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TIME DEPENDENT CENTER MANIFOLD IN PDES

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Abstract. We consider externally forced equations in an evolution form. Mathematically, these are skew systems driven by a finite dimensional dynamical system. Two very common cases included in our treatment are quasi-periodic forcing and forcing by a stochastic process. We allow that the evolution is a PDE and even that it is not well-posed and that it does not define a flow (not all initial conditions lead to a solution).

We first establish a general abstract theorem which, under suitable (spectral, non-degeneracy, smoothness, etc) assumptions, establishes the existence of a "time-dependent invariant manifold" (TDIM). These manifolds evolve with the forcing. They are such that the original equation is always tangent to a vector field in the manifold. Hence, for initial data in the TDIM, the original equation is equivalent to an ordinary differential equation. This allows us to define families of solutions of the full equation by studying the solutions of a finite dimensional system. Note that this strategy may apply even if the original equation is ill posed and does not admit solutions for arbitrary initial conditions (the TDIM selects initial conditions for which solutions exist). It also allows that the TDIM is infinite dimensional.

Secondly, we construct the center manifold for skew systems driven by the external forcing.

Thirdly, we present concrete applications of the abstract result to the differential equations whose linear operators are exponential trichotomy subject to quasi-periodic perturbations. The use of TDIM allows us to establish the existence of quasi-periodic solutions and to study the effect of resonances.

1. **Introduction and organization of the paper.** The goals of this paper are to present two "reduction principles" for some abstract evolution equations subject to quasi-periodic forcing or forcings with some more complicated time dependence, including forcing by stochastic processes. The main novelty is that we do not need to

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assume that the equation is well posed nor that it defines an evolution. We assume only the existence of forwards or backwards evolutions in subspaces. This forbids to use methods such as graph transform that assume the existence of the evolution. Furthermore, the functional equations satisfied by the parameterization of the center manifold need to capture not only that it is an invariant set, but also that it consists of points where the evolution can be defined. We will construct some smooth families of special solutions that are obtained by studying functional equations for invariant manifolds. These invariant manifolds will be finite dimensional in some examples. In them, we can define the dynamics.

In the spirit of the reduction results, we show that the dynamics of the full system can be understood in terms of the dynamics in the reduced system in the center manifold. Roughly, the complicated dynamics can only happen in the reduced system and all the other solutions converge – exponentially fast with a large rate – to the solutions of the reduced system either in the future or in the past.

The reduction principles were studied originally for finite dimensional systems in [46]. In general, reduction principles state that the bounded solution of a (possibly infinite dimensional) problem can be found in a manifold of smaller dimensions. When the reduced manifold is finite dimensional, the restriction of the system to the invariant manifold may be studied with the techniques from ordinary differential equations even if the original problem is infinite dimensional. In modern times reduction principles are formulated as the existence of a center manifold and proved by different methods [11, 34, 37, 43, 2, 35, 4, 28, 9, 33, 53, 7, 41, 16, 17, 29] and, closer to our point of view, especially [15, 24].

To obtain an invariant manifold, we derive a functional equation for the time-dependent equation and transform it into a fixed point problem. This is very different from the graph transform method since we do not need to involve the evolution (which may fail to exist in some of the examples we consider). Indeed, one of the complications is that we need to ensure that the invariant manifolds consist of initial conditions for which the evolution can be defined.

The application of these reduction principles allows us to study the effect of resonances by using methods of ordinary differential equations and find many different types of solutions besides the quasi-periodic ones. For example, we can produce chaotic solutions or quasi-periodic solutions. As we will see, reduction principles can be applied also to stochastic forcings of PDEs so that one can apply Fokker-Plank equation's methods (even if the original phase space is infinite dimensional). Invariant manifold for stochastic perturbation have been considered in [4], which also includes an extensive review of existing literature.

The main novelty of this paper is that we consider the situation where the evolution equation is subject to external forces and we obtain a non-autonomous center manifold. The external forces we consider can be very general, they can be either a deterministic system (for example quasi-periodic forcing) or a stochastic process satisfying some mild conditions. Even our conditions do not apply to general addition of white noise, they apply to Ornstein-Uhlenbeck processes. Of course, in the case of smooth forcing we obtain more regularity of the manifold. In the quasi-periodic case, we can use the time-dependent center manifold to study the effect of resonances of the external forcing using methods of finite dimensional system.

It is important to note that we do not need to assume that the equations we consider are well-posed nor define an evolution. We assume only that the linearized equations define forward and backwards evolution in the corresponding closed subspaces (these closed subspaces span the whole space and have a finite dimensional intersection). We will conclude that there is a finite dimensional manifold in which the evolution of the full equation can be defined. We also allow that the nonlinear terms are unbounded, but we require that they are lower order than the linear terms. We can think that our result is a perturbation of the dynamics of the linearized system.

1.1. **Some motivating examples.** Now we give some applications as motivations for this work, which we explain in detail in Section 4. Of course, the applications mentioned here are only for motivation and they can be skipped without affecting the reading of the paper. The list of applications is certainly not meant to be an exhaustive list, indeed, we are hoping to extend the techniques to other problems such as state dependent delay equations, mean field games, which share some of the problems of ill-posed equations.

The first example to keep in mind is complex Ginzburg-Landau equation. More concretely, the external forced complex Ginzburg-Landau equation (henceforth CGL)

$$\begin{cases} \theta_t = B(\theta), \\ u_t = ru + (b_1 + \mathrm{i}b_2)\Delta u + N_{10}(\theta, x) + N_{11}(\theta, x)u + N_{12}(\theta, x, u), \\ t \in \mathbb{R}, \quad \theta \in \mathscr{C}, \;\; x \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d, \end{cases} \tag{1.1}$$

and the derivative complex Ginzburg-Landau equation (henceforth DCGL)

$$\begin{cases} \theta_t = B(\theta), \\ u_t = ru + (b_1 + \mathrm{i}b_2)\Delta u + N_{20}(\theta, x) + N_{21}(\theta, x)u + N_{22}(\theta, x, u, \triangledown u), \\ t \in \mathbb{R}, \quad \theta \in \mathscr{C}, \quad x \in \mathbb{T}^d, \end{cases} \tag{1.2}$$

where $d \geq 1$ and the functions $u(t, x, \theta)$ are unknown, $N_{i,0}$(resp. $N_{i,1}u$), $i = 1, 2$, are the inhomogeneous perturbations (resp. linear perturbations). Moreover, we assume N_{12} (resp. N_{22}) is higher (at least two) order in u (resp. $(u, \triangledown u)$), that is $N_{12}(\theta, x, 0) = \partial_3 N_{12}(\theta, x, 0) = 0$ and

$$N_{22}(\theta, x, 0, 0) = \partial_3 N_{22}(\theta, x, 0, 0) = \partial_4 N_{22}(\theta, x, 0, 0) = 0.$$

Here $\mathscr{C}$ is a manifold and B a vector field in it. A very important particular case to keep in mind is $\mathscr{C} = \mathbb{T}^d$ and $B(\theta) = \omega$.

The second and third models of this paper are the externally forced Boussinesq equation

$$\begin{cases} \theta_t = B(\theta), \\ u_{tt} = \mu\Delta^2 u + \Delta u + N_{30}(\theta, x) + N_{31}(\theta, x)u + N_{32}(\theta, x, u, \triangledown u, \Delta u), \\ t \in \mathbb{R}, \quad \theta \in \mathscr{C}, \quad x \in \mathbb{T}^d, \end{cases} \tag{1.3}$$

and the external forced nonlinear elliptic differential equation defined on a cylindrical domain

$$\begin{cases} \theta_t = B(\theta), \\ u_{tt} = -\Delta u - \alpha u + N_{40}(\theta, x) + N_{41}(\theta, x)u + N_{42}(\theta, x, u, \triangledown u), \\ t \in \mathbb{R}, \quad \theta \in \mathscr{C}, \quad x \in \mathbb{T}^d, \end{cases} \tag{1.4}$$

where $d \geq 1$ and the functions $u(t, x, \theta)$ are unknown, N_{32} (resp. N_{42}) is higher (at least two) order in $(u, \triangledown u, \Delta u)$ (resp. $(u, \triangledown u)$).

The reason that we separate the general complex Ginzburg-Landau equation into the systems (1.1) and (1.2) is that we want to stress the term $\triangledown u$ in (1.2), which makes perturbation N_{22} unbounded (in the sense of operator theory). It is well known that the unbounded perturbation will make the system itself more complicated and we have to do more work such as apply the *"two spaces approach"* to deal with this term. The functions $N_{i,j}, i = 1, 2, 3, 4, j = 0, 1, 2$, are assumed to be C^r from a Banach space X to another space Y (that satisfy some conditions).

Besides the infinite dimensional examples, mentioned above, we want to mention a finite dimensional example that served us as motivation. The study of chemical reactions in vibrating molecules can be understood extending the *transition state theory* in time-dependent systems [5]. In [54], one can find a reformulation of the time dependent transition state theory in a formalism similar to that of this paper – it is, of course, much simpler, since the models in [54] are just ODE's. The paper [54] includes numerical implementations of the formalism of these paper. These implementations are very efficient compared with methods based on normal forms. It would be very interesting to develop numerical implementations of the infinite dimensional problems mentioned above.

1.2. **Some informal description of the results.** The linear operators in the four differential equations above just have the discrete spectrum, but this is not essential. The general results Theorems 3.1 and 3.2 allow that the operators have continuous spectrum.

The essential aspect about the spectrum of the examples above is that it can be divided into three parts: One that allows to take exponentials in the future, another that can take exponentials in the past, and another one that allows to take exponentials for all time. These regions have well separated rates. This is formulated more precisely as the *exponential trichotomy*. See Assumption **(H2.2)** and (3.1), (3.2).

For example, the external forced nonlinear elliptic differential equation (1.4) can be replaced by

$$\begin{cases} \theta_t = B(\theta), \\ u_{tt} = -\partial^2_{xx} u - \alpha u + V(t,x)u + N(\theta, x, u, u_x), \ x \in \mathbb{R}^n, \end{cases}$$

where $\alpha \le 0$, $V(t,x) = W(x) + T(t,x)$ with W such that $-\partial^2_{xx} - \alpha + W$ has a finite number of bound states and the rest of the spectrum is positive (See [47] for sufficient conditions for W). The term T is part of the perturbations we consider. Similar modifications can be done to the other three examples above. We omit the details.

In (1.1)-(1.4), if we assume the compact manifold $\mathscr{C} = \mathbb{T}^d$, this means we allow these equations to be subject to the quasi-periodic forcing. On the other hand, we want to cover Brownian motions, and in this case natural manifold $\mathscr{C} = \mathbb{R}^d$, is not compact since, it is well known, the Brownian motions, generally, will explore all phase space. That is there are large excursions.

Moreover, in both cases we will assume that the perturbations in the four equations above belong to $C^r(\mathscr{C}, (C^r(X, Y))$, *i.e.*, $N_{i,j} \in C^r(\mathscr{C}, (C^r(X, Y))$, $i = 1, 2, 3, 4$, $j = 0, 1, 2$, where X, Y are suitable Banach spaces which ensure that the C^r norms of $N_{i,j}$, $i = 1, 2, 3, 4, j = 0, 1, 2$, are bounded.

For the Brownian motions case, we know that for all l with $0 \le l \le r$, the $l-$order derivatives of the perturbations are functions that depict the Brownian motions. Moreover, by the definition of C^r space (see Definition 3 for details), we know that these $l-$order derivatives are uniformly bounded. That is, we only include Brownian motion in a forcing provided that the large excursions do not have too much effect. This happens, for example, in mechanical systems where the white noise is added to the velocity (and not the position); this is the Orstein-Uhlenbeck model.

In models for which this suppression of large effects is not true, we can introduce cut-offs (such as in the prepared equation (2.11)). The cut-off model will agree with the real model except for low probability events. In this paper, we will not consider these probabilistic considerations and consider only the geometric aspects of the well behaved models.

Informally, the result that will be proved is that when we consider the full equation, there is manifold, – analogue to the center space of the linear approximation – in which the dynamics can be defined. The dynamics of the full equation remains in the manifold, which will be referred as an invariant manifold.

The fact that the dynamics is non-autonomous, leads to the fact that the appropriate center manifold of the linear problem is really a family of center spaces so that the forcing moves along them. Of course, due to the unbounded nature of the perturbation and the fact that we will need to ensure that the dynamics is defined, we will need a sophisticated argument that requires some technical assumptions. The technical assumptions will be presented in Section 2 and the statement of the main results is in Section 3.

1.3. **Organization of this paper.**

This paper is organized as follows: In section 2, we present an informal formulation of the invariant manifold problem and introduce two different partial differential equations, one with smooth forcing and the other with stochastic forcing. The cases of deterministic forcing and of stochastic forcing lead to the same fixed point equations. The differences between smooth forcing and stochastic forcing appears mainly in the regularity assumed on the dependence on the forcing variables and the regularity obtained for the manifold. In section 3, we formulate precise hypotheses and present our main results. Then we formulate equations that are equivalent to the invariance and transform them into a fixed point problem, by proving this operator is a contraction we give the proof of our main result, Theorem 3.1. By Hadamard interpolation theorem (Theorem 1.1 in the Appendix), we get an improved estimate on the regularity of the fixed point. Finally, for the case of smooth forcing, we discuss the regularity with respect to θ of the manifold. In section 4, we formulate and prove the results for the external forced nonlinear elliptic differential equations (1.4).

2. **Overview of the argument.**

We will consider the following non-autonomous system

$$\frac{d}{dt}y = \mathscr{X}(\theta, y), \tag{2.1}$$

where y belongs to a Banach space X, and $\theta \in \mathscr{C}$, is the configuration space of external forcings which evolve under its own law of evolution independent of y. The vector field $\mathscr{X}$, taking values in X, may be a partial differential operator, so that questions of domains, existence of the evolution are subtle. The precise assumptions will be formulated after being motivated by the informal presentation here.

We will assume that the equation (2.1) is basically linear. That is

$$\mathscr{X}(\theta, y) = \mathscr{A}y + f(\theta, y) + N(\theta, y) \tag{2.2}$$

where $\mathscr{A}$ is a linear operator (possibly unbounded) and for a fixed θ we assume $f(\theta,\cdot)$ and $N(\theta,\cdot)$ are non-linear operators which could also be unbounded. Following [34], we will assume that $f(\theta,\cdot)$ and $N(\theta,\cdot)$ are differentiable from the space X to another space Y with X consisting of smooth functions and Y constituting of less smooth functions. More detailed assumptions on f and N will be given later.

We will not assume that the linear equation

$$\frac{d}{dt}y = \mathscr{A}y \tag{2.3}$$

defines an evolution, but only that there are spaces where (2.3) defines an evolution in the future or in the past and a subspace where (2.3) defines an evolution for all time. We call these spaces as stable, unstable and center subspaces respectively (very often the center subspace is finite dimensional). Note that the equation (2.3) is a constant equation, so we can use the techniques of semigroup theory [45, 30] to study the existence of these partial evolutions. See Section 2.1.

For the perturbations f and N, we will assume that for fixed $\theta \in \mathscr{C}$, $N(\theta,0)=0$, $D_2N(\theta,0)=0$ and that $f(\theta,\cdot)$ has a small Lipschitz norm as a function from X to Y.

For the external forcing, there are two cases that are of particular interest:

A) The case of smooth forcing:
   In this case, the set $\mathscr{C}$ is a manifold and the evolution $\theta(t)$ is generated by a smooth differential equation
$$\frac{d}{dt}\Phi_t(\theta) = B(\Phi_t(\theta)), \quad \Phi_0(\theta)=\theta, \tag{2.4}$$
   where B is a smooth vector field. Hence, we can rewrite the equation (2.1) in an autonomous way:
$$\frac{d}{dt}(\theta,y) = \widetilde{\mathscr{X}}(\theta,y) = \big(B(\theta),\mathscr{X}(\theta,y)\big). \tag{2.5}$$
B) The case of stochastic forcing:
   In this case, the set $\mathscr{C}$ is a measure space with measure μ and the evolution Φ_t is a Markov stochastic process. The $\Phi_t(\theta)$ is only measurable in θ and it is continuous (or even less regular) in t. There will be no regularity assumed in θ beyond measurability and some boundedness. In this case the equation (2.1) can be rewritten as
$$\frac{d}{dt}y = \mathscr{X}(\Phi_t(\theta),y). \tag{2.6}$$

Note that in (2.6) the function $\mathscr{X}$ is much less regular in the time t than in x. The existence, uniqueness and regularity of solutions of equations with this regularity were studied systematically by Caratheodory and is, by now considered in textbooks [32, Theorem 5.3, Chapter 1].

The goal of this paper is to show that the space where solutions of (2.3) are defined for all time is analogous to the one on which the solutions of the full equation (2.1) are defined for all time. Namely, under suitable smallness assumptions on the nonlinear terms, there are manifolds W_{θ_0} in X such that the initial value problem for (2.5) with initial conditions $\theta(0)=\theta_0$ and $y(0)\in W_{\theta_0}$ has a solution defined for all time. We note, however, that assuming that y is given by an measurable path.

We recall that the definition of C^r spaces that we use in this paper is Definition 3, which includes that the function and its derivatives are uniformly bounded. This can be arranged by cut-off the functions. See Section 2.2 and the discussion of forcing by unbounded stochastic forcings such as Brownian motion.

Definition 1. Let Z be a Banach space. We say that a C^r function $\varphi : Z \to \mathbb{R}$ is a cut-off function when $\varphi(u) = 1$ when $||u||_Z \le 1$ and $\varphi(u) = 0$ when $\|u\|_Z \ge 2$ and all the derivatives of φ of order up to r are uniformly bounded.

See [Remark 3.28] [13] for more discussions about the cut-off functions. It is somewhat surprising that there are spaces that do not admit C^2 cut-off functions. Finite dimensional spaces or Hilbert spaces do admit cut-off functions but $C^r[0,1]$ does not.

2.1. **Invariant splittings of the linear part of the evolution equation and their semigroups.** In this section we present the assumptions on $\mathscr{A}$ in (2.2). We will assume that there is a decomposition

$$X = X_u \oplus X_c \oplus X_s, \tag{2.7}$$

where X_σ are closed subspaces, $\sigma = u, c, s$, which are invariant for $\mathscr{A}$. We allow the possibility that X_s or X_u are trivial. In this case, the center manifold theorem stated in this paper will become a slow invariant manifold theorem.

We will introduce the notations y_s, y_c, y_u to denote the components of a point $y \in X$ over the corresponding spaces and will not distinguish between y_σ as an element of X_σ and as an element of X. We denote by Π_σ the projection operator over X_σ associated to the splitting (2.7) and assume that the decomposition (2.7) is invariant under A, this means that

$$\mathscr{A}(D(\mathscr{A}) \bigcap X_\sigma) \subset X_\sigma \tag{2.8}$$

for $\sigma = s, c, u$. We will denote by $\mathscr{A}_\sigma$ the restriction of $\mathscr{A}$ to X_σ and not distinguish between $\mathscr{A}_\sigma$ as an operator taking values in X_σ or as an operator taking values in X. As a consequence of (2.8) we have that $D(\mathscr{A}_\sigma) = D(\mathscr{A}) \bigcap X_\sigma$.

Under the splitting assumption and using the notation for the splitting, the equation (2.1) can be rewritten as the system of equations as:

$$\begin{aligned} \frac{dy_s}{dt} &= \mathscr{A}_s y_s + f_s(\theta, y_s, y_c, y_u) + N_s(\theta, y_s, y_c, y_u), \\ \frac{dy_c}{dt} &= \mathscr{A}_c y_c + f_c(\theta, y_s, y_c, y_u) + N_c(\theta, y_s, y_c, y_u), \\ \frac{dy_u}{dt} &= \mathscr{A}_u y_u + f_u(\theta, y_s, y_c, y_u) + N_u(\theta, y_s, y_c, y_u), \end{aligned} \tag{2.9}$$

where $f_\sigma = \Pi_\sigma f$ and $N_\sigma = \Pi_\sigma N, \sigma = u, c, s$.

The most important assumption for us will be that

(T0):

- $\mathscr{A}_s$ defines a semigroup $\{e^{t\mathscr{A}_s}\}_{t\in\mathbb{R}^+}$ on X_s for positive time.
- $\mathscr{A}_u$ defines a semigroup $\{e^{t\mathscr{A}_u}\}_{t\in\mathbb{R}^-}$ on X_u for negative time.
- $\mathscr{A}_c$ defines a group $\{e^{t\mathscr{A}_c}\}_{t\in\mathbb{R}}$ on X_c for all times.

Later, we will make more precise assumptions which include that $e^{t\mathscr{A}_s}$, for $t \ge 0, e^{t\mathscr{A}_u}$ for $t \le 0$ are smoothing semigroups and that the contraction rates of the semigroups on different spaces are precisely related. See hypothesis **(H2.1)**, (3.1) and (3.2) for details.

We present some concrete hypotheses about the equation (2.1) whose vector field defined by (2.2) and on the space X. More hypotheses will be introduced later.

(T1): The functions $f, N \in C^r(\mathscr{C}, C^r(X,Y))$. Moreover, $\|f(\theta,\cdot)\|_{C^r(U,Y)}$ is small enough and

$$N(\theta,0)=0, DN(\theta,0)=0, \|N(\theta,\cdot)\|_{C^r(U,Y)} \leq L,$$

where L is a bounded number.

(T2): The space X_c admits C^r-cut-off functions.

2.2. **The prepared equations.** We assume the space X_c admits C^r-cut-off functions, *i.e.*, (**T2**), then for the full equation (2.9), the prepared equations ([40]) are obtained by scaling the variables y_σ,

$$y_\sigma = \epsilon \tilde{y}_\sigma, \quad \sigma = s, c, u \tag{2.10}$$

and by multiplying the cut-off function φ, *i.e.*, the prepared equation is:

$$\begin{aligned} \frac{d}{dt}\tilde{y}_s &= \mathscr{A}_s \tilde{y}_s + \epsilon^{-1} F_s(\theta, \epsilon\tilde{y}_s, \epsilon\tilde{y}_c, \epsilon\tilde{y}_u), \\ \frac{d}{dt}\tilde{y}_c &= \mathscr{A}_c \tilde{y}_c + \varphi(\tilde{y}_c)\epsilon^{-1} F_c(\theta, \epsilon\tilde{y}_s, \epsilon\tilde{y}_c, \epsilon\tilde{y}_u), \\ \frac{d}{dt}\tilde{y}_u &= \mathscr{A}_u \tilde{y}_u + \epsilon^{-1} F_u(\theta, \epsilon\tilde{y}_s, \epsilon\tilde{y}_c, \epsilon\tilde{y}_u), \end{aligned} \tag{2.11}$$

where we denote

$$F(\theta, y) \equiv f(\theta, y) + N(\theta, y). \tag{2.12}$$

By the definition of C^r-cut-off functions we know that $\varphi(u)=1$ when $||u||_{X_c} \leq 1$ and $\varphi(u)=0$ when $\|u\|_{X_c} \geq 2$, and all the derivatives of φ of order up to r are uniformly bounded. Thus, (2.11) is equivalent to the original equation, (2.9), when $\tilde{y}_c \in E_\epsilon(X_c) \equiv \{\tilde{y}_c : \ \|\tilde{y}_c\|_{X_c} < 1\}$ (that is $\|y_c\|_{X_c} < \epsilon$) under the change of variables (2.10).

Therefore, an invariant manifold for the prepared equations will be a locally invariant manifold for the original equations.

The cut-off function φ in (2.11) is not unique, consequently, the center manifold we construct is not unique even if it is unique after the prepared equations are chosen.

Of course, the existence of a cut-off function is a sufficient condition to be able to work with the prepared equations, but it is by no means necessary. In many cases, one can take advantage of the structure of the non-linearity to construct the prepared equations.

It is a simple calculation in [40] that, if we denote

$$\widetilde{F}(\tilde{y}_s, \tilde{y}_u, \tilde{y}_c) \equiv \epsilon^{-1}\big(F_s(\epsilon\tilde{y}_s, \epsilon\tilde{y}_c, \epsilon\tilde{y}_u), \varphi(\tilde{y}^c) F_c(\epsilon\tilde{y}_s, , \epsilon\tilde{y}_c \epsilon\tilde{y}_u), F_u(\epsilon\tilde{y}_s, \epsilon\tilde{y}_c, \epsilon\tilde{y}_u)\big),$$

with the functions f and N satisfying the hypothesis **(T1)**, then from (2.12) we know that

$$\|\widetilde{F}\|_{C^r(B(0,1,X),Y)} \tag{2.13}$$

is as small as desired for ϵ sufficiently small. The norm in (2.13) involves domains which are unit balls in the X_s, X_u spaces but the whole space in X_c. We refer to [40] for the details.

The results of [24] showed that for the prepared equation in the autonomous case, one can obtain a unique center manifold which is expressed as the graph of a function from X_c to $X_s \oplus X_u$. This center manifold is invariant under the prepared equations and is tangent to the center space at the origin. Similarly, in this paper we will obtain the time-dependent center manifold for the prepared equation (2.11).

With the similar tricks to get (3.12) from full equation (2.9) we can also get the evolution equation of (2.11) which has the same formula with (3.12) since we have replaced the perturbation of (2.11) with the one in (2.9). So we still use (3.12) to denote the evolution of the prepared equation.

2.3. **The case of smooth evolution of forcing.** We first consider the case when the external forcing is evolving according to a differential equation. For example, we may consider the effect of large celestial bodies on a small satellite, the effects of weather in a biological population or the effect of a heat bath in a molecule, etc. We formulate the notion of non-autonomous manifolds and reduce (formally) the existence of these invariant manifolds to a fixed point equation. (See (3.11)). We anticipate that the formulation makes perfect sense and it is natural in the case of stochastic forcing with some mild regularities, so the treatment of stochastic forcing will start also from the fixed point equation (3.11). Of course, in case that $\mathscr{C}$ is a manifold and that the evolution with respect to θ is smooth, we will also obtain the regularity with respect to θ in the conclusions, but in the stochastic forcing no regularity in θ will be obtained.

We will indicate the formulation of the existence of invariant sets in the case when $\mathscr{C}$ is a manifold and its evolution is given by a differential equation as in (2.5), where $\mathscr{C}$ is a finitely dimensional manifold which is also called "clock manifold", X is a Banach space and B is a smooth vector field on a manifold $\mathbb{T}^d$. A particularly case of the set-up which is important to us is $\mathscr{C} = \mathbb{T}^m$ and $B(\theta) = \omega$. In this case, the system is said to be subject to quasi-periodic forcing.

Another important case that appears in applications is when $\mathscr{C} = \mathbb{R}$, the dynamics on it is given by:

$$\dot{\theta} = -\lambda\theta, \quad \lambda > 0$$

and the forcing vanishes when $\theta = 0$.

Since θ is decreasing exponentially fast towards zero and the forcing vanishes at $\theta = 0$, the models (2.3) describe a perturbation which dies off exponentially as time goes to infinity. These problems appear in practice when systems are subject to sudden transient forcings.

Assume that $\Phi_t(\theta) \in \mathscr{C}$ is the flow of the following differential equation

$$\frac{d}{dt}\theta(t) = B(\theta(t)), \theta(0) = \theta \in \mathscr{C}. \tag{2.14}$$

Then (2.5) can be rewritten as

$$\begin{aligned} &\frac{d}{dt}\Phi_t(\theta) = B(\Phi_t(\theta)), \quad \Phi_0(\theta) = \theta \in \mathscr{C}, \\ &\frac{d}{dt}y = \mathscr{A}y + F(\Phi_t(\theta), y). \end{aligned} \tag{2.15}$$

The equation (2.15) is referred in the mathematical literature as an skew flow. It is can also be considered as a non-autonomous evolution equation in a Banach space X. The equation depends on parameters defined on another set which we will denote by $\mathscr{C}$ and which are evolving independent of the state of y.

By a solution of (2.15) we always mean a classical solution, that is, a continuous function u of (θ, y) satisfying

$$\theta : \mathbb{R} \to \mathscr{C}, \quad y : \mathscr{C} \times \mathbb{R} \to X,$$

such that, $\forall t \in \mathbb{R}, \theta \in \mathscr{C}, \theta$ is smooth with respect to t, y is such that the operator $\mathscr{X}$ can be interpreted in the classical sense (this is guaranteed by using spaces X, Y consisting of functions which are enough times differentiable), y is continuously differentiable with respect to t and finitely differentiable with respect to θ. Moreover, (2.15) is verified in the claimed set.

From (2.9) and (2.11) we know that the prepared equation of (2.15) is

$$\begin{aligned} &\frac{d}{dt}\Phi_t(\theta) = B(\Phi_t(\theta)), \quad \Phi_0(\theta) = \theta \in \mathscr{C}, \\ &\frac{d}{dt} y_s = \mathscr{A}_s y_s + F_s(\Phi_t(\theta), y_s, y_c, y_u), \\ &\frac{d}{dt} y_c = \mathscr{A}_c y_c + F_c(\Phi_t(\theta), y_s, y_c, y_u), \\ &\frac{d}{dt} y_u = \mathscr{A}_u y_u + F_u(\Phi_t(\theta), y_s, y_c, y_u), \end{aligned} \tag{2.16}$$

where, to avoid cluttering the typography, we write $\widetilde{F}$ and $\tilde{y}$ as F and y, respectively.

The definition of non-autonomous invariant manifold associated to the space X_c is the following.

Definition 2. Let $\Theta : \mathscr{C} \times X_c \to X$ be a C^1 embedding. We say that it defines a non-autonomous invariant manifold for (2.15) if there exists L, a vector field defined on $\mathscr{C} \times X_c$ taking values in TX_c, since X_c is a Banach space we can replace TX_c with X_c, such that

$$D_\theta \Theta(\theta, y_c) B(\theta) + D_{y_c} \Theta(\theta, y_c) L(\theta, y_c) = \mathscr{X}(\theta, \Theta(\theta, y_c)). \tag{2.17}$$

Then, the manifold

$$\widetilde{\mathscr{W}} \equiv \Big\{ \Theta(\theta, y_c) \in X : \quad \theta \in \mathscr{C}, y_c \in X_c \Big\}$$

is referred to as a non-autonomous invariant manifold of the system (2.15).

The key point is that if (2.17) holds and $\big(\theta(t), y_c(t)\big)$ is a solution of

$$\frac{d}{dt} \begin{pmatrix} \theta \\ y_c \end{pmatrix} = \begin{pmatrix} B(\theta) \\ L(\theta, y_c) \end{pmatrix} \tag{2.18}$$

then $\Theta(\theta(t), y_c(t))$ is a solution of the full equation (e.g.(2.1)).

Note that $\Phi_t(\theta)$ is the evolution of the vector field $B(\Phi_t(\theta))$, then (2.18) can be written in a non-autonomous form

$$\frac{d}{dt} y_c = L(\Phi_t(\theta), y_c). \tag{2.19}$$

We anticipate that (2.19) also makes sense in the case that Φ_t is not derived from a differential equation but is a stochastic process under some mild regularity assumptions.

In the case that the vector field $\mathscr{X}$ is unbounded, the formal argument above requires to justify the sense in which the equations are satisfied. In our applications, we will be able to take X as spaces of very differentiable functions, so that the equations (2.1) will hold in the classical sense for the solutions in the center manifold we construct. It seems that by formulating some operators as stochastic integral we could generalize the processes considered.

Hence, the non-autonomous invariant manifolds provide with a way to produce solutions of the problem by solving the "reduced" system (2.18). Notice that the reduced system may be finite dimensional even if the original system was infinite dimensional (and possible unbounded).

The geometric meaning of the equation (2.17) is that the vector field $(B, \mathscr{X})$ restricted to the manifold $\widetilde{\mathscr{W}}$ is tangent to the manifold. If we pull back the vector field by the embedding Θ we obtain that (B, L) is the vector field corresponding to $\widetilde{\mathscr{X}}$ in coordinates.

In the case that the extended vector field $\widetilde{\mathscr{X}}$ (defined in (2.5)) defines a flow, if (2.17) is satisfied, the trajectories of the extended vector field remain in the manifold $\widetilde{\mathscr{W}}$ and that the motion in the coordinates provided by Θ is precisely $\widetilde{L} \equiv (B, L)$.

It will be important for us that we can make sense of (2.17) even in some cases when the extended vector field $\widetilde{\mathscr{X}}$ does not define a flow. That is not all the possible initial conditions admit a solution of the full equation. Nevertheless, we will construct a manifold of initial conditions which indeed lead to solutions. The solutions will remain in this manifold. Hence, we can consider the reduced flow in this invariant manifold. The key point of the present results is that, even for ill-posed equations, provided that the equations satisfy our assumptions, we can find manifolds of solutions. These manifolds of solutions can be constructed systematically as perturbations from some families of solutions in linear equations. Furthermore, it will turn out that there are other solutions that converge to them either in the future or in the past.

The manifold $\mathcal{W}$ we need will be presented as the graph of a function $w : \mathscr{C} \times X_c \to X_s \oplus X_u$, *i.e.*,

$$\mathcal{W} = \left\{ (\theta, w_u(\theta, y_c), y_c, w_s(\theta, y_c)) | \theta \in \mathscr{C}, y_c \in X_c \right\}, \tag{2.20}$$

we need to solve some equations that express invariance. Note that in the case of ill-posed equations, the graph will be determined not only by the requirement that it is an invariant set but also by requiring that its points admit a forward evolution. More concretely, we need to derive heuristically an equation (see equation (3.11)) for the functions w_s, w_u that ensure that $\mathcal{W}$ is invariant under (2.15), equivalently, (2.9) since we want to allow ill-posed equation. We will also need to impose that the points of $\mathcal{W}$ admit an evolution.

From the point of geometry, the invariance of the manifold $\mathcal{W}$ will mean that $\mathcal{W} \subset D(\widetilde{\mathscr{X}})$ that is if $z \in \mathcal{W}$, $\widetilde{\mathscr{X}}(z) \in T_z\mathcal{W}$ and $F|_{\mathcal{W}}$ defines an evolution equation, moreover, that there are some precise estimates, which we will make explicit later, on the dependence on parameters. This last point is automatic in the applications we consider in Section 4, since $\mathcal{W}$ is a finite dimensional smooth manifold and $F|_{\mathcal{W}}$ is a smooth and bounded vector field.

From the point of mathematics, as it is standard in invariant manifold theory, see [32, 40], we will formulate heuristically the invariance condition as a fixed point problem, see equation (3.11). Then, we show that indeed one can find a fixed point of (2.9) enjoying several extra properties (regularity, asymptotic behavior, etc). Once we have a well behaved fixed point solving (3.5), it will be easy to check that the set (2.20) is invariant under the equation (2.9) when w is the solution of the fixed point of equation (3.11).

In the heuristic derivation of the fixed point equations (2.20) to be solved, we will use, with abandon assumptions, that the differential equations we write have solutions, etc. This will indicate that the equation derived is a necessary condition for the invariance equation. Once the solutions for this invariance equation are obtained, we will be able to show that the existence of solutions for the initial conditions in the invariant manifold. Therefore, we will, for the moment, assume that w is such that its graph is an invariant manifold and manipulate the invariance equation to derive necessary conditions that w has to satisfy.

Since we do not assume that our equations are well posed, our procedure differs from that in [46]. The functional equations have to select the initial conditions that allow us to construct solutions. We will have one functional equation in each of the components. These three equations are coupled, we formulate them as a fixed point equation for an operator $\mathscr{T}$. Then we show this operator is a contraction in appropriate spaces. See the system (3.12) for details.

As we can only construct the evolutions of the equations whose perturbations are small enough, we can not consider the original equation (2.15) directly, we have to turn to its "prepared equation", (2.16). As it is standard in Center Manifold theory, the prepared equations agree with the original ones in a neighborhood of original point but globally mild. The invariant manifolds for the prepared equations will be locally invariant for the original equation. The manifolds we construct will be unique once the prepared equations are selected, but they can depend on the prepared equation. Hence, the locally invariant manifolds are not unique.

Therefore, in the rest of the paper, we will just construct the global center manifold which is global invariant by equation (2.16). We will go back to (2.15) by the inverse of (2.10) inside the set of $E_\epsilon(X_c)$, that is we obtain the local center manifold which is global invariant by equation (2.15). For this reason we will only construct the center manifold for the prepared equation (2.16).

2.4. **Stochastic forcing.** The set-up described in the previous section is very general and a slight modification also accommodates the case of stochastic forcing.

When we consider stochastic forcing, $\mathscr{C}$ is a measure space endowed with a measure μ. We will assume that the variables θ evolve according to an stationary stochastic process $\Phi_t(\theta)$. The dependence on θ need only be measurable, not even continuous since $\mathscr{C}$, in general, does not need to be assumed to be a metric space (much less a manifold).

In the stochastic processes setting, we assume that Φ_t preserves the probability measure μ, which will be used to discuss expectations, etc. In this paper, however, we will concentrate on geometric properties and will not discuss probabilistic aspects of the manifold.

We will assume that the evolution satisfies the Markov property $\Phi_{t+s}(\theta) = \Phi_t(\Phi_s(\theta))$.

A key observation is that, the usual theory of existence, uniqueness, smooth dependence on initial conditions, etc. of the Cauchy problem for skew equations

$$\dot{y} = F(\Phi_t(\theta), y), \quad y(0) = y_0 \tag{2.21}$$

is based on the integral equation $y(t) = y_0 + \int_0^t F(\Phi_s(\theta), y(s))\, ds$. Hence, the theory of this paper can be developed assuming only regularity of F with respect to the y variable. For the dependence on t continuity (or even less regularity) is enough. Any regularity that allows to derive Duhamel variation of parameters formulas are enough.

In this paper, we will assume that the integral $y(t) = y_0 + \int_0^t F(\Phi_s(\theta), y(s))\, ds$ is a standard $1 - D$ integral and not an stochastic integral. This will require that our stochastic process have measurable paths. This assumption is verified in several applications [6, 27] but it excludes important applications such as white noise which cannot be described by continuous forcings. The assumption applies to Ornstein-Uhlenbeck processes, which are white noise added to the velocity. We also recall that in **(T1)** we are assuming that the effects of noise are bounded at ∞. We hope to be able to come back to the study of (2.21) as an stochastic integral.

Using the above discussion, we can start our formulation by using the weak form of the evolution, (2.6), which is natural when the external forcing is evolving according to an stochastic process rather than according to the differential equation (2.4).

Proceeding as in the previous section, we proceed to derive again (3.11). Hence, our goal is again to establish the existence of solutions of the fixed point equation (3.11), but the regularity assumptions (and the conclusions) will be different from the case when the external forcing is evolving under a differential equation.

Remark 2.1. The physical interpretation of the above setup for stochastic forcing is that, for each realization of the noise, satisfying our hypotheses, we obtain a manifold satisfying the invariance equation (3.11). Hence, the manifold becomes a random variable and it is possible to study statistical properties such as expected values of several events such as the manifold going trough a point, deviations of the above etc.

This formulation of the invariant manifold problem has been standard in the literature. See [9, 4] and it usually receives the name *quenched analysis*.

We point out that in the literature, there are other approaches of the analysis of random systems. A very important one is the study of the evolution of probability densities, [18, 52]. In finite dimensional systems, the density with respect to the Lebesgue measure satisfies a PDE (the Fokker-Plank equation), which is a very powerful tool since it can be analysed using PDE tools. However, for infinite dimensional problems, since there is no Lebesgue measure, even if the evolution of measures can be defined, there is no analogue of the Fokker-Plank equation. Of course, for infinite dimensional problems, once a finite dimensional manifold has been identified, it is possible to use Fokker-Plank equations to the reduced system in the center manifolds. Hence, the reduction principles established here are useful both in the quenched and in the Fokker-Plank approaches.

Remark 2.2. We note that the functions we produce are solutions in the sense that they satisfy the equations in integral form, thus the solution we obtain is the "mild solution". If we take space X as an space of functions which consists of sufficiently differentiable functions in the applications we will take $X = H^r$ for $r \geq r_0$, so that this includes C^ℓ functions for arbitrarily large ℓ, we can conclude that the solutions thus produced are very smooth and that they satisfy the equations in the classical sense.

3. **Formulation of the results.** In this section we formulate precisely the notation and hypotheses as well as the main result, Theorem 3.1. We will start formulating the hypotheses and state the Theorem 3.1.

In the case of the smooth forcing if we assume the regularity of F with respect to θ, then we get the regularity of w with respect to θ. However, in our second case (see Section 2.4) even we assume the regularity of F with respect to θ we can not conclude regularity of w with respect to θ.

Our result will be the existence and regularity of (3.11). We will first present results that assume that F is regular in the initial value $\xi \in X_c$ (but not in θ). These results apply both to stochastic forcing and to smooth forcing. It is well known that the invariant manifolds for the prepared equations are unique (under some conditions), but since the bump process involved in deriving the prepared equations involves arbitrary choices the locally invariant manifolds for the full equation are not unique. The lack of the uniqueness of the center manifolds is a real effect, which has been studied in great detail in [49].

3.1. **Formulation of the hypotheses and the statement of the main result.** Now, we turn to formulating precisely the hypotheses for our result. We note that these hypotheses are verified in several examples in the literature. In Section 4, we will present the cases of the external forced nonlinear elliptic differential equations. The same framework applies to the examples discussed in [24] and [13].

The first group of Hypotheses (which we will refer collectively as **(H1)**) concerns the splitting assumed in (2.7). As we have use the *"two spaces approach"* method of [34] to guarantee the boundedness of perturbation F, we assume that F is differentiable from the space X to another space Y consisting of less smooth functions.

In this paper we will define $C^r(X,Y)$ to be the set functions from X to Y that have continuous derivatives of order less than or equal to r. We will assume that the derivatives are uniformly bounded and uniformly continuous. We endow this space with the supremum norm of the domain which makes it a Banach space.

(H1.1): The decomposition (2.7) of the space X is invariant under $\mathscr{A}$. That is, if $x \in D(\mathscr{A})\bigcap X_\sigma$ then $\mathscr{A}x \in X_\sigma, \sigma = s,c,u$.

(H1.2): The projections Π_s, Π_c, Π_u are bounded in X.

(H1.3): There are Banach spaces $Y_\sigma, \sigma = s,c,u$ $(Y = Y_s \oplus Y_c \oplus Y_u)$ such that:

– $X_\sigma \subset Y_\sigma$.

– X_σ is dense in Y_σ and the embedding is continuous. Actually, X_c is isomorphic to Y_c.

The next set of hypotheses considers the existence of evolution semigroups in the corresponding subspaces and making quantitative assumptions on the contraction rates of the operators, we set as **(H2)**.

(H2.1):

– For $t > 0$, $e^{t\mathscr{A}_s} : Y_s \to X_s$ is a contraction semigroup for positive time, we will denote this semigroup by $\{e^{t\mathscr{A}_s}\}_{t\geq 0}$.

– For $t < 0$, $e^{t\mathscr{A}_u} : Y_u \to X_u$ is a contraction semigroup for negative time, we will denote this semigroup by $\{e^{t\mathscr{A}_u}\}_{t<0}$.

– For $t \in \mathbb{R}$, $e^{t\mathscr{A}_c} : Y_c \to X_c$ generates a group on X_c, which we will denote by $\{e^{t\mathscr{A}_c}\}_{t\in\mathbb{R}}$.

(H2.2): For some $\beta_3^{\pm}, \beta_1, \beta_2 > 0, C_h > 1, \alpha_1, \alpha_2 \in [0,1)$,

$$\begin{aligned} \|e^{t\mathscr{A}_c}\|_{X_c,X_c} &\leq C_h e^{\beta_3^+ t}, \quad t \geq 0,\\ \|e^{t\mathscr{A}_c}\|_{X_c,X_c} &\leq C_h e^{\beta_3^- |t|}, \quad t \leq 0,\\ \|e^{t\mathscr{A}_s}\|_{Y_s,X_s} &\leq C_h e^{-\beta_1 t} t^{-\alpha_1}, \quad t \geq 0,\\ \|e^{t\mathscr{A}_u}\|_{Y_u,X_u} &\leq C_h e^{-\beta_2 |t|} |t|^{-\alpha_2}, \quad t \leq 0 \end{aligned} \tag{3.1}$$

with $0 \leq \beta_3^- < \beta_1$ and $0 \leq \beta_3^+ < \beta_2$. Moreover, we will also use the standard trichotomy assumption

$$\begin{aligned} \|e^{t\mathscr{A}_c}\|_{X^c,X^c} &\leq C_h e^{\beta_3^+ t}, \quad t \geq 0,\\ \|e^{t\mathscr{A}_c}\|_{X^c,X^c} &\leq C_h e^{\beta_3^- |t|}, \quad t \leq 0, \end{aligned}$$

$$\begin{aligned}\|e^{t\mathscr{A}_s}\|_{X^s,X^s} &\le C_h e^{-\beta_1 t}, \quad t>0,\\ \|e^{t\mathscr{A}_u}\|_{X^u,X^u} &\le C_h e^{-\beta_2 |t|}, \quad t<0.\end{aligned} \tag{3.2}$$

If the operator $\mathscr{A}$ possesses the estimates defined by (3.2), we say that $\mathscr{A}$ has exponential trichotomy, see [48] about exponential trichotomy for details.

We will also make a regularity assumption about the perturbation F. For the case of forcing by smoothing function.

(H3): The function $F : \mathscr{C} \times X \to X$ is C^r when considered as a function from $U \subset X$ to Y, where U is an open set containing the origin and $\|F\|_{C^r(\mathscr{C}\times U,Y)}$ is sufficiently small, where $1 < r < \frac{\delta}{\beta}$, for $\delta > \beta$ with

$$\delta = \min\{\beta_1, \beta_2\},\ \beta > \max\{\beta_3^+, \beta_3^-\}, \tag{3.3}$$

several subsequent hypotheses (such as the sizes of balls to be considered) will be depended on this choice of β.

In the case of forcing by stochastic processes we make the following regularity assumption.

(H4): $\mathscr{C}$ is a measure space. The function $F \in L^\infty(\mathscr{C}, C^r(X,Y))$ when considered as functions from $U \subset X$ to Y, where U is an open set containing the origin, and $\|F(\theta,\cdot)\|_{C^r(U,Y)}$ is small enough.

Instead of assuming **(H3)**, **(H4)**, we will assume the global versions of the above local hypotheses. Hence, we will replace U, the domain of the nonlinearities assumed before, with the whole space X.

(H3)': The function $F : \mathscr{C} \times X \to X$ is C^r and $\|F\|_{C^r(\mathscr{C}\times X,Y)}$ is sufficiently small.

(H4)': $\mathscr{C}$ is a measure space. The function $F \in L^\infty(\mathscr{C}, C^r(X,Y))$ and $\|F(\theta,\cdot)\|_{C^r(X,Y)}$ is small enough.

Note that in this paper $F = f + N$, so if we use the functions f and N to formulate the assumptions above we have the following:

(H5.1): The function $f : \mathscr{C} \times X \to X$ is C^r when considered as a function from $U \subset X$ to Y, where U is an open set containing the origin and $\|f\|_{C^r(\mathscr{C}\times U,Y)}$ is sufficiently small enough. Moreover, $\mathcal{N}(\theta,0) = 0, D_2\mathcal{N}(\theta,0) = 0$.

(H5.2): $\mathscr{C}$ is a measure space. The function $f \in L^\infty(\mathscr{C}, C^r(X,Y))$ when considered as functions from $U \subset X$ to Y, where U is an open set containing the origin, and $\|f(\theta,\cdot)\|_{C^r(U,Y)}$ is small enough. Moreover, $\mathcal{N}(\theta,0) = 0, D_2\mathcal{N}(\theta,0) = 0$.

(H5.3): The function $f : \mathscr{C} \times X \to X$ is C^r and $\|f\|_{C^r(\mathscr{C}\times X,Y)}$ is sufficiently small. Moreover, $\mathcal{N}(\theta,0) = 0, D_2\mathcal{N}(\theta,0) = 0$.

(H5.4): $\mathscr{C}$ is a measure space. The function $f \in L^\infty(\mathscr{C}, C^r(X,Y))$ and $\|F(\theta,\cdot)\|_{C^r(X,Y)}$ is small enough. Moreover, $\mathcal{N}(\theta,0) = 0, D_2\mathcal{N}(\theta,0) = 0$.

In the case of forcing by a smooth vector field we will make extra assumptions on the regularity of the vector field generating the forcing. Then, we will obtain extra regularity of the manifolds in the conclusions, we set this hypothesis as **(H6)**.

(H6.1): The set $\mathscr{C}$ is a C^r manifold.

(H6.2): The vector field B appearing in (2.4) is a C^r vector field.

(H6.3): Denoting by $\Phi_t(\theta)$ the evolution of the equation (2.14), we have

$$\begin{aligned}\|D\Phi_t(\theta)\|_{\mathscr{C},\mathscr{C}} &\le Ce^{\beta_3^+ t}, \quad t>0,\\ \|D\Phi_t(\theta)\|_{\mathscr{C},\mathscr{C}} &\le Ce^{\beta_3^- |t|}, \quad t<0.\end{aligned} \tag{3.4}$$

The following is the main result of this paper.

Theorem 3.1. *Assume that X, Y are Banach spaces satisfying* **(H1)** *and the operator A and F of* (2.16) *satisfying* **(H2)** *and* **(H3)** *respectively.*

Then, for every $\theta \in \mathscr{C}$, there exists a $C^{r-1+Lip}$ function $w \in \mathcal{X}_1$ defined on X_c and mapping X_c to X_{su}. $\mathscr{W}$, the graph of w, is globally invariant by (2.16).

Furthermore, if the forcing is given by a smooth vector field satisfying **(H6)***, then for the each initial condition ξ, the manifold $\mathscr{W}$ is $C^{r-1+Lip}$ in θ.*

Theorem 3.2. *If hypotheses* (**H1**), (**H2**), (**H3**), (**H4**) *and* (**H5**) *are fulfilled, $r \geq 1$, there exists a $C^{r-1+Lip}-$ function w defined on $\mathscr{C} \times X_c$ and mapping $\mathscr{C} \times X_c$ to $X_s \oplus X_u$. Furthermore, W, the graph of w, is globally invariant by* (2.9).

Here in Theorems 3.1, 3.2, a function f is C^{l+Lip}, $l \in \mathbb{N}$ (resp. C^l, $l \in \mathbb{N}$) means that the $l-$order derivative $D^l f$ is Lip (resp. $D^l f$ is continuous and bounded). Please see Definitions 3, 4 about the $C^l(l \in \mathbb{N})-$space and $C^{l+Lip}(l \in \mathbb{N})-$space, in Appendix for details.

The proof of Theorem 3.1 will be obtained by formulating a fixed point equation (see system (3.12)) for a function whose graph gives the desired invariant manifold. The operator involved in (3.12) will be a contraction in the C^0- norm, but it will preserve a set of differentiable functions. This is a standard situation in the theory of center manifolds and in the theory of normally hyperbolic invariant manifolds. See [40]. The extension of the smooth dependence on parameter θ for the case of smooth forcing will be given in Section 3.6.

3.2. **Derivation the invariance of equations.** We first proceed to derive, heuristically, the equations satisfied by an invariant graph. Note that the equations are ill-posed, we also need to impose that the evolution can be defined. We note that if $\mathscr{W}$, the graph of w, is invariant under (2.16), then the component along center direction in (2.16) gives an evolution equation for y_c, namely

$$\begin{aligned}&\frac{d}{dt}\Phi_t(\theta) = B(\Phi_t(\theta)), \quad \Phi_0(\theta) = \theta,\\&\frac{d}{dt}y_c(t) = \mathscr{A}_c y_c(t) + F_c(\Phi_t(\theta), y_c(t), w(\Phi_t(\theta), y_c(t))), \quad y_c(0) = \xi \in X_c, \quad t \in \mathbb{R}.\end{aligned}$$

First, following the standard practice, we study the evolution of the projection in the center directions for the vector field in a graph of a function w (even if the graph of w is not invariant).

For any w we denote $\Gamma_t^w(\theta, \xi) := (\Phi_t(\theta), J_t^w(\theta, \xi))$ be the solution of the above system with the initial datum $\Gamma_0^w(\theta, \xi) = (\theta, \xi)$. From Duhamel principle (see e.g., [34, 51]), we have

$$J_t^w(\theta, \xi) = e^{\mathscr{A}_c t}\xi + \int_0^t e^{\mathscr{A}_c(t-\tau)} F_c(\Phi_\tau(\theta), J_\tau^w(\theta, \xi), w(\Phi_\tau(\theta), J_\tau^w(\theta, \xi)))d\tau. \tag{3.5}$$

Furthermore, the components along y_s and y_u in (2.16) satisfy

$$\begin{aligned}y_s(t) =& e^{t\mathscr{A}_s} y_s(0)\\&+ \int_0^t e^{(t-\tau)\mathscr{A}_s} F_s(\Phi_\tau(\theta), J_\tau^w(\theta, \xi), w(\Phi_\tau(\theta), J_\tau^w(\theta, \xi)))d\tau, \ t \geq 0,\\y_u(t) =& e^{t\mathscr{A}_u} y_u(0)\\&+ \int_0^t e^{(t-\tau)\mathscr{A}_u} F_u(\Phi_\tau(\theta), J_\tau^w(\theta, \xi), w(\Phi_\tau(\theta), J_\tau^w(\theta, \xi)))d\tau, \ t \leq 0.\end{aligned} \tag{3.6}$$

To derive heuristically (in particular, we will assume that the evolution is defined for the initial data in the graph of w, but this will be another equation later) an equation for y_s and y_u, we note that if the graph of w were invariant, we should have

$$(y_s(t),\ y_u(t)) = \Big(w_s(\Phi_t(\theta), J_t^w(\theta,\xi)),\ w_u(\Phi_t(\theta), J_t^w(\theta,\xi))\Big). \tag{3.7}$$

Hence, y_s and y_u, the components of (3.6), should read (after making the change of variables $\tau = t - r$ in the integral)

$$\begin{aligned} w_s(\Phi_t(\theta),&J_t^w(\theta,\xi)) = e^{t\mathscr{A}_s} w_s(\theta,\xi) \\ &+ \int_0^t e^{r\mathscr{A}_s} F_s(\Phi_{t-r}(\theta), J_{t-r}^w(\theta,\xi), w(\Phi_{t-r}(\theta), J_{t-r}^w(\theta,\xi)))dr,\ t \geq 0, \\ w_u(\Phi_t(\theta),&J_t^w(\theta,\xi)) = e^{t\mathscr{A}_u} w_u(\theta,\xi) \\ &+ \int_0^t e^{r\mathscr{A}_u} F_u(\Phi_{t-r}(\theta), J_{t-r}^w(\theta,\xi), w(\Phi_{t-r}(\theta), J_{t-r}^w(\theta,\xi)))dr,\ t \leq 0. \end{aligned} \tag{3.8}$$

Note that in the first equation of (3.8) which involves w_s, we assume $t \geq 0$. Whereas, in the second equation of (3.8) which involves w_u, we assume $t \leq 0$. Using that $\Gamma_t^w(\Phi_t(\theta), J_t^w(\theta,\xi))$ is invertible, we find that (3.8) is equivalent to

$$\begin{aligned} w_s(\theta,\xi) =& e^{t\mathscr{A}_s} w_s(\Phi_{-t}(\theta), J_{-t}^w(\theta,\xi)) \\ &+ \int_0^t e^{r\mathscr{A}_s} F_s(\Phi_{-r}(\theta), J_{-r}^w(\theta,\xi), w(\Phi_{-r}(\theta), J_{-r}^w(\theta,\xi)))dr,\ t \geq 0, \\ w_u(\theta,\xi) =& e^{t\mathscr{A}_u} w_u(\Phi_{-t}(\theta), J_{-t}^w(\theta,\xi)) \\ &+ \int_0^t e^{r\mathscr{A}_u} F_u(\Phi_{-r}(\theta), J_{-r}^w(\theta,\xi), w(\Phi_{-r}(\theta), J_{-r}^w(\theta,\xi)))d\tau,\ t \leq 0. \end{aligned} \tag{3.9}$$

Moreover, taking $r = -\tau$ and changing t to $-t$, we get

$$\begin{aligned} w_s(\theta,\xi) =& e^{-t\mathscr{A}_s} w_s(\Phi_t(\theta), J_t^w(\theta,\xi)) \\ &+ \int_t^0 e^{-\tau\mathscr{A}_s} F_s(\Phi_\tau(\theta), J_\tau^w(\theta,\xi), w(\Phi_\tau(\theta), J_\tau^w(\theta,\xi)))d\tau, t \leq 0, \\ w_u(\theta,\xi) =& e^{-t\mathscr{A}_u} w_u(\Phi_t(\theta), J_t^w(\theta,\xi)) \\ &- \int_0^t e^{-\tau\mathscr{A}_u} F_u(\Phi_\tau(\theta), J_\tau^w(\theta,\xi), w(\Phi_\tau(\theta), J_\tau^w(\theta,\xi)))d\tau, t \geq 0. \end{aligned} \tag{3.10}$$

Since we assume that w_s and w_u are bounded, then

$$\lim_{t\to-\infty} e^{-t\mathscr{A}_s} w_s(\Phi_t(\theta), J_t^w(\theta,\xi)) = 0,\ \lim_{t\to\infty} e^{-t\mathscr{A}_u} w_u(\Phi_t(\theta), J_t^w(\theta,\xi)) = 0,$$

then, taking limits when $t \to \pm\infty$ in (3.10) we are lead heuristically to the equations:

$$\begin{aligned} w_s(\theta,\xi) &= \int_{-\infty}^0 e^{-\tau\mathscr{A}_s} F_s(\Phi_\tau(\theta), J_\tau^w(\theta,\xi), w(\Phi_\tau(\theta), J_\tau^w(\theta,\xi)))d\tau, \\ w_u(\theta,\xi) &= -\int_0^\infty e^{-\tau\mathscr{A}_u} F_u(\Phi_\tau(\theta), J_\tau^w(\theta,\xi), w(\Phi_\tau(\theta), J_\tau^w(\theta,\xi)))d\tau. \end{aligned} \tag{3.11}$$

By combining the RHS of (3.5) and (3.11) we obtain an operator $\mathscr{T}$ defined in some metric space $\mathcal{L}_\delta \times \mathcal{X}_1$ that we will define precisely in Section 3.3,

$$\begin{pmatrix} J \\ w \end{pmatrix}(t,\theta,\xi) = \mathscr{T}[J,w](t,\theta,\xi) = \begin{pmatrix} \mathscr{T}_c[J,w] \\ \mathscr{T}_s[J,w] \\ \mathscr{T}_u[J,w] \end{pmatrix}(t,\theta,\xi) \tag{3.12}$$

with

$$\mathscr{T}_c[J,w](t,\theta,\xi) = e^{\mathscr{A}_c t}\xi + \int_0^t e^{\mathscr{A}_c(t-\tau)} F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))d\tau, \tag{3.13}$$

$$\mathscr{T}_s[J,w](\theta,\xi) = \int_{-\infty}^0 e^{-\tau\mathscr{A}_s} F_s(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))d\tau, \tag{3.14}$$

and

$$\mathscr{T}_u[J,w](\theta,\xi) = -\int_0^{\infty} e^{-\tau\mathscr{A}_u} F_u(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))d\tau. \tag{3.15}$$

Note that the RHS of (3.12) contains three equations: (3.13)-(3.15), then the function w, which is the fixed point of the operator, is also fixed, thus for typographical reason we omit the symbol "w" on the superscript of J.

Since the integrand that appears in the RHS of (3.12) is continuous, therefore the fixed points of the operator (3.12) correspond to classical solutions. As said in the Section 2.4 the same equation appears in the case of the stochastic forcing.

3.3. **Function spaces for the operator $\mathscr{T}$.** Now we define the space on which $\mathscr{T}_c$ acts, which we set as

$$\begin{aligned} \mathcal{L}_\delta = \Big\{ &\phi : \mathbb{R}\times\mathscr{C}\times X_c \to X_c |\ \phi(0,\theta,\xi) = \xi, \forall\theta\in\mathscr{C}, \xi\in X_c, \\ &\text{for fixed } \theta,\ \phi(\cdot_1,\theta,\cdot_2) \text{ is continuous in t and r order differentiable in } \xi, \\ &\|\phi(t,\theta,\xi)\|_{X_c} \le 2C_h(1+\|\xi\|_{X_c})e^{\beta|t|}, \quad \|D_\xi\phi(t,\theta,\xi)\|_{X_c,X_c} \le 2C_h e^{\beta|t|}, \\ &\|D_\xi^i\phi(t,\theta,\xi)\|_{X_c^{\otimes i},X_c} \le C_h e^{i\beta|t|}, \quad i=2,\cdots,r,\ t\in\mathbb{R} \Big\} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \mathcal{X}_1 = \Big\{ &w : \mathscr{C}\times X_c \to X_{su} :\ \text{for fixed } \theta,\ w \text{ is r order} \\ &\text{differentiable in } \xi,\ \|D_\xi^i w(\theta,\xi)\|_{X_c^{\otimes i},X_{cu}} \le 1, \quad i=0,1,\cdots,r \Big\}. \end{aligned} \tag{3.17}$$

For any β defined in (3.3), we define the weighted norms

$$\begin{aligned} \|J\|_{C^0}^{(\beta} &\equiv \sup_{t\in\mathbb{R}}\sup_{\theta\in\mathscr{C}}\sup_{\xi\in X_c} \|J_t(\theta,\xi)\|_{X_c} e^{-\beta|t|}, \\ \|w\|_{C^0} &\equiv \sup_{\theta\in\mathscr{C}}\sup_{\xi\in X_c} \|w(\theta,\xi)\|_{X_{su}}, \\ \|(J,w)\|_{C^0} &= \max\{\|J\|_{C^0}^{(\beta}, \quad \|w\|_{C^0}\}. \end{aligned} \tag{3.18}$$

The induced metric on $\mathcal{L}_\delta\times\mathcal{X}_1$ is

$$d((J,w),(\widetilde{J},\widetilde{w})) = \|(J-\widetilde{J}, w-\widetilde{w})\|_{C^0}. \tag{3.19}$$

The notation "$(\beta$" denotes the weight in the weighted norm $\|\cdot\|_{C^0}^{(\beta}$, which is $e^{-\beta|t|}$. This weight depicts the max-speed of increasing of the function with respect to t.

Note that in the study of the J, we have used a exponentially norm. This is a standard device to obtain existence of solution for all time.

3.4. **Proof of Theorem 3.1.** In this section we will prove our main results. The proof is based on a version of the Banach contraction fixed point theorem. One consequence of the proof by Banach contraction fixed point theorem is that we can have an a-posteriori version of the result. That is, if we have an approximate fixed point for (3.12), then we can conclude that there is a true solution close to it. This is important since we can use such result to validate numerical approximations or asymptotic expansions that can lead to rigorous proofs. Similar results had lead to computer assisted-proofs of existence of invariant manifolds in other contexts [19, 10, 22]. With these possible applications in view, we give two a-posteriori versions of the result. One estimate is under the distance defined by the C^0- norm and the other one is under the distance defined by $C^{rs}-$ norm, $0 < s \leq 1$, (the higher order regularity).

We will check that $\mathscr{T} = (\mathscr{T}_c, \mathscr{T}_s, \mathscr{T}_u)$ in (3.12) indeed defines a contraction operator in the function space $\mathcal{L}_\delta \times \mathcal{X}_1$ under the $d-$distance defined in (3.19).

We separate the proof of Theorem 3.1 into two steps. First, we prove that $\mathscr{T}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset (\mathcal{L}_\delta \times \mathcal{X}_1)$ (Step 1) and then we prove that $\mathscr{T}$ is a contraction in $\mathcal{L}_\delta \times \mathcal{X}_1$ under the $d-$distance defined by (3.19), (Step 2).

Step 1. $\mathscr{T}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset (\mathcal{L}_\delta \times \mathcal{X}_1)$.

The fact that $\mathscr{T}[J,w]$ is a C^1 function in t and C^r in ξ is a direct consequence of the fact that, for any $l \in \mathbb{N}$, the composition of C^l functions is a C^l function.

The inequality

$$\int_0^t e^{-\beta(t-\tau)}(t-\tau)^{-\alpha} d\tau = \int_0^t e^{-\beta\tau}\tau^{-\alpha} d\tau \leq \frac{1}{1-\alpha} + \frac{1}{\beta}$$

will be used in many places, see the elementary proof in [13]. Moreover, we will abbreviate $C^i(\mathscr{C} \times X_c, X_\sigma)$, $\sigma = s, u, c$, $i = 0, \cdots, r$, and $C^i(\mathscr{C} \times X_c, X)$, $i = 0, \cdots, r$, into C^i, $i = 0, \cdots, r$.

We will only give the proof of the boundedness of (3.13) for $t \geq 0$ and omit the case $t < 0$ since the later case can be done in an identical manner or deduced from former case by a change of the direction of time. From equations (3.1), (3.2) and (3.13) we get

$$\begin{aligned}
\|\mathscr{T}_c[J,w](t,\theta,\xi)\|_{X_c} &\leq C_h e^{\beta_3^+ t}\|\xi\|_{X_c} + C_h \int_0^t e^{\beta_3^+(t-\tau)}\|F_c\|_{C^0} d\tau \\
&= C_h e^{\beta_3^+ t}\|\xi\|_{X_c} + C_h \frac{1}{\beta_3^+}\|F_c\|_{C^0}(e^{\beta_3^+ t} - 1) \\
&\leq 2C_h e^{\beta_3^+ t}(\|\xi\|_{X_c} + 1),
\end{aligned}$$

the last inequality is by the smallness of $\|F_c\|_{C^0}$. So

$$\|\mathscr{T}_c[J,w](t,\theta,\xi)\|_{X_c} \leq 2C_h e^{\beta t}(\|\xi\|_{X_c} + 1).$$

Taking derivatives under the integral sign (which is justified by the uniform integrability of the result) and applying the chain rule of derivatives, we obtain (for any time $t \in \mathbb{R}$).

$$
\begin{aligned}
D_\xi(\mathscr{T}_c[J,w])(t,\theta,\xi) = e^{\mathscr{A}_c t} \\
&+ \int_0^t e^{\mathscr{A}_c(t-\tau)}\Big\{ D_2F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\quad \cdot D_\xi J_\tau(\theta,\xi) + D_3F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\quad \cdot D_2 w(\Phi_\tau(\theta), J_\tau(\theta,\xi)) D_\xi J_\tau(\theta,\xi)\Big\} d\tau.
\end{aligned}
\tag{3.20}
$$

From (3.1) and the fact that $J \in \mathcal{L}_\delta, w \in \mathscr{X}_1$ we have

$$
\begin{aligned}
&\Big\| \int_0^t e^{\mathscr{A}_c(t-\tau)}\Big\{ D_2F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) D_\xi J_\tau(\theta,\xi) \\
&\quad + D_3F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\qquad \cdot D_2 w(\Phi_\tau(\theta), J_\tau(\theta,\xi)) D_\xi J_\tau(\theta,\xi)\Big\} d\tau \Big\|_{X_c,X_c} \\
&\quad \le \int_0^t C_h e^{\beta_3^+(t-\tau)} \|DF_c\|_{C^0} \Big[\|D_\xi J_\tau\|_{X_c,X_c} + \|D_2 w\|_{C^0} \|D_\xi J\|_{X_c,X_c} \Big] d\tau \\
&\quad \le 4C_h^2 \|F_c\|_{C^1} e^{\beta t} \int_0^t e^{-(\beta-\beta_3^+)(t-\tau)} d\tau \\
&\quad = \frac{2\|F_c\|_{C^1}}{\beta-\beta_3^+}(1-e^{(\beta_3^+-\beta)t}) e^{\beta t} \le C_h e^{\beta t},
\end{aligned}
$$

the last inequality is from the assumption that $\|F_c\|_{C^1}$ is small enough $(\le \frac{\beta-\beta_3^+}{2})$. All the estimates above yield

$$
\|D_\xi(\mathscr{T}_c[J,w])(t,\theta,\xi)\|_{X_c,X_c} \le \|e^{\mathscr{A}_c t}\|_{X_c,X_c} + C_h e^{\beta t} \le 2C_h e^{\beta t}.
$$

Then we give the estimate about $D_2^i(\mathscr{T}_c[J,w])$ with $2 \le i \le r$. Note that

$$
\begin{aligned}
D_\xi^i(\mathscr{T}_c[J,w])(t,\theta,\xi) &= \int_0^t e^{\mathscr{A}_c(t-\tau)}\Big\{ D_2F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\quad \cdot D_2^i J_\tau(\theta,\xi) + D_3F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\quad \cdot D_2 w(\Phi_\tau(\theta), J_\tau(\theta,\xi)) D_2^i J_\tau(\theta,\xi) + R_i^{J,w}(\tau,\theta,\xi)\Big\} d\tau,
\end{aligned}
\tag{3.21}
$$

where $R_i^{J,w}(\tau,\theta,\xi)$ is a sum of monomials, e.g., $(D_\xi^{i_1}J)^{\otimes j_1}\cdots(D_\xi^{i_m}J)^{\otimes j_m}$ with $i_1j_1+i_2j_2+\cdots+i_mj_m = i$, except for the two terms we give in the integral, whose factors are derivatives of F (evaluated at J, w) and of w up to order i. This last statement about the order of the derivatives of J is a consequence of the Faa Di Bruno formula (A.3).

It is easy to see that all the derivatives of J appearing in the derivatives of $\mathscr{T}$ are of order at most i and we have pulled out explicitly the terms containing derivatives of J of order i. We call also attention to the fact all the monomials in $R_i^{J,w}(\tau,\theta,\xi)$ contain at least one factor which is a derivative of F. Hence, by taking into account that $J \in \mathcal{L}_\delta, w \in \mathscr{X}_1$ and assuming $\|F\|_{C^i}$ is sufficiently small and applying (3.1), we get the the following estimate

$$\begin{aligned}
&\Big\| \int_0^t e^{\mathscr{A}_c(t-\tau)}\Big\{ D_2F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))\\
&\qquad \cdot D_2^i J_\tau(\theta,\xi) + D_3F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))\\
&\qquad \cdot D_2 w(\Phi_\tau(\theta), J_\tau(\theta,\xi)) D_2^i J_\tau(\theta,\xi) + R_i^{J,w}(\tau,\theta,\xi)\Big\} d\tau \Big\|_{X_c^{\otimes i}, X_c} \qquad (3.22)\\
&\le c_{i,h}\|F_c\|_{C^i} e^{i\beta t}\int_0^t e^{-(i\beta-\beta_3^+)(t-\tau)}d\tau\\
&\le \frac{c_{i,h}\|F_c\|_{C^i}}{i\beta-\beta_3^+}(1-e^{(\beta_3^+-i\beta)t})e^{i\beta t} \le C_h e^{i\beta t},
\end{aligned}$$

where $c_{i,h}$ is a constant depending only on i, C_h and the last inequality of (3.22) is again from assumption on $\|F_c\|_{C^r}$. That is

$$\|D_\xi^i(\mathscr{T}_c[J,w])(t,\theta,\xi)\|_{X_c^{\otimes i},X_c} \le C_h e^{i\beta t}, \quad 2\le i\le r.$$

This finishes the verification of $\mathscr{T}_c(\mathcal{L}_\delta)\subset\mathcal{L}_\delta$.

We also note that if the derivatives of order up to r of w are uniformly continuous, so is the $D^r(\mathscr{T}[J,w])$.

Remark 3.1. Note that, when r grows, the smallness assumption required for $\|F\|_{C^r}$ becomes more severe. Hence, we can not prove that the slow manifolds constructed here are C^∞. Indeed, it is well known ([11, 40]) that even in finite-dimensional autonomous systems there are cases where the center manifolds are only finitely differentiable.

Now we give the estimate about $\mathscr{T}_s[J,w]$. For (3.14) from the smallness of $\|F_s\|_{C^0}$ we have

$$\begin{aligned}
\|\mathscr{T}_s[J,w](\theta,\xi)\|_{X_s} &\le \|F_s\|_{C^0}\int_{-\infty}^0 e^{\beta_1 t}|t|^{-\alpha_1}dt,\\
&= \|F_s\|_{C^0}\int_0^\infty e^{-\beta_1 t}|t|^{-\alpha_1}dt,\\
&\le \|F_s\|_{C^0}\Big(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1}\Big)\le 1.
\end{aligned}$$

For $1\le i\le r$, we have

$$\begin{aligned}
D_\xi^i\Big(\mathscr{T}_s[J,w](\theta,\xi)\Big) = \int_{-\infty}^0 e^{-\mathscr{A}_s t}\Big\{& D_2F_s(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))\\
&\cdot D_2^i J_\tau(\theta,\xi) + D_3F_s(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi)))\\
&\cdot D_2 w(\Phi_\tau(\theta), J_\tau(\theta,\xi)) D_2^i J_\tau(\theta,\xi) + Q_i^{J,w}(t,\theta,\xi)\Big\}dt,
\end{aligned}$$

where $Q_i^{J,w}(t,\theta,\xi)$ is a sum of monomials like $R_i^{J,w}(\tau,\theta,\xi)$ in (3.21). Note that $\beta_1 > r\beta$, then with the same tricks to get (3.22) we obtain

$$\begin{aligned}
\|D_\xi^i\Big(\mathscr{T}_s[y^s,w](\theta,\xi)\Big)\|_{X_c^{\otimes i},X_s} &\le C_{i,h}\|F_s\|_{C^i}\int_{-\infty}^0 e^{\beta_1 t}|t|^{-\alpha_1}e^{i\beta|t|}dt\\
&\le C_{i,h}\|F_s\|_{C^i}\Big(\frac{1}{\beta_1-i\beta}+\frac{1}{1-\alpha_2}\Big)\le 1,
\end{aligned}$$

the last inequality is from the smallness of $\|F_s\|_{C^i},\ 1\le i\le r$.

We omit the calculations about $\mathscr{T}_u[J,w]$ since we just need to copy the calculations about the stable direction by a change of the direction of time and replacing the parameter β_1 with β_2 and noting $\beta_2 > r\beta$.

Remark 3.2. As remarked in [13], it is amusing to note that, since the sets $\mathcal{L}_\delta$ and $\mathcal{X}_1$ are convex and compact in the C^0 topology, which makes $\mathscr{T}$ continuous, we can apply the Schauder fixed point theorem and obtain the existence (but not the uniqueness) of a fixed point at this stage.

This remarks applies to many of the textbook proofs the invariant manifold theorem based on functional analysis since many of them involve some propagated bounds in the proofs in [7, 11, 40].

Of course, Step 2, provides uniqueness, gives a constructive algorithm to find the fixed point, allows to validate approximate calculations, gives slightly better regularity. One can argue that the contraction mapping is more elementary than Schauder fixed point theorem even if it requires more work.

Step 2. $\mathscr{T}$ is a contraction under the metric $d-$distance defined by (3.19). Adding and subtracting terms and using the triangle inequality, for fixed τ we obtain

$$
\begin{aligned}
\big\| F_\sigma(\Phi_\tau(\theta), J_\tau(\theta,\xi), & w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&- F_\sigma(\Phi_\tau(\theta), \widetilde{J}_\tau(\theta,\xi), \widetilde{w}(\Phi_\tau(\theta), \widetilde{J}_\tau(\theta,\xi)))\big\|_{X_\sigma} \\
&\le \|F_\sigma\|_{C^1} e^{\beta|\tau|}\big[2\|J_\tau - \widetilde{J}_\tau\|_{C^0}^{(\beta} + \|J_\tau\|_{C^0}^{(\beta}\|w-\widetilde{w}\|_{C^0}\big] \\
&\le 4C_h\|F_\sigma\|_{C^1} e^{\beta|\tau|}\|(J-\widetilde{J}, w-\widetilde{w})\|_{C^0}.
\end{aligned}
\tag{3.23}
$$

For the operator $\mathscr{T}_c$, from (3.23) we get (for $t>0$)

$$
\begin{aligned}
\|\mathscr{T}_c&[J,w](t,\theta,\xi) - \mathscr{T}_c[\widetilde{J},\widetilde{w}](t,\theta,\xi)\|_{X_c} \\
&\le \bigg\| \int_0^t e^{\mathscr{A}_c(t-\tau)}\Big[F_c(\Phi_\tau(\theta), J_\tau(\theta,\xi), w(\Phi_\tau(\theta), J_\tau(\theta,\xi))) \\
&\qquad - F_c(\Phi_\tau(\theta), \widetilde{J}_\tau(\theta,\xi), \widetilde{w}(\Phi_\tau(\theta), \widetilde{J}_\tau(\theta,\xi)))\Big] d\tau \bigg\|_{X_c} \\
&\le 4\|F_c\|_{C^1}\|(J-\widetilde{J}, w-\widetilde{w})\|_{C^0} e^{\beta t}\int_0^t e^{-(\beta-\beta_3^+)(t-\tau)} d\tau \\
&\le \frac{4\|F_c\|_{C^1}}{\beta-\beta_3^+} e^{\beta t}\|(J-\widetilde{J}, w-\widetilde{w})\|_{C^0},
\end{aligned}
$$

that is

$$
\|\mathscr{T}_c[J,w] - \mathscr{T}_c[\widetilde{J},\widetilde{w}]\|_{C^0}^{(\beta} \le \frac{4\|F_c\|_{C^1}}{\beta-\beta_3^+}\|(J-\widetilde{J}, w-\widetilde{w})\|_{C^0}. \tag{3.24}
$$

Similarly, for the operators $\mathscr{T}_s$ and $\mathscr{T}_u$, from (3.23), we get

$$
\begin{aligned}
\|\mathscr{T}_s&[J,w](\theta,\xi) - \mathscr{T}_s[\widetilde{J},\widetilde{w}](\theta,\xi)\|_{X_s} \\
&\le \bigg\| \int_{-\infty}^0 e^{\mathscr{A}_s t}\Big[F_s(\Phi_t(\theta), J_t(\theta,\xi), w(\Phi_t(\theta), J_t(\theta,\xi))) \\
&\qquad - F_s(\Phi_t(\theta), \widetilde{J}_t(\theta,\xi), \widetilde{w}(\Phi_t(\theta), \widetilde{J}_t(\theta,\xi)))\Big] dt \bigg\|_{X_s}
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\leq 4\|F_s\|_{C^1}\|(J-\widetilde{J},w-\widetilde{w})\|_{C^0}\int_0^{\infty}e^{-(\beta_1-\beta)t}|t|^{-\alpha_1}dt \\
&\leq 4\|F_s\|_{C^1}\Big(\frac{1}{\beta_1-\beta}+\frac{1}{1-\alpha_1}\Big)\|(J-\widetilde{J},w-\widetilde{w})\|_{C^0},
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
&\|\mathscr{T}_u[J,w](\theta,\xi)-\mathscr{T}_u[\widetilde{J},\widetilde{w}](\theta,\xi)\|_{X_u} \\
&\qquad\leq 4\|F_u\|_{C^1}\Big(\frac{1}{\beta_2-\beta}+\frac{1}{1-\alpha_2}\Big)\|(J-\widetilde{J},w-\widetilde{w})\|_{C^0}.
\end{aligned}
\tag{3.27}
$$

From (3.24)-(3.27) we obtain

$$
d(\mathscr{T}[J,w],\mathscr{T}[\widetilde{J},\widetilde{w}])\leq \kappa d((J,w),(\widetilde{J},\widetilde{w})),
$$

where $\kappa=4c\|F\|_{C^1}$ with

$$
c=\max\Big\{\frac{1}{\beta-\beta_3^{\pm}},\frac{1}{\beta_1-\beta}+\frac{1}{1-\alpha_1},\frac{1}{\beta_2-\beta}+\frac{1}{1-\alpha_2}\Big\}.
$$

The smallness of $\|F\|_{C^1}$ guarantees that $0<\kappa<1$, *i.e.*, the operator $\mathscr{T}$ is a contraction under the $d-$distance defined by (3.19). Then from the contraction fixed theorem we know that there is a unique solution of (3.12), (J_*,w_*), which is in the C^0 closure of $\mathcal{L}_\delta\times\mathcal{X}_1$. From Lemma 1.1 and the definition of $\mathcal{X}_1$ we know that the function w_* is $C^{r-1+Lip}$ derivatives in the variables ξ.

3.5. **Two a-*posteriori* estimates.** In this subsection we will give two a-*posteriori* estimates, the first estimate is under the distance defined by C^0- norm and the second estimate is under the distance defined by the $C^{rs}-$ norm $(0<s\leq 1)$.

3.5.1. *The a−posteriori estimate under the distance defined by C^0- norm.* Denote $(J,w)\in\mathcal{L}_\delta\times\mathcal{X}_1$ be any approximated solution of the functional functions defined by (3.12), that is

$$
\|\mathscr{T}[J,w]-(J,w)\|_{C^0}^{(\beta}<\epsilon. \tag{3.28}
$$

By the discussions in Subsection 3.4 we know that the operator $\mathscr{T}$ defined by (3.12) is a contract operator on $\mathcal{L}_\delta\times\mathcal{X}_1$, that is

$$
\|\mathscr{T}[J,w]-\mathscr{T}[\widetilde{J},\widetilde{w}]\|_{C^0}^{(\beta}\leq\kappa\|(J,w)-(\widetilde{J},\widetilde{w})\|_{C^0}^{(\beta}=\kappa\|(J-\widetilde{J},w-\widetilde{w})\|_{C^0}^{(\beta}, \tag{3.29}
$$

where $0<\kappa<1$ is the one in Subsection 3.4. Denote, moreover, that (J_*,w_*) is the true solution of the functional functions defined by (3.12). Then by the inequalities (3.28) and (3.29) we have

$$
\|(J-J_*,w-w_*)\|_{C^0}^{(\beta}<\frac{\epsilon}{1-\kappa},
$$

which implies

$$
\|w-w_*\|_{C^0}^{(\beta}<\frac{\epsilon}{1-\kappa}.
$$

We refer [24] or the discussions below for details.

3.5.2. *The $a-posteriori$ estimate under the distance defined by $C^{rs}-$ norm.* In this section we give estimates for higher derivatives of $(w-w_*)$ by using Hadamard interpolation given in Theorem 1.1 in the Appendix, (see [22] and [13] for similar arguments).

In (3.18), we define the $\|\cdot\|^{(\beta}_{C^0}$ for any β given in (3.3). For any $0<s<1$, we also define the norms $\|\cdot\|^{(r\beta}_{C^r}$ and $\|\cdot\|^{(rs\beta}_{C^{rs}}$ with the same definition of $\|\cdot\|^{(\beta}_{C^0}$ with C^0 and β replaced by C^r (resp. C^{rs}) and $r\beta$ (resp. $rs\beta$), respectively.

Let $(J,w)\in\mathcal{L}_\delta\times\mathcal{X}_1$ be the approximated solution of the functional functions defined by (3.12), and satisfying the inequality (3.28). From (3.29) we obtain

$$\|\mathscr{T}[J,w]-\mathscr{T}^2[J,w]\|^{(\beta}_{C^0}\le\kappa\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0}.$$

Inductively,

$$\|\mathscr{T}^n[J,w]-\mathscr{T}^{n+1}[J,w]\|^{(\beta}_{C^0}\le\kappa^n\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0},\ n=1,2,\cdots. \tag{3.30}$$

Moreover, note that $\mathscr{T}^n[J,w]\in\mathcal{L}_\delta\times\mathcal{X}_1,\ n=1,2,\cdots$, then we have

$$\|\mathscr{T}^n[J,w]\|^{(r\beta}_{C^r}\le c,$$

which implies

$$\|\mathscr{T}^n[J,w]-\mathscr{T}^{n+1}[J,w]\|^{(r\beta}_{C^r}\le 2c. \tag{3.31}$$

For any $0\le s\le1$, by (3.30), (3.31) and the Hadamard interpolation theorem, Theorem 1.1, we have

$$\begin{aligned}
&\|\mathscr{T}^n[J,w]-\mathscr{T}^{n+1}[J,w]\|^{(rs\beta}_{C^{rs}}\\
&\qquad\le C\Big(\|\mathscr{T}^n[J,w]-\mathscr{T}^{n+1}[J,w]\|^{(r\beta}_{C^r}\Big)^s\cdot\Big(\|\mathscr{T}^n[J,w]-\mathscr{T}^{n+1}[J,w]\|^{(\beta}_{C^0}\Big)^{1-s}\\
&\qquad\le C(2c)^s\Big(\kappa^n\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0}\Big)^{1-s}\\
&\qquad= C(2c)^s\kappa^{n(1-s)}\Big(\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0}\Big)^{1-s}.
\end{aligned}$$

Let (J_*,w_*) be the fixed point of the operator $\mathscr{T}$ obtained in Subsection 3.4, that is $(J_*,w_*)=\lim_{n\to\infty}\mathscr{T}^n[J,w]$. By the inequality above we obtain

$$\begin{aligned}
\|(J_*,w_*)-(J,w)\|^{(rs\beta}_{C^{rs}}&=\|\lim_{n\to\infty}\mathscr{T}^n[J,w]-(J,w)\|^{(rs\beta}_{C^{rs}}\\
&\le\sum_{n=0}^{\infty}\|\mathscr{T}^{n+1}[J,w]-\mathscr{T}^n[J,w]\|^{(rs\beta}_{C^{rs}}\\
&\le\sum_{n=0}^{\infty}C(2c)^s\kappa^{n(1-s)}\Big(\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0}\Big)^{1-s}\\
&\le\ \frac{C(2c)^s}{1-\kappa^{(1-s)}}\Big(\|(J,w)-\mathscr{T}[J,w]\|^{(\beta}_{C^0}\Big)^{1-s}\le\frac{C(2c)^s\epsilon^{1-s}}{1-\kappa^{(1-s)}},
\end{aligned}$$

where the last inequality is from (3.28). The inequality above implies

$$\|w_*-w\|^{(rs\beta}_{C^{rs}}\le\ \frac{C(2c)^s\epsilon^{1-s}}{1-\kappa^{(1-s)}}.$$

3.6. **The differentiability in θ.** In this section, we give the estimates about the derivatives in θ of w_*. These, of course, require that the flow B is differentiable in θ.

Lemma 3.1. *Let $\Phi_t(\theta)$ be a semi-flow defined in (2.4) which satisfies the hypothesis* **(H6)***, then for any $1 \le j \le r$ we have*

$$\begin{aligned} \sup_{t\in[0,\infty),\theta\in\mathscr{C}} \|D^j_\theta \Phi_t(\theta)\|_{\mathscr{C}^{\otimes j},\mathscr{C}} \le Ce^{j\beta_3^+ t} t^j, \\ \sup_{t\in(-\infty,0],\theta\in\mathscr{C}} \|D^j_\theta \Phi_t(\theta)\|_{\mathscr{C}^{\otimes j},\mathscr{C}} \le Ce^{j\beta_3^- |t|} |t|^j. \end{aligned} \tag{3.32}$$

For simplicity of notation we will prefer to deal only with exponentials and note that the polynomial factors $|t|^j,\ j=1,\cdots,r$, estimated by an exponential times a constant, so we worsen slightly the exponents and add a multiplicative constant.

$$\begin{aligned} \sup_{\theta\in X} \|D^j_\theta \Phi_t(\theta)\|_{\mathscr{C}^{\otimes j},Y} &\le Ce^{-j}\epsilon^{\frac{-j}{4r}} e^{j(\beta_3^+ + \epsilon^{\frac{1}{4r}})t} \\ &\le c\epsilon^{\frac{-1}{4}} e^{j(\beta_3^+ + \epsilon^{\frac{1}{4r}})t},\ t\in[0,\infty),\ 1\le j\le r, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \sup_{\theta\in X} \|D^j_\theta \Phi_t(\theta)\|_{\mathscr{C}^{\otimes j},Y} &\le Ce^{-j}\epsilon^{\frac{-j}{4r}} e^{j(\beta_3^- + \epsilon^{\frac{1}{4r}})|t|} \\ &\le c\epsilon^{\frac{-1}{4}} e^{j(\beta_3^- + \epsilon^{\frac{1}{4r}})|t|},\ t\in(-\infty,0],\ 1\le j\le r, \end{aligned} \tag{3.34}$$

where $c = \max\{C_h,\ C\}$.

Proof. This is just the Lemma 5.2 of [26]. □

Let us go back to the functional equation (3.12). We define the set on which the operator $\mathscr{T}$ defined in (3.12) act

$$\begin{aligned} \mathcal{H} = \Big\{ &(J,w) : J : \mathbb{R}\times\mathscr{C}\times X_c \to X_c,\quad w : \mathscr{C}\times X_c \to X_{su}, \\ &\|J_t(\theta,\xi)\|_{X_c} \le 2C_h e^{\beta|t|}(\|\xi\|_{X_s}+1), \|D^j_\theta J_t(\theta,\xi)\|_{\mathscr{C}^{\otimes j},X_c} \le C_h e^{\beta|t|}, 1\le j\le r, \\ &\|D^j_\theta w(\theta,\xi)\|_{\mathscr{C}^{\otimes j},X_{su}} \le 1, 0\le j\le r \Big\}. \end{aligned}$$

We adopt the weighted norms $\|\cdot\|^{(\beta}_{C^0}, \|\cdot\|_{C^0}$ and $\|(\cdot_1,\cdot_2)\|_{C^0}$ which are defined in (3.18) for the functions J, w and (J,w), respectively. The induced metric on $\mathcal{H}$ is also the $d-$distance defined in (3.19).

Following the standard strategy in center manifold theory, we will prove that the operator $\mathscr{T} = (\mathscr{T}_c, \mathscr{T}_{su})$ is a contraction in $\mathcal{H}$. In this situation, we can appeal to [40, Proposition A2] (which shows that the C^0 closure of functions with uniformly bounded C^r norms is $C^{r-1+Lip}$) or to Hadamard's interpolation theorem (See Theorem 1.1) which shows that a C^0 contraction in spaces of uniformly bounded C^r functions also converges in C^{r-1}.

Step 1. $\mathscr{T}(\mathcal{H}) \subset \mathcal{H}$.

The estimate about $\mathscr{T}(J,w)$ has been given in Section 3.4, we omit the details. From (3.13) -(3.15) we obtain

$$
\begin{aligned}
&D_\theta \mathscr{T}_s[J,w](\theta,\xi)\\
&=\int_{-\infty}^{0} e^{-tA_s}\Big\{D_1F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta\Phi_t(\theta)\\
&\quad + D_2F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta J_t(\theta,\xi)\\
&\quad + D_3F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))\\
&\quad \cdot\big[D_1w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta\Phi_t(\theta)+D_2w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta J_t(\theta,\xi)\big]\Big\}dt,
\end{aligned}
$$

$$
\begin{aligned}
&D_\theta \mathscr{T}_u[J,w](\theta,\xi)\\
&=-\int_{0}^{\infty} e^{-tA_u}\Big\{D_1F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta\Phi_t(\theta)\\
&\quad + D_2F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta J_t(\theta,\xi)\\
&\quad + D_3F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))\\
&\quad \cdot\big[D_1w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta\Phi_t(\theta)+D_2w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta J_t(\theta,\xi)\big]\Big\}dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&D_\theta \mathscr{T}_c[J,w](t,\theta,\xi)\\
&\quad=\int_{0}^{t} e^{A_c(t-\tau)}\Big\{D_1F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi)))D_\theta\Phi_\tau(\theta)\\
&\quad\quad + D_2F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi)))D_\theta J_\tau(\theta,\xi)\\
&\quad\quad + D_3F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi)))\\
&\quad\quad \cdot[D_1w(\Phi_\tau(\theta),J_\tau(\theta,\xi))D_\theta\Phi_\tau(\theta)+D_2w(\Phi_\tau(\theta),J_\tau(\theta,\xi))D_\theta J_\tau(\theta,\xi)]\Big\}d\tau.
\end{aligned}
$$

With (3.17), (3.33) and note that $(J,w)\in\mathcal{H}$, we obtain, (we just consider the case $t\geq 0$),

$$
\begin{aligned}
\|D_\theta\mathscr{T}_s[J,w](\theta,\xi)\|_{\mathscr{C},X_s} &\leq \int_{-\infty}^{0} C_h e^{\beta_1 t}|t|^{-\alpha_1}\big[2c\epsilon^{\frac{-1}{4}}e^{(\beta_3^-+\epsilon^{\frac{-1}{4r}})|t|}+2C_he^{\beta|t|}\big]dt\|F_s\|_{C^r}\\
&\leq \int_{-\infty}^{0} 4cC_h\epsilon^{\frac{-1}{4}}e^{(\beta_1-\beta)t}|t|^{-\alpha_1}dt\|F_s\|_{C^r}\\
&=\int_{0}^{\infty} 4cC_h\epsilon^{\frac{-1}{4}}e^{-(\beta_1-\beta)t}t^{-\alpha_1}dt\|F_s\|_{C^1}\\
&\leq 4cC_h\epsilon^{\frac{-1}{4}}\|F\|_{C^1}\Big(\frac{1}{\beta_1-\beta}+\frac{1}{1-\alpha_1}\Big)\leq 1,
\end{aligned}
$$

where the last inequality follows from $\|F\|_{C^1}$ sufficiently small. Similarly, we also have

$$\|D_\theta\mathscr{T}_u[J,w](\theta,\xi)\|_{\mathscr{C},X_u}\leq 1,$$

and

$$
\begin{aligned}
\|D_\theta\mathscr{T}_c[J,w](t,\theta,\xi)\|_{\mathscr{C},X_c} &\leq \int_{0}^{t} C_h e^{\beta_3^+(t-\tau)}\big[2c\epsilon^{\frac{-1}{4}}e^{(\beta_3^++\epsilon^{\frac{-1}{4r}})|\tau|}+2C_he^{\beta|\tau|}\big]d\tau\|F_s\|_{C^r}\\
&\leq \int_{0}^{t} 4cC_h\epsilon^{\frac{-1}{4}}e^{\beta_3^+(t-\tau)}e^{\beta\tau}d\tau\|F_c\|_{C^1}
\end{aligned}
$$

$$= e^{\beta t}\int_0^t 4cC_h\epsilon^{\frac{-1}{4}}e^{-(\beta-\beta_3^+)t}dt\|F\|_{C^1}$$

$$\le 4cC_h\epsilon^{\frac{-1}{4}}e^{\beta t}\frac{\|F\|_{C^1}}{\beta-\beta_3^+},\ t>0.$$

Moreover, for $2 \le j \le r$ we have

$$\begin{aligned} D_\theta^j\mathscr{T}_s[J,w](\theta,\xi) = \int_{-\infty}^0 e^{-tA_s}\Big\{ & D_1F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta^j\Phi_t(\theta) \\ &+ D_2F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta^jJ_t(\theta,\xi) \\ &+ D_3F_s(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi))) \\ &\cdot D_2w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta^jJ_t(\theta,\xi) + P_j^{J,w}(t,\theta,\xi)\Big\}dt, \end{aligned}$$

$$\begin{aligned} D_\theta^j\mathscr{T}_u[J,w](\theta,\xi) = -\int_0^\infty e^{-tA_u}\Big\{ & D_1F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta^j\Phi_t(\theta) \\ &+ D_2F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi)))D_\theta^jJ_t(\theta,\xi) \\ &+ D_3F_u(\Phi_t(\theta),J_t(\theta,\xi),w(\Phi_t(\theta),J_t(\theta,\xi))) \\ &\cdot D_2w(\Phi_t(\theta),J_t(\theta,\xi))D_\theta^jJ_t(\theta,\xi) + X_j^{J,w}(t,\theta,\xi)\Big\}dt, \end{aligned}$$

and

$$\begin{aligned} D_\theta^j\mathscr{T}_c[J,w](t,\theta,\xi) = \int_0^t e^{A_c(t-\tau)}\Big\{ & D_1F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi)))D_\theta^j\Phi_\tau(\theta) \\ &+ D_2F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi)))D_\theta^jJ_\tau(\theta,\xi) \\ &+ D_3F_c(\Phi_\tau(\theta),J_\tau(\theta,\xi),w(\Phi_\tau(\theta),J_\tau(\theta,\xi))) \\ &\cdot D_2w(\Phi_\tau(\theta),J_\tau(\theta,\xi))D_\theta^jJ_\tau(\theta,\xi) + Y_j^{J,w}(\tau,\theta,\xi)\Big\}d\tau, \end{aligned}$$

where $P_j^{J,w}(t,\theta,\xi)$, $X_j^{J,w}(t,\theta,\xi)$, $Y_j^{J,w}(t,\theta,\xi)$ are sum of monomials like $R_j^{J,w}(\tau,\theta,\xi)$ in (3.21). Then, with the same tricks in (3.22), we obtain

$$\|D_\theta^jw_s(\theta,\xi)\|_{\mathscr{C}^{\otimes j},X_s} \le 1,\quad \|D_\theta^jw_s(\theta,\xi)\|_{\mathscr{C}^{\otimes j},X_u} \le 1,\quad 2\le j\le r,$$

$$\|D_\theta^jJ(\theta,\xi)\|_{\mathscr{C}^{\otimes j},X_c} \le C_he^{j\beta t},\ t>0, 2\le j\le r.$$

The calculations above yield $\mathscr{T}(\mathcal{H}) \subset \mathcal{H}$.

Step 2. $\mathscr{T}$ is a contraction in $\mathcal{H}$. Take any $(J,w),(\widetilde{J},\widetilde{w}) \in \mathcal{H}$, with the same calculations in Section 3.4, we obtain

$$d\big(\mathscr{T}[J,w],\mathscr{T}[\widetilde{J},\widetilde{w}]\big) < cd\big((J,w),(\widetilde{J},\widetilde{w})\big),$$

where $\kappa = 4c_1cC_h\epsilon^{-\frac{1}{4}}\|F\|_{C^1}$ with

$$c_1 = \max\left\{\frac{1}{\beta-\beta_3^\pm},\frac{1}{\beta_1-\beta}+\frac{1}{1-\alpha_1},\frac{1}{\beta_2-\beta}+\frac{1}{1-\alpha_2}\right\}.$$

The smallness of $\|F\|_{C^1}$ guarantees that $0<\kappa<1$, *i.e.*, the operator $\mathscr{T}$ is a contraction under the $d-$distance defined by (3.19). Then from the contraction fixed theorem we know that there is a unique solution of (3.12), (J_{**},w_{**}), which is in the C^0 closure of $\mathcal{H}$. From Lemma 1.1 and the definition of $\mathcal{H}$ we know that the function w_{**} is $C^{r-1+Lip}$ derivatives in the variables θ.

Note that (J_*, w_*) is also the fixed point of the functional equation (3.12) constructed in Section 3.4. By the uniqueness of the fixed point we know that $(J_*, w_*) = (J_{**}, w_{**})$. That is the function w_* is $C^{r-1+Lip}$ derivatives in the variables θ for the fixed $\xi \in X_c$ and $C^{r-1+Lip}$ derivatives in the variables ξ for the fixed $\theta \in \mathscr{C}$. That is, w_* satisfies the second conclusions of the Theorem 3.1.

4. **Applications.** This section is devoted to an application of Theorem 3.1 to concrete equations mentioned in the introduction.

4.1. **Elliptic equations in cylindrical domains.** The study of the deformation of beams or viscid channel flows leads to the study of elliptic problem (1.4) in a cylindrical domain $\widetilde{\Omega} = \mathbb{R} \times \Omega$ with Ω a bounded domain. Following [37, 42, 43, 44] we think of the the elliptic problem as a evolution equation when the role time is taken by the x variable. This problem, of course, is ill-posed.

Consider the externally forced nonlinear elliptic differential equation defined on a cylindrical domain defined in (1.4) under the periodic boundary conditions, note that this system is a second order differential equation, by setting $u_t = v$ and $z = (u, v)^T$, we obtain the evolution of z, which is a first order system,

$$\dot{z} = \mathcal{A}_\alpha z + f(\theta, x) + \mathcal{N}(\theta, x, z),\ z(t, x+1) = z(t, x),\ \alpha \geq 0, \tag{4.1}$$

with

$$\mathcal{A}_\alpha = \begin{pmatrix} 0 & 1 \\ -\partial_x^2 - \alpha & 0 \end{pmatrix}$$

and

$$f(\theta, x, z) = (0, \quad N_{40}(\theta, x) + N_{41}(\theta, x)u)^T,\ \ \mathcal{N}(z) = (0, \quad N_{42}(\theta, x, u, u_x))^T.$$

For $\zeta > 0$ and $m \in \mathbb{N}$, we denote by $H^{\zeta,m}$ the analytic functions u from $\mathbb{T}_\zeta$ to $\mathbb{C}$ with Fourier expansion $u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi i k x}$ such that the norm

$$\|u\|^2_{\zeta,m} = \sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \exp(4\pi\zeta|k|)(|k|^2+1)^m$$

is finite. Obviously, for any $\zeta > 0$ and $m \in \mathbb{N}$ the space $\left(H^{\zeta,m}, \|\cdot\|_{\zeta,m}\right)$ is a Hilbert space. It is a Banach algebra for $m > 1/2$. In such a case, the operator of composition on the right with an analytic function is an analytic operator in the space $H^{\zeta,m}$. For a longer discussion on properties of these spaces see [13].

Even though the perturbation N_{42} contains u_x, the perturbation in the first order system is bounded from X to Y if we set the Banach spaces X and Y as

$$X = Y = H^{\zeta,m} \times H^{\zeta,m-1},\ \zeta > 0,\ \ m > \frac{3}{2}. \tag{4.2}$$

The restriction $m > 3/2$ ensures that both components of the space are Banach algebras under space multiplication.

Remark 4.1. Note that $\psi_{k,\tau}(x) = \left(e^{\mathrm{i}2\pi kx},\ \lambda_{k,\tau} e^{\mathrm{i}2\pi kx}\right)^T$ is the eigenvector of the operator $\mathcal{A}_\alpha$ belongs to the eigenvalue $\lambda_{k,\tau} = \tau\sqrt{(2\pi k)^2 - \alpha},\ \tau \in \{1,\ -1\}, k \in \mathbb{Z}$. Thus we take $\{\psi_{k,\tau}(x)\}_{k\in\mathbb{Z},\tau\in\{1,\ -1\}}$ as a basis of the space X.

If there exists α such that $\alpha = (2\pi k_*)^2$ $(k_* \in \mathbb{Z})$, then this α is called a resonance and very interesting phenomena happen when α changes around these values [12].

Proposition 4.1. *Assume that the function* $(N_{40} + N_{41}u)$ *is analytic and small enough under the analytic topology. Moreover,* N_{42} *is higher order in* u *and analytic. Then the nonlinearity* $\mathcal{N}$ *and the function* f *defined above are analytic from* X *into* X *when* $m > 3/2$.

Of course, in our main theorem, we use C^r regularity, which is implied by the analytic regularity. On the other hand, even with analytic regularity in the nonlinearity we will not obtain analytic center manifolds, only finitely differentiable ones. See Remark 3.1.

Note that the regularity of a manifold in a Banach space is very different from the regularity of the functions that constitute the space. Using Banach spaces of functions analytic in x allows us to obtain solutions which are analytic in x. In the examples we know of in the literature, the nonlinearities are analytic. A theory with finitely differentiable nonlinearities is, of course, possible, but then, it would be better to use Banach spaces of functions that are Sobolev regular in their arguments.

Proof. We note that the function $u \to u$ is linear and bounded from $H^{\zeta,m}$ to $H^{\zeta,m}$ and the function $u \to u_x$ is bounded from $H^{\zeta,m}$ to $H^{\zeta,m-1}$. Since the function N_{42} : $u \mapsto N_{42}(\theta, x, u, u_x)$ is analytic in its all variable we know that $N_{42}(\theta, x, u, u_x) \in H^{\zeta,m-1}$. Then the nonlinearity $\mathcal{N}(\theta, x, u, u_x) \in X$. The discussions about f is the same.

Form the proposition above we know that the perturbation in the system (4.2) is bounded from the Banach space X to itself. Then for the operators U^σ, $\sigma = u, s, c$, generated by the linear operator $\mathcal{A}_\alpha$, we just give the estimates about the standard trichotomy estimates (3.2) and we do not need to give the calculations to get the estimates in (3.1).

Lemma 4.1. *Fix* $\alpha > 0$. *Then, there exist,* $\beta_1 > \beta_3^- \geq 0$ *and* $\beta_2 > \beta_3^+ \geq 0$ *and a splitting of spectrum of linear operator* $\mathcal{A}_\alpha, \lambda_{k,\tau} = \left\{ \ \tau\sqrt{(2\pi k)^2 - \alpha} \ \right\}_{k\in\mathbb{Z},\tau\in\{-1,1\}}$, *i.e.,*

$$Spec(\mathcal{A}_\alpha) = \sigma_s \cup \sigma_u \cup \sigma_c,$$

where

$$\begin{aligned}
\sigma_s &= \{\lambda_{k,\tau} : \ Re\lambda_{k,\tau} < -\beta_1, \quad k \in \mathbb{Z}\}, \\
\sigma_u &= \{\lambda_{k,\tau} : \ Re\lambda_{k,\tau} > \beta_2, \quad k \in \mathbb{Z}\}, \\
\sigma_c &= \left\{\lambda_{k,\tau} : \ -\beta_3^- \leq Re\lambda_{k,\tau} \leq \beta_3^+, \quad j \in \mathbb{Z}\right\},
\end{aligned}$$

with $\beta_1 > \beta_3^- \geq 0$ *and* $\beta_2 > \beta_3^+ \geq 0$.

Proof. The eigenvalues are discrete and go to infinity with $k \to \infty$. Hence, it suffices to choose the β's in such a way that there is no eigenvalue with these real parts. Then the sets σ_s, σ_u and σ_c are disjoint and cover all the eigenvalues. □

We call the spectrum belongs to $\sigma_s \cup \sigma_u$ and σ_c as the hyperbolic spectrum and the center spectrum, respectively. Obviously, the center spectrum is finite and the hyperbolic spectrum is well separated from the center spectrum.

We now come to the evolution operators and their smoothing properties. We have:

Lemma 4.2. *For* $\alpha \geq 0$, *the operator* $\mathcal{A}_\alpha$ *generates semi-group operators* $U^{s,u}$ *in positive and negative times and group operator* U^c *for all* t. *Furthermore, the* *following estimates hold*

$$\|U^s(t)\|_{X,X} \le e^{-\beta_1 t}, \quad t > 0,$$
$$\|U^u(t)\|_{X,X} \le e^{-\beta_2 |t|}, \quad t < 0,$$
$$\|U^c(t)\|_{X,X} \le e^{\beta_3^+ |t|}, \quad t \ge 0,$$
$$\|U^c(t)\|_{X,X} \le e^{\beta_3^- |t|}, \quad t \le 0,$$

where the parameters $\beta_i,\ i = 1, 2,$ *and* $\beta_3^\pm$ *are the ones in Lemma 4.1.*

Proof. Assume that $u \in X$ with the Fourier expansion

$$u = \sum_{k \in \mathbb{Z}} \widehat{u}_k(t) \psi_k(x)$$

and the norm

$$\|u\|_X^2 = \sum_{k \in \mathbb{Z}} |\widehat{u}_k(t)|^2 \exp(4\pi\zeta|k|)(|k|^2 + 1)^r.$$

Then

$$U^s(t)u = \sum_{k \in \mathbb{Z}_1} e^{\lambda_k t} \widehat{u}_k(t) \psi(x),\ t > 0,$$

so we have

$$\begin{aligned} \|U^s(t)u\|_X^2 &= \sum_{k \in \mathbb{Z}_1} |e^{\lambda_k t} \widehat{u}_k(t)|^2 \exp(4\pi\zeta|k|)(|k|^2 + 1)^r, \\ &= \sum_{k \in \mathbb{Z}_1} e^{2\lambda_k t} |\widehat{u}_k(t)|^2 \exp(4\pi\zeta|k|)(|k|^2 + 1)^r \qquad (4.3) \\ &\le e^{-2\beta_1 t} \|u\|_X^2, \quad t > 0, \end{aligned}$$

the last inequality is from Lemma 4.1, that is

$$\|U^s(t)\|_{X,X} \le e^{-\beta_1 t}, \quad t > 0.$$

Similarly, we also have

$$\|U^u(t)\|_{X,X} \le e^{-\beta_2 |t|}, \quad t < 0.$$

and

$$\|U^c(t)\|_{X,X} \le e^{\beta_3^+ |t|}, \quad t \ge 0, \quad \|U^c(t)\|_{X,X} \le e^{\beta_3^- |t|}, \quad t < 0.$$

□

4.2. **Some discussions about other models mentioned in the introduction.** For the CGL defined in (1.1), note that the perturbation N_{12} is bounded when we set the Banach spaces X and Y as

$$X = Y = H^{\zeta,m},\ \zeta > 0,\ m > \frac{1}{2}.$$

For the DCGL defined in (1.2), note that the perturbation N_{22} contains the term ∇u. Thus the perturbation N_{22} is unbounded form the Banach space X to itself, we have to use the "*two spaces approach*". We need to introduce the Banach space Y and give the calculations to get the estimates in (3.1). In this case, we set the Banach spaces X and Y as

$$X = H^{\zeta,m},\ Y = H^{\zeta,m-1},\ \zeta > 0,\ \ m > \frac{3}{2}.$$

There are more discussions about the DCGL (1.2) in Section 4.2 of the paper [13], we omit the details here.

Remark 4.2. Consider the DCGL (1.2) with the quasi-periodic forcing, that is $\mathscr{C} = \mathbb{T}^d$ and

$$\dot{\theta} = \omega,\ \omega \in \mathbb{R}^d.$$

Using the center reduction to the time dependent center manifold, it is possible to produce quasi-periodic solutions using finite dimensional KAM theory. Since the center manifold is only finitely differentiable, we can only obtain finitely differentiable solutions by using methods from finite dimensional KAM theory. It is well known that some number theoretical condition of the frequency ω is crucial for the KAM iteration. For example, the Diophantine condition, i.e.:

$$|\langle k, \omega\rangle| \geq \gamma |k|^{\tau},\quad \forall\ k \in \mathbb{Z}^d \setminus \{0\},\quad \gamma > 0,\ \tau > d,$$

or the Brjuno condition, i.e.:

$$\sum_{n\geq 0} \frac{1}{2^n} \max_{0<|k|\leq 2^n,\ k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} < \infty.$$

By assuming that the parameters r and b_1 in the linear part satisfy appropriate relations, the dimension of center manifold of the linear operator $r + (b_1 + \mathrm{i}b_2)\Delta$ is 2. By constructing an infinite-dimensional KAM Theorem, in [14] the authors constructed a class of $(d+2)-$ dimensional quasi-periodic solutions. The solutions produced in [14] are analytic – and can have Brjuno frequencies – in contract with the ones produced using our method which are finite differentiable and require Diophantine frequencies.

For the Boussinesq equation defined in (1.3), note that this system is the two order differential equation, by setting $u_t = \partial_x v$ and $w = (u, v)^T$, we obtain the evolution of w. Even though the perturbation N_{32} contains the term u_{xx}, the perturbation in the new system is bounded. In this case, we set the Banach spaces X and Y as

$$X = Y = H^{\zeta,m} \times H^{\zeta,m-1},\ \zeta > 0,\quad m > \frac{3}{2}.$$

The readers can refer to [25] for more details on (1.3).

Of course, there are many more possible examples in the literature. Notably, [25, 8] include also a discussion of the Boussinesq system which is more singular than the Boussinesq equation.

Appendix A. **Basic definitions.** In this appendix, we collect some basic definitions and results to make the paper more self-contained. The material is quite standard and indeed a very similar Appendix can be found in [13].

We call attention to Definition 3 which is a common definition in the field of invariant manifolds, but which is different from other definitions (e.g. Whitney definition) used in other areas of Mathematics. Note that Definition 3 involves uniform boundedness of the derivatives, which makes it into a Banach space, which is convenient for us. In contrast, the Whitney definition leads only to Frechet spaces.

Definition 3. Let X, Y be two Banach spaces. Let $O \subset X$ be an open set. We will denote by $C^r(O, Y)$ the space of all functions from X to Y which possess uniformly bounded continuous derivatives of orders $0, 1, ..., r$. We endow $C^r(O, Y)$ with the norm of the supremum of all the derivatives, e.g.

$$\|f\|_{C^r(O,Y)} = \max_{0\leq i\leq r} \sup_{\xi\in O} |[D^i f](\xi)|_{X^{\otimes i}, Y}. \tag{A.1}$$

The $|A|_{X^{\otimes i},Y} \equiv \sup_{|\xi_1|_X=1,\ldots|\xi_i|_X=1} |A(\xi_1,\ldots,\xi_i)|_Y$ is the usual norm of symmetric multilinear functions from X taking values in Y. As it is well known, the norm (A.1) makes $C^r(O,Y)$ a Banach space.

Definition 4. We will denote by $C^{r-1+Lip}(O,Y)$ the space of functions in $C^{r-1}(O, Y)$ whose $(r-1)-th$ derivative is Lipschitz. The Lipschitz constant is

$$Lip_{O,Y} D^{r-1} f = \sup_{\xi\neq\zeta} \frac{|D_f(\xi) - D_2^{r-1} f(\zeta)|_{X^{\otimes(r-1)},Y}}{\|\xi-\zeta\|_X}$$

and the norm in $C^{r-1+Lip}(O,Y)$ is the max of the C^{r-1} norm and $Lip_{O,Y} D^{r-1} f$.
Again this norm makes $C^{r-1+Lip}$ into a Banach space.

We note that since O may be not compact, this definition is different from the Whitney definition in which the topology is given by seminorms of suprema in compact sets. We will not use the Whitney definition of C^r in this paper.

Definition 5. An open set O is called a compensated domain if there is a constant such given $x, y \in O$ there is a C^1 path γ contained in O joining x, y such that $|\gamma| \leq C\|x-y\|$.

For O a compensated domain, we have the mean value theorem

$$\|f(x) - f(y)\|_Y \leq C\|f\|_{C^1(O,Y)}\|x-y\|_X. \tag{A.2}$$

In particular, C^1 functions in a compensated domain are Lipschitz. It is not difficult to construct non-compensated domains with C^1 functions which are not Lipschitz.

Of course a convex set is compensated and the compensation constant is 1. In our paper, we will just be considering domains which are balls or full spaces. See [23] for the effects of the compensation constants in many problems of the function theory.

A.1. **Hadamard interpolation theorem.** We have the following result:

Theorem 1.1. *Let O be a compensated domain. Let $f \in C^r(O,Y)$. Then defining $\eta(r) \equiv \|f\|_{C^r(O,Y)}$, we have that $\log(\eta(r))$ is convex in r.*
That is for $0 \leq \theta \leq 1$, $0 \leq a, b \leq r$, we have

$$||f||_{C^{\theta a+(1-\theta)b}(O,Y)} \leq C\|f\|^{\theta}_{C^a(O,Y)}\|f\|^{1-\theta}_{C^b(O,Y)}.$$

A proof of Theorem 1.1 extending for non-integer values of r for suitable definitions of C^r can be found in [23]. In finite dimensional spaces it was proved in [31]. See also [38]. We also note that the interpolation is a consequence of the existence of *Smoothing operators*.

For us, the following corollary will be important.

Corollary 1.1. *Assume that $\{f_n\}_{n=1}^{\infty} \subset C^r(O,Y)$ is such that $\|f_n\|_{C^r(O,Y)} \leq M$. Assume that $\|f_n - f_{n+1}\|_{C^0(O,Y)} \leq C\kappa^n$. Then,*

$$\|f_n - f_{n+1}\|_{C^{r-1}(O,Y)} \leq (2M)^{(r-1)/r} C^{1/r} \kappa^{n/r}.$$

Of course, even if the corollary is true for all values of κ, it is more interesting for $k < 1$ as it happens in contraction mapping principles.

Remark A.1. As we mentioned above, the interpolation Theorem 1.1 extends for non-integer values of r' with a suitable definition of the norm. With this definition, we have Corollary 1.1 for all values of $r' < r$. The same applies to the following result Corollary 1.2.

A further corollary of Corollary 1.1 is

Corollary 1.2. *Assume that* $\{f_n\}_{n=1}^{\infty} \subset C^r(O,Y)$ *is such that* $\|f_n\|_{C^r(O,Y)} \le M$.

Assume that for some $f_\infty \in C^0(O,Y)$ $\|f_n - f_\infty\|_{C^0(O,Y)} \to 0$. *Then, for all* $r' < r$,

$$f_\infty \in C^{r'}$$

and $f_n \to f_\infty$ *in* $C^{r'}$.

Proof. Given a subsequence f_{n_i} of f_n we can obtain a further subsequence $f_{n_{i_j}}$ which satisfies $||f_{n_{i_j}} - f_{n_{i_{j+1}}}||_C^0 \le (1/2)^j$. By Corollary 1.1 we get that $f_{n_{i_j}}$ converges in $C^{r'}$. This limit – in $C^{r'}$ sense – has to be f_∞.

It is an exercise in metric space topology that, if for all subsequences we can obtain a subsequence that converges and all these limits are the same, then, the original sequence converges. □

A.2. **Lanford's closure lemma.** The following result is [40, Proposition A2].

Lemma 1.1. *Let* O *be a convex set inside of a Banach space* X. *Let* Y *be another Banach space.*

Denote by $\mathcal{B}$ *the unit ball in* $C^r(O,Y)$.

Assume that $\{f_n\} \subset \mathcal{B}$ *and that for each value* x, $f_n(x)$ *converges weakly to* $f_\infty(x)$.

Then, $f_\infty \in C^{r-1+\mathrm{Lipschitz}}$ *and for* $1 \le j \le r-1$, $D^j f_n$ *converges uniformly to* $D^j f_\infty$.

The assumption of weak pointwise converge is, of course, much weaker than the assumption of uniform convergence, which is what will appear in our applications.

Note that, in finite dimensional spaces, Lemma 1.1 would be a consequence of Ascoli-Arzel'a theorem (indeed, the proof of [40, Proposition A2] relies in Ascoli-Arzel'a theorem for finite dimensional functions.

Even if the proof in [40] uses the fact that points are joined by straight lines, the result seems to generalize to compensated sets.

A.3. **Faa Di Bruno formula.**

Lemma 1.2. *Let* $g(x)$ *be defined on a neighborhood of* x_0 *and have derivatives up to order* n *at* x_0. *Let* $f(y)$ *be defined on a neighborhood of* $y_0 = g(x_0)$ *and have derivatives up to order* n *at* y_0. *Then, the* n*-th derivative of the composition* $h(x) = f[g(x)]$ *at* x_0 *is given by the formula*

$$h_n = \sum_{k=1}^{n} f_k \sum_{p(n,k)} n!\Pi_{i=1}^{n} \frac{g_i^{\lambda_i}}{(\lambda_i!)(i!)^{\lambda_i}}. \tag{A.3}$$

In the above expression, we set

$$h_n = \frac{d^n}{dx^n} h(x_0), \quad f_k = \frac{d^k}{dy^k} f(y_0), \quad g_i = \frac{d^i}{dy^i} g(x_0)$$

and

$$p(n,k) = \left\{ (\lambda_1, \cdots, \lambda_n) : \lambda_i \in \mathbb{N}, \sum_{i=1}^{n} \lambda_i = k, \sum_{i=1}^{n} i\lambda_i = n \right\}.$$

The formula (A.3) without an explicit expression of the combinatorial coefficients was obtained in [3].

The explicit computation of the combinatorial coefficients is less straightforward, but can be found in [1].

A.4. **Functions of several variables and partial regularity.** In several applications, we have to consider functions of several variables. One can think of one as the regularity of the function and the other is the regularity with respect to parameters.

In some of our applications it is easy to estimate the regularity in each of the variables since they play a different role.

The following result shows that if we can estimate the derivatives in each of the variables, we can obtain automatically also the mixed derivatives.

Lemma 1.3. *Let X_1, X_2, Y be Banach spaces. $O_1 \subset X_1, O_2 \subset X_2$ be convex, bounded sets.*

Let $f : O_1 \times O_2 \to Y$ be a continuous function.

Assume that for all $x_1 \in O_1, x_2 \in O_2$, $i, j \le r$, we have

$$\begin{aligned} ||\partial^i_{x_1} f(x_1, x_2)|| &\le M < \infty, \\ ||\partial^j_{x_2} f(x_1, x_2)|| &\le M < \infty. \end{aligned} \tag{A.4}$$

Then, for every n, m such that $n+m < r$, we have that the function f admits mixed partial derivatives $\partial^n_{x_1} \partial^m_{x_2} f$. Furthermore, we have

$$\sup_{x_1 \in O_1, x_2 \in O_2} \partial^n_{x_1} \partial^m_{x_2} f(x_1, x_2) \le \Gamma(M, O_1, O_2).$$

Of course, in analytic regularity, the fact that analyticity in several complementary directions is the celebrated Hartog's theorem [39]. In our case, we are assuming that the functions are bounded, but the Hartog's theorem does not need that assumption. The Hartog's theorem is much easier under the assumption that the functions are bounded.

For finite dimensional spaces X_1, X_2 this result is a classical result in the theory of Riesz potentials. A modern proof can be found in [38, Lemma 9.1] and [50]. This result is the basis of many results in the regularity theory of elliptic equations. There are also results when the number of derivatives is asymmetric and also for fractional derivatives.

Results of this type were found useful in the theory of Anosov systems when the partial derivatives along the coordinate axis are generalized to be partial derivatives along stable and unstable foliations [21, Lemma 2.5]. A more elementary and more general proof based on the theory of Morrey-Campanato spaces is in [36]. A very elementary proof using just the converse Taylor theorem and generalizing to some fractal sets is in [20]. To go from the finite dimensional proofs above to the infinite dimensional case, it suffices to take finite dimensional sections and observe that the bounds obtained are independent of the finite dimensional space considered.

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