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Blow-up in a network mutualistic model ☆

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ABSTRACT

A graph Laplacian reaction–diffusion system is introduced to describe a network mutualistic model of population ecology. By the approach of upper and lower solutions, we show that the strong mutualistic system occurs blow-up if the intrinsic growth rates of population are large.

1. Introduction

Reaction–diffusion systems on complex networks have been used to study population ecology [1–3]. Actually a network is mathematically a graph $G = (V, E)$ consisting of vertices $V = \{1, 2, \ldots, n\}$ and edges E connecting them. If V has finite vertices, G is called finite graph. If vertex y is adjacent to vertex x, we write $y \sim x$. A graph is weighted if each adjacent x and y is assigned a weight function ω_{xy}. Here $\omega : V \times V \to [0, \infty)$ satisfies that $\omega_{xy} = \omega_{yx}$ and $\omega_{xy} > 0$ if and only if $x \sim y$. By considering G be the connected finite graph, we extend some conceptions from continuous space to graph as follows:

$$d_\omega x := \sum_{y\sim x,\ y\in V} \omega(x, y) \tag{1.1}$$

$$\int_V f d_\omega \ (\text{or simply} \int_V f) := \sum_{x\in V} f(x) d_\omega x \tag{1.2}$$

$$D_{\omega,y} f(x) := (f(y) - f(x))\sqrt{\frac{\omega(x, y)}{d_\omega x}} \tag{1.3}$$

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$$\nabla_\omega f(x) := \quad \left(D_{\omega,y} f(x)\right)_{y\sim x,\ y\in V} \tag{1.4}$$

$$\Delta_\omega f(x) := \sum_{y\sim x} (f(y) - f(x)) \frac{\omega(x,y)}{d_\omega x} \tag{1.5}$$

where $f: V \to \mathbb{R}$, (1.1)–(1.5) are called graph differential, graph integral, graph directional derivative, graph gradient, and graph Laplacian, respectively.

For population dynamical models on networks, various methods and techniques have been used to study the existence and qualitative properties of solutions [4–10]. We develop the upper and lower solutions to deal with the blow-up problems on networks. We consider a mutualistic model on complex networks:

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta_\omega u_1 = u_1(a_1 - b_1 u_1 + c_1 u_2), & (x,t) \in V \times (0,+\infty), \\ \frac{\partial u_2}{\partial t} - d_2 \Delta_\omega u_2 = u_2(a_2 + b_2 u_1 - c_2 u_2), & (x,t) \in V \times (0,+\infty), \\ u_1(x,0) = u_{10}(x) \geq (\not\equiv) 0,\ u_2(x,0) = u_{20}(x) \geq (\not\equiv) 0, & x \in V. \end{cases} \tag{1.6}$$

When $b_1 c_2 < b_2 c_1$, this system is called strong mutualistic. For classical Laplacian diffusion system, the strong mutualistic population dynamical system occurs blow-up if the intrinsic growth rates of population are large or the initial data is sufficiently large [11]. Our main aim is to extend the results of classical Laplacian diffusion system to networks.

2. Preliminaries

Lemma 2.1 (*Green Formula*)**.** *For any pair of functions $f: V \to \mathbb{R}$ and $g: V \to \mathbb{R}$, the graph Laplacian Δ_ω satisfies that*

$$2\int_V f(-\Delta_\omega) g = \int_V \nabla_\omega f \cdot \nabla_\omega g = 2\int_V g(-\Delta_\omega) f. \tag{2.1}$$

In particular, in the case $f = g$, we have

$$2\int_V f(-\Delta_\omega) f = \int_V |\nabla_\omega f|^2. \tag{2.2}$$

Proof. In view of (1.3) and (1.5), we have

$$\begin{aligned} \int_V \nabla_\omega f \cdot \nabla_\omega g &= \int_V \sum_{y\sim x, y\in V} (f(y) - f(x)) \sqrt{\frac{\omega(x,y)}{d_\omega x}} (g(y) - g(x)) \sqrt{\frac{\omega(x,y)}{d_\omega x}} \\ &= \sum_{x\in V} \sum_{y\sim x, y\in V} (f(y) - f(x))(g(y) - g(x)) \omega(x,y) \\ &= 2 \sum_{x,y\in V} (f(y) - f(x))(g(y) - g(x)) \omega(x,y). \end{aligned} \tag{2.3}$$

Consequently, combining (1.5) and (2.3) yields

$$\begin{aligned} \int_V f(-\Delta_\omega) g &= -\int_V f(x) \sum_{y\sim x, y\in V} (g(y) - g(x)) \frac{\omega(x,y)}{d_\omega x} \\ &= \sum_{x\in V} \sum_{y\sim x, y\in V} (-f(x))(g(y) - g(x)) \omega(x,y) \\ &= \sum_{x,y\in V} (f(y) - f(x))(g(y) - g(x)) \omega(x,y) = \int_V g(-\Delta_\omega) f. \end{aligned} \tag{2.4}$$

The two equalities (2.3) and (2.4) complete the proof. □

Lemma 2.2. *Consider the eigenvalue problem*

$$\begin{cases} -\Delta_\omega \phi(x) = \lambda\phi(x), & x \in V, \\ \sum_{x\in V} \phi(x) d_\omega x = 1. \end{cases} \tag{2.5}$$

There exists

$$\lambda_1 = \min_{u\neq 0} \frac{\int_V |\nabla_\omega u|^2}{2\int_V u^2}, \text{ here } u: \ V \to \mathbb{R} \tag{2.6}$$

and $\Phi_1(x) > 0$ satisfies the above equation. λ_1 and Φ_1 are called the first eigenvalue and eigenfunction of (2.5). Moreover, $\lambda_1 = 0$.

Proof. Let us multiply the first equation of (2.5) by u, and integrate over V. We have $-\int_V u\Delta_\omega u = \int_V \lambda u^2$. By (2.2), we obtain $\lambda = \frac{\int_V |\nabla_\omega u|^2}{2\int_V u^2}$. Hence we deduce that $\lambda_1 = \min_{u\neq 0} \frac{\int_V |\nabla_\omega u|^2}{2\int_V u^2}$ where the minimum can be attained by taking $u(x) = \frac{1}{\sum_{x\in V} d_\omega x}$. Therefore by taking $\lambda_1 = 0$ and $\Phi_1 = \frac{1}{\sum_{x\in V} d_\omega x}$ such that λ_1 and Φ_1 satisfy (2.5). □

Definition 2.1. Assume that $\hat{u}_1$ and $\hat{u}_2$ are continuous with respect to t in $V \times [0, T]$, and differentiable with respect to t in $V \times (0, T]$. If $(\hat{u}_1, \hat{u}_2)$ satisfies

$$\begin{cases} \frac{\partial \hat{u}_1}{\partial t} - d_1\Delta_\omega \hat{u}_1 \le \hat{u}_1(a_1 - b_1\hat{u}_1 + c_1\hat{u}_2), & (x,t) \in V \times (0, T], \\ \frac{\partial \hat{u}_2}{\partial t} - d_2\Delta_\omega \hat{u}_2 \le \hat{u}_2(a_2 + b_2\hat{u}_1 - c_2\hat{u}_2), & (x,t) \in V \times (0, T], \\ \hat{u}_1(x,0) \le u_{10}(x), \ \hat{u}_2(x,0) \le u_{20}(x), & x \in V, \end{cases} \tag{2.7}$$

$(\hat{u}_1, \hat{u}_2)$ is called a lower solution of (1.6). Moreover, if $(\tilde{u}_1, \tilde{u}_2)$ satisfies (2.7) by reversing all the inequalities, $(\tilde{u}_1, \tilde{u}_2)$ is called an upper solution of (1.6).

Lemma 2.3 (*Comparison Principle*)**.** *Assume that (u_1, u_2) is a solution of (1.6). If $(\hat{u}_1, \hat{u}_2)$ is a lower solution of (1.6), then $(u_1, u_2) \ge (\hat{u}_1, \hat{u}_2)$ in $V \times [0, T]$.*

Proof. By setting $z_1 = (u_1 - \hat{u}_1)e^{-Kt}$, $z_2 = (u_2 - \hat{u}_2)e^{-Kt}$, we have

$$\frac{\partial z_1}{\partial t} - d_1\Delta_\omega z_1 \ge (-K + b_{11})z_1 + b_{12}z_2, \text{ for } (x,t) \in V \times (0, T], \tag{2.8}$$

$$\frac{\partial z_2}{\partial t} - d_2\Delta_\omega z_2 \ge b_{21}z_1 + (-K + b_{22})z_2, \text{ for } (x,t) \in V \times (0, T], \tag{2.9}$$

where

$$\begin{aligned} b_{11} &= a_1 - b_1(u_1 + \hat{u}_1) + c_1u_2, \ b_{12} = c_1\hat{u}_1, \\ b_{21} &= b_2\hat{u}_2, \ b_{22} = a_2 - c_2(u_2 + \hat{u}_2) + b_2u_1. \end{aligned} \tag{2.10}$$

Notice that $z_i(x,t)$ $(i = 1, 2)$ are continuous on $[0, T]$ for each $x \in V$ and V is finite, we can find $(x_0, t_0) \in V \times [0, T]$ such that

$$z_1(x_0, t_0) = \min_{x\in V} \min_{t\in[0,T]} z_1(x,t).$$

The above equation implies

$$z_1(x_0, t_0) \le z_1(y, t_0), \text{ for any } y \in V.$$

In view of the definition of Δ_ω, we have

$$\Delta_\omega z_1(x_0, t_0) \ge 0. \tag{2.11}$$

Meanwhile it follows from the differentiability of $z_1(x,t)$ in $(0,T]$ that

$$\frac{\partial z_1}{\partial t}(x_0,t_0) \le 0, \text{ for } i=1,2. \tag{2.12}$$

By substituting (2.11) and (2.12) into (2.8), we have

$$\left((-K+b_{11})z_1 + b_{12}z_2\right)(x_0,t_0) \le 0. \tag{2.13}$$

Suppose that $z_1(x_0,t_0) = -\delta < 0$ on the contrary. By choosing

$$K = \frac{|z_2(x_0,t_0)|}{\delta}|b_{12}(x_0,t_0)| + |b_{11}(x_0,t_0)| + 1,$$

we deduce that $\left((-K+b_{11})z_1 + b_{12}z_2\right)(x_0,t_0) > 0$, which contradicts (2.13). Hence, we have $z_1(x_0,t_0) \ge 0$, which means $\min_{x\in V}\min_{t\in[0,T]} z_1(x,t) \ge 0$. Therefore, we obtain $z_1(x,t) \ge 0$ for $(x,t) \in V \times [0,T]$. On the other hand, by employing a similar argument to (2.9) we can obtain $z_2(x,t) \ge 0$ for $(x,t) \in V \times [0,T]$. The above conclusions imply that

$$u_i \ge \hat{u}_i \ (i=1,2) \text{ for } (x,t) \in V \times [0,T]. \quad \square \tag{2.14}$$

3. Blow-up

Theorem 3.1. *Let $w(x,t)$ be a solution of*

$$\begin{cases} d\frac{\partial w}{\partial t} - \Delta_\omega w = w(a+bw), & (x,t) \in V \times (0,T], \\ w(x,0) \ge (\not\equiv) 0, & x \in V, \end{cases} \tag{3.1}$$

where d, a, and b are constants and satisfying $d > 0$ and $b \ge 0$. Then we have the following blowup properties:

(i) If $a \ge \lambda_1$, the solution of (3.1) blows up for any nontrivial initial data.
(ii) If $a < \lambda_1$, the solution of (3.1) blows up for large enough initial data.

Proof. We define $F(t) = \int_V \Phi_1(x)w(x,t)$. Deriving $F(t)$ with respect to t and using (3.1), we have

$$\begin{aligned} dF'(t) &= \int_V \Phi_1(x)\big(\Delta_\omega w + w(a+bw)\big) \\ &= \int_V w\Delta_\omega \Phi_1(x) + \int_V \Phi_1(x)w(a+bw) \qquad (3.2) \\ &= (a-\lambda_1)F(t) + b\int_V \Phi_1(x)w^2 \\ &= (a-\lambda_1)F(t) + b\int_V \Phi_1(x)w^2 \int_V \Phi_1(x) \\ &\ge (a-\lambda_1)F(t) + bF^2(t), \qquad (3.3) \end{aligned}$$

here (3.2) is due to the Green Formula of Lemma 2.1, (3.3) follows from Hölder's inequality.

(i) In the case $a \ge \lambda_1$, from (3.3), we immediately obtain that $F(t)$ blows up for any nontrivial initial data.

(ii) In the case $a < \lambda_1$, we can choose sufficiently large initial data $w(x,0)$ such that

$$F(0) = \int_V \Phi_1(x)w(x,0) > \frac{\lambda_1 - a}{b}.$$

It follows from (3.3) that $F(t)$ blows up for sufficiently large initial data. $\square$

Theorem 3.2. *If $b_1c_2 < b_2c_1$ holds, then we have the following blowup properties: If* $\min\left\{\frac{a_1}{d_1}, \frac{a_2}{d_2}\right\} \geq 1$*, the solution of (1.6) blows up for any nontrivial initial data satisfying that for all* $x \in V$*,* $\min\{u_{10}(x), u_{20}(x)\} \not\equiv 0$.

Proof. We define $(\hat{u}_1(x,t), \hat{u}_2(x,t)) = (\delta_1 w(x,t), \delta_2 w(x,t))$. In order to guarantee $(\hat{u}_1(x,t), \hat{u}_2(x,t))$ to be a lower solution of (1.6), we need to show $(\hat{u}_1(x,0), \hat{u}_2(x,0)) \leq (u_{10}(x), u_{20}(x))$ and

$$\begin{cases} \frac{\partial w}{\partial t} - d_1 \Delta_\omega w \leq w(a_1 - b_1\delta_1 w + c_1\delta_2 w), & (x,t) \in V \times (0,T], \\ \frac{\partial w}{\partial t} - d_2 \Delta_\omega w \leq w(a_2 + b_2\delta_1 w - c_2\delta_2 w), & (x,t) \in V \times (0,T]. \end{cases} \tag{3.4}$$

Since $b_1c_2 < b_2c_1$, we have $\frac{b_1}{c_1} < \frac{b_2}{c_2}$. By setting sufficiently small positive δ_1, δ_2, for example, $\delta_1 = \varepsilon$, $\delta_2 = \frac{1}{2}(\frac{b_2}{c_2} + \frac{b_1}{c_1})\varepsilon$ such that

$$\begin{cases} -b_1\delta_1 + c_1\delta_2 > 0, \\ b_2\delta_1 - c_2\delta_2 > 0. \end{cases} \tag{3.5}$$

We set

$$d = \max\left\{{d_1}^{-1}, {d_2}^{-1}\right\}, \ a = \min\left\{\frac{a_1}{d_1}, \frac{a_2}{d_2}\right\}, \ b = \min\left\{\frac{-b_1\delta_1 + c_1\delta_2}{d_1}, \frac{b_2\delta_1 - c_2\delta_2}{d_2}\right\}. \tag{3.6}$$

To show (3.4), it suffices to show

$$\begin{cases} d_1^{-1}\frac{\partial w}{\partial t} - \Delta_\omega w \leq w(a + bw), & (x,t) \in V \times (0,T], \\ d_2^{-1}\frac{\partial w}{\partial t} - \Delta_\omega w \leq w(a + bw), & (x,t) \in V \times (0,T]. \end{cases} \tag{3.7}$$

Hence we only need to show if w is a solution of

$$\begin{cases} d\frac{\partial w}{\partial t} - \Delta_\omega w = w(a + bw), & (x,t) \in V \times (0,T], \\ w(x,0) \geq (\not\equiv) 0, & t \in V, \end{cases} \tag{3.8}$$

then w satisfies $\frac{\partial w}{\partial t} \geq 0$ for $t \in (0,T]$. By applying comparison principle (Lemma 2.3), we can obtain that $w(x,t) \geq 0$ for $t \in (0,T]$ and $x \in V$. Then using the definition of Δ_ω in (1.5),

$$\begin{aligned} \Delta_\omega w(x,t) &= \sum_{y\sim x}(w(y,t) - w(x,t))\frac{\omega(x,y)}{d_\omega x} \\ &\geq \sum_{y\sim x} -w(x,t)\frac{\omega(x,y)}{d_\omega x} = -w(x,t)\sum_{y\sim x}\frac{\omega(x,y)}{d_\omega x} = -w(x,t). \end{aligned} \tag{3.9}$$

In view of the condition $a \geq 1$ and Lemma 2.2 $\lambda_1 = 0$, we have $a \geq \max\{\lambda_1, 1\}$. For $t \in (0,T]$, we can compute

$$d\frac{\partial w}{\partial t}(x,t) = \Delta_\omega w(x,t) + w(x,t)(a + bw(x,t)) \geq 0. \tag{3.10}$$

Hence $(\hat{u}_1(x,t), \hat{u}_2(x,t))$ is a lower solution of (1.6) provided $(\hat{u}_1(x,0), \hat{u}_2(x,0)) \leq (u_{10}(x), u_{20}(x))$. We choose the initial data of w be the function $w(x,0) = \min\{u_{10}(x), u_{20}(x)\}$ and ε enough small such that $\delta_1 < 1$ and $\delta_2 < 1$, which guarantees that $(\hat{u}_1(x,0), \hat{u}_2(x,0)) \leq (u_{10}(x), u_{20}(x))$ holds. By applying Theorem 3.1 to w, we have $(\hat{u}_1(x,t), \hat{u}_2(x,t))$ blows up. By Lemma 2.3, the system (1.6) blows up. □

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