etc. introduced in Section 1.

For $\theta \in Q_+$ of height d, we define $I^\theta := \{\boldsymbol{i} = i_1 \cdots i_d \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \theta\}$. Let $\mathbb{Z}((q)) \cdot I^\theta := \bigoplus_{\boldsymbol{i} \in I^\theta} \mathbb{Z}((q)) \cdot \boldsymbol{i}$. For $1 \leq k \leq t$, suppose $\theta_k \in Q_+$ are such that $\theta_1 + \cdots + \theta_t = \theta$, and set $d_k := \mathrm{ht}(\theta_k)$. If $\boldsymbol{i}^k \in I^{\theta_k}$, then the concatenation $\boldsymbol{i}^1 \cdots \boldsymbol{i}^t$ is considered as an element of I^θ. Set $\boldsymbol{i}^1 \cdots \boldsymbol{i}^t =: i_1 \cdots i_d$. Then the *quantum shuffle product* is

$$\boldsymbol{i}^1 \circ \cdots \circ \boldsymbol{i}^t := \sum_{w \in \mathscr{D}^{(d_1, \dots, d_t)}} q^{-e(w)} w \cdot (\boldsymbol{i}^1 \cdots \boldsymbol{i}^t) \in \mathbb{Z}((q)) \cdot I^\theta, \tag{2.9}$$

where $e(w) := \sum_{n<m,\, w(n)>w(m)} \mathsf{c}_{i_n,i_m}$. If $a_k \in \mathbb{Z}((q)) \cdot I^{\theta_k}$, we define $a_1 \circ \cdots \circ a_t \in \mathbb{Z}((q)) \cdot I^{\theta}$ extending (2.9) by linearity.

The *KLR algebra* [7, 16] corresponding to θ as above is the unital $\Bbbk$-algebra R_θ (with identity denoted 1_θ) with generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\}$$

and defining relations

$$y_r y_s = y_s y_r; \tag{R1}$$

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}} \quad \text{and} \quad \sum_{\boldsymbol{i} \in I^\theta} 1_{\boldsymbol{i}} = 1_\theta; \tag{R2}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r \quad \text{and} \quad \psi_r 1_{\boldsymbol{i}} = 1_{s_r \cdot \boldsymbol{i}} \psi_r; \tag{R3}$$

$$(\psi_r y_t - y_{s_r(t)} \psi_r) 1_{\boldsymbol{i}} = \delta_{i_r, i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) 1_{\boldsymbol{i}}; \tag{R4}$$

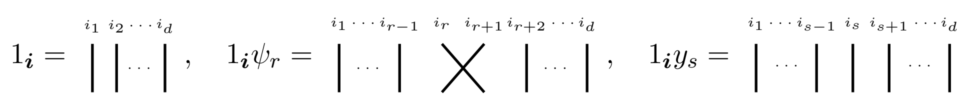
$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ \varepsilon_{i_r,i_{r+1}} (y_r - y_{r+1}) 1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1, \\ 1_{\boldsymbol{i}} & \text{otherwise}; \end{cases} \tag{R5}$$

$$\psi_r \psi_s = \psi_s \psi_r \text{ if } |r - s| > 1; \tag{R6}$$

$$(\psi_{r+1} \psi_r \psi_{r+1} - \psi_r \psi_{r+1} \psi_r) 1_{\boldsymbol{i}} = \begin{cases} \varepsilon_{i_r,i_{r+1}} 1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1 \text{ and } i_r = i_{r+2}, \\ 0 & \text{otherwise}. \end{cases} \tag{R7}$$

The right-hand sides of relations (R4) and (R7), when they are nonzero, will be referred to as *error terms*. The algebra R_θ is graded with $\deg 1_{\boldsymbol{i}} = 0$; $\deg(y_s) = 2$; $\deg(\psi_r 1_{\boldsymbol{i}}) = -\mathsf{c}_{i_r,i_{r+1}}$.

We will use the Khovanov-Lauda [7] diagrammatic notation for elements of R_θ. In particular, for $\boldsymbol{i} = i_1 \cdots i_d \in I^\theta$, $1 \le r < d$ and $1 \le s \le d$, we denote



For each element $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{r_1} \cdots s_{r_l}$ which determines an element $\psi_w = \psi_{r_1} \cdots \psi_{r_l}$. This element depends on the reduced expression of w.

Theorem 2.10. *[7, Theorem 2.5], [16, Theorem 3.7] Let $\theta \in Q_+$ and $d = \mathrm{ht}(\theta)$. Then the following sets are $\Bbbk$-bases of R_θ:*

$$\{\psi_w y_1^{k_1} \dots y_d^{k_d} 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\},$$
$$\{y_1^{k_1} \dots y_d^{k_d} \psi_w 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1, \dots, k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\}.$$

We identify the polynomial algebra

$$\mathcal{Y}_d := \Bbbk[y_1, \dots, y_d] \tag{2.11}$$

with the subalgebra of R_θ generated by $\{y_1, \dots, y_d\}$ according to Theorem 2.10.

The following lemma often simplifies calculations in R_θ.

Lemma 2.12. *Let $i_1, \dots, i_l \in I$ be distinct, $\theta := \alpha_{i_1} + \cdots + \alpha_{i_l}$, $\boldsymbol{i} = i_1 \cdots i_l \in I^\theta$, and $w, w' \in \mathfrak{S}_l$. In R_θ, we have $\psi_{w'} \psi_w 1_{\boldsymbol{i}} = \psi_{w'w} 1_{\boldsymbol{i}}$ unless there are $r, s \in [1, l]$ such that $|i_r - i_s| = 1$, $r < s$, $w(r) > w(s)$, and $w'w(r) < w'w(s)$.*

Proof. As $i_1, \dots, i_l$ are distinct, the braid relation (R7) holds without error term in R_θ. Moreover, so long as there is no pair r, s as in the statement, the only quadratic relations we need to use are of the form $\psi_t^2 1_{\boldsymbol{j}} = 1_{\boldsymbol{j}}$. Therefore $\psi_w \psi_{w'} 1_{\boldsymbol{i}}$ simplifies directly to $\psi_{ww'} 1_{\boldsymbol{i}}$ as claimed. □

2.4. **Parabolic subalgebras and divided power idempotents.** Let $\theta_1, \dots, \theta_t \in Q_+$ and set $\theta := \theta_1 + \cdots + \theta_t$. Let

$$1_{\theta_1,\dots,\theta_t} := \sum_{\boldsymbol{i}^1 \in I^{\theta_1},\dots,\boldsymbol{i}^t \in I^{\theta_t}} 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t} \in R_\theta.$$

Then we have an algebra embedding

$$\iota_{\theta_1,\dots,\theta_t} : R_{\theta_1} \otimes \cdots \otimes R_{\theta_t} \hookrightarrow 1_{\theta_1,\dots,\theta_t} R_{\theta_1+\cdots+\theta_t} 1_{\theta_1,\dots,\theta_t} \tag{2.13}$$

obtained by horizontal concatenation of the Khovanov-Lauda diagrams. For $r_1 \in R_{\theta_1}, \dots, r_t \in R_{\theta_t}$ we often write

$$r_1 \circ \cdots \circ r_t := \iota_{\theta_1,\dots,\theta_t}(r_1 \otimes \cdots \otimes r_t).$$

For example,

$$1_{\boldsymbol{i}^1} \circ \cdots \circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1}, \dots, \boldsymbol{i}^t \in I^{\theta_t}). \tag{2.14}$$

We fix for the moment $i \in I$, $d \in \mathbb{Z}_{\geq 0}$ and take $\theta = d\alpha_i$. Then we have an isomorphism $\varphi : \mathcal{NH}_d \xrightarrow{\sim} R_{d\alpha_i},\ x_r \mapsto y_r,\ \tau_s \mapsto \psi_s$. Recalling (2.7) and Lemma 2.8, the following element is an idempotent in $R_{d\alpha_i}$:

$$1_{i^{(d)}} := \varphi(e'_d).$$

Now let $\theta \in Q_+$ be arbitrary. We define I^θ_{div} to be the set of all expressions of the form $i_1^{(d_1)} \cdots i_r^{(d_r)}$ with $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$, $i_1, \dots, i_r \in I$ and $d_1\alpha_{i_1} + \cdots + d_r\alpha_{i_r} = \theta$. We refer to such expressions as *divided power words*. We identify I^θ with the subset of I^θ_{div} which consists of all divided power words as above with all $d_k = 1$. We use the same notation for concatenation of divided power words as for concatenation of words. For $\boldsymbol{i} = i_1^{(d_1)} \cdots i_r^{(d_r)} \in I^\theta_{\mathrm{div}}$, we define

$$\boldsymbol{i}^!_{(\pm)} := [d_1]^!_{(\pm)} \cdots [d_r]^!_{(\pm)}, \quad \text{and} \quad \hat{\boldsymbol{i}} := i_1^{d_1} \cdots i_r^{d_r} \in I^\theta, \tag{2.15}$$

and the corresponding *divided power idempotent* is

$$1_{\boldsymbol{i}} = 1_{i_1^{(d_1)} \cdots i_r^{(d_r)}} := 1_{i_1^{(d_1)}} \circ \cdots \circ 1_{i_r^{(d_r)}} \in R_\theta.$$

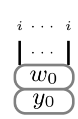
We have the following generalization of (2.14):

$$1_{\boldsymbol{i}^1} \circ \cdots \circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1}_{\mathrm{div}}, \dots, \boldsymbol{i}^t \in I^{\theta_t}_{\mathrm{div}}).$$

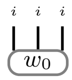
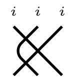
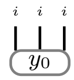
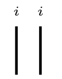
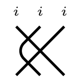
Lemma 2.16. *In the algebra $R_{d\alpha_i}$, if $r_1 + \cdots + r_t = d$ then $1_{i^{(r_1)} \cdots i^{(r_t)}} \psi_{w_{0,d}} = \psi_{w_{0,d}}$ and $1_{i^{(r_1)} \cdots i^{(r_t)}} 1_{i^{(d)}} = 1_{i^{(d)}}$.*

Proof. Write $\psi_{w_{0,d}} = (\psi_{w_{0,r_1}} \circ \cdots \circ \psi_{w_{0,r_t}})\psi_u$ for some $u \in \mathfrak{S}_d$ and use Lemma 2.8. □

To be used as part of the Khovanov-Lauda diagrammatics, we denote

$\psi_{w_{0,d}} =:$ [w_0], $\quad y_{0,d} =:$ [y_0], $\quad 1_{i^{(d)}} =$  $=:$ [i^d].

For example, if $d = 3$, we have

$1_{i^3}\psi_{w_0} =$  $=$  , $1_{i^3}y_0 =$  $=$  , $1_{i^{(3)}} =$  .

More generally, we denote

$$1_{i_1^{(d_1)}\cdots i_r^{(d_r)}} =: \boxed{i_1^{d_1}} \cdots \boxed{i_r^{d_r}}.$$

2.5. **Modules over** R_θ**.** Let $\theta \in Q_+$. We denote by R_θ–Mod the category of graded left R_θ-modules. The morphisms in this category are all homogeneous degree zero R_θ-homomorphisms, which we denote $\hom_{R_\theta}(-,-)$. For $V \in R_\theta$–Mod, let q^dV denote its grading shift by d, so if V_m is the degree m component of V, then $(q^dV)_m = V_{m-d}$. More generally, for a Laurent series $a = a(q) = \sum_d a_d q^d \in \mathbb{Z}((q))$ with non-negative coefficients, we set $aV := \bigoplus_d (q^dV)^{\oplus a_d}$. For $U, V \in R_\theta$–Mod, we set $\mathrm{Hom}_{R_\theta}(U,V) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{R_\theta}(U,V)_d$, where

$$\mathrm{Hom}_{R_\theta}(U,V)_d := \hom_{R_\theta}(q^dU,V) = \hom_{R_\theta}(U,q^{-d}V).$$

We define $\mathrm{Ext}^m_{R_\theta}(U,V)$ and $\mathrm{End}_{R_\theta}(U)$ similarly from $\mathrm{ext}^m_{R_\theta}(U,V)$ and $\mathrm{end}_{R_\theta}(U)$.

For a free $\Bbbk$-module V of finite rank we denote the rank of V by $\dim V$. A graded $\Bbbk$-module $V = \bigoplus_{m\in\mathbb{Z}} V_m$ is called *Laurentian* if the graded components V_m are free of finite rank for all $m \in \mathbb{Z}$ and $V_m = 0$ for $m \ll 0$. For example R_θ itself is Laurentian by Theorem 2.10. The *graded rank* of a Laurentian $\Bbbk$-module V is

$$\dim_q V := \sum_{m\in\mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}((q)).$$

Recall that the ground ring $\Bbbk$ is assumed to be a PID. The following standard result often allows us to reduce to the case where $\Bbbk$ is a field.

Lemma 2.17. *If $\varphi : V \to W$ is a degree 0 homomorphism of Laurentian $\Bbbk$-modules such that the induced map $\overline{\varphi} : V/JV \to W/JW$ is an isomorphism of $\Bbbk/J$-vector spaces for every maximal ideal J, then φ is an isomorphism.*

We say that an R_θ-module V is *Laurentian* if it is so as a $\Bbbk$-module. Recalling (2.15), for a Laurentian R_θ-module V and $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$, by [7, §2.5], we have

$$\dim_q(1_{\boldsymbol{i}}V) = \frac{1}{\boldsymbol{i}^!_+} \dim_q(1_{\hat{\boldsymbol{i}}}V), \tag{2.18}$$

which explains the usage of the term "divided power word" for $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$. If V is a Laurentian R_θ-module then each $1_{\boldsymbol{i}}V$ is a Laurentian $\Bbbk$-module, and so we can define the *formal character* of V as follows:

$$\mathrm{ch}_q V := \sum_{\boldsymbol{i}\in I^\theta} (\dim_q 1_{\boldsymbol{i}}V) \cdot \boldsymbol{i} \in \mathbb{Z}((q)) \cdot I^\theta.$$

Note that $\mathrm{ch}_q(q^dV) = q^d \mathrm{ch}_q(V)$.

For $\theta_1, \dots, \theta_t \in Q_+$ and $\theta := \theta_1 + \cdots + \theta_t$, recalling (2.13), we have a functor

$$\mathrm{Ind}_{\theta_1,\dots,\theta_t} = R_\theta 1_{\theta_1,\dots,\theta_t} \otimes_{R_{\theta_1}\otimes\cdots\otimes R_{\theta_t}} - : (R_{\theta_1} \otimes \cdots \otimes R_{\theta_t})\text{–Mod} \to R_\theta\text{–Mod}.$$

For $V_1 \in R_{\theta_1}$–Mod, $\dots, V_t \in R_{\theta_t}$–Mod, we denote by $V_1 \boxtimes \cdots \boxtimes V_t$ the $\Bbbk$-module $V_1 \otimes \cdots \otimes V_t$, considered naturally as an $(R_{\theta_1} \otimes \cdots \otimes R_{\theta_t})$-module, and set

$$V_1 \circ \cdots \circ V_t := \mathrm{Ind}_{\theta_1,\dots,\theta_t} V_1 \boxtimes \cdots \boxtimes V_t.$$

By Theorem 2.10, setting $d_k := \mathrm{ht}(\theta_k)$, we have

$$V_1 \circ \cdots \circ V_t = \bigoplus_{w \in \mathscr{D}^{(d_1,\dots,d_t)}} \psi_w \otimes V_1 \otimes \cdots \otimes V_t. \tag{2.19}$$

If $V_1, \dots, V_t$ are Laurentian, then by [7, Lemma 2.20], recalling (2.9), we have

$$\mathrm{ch}_q(V_1 \circ \cdots \circ V_t) = \mathrm{ch}_q(V_1) \circ \cdots \circ \mathrm{ch}_q(V_t). \tag{2.20}$$

For $v_1 \in V_1, \dots, v_t \in V_t$, we denote

$$v_1 \circ \cdots \circ v_t := 1_{\theta_1,\dots,\theta_t} \otimes v_1 \otimes \cdots \otimes v_t \in V_1 \circ \cdots \circ V_t.$$

If $\boldsymbol{i}^1 \in I^{\theta_1}_{\mathrm{div}}, \dots, \boldsymbol{i}^t \in I^{\theta_t}_{\mathrm{div}}$, it is easy to check that

$$R_{\theta_1} 1_{\boldsymbol{i}^1} \circ \cdots \circ R_{\theta_t} 1_{\boldsymbol{i}^t} \cong R_\theta 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t}. \tag{2.21}$$

Since $R_\theta 1_{\theta_1,\dots,\theta_t}$ is a free right $R_{\theta_1} \otimes \cdots \otimes R_{\theta_t}$-module of finite rank by Theorem 2.10, we get the following well-known properties:

Proposition 2.22. *The functor* $\mathrm{Ind}_{\theta_1,\dots,\theta_t}$ *is exact and sends finitely generated projectives to finitely generated projectives.*

Let again $\theta_1, \dots, \theta_t \in Q_+$ and $\theta = \theta_1 + \cdots + \theta_t$. Suppose $(C^k_\bullet, d_k)$ is a chain complexes of R_{θ_k}-modules for $1 \le k \le t$. Let $(C^1_\bullet \circ \cdots \circ C^t_\bullet)_n := \bigoplus_{p_1+\cdots+p_t=n} C^1_{p_1} \circ \cdots \circ C^t_{p_t}$, and for $x_1 \in C^1_{p_1}, \dots, x_t \in C^t_{p_t}$, define

$$d(x_1 \circ \cdots \circ x_t) := \sum_{k=1}^{t} (-1)^{p_{k+1}+\cdots+p_t} x_1 \circ \cdots \circ x_{k-1} \circ d_k(x_k) \circ x_{k+1} \circ \cdots \circ x_t.$$

Then $(C^1_\bullet \circ \cdots \circ C^t_\bullet, d)$ is a chain complex of R_θ-modules. Proposition 2.22 and [17, Lemma 2.7.3] immediately imply the following.

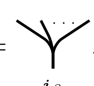
Lemma 2.23. *If* $C^k_\bullet$ *is a projective resolution of* $M_k \in R_{\theta_k}$–Mod *for* $1 \le k \le t$, *then* $C^1_\bullet \circ \cdots \circ C^t_\bullet$ *is a projective resolution of* $M_1 \circ \cdots \circ M_t \in R_\theta$–Mod.

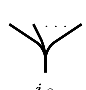
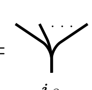
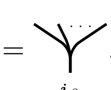
2.6. **Standard modules.** The algebra R_θ is affine quasihereditary in the sense of [9]. In particular, it comes with an important class of *standard modules*, which we now describe explicitly following [1].

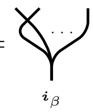
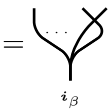
Fix $\beta = \alpha_i + \cdots + \alpha_j \in \Phi_+$ of height $l := j - i + 1$, and set $\boldsymbol{i}_\beta := i\,(i+1) \cdots j \in I^\beta$. We define the R_β-module $\Delta(\beta)$ to be a cyclic R_β-module generated by a vector v_β of degree 0 with defining relations

- $1_{\boldsymbol{i}} v_\beta = \delta_{\boldsymbol{i},\boldsymbol{i}_\beta} v_\beta$ for all $\boldsymbol{i} \in I^\beta$;
- $\psi_r v_\beta = 0$ for all $1 \le r < l$;
- $y_r v_\beta = y_s v_\beta$ for all $1 \le r, s \le l$.

The module $\Delta(\beta)$ can be considered as an $(R_\beta, \Bbbk[x])$-bimodule with the right action given by $v_\beta x := y_1 v_\beta$. Diagrammatically, we represent

$v_\beta =$  ,

$v_\beta x =$  $= y_1 v_\beta =$  $= \cdots = y_l v_\beta =$  ,

$\psi_1 v_\beta =$  $= \cdots = \psi_{l-1} v_\beta =$  $= 0.$

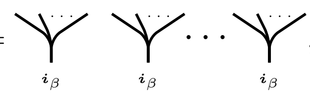
The following lemma is easy to check.

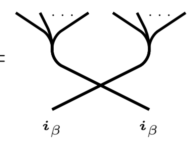
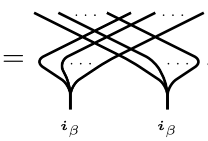
Lemma 2.24. *Let $\beta \in \Phi_+$. Then there is an isomorphism of right $\Bbbk[x]$-modules $\Bbbk[x] \to \Delta(\beta),\ 1 \mapsto v_\beta$.*

For $m \in \mathbb{Z}_{\geq 0}$ and $\beta \in \Phi_+$, the $R_{m\beta}$-module $\Delta(\beta)^{\circ m}$ is cyclicly generated by $v_\beta^{\circ m}$. As explained in [1, §3.2], $\mathcal{NH}_m$ acts on $\Delta(\beta)^{\circ m}$ on the right so that

$$v_\beta^{\circ m} x_r = v_\beta^{\circ(r-1)} \circ (v_\beta x) \circ v_\beta^{\circ(m-r)}, \quad v_\beta^{\circ m} \tau_s = v_\beta^{\circ(s-1)} \circ (\psi_{w_{l,l}}(v_\beta \circ v_\beta)) \circ v_\beta^{\circ(m-s-1)},$$

where $w_{l,l}$ is the longest element of $\mathscr{D}^{(l,l)}$. Diagrammatically, we represent

$v_\beta^{\circ m} =$  ,

$(v_\beta \circ v_\beta)\tau_1 =$  $= \psi_{w_{l,l}}(v_\beta \circ v_\beta) =$  .

Let $\leq$ be the lexicographic total order on Φ_+, i.e. for $\beta = \alpha_i + \cdots + \alpha_j \in \Phi_+$ and $\beta' = \alpha_{i'} + \cdots + \alpha_{j'} \in \Phi_+$, we have $\beta < \beta'$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. Given $\theta \in Q_+$, a *Kostant partition* of θ is a sequence $\pi = (\beta_1^{m_1}, \ldots, \beta_t^{m_t})$ such that $m_1, \ldots, m_t \in \mathbb{Z}_{>0}$, $\beta_1 > \cdots > \beta_t$ are positive roots, and $m_1\beta_1 + \cdots + m_t\beta_t = \theta$. We denote by $\mathrm{KP}(\theta)$ the set of all Kostant partitions of θ. For $\pi = (\beta_1^{m_1}, \ldots, \beta_t^{m_t}) \in \mathrm{KP}(\theta)$,

$$\hat{\Delta}(\pi) := q^{\binom{m_1}{2} + \cdots + \binom{m_t}{2}} \Delta(\beta_1)^{\circ m_1} \circ \cdots \circ \Delta(\beta_t)^{\circ m_t} \tag{2.25}$$

can now be considered as an $(R_\theta, \mathcal{NH}_{m_1} \otimes \cdots \otimes \mathcal{NH}_{m_t})$-bimodule. Recalling (2.7), we define the corresponding *standard module* as

$$\Delta(\pi) := \hat{\Delta}(\pi)(e_{m_1} \otimes \cdots \otimes e_{m_t}).$$

Setting

$$\Lambda_\pi := \Lambda_{m_1, \ldots, m_t}, \tag{2.26}$$

by Theorem 2.5, $\Delta(\pi)$ is naturally an (R_θ, Λ_π)-bimodule. In fact, by [11, Theorem 2.17], the bimodule structure yields the isomorphism

$$\mathrm{End}_{R_\theta}(\Delta(\pi))^{\mathrm{op}} \cong \Lambda_\pi. \tag{2.27}$$

The module $\Delta(\pi)$ is cyclic as a left R_θ-module with *standard generator*

$$v_\pi := (v_\beta^{\circ m_1} \circ \cdots \circ v_\beta^{\circ m_t})(e_{m_1} \otimes \cdots \otimes e_{m_t}). \tag{2.28}$$

Noting that $\Delta(\pi) = \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_1^{m_t})$, by [1, Lemma 3.10], we have an isomorphism of R_θ-modules

$$\hat{\Delta}(\pi) \cong [m_1]^!_+ \cdots [m_t]^!_+ \Delta(\pi). \tag{2.29}$$

If $\Bbbk$ is a field, the modules $\{\Delta(\pi) \mid \pi \in \mathrm{KP}(\theta)\}$ are the standard modules for an affine quasihereditary structure on the algebra R_θ, see [1,9]. If $\Bbbk = \mathbb{Z}$ or $\mathbb{Z}_p$, they can be thought of as integral forms of the standard modules, see [11, §4].

For $\rho \in \mathrm{KP}(\theta)$, suppose we have a projective resolution of $\Delta(\rho)$ of the form $P^\rho_\bullet = \cdots \xrightarrow{d_1} P^\rho_1 \xrightarrow{d_0} P^\rho_0 \xrightarrow{\varepsilon_\rho} \Delta(\rho)$ with $P^\rho_n = \bigoplus_{x\in X_n} q^{s_x} R_\theta 1_x$ for some index set X_n, integers s_x, and idempotents 1_x. The map $d_n : P^\rho_{n+1} \to P^\rho_n$ can be described as right multiplication by an $X_{n+1} \times X_n$ matrix $D_n = (d^{y,x}_n)$ for some $d^{y,x}_n \in 1_y R_\theta 1_x$. Using the isomorphism $\mathrm{Hom}_{R_\theta}(q^n R_\theta 1_x, \Delta(\sigma)) \xrightarrow{\sim} q^{-n} 1_x \Delta(\sigma)$ and recalling (2.27), we obtain:

Lemma 2.30. *There is an isomorphism of complexes of (right) Λ_σ-modules*

$$\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, \Delta(\sigma)) \cong T^{P^\rho_\bullet}_\bullet(\Delta(\sigma)),$$

where $T^{P^\rho_\bullet}_\bullet(\sigma) = \cdots \xleftarrow{d^1} T^{P^\rho_\bullet}_1(\sigma) \xleftarrow{d^0} T^{P^\rho_\bullet}_0(\sigma)$ with $T^{P^\rho_\bullet}_n(\sigma) = \bigoplus_{x\in X_n} q^{-s_x} 1_x \Delta(\sigma)$ and d^n given by left multiplication with the $X_{n+1} \times X_n$ matrix D_n.

This yields an isomorphism $\mathcal{E}_\theta(\rho,\sigma) \cong H(T^{P^\rho_\bullet}_\bullet(\sigma))$ of Λ_σ-modules. One can also use the resolutions $P^\rho_\bullet$, $P^\sigma_\bullet$, and $P^\tau_\bullet$ to describe the composition map $\mathcal{E}_\theta(\sigma,\tau) \otimes \mathcal{E}_\theta(\rho,\sigma) \to \mathcal{E}_\theta(\rho,\tau)$. Indeed, let $\mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$ denote the homological degree m homomorphisms. Then $\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$ is a complex with respect to the differential δ given by

$$\delta(\varphi) := d\varphi - (-1)^m \varphi d \tag{2.31}$$

for $\varphi \in \mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$. We have an isomorphism

$$H(\mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)) \cong \mathcal{E}_\theta(\rho,\sigma) \tag{2.32}$$

induced by the maps

$$\mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet) \to \mathrm{Hom}_{R_\theta}(P^\rho_m, \Delta(\sigma)),\ \varphi \mapsto (-1)^{\frac{m(m+1)}{2}} \varepsilon_\sigma(\varphi|_{P^\rho_m}). \tag{2.33}$$

Now the composition map $\mathcal{E}^n_\theta(\sigma,\tau)\otimes\mathcal{E}^m_\theta(\rho,\sigma) \to \mathcal{E}^{n+m}_\theta(\rho,\tau)$ is induced from the composition of homomorphisms $\mathrm{Hom}^n_{R_\theta}(P^\sigma_\bullet, P^\tau_\bullet)\otimes\mathrm{Hom}^m_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet) \to \mathrm{Hom}^{n+m}_{R_\theta}(P^\rho_\bullet, P^\tau_\bullet)$.

3. The case where θ is a root

Throughout the section, $\theta = \alpha_a + \cdots + \alpha_{b+1}$ (with $a \le b+1$) is a positive root of height $l = b+2-a$. There is a bijection from $\mathrm{KP}(\theta)$ to the set of all subsets of $[a,b]$

$$\pi = (\pi_1,\dots,\pi_u) \mapsto C_\pi := \{\max(\mathrm{supp}\,\pi_2), \max(\mathrm{supp}\,\pi_3),\dots,\max(\mathrm{supp}\,\pi_u)\},$$

where, for a root $\alpha = \alpha_i + \cdots + \alpha_j$, we let $\mathrm{supp}\,\alpha := [i,j]$. For $\pi,\tau \in \mathrm{KP}(\theta)$, if $C_\tau \supseteq C_\pi$, we say that τ is a *refinement* of π and write $\tau \supseteq \pi$. If, in addition, $|C_\tau \backslash C_\pi| = n$, we write $\tau \supseteq_n \pi$ and say that τ is an *n-refinement* of π. If $C_\tau\backslash C_\pi = \{i\}$ for some $i \in [a,b]$, we write $\mathrm{ref}^i(\pi) := \tau$. For example, we have $\mathrm{ref}^i((\theta)) = (\alpha_{i+1} + \cdots + \alpha_{b+1}, \alpha_a + \cdots + \alpha_i)$.

If $\tau = (\tau_1,\dots,\tau_t) \in \mathrm{KP}(\theta)$, the elements of

$$\mathscr{D}^\tau := \mathscr{D}^{(\mathrm{ht}(\tau_1),\dots,\mathrm{ht}(\tau_t))} \tag{3.1}$$

are called *τ-shuffles.* Set

$$d_v := \mathrm{ht}(\tau_1) + \cdots + \mathrm{ht}(\tau_v) \qquad (0 \le v \le t).$$

We say that integers $r,s \in [1,l]$ are *τ-equivalent* if there is some $v \in [1,t]$ with $d_{v-1} < r, s \le d_v$. Recalling (2.26) and (2.27), we have

$$\Lambda_\tau = \Bbbk[x_1,\dots,x_t] \cong \mathrm{End}_{R_\theta}(\Delta(\tau))^{\mathrm{op}}.$$

We have a surjection

$$\mathtt{p}_\tau : \mathcal{Y}_l \twoheadrightarrow \Lambda_\tau,\ y_r \mapsto x_v \text{ if } d_{v-1} < r \le d_v, \tag{3.2}$$

so that $\mathtt{p}_\tau(y_r) = \mathtt{p}_\tau(y_s)$ if and only if r and s are τ-equivalent. If, in addition, $\tau \supseteq \pi = (\pi_1, \dots, \pi_u)$, then $\mathtt{p}_\pi$ factors as $\mathtt{p}_\pi = \mathtt{p}^\tau_\pi \mathtt{p}_\tau$, where the surjection $\mathtt{p}^\tau_\pi$ is defined as follows: for each $m \in [0, u]$, we have $\mathrm{ht}(\pi_1) + \cdots + \mathrm{ht}(\pi_m) = d_{v_m}$ for some $v_m \in [0, t]$; now

$$\mathtt{p}^\tau_\pi : \Lambda_\tau \twoheadrightarrow \Lambda_\pi,\ x_r \mapsto x_m \text{ if } v_{m-1} < r \le v_m. \tag{3.3}$$

3.1. **The resolution $S^\rho_\bullet$.** In this subsection, we fix $\rho = (\rho_1, \dots, \rho_t) \in \mathrm{KP}(\theta)$. For $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \in \Phi_+$, let

$$\boldsymbol{j}_\alpha := i(i+1)\cdots j \in I^\alpha, \quad e_\alpha := 1_{\boldsymbol{j}_\alpha} \in R_\alpha.$$

Then set

$$\boldsymbol{j}_\rho := \boldsymbol{j}_{\rho_1} \cdots \boldsymbol{j}_{\rho_t} \in I^\theta, \quad e_\rho := 1_{\boldsymbol{j}_\rho} \in R_\theta. \tag{3.4}$$

For $\pi, \tau \in \mathrm{KP}(\theta)$, let $w(\tau, \pi) \in \mathfrak{S}_l$ be the unique permutation with

$$w(\tau, \pi) \cdot \boldsymbol{j}_\pi = \boldsymbol{j}_\tau, \tag{3.5}$$

so that $e_\tau \psi_{w(\tau,\pi)} = \psi_{w(\tau,\pi)} e_\pi$. If $\tau = \mathrm{ref}^i(\pi)$, we set

$$s(\tau, \pi) := (-1)^{|C_\pi \cap [a,i)|}.$$

For $n \in \mathbb{Z}_{\ge 0}$, we set

$$S^\rho_n := \bigoplus_{\pi \supseteq_n \rho} q^n R_\theta e_\pi.$$

The boundary map $S^\rho_{n+1} \to S^\rho_n$ is defined to be right multiplication with the matrix

$$d_n := (d_n^{\tau,\pi})_{\substack{\tau \supseteq_{n+1} \rho \\ \pi \supseteq_n \rho}}, \text{ where } d_n^{\tau,\pi} := \begin{cases} s(\tau,\pi) e_\tau \psi_{w(\tau,\pi)} e_\pi & \text{if } \tau \supseteq \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

We define the augmentation map by

$$\varepsilon_\rho : S^\rho_0 = R_\theta e_\rho \to \Delta(\rho),\ h e_\rho \mapsto h v_\rho,$$

where v_ρ is the standard generator for $\Delta(\rho)$, see (2.28).

Lemma 3.7. *The following is a projective resolution of $\Delta(\rho)$:*

$$0 \longrightarrow S^\rho_{l-t} \longrightarrow \cdots \longrightarrow S^\rho_{n+1} \xrightarrow{d_n} S^\rho_n \longrightarrow \cdots \longrightarrow S^\rho_0 \xrightarrow{\varepsilon_\rho} \Delta(\rho) \longrightarrow 0.$$

Proof. Given a complex $C_\bullet$, we denote by $\overline{C}_\bullet$ the same complex but with all the boundary maps negated. If $v \in [1, t]$, we let

$$\widehat{S}^{(\rho_v)}_\bullet = \begin{cases} \overline{S}^{(\rho_v)}_\bullet & \text{if } t - v \text{ is odd,} \\ S^{(\rho_v)}_\bullet & \text{if } t - v \text{ is even.} \end{cases}$$

Using (2.21) and the fact that the resolutions $\overline{S}^{(\rho_v)}_\bullet$ and $S^{(\rho_v)}_\bullet$ are isomorphic, it is easy to note that

$$S^\rho_\bullet = \widehat{S}^{(\rho_1)}_\bullet \circ \cdots \circ \widehat{S}^{(\rho_t)}_\bullet \cong S^{(\rho_1)}_\bullet \circ \cdots \circ S^{(\rho_t)}_\bullet,$$

so by Lemma 2.23, we have reduced to the case $t = 1$, i.e. $\rho = (\rho_1) = (\theta)$.

To complete the proof, we show that $S_\bullet^{(\theta)} \cong P_\bullet$, where $P_\bullet$ is a resolution of $\Delta(\theta)$ constructed in [1, §4.5] (see also [2]) and which we now recall. For $\pi \in \mathrm{KP}(\theta)$, put

$$\boldsymbol{i}^\pi := a^{\delta_{a\notin C_\pi}}(a+1)^{\delta_{a+1\notin C_\pi}} \cdots b^{\delta_{b\notin C_\pi}}(b+1)b^{\delta_{b\in C_\pi}} \cdots (a+1)^{\delta_{a+1\in C_\pi}} a^{\delta_{a\in C_\pi}} \in I^\theta.$$

For $n \in \mathbb{Z}_{\geq 0}$, set $P_n := \bigoplus_{\pi \supseteq_n (\theta)} q^n R_\theta 1_{\boldsymbol{i}^\pi}$. If $\pi, \tau \in \mathrm{KP}(\theta)$ with $\tau = \mathrm{ref}^i(\pi)$, let $u(\tau,\pi) \in \mathfrak{S}_l$ be determined from $u(\tau,\pi)\cdot \boldsymbol{i}^\pi = \boldsymbol{i}^\tau$, and define the matrix

$$\partial_n := (\partial_n^{\tau,\pi})_{\substack{\tau\supseteq_{n+1}(\theta)\\ \pi\supseteq_n(\theta)}}, \text{ where } \partial_n^{\tau,\pi} := \begin{cases} s(\tau,\pi)1_{\boldsymbol{i}^\tau}\psi_{u(\tau,\pi)}1_{\boldsymbol{i}^\pi} & \text{if } \tau \supseteq \pi,\\ 0 & \text{otherwise.}\end{cases}$$

Right multiplication with ∂_n defines a map $P_{n+1} \to P_n$. By [1, Theorem 4.12] (see also [2, Theorem A]), noting that $P_0 = S_0^{(\theta)}$, we have that

$$0 \longrightarrow P_{b+1-a} \longrightarrow \cdots \longrightarrow P_{n+1} \xrightarrow{\partial_n} P_n \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon(\theta)} \Delta(\theta) \longrightarrow 0$$

is a projective resolution of $\Delta(\theta)$.

For $\pi \in \mathrm{KP}(\theta)$, let $w(\pi) \in \mathfrak{S}_l$ be the unique permutation with $w(\pi)\cdot \boldsymbol{i}^\pi = \boldsymbol{j}_\pi$ so that $e_\pi \psi_{w(\pi)} = \psi_{w(\pi)} 1_{\boldsymbol{i}^\pi}$. We have the map

$$S_n^{(\theta)} = \bigoplus_{\pi\supseteq_n(\theta)} q^n R_\theta e_\pi \to \bigoplus_{\pi\supseteq_n(\theta)} q^n R_\theta 1_{\boldsymbol{i}^\pi} = P_n,$$
$$(h_\pi e_\pi)_{\pi\supseteq_n(\theta)} \mapsto (h_\pi e_\pi \psi_{w(\pi)} 1_{\boldsymbol{i}^\pi})_{\pi\supseteq_n(\theta)}.$$

To check that this yields an isomorphism of complexes, let $\tau := \mathrm{ref}^i(\pi)$ for some $i \in [a,b]\setminus C_\pi$, and check the following using Lemma 2.12:

- $e_\tau \psi_{w(\tau)} \psi_{u(\tau,\pi)} = e_\tau \psi_{w(\tau,\pi)} \psi_{w(\pi)}$,
- $e_\pi \psi_{w(\pi)} \psi_{w(\pi)^{-1}} = e_\pi$,
- $1_{\boldsymbol{i}^\pi} \psi_{w(\pi)^{-1}} \psi_{w(\pi)} = 1_{\boldsymbol{i}^\pi}$. □

3.2. **The $\Bbbk$-module $\mathcal{E}_\theta(\rho,\sigma)$.** Throughout this subsection, we fix $\rho, \sigma \in \mathrm{KP}(\theta)$. Recall the word $\boldsymbol{j}_\sigma \in I^\theta$ and the idempotent e_σ from (3.4). Writing $\boldsymbol{j}_\sigma = j_1 j_2 \cdots j_l$, for $i \in [a, b+1]$, there exists a unique $r \in [1,l]$ such that $j_r = i$, and we denote $r_\sigma(i) := r$.

We have the standard module $\Delta(\sigma)$ with generator $v_\sigma \in e_\sigma \Delta(\sigma)$, see (2.28). As in §2.6, we consider $\Delta(\sigma)$ as an $(R_\theta, \Lambda_\sigma)$-bimodule. Recalling Lemma 2.30, we write $T_\bullet^\rho(\sigma) := T_\bullet^{S_\bullet^\rho}(\Delta(\sigma))$, so that $T_n^\rho(\sigma) = \bigoplus_{\pi\supseteq_n \rho} q^{-n} e_\pi \Delta(\sigma)$. Recall (3.2) and (3.1).

Lemma 3.8. *In $\Delta(\sigma)$, we have:*

(i) $y_r v_\sigma = v_\sigma \mathtt{p}_\sigma(y_r)$*; in particular,* $y_r v_\sigma = y_s v_\sigma$ *if r and s are σ-equivalent;*
(ii) $\psi_w v_\sigma = 0$ *whenever $w \in \mathfrak{S}_l$ is not a σ-shuffle.*

Moreover, $\Delta(\sigma)$ is free as a right Λ_σ-module with basis $\{\psi_w v_\sigma \mid w \in \mathscr{D}^\sigma\}$.

Proof. Use Lemma 2.24 and (2.19). □

Recalling (3.5), we now get:

Lemma 3.9. *Let $\pi, \sigma \in \mathrm{KP}(\theta)$. If $\sigma \not\supseteq \pi$, then $e_\pi \Delta(\sigma) = 0$. If $\sigma \supseteq_n \pi$, then there is an isomorphism of right Λ_σ-modules*

$$q^n \Lambda_\sigma \xrightarrow{\sim} e_\pi \Delta(\sigma), \quad f \mapsto \psi_{w(\pi,\sigma)} v_\sigma f.$$

Proof. If σ is a refinement of π, then every pair of σ-equivalent integers is also π-equivalent, so $w(\pi,\sigma)$ is a σ-shuffle and the result follows from Lemma 3.8 since in this case we have $\deg(\psi_{w(\pi,\sigma)}e_\sigma) = n$.

On the other hand, if $\sigma \not\supseteq \pi$, then there is some $i \in C_\pi$ with $i \notin C_\sigma$. It follows that $r_\sigma(i)$ and $r_\sigma(i)+1$ are σ-equivalent but $w(\pi,\sigma)(r_\sigma(i)) > w(\pi,\sigma)(r_\sigma(i)+1)$, so $w(\pi,\sigma)$ is not a σ-shuffle, and $e_\pi\Delta(\sigma) = 0$ by Lemma 3.8(ii). □

Corollary 3.10. *We have:*

(i) *If $\sigma \not\supseteq \rho$, then $T^\rho_\bullet(\sigma) = 0$.*

(ii) *If $\sigma \supseteq_m \rho$, then as right Λ_σ-modules,*

$$T^\rho_n(\sigma) = \bigoplus_{\sigma\supseteq\pi\supseteq_n\rho} q^{-n}\psi_{w(\pi,\sigma)}v_\sigma \cdot \Lambda_\sigma \cong \bigoplus_{\sigma\supseteq\pi\supseteq_n\rho} q^{m-2n}\Lambda_\sigma.$$

In particular, $T^\rho_n(\sigma) = 0$ for $n > m$ and

$$T^\rho_m(\sigma) = q^{-m}e_\sigma\Delta(\sigma) = v_\sigma \cdot \Lambda_\sigma \cong q^{-m}\Lambda_\sigma.$$

Proof. If σ is not a refinement of ρ, then it cannot be a refinement of any $\pi \supseteq \rho$, which implies (i). Let $\sigma \supseteq_m \rho$. Recall that $T^\rho_n(\sigma) = \bigoplus_{\pi\supseteq_n\rho} q^{-n}e_\pi\Delta(\sigma)$. Let $\pi \supseteq_n \rho$. If $\sigma \not\supseteq \pi$, then $e_\pi\Delta(\sigma) = 0$. Otherwise $e_\pi\Delta(\sigma) = \psi_{w(\pi,\sigma)}v_\sigma \cdot \Lambda_\sigma$. □

Lemma 3.11. *Let $\pi \in \mathrm{KP}(\theta)$. Suppose that $\sigma \supseteq \mathrm{ref}^i(\pi)$ for some $i \in [a,b]$. Then*

$$\psi_{w(\mathrm{ref}^i(\pi),\pi)}\psi_{w(\pi,\sigma)}e_\sigma = \psi_{w(\mathrm{ref}^i(\pi),\sigma)}(y_{r_\sigma(i)} - y_{r_\sigma(i+1)})e_\sigma.$$

Proof. Let $\tau = \mathrm{ref}^i(\pi)$. We have that i and $i+1$ are the only adjacent elements of $[a,b+1]$ with $r_\tau(i) > r_\tau(i+1)$ and $r_\pi(i) < r_\pi(i+1)$. Since $\sigma \supseteq \tau = \mathrm{ref}^i(\pi)$, we have $r_\sigma(i) > r_\sigma(i+1)$. Now we compute:

$$\begin{aligned}\psi_{w(\tau,\pi)}\psi_{w(\pi,\sigma)}e_\sigma &= \psi_{w(\tau,\pi)s_{r_\pi(i)}}\psi^2_{r_\pi(i)}\psi_{s_{r_\pi(i)}w(\pi,\sigma)}e_\sigma \\ &= \psi_{w(\tau,\pi)s_{r_\pi(i)}}(y_{r_\pi(i+1)} - y_{r_\pi(i)})\psi_{s_{r_\pi(i)}w(\pi,\sigma)}e_\sigma \\ &= \psi_{w(\tau,\pi)s_{r_\pi(i)}}\psi_{s_{r_\pi(i)}w(\pi,\sigma)}(y_{r_\sigma(i)} - y_{r_\sigma(i+1)})e_\sigma \\ &= \psi_{w(\tau,\sigma)}(y_{r_\sigma(i)} - y_{r_\sigma(i+1)})e_\sigma,\end{aligned}$$

where the first equality is obtained using the fact that the braid relations (R7) hold without error term in R_θ, the second comes by applying a non-trivial quadratic relation (R5) on strands colored i and $i+1$, the third is obtained using the relation (R4), and the last comes from Lemma 2.12. □

The proof of Theorem 3.12 amounts to showing that $T^\rho_\bullet(\sigma)$ is isomorphic to a certain Koszul complex (see [17, §4.5]) which we now define. Suppose $\sigma \supseteq_m \rho$, and write $C_\sigma \setminus C_\rho = \{i_1,\dots,i_m\}$ with $i_1 < \cdots < i_m$. Let N be the free right Λ_σ-module of graded rank mq^{-2}, that is, $N := q^{-2}\Lambda_\sigma^{\oplus m}$. For $k \in [1,m]$, we denote $\epsilon_k := (0,\dots,0,1,0,\dots,0) \in N$ (with "1" in the kth entry). Recalling (3.2), define

$$\begin{aligned} z_k &:= s(\mathrm{ref}^{i_k}(\rho),\rho)\mathrm{p}_\sigma(y_{r_\sigma(i_k)} - y_{r_\sigma(i_k+1)}) \in \Lambda_\sigma \qquad (k = 1,\dots,m), \\ Z &:= (z_1,\dots,z_m) = \epsilon_1 z_1 + \cdots + \epsilon_m z_m \in N.\end{aligned}$$

Note that Z is a homogeneous degree 0 element of N. We consider the Koszul complex $q^m\bigwedge^\bullet N$ associated to the regular sequence Z for the ring Λ_σ:

$$0 \longleftarrow q^m{\textstyle\bigwedge}^m N \longleftarrow \cdots \longleftarrow q^m{\textstyle\bigwedge}^{n+1} N \longleftarrow q^m{\textstyle\bigwedge}^n N \longleftarrow \cdots \longleftarrow q^m{\textstyle\bigwedge}^0 N \longleftarrow 0$$

$$Z \wedge a \longleftarrow a$$

where $\bigwedge^n N$ is the nth exterior power of the free Λ_σ-module N. Note that $\bigwedge^n N$ has a Λ_σ-basis $\{\epsilon_{k_1} \wedge \cdots \wedge \epsilon_{k_n} \mid 1 \leq k_1 < \cdots < k_n \leq m\}$.

Let $\sigma \supseteq_m \rho$. By Corollary 3.10(ii), the mth component of $\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet, \Delta(\sigma))$ is the last nonzero component and it can be identified with $\mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma))$. Thus, every element of the mth component is a cocycle, so there is a surjective map

$$[-] : \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)) \twoheadrightarrow \mathcal{E}^m_\theta(\rho, \sigma) = H^m(\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet, \Delta(\sigma))),\ \varphi \mapsto [\varphi],$$

where $[\varphi]$ is the cohomology class of φ. Moreover, by Lemma 3.9, we have an isomorphism

$$\xi : q^{-m}\Lambda_\sigma \xrightarrow{\sim} \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)),\ f \mapsto (e_\sigma \mapsto v_\sigma f).$$

We consider Λ_ρ to be a Λ_σ-module via the homomorphism $\mathtt{p}^\sigma_\rho : \Lambda_\sigma \twoheadrightarrow \Lambda_\rho$, see (3.3).

Theorem 3.12. *Let $\rho, \sigma \in \mathrm{KP}(\theta)$. If $\sigma \not\supseteq \rho$, then $\mathcal{E}_\theta(\rho, \sigma) = 0$. If $\sigma \supseteq_m \rho$, then $\mathcal{E}_\theta(\rho, \sigma) = \mathcal{E}^m_\theta(\rho, \sigma)$ and there is an isomorphism of Λ_σ-modules $\mathcal{E}^m_\theta(\rho, \sigma) \xrightarrow{\sim} q^{-m}\Lambda_\rho$ which makes the following diagram of Λ_σ-modules commute:*

$$\begin{array}{ccc} q^{-m}\Lambda_\sigma & \xrightarrow[\xi]{\sim} & \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)) \\ \big\downarrow \mathtt{p}^\sigma_\rho & & \big\downarrow [-] \\ q^{-m}\Lambda_\rho & \overset{\sim}{\dashleftarrow} & \mathcal{E}^m_\theta(\rho, \sigma). \end{array}$$

Proof. If $\sigma \not\supseteq \rho$, then $\mathcal{E}_\theta(\rho, \sigma) = 0$ by Lemma 2.30 and Corollary 3.10(i).

Assume now that $\sigma \supseteq_m \rho$ and write $C_\sigma \setminus C_\rho = \{i_1 < \cdots < i_m\}$. If $\sigma \supseteq \pi \supseteq_n \rho$, we set $B(\pi) := \{k \in [1, m] \mid i_k \in C_\pi\}$. Note that $|B(\pi)| = n$. We define a map $\Theta_n : T^\rho_n(\sigma) \to q^m \bigwedge^n N$ of Λ_σ-modules by defining it on the Λ_σ-basis $\{\psi_{w(\pi,\sigma)} v_\sigma \mid \sigma \supseteq \pi \supseteq_n \rho\}$ of $T^\rho_n(\sigma)$, see Corollary 3.10. If $B(\pi) = \{k_1 < \cdots < k_n\}$, then define

$$\Theta_n(\psi_{w(\pi,\sigma)} v_\sigma) := \epsilon_{k_1} \wedge \cdots \wedge \epsilon_{k_n}.$$

It is easy to see that Θ_n is an isomorphism of (graded) Λ_σ-modules. To show that Θ_n defines an isomorphism of complexes $T^\rho_\bullet(\sigma) \to q^m \bigwedge^\bullet N$, we must verify that the following square commutes:

$$\begin{array}{ccc} T^\rho_{n+1}(\sigma) & \xleftarrow{d_n \cdot -} & T^\rho_n(\sigma) \\ \big\downarrow \Theta_{n+1} & & \big\downarrow \Theta_n \\ q^m \bigwedge^{n+1} N & \xleftarrow{Z \wedge -} & q^m \bigwedge^n N, \end{array}$$

where d_n is the matrix defined in (3.6). We check this using an arbitrary basis element $\psi_{w(\pi,\sigma)} v_\sigma \in T^\rho_n(\sigma)$. We have

$$\begin{aligned} \Theta_{n+1}(d_n \psi_{w(\pi,\sigma)} v_\sigma) &= \sum_{\sigma \supseteq \tau \supseteq_1 \pi} s(\tau, \pi) \Theta_{n+1}(\psi_{w(\tau,\pi)} \psi_{w(\pi,\sigma)} v_\sigma) \\ &= \sum_{k \in [1,m] \setminus B(\pi)} s(\mathrm{ref}^{i_k}(\pi), \pi) \Theta_{n+1}(\psi_{w(\mathrm{ref}^{i_k}(\pi),\pi)} \psi_{w(\pi,\sigma)} v_\sigma) \\ &= \sum_{k \in [1,m] \setminus B(\pi)} (-1)^{B(\pi) \cap [1,k)} \Theta_{n+1}(\psi_{w(\mathrm{ref}^{i_k}(\pi),\sigma)} v_\sigma) z_k \end{aligned}$$

$$= \sum_{k\in[1,m]\setminus B(\pi)} (-1)^{B(\pi)\cap[1,k)}(\epsilon_{k_1}\wedge\cdots\wedge\epsilon_k\wedge\cdots\wedge\epsilon_{k_n})z_k$$
$$= Z\wedge(\epsilon_{k_1}\wedge\cdots\wedge\epsilon_{k_n})$$
$$= Z\wedge\Theta_n(\psi_{w(\pi,\sigma)}v_\sigma)$$

where the second equality follows by noting that $k\mapsto \mathrm{ref}^{i_k}(\pi)$ defines a bijection from $[1,m]\setminus B(\pi)$ to the set of 1-refinements of π which are refined by σ, the third by Lemma 3.11 and the observation that $s(\mathrm{ref}^{i_k}(\pi),\pi) = (-1)^{B(\pi)\cap[1,k)}s(\mathrm{ref}^{i_k}(\rho),\rho)$, and the remaining equalities follow from the definitions.

Since $q^m\bigwedge^\bullet N$ is a Koszul complex corresponding to a regular sequence, we now have that $\mathcal{E}^n_\theta(\rho,\sigma)\cong H^n(q^m\bigwedge^\bullet N)=0$ unless $n=m$. The proof is complete in view of Lemma 2.30 upon noting that the kernel of $\mathtt{p}^\sigma_\rho:\Lambda_\sigma\twoheadrightarrow\Lambda_\rho$ is the ideal generated by $(z_1,\dots,z_m)$, so $\mathtt{p}^\sigma_\rho$ induces an isomorphism

$$H^m(q^m\textstyle\bigwedge^\bullet N) = \dfrac{q^{-m}\Lambda_\sigma}{(z_1,\dots,z_m)} \xrightarrow{\sim} q^{-m}\Lambda_\rho. \qquad\square$$

3.3. **The category $\mathcal{E}_\theta$.** Throughout this subsection, we use Theorem 3.12 to identify $\mathcal{E}_\theta(\rho,\sigma)=\mathcal{E}^m_\theta(\rho,\sigma)$ with $q^{-m}\Lambda_\rho$ whenever $\sigma\supseteq_m\rho\in\mathrm{KP}(\theta)$.

Let $\sigma\supseteq_m\rho\in\mathrm{KP}(\theta)$. For any $\hat f\in q^{-m}\mathcal{Y}_l$ and $\pi\supseteq_k\sigma$, set

$$\hat f^\pi_{\rho,\sigma} := (-1)^{\frac{m(m+1)}{2}+mk}w(\pi,\sigma)\cdot\hat f.$$

We define an element of $\mathrm{Hom}^m_{R_\theta}(S^\rho_\bullet,S^\sigma_\bullet)$ by

$$\varphi^{\hat f}_{\rho,\sigma}: S^\rho_{m+k} = \bigoplus_{\pi\supseteq_{m+k}\rho} q^{m+k}R_\theta e_\pi \to \bigoplus_{\tau\supseteq_k\sigma} q^kR_\theta e_\tau = S^\sigma_k$$
$$(h_\pi e_\pi)_{\pi\supseteq_{m+k}\rho}\mapsto (h_\tau\hat f^\tau_{\rho,\sigma}e_\tau)_{\tau\supseteq_k\sigma}. \tag{3.13}$$

Recalling the differential (2.31) on $\mathrm{Hom}^\bullet_{R_\theta}(S^\rho_\bullet,S^\sigma_\bullet)$, we have:

Lemma 3.14. *Let $\sigma\supseteq_m\rho\in\mathrm{KP}(\theta)$. If $f\in q^{-m}\Lambda_\rho=\mathcal{E}^m_\theta(\rho,\sigma)$ and $\hat f\in q^{-m}\mathcal{Y}_l$ are such that $\mathtt{p}_\rho(\hat f)=f$, then*

(i) *$\delta(\varphi^{\hat f}_{\rho,\sigma})=0$, and*

(ii) *the isomorphism (2.32) sends the cohomology class of $\varphi^{\hat f}_{\rho,\sigma}$ to f.*

Proof. We prove (i) by checking that the following diagram either commutes (if m is even) or anticommutes (if m is odd) whenever $\tau\supseteq_1\pi\supseteq_k\sigma$:

$$\begin{array}{ccc} q^{k+m+1}R_\theta e_\tau & \xrightarrow{-\cdot d^{\tau,\pi}_{k+m}} & q^{k+m}R_\theta e_\pi \\ {\scriptstyle -\cdot\hat f^\tau_{\rho,\sigma}}\downarrow & & \downarrow{\scriptstyle -\cdot\hat f^\pi_{\rho,\sigma}} \\ q^{k+1}R_\theta e_\tau & \xrightarrow{-\cdot d^{\tau,\pi}_{k}} & q^kR_\theta e_\pi. \end{array}$$

Modulo the signs, this is checked by the computation:

$$\psi_{w(\tau,\pi)}(w(\pi,\sigma)\cdot\hat f)e_\pi = (w(\tau,\pi)w(\pi,\sigma)\cdot\hat f)\psi_{w(\tau,\pi)}e_\pi = (w(\tau,\sigma)\cdot\hat f)\psi_{w(\tau,\pi)}e_\pi,$$

and the signs are taken care of by

$$(-1)^{\frac{m(m+1)}{2}+mk}s(\tau,\pi) = (-1)^m(-1)^{\frac{m(m+1)}{2}+m(k+1)}s(\tau,\pi).$$

To prove (ii), first note that the restriction of $\varphi^{\hat{f}}_{\rho,\sigma}$ to S^ρ_m has image in S^σ_0 and can therefore be realized as

$$\varphi^{\hat{f}}_{\rho,\sigma}|_{S^\rho_m} : S^\rho_m = \bigoplus_{\pi \supseteq_m \rho} q^m R_\theta e_\pi \to R_\theta e_\sigma = S^\sigma_0$$
$$(h_\pi e_\pi)_{\pi \supseteq_m \rho} \mapsto h_\sigma \hat{f}^\sigma_{\rho,\sigma} e_\sigma = (-1)^{\frac{m(m+1)}{2}} h_\sigma \hat{f} e_\sigma.$$

By Corollary 3.10(ii), we identify $\mathrm{Hom}_{R_\theta}(S^\rho_m, \Delta(\sigma))$ with $\mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma))$. Then the image of $\varphi^{\hat{f}}_{\rho,\sigma}$ under the map (2.33) is

$$(-1)^{\frac{m(m+1)}{2}} \varepsilon_\sigma(\varphi^{\hat{f}}_{\rho,\sigma}|_{S^\rho_m}) = (e_\sigma \mapsto v_\sigma f) \in \mathrm{Hom}_{R_\theta}(q^m R_\theta e_\sigma, \Delta(\sigma)).$$

An application of Theorem 3.12 completes the proof. □

Lemma 3.15. *Let $\tau \supseteq \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. If $f \in q^{-m}\Lambda_\rho$ and $\hat{f} \in q^{-m}\mathcal{Y}_l$ are such that $\mathtt{p}_\rho(\hat{f}) = f$, then we also have $\mathtt{p}_\rho(w(\tau,\sigma)\cdot \hat{f}) = f$.*

Proof. Since $\tau \supseteq \sigma \supseteq \rho$, we have that r and $w(\tau,\sigma)(r)$ are ρ-equivalent for any $r \in [1,l]$, so $\mathtt{p}_\rho(w(\tau,\sigma)\cdot y_r) = \mathtt{p}_\rho(y_r)$. This implies the result. □

Lemma 3.16. *Let $\tau \supseteq_n \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. If $f \in q^{-m}\Lambda_\rho$, and $g \in q^{-n}\Lambda_\sigma$, then there exist $\hat{f} \in q^{-m}\mathcal{Y}_l$, $\hat{g} \in q^{-n}\mathcal{Y}_l$, and $\widehat{\mathtt{p}^\sigma_\rho(g)f} \in q^{-(m+n)}\mathcal{Y}_l$ with $\mathtt{p}_\rho(\hat{f}) = f$, $\mathtt{p}_\sigma(\hat{g}) = g$, and $\mathtt{p}_\rho(\widehat{\mathtt{p}^\sigma_\rho(g)f}) = \mathtt{p}^\sigma_\rho(g)f$, such that*

$$\varphi^{\hat{g}}_{\sigma,\tau}\varphi^{\hat{f}}_{\rho,\sigma} = \varphi^{\widehat{\mathtt{p}^\sigma_\rho(g)f}}_{\rho,\tau}.$$

Proof. Choose any two lifts $\hat{f} \in q^{-m}\mathcal{Y}_l$ and $\hat{g} \in q^{-n}\mathcal{Y}_l$ of f and g, respectively. By Lemma 3.15, since $\tau \supseteq \sigma$, we have that $w(\tau,\sigma)\cdot\hat{f}$ is also a lift of f, and so $\widehat{\mathtt{p}^\sigma_\rho(g)f} := \hat{g}(w(\tau,\sigma)\cdot\hat{f}) \in q^{-(m+n)}\mathcal{Y}_l$ is a lift of $\mathtt{p}^\sigma_\rho(g)f$. By (3.13), it suffices to show that for any $\pi \supseteq \tau$ we have $(\widehat{\mathtt{p}^\sigma_\rho(g)f})^\pi_{\rho,\tau} = \hat{g}^\pi_{\sigma,\tau}\hat{f}^\pi_{\rho,\sigma}$. Modulo the signs, this holds by the following computation:

$$\begin{aligned} w(\pi,\tau)\cdot(\hat{g}(w(\tau,\sigma)\cdot\hat{f})) &= (w(\pi,\tau)\cdot\hat{g})(w(\pi,\tau)w(\tau,\sigma)\cdot\hat{f}) \\ &= (w(\pi,\tau)\cdot\hat{g})(w(\pi,\sigma)\cdot\hat{f}), \end{aligned}$$

and if $\pi \supseteq_k \tau$, the signs are taken care of by

$$(-1)^{\frac{(m+n)(m+n+1)}{2}+(m+n)k} = (-1)^{\frac{m(m+1)}{2}+m(k+n)}(-1)^{\frac{n(n+1)}{2}+nk}.$$ □

We combine Lemmas 3.16 and 3.14 to obtain the following theorem.

Theorem 3.17. *Let $\tau \supseteq_n \sigma \supseteq_m \rho \in \mathrm{KP}(\theta)$. The composition in the category $\mathcal{E}_\theta$ is given by*

$$\begin{array}{ccc} \mathcal{E}^n_\theta(\sigma,\tau)\otimes\mathcal{E}^m_\theta(\rho,\sigma) & \longrightarrow & \mathcal{E}^{m+n}_\theta(\rho,\tau) \\ \wr\Vert & & \wr\Vert \\ q^{-n}\Lambda_\sigma \otimes q^{-m}\Lambda_\rho & \longrightarrow & q^{-(m+n)}\Lambda_\rho \\ g\otimes f & \longmapsto & \mathtt{p}^\sigma_\rho(g)f. \end{array}$$

4. The $\mathtt{A}_2$ case

Throughout this section, we use a special notation

$$\alpha := \alpha_1,\ \beta := \alpha_2,\ \gamma := \alpha_1 + \alpha_2,$$

so that α, β, γ are now the positive roots of the root system of type $\mathtt{A}_2$. We fix

$$\theta := a\alpha + b\beta$$

with $a, b \in \mathbb{Z}_{\geq 0}$. There is a bijection

$$\sigma : [0, \min\{a,b\}] \xrightarrow{\sim} \mathrm{KP}(\theta),\ s \mapsto \sigma(s) := (\beta^{b-s}, \gamma^s, \alpha^{a-s}).$$

The standard R_θ-modules are

$$\Delta(s) := \Delta(\sigma(s)) \qquad (0 \leq s \leq \min\{a,b\}).$$

We denote the standard generator of $\Delta(s)$ by $v_s := v_{\sigma(s)}$, see (2.28). Recall that

$$\mathrm{End}_{R_\theta}(\Delta(s))^{\mathrm{op}} \cong \Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} = \Lambda_{b-s} \otimes \Lambda_s \otimes \Lambda_{a-s},$$

see (2.27).

4.1. **The resolution $P^r_\bullet$.** Let $0 \leq r \leq \min\{a,b\}$. We recall the resolution of $\Delta(r)$ defined in [2]. For $n \in [0,r]$, recalling (2.2), we define:

$$\begin{aligned}
\boldsymbol{j}_{r,n} &:= 2^{b-r}1^{r-n}2^r1^{n+a-r} \in I^\theta,\\
\boldsymbol{i}_{r,n} &:= 2^{(b-r)}1^{(r-n)}2^{(r)}1^{(n)}1^{(a-r)} \in I^\theta_{\mathrm{div}},\\
e_{r,n} &:= 1_{\boldsymbol{i}_{r,n}} \in R_\theta,\\
\mathtt{s}_{r,n} &:= n(r-n+1) - \binom{a-r}{2} - \binom{b-r}{2} - r(r-1),\\
x_{r,n} &:= U_{b-n-1;1,r+n} = (b+r, b+r-1, \dots, b-n) \in \mathfrak{S}_{a+b},\\
d_{r,n} &:= e_{r,n+1}\psi_{x_{r,n}}e_{r,n} \in R_\theta,\\
P^r_n &:= q^{\mathtt{s}_{r,n}} R_\theta e_{r,n}.
\end{aligned}$$

Note that the right multiplication with $d_{r,n}$ yields the degree zero R_θ-homomorphism $- \cdot d_{r,n} : P^r_{n+1} \to P^r_n$. Define $u_r \in \mathfrak{S}_{2r}$ by

$$u_r(i) = \begin{cases} r - \frac{i-1}{2} & \text{if } i \text{ is odd},\\ 2r - \frac{i-2}{2} & \text{if } i \text{ is even}. \end{cases} \tag{4.1}$$

Let

$$\varepsilon_r : P^r_0 \twoheadrightarrow \Delta(r),\ xe_{r,0} \mapsto x(\psi_{w_{0,b-r}} \circ \psi_{u_r} \circ \psi_{w_{0,a-r}})v_r. \tag{4.2}$$

By [2, Theorem A] and Lemma 2.23, we have:

Lemma 4.3. *The following sequence is a projective resolution of $\Delta(r)$:*

$$0 \longrightarrow P^r_r \longrightarrow \cdots \longrightarrow P^r_{n+1} \xrightarrow{-\cdot d_{r,n}} P^r_n \longrightarrow \cdots \longrightarrow P^r_0 \xrightarrow{\varepsilon_r} \Delta(r) \longrightarrow 0.$$

4.2. **Weight spaces of standard modules.** The following lemmas are useful for finding bases for certain weight spaces of the standard modules $\Delta(s)$. The first of them concerns the nil-Hecke algebra and is well-known and easy to check. Recall the notation from §2.2 and the $(R_{c\alpha_i}, \mathcal{NH}_c)$-bimodule structure on $\hat{\Delta}(\alpha_i^c)$ from §2.6.

Lemma 4.4. *Let $i \in I$ and $c \in \mathbb{Z}_{\geq 0}$. The map*

$$q^{-\binom{c}{2}}\mathcal{X}_c \to \hat{\Delta}(\alpha_i^c), \ f \mapsto v_{\alpha_i}^{\circ c} f \tau_{w_0}$$

is an injective map of right Λ_c-modules with image $\Delta(\alpha_i^c)$.

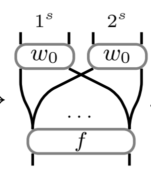
Recalling (4.1), we have:

Lemma 4.5. *The map*

$$q^{-2\binom{s}{2}}\Lambda_s \to \hat{\Delta}(\gamma^s), \ f \mapsto \psi_{u_s} v_\gamma^{\circ s} f$$

is an injective map of right Λ_s-modules with image $1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$.

Diagrammatically, the map in the lemma is given by

$f \mapsto$ 

Proof. In view of Lemma 2.17, we may assume that $\Bbbk$ is a field. Let M be the free graded $\Bbbk$-module with basis $\mathscr{D}^{(2^s)}$, where the degree of the basis element $w \in \mathscr{D}^{(2^s)}$ is set to equal to the degree of $\psi_w 1_{(12)^s}$ in $R_{s\alpha+s\beta}$. Lemma 2.24 and (2.19) show that the map $\xi : q^{\binom{s}{2}} M \otimes_\Bbbk \mathcal{X}_s \to \hat{\Delta}(\gamma^s), \ w \otimes f \mapsto \psi_w v_\gamma^{\circ s} f$ is an isomorphism of right $\mathcal{X}_s$-modules.

Let φ be the map in the statement. Then φ is the composition of the inclusion $q^{-2\binom{s}{2}}\Lambda_s \hookrightarrow q^{\binom{s}{2}} M \otimes_\Bbbk \mathcal{X}_s, \ f \mapsto u_s \otimes f$ with ξ, so φ is injective. It is now enough to show that $\varphi(f) \in 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$ and that $\dim_q(1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)) = q^{-2\binom{s}{2}} \dim_q(\Lambda_s)$. For the first claim, note that $\psi_{u_s} v_\gamma^{\circ s}$ is a nonzero element of smallest possible degree in $1_{1^s 2^s}\hat{\Delta}(\gamma^s)$, so recalling (2.7), we have

$$\begin{aligned}
1_{1^{(s)}2^{(s)}}\psi_{u_s} v_\gamma^{\circ s} f e_s &= \psi_{u_s} v_\gamma^{\circ s} f x_0 \tau_{w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} f x_0 \tau_1 \tau_{s_1^{-1} w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} \tau_1 f x_0 \tau_{s_1^{-1} w_0} + \psi_{u_s} v_\gamma^{\circ s} \partial_1(f x_0) \tau_{s_1^{-1} w_0} \\
&= \psi_{u_s} v_\gamma^{\circ s} \partial_1(f x_0) \tau_{s_1^{-1} w_0} \\
&\ \ \vdots \\
&= \psi_{u_s} v_\gamma^{\circ s} \partial_{w_0}(f x_0) \\
&= \psi_{u_s} v_\gamma^{\circ s} f,
\end{aligned}$$

where the first equality comes from Lemma 2.16 and (2.7), the third comes from the relations in $\mathcal{NH}_s$, the fourth follows because $\deg(\psi_{u_s} v_\gamma^{\circ s} \tau_1) = \deg(\psi_{u_s} v_\gamma^{\circ s}) - 2 < \deg(\psi_{u_s} v_\gamma^{\circ s})$, and the last holds by Lemma 2.4.

As for graded dimension, we have

$$\begin{aligned}\dim_q(1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)) &= \frac{1}{\left([s]^!_+\right)^3}\dim_q(1_{1^s2^s}\hat{\Delta}(\gamma^s))\\ &= \frac{q^{2\binom{s}{2}}\left([s]^!_-\right)^2}{\left([s]^!_+\right)^3}(\dim_q \mathcal{X}_1)^s\\ &= q^{-2\binom{s}{2}}\dim_q \Lambda_s.\end{aligned}$$

where the first equality follows from (2.29) and (2.18), the second from (2.25), (2.20), and Lemma 2.24, and the last from an elementary computation. □

4.3. **The $\Bbbk$-module $\mathcal{E}_\theta(r,s)$.** In this subsection we fix $r, s \in [0, \min\{a,b\}]$. Recalling Lemma 2.30, we write $T^r_\bullet(s) := T^{P^r}_\bullet(\Delta(s))$. The terms of this complex are of the form

$$T^r_n(s) = q^{-\mathtt{s}_{r,n}} e_{r,n}\Delta(s) \qquad (n = 0,\dots,r).$$

If $r \geq s$ and $0 \leq n \leq r-s$, we define

$$\begin{aligned}\omega_n(r,s) &:= -(r-s)(1+(a-r)+(b-r)) + (r-s-n+1)(r-s-n),\\ K_n &:= q^{\omega_n(r,s)}\Lambda_{b-r,r-s,s,r-s-n,n,a-r},\\ w_n &:= U_{b-r;s,r-s-n}U_{b-n;r-s,s}U_{b-r+s;r-s,r-s-n}U_{b-r;r-s,s}U_{b;s,r-s-n}.\end{aligned}$$

The diagram for $1_{\boldsymbol{j}_{r,n}}\psi_{w_n}1_{\boldsymbol{j}_{s,0}}$ is the top part of the diagram below. Observe that $\Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} \subseteq K_n$ in a natural way, so we may consider K_n as a $\Lambda_{\sigma(s)}$-module. Recalling (4.1), we have

Lemma 4.6. *Suppose* $0 \leq n \leq r$.

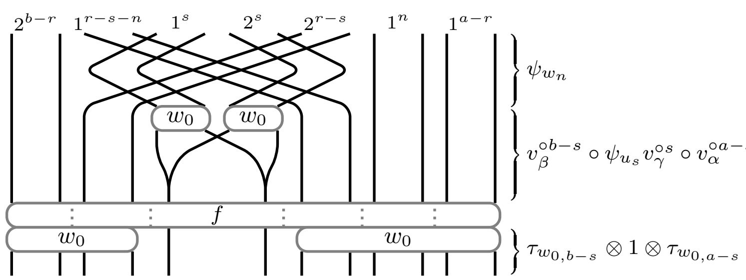
(i) *If* $n > r-s$, *then* $T^r_n(s) = 0$.

(ii) *If* $n \leq r-s$, *then the map*

$$\begin{aligned}\Xi_n : K_n &\to q^{-\mathtt{s}_{r,n}}\hat{\Delta}(s),\\ f &\mapsto \psi_{w_n}(v^{\circ b-s}_\beta \circ \psi_{u_s}v^{\circ s}_\gamma \circ v^{\circ a-s}_\alpha)\, f\,(\tau_{w_{0,b-s}}\otimes 1\otimes \tau_{w_{0,a-s}})\end{aligned}$$

is an injective degree zero map of $\Lambda_{\sigma(s)}$*-modules with image* $T^r_n(s)$.

Diagrammatically, Ξ_n is given by

$f \mapsto$ 

Proof. (i) The condition $n > r-s$ is equivalent to the condition $a-r+n > a-s$, which easily implies, by (2.20), that $1_{\boldsymbol{j}_{r,n}}\Delta(s) = 0$ hence $e_{r,n}\Delta(s) = 0$.

(ii) It is straightforward to check that Ξ_n is homogeneous and $\Lambda_{\sigma(s)}$-equivariant. Define

$$\xi_\alpha : q^{-\binom{a-s}{2}}\Lambda_{r-s-n,n,a-r} \hookrightarrow q^{-\binom{a-s}{2}}\mathcal{X}_{a-s} \xrightarrow{\sim} \Delta(\alpha^{a-s})$$

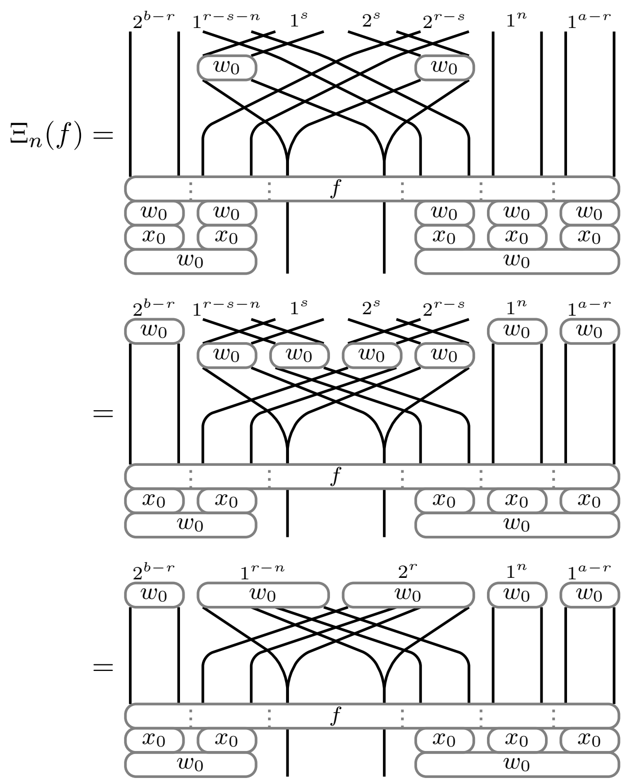
$$\xi_\beta : q^{-\binom{b-s}{2}}\Lambda_{b-r,r-s} \hookrightarrow q^{-\binom{b-s}{2}}\mathcal{X}_{b-s} \xrightarrow{\sim} \Delta(\beta^{b-s})$$
$$\xi_\gamma : q^{-2\binom{s}{2}}\Lambda_s \xrightarrow{\sim} 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)$$

where the isomorphisms are from Lemmas 4.4 and 4.5. Let $d = (r-s)(r-s-n) - s(r-s-n) - s(r-s)$ and define

$$\xi : \Delta(\beta^{b-s}) \otimes 1_{1^{(s)}2^{(s)}}\Delta(\gamma^s) \otimes \Delta(\alpha^{a-s}) \to q^{-d}1_{\boldsymbol{j}_{r,n}}\Delta(s),\ x \otimes y \otimes z \mapsto \psi_{w_n}(x \circ y \circ z).$$

Since $w_n \in \mathscr{D}^{(b-s,2s,a-s)}$, (2.19) shows that ξ is injective. Observing that $\Xi_n = \xi(\xi_\beta \otimes \xi_\gamma \otimes \xi_\alpha)$, we see that Ξ_n is injective with image in $1_{\boldsymbol{j}_{r,n}}\Delta(s)$.

Now we prove that $\operatorname{im}\Xi_n \subseteq e_{r,n}\Delta(s)$ by showing that $e_{r,n}\Xi_n(f) = \Xi_n(f)$. We have, using the relations in R_θ,



where the first equality follows from Lemma 2.16, the second follows because f is symmetric in the variables as indicated by the vertical dotted lines, hence, by Theorem 2.5, commutes with the parabolic subalgebra $\mathcal{NH}_{b-r} \otimes \mathcal{NH}_{r-s} \otimes \mathcal{NH}_s \otimes \mathcal{NH}_{r-s-n} \otimes \mathcal{NH}_n \otimes \mathcal{NH}_{a-r}$, and the last equality is straightforward. Now we have $e_{r,n}\Xi_n(f) = \Xi_n(f)$ by Lemma 2.16.

To complete the proof, in view of Lemma 2.17, we assume that $\Bbbk$ is a field and check that $\dim_q K_n = \dim_q q^{-\mathbf{s}_{r,n}}e_{r,n}\Delta(s)$. For brevity, we denote $[M] := \dim_q M$. Let $z = (r-s)s + (r-s-n)s + (r-s)(r-s-n)$. We have

$$\begin{aligned}\left[1_{\boldsymbol{j}_{r,n}}\Delta(s)\right] &= q^z \begin{bmatrix} r-n \\ s \end{bmatrix}_- \begin{bmatrix} r \\ s \end{bmatrix}_- \left[\Delta(\beta^{b-s})\right] \left[1_{1^s2^s}\Delta(\gamma^s)\right] \left[\Delta(\alpha^{a-s})\right] \\ &= q^z \begin{bmatrix} r-n \\ s \end{bmatrix}_- \begin{bmatrix} r \\ s \end{bmatrix}_- ([s]^!_+)^2 \left[\Delta(\alpha^{a-s})\right] \left[1_{1^{(s)}2^{(s)}}\Delta(\gamma^s)\right] \left[\Delta(\alpha^{a-s})\right]\end{aligned}$$

$$= q^{\omega_n(r,s)+\mathtt{s}_{r,n}} \frac{[r-n]^!_+[r]^!_+}{[r-n-s]^!_+[r-s]^!_+}[\mathcal{X}_{b-s}][\Lambda_s][\mathcal{X}_{a-s}],$$

where the first equality follows from (2.20), then second from (2.18), and the last from Lemmas 4.4 and 4.5. Thus,

$$\begin{aligned} q^{-\mathtt{s}_{r,n}}\,[e_{r,n}\Delta(s)] &= \frac{q^{-\mathtt{s}_{r,n}}}{[b-r]^!_+[r-n]^!_+[r]^!_+[n]^!_+[a-r]^!_+}\left[1_{\boldsymbol{j}_{r,n}}\Delta(s)\right] \\ &= q^{\omega_n(r,s)}\frac{[\mathcal{X}_{b-s}]}{[b-r]^!_+[r-s]^!_+}[\Lambda_s]\frac{[\mathcal{X}_{a-s}]}{[r-s-n]^!_+[n]^!_+[a-r]^!_+} \\ &= q^{\omega_n(r,s)}[\Lambda_{b-r}][\Lambda_{r-s}][\Lambda_s][\Lambda_{r-s-n}][\Lambda_n][\Lambda_{a-r}], \end{aligned}$$

where the first equality is (2.18), the second is by the above computation, and the last is by an elementary computation. □

Because of Lemma 4.6(i), we assume for the rest of the subsection that $r \geq s$. We use Lemma 4.6(ii) to understand the complex $T^r_\bullet(s)$ and compute its cohomology. First, we re-express the coboundary map of $T^r_\bullet(s)$. For $0 \leq n < r-s$, set

$$g_n := \prod_{k=1}^{r-s}(x_{b+r-s-n} - x_{b-r+k}) \in \mathcal{X}_{a+b-s}.$$

If $f \in K_n$, observe that $fg_n \in \Lambda_{b-r,r-s,s,r-s-n-1,1,n,a-r}$ so by Proposition 2.3, we have a map

$$\delta_n : K_n \to K_{n+1},\ f \mapsto \partial_{U_{b+r-s-n-1;1,n}}(fg_n).$$

Recalling the isomorphisms $\Xi_n : K_n \xrightarrow{\sim} T^r_n(s)$ from Lemma 4.6(ii), we have:

Lemma 4.7. *If $0 \leq n < r-s$, then the following diagram commutes:*

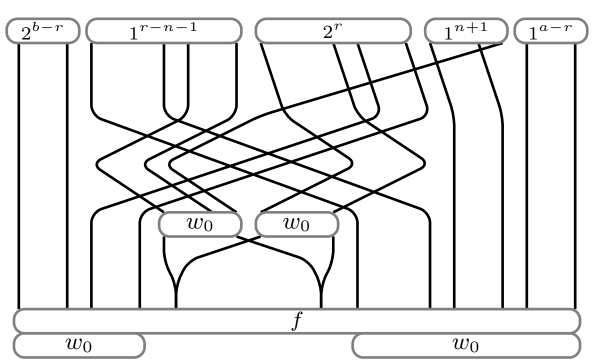
$$\begin{array}{ccc} K_{n+1} & \xleftarrow{\ \delta_n\ } & K_n \\ \big\downarrow{\scriptstyle \Xi_{n+1}} & & \big\downarrow{\scriptstyle \Xi_n} \\ T^r_{n+1}(s) & \xleftarrow{\ d_{r,n}\cdot -\ } & T^r_n(s). \end{array}$$

In particular, the maps δ_n make $K_\bullet$ into a complex isomorphic to $T^r_\bullet(s)$.

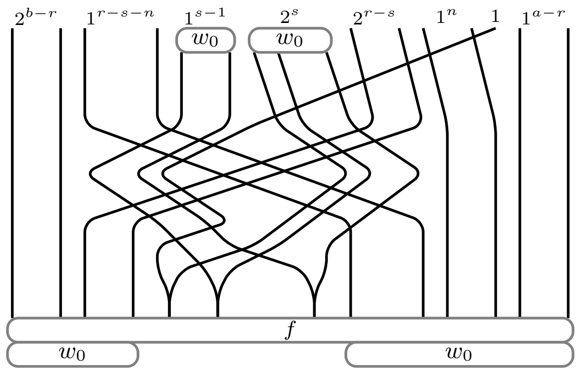
Proof. There exist polynomials $h_j \in \Bbbk[y_{b-r+1}, \dots, y_{b-s}]$ and $k_j \in \Bbbk[y_{b+r-n}]$ such that $\prod_{i=1}^{r-s}(y_{b+r-n} - y_{b-r+i}) = \sum_j h_j k_j$. Let $f \in K_n$ and note that

$$d_{r,n}\Xi_n(f) = e_{r,n+1}\psi_{x_{r,n}}e_{r,n}\Xi_n(f) = e_{r,n+1}\psi_{x_{r,n}}\Xi_n(f)$$

since $\operatorname{im}\Xi_n \subseteq T^r_n(s) = q^{-\mathtt{s}_{r,n}}e_{r,n}\Delta(s)$ by Lemma 4.6. We compute $e_{r,n+1}\psi_{x_{r,n}}\Xi_n(f)$:

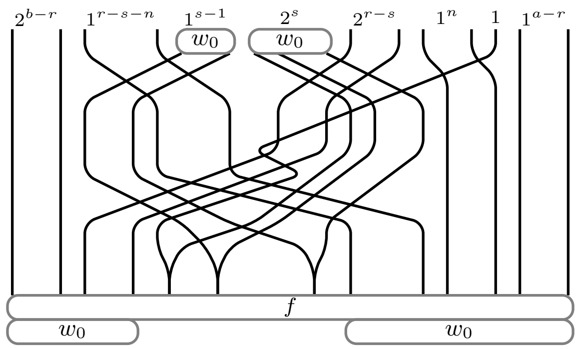


=



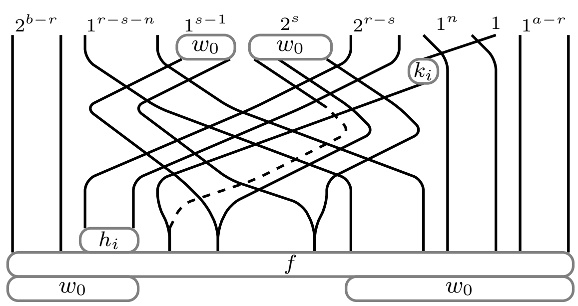
by the relation (R7) and Lemma 2.16

=



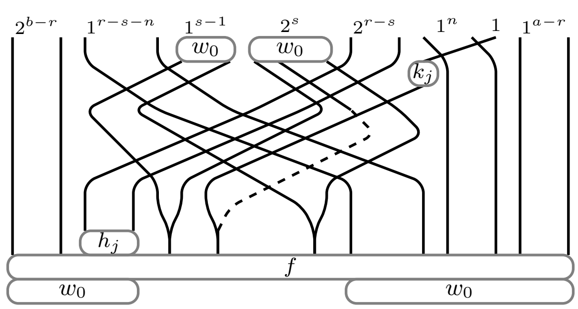
by the relation (R7)

$= \sum_j$

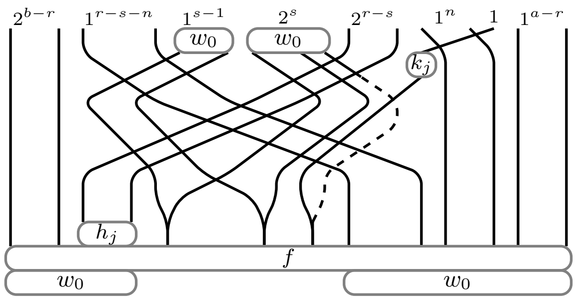


by several applications of the relation (R5)

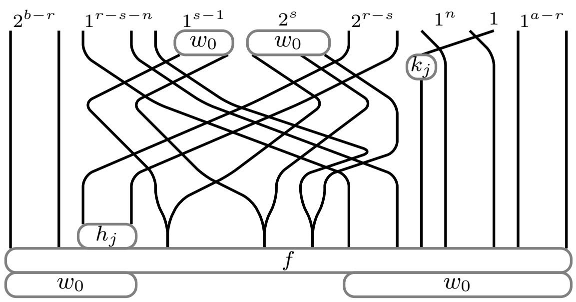
$= \sum_j$



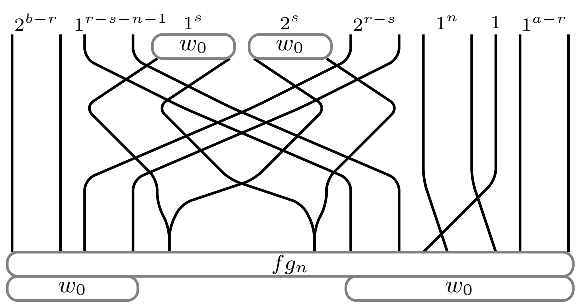
(see (∗) below)

$= \sum_j$ 

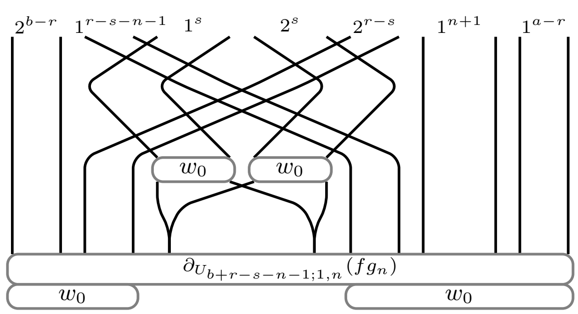
repeat the argument from the previous step several times

$= \sum_j$ 

(see (∗∗) below)

$=$ 

by the relation (R7)

$=$ 

by Lemma 2.6 and the relation (R7).

(∗) To obtain this equality, we attempt to move the portion of dashed strand in the previous diagram to the left by applying a special case of relation (R7):

$$\psi_{t+1}\psi_t\psi_{t+1}1_{\boldsymbol{i}} = \psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}} + 1_{\boldsymbol{i}} \text{ (if } i_{t+1} = i_t + 1 \text{ and } i_t = i_{t+2}) \tag{4.8}$$

several times. In all except the last application, the error term $1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0, so we only keep the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$. In the last application, because of the defining relations in $\Delta(s)$, the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0, so we only keep the error term $1_{\boldsymbol{i}}$, which yields the desired diagram.

(∗∗) To obtain this equality, we again attempt to move the portion of dashed strand in the previous diagram to the left by applying the relation (4.8) several times. In the first application, the error term $1_{\boldsymbol{i}}$ yields the desired diagram, so we wish to show that the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ causes the rest of the diagram to become 0. Since f is symmetric in the variables $x_{b+1}, \dots, x_{b+r-s-n}$, the error term in any application of the relation (4.8) other than the first causes the rest of the diagram to become 0. However, after the last application of the relation, the term $\psi_t\psi_{t+1}\psi_t 1_{\boldsymbol{i}}$ also causes the rest of the diagram to become 0 because of the defining relations in $\Delta(s)$.

The expression represented by the last diagram above is $\Xi_{n+1}(\delta_n(f))$, which completes the proof of the lemma. □

Given an interval $(c, d]$ and a polynomial in $d - c$ variables, we denote

$$f(\underline{x}_{(c,d]}) := f(x_{c+1}, \dots, x_d).$$

For example, if $0 \le m \le d - c$ then we have the mth elementary symmetric function

$$E_m(\underline{x}_{(c,d]}) = \sum_{c<i_1<\dots<i_m\le d} x_{i_1} \cdots x_{i_m}.$$

Now, for $0 \le k < r - s$, we define

$$z_k := (-1)^{r-s-k}(E_{r-s-k}(\underline{x}_{(b-r,b-s]}) - E_{r-s-k}(\underline{x}_{(b,b+r-s]})).$$

These elements are considered as elements of the algebra

$$\Lambda^{r,s} := \Lambda_{b-r,r-s,s,r-s,a-r}. \tag{4.9}$$

Note that since $\Lambda^{r,s} \subseteq \Lambda_{b-r,r-s,s,r-s-n,n,a-r}$, each K_n is naturally a (right) $\Lambda^{r,s}$-module. We use this to interpret the right-hand side of the lemma below as an element of K_{n+1}.

Lemma 4.10. *For $0 \le n < r - s$ and a symmetric polynomial f in n variables, we have*

$$\delta_n(f(\underline{x}_{(b+r-s-n,b+r-s]})) = \sum_{k=0}^{r-s-1} \left(x^k_{b+r-s-n} \star f(\underline{x}_{(b+r-s-n,b+r-s]})\right) z_k.$$

Proof. For brevity, write $f = f(\underline{x}_{(b+r-s-n,b+r-s]})$. We observe

$$g_n = \prod_{k=1}^{r-s}(x_{b+r-s-n} - x_{b-r+k}) = \sum_{k=0}^{r-s}(-1)^{r-s-k} x^k_{b+r-s-n} E_{r-s-k}(\underline{x}_{(b-r,b-s]}),$$

so that

$$\delta_n(f) = \sum_{k=0}^{r-s}(-1)^{r-s-k} \partial_{U_{b+r-s-n-1;1,n}}(x^k_{b+r-s-n} f) E_{r-s-k}(\underline{x}_{(b-r,b-s]}). \tag{4.11}$$

Next, using the identity

$$x^{r-s}_{b+r-s-n} = -\sum_{k=0}^{r-s-1} (-1)^{r-s-k} x^k_{b+r-s-n} E_{r-s-k}(\underline{x}_{(b,b+r-s]}),$$

(4.11) becomes

$$\delta_n(f) = \sum_{k=0}^{r-s-1} \partial_{U_{b+r-s-n-1;1,n}}(x^k_{b+r-s-n} f) z_k,$$

and the result follows from Proposition 2.3. □

The proof of Theorem 4.13 below amounts to showing that $T^{\rho}_{\bullet}(\sigma) \cong K_{\bullet}$ is isomorphic to a certain Koszul complex which we now define. Let N be the free right $\Lambda^{r,s}$-module of graded rank $\sum_{k=0}^{r-s-1} q^{2k-2(r-s)}$. For $k = 0, \dots, r-s-1$, we have the basis element $\epsilon_k := 1 \in q^{2k-2(r-s)}\Lambda^{r,s} \subseteq N$. We set

$$Z := (z_0, \dots, z_{r-s-1}) = \epsilon_0 z_0 + \cdots + \epsilon_{r-s-1} z_{r-s-1} \in N.$$

Note that Z is a homogeneous degree 0 element of N. We consider the Koszul complex $q^{\omega_0(r,s)}\bigwedge^{\bullet} N$ associated to the regular sequence Z for the algebra $\Lambda^{r,s}$ (see [17, §4.5]), which has the form

$$\begin{array}{c} \cdots \longleftarrow q^{\omega_0(r,s)}\bigwedge^{n+1} N \longleftarrow q^{\omega_0(r,s)}\bigwedge^{n} N \longleftarrow \cdots \\ Z \wedge a \longleftarrow\!\shortmid\, a \end{array}$$

where $\bigwedge^n N$ is the nth exterior power of the free $\Lambda^{r,s}$-module N. Note that $\bigwedge^n N$ has basis $\{\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_n} \mid 0 \le i_1 < \cdots < i_n < r-s\}$. Recall (1.2).

By Lemmas 2.30 and 4.6(i), the complex $\mathrm{Hom}^{\bullet}_{R_\theta}(P^r_{\bullet}, \Delta(s))$ is zero in degrees larger than $r-s$, so every element of the $r-s$ component is a cocycle and there is a surjective map

$$[-] : \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)) \twoheadrightarrow \mathcal{E}^{r-s}_{\theta}(r,s) = H^{r-s}(\mathrm{Hom}^{\bullet}_{R_\theta}(P^r_{\bullet}, \Delta(s))),\ \varphi \mapsto [\varphi],$$

where $[\varphi]$ is the cohomology class of φ. Moreover, by Lemma 4.6(ii), we have an isomorphism

$$\xi : K_{r-s} \xrightarrow{\sim} \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)),\ f \mapsto (e_{r-s,s} \mapsto v_s f).$$

For $r, s \in \mathbb{Z}_{\ge 0}$ with $\min\{a,b\} \ge r \ge s$, we define

$$\Lambda(r,s) := q^{\omega_{r-s}(r,s)} \Lambda_{b-r,r-s,s,a-r}.$$

Note that there is a surjection

$$\mathsf{p}_{r,s} : q^{\omega_{r-s}(r,s)}\Lambda^{r,s} \twoheadrightarrow \Lambda(r,s),\ f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 \mapsto f_1 \otimes f_2 f_4 \otimes f_3 \otimes f_5, \quad (4.12)$$

obtained by identifying the two Λ_{r-s} components. Since $\Lambda_{\sigma(s)} = \Lambda_{b-s,s,a-s} \subseteq \Lambda^{r,s}$, we consider $\Lambda^{r,s}$ to be a right $\Lambda_{\sigma(s)}$-module, and we consider $\Lambda(r,s)$ to be a right $\Lambda_{\sigma(s)}$-module via the composition of algebra homomorphisms

$$\Lambda_{\sigma(s)} \hookrightarrow \Lambda^{r,s} \overset{\mathsf{p}_{r,s}}{\twoheadrightarrow} \Lambda_{b-r,r-s,s,a-r} = q^{-\omega_{r-s}(r,s)}\Lambda(r,s)$$

and then degree shift. Note that $\mathsf{p}_{r,s}$ is a $\Lambda_{\sigma(s)}$-homomorphism.

Theorem 4.13. *Let $0 \le r, s \le \min\{a,b\}$. If $r < s$, then $\mathcal{E}_\theta(r,s) = 0$. If $r \ge s$, then $\mathcal{E}_\theta(r,s) = \mathcal{E}^{r-s}_\theta(r,s)$ and there is an isomorphism of right $\Lambda_{\sigma(s)}$-modules $\mathcal{E}^{r-s}_\theta(r,s) \xrightarrow{\sim} \Lambda(r,s)$ such that the following diagram of right $\Lambda_{\sigma(s)}$-modules commutes:*

$$\begin{array}{ccc} q^{\omega_{r,s}(r-s)}\Lambda^{r,s} & \xrightarrow[\xi]{\sim} & \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)) \\ \big\downarrow{\scriptstyle \mathsf{p}_{r,s}} & & \big\downarrow{\scriptstyle [-]} \\ \Lambda(r,s) & \overset{\sim}{\dashleftarrow} & \mathcal{E}^{r-s}_\theta(r,s). \end{array}$$

Proof. If $r < s$, then $\mathcal{E}_\theta(r,s) = 0$ by Lemmas 2.30 and 4.6(i), so assume $r \geq s$. For $0 \leq n \leq r-s$ and $\lambda \in \mathscr{P}(n, r-s-n)$, let $s_\lambda := s_\lambda(\underline{x}_{(b+r-s-n,b+r-s]}) \in K_n$. By Proposition 2.1, $\{s_\lambda \mid \lambda \in \mathscr{P}(n, r-s-n)\}$ is a basis of K_n as an $\Lambda^{r,s}$-module, so there exists an isomorphism $\Theta_n : K_n \to q^{\omega_0(r,s)}\bigwedge^n N$ of $\Lambda^{r,s}$-modules such that

$$\Theta_n(s_\lambda) = \epsilon_{\lambda_n} \wedge \epsilon_{\lambda_{n-1}+1} \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}.$$

We claim that the maps Θ_n define an isomorphism of complexes between $K_\bullet$ and $q^{\omega_0(r,s)}\bigwedge^\bullet N$. We must verify that the following square commutes:

$$\begin{array}{ccc} K_{n+1} & \xleftarrow{\ \delta_n\ } & K_n \\ \downarrow{\scriptstyle \Theta_{n+1}} & & \downarrow{\scriptstyle \Theta_n} \\ q^{\omega_0(r,s)}\bigwedge^{n+1} N & \xleftarrow{\ Z\wedge -\ } & q^{\omega_0(r,s)}\bigwedge^n N. \end{array}$$

Fix some $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathscr{P}(n, r-s-n)$ and set $X := \{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1 + n - 1\}$. We then have by Lemma 4.10 and (1.2):

$$\begin{aligned} \Theta_{n+1}(\delta_n(s_\lambda)) &= \sum_{k \in [0,r-s)\setminus X} (-1)^{|X \cap [0,k)|}(\epsilon_{\lambda_n} \wedge \cdots \wedge \epsilon_k \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}) z_k \\ &= \sum_{k \in [0,r-s)\setminus X} (\epsilon_k \wedge \epsilon_{\lambda_n} \wedge \cdots \wedge \epsilon_{\lambda_1+n-1}) z_k \\ &= Z \wedge \Theta_n(s_\lambda). \end{aligned}$$

Since Z is a regular sequence, we have $\mathcal{E}^n_\theta(r,s) \cong H^n(q^{\omega_0(r,s)}\bigwedge^\bullet N) = 0$ unless $n = r-s$. We complete the proof using Lemmas 2.30 and 4.7 and the observation that by the fundamental theorem of elementary symmetric polynomials, the kernel of $\mathtt{p}_{r,s} : q^{\omega_{r-s}(r,s)}\Lambda^{r,s} \twoheadrightarrow \Lambda(r,s)$ is the ideal generated by $(z_0, \dots, z_{r-s-1})$, so $\mathtt{p}_{r,s}$ induces an isomorphism of $\Lambda^{r,s}$-modules (and therefore of $\Lambda_{\sigma(s)}$-modules)

$$H^{r-s}(q^{\omega_0(r,s)}\bigwedge^\bullet N) = q^{\omega_{r-s}(r,s)}\Lambda^{r,s}/(z_0, \dots, z_{r-s-1}) \xrightarrow{\sim} \Lambda(r,s). \qquad \square$$

4.4. **The category $\mathcal{E}_\theta$.** Throughout this subsection, we use Theorem 4.13 to identify $\mathcal{E}^{r-s}_\theta(r,s)$ with $\Lambda(r,s) = q^{\omega_{r-s}(r,s)}\Lambda_{b-r,r-s,s,s,a-r}$ whenever $\min\{a,b\} \geq r \geq s \geq 0$.

In this subsection, we will need to consider not only partially symmetric polynomials in the variables x but also partially symmetric polynomials in the variables y. This will be important since elements of $\mathcal{Y}_{a+b}$ will be considered as elements of R_θ, cf. (2.11). For any d we have an isomorphism

$$\iota_{y\to x} : \mathcal{Y}_d \xrightarrow{\sim} \mathcal{X}_d,\ y_r \mapsto x_r.$$

We will use the notation $\Lambda^{\mathcal{Y}}_m := \mathcal{Y}_m^{\mathfrak{S}_m}$ for the symmetric polynomials in $y_1, \dots, y_m$. More generally, given a composition $\mu = (\mu_1, \dots, \mu_k)$ of d, we have the algebra of μ-partially symmetric polynomials $\Lambda^{\mathcal{Y}}_\mu := \mathcal{Y}_d^{\mathfrak{S}_\mu}$. We often write $\Lambda^{\mathcal{Y}}_{\mu_1,\dots,\mu_k}$ for $\Lambda^{\mathcal{Y}}_\mu$ and identify it with $\Lambda^{\mathcal{Y}}_{\mu_1} \otimes \cdots \otimes \Lambda^{\mathcal{Y}}_{\mu_k}$. The isomorphism $\iota_{y\to x}$ restricts to the isomorphism $\iota_{y\to x} : \Lambda^{\mathcal{Y}}_\mu \xrightarrow{\sim} \Lambda_\mu$.

For integers r, s, t with $\min\{a,b\} \geq r \geq s \geq t \geq 0$, define the following:

$$\begin{aligned} \hat{\Lambda}(r,s) &:= q^{\omega_{r-s}(r,s)}\Lambda^{\mathcal{Y}}_{b-r,r-s,s,s,r-s,a-r} \subseteq \mathcal{Y}_{a+b}, \\ \hat{\Lambda}(r,s,t) &:= q^{\omega_{r-t}(r,t)-4(r-s)(s-t)}\Lambda^{\mathcal{Y}}_{b-r,r-s,s-t,t,t,s-t,r-s,a-r} \subseteq \mathcal{Y}_{a+b}. \end{aligned}$$

Recalling (4.9), there is a surjection

$$\hat{\mathsf{p}}_{r,s} : \hat{\Lambda}(r,s) \twoheadrightarrow q^{\omega_{r-s}(r,s)}\Lambda^{r,s},\ f_1\otimes f_2\otimes f_3\otimes f_4\otimes f_5\otimes f_6 \mapsto \iota_{y\to x}\big(f_1\otimes f_2\otimes f_3 f_4\otimes f_5\otimes f_6\big).$$

Recalling (4.12), let

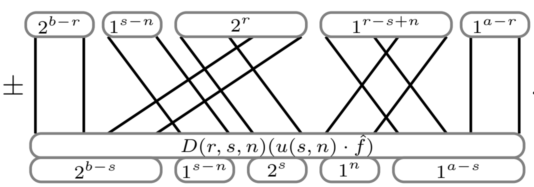
$$\mathsf{q}_{r,s} := \mathsf{p}_{r,s}\hat{\mathsf{p}}_{r,s} : \hat{\Lambda}(r,s) \to \Lambda(r,s).$$

Let again $\min\{a,b\} \geq r \geq s \geq t \geq 0$, and $0 \leq n \leq r-s$. Set

$$\begin{aligned} D(r,s,n) &:= \prod_{\substack{i\in(b+s-n,b+s]\\ j\in(b-r,b-s]}} (y_i - y_j) \in \mathcal{Y}_{a+b},\\ u(s,n) &:= U_{b-n;n,s} \in \mathfrak{S}_{a+b},\\ v(r,s,n) &:= U_{b-r;r-s,2s-n}U_{b+s-n;n,r-s} \in \mathfrak{S}_{a+b},\\ w(s,t) &:= U_{b-s;t,s-t}U_{b;s-t,t}U_{b-s+t;s-t,s-t} \in \mathfrak{S}_{a+b},\\ x(r,s,t) &:= U_{b-r;r-s,s-t}U_{b+t;s-t,r-s} \in \mathfrak{S}_{a+b}. \end{aligned}$$

Let $\min\{a,b\} \geq r \geq s \geq n \geq 0$. For any $\hat{f} \in \hat{\Lambda}(r,s)$, set

$$\hat{f}^n_{r,s} := (-1)^{\binom{r-s+1}{2}+n(r-s)} e_{r,r-s+n}\psi_{v(r,s,n)}D(r,s,n)(u(s,n)\cdot\hat{f})e_{s,n}$$

$= \pm$ 

.

Define

$$\varphi^{\hat{f}}_{r,s} : P^r := \bigoplus_{m=0}^{r} q^{\mathsf{s}_{r,m}} R_\theta e_{r,m} \to \bigoplus_{n=0}^{s} q^{\mathsf{s}_{s,n}} R_\theta e_{s,n} =: P^s,$$

$$(h_m e_{r,m})_{m=0}^r \mapsto (h_{r-s+n}\hat{f}^n_{r,s}e_{s,n})_{n=0}^s.$$

We think of $\varphi^{\hat{f}}_{r,s}$ as an element of $\mathrm{Hom}^{r-s}_{R_\theta}(P^r_\bullet, P^s_\bullet)$. Recalling the differential (2.31) on $\mathrm{Hom}^\bullet_{R_\theta}(P^r_\bullet, P^s_\bullet)$, we have:

Lemma 4.14. *Suppose* $0 \leq s \leq r \leq \min\{a,b\}$ *and let* $f \in \Lambda(r,s) = \mathcal{E}_\theta(r,s)$. *If* $\hat{f} \in \hat{\Lambda}(r,s)$ *is such that* $\mathsf{q}_{r,s}(\hat{f}) = f$, *then*

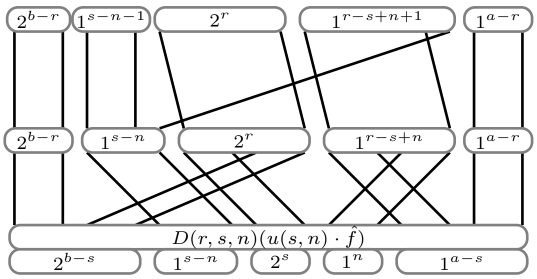
(i) $\delta(\varphi^{\hat{f}}_{r,s}) = 0$, *and*

(ii) *the isomorphism (2.32) sends the cohomology class of* $\varphi^{\hat{f}}_{r,s}$ *to* f.

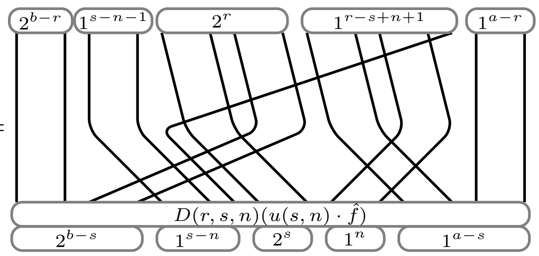
Proof. We prove (i) by checking that the following diagram either commutes (if $r-s$ is even) or anticommutes (if $r-s$ is odd) whenever $0 \leq n < r-s$.

$$\begin{array}{ccc} q^{\mathsf{s}_{r,n+1+r-s}}R_\theta e_{r,n+1+r-s} & \xrightarrow{-\cdot d_{r,r-s+n}} & q^{\mathsf{s}_{r,r-s+n}}R_\theta e_{r,r-s+n} \\ \big\downarrow{\scriptstyle -\cdot\hat{f}_{n+1}} & & \big\downarrow{\scriptstyle -\cdot\hat{f}_n} \\ q^{\mathsf{s}_{s,n+1}}R_\theta e_{s,n+1} & \xrightarrow{-\cdot d_{s,n}} & q^{\mathsf{s}_{s,n}}R_\theta e_{s,n}. \end{array}$$

We compute $\pm d_{r,r-s+n}\hat{f}_n$:

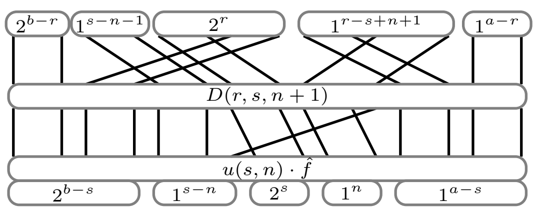


$=$



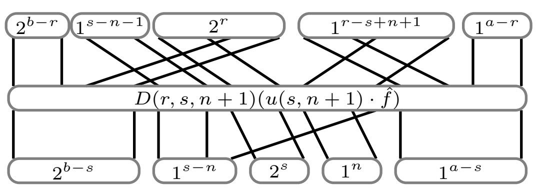
by Lemma 2.16

$=$



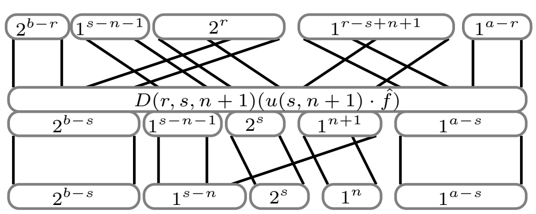
by several applications of the relations (R5) and (R4), and by Lemma 2.5

$=$



by Lemma 2.5 and the relation (R4)

$=$

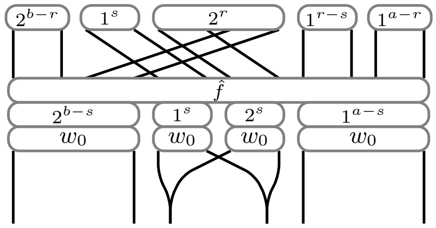


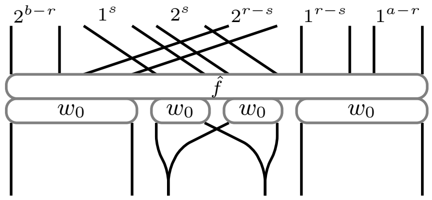
by Lemma 2.16.

The last diagram represents $\pm\hat{f}_{n+1}d_{s,n}$. The signs are taken care of by

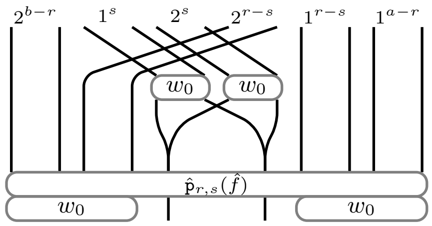
$$(-1)^{\binom{r-s+1}{2}+(n+1)(r-s)} = (-1)^{r-s}(-1)^{\binom{r-s+1}{2}+n(r-s)}.$$

To prove (ii), recalling (4.2), we compute $(-1)^{\frac{(r-s)(r-s+1)}{2}}\varepsilon_s(e_{r,r-s}\hat{f}^0_{r,s})$:



= 

by Lemma 2.16

= 

by Lemma 2.5, the relation (R4), and the defining relations of $\Delta(s)$.

The last diagram represents $\Xi_{r-s}(\hat{\mathsf{p}}_{r,s}(\hat{f}))$, where Ξ_{r-s} is as in Lemma 4.6(ii). Thus, the image of $\varphi^{\hat{f}}_{r,s}$ under the map (2.33) is

$$(e_{r,r-s} \mapsto \Xi_{r-s}(\hat{\mathsf{p}}_{r,s}(\hat{f}))) \in \mathrm{Hom}_{R_\theta}(q^{\mathsf{s}_{r,r-s}} R_\theta e_{r,r-s}, \Delta(s)).$$

The proof is complete upon an application of Theorem 4.13. □

For $\hat{f} \in \hat{\Lambda}(r,s)$ and $\hat{g} \in \hat{\Lambda}(s,t)$, define

$$-\,\hat{\diamond}\,- : \hat{\Lambda}(s,t) \otimes \hat{\Lambda}(r,s) \to \hat{\Lambda}(r,t),\ \hat{g} \otimes \hat{f} \mapsto \partial_{x(r,s,t)}\big(D(r,s,s-t)\hat{g}(w(s,t)\cdot \hat{f})\big). \quad (4.15)$$

Lemma 4.16. *Let* $0 \le t \le s \le r \le \min\{a,b\}$. *If* $\hat{f} \in \hat{\Lambda}(r,s)$ *and* $\hat{g} \in \hat{\Lambda}(s,t)$, *then* $\varphi^{\hat{g}}_{s,t}\varphi^{\hat{f}}_{r,s} = \varphi^{\hat{g}\,\hat{\diamond}\,\hat{f}}_{r,t}$.

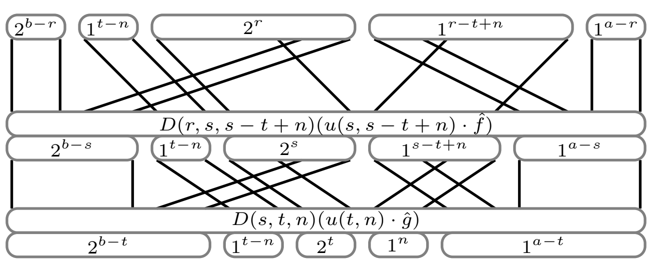
Proof. Let $y(s,t,n) := U_{b-s;t-n,s-t} \in \mathfrak{S}_d$ and note that

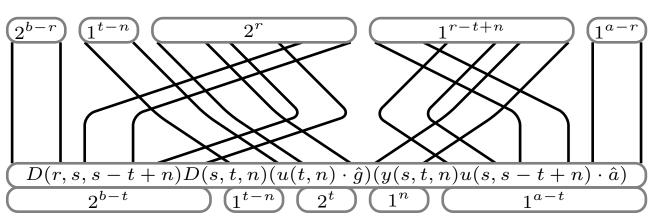
$$u(t,n)^{-1} y(s,t,n) u(s,s-t+n) = w(s,t) \quad (4.17)$$

and

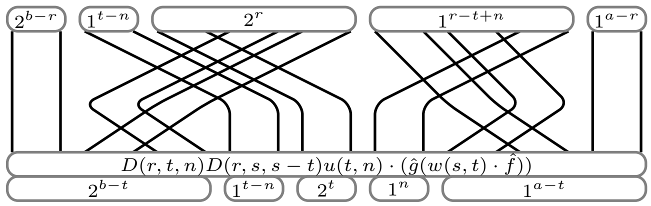
$$D(r,s,s-t+n)D(s,t,n) = D(r,t,n)D(r,s,s-t). \quad (4.18)$$

We compute $\hat{f}^{s-t+n}_{r,s}\hat{g}^{n}_{s,t}$:

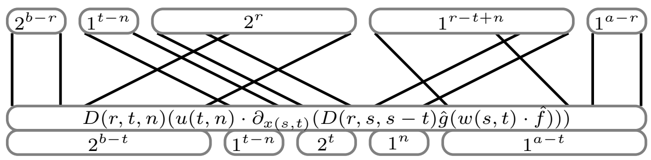


= 

by (R4), Lemma 2.16, and Theorem 2.5

$=$ 

by (4.17), (4.18), and the relation (R7)

$=$ 

by Lemma 2.6.

The last diagram represents $(\hat{g}\diamond\hat{f})^n$. □

Lemma 4.19. *Let $0\leq t\leq s\leq r\leq\min\{a,b\}$, $\lambda\in\mathscr{P}(r-s,s)$, and $\mu\in\mathscr{P}(s-t,t)$, and set $\hat{f}:=1\otimes s_\lambda\otimes 1\otimes 1\otimes 1\otimes 1\in\hat{\Lambda}(r,s)$ and $\hat{g}:=1\otimes s_\mu\otimes 1\otimes 1\otimes 1\otimes 1\in\hat{\Lambda}(s,t)$. Then*

$$\hat{g}\diamond\hat{f}=1\otimes(s_\lambda\star s_\mu)\otimes 1\otimes 1\otimes 1\otimes 1\in\hat{\Lambda}(r,t).$$

Proof. We use the notation λ^c for the conjugate of a partition λ. For a partition $\kappa=(\kappa_1,\dots,\kappa_{s-t})\in\mathscr{P}(s-t,r-s)$, let $\hat{\kappa}:=(r-s-\kappa_{s-t},\dots,r-s-\kappa_1)^c\in\mathscr{P}(r-s,s-t)$. We have $\hat{f}=w(s,t)\hat{f}$ and

$$D(r,s,s-t)=\sum_{\kappa\in\mathscr{P}(s-t,r-s)}(-1)^{|\hat{\kappa}|}1\otimes s_{\hat{\kappa}}\otimes 1\otimes 1\otimes 1\otimes s_\kappa\otimes 1\otimes 1\in\Lambda(r,s,t)$$

by [14, I.4 Example 5], so that

$$\begin{aligned}\hat{g}\diamond\hat{f}&=\sum_{\kappa\in\mathscr{P}(s-t,r-s)}(-1)^{|\hat{\kappa}|}\partial_{x(r,s,t)}(1\otimes(s_{\hat{\kappa}}s_\lambda)\otimes s_\mu\otimes 1\otimes 1\otimes s_\kappa\otimes 1\otimes 1)\\&=\sum_{\kappa\in\mathscr{P}(s-t,r-s)}(-1)^{|\hat{\kappa}|}1\otimes((s_{\hat{\kappa}}s_\lambda)\star s_\mu)\otimes 1\otimes 1\otimes(s_\kappa\star 1)\otimes 1\\&=1\otimes(s_\lambda\star s_\mu)\otimes 1\otimes 1\otimes(s_{(r-s)^{(s-t)}}\star 1)\otimes 1\\&=1\otimes(s_\lambda\star s_\mu)\otimes 1\otimes 1\otimes 1\otimes 1.\end{aligned}$$ □

We consider $\Lambda(r,s)$ as a right $\Lambda(r,r)$ module via the natural algebra embedding

$$\Lambda(r,r)=\Lambda_{b-r,r,a-r}\hookrightarrow\Lambda_{b-r,r-s,s,a-r}=q^{-\omega_{r-s}(r,s)}\Lambda(r,s).$$

If $f\in\Lambda_{r-s}$, we write

$$f^{r,s}:=1_{\Lambda_{b-r}}\otimes f\otimes 1_{\Lambda_r}\otimes 1_{\Lambda_{a-r}}\in\Lambda(r,s).$$

By Proposition 2.1, $\Lambda(r,s)$ is a free right $\Lambda(r,r)$-module with basis

$$\{s_\lambda^{r,s}\mid\lambda\in\mathscr{P}(r-s,s)\}.$$

We also make $\Lambda(r,s)$ into a left $\Lambda(s,s)$-module via the composition of algebra homomorphisms:

$$\Lambda(s,s)=\Lambda_{b-s,s,a-s}\hookrightarrow\Lambda_{b-r,r-s,s,r-s,a-r}=\Lambda^{r,s}\xrightarrow{\mathrm{p}_{r,s}}q^{-\omega_{r-s}(r,s)}\Lambda(r,s),\qquad(4.20)$$

the first map being the natural embedding. (This is similar to the definition of the *right* $\Lambda_{\sigma(s)}$-module structure on $\Lambda(r,s)$ used in the previous subsection; in fact $\Lambda(s,s)$ can be identified with $\mathrm{End}_{R_\theta}(\Delta(s))=\Lambda_{\sigma(s)}^{\mathrm{op}}$, see the proof of Theorem 4.22.)

If $0 \leq t \leq s \leq r \leq \min\{a,b\}$, then the tensor product $\Lambda(s,t) \otimes_{\Lambda(s,s)} \Lambda(r,s)$ is now a free right $\Lambda(r,r)$-module with basis

$$\{s^{s,t}_\mu \otimes s^{r,s}_\lambda \mid \mu \in \mathscr{P}(s-t,t), \lambda \in \mathscr{P}(r-s,s)\} \tag{4.21}$$

and we define a map of right $\Lambda(r,r)$-modules

$$\Theta : \Lambda(s,t) \otimes_{\Lambda(s,s)} \Lambda(r,s) \to \Lambda(r,t),\ s^{s,t}_\mu \otimes s^{r,s}_\lambda \mapsto (s_\mu \star s_\lambda)^{r,t}.$$

Let

$$- \diamond - : \Lambda(s,t) \otimes_{\Bbbk} \Lambda(r,s) \to \Lambda(r,t),\ g \otimes f \mapsto \Theta(g \otimes f).$$

Theorem 4.22. *Let $0 \leq t \leq s \leq r \leq \min\{a,b\}$. The composition in the category $\mathcal{E}_\theta$ is given by*

$$\begin{array}{ccc} \mathcal{E}_\theta(s,t) \otimes \mathcal{E}_\theta(r,s) & \longrightarrow & \mathcal{E}_\theta(r,t) \\ \wr\Vert & & \wr\Vert \\ \Lambda(s,t) \otimes \Lambda(r,s) & \longrightarrow & \Lambda(r,t) \\ g \otimes f & \longmapsto & g \diamond f. \end{array}$$

Proof. According to Lemmas 4.14 and 4.16, the composition of $g \in \Lambda(s,t)$ with $f \in \Lambda(r,s)$ is given by $\mathsf{q}_{r,t}(\hat{g} \mathbin{\hat{\diamond}} \hat{f})$, where $\hat{g} \in \hat{\Lambda}(s,t)$ and $\hat{f} \in \hat{\Lambda}(r,s)$ are such that $\mathsf{q}_{s,t}(\hat{g}) = g$ and $\mathsf{q}_{r,s}(\hat{f}) = f$.

First suppose $r = s$. Since $D(s,s,s-t) = 1$ and $x(s,s,t) = 1$, (4.15) shows that $\mathsf{q}_{s,t}(\hat{g} \mathbin{\hat{\diamond}} \hat{f}) = gf \in \Lambda(s,t)$. Thus, the composition map $\Lambda(s,t) \otimes \Lambda(s,s) \to \Lambda(s,s)$ coincides with the right $\Lambda(s,s)$-module structure of $\Lambda(s,t)$.

Now suppose $s = t$. Since $D(r,s,0) = 1$ and $w(s,s) = x(r,s,s) = 1$, (4.15) shows that $\hat{g} \mathbin{\hat{\diamond}} \hat{f} = \hat{g}\hat{f}$, so that $\mathsf{q}_{r,s}(\hat{g} \mathbin{\hat{\diamond}} \hat{f}) = \varphi(g)f \in \Lambda(r,s)$ where φ is the composition in (4.20). Thus, the composition map $\Lambda(s,s) \otimes \Lambda(r,s) \to \Lambda(r,s)$ coincides with the left $\Lambda(s,s)$-module structure of $\Lambda(r,s)$.

Associativity in $\mathcal{E}_\theta$ implies that the composition map $\Lambda(s,t) \otimes \Lambda(r,s) \to \Lambda(r,t)$ is $\Lambda(s,s)$-balanced and $\Lambda(r,r)$-equivariant, and is therefore completely determined by the image of the basis elements in (4.21). Lemma 4.19 now completes the proof. □

5. Non-formality of the A_∞ structure

We provide an example to show that the A_∞-category structure on $\mathcal{E}_\theta$ is, in general, non-formal. First, we recall some basic definitions and an important theorem, see [5,6].

A ($\Bbbk$-linear) *A_∞-category* $\mathcal{A}$ consists of a class of objects $\mathrm{ob}(\mathcal{A})$, a $\mathbb{Z}$-graded $\Bbbk$-module $\mathcal{A}^\bullet(\rho,\sigma)$ for every pair of objects $\rho,\sigma \in \mathrm{ob}(\mathcal{A})$, and for each $n \in \mathbb{Z}_{>0}$ and objects $\pi_0,\dots,\pi_n \in \mathrm{ob}(\mathcal{A})$, a degree $2-n$ $\Bbbk$-linear map

$$m_n^{\pi_0,\dots,\pi_n} : \mathcal{A}^\bullet(\pi_{n-1},\pi_n) \otimes \mathcal{A}^\bullet(\pi_{n-2},\pi_{n-1}) \otimes \cdots \otimes \mathcal{A}^\bullet(\pi_0,\pi_1) \to \mathcal{A}^\bullet(\pi_0,\pi_n),$$

such that for each $n \in \mathbb{Z}_{>0}$, we have

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0, \tag{$*_n$}$$

where $m_n := \bigoplus_{\pi_0,\dots,\pi_n \in \mathrm{ob}(\mathcal{A})} m_n^{\pi_0,\dots,\pi_n}$. An *$A_\infty$-functor* F from $\mathcal{B} = (\mathcal{B}, M_1, M_2, \dots)$ to $\mathcal{A} = (\mathcal{A}, m_1, m_2, \dots)$ consists of a function $F : \mathrm{ob}(\mathcal{B}) \to \mathrm{ob}(\mathcal{A})$, and for each $n \in \mathbb{Z}_{\geq 0}$ and objects $\pi_0,\dots,\pi_n \in \mathrm{ob}(\mathcal{A})$, a degree $1-n$ map

$$F_n^{\pi_0,\dots,\pi_n} : \mathcal{B}^\bullet(\pi_{n-1},\pi_n) \otimes \mathcal{B}^\bullet(\pi_{n-2},\pi_{n-1}) \otimes \cdots \otimes \mathcal{B}^\bullet(\pi_0,\pi_1) \to \mathcal{A}^\bullet(F(\pi_0),F(\pi_n)),$$

such that for each $n \in \mathbb{Z}_{>0}$, we have

$$\sum_{r+s+t=n} (-1)^{r+st} F_{r+1+t}(1^{\otimes r} \otimes M_s \otimes 1^{\otimes t}) = \sum_{i_1+\cdots+i_k=n} (-1)^{\mathtt{s}} m_k(F_{i_1} \otimes \cdots \otimes F_{i_k}), \quad (**_n)$$

where $\mathtt{s} := \sum_{j=1}^{k-1}(k-j)(i_j-1)$ and $F_n := \bigoplus_{\pi_0,\dots,\pi_n \in \mathrm{ob}(\mathcal{A})} F_n^{\pi_0,\dots,\pi_n}$. The *composition* GF of two A_∞-functors $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{C} \to \mathcal{B}$ is given on objects by $(FG)(\pi) = F(G(\pi))$ for $\pi \in \mathrm{ob}(\mathcal{C})$ with

$$(FG)_n := \sum_{i_1+\cdots+i_k=n} (-1)^{\mathtt{s}} F_k(G_{i_1} \otimes \cdots G_{i_k}),$$

where $\mathtt{s}$ is as above. The *identity* functor $1_\mathcal{A}$ on $\mathcal{A}$ is the A_∞-functor $F : \mathcal{A} \to \mathcal{A}$ with $F(\pi) = \pi$ for each $\pi \in \mathrm{ob}(\mathcal{A})$, $F_1^{\rho,\sigma} = \mathrm{id}_{\mathcal{A}^\bullet(\rho,\sigma)}$ for each $\rho, \sigma \in \mathrm{ob}(\mathcal{A})$, and $F_n = 0$ for $n > 1$. An *isomorphism* of A_∞-categories is an A_∞-functor $F : \mathcal{B} \to \mathcal{A}$ such that there exists an A_∞-functor $G : \mathcal{A} \to \mathcal{B}$ such that $GF = 1_\mathcal{B}$ and $FG = 1_\mathcal{A}$.

When we speak of the *homology* $H\mathcal{A}$ of an A_∞-category $\mathcal{A} = (\mathcal{A}, m_1, m_2, \dots)$, we mean the homology with respect to m_1. It is a graded category with the same object class as $\mathcal{A}$. In fact, we have:

Theorem 5.1. *[3, Theorem 1] Let $(\mathcal{A}, m_1, m_2, \dots)$ be an A_∞ category and $H\mathcal{A}$ its homology. If each morphism space in $H\mathcal{A}$ is a free graded $\Bbbk$-module, then $H\mathcal{A}$ carries the structure of an A_∞-category $H\mathcal{A} = (H\mathcal{A}, M_1, M_2, \dots)$ such that*

(i) *$M_1 = 0$ and $M_2 = [m_2]$,*
(ii) *there exists an A_∞-functor $F : H\mathcal{A} \to \mathcal{A}$ such that $F(\pi) = \pi$ for each $\pi \in \mathrm{ob}(\mathcal{A})$ and $[F_1] = \mathrm{id}_{H\mathcal{A}}$.*

Moreover, the A_∞-category structure on $H\mathcal{A}$ satisfying (i) and (ii) is unique up to (non-unique) isomorphism of A_∞-categories.

Such an A_∞-category structure on $H\mathcal{A}$ is called a *minimal model* of $\mathcal{A}$. An A_∞-category $\mathcal{A}$ is called *formal* if its minimal model can be chosen so that $M_n = 0$ for $n \neq 2$. A graded category $\mathcal{B}$ is *intrinsically formal* if every A_∞-category $\mathcal{A} = (\mathcal{A}, m_1, m_2, \dots)$ whose homology is isomorphic to $\mathcal{B}$ as a graded category is formal. For example, the graded category $\mathcal{E}_\theta$ in the situation of either Theorems A or B is intrinsically formal because by homological degree consideration, there is no way to impose an A_∞-category structure on $\mathcal{E}_\theta$ with $M_n \neq 0$ for any $n \neq 2$.

Kadeishvili's original proof is constructive and yields an inductive algorithm for producing a minimal model in the special case where $\mathcal{A}$ is a differential-graded algebra, i.e., $m_n = 0$ for $n > 2$:

Algorithm 5.2. [3, Proof of Theorem 1] Let $\mathcal{A} = (\mathcal{A}, m_1, m_2, \dots)$ be an A_∞-category with $m_k = 0$ for $k > 2$ and $H\mathcal{A}$ its homology. The following algorithm produces an A_∞-category structure $(H\mathcal{A}, M_1, M_2, \dots)$ and an A_∞-functor $F : H\mathcal{A} \to \mathcal{A}$ which satisfies the conditions in Theorem 5.1.

Step 1: Let $M_1 = 0$ and take $F_1^{\mu,\nu} : \mathcal{E}_\theta(\mu, \nu) \to \mathcal{H}_\theta(\mu, \nu)$ to be a cycle-choosing homomorphism of $\Bbbk$-modules. Set $n := 2$.

Step 2: Since $m_k = 0$ for $k > 2$, we may rewrite $(**_n)$ as

$$m_1 F_n = F_1 M_n - U_n, \tag{5.3}$$

where

$$U_n := m_2\sum_{i=1}^{n-1}(-1)^{i-1}(F_i\otimes F_{n-i}) - \sum_{s=2}^{n-1}\sum_{t=0}^{n-2}(-1)^{n-s-t+st}F_{n-s+1}(1^{\otimes n-s-t}\otimes M_s\otimes 1^{\otimes t}).$$

We will also use the restriction to $H\mathcal{A}(\pi_{n-1},\pi_n)\otimes\cdots\otimes H\mathcal{A}(\pi_0,\pi_1)$:

$$U_n^{\pi_0,\dots,\pi_n} : H\mathcal{A}(\pi_{n-1},\pi_n)\otimes\cdots\otimes H\mathcal{A}(\pi_0,\pi_1)\to\mathcal{A}(\pi_0,\pi_n).$$

One can check that $m_1U_n = 0$. Thus, since M_k, F_k have been defined for $k < n$, we take M_n to be the (well-defined) homology class $[U_n]$ of U_n.

Step 3: Note that $[F_1M_n - U_n] = [F_1M_n] - [U_n] = M_n - M_n = 0$, so $F_1M_n - U_n$ is a boundary, and choose F_n such that $m_1F_n = F_1M_n - U_n$. Increment n and return to Step 2.

Let $\theta\in Q_+$. For each $\pi\in\mathrm{KP}(\theta)$, fix a projective resolution $P^\pi_\bullet$ of $\Delta(\pi)$. Consider the differential-graded category $\mathcal{H}_\theta$ whose objects are the Kostant partitions of θ with morphism spaces $\mathcal{H}^\bullet_\theta(\rho,\sigma) := \mathrm{Hom}^\bullet_{R_\theta}(P^\rho_\bullet, P^\sigma_\bullet)$. We denote by

$$m_1^{\rho,\sigma} : \mathcal{H}^\bullet_\theta(\rho,\sigma)\to\mathcal{H}^\bullet_\theta(\rho,\sigma)$$
$$m_2^{\rho,\sigma,\tau} : \mathcal{H}^\bullet_\theta(\sigma,\tau)\otimes\mathcal{H}^\bullet_\theta(\rho,\sigma)\to\mathcal{H}^\bullet_\theta(\rho,\tau)$$

the differential and composition in $\mathcal{H}_\theta$, respectively. Note that $m_1^{\rho,\sigma}$ is precisely δ from (2.31). Being a differential-graded category, $\mathcal{H}_\theta$ is also an A_∞-category (see [5]), so its homology $\mathcal{E}_\theta$ carries a structure of an A_∞-category according to Theorem 5.1.

For the rest of this section, we let $\theta := \alpha_1 + 2\alpha_2 + \alpha_3 \in Q_+$ and set

$$\pi := (\alpha_2, \alpha_1+\alpha_2+\alpha_3) \qquad \rho := (\alpha_2+\alpha_3, \alpha_1+\alpha_2)$$
$$\sigma := (\alpha_3, \alpha_2, \alpha_1+\alpha_2) \qquad \tau := (\alpha_3, (\alpha_2)^2, \alpha_1).$$

Note that there is one other Kostant partition, $(\alpha_2+\alpha_3, \alpha_2, \alpha_1)$, which will not play a role in our construction. Recall the standard generators (2.28) of the standard modules and the idempotents (2.7) in the nil-Hecke algebra. We define

$$\hat v_\tau := v_{\alpha_3}\circ v_{\alpha_2}\circ v_{\alpha_2}\circ v_{\alpha_1}\in\hat\Delta(\tau),$$
$$e_\tau := e_1\otimes e_2\otimes e_1\in\mathcal{NH}_1\otimes\mathcal{NH}_2\otimes\mathcal{NH}_1,$$

so that $v_\tau = \hat v_\tau e_\tau$. We list the resolutions of the corresponding standard modules from [1, Theorem A] and Lemma 2.23 below:

$$0\to q^2R_\theta 1_{2321}\xrightarrow{d_1^\pi} qR_\theta 1_{2231}\oplus qR_\theta 1_{2132}\xrightarrow{d_0^\pi} R_\theta 1_{2123}\xrightarrow{\epsilon_\pi}\Delta(\pi)\to 0,$$

$$0\to q^2R_\theta 1_{3221}\xrightarrow{d_1^\rho} qR_\theta 1_{2321}\oplus qR_\theta 1_{3212}\xrightarrow{d_0^\rho} R_\theta 1_{2312}\xrightarrow{\epsilon_\rho}\Delta(\rho)\to 0,$$

$$0\longrightarrow qR_\theta 1_{3221}\xrightarrow{d_0^\sigma} R_\theta 1_{3212}\xrightarrow{\epsilon_\sigma}\Delta(\sigma)\to 0,$$

$$0\longrightarrow q^{-1}R_\theta 1_{32^{(2)}1}\xrightarrow{\epsilon_\tau}\Delta(\tau)\to 0,$$

where a matrix label stands for right multiplication with that matrix, and

$$d_1^\pi := \begin{bmatrix}-\psi_2 1_{2231} & \psi_3\psi_2 1_{2132}\end{bmatrix}, \quad d_0^\pi := \begin{bmatrix}\psi_3\psi_2 1_{2123}\\ \psi_3 1_{2123}\end{bmatrix}, \quad \epsilon_\pi := \begin{bmatrix}v_\pi\end{bmatrix},$$

$$d_1^\rho := \begin{bmatrix} -\psi_1 1_{2321} & \psi_3 1_{3212} \end{bmatrix}, \qquad d_0^\rho := \begin{bmatrix} \psi_3 1_{2312} \\ \psi_1 1_{2312} \end{bmatrix}, \qquad \epsilon_\rho := \begin{bmatrix} v_\rho \end{bmatrix},$$
$$d_0^\sigma := \begin{bmatrix} \psi_3 1_{3212} \end{bmatrix}, \qquad \epsilon_\sigma := \begin{bmatrix} v_\sigma \end{bmatrix},$$
$$\epsilon_\tau := \begin{bmatrix} \psi_2 v_\tau \end{bmatrix} = \begin{bmatrix} \hat{v}_\tau \tau_2 \end{bmatrix}.$$

One can easily check, using (2.19), that the complexes $T_\bullet^\pi(\rho)$, $T_\bullet^\rho(\sigma)$, $T_\bullet^\sigma(\tau)$, and $T_\bullet^\pi(\tau)$ are, respectively, the top complexes in the four diagrams below. Moreover, the diagrams define isomorphisms of complexes.

$$\begin{array}{ccccccc}
0 & \longleftarrow & q^{-1}1_{2132}\Delta(\rho) & \xleftarrow{\;[(d_0^\pi)_{2,1}]\;} & 1_{2123}\Delta(\rho) & \longleftarrow & 0 \\
 & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_2 v_\rho f} & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_3\psi_2 v_\rho f} & & \\
0 & \longleftarrow & q^{-1}\mathcal{X}_2 & \xleftarrow{\;[x_2-x_1]\;} & q\mathcal{X}_2 & \longleftarrow & 0,
\end{array}$$

$$\begin{array}{ccccccc}
0 & \longleftarrow & q^{-1}1_{3212}\Delta(\sigma) & \xleftarrow{\;[(d_0^\rho)_{2,1}]\;} & 1_{2312}\Delta(\sigma) & \longleftarrow & 0 \\
 & & \wr\Big\uparrow {\scriptstyle f\mapsto v_\sigma f} & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_1 v_\sigma f} & & \\
0 & \longleftarrow & q^{-1}\mathcal{X}_3 & \xleftarrow{\;[x_2-x_1]\;} & q\mathcal{X}_3 & \longleftarrow & 0,
\end{array}$$

$$\begin{array}{ccccccc}
0 & \longleftarrow & 1_{3221}\Delta(\tau) & \xleftarrow{\;[(d_0^\sigma)_{1,1}]\;} & q1_{3212}\Delta(\tau) & \longleftarrow & 0 \\
 & & \wr\Big\uparrow {\scriptstyle f\mapsto \hat{v}_\tau f\tau_2} & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_3\hat{v}_\tau f\tau_2} & & \\
0 & \longleftarrow & q^{-2}\mathcal{X}_4 & \xleftarrow{\;[x_4-x_3]\;} & \mathcal{X}_4 & \longleftarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longleftarrow & q^{-1}1_{2321}\Delta(\tau) & \xleftarrow{\;d_1^\pi\;} & 1_{2231}\Delta(\tau)\oplus 1_{2132}\Delta(\tau) & \xleftarrow{\;d_0^\pi\;} & q1_{2123}\Delta(\tau) & \longleftarrow & 0 \\
 & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_1\hat{v}_\tau f\tau_2} & & \wr\Big\uparrow {\scriptstyle (f,g)\mapsto(\psi_2\psi_1\hat{v}_\tau f\tau_2,\,\psi_2\psi_1\psi_3\hat{v}_\tau g\tau_2)} & & \wr\Big\uparrow {\scriptstyle f\mapsto \psi_3\psi_2\psi_3\psi_1\hat{v}_\tau f\tau_2} & & \\
0 & \longleftarrow & q^{-2}\mathcal{X}_4 & \xleftarrow[\begin{bmatrix} -(x_3-x_1) & x_4-x_3 \end{bmatrix}]{} & \mathcal{X}_4\oplus\mathcal{X}_4 & \xleftarrow[\begin{bmatrix} x_4-x_3 \\ x_3-x_1 \end{bmatrix}]{} & q^2\mathcal{X}_4 & \longleftarrow & 0
\end{array}$$

Thus, denoting $\mathcal{Z}_k := \Bbbk[z_1,\dots,z_k]$, we have

$$\mathcal{E}_\theta(\pi,\rho) = \mathcal{E}_\theta^1(\pi,\rho) \cong q^{-1}\mathcal{X}_2/(x_1=x_2) \xrightarrow{\sim} q^{-1}\mathcal{Z}_1,\ \bar{x}_1\mapsto z_1$$
$$\mathcal{E}_\theta(\rho,\sigma) = \mathcal{E}_\theta^1(\rho,\sigma) \cong q^{-1}\mathcal{X}_3/(x_1=x_2) \xrightarrow{\sim} q^{-1}\mathcal{Z}_2,\ \bar{x}_1\mapsto z_1, \bar{x}_3\mapsto z_2$$
$$\mathcal{E}_\theta(\sigma,\tau) = \mathcal{E}_\theta^1(\sigma,\tau) \cong q^{-2}\mathcal{X}_4/(x_3=x_4) \xrightarrow{\sim} q^{-2}\mathcal{Z}_3,\ \bar{x}_1\mapsto z_1, \bar{x}_2\mapsto z_2, \bar{x}_3\mapsto z_3$$
$$\mathcal{E}_\theta(\pi,\tau) = \mathcal{E}_\theta^2(\pi,\tau) \cong q^{-2}\mathcal{X}_4/(x_1=x_3=x_4) \xrightarrow{\sim} q^{-2}\mathcal{Z}_2,\ \bar{x}_1\mapsto z_1, \bar{x}_2\mapsto z_2.$$

Example 5.4. There is an A_∞-category structure $\mathcal{E}_\theta = (\mathcal{E}_\theta, M_1, M_2, \ldots)$ satisfying the conditions in Theorem 5.1 such that

$$\begin{array}{ccc}
\mathcal{E}^1_\theta(\sigma,\tau)\otimes\mathcal{E}^1_\theta(\rho,\sigma)\otimes\mathcal{E}^1_\theta(\pi,\rho) & \xrightarrow{M_3^{\pi,\rho,\sigma,\tau}} & \mathcal{E}^2_\theta(\pi,\tau) \\
\wr\Vert & & \wr\Vert \\
q^{-2}\mathcal{Z}_3\otimes q^{-1}\mathcal{Z}_2\otimes q^{-1}\mathcal{Z}_1 & \longrightarrow & q^{-2}\mathcal{Z}_2 \\
z_1^a z_2^b z_3^c \otimes z_1^m z_2^n \otimes z_1^w & \longmapsto & z_1^{c+m+w} z_2^{b+n}\dfrac{z_2^a - z_1^a}{z_2 - z_1}.
\end{array}$$

Moreover, there is no A_∞-category structure on $\mathcal{E}_\theta$ satisfying the conditions in Theorem 5.1 with $M_3 = 0$.

Proof. We apply Algorithm 5.2. We will need to examine the complexes $T^\pi_\bullet(\sigma)$ and $T^\rho_\bullet(\tau)$. They are, respectively, the top complexes in the two diagrams below, and the diagrams define isomorphisms of complexes.

$$\begin{array}{ccccccc}
0 & \longleftarrow & q^{-1}1_{2132}\Delta(\sigma) & \xleftarrow{[(d_0^\pi)_{2,1}]} & 1_{2123}\Delta(\sigma) & \longleftarrow & 0 \\
 & & \wr\big\uparrow f\mapsto \psi_2\psi_1 v_\sigma f & & \wr\big\uparrow f\mapsto\psi_3\psi_2\psi_1 v_\sigma f & & \\
0 & \longleftarrow & \mathcal{X}_3 & \xleftarrow{[x_3 - x_1]} & q^2\mathcal{X}_3 & \longleftarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longleftarrow & q^{-1}1_{3221}\Delta(\tau) & \xleftarrow{d_1^\rho} & 1_{2321}\Delta(\tau)\oplus 1_{3212}\Delta(\tau) & \xleftarrow{d_0^\rho} & q1_{2312}\Delta(\tau) & \longleftarrow & 0 \\
 & & \wr\big\uparrow f\mapsto \hat{v}_\tau f\tau_2 & & \wr\big\uparrow (f,g)\mapsto(\psi_1\hat{v}_\tau f\tau_2, \psi_3\hat{v}_\tau g\tau_2) & & \wr\big\uparrow f\mapsto\psi_1\psi_3\hat{v}_\tau f\tau_2 & & \\
0 & \longleftarrow & q^{-3}\mathcal{X}_4 & \xleftarrow[{\begin{bmatrix}-(x_2-x_1) & x_4-x_3\end{bmatrix}}]{} & q^{-1}\mathcal{X}_4\oplus q^{-1}\mathcal{X}_4 & \xleftarrow[{\begin{bmatrix}x_4-x_3\\ x_2-x_1\end{bmatrix}}]{} & q\mathcal{X}_4 & \longleftarrow & 0
\end{array}$$

so that

$$\mathcal{E}_\theta(\pi,\sigma) = \mathcal{E}^1_\theta(\pi,\sigma)\cong\mathcal{X}_3/(x_1 = x_3)\xrightarrow{\sim}\mathcal{Z}_2,\ \bar{x}_1\mapsto z_1, \bar{x}_2\mapsto z_2$$
$$\mathcal{E}_\theta(\rho,\tau) = \mathcal{E}^2_\theta(\rho,\tau)\cong q^{-3}\mathcal{X}_4/(x_1 = x_2, x_3 = x_4)\xrightarrow{\sim} q^{-3}\mathcal{Z}_2, \bar{x}_1\mapsto z_1, \bar{x}_3\mapsto z_2.$$

Instead of F_1, it will only be relevant to determine the restrictions $F_1^{\pi,\rho}$, $F_1^{\rho,\sigma}$, $F_1^{\sigma,\tau}$, and $F_1^{\rho,\tau}$. According to the algorithm, we are free to take for these any cycle-choosing homomorphisms. We define

$$F_1^{\pi,\rho}(z_1^a) := \begin{array}{ccccccccc}
0 & \longrightarrow & q^2R_\theta 1_{2321} & \xrightarrow{d_1^\pi} & qR_\theta 1_{2231}\oplus qR_\theta 1_{2132} & \xrightarrow{d_0^\pi} & R_\theta 1_{2123} & \longrightarrow & 0, \\
 & & & \searrow{\scriptstyle\begin{bmatrix}y_4^a 1_{2321} & 0\end{bmatrix}} & & \searrow{\scriptstyle\begin{bmatrix}0\\ -\psi_2 y_3^a 1_{2312}\end{bmatrix}} & & & \\
0 & \longrightarrow & q^2R_\theta 1_{3221} & \xrightarrow[d_1^\rho]{} & qR_\theta 1_{2321}\oplus qR_\theta 1_{3212} & \xrightarrow[d_0^\rho]{} & R_\theta 1_{2312} & \longrightarrow & 0,
\end{array}$$

$$F_1^{\rho,\sigma}(z_1^a z_2^b) := \begin{array}{ccccccccc}
0 & \longrightarrow & q^2R_\theta 1_{3221} & \xrightarrow{d_1^\rho} & qR_\theta 1_{2321}\oplus qR_\theta 1_{3212} & \xrightarrow{d_0^\rho} & R_\theta 1_{2312} & \longrightarrow & 0, \\
 & & & \searrow{\scriptstyle\begin{bmatrix}y_2^a y_4^b 1_{3221}\end{bmatrix}} & & \searrow{\scriptstyle\begin{bmatrix}0\\ -y_2^a y_3^b 1_{3212}\end{bmatrix}} & & & \\
 & & 0 & \longrightarrow & qR_\theta 1_{3221} & \xrightarrow[d_0^\sigma]{} & R_\theta 1_{3212} & \longrightarrow & 0,
\end{array}$$

$$F_1^{\sigma,\tau}(z_1^a z_2^b z_3^c) := \begin{array}{c} 0 \longrightarrow qR_\theta 1_{3221} \xrightarrow{d_0^\sigma} R_\theta 1_{3212} \longrightarrow 0, \\ \searrow \left[-y_1^a y_2^b y_4^c 1_{32^{(2)}1}\right] \\ 0 \longrightarrow q^{-1} R_\theta 1_{32^{(2)}1} \longrightarrow 0, \end{array}$$

$$F_1^{\rho,\tau}(z_1^a z_2^b) := \begin{array}{c} 0 \to q^2 R_\theta 1_{3221} \xrightarrow{d_1^\rho} qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \xrightarrow{d_0^\rho} R_\theta 1_{2312} \longrightarrow 0, \\ \searrow \left[-y_2^a y_4^b 1_{32^{(2)}1}\right] \\ 0 \longrightarrow q^{-1} R_\theta 1_{32^{(2)}1} \to 0. \end{array}$$

Using the above, we now show that the following choices are in accordance with the algorithm:

$$M_2^{\pi,\rho,\sigma} = 0, \quad F_2^{\pi,\rho,\sigma} = 0, \quad M_2^{\pi,\rho,\tau} = 0, \quad F_2^{\pi,\rho,\tau} = 0 \tag{5.5}$$

By our choices of $F_1^{\pi,\rho}$ and $F_1^{\rho,\sigma}$, we have $U_2^{\pi,\rho,\sigma} = 0$, so $M_2^{\pi,\rho,\sigma} = 0$, and according to (5.3), we may take $F_2^{\pi,\rho,\sigma} = 0$. Since $\mathcal{E}_\theta(\pi,\rho)$ is concentrated in homological degree 1 and $\mathcal{E}_\theta(\rho,\tau)$ is concentrated in homological degree 2, the image of $U_2^{\pi,\rho,\tau} = m_2^{\pi,\rho,\tau}(F_1^{\rho,\tau} \otimes F_1^{\pi,\rho})$ is in $\mathcal{H}_\theta^3(\pi,\tau)$, which is zero since $P_\bullet^\pi$ has length 2. Thus $M_2^{\pi,\rho,\tau} = 0$ and according to (5.3), we may take $F_2^{\pi,\rho,\tau} = 0$.

We now have

$$\begin{aligned} U_3^{\pi,\rho,\sigma,\tau} &= m_2^{\pi,\sigma,\tau}(F_1^{\sigma,\tau} \otimes F_2^{\pi,\rho,\sigma}) - m_2^{\pi,\rho,\tau}(F_2^{\rho,\sigma,\tau} \otimes F_1^{\pi,\rho}) + F_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes M_2^{\pi,\rho,\sigma}) \\ &\qquad - F_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) \\ &= -m_2^{\pi,\rho,\tau}(F_2^{\rho,\sigma,\tau} \otimes F_1^{\pi,\rho}), \end{aligned}$$

so we only need to make a choice for $F_2^{\rho,\sigma,\tau}$. We have $U_2^{\rho,\sigma,\tau} = m_2^{\rho,\sigma,\tau}(F_1^{\sigma,\tau} \otimes F_1^{\rho,\sigma})$ so that $U_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by

$$\begin{array}{c} 0 \longrightarrow q^2 R_\theta 1_{3221} \xrightarrow{d_1^\rho} qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} \xrightarrow{d_0^\rho} R_\theta 1_{2312} \longrightarrow 0 \\ \searrow \left[-y_1^a y_2^{b+m} y_4^{c+n} 1_{32^{(2)}1}\right] \\ 0 \longrightarrow q^{-1} R_\theta 1_{32^{(2)}1} \longrightarrow 0. \end{array}$$

Thus

$$M_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n) = z_1^{a+b+m} z_2^{c+n}, \tag{5.6}$$

and $(F_1^{\rho,\tau} M_2^{\rho,\sigma,\tau} - U_2^{\rho,\sigma,\tau})(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by the above diagram, except the diagonal arrow is right multiplication with

$$\left[-y_2^{b+m} y_4^{c+n}(y_2^a - y_1^a) 1_{32^{(2)}1}\right].$$

We may now take $F_2^{\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ to be:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & q^2 R_\theta 1_{3221} & \xrightarrow{d_1^\rho} & qR_\theta 1_{2321} \oplus qR_\theta 1_{3212} & \xrightarrow{d_0^\rho} & R_\theta 1_{2312} & \longrightarrow & 0 \\
 & & & & & \searrow^{\left[\begin{smallmatrix} \psi_1 y_2^{b+m} y_4^{c+n} \frac{y_2^a - y_1^a}{y_2 - y_1} 1_{32^{(2)}1} \\ 0 \end{smallmatrix}\right]} & & & \\
 & & & & 0 & \longrightarrow & q^{-1} R_\theta 1_{32^{(2)}1} & \longrightarrow & 0.
\end{array}$$

Now $U_3^{\pi,\rho,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n \otimes z_1^w)$ is given by:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & q^2 R_\theta 1_{2321} & \xrightarrow{d_1^\pi} & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} & \xrightarrow{d_0^\pi} & R_\theta 1_{2123} & \longrightarrow & 0, \\
 & & & \searrow^{\left[-\psi_1 y_2^{b+m} y_4^{c+n+w} \frac{y_2^a - y_1^a}{y_2 - y_1} 1_{32^{(2)}1}\right]} & & & & & \\
 & & & & 0 & \longrightarrow & q^{-1} R_\theta 1_{32^{(2)}1} & \longrightarrow & 0,
\end{array}$$

so that $M_3^{\pi,\rho,\sigma,\tau} = [U_3^{\pi,\rho,\sigma,\tau}]$ is as in the theorem statement.

For the second assertion, note that the existence of a second A_∞-category structure $\mathcal{E}_\theta = (\mathcal{E}_\theta, N_1, N_2, \ldots)$ satisfying the conditions of Theorem 5.1 implies the existence of an isomorphism of A_∞-categories $G : (\mathcal{E}_\theta, M_1, M_2, \ldots) \to (\mathcal{E}_\theta, N_1, N_2, \ldots)$ with G_1 being the identity on each morphism space. Assume, toward a contradiction, that such an isomorphism exists, and that $N_3 = 0$.

Recall that we take $M_1 = N_1 = 0$, so $(**_2)$ applied to G reads $G_1 M_2 = N_2(G_1 \otimes G_1)$, and since G_1 is the identity, we have $M_2 = N_2$. Now since $N_3 = 0$ by assumption, $(**_3)$ reads

$$G_2(M_2 \otimes 1 - 1 \otimes M_2) + M_3 = M_2(1 \otimes G_2 - G_2 \otimes 1). \tag{5.7}$$

The restriction of (5.7) to $\mathcal{E}_\theta(\sigma,\tau) \otimes \mathcal{E}_\theta(\rho,\sigma) \otimes \mathcal{E}_\theta(\pi,\rho)$ is

$$\begin{aligned} G_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) - G_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes M_2^{\pi,\rho,\sigma}) + M_3^{\pi,\rho,\sigma,\tau} \\ = M_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes G_2^{\pi,\rho,\sigma}) - M_2^{\pi,\rho,\tau}(G_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}), \end{aligned} \tag{5.8}$$

so according to (5.5), (5.8) becomes

$$G_2^{\pi,\rho,\tau}(M_2^{\rho,\sigma,\tau} \otimes 1^{\pi,\rho}) + M_3^{\pi,\rho,\sigma,\tau} = M_2^{\pi,\sigma,\tau}(1^{\sigma,\tau} \otimes G_2^{\pi,\rho,\sigma}). \tag{5.9}$$

We have a formula for $M_2^{\rho,\sigma,\tau}$ given by (5.6). We define

$$F_1^{\pi,\sigma}(z_1^m z_2^n) := \begin{array}{ccccccccc}
0 & \longrightarrow & q^2 R_\theta 1_{2321} & \xrightarrow{d_1^\pi} & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} & \xrightarrow{d_0^\pi} & R_\theta 1_{2123} & \longrightarrow & 0, \\
 & & & \searrow^{\left[\psi_1 y_2^n y_4^m 1_{3221}\right]} & & \searrow^{\left[\begin{smallmatrix} 0 \\ -\psi_2 \psi_1 y_2^n y_3^m 1_{3212} \end{smallmatrix}\right]} & & & \\
 & & 0 & \longrightarrow & qR_\theta 1_{3221} & \xrightarrow[d_0^\sigma]{} & R_\theta 1_{3212} & \longrightarrow & 0,
\end{array}$$

By our choices of $F_1^{\pi,\sigma}$ and $F_1^{\sigma,\tau}$, we have that $U_2^{\pi,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n)$ is given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & q^2 R_\theta 1_{2321} & \xrightarrow{d_1^\pi} & qR_\theta 1_{2231} \oplus qR_\theta 1_{2132} & \xrightarrow{d_0^\pi} & R_\theta 1_{2123} & \longrightarrow & 0, \\
 & & & \searrow^{\left[-\psi_1 y_1^a y_2^{b+n} y_4^{c+m} 1_{32^{(2)}1}\right]} & & & & & \\
 & & & & 0 & \longrightarrow & q^{-1} R_\theta 1_{32^{(2)}1} & \longrightarrow & 0
\end{array}$$

so that $M_2^{\pi,\sigma,\tau}(z_1^a z_2^b z_3^c \otimes z_1^m z_2^n) = z_1^{a+c+m} z_2^{b+n}$.

Denote the left- and right-hand sides of (5.9) by L and R, respectively. We apply L and R to two elements: $z_1 \otimes 1 \otimes 1$ and $z_2 \otimes 1 \otimes 1$. Note that the map

$$G_2^{\pi,\rho,\sigma} : q^{-1}\mathcal{Z}_2 \otimes q^{-1}\mathcal{Z}_1 \cong \mathcal{E}_\theta^1(\rho,\sigma) \otimes \mathcal{E}_\theta^1(\pi,\rho) \to \mathcal{E}_\theta^1(\pi,\sigma) \cong \mathcal{Z}_2,$$

is a KLR degree 0 map, so we must have $G_2^{\pi,\rho,\sigma}(1 \otimes 1) = 0$. Thus, $R(z_1 \otimes 1 \otimes 1) = R(z_2 \otimes 1 \otimes 1) = 0$. On the other hand, we have $M_3^{\pi,\rho,\sigma,\tau}(z_1 \otimes 1 \otimes 1) = 1$, $M_3^{\pi,\rho,\sigma,\tau}(z_2 \otimes 1 \otimes 1) = 0$, and $M_2^{\rho,\sigma,\tau}(z_1 \otimes 1) = M_2^{\rho,\sigma,\tau}(z_2 \otimes 1) = z_1$, so by (5.9), we have

$$\begin{aligned}
0 = R(z_1 \otimes 1 \otimes 1) = L(z_1 \otimes 1 \otimes 1) = G_2(z_1 \otimes 1) + 1, \\
0 = R(z_2 \otimes 1 \otimes 1) = L(z_2 \otimes 1 \otimes 1) = G_2(z_1 \otimes 1),
\end{aligned}$$

a clear contradiction. □

References

[1] J. Brundan, A. Kleshchev, and P. J. McNamara. Homological properties of finite type Khovanov-Lauda-Rouquier algebras. *Duke Math. J.*, 163:1353–1404, 2014.

[2] D. Buursma, A. Kleshchev, and D. J. Steinberg. Resolutions of standard modules over KLR algebras in type A. 2019.

[3] T. Kadeishvili. On the homology theory of fiber spaces. *Uspekhi Mat. Nauk*, 35:183–188, 1980. Translated in Russ. Math. Surv. 35:231-238, 1980.

[4] S. Kato. Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras. *Duke Math. J.*, 163:619–663, 2014.

[5] B. Keller. Introduction to A-infinity algebras and modules. *Homology, Homotopy Appl.*, 3(1):1–35, 2001.

[6] B. Keller. a-infinity algebras in representation theory. *Representations of Algebras*, I:64–86, 2002.

[7] M. Khovanov and A. Lauda. A diagrammatic approach to categorification of quantum groups I. *Represent. Theory*, 13:309–347, 2009.

[8] M. Khovanov, A. Lauda, M. Mackaay, and M. Stošić. Extended graphical calculus for categorified quantum sl(2). *Mem. Amer. Math. Soc.*, 219(1029), 2012.

[9] A. Kleshchev. Affine highest weight categories and affine quasihereditary algebras. *Proc. Lond. Math. Soc.*, 110:841–882, 2015.

[10] A. Kleshchev and J. Loubert. Affine cellularity of Khovanov-Lauda-Rouquier algebras in finite types. *Int. Math. Res. Not.*, 14:5659–5709, 2015.

[11] A. Kleshchev and D. J. Steinberg. Homomorphisms between standard modules over finite-type KLR algebras. *Compos. Math.*, 153:621–646, 2017.

[12] S. Koenig, J. Külshammer, and S. Ovsienko. Quasi-hereditary algebras, exact Borel subalgebras, A_∞-categories and boxes. *Adv. Math.*, 262:546–592, 2014.

[13] D. Laksov, A. Lascoux, P. Pragacz, and A. Thorup. The LLPT notes, 1995.

[14] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Clarendon Press, 1995.

[15] L. Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci.* Société Mathématique de France, 1998.

[16] R. Rouquier. 2-Kac-Moody algebras. 2009. arXiv:0812.5023.

[17] C. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.
[18] G. Williamson. On an analogue of the James conjecture. *Represent. Theory*, 18:15–27, 2014.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: dbuursma@uoregon.edu

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: klesh@uoregon.edu

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: dsteinbe@uoregon.edu