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The financial value of knowing the distribution of stock prices in discrete market models

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An explicit formula is derived for the value of weak information in a discrete-time model that works for a wide range of utility functions, including the logarithmic utility and power utility. We assume a complete market with a finite number of assets and a finite number of possible outcomes. Explicit calculations are performed for a binomial model with two assets.

1. Introduction

Suppose an investor knows the distribution of the prices of the stocks in the market at a future time and this investor wants to optimize her or his expected utility from wealth at that future time. Our basic question is: *What is the financial value of this information*?

Much of the research into utility optimization and the financial value of weak information has been looked at previously in a continuous time setting [Baudoin 2003; Baudoin and Nguyen-Ngoc 2004]. The purpose of this paper is to investigate how to optimize a stock portfolio given weak information in a discrete-time setting. It should be stressed that the results we obtain are new and cannot be obtained as a consequence of the results in [Baudoin 2003; Baudoin and Nguyen-Ngoc 2004].

We will assume that the market is complete. We will also assume that there are no transactions costs. For a definition of complete markets, see [Björk 2009]. The main tool we use in finding the optimal expected utility given the weak information on future stock prices is the martingale method; see [Shreve 2004]. The reader might recognize that the problem treated here is related to robust utility maximization problems, as discussed in [Gilboa and Schmeidler 1989] and later works in mathematical finance by H. Föllmer, A. Gundel and S. Weber.

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As with classical results in this field, we will be looking at the expected utility as opposed to the expected wealth. This is an important difference to note since utility functions allow us to include an individual's attitude towards risk.

2. Utility functions

There are many different utility functions used in mathematics and economics to measure an individual's happiness or satisfaction. We denote our utility functions by U. We require that a utility function is strictly concave, strictly increasing, and continuously differentiable. We assume as in [Baudoin 2003] that

$$\lim_{x\to 0} U'(x) = +\infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0. \tag{1}$$

These conditions are sufficient for a utility function to exhibit risk aversion, to satisfy the law of diminishing marginal utility, and to guarantee that an increase in wealth results in an increase in utility. Further, when discussing the risk aversion of our utility functions, we use the absolute and relative risk aversion functions; see [Meyer and Meyer 2005]. We will be looking specifically at three different types of utility functions:

(i) Log utility: $U(x) = \ln(x),\ x > 0$. The log utility function has a constant relative risk aversion of 1. This implies the individual will always take on a constant proportion of risk with respect to their wealth.

(ii) Power utility: $U(x) = x^\gamma/\gamma$ for $-\infty < \gamma < 0$ and $0 < \gamma < 1$ and $x > 0$. The power utility function also has a constant relative risk aversion, but the constant value is $1-\gamma$. Thus, the power utility function is less risk-averse compared to the log utility function for $0 < \gamma < 1$. In this case, the constant γ reflects the relative risk aversion with the individual becoming more risk-averse as γ approaches 0. If $-\infty < \gamma < 0$, the individual is more risk-averse than an individual whose preferences can be described by the logarithmic utility function. As γ approaches $-\infty$, the individual becomes more and more risk-averse.

(iii) Exponential utility: $U(x) = -e^{-\alpha x}$ for $\alpha > 0$ and $x \in \mathbb{R}$. The exponential utility function has a constant absolute risk aversion of 1. Thus, the individual with an exponential utility function will assume a constant amount of risk rather than a constant proportion of risk with respect to their wealth. Notice that the exponential utility function does not satisfy the condition (1), but it is still an interesting function to note, and our results still hold true for this function.

3. Modeling the financial value of weak information on discrete-time complete markets with a discrete state space

Setup. Suppose we have a market with d financial assets, and a sample space $\Omega_1 = \{\omega_1, \dots, \omega_M\}$ of possible outcomes of all the asset prices after one time period. For all probability measures $\mathbb{P}$, we always assume $\mathbb{P}(\omega_j) > 0$ for all $j \in \{1, \ldots, M\}$. This is not a restriction since if $\mathbb{P}(\omega_j) = 0$, then we exclude ω_j from Ω_1. Let N be our final time period, and let $\vec{S}_n \in \mathbb{R}^d$ denote the asset prices at time n where $n \in \{0, 1, \ldots, N\}$. Further, let the random variable V_n denote the value of the portfolio at time n. Denote the initial wealth of the investor V_0 by υ. Without loss of generality we can assume one of the assets is a risk-free asset. We define r to be the rate of return of the risk-free asset. We will denote by $\mathcal{M}$ the set of equivalent[1] probability measures under which discounted stock prices are martingales. Furthermore, we will assume our market is free from arbitrage. Thus, we can assume that the set $\mathcal{M}$ is nonempty. For a complete market, $\mathcal{M}$ is a singleton, say $\mathcal{M} = \{\widetilde{\mathbb{P}}\}$, where $\widetilde{\mathbb{P}}$ is the unique probability measure under which discounted stock prices are martingales; see [Björk 2009] for more details about arbitrage, completeness, and equivalent martingale measures. We denote by Ψ^υ the set of self-financing portfolios given initial wealth υ. The probability measure $\widetilde{\mathbb{P}}$ basically represents the "knowledge" of the uninformed investor. Notice that by Jensen's inequality this is the same as having no information at all, since it is optimal to invest in the risk-free asset only.

3.1. ***Weak anticipation.*** Now suppose we have some weak anticipation (weak information) regarding the prices of assets at our final time period. That is to say, we know the distribution of $\vec{S}_N$. We will denote this distribution by ν. Let Ω denote the path space of the (M-dimensional) stock price process $\{\vec{S}_n\}_{1 \le n \le N}$. Further, let $\mathcal{A}$ be the (finite) set of possible asset prices at time N. Note $|\mathcal{A}| \le M^N$.

Definition. The probability measure $\mathbb{P}^\nu$ defined by

$$\mathbb{P}^\nu(\omega) := \sum_{\vec{x} \in \mathcal{A}} \widetilde{\mathbb{P}}(\omega \mid \vec{S}_N = \vec{x}) \, \nu(\vec{S}_N = \vec{x})$$

is called the minimal probability measure associated with the weak information ν, where $\widetilde{\mathbb{P}} \in \mathcal{M}$ is an (remember $\mathcal{M}$ is a singleton in a complete market) equivalent martingale measure.

In the sense of the following proposition, $\mathbb{P}^\nu$ is minimal in the set of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\mathbb{Q}(\vec{S}_N = \vec{x}) = \nu(\vec{S}_N = \vec{x})$ for all $\vec{x} \in \mathcal{A}$. We denote this set by $\mathcal{E}^\nu$.

Proposition 3.1. *Let ϕ be a convex function. Then*

$$\min_{\mathbb{Q} \in \mathcal{E}^\nu} \widetilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\right)\right] = \widetilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\right)\right],$$

where $d\mathbb{Q}/d\widetilde{\mathbb{P}}$ denotes the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\widetilde{\mathbb{P}}$.

[1] In our finite discrete sample space, by equivalent we simply mean, for all $i \in \{1, 2, \ldots, M\}$, $\mathbb{Q}(\omega_i) > 0$.

Proof. Let $\vec{x} \in \mathcal{A}$ and $\mathbb{Q} \in \mathcal{E}^\nu$ be given. Then,

$$\widetilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \,\middle|\, \vec{S}_N = \vec{x}\right] = \frac{\nu(\vec{S}_N = \vec{x})}{\widetilde{\mathbb{P}}(\vec{S}_N = \vec{x})}.$$

Let ϕ be a convex function. Then from the conditional version of Jensen's inequality

$$\phi\left(\frac{\nu(\vec{S}_N = \vec{x})}{\widetilde{\mathbb{P}}(\vec{S}_N = \vec{x})}\right) = \phi\left(\widetilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \,\middle|\, \vec{S}_N = \vec{x}\right]\right) \le \widetilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\right) \,\middle|\, \vec{S}_N = \vec{x}\right].$$

Taking the expected value on both sides, we get

$$\widetilde{\mathbb{E}}\left[\phi\left(\frac{\nu(S_N)}{\widetilde{\mathbb{P}}(S_N)}\right)\right] = \widetilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\right)\right] \le \widetilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\right)\right],$$

and the result is proved. □

3.2. *Value of weak information.* Since an insider's anticipation has a different final time distribution than an uninformed investor's, it is natural to find a way to characterize the value of this information. Since we focused on maximizing our utility of wealth rather than the monetary value of wealth, we will define our value accordingly.

Definition. The *financial value of weak information* is the lowest expected utility that can be gained from anticipation. We write

$$u(v, \nu) = \min_{\mathbb{Q} \in \mathcal{E}^\nu} \max_{\psi \in \Psi^\nu} \mathbb{E}^{\mathbb{Q}}[U(V_N)].$$

Our main theorem is the following:

Theorem 3.2. *The financial value of weak information in a complete market is*

$$u(v, \nu) = \max_{\psi \in \Psi^\nu} \mathbb{E}^\nu[U(V_N)] = \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right],$$

where $\lambda(v)$ is determined by

$$\widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v,$$

where $\widetilde{\mathbb{P}} \in \mathcal{M}$ is the unique probability measure under which the prices are martingales. Moreover, the optimal wealth at time n, $\widehat{V}_n$, is given by

$$\widehat{V}_n = \frac{1}{(1+r)^{N-n}} \sum_{\omega \in \Omega} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}(\omega)\right)\widetilde{\mathbb{P}}(\omega \mid \vec{S}_n) \quad \text{for } n \in \{0, 1, \dots, N\}.$$

At time n, the optimal amount to purchase of the i-th linearly independent asset is

$$\delta_n^i = \sum_{j=1}^{M} (D_{n+1}^{-1})_{i,j} \widehat{V}_{n+1}(\omega_j) \quad \text{for } n \in \{0, 1, \dots, N-1\},$$

where

$$D_{n+1} = \begin{bmatrix} S^1_{n+1}(\omega_1) & S^2_{n+1}(\omega_1) & \cdots & S^M_{n+1}(\omega_1) \\ S^1_{n+1}(\omega_2) & S^2_{n+1}(\omega_2) & \cdots & S^M_{n+1}(\omega_2) \\ \vdots & \vdots & & \vdots \\ S^1_{n+1}(\omega_M) & S^2_{n+1}(\omega_M) & \cdots & S^M_{n+1}(\omega_M) \end{bmatrix}$$

is the matrix of M linearly independent asset prices at time $n+1$, $(D^{-1}_{n+1})_{i,j}$ represents the element (i, j) of the matrix D^{-1}_{n+1}, and $\widehat{V}_{n+1}$ comes from the above equation.

Proof. We will proceed by rewriting $\max_{\psi\in\Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$. In order to do this, we need the convex conjugate $\widetilde{U}(y) := \max_{x>0}[U(x) - xy]$; see [Karatzas et al. 1991]. We form the Lagrangian for solving $\max_{\psi\in\Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$ by

$$\mathcal{L}(\lambda) = \mathbb{E}^{\mathbb{Q}}[U(V_N)] + \lambda\left[v - \mathbb{E}^{\mathbb{Q}}\left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\frac{V_N}{(1+r)^N}\right]\right].$$

Now using $\widetilde{U}$, substituting in for V_N from the martingale method (see the Appendix), and doing algebra, we can rewrite our Lagrangian as

$$\mathcal{L}(\lambda) = \lambda v + \widetilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\widetilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right].$$

Thus, we deduce

$$\begin{aligned} u(v, \nu) &= \min_{\mathbb{Q}\in\mathcal{E}^\nu}\min_{\lambda>0}\left[\lambda v + \widetilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\widetilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right] \\ &= \min_{\lambda>0}\left[\lambda v + \min_{\mathbb{Q}\in\mathcal{E}^\nu}\widetilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}\widetilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right]. \end{aligned}$$

Since the convexity of $\widetilde{U}$ implies the function mapping $z \mapsto z\widetilde{U}\big(\lambda/((1+r)^N z)\big)$ is convex, we can use Proposition 3.1 to get

$$u(v, \nu) = \min_{\lambda>0}\left[\lambda v + \widetilde{\mathbb{E}}\left[\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\widetilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]\right].$$

Taking the derivative now with respect to λ and setting it equal to 0, we find

$$v = \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda^*(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right],$$

where $\lambda^*(v)$ is the minimizer. Now,

$$u(v, \nu) = \lambda^*(v)v + \widetilde{\mathbb{E}}\left[\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\widetilde{U}\left(\frac{\lambda^*(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda^*(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right].$$

Thus, we have shown the first part of the theorem. Now note that the discounted optimal wealth process $\{\widehat{V}_n/(1+r)^n\}_{0\le n\le N}$ is a martingale under $\widetilde{\mathbb{P}}$ (see the Appendix). As a result,

$$\widehat{V}_n = \frac{1}{(1+r)^{N-n}}\widetilde{\mathbb{E}}[\widehat{V}_N \mid \vec{S}_n] = \frac{1}{(1+r)^{N-n}}\sum_{\omega\in\Omega} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}(\omega)\right)\widetilde{\mathbb{P}}(\omega \mid \vec{S}_n)$$

for all $n \in \{0, 1, \dots, N\}$. Further, note that wealth is determined by your portfolio from the previous time period and the current prices. Thus,

$$\widehat{V}_{n+1} = D_{n+1}\vec{\delta}_n,$$

so we have

$$\mathrm{D}_{n+1}^{-1}\widehat{V}_{n+1} = \vec{\delta}_n. \qquad \square$$

Remark. We know from [Björk 2009] that the matrix of all asset prices in the complete market has rank M. Therefore, we can choose M linearly independent assets to invest in. Further, note that the optimal amount to purchase for each asset is only unique when $M = d$.

Definition. We define the *additional value of weak information* as the extra utility gained from investing with anticipation instead of just putting all of your wealth in the risk-free asset, which we define by

$$F(v, \nu) = u(v, \nu) - U(v(1+r)^N).$$

Definition. We also define the *ratio of added value to the total value* by

$$\pi(v, \nu) = \frac{F(v, \nu)}{u(v, \nu)} = 1 - \frac{U(v(1+r)^N)}{u(v, \nu)}.$$

As a consequence of Theorem 3.2 we obtain the following interpretation of the additional value of weak information for the log utility function.

Corollary 3.3. *The additional value of weak information for the log utility function is given by the relative entropy of ν with respect to $\widetilde{\mathbb{P}}_{\vec{S}_N}$:*

$$F(v, \nu) = \mathbb{E}^\nu\left[\ln\left(\frac{d\nu}{d\widetilde{\mathbb{P}}_{\vec{S}_N}}\right)\right].$$

Proof. We first solve for λ:

$$v = \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\frac{(1+r)^N}{\lambda}\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\right] \quad\Longrightarrow\quad \lambda = \frac{1}{v}.$$

Substituting for λ in our value of weak information equation, we thus have

$$\begin{aligned} u(v,\nu) &= \mathbb{E}^{\nu}\Big[U\Big(I\Big(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)\Big)\Big] \\ &= \mathbb{E}^{\nu}\Big[\ln\Big(\frac{(1+r)^N}{1/v}\frac{d\mathbb{P}^{\nu}}{d\widetilde{\mathbb{P}}}\Big)\Big] = \ln(v(1+r)^N) + \mathbb{E}^{\nu}\Big[\ln\Big(\frac{d\mathbb{P}^{\nu}}{d\widetilde{\mathbb{P}}}\Big)\Big]. \end{aligned}$$

This implies the additional value of weak information for the log utility is

$$F(v,\nu) = \mathbb{E}^{\nu}\Big[\ln\Big(\frac{d\mathbb{P}^{\nu}}{d\widetilde{\mathbb{P}}}\Big)\Big] = \mathbb{E}^{\nu}\Big[\ln\Big(\frac{d\nu}{d\widetilde{\mathbb{P}}_{\vec{S}_N}}\Big)\Big],$$

where the last equality follows from the definition of $\mathbb{P}^{\nu}$. □

Just like the log utility, we can also find the financial value of weak information for the power utility.

Corollary 3.4. *The value of weak information for the power utility function is given by*

$$u(v,\nu) = \frac{v^{\gamma}(1+r)^{N\gamma}}{\gamma(\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^{\nu})^{1/(\gamma-1)}])^{\gamma-1}}\,\mathbb{E}^{\nu}\Big[\Big(\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)^{\gamma/(\gamma-1)}\Big].$$

Proof. We now will solve for the value of λ:

$$\widetilde{\mathbb{E}}\Big[\frac{1}{(1+r)^N}\Big(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)^{1/(\gamma-1)}\Big] = v \quad\Longrightarrow\quad \lambda = \Big(\frac{v(1+r)^{N\gamma/(\gamma-1)}}{\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^{\nu})^{1/(\gamma-1)}]}\Big)^{\gamma-1}.$$

Substituting in for λ, we get

$$\begin{aligned} u(v,\nu) &= \mathbb{E}^{\nu}\Big[U\Big(I\Big(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)\Big)\Big] \\ &= \mathbb{E}^{\nu}\Big[\frac{1}{\gamma}\Big(\Big(\frac{v(1+r)^{N\gamma/(\gamma-1)}}{\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^{\nu})^{1/(\gamma-1)}]}\Big)^{\gamma-1}\frac{1}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)^{\gamma/(\gamma-1)}\Big] \\ &= \frac{v^{\gamma}(1+r)^{N\gamma}}{\gamma(\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^{\nu})^{1/(\gamma-1)}])^{\gamma-1}}\,\mathbb{E}^{\nu}\Big[\Big(\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\Big)^{\gamma/(\gamma-1)}\Big]. \end{aligned}$$

□

4. Complete markets: the binomial model

Single-period binomial model. We first will focus on a single-period binomial model with two assets: one risk-free asset with payoff $1+r$, and one risky asset with payoffs $S_0(1+h)$ if the stock goes up, and $S_0(1-k)$ if the stock goes down, where we assume $S_0 > 0$ and $k < 1$. In order to have an arbitrage-free market, we

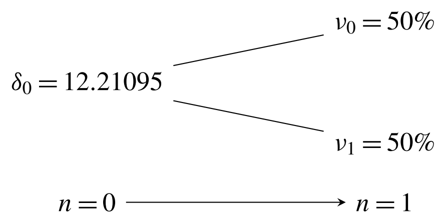

Figure 1. An example of a single-period binomial model using the log utility, where the parameter values are $r = .032,\ h = .09,$ $k = .019,\ v = 200.0$, and $s = 20.0$.

require $h > r > -k$. Since there is only one risky asset, we will denote the amount of units owned of the risky asset at time n by δ_n.

Figure 1 shows a basic single-period binomial using the log utility. It represents the amount of stock you should buy initially, δ_0. From here there are only two outcomes for our final time; the stock price will either go up or down.

Example 1 (log utility). When looking at the log utility function, we begin by maximizing $\mathbb{E}[U(V_N)]$ with respect to δ. We then are able to obtain our equation for the optimal number of shares with respect to wealth, $\hat{\delta}$, in a single-period model:

$$\hat{\delta}_0 = \frac{v(1+r)(\nu_0(h-r)+\nu_1(-k-r))}{-s(h-r)(-k-r)}.$$

Example 2 (power utility). As in the log utility case, we solve for our optimal number of shares with respect to wealth, $\hat{\delta}_0$, in a single-period model:

$$\hat{\delta}_0 = \frac{((\nu_0(h-r))^{1/(\gamma-1)} - (\nu_1(-k-r))^{1/(\gamma-1)})(1+r)v}{(\nu_1(-k-r))^{1/(\gamma-1)}s(-k-r) - (\nu_0(h-r))^{1/(\gamma-1)}s(h-r)}.$$

Example 3 (exponential utility). Similarly to the previously examined utilities, we solve for the optimal number of shares with respect to wealth, $\hat{\delta}$, in a single-period model for the exponential utility:

$$\hat{\delta}_0 = \frac{\ln(\nu_0(h-r)) - \ln(-\nu_1(-k-r))}{s(h+k)}.$$

N-period binomial model. In binomial models, everything can be explicitly computed. For instance, the following proposition gives the formula for the transition probabilities of the minimal probability $\mathbb{P}^\nu$. It is easy to establish by using the formula for conditional probabilities and straightforward combinatorial arguments. We note that $\{S_n\}_{1\le n\le N}$ is a Markov chain under the probability $\mathbb{P}^\nu$.

Proposition 4.1. *Let* $l \in \{1, \ldots, N-1\}$ *and* $i \in \{0, \ldots, N-l\}$. *Then*

$$\mathbb{P}^\nu\big(S_{N-l+1} = (1+h)S_{N-l} \mid S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0\big) \\ = \frac{\sum_{j=0}^{l-1}\binom{l-1}{j}(N-i-j)\cdots(N-i-(l-1))(i+1)(i+2)\cdots(i+j)\nu_{i+j}}{\sum_{j=0}^{l}\binom{l}{j}(N-i-j)\cdots(N-i-(l-1))(i+1)(i+2)\cdots(i+j)\nu_{i+j}}$$

and

$$\mathbb{P}^\nu\big(S_{N-l+1} = (1-k)S_{N-l} \mid S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0\big) \\ = \frac{\sum_{j=0}^{l-1}\binom{l-1}{j}(N-i-j-1)\cdots(N-i-(l-1))(i+1)\cdots(i+j+1)\nu_{i+j+1}}{\sum_{j=0}^{l}\binom{l}{j}(N-i-j)\cdots(N-i-(l-1))(i+1)(i+2)\cdots(i+j)\nu_{i+j}}.$$

Example 4 (log utility). Figure 5 shows an example of two different 3-period binomial trees with set values. The first tree shows the values of δ at time n when the anticipation has a uniform distribution. The second tree, however, shows an optimistic anticipation example. One can see how the amount of stocks in which one should invest changes depending on the distribution of the anticipation. For example, one would want to buy more stocks in an optimistic model because there is a better chance of the stock increasing in price as time goes on than in the model where all of the probabilities are the same. Negative values of δ correspond to short-selling the asset.

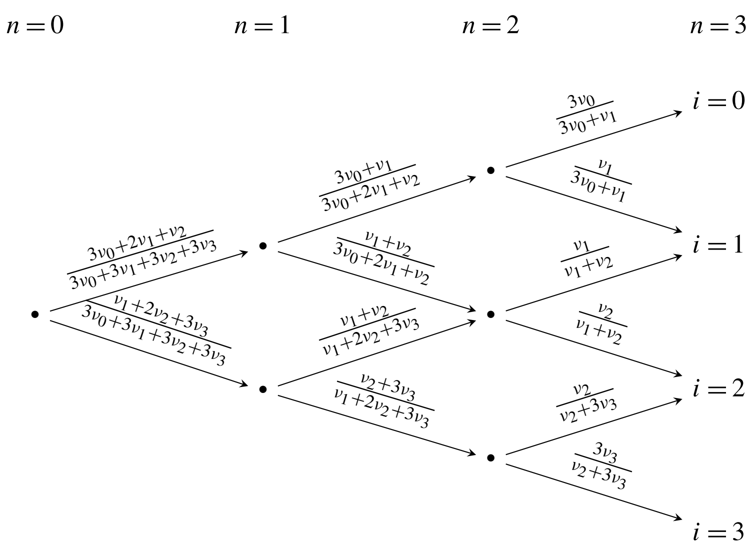

Figure 2. $\mathbb{P}^\nu$ for a 3-period binomial model.

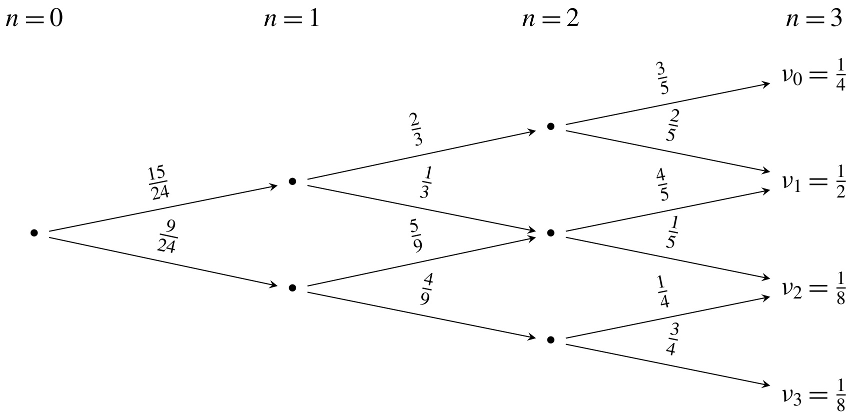

Figure 3. $\mathbb{P}^\nu$ for a 3-period binomial model for a specific choice of ν.

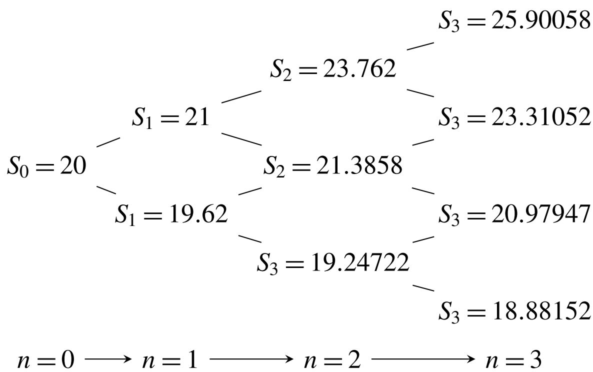

Figure 4. A 3-period binomial tree showing the values of S_n, where the parameters are $r = .032,\ h = .09,\ k = .019,\ v = 200.0$, and $s = 20.0$.

Recall from Corollary 3.3 the additional value of weak information for the log utility is

$$F(v, \nu) = \mathbb{E}^\nu\left[\ln\left(\frac{d\mathbb{P}^\nu}{d\widetilde{\mathbb{P}}}\right)\right],$$

and the proportion is

$$\pi(v, \nu) = \frac{\mathbb{E}^\nu[\ln(d\mathbb{P}^\nu/d\widetilde{\mathbb{P}})]}{\ln(v(1+r)^N) + \mathbb{E}^\nu[\ln(d\mathbb{P}^\nu/d\widetilde{\mathbb{P}})]}.$$

Note that $F(v, \nu)$ is only a function of ν, so for any fixed ν, we have that $F(v, \nu)$ is constant. Furthermore, $\pi(v, \nu)$ is a decreasing function of v for any fixed ν. As

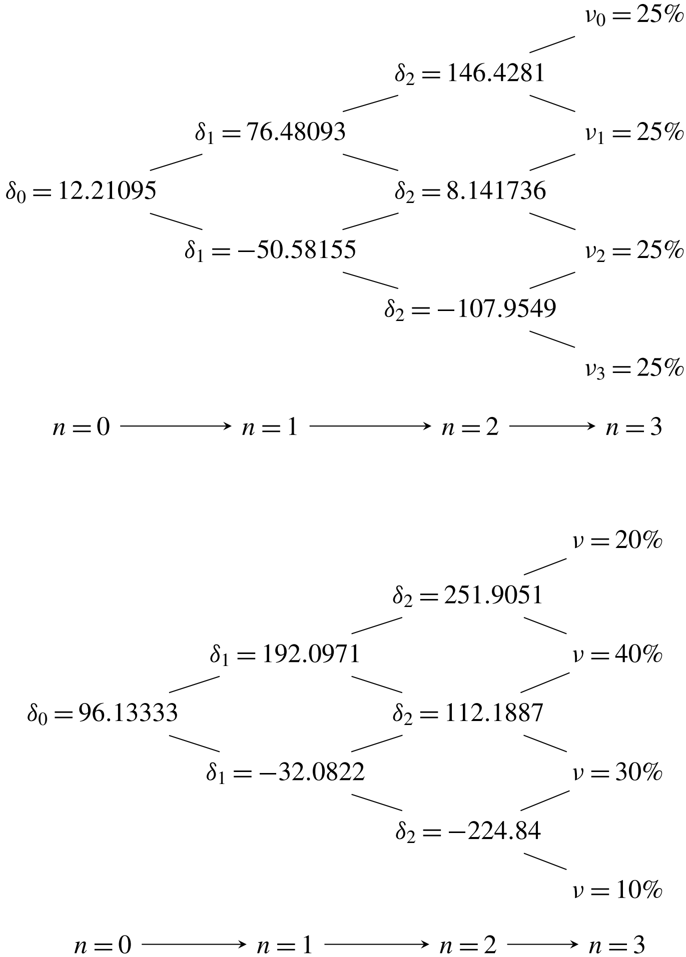

Figure 5. 3-period binomial trees showing the values of δ for various anticipations of ν using the log utility, where the parameters are $r = .032,\ h = .09,\ k = .019,\ v = 200.0$, and $s = 20.0$.

a result, the wealthier you are, the less proportion of utility you are gaining as a result of anticipation. In a 5-period binomial model, with the four anticipations below, we can look at the above functions as functions of v:

- Precise: $\{0.01, 0.01, 0.01, 0.95, 0.01, 0.01\}$.
- Uniform distribution: $\left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$.
- Conservative: $\{0.1, 0.2, 0.2, 0.2, 0.2, 0.1\}$.
- Risk-neutral: $\nu = \widetilde{\mathbb{P}}$.

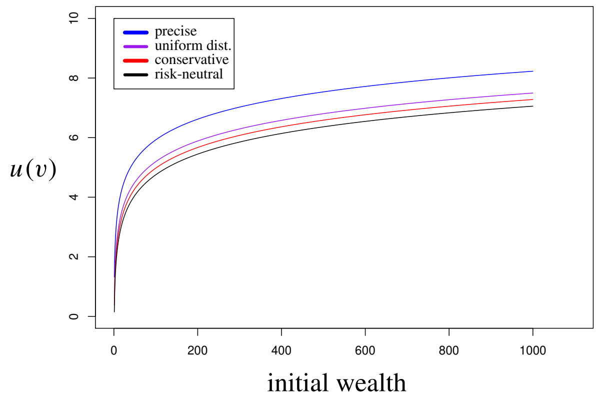

Figure 6. Value of weak information, given $r = 3\%$, $h = 8\%$, $k = 4\%$, using the log utility. The legend labels the curves in order, top to bottom.

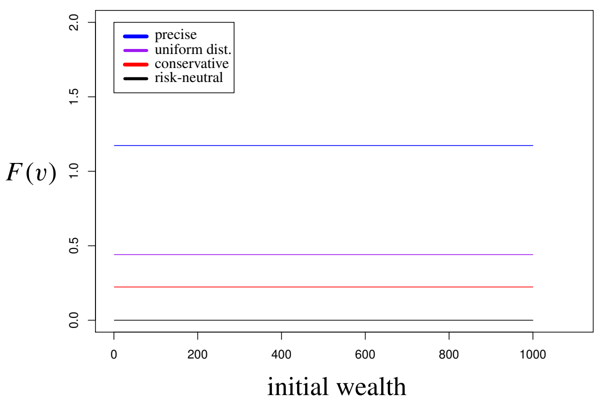

Figure 7. Additional value of weak information, given $r = 3\%$, $h = 8\%$, $k = 4\%$, using the log utility. Legend labels curves in order.

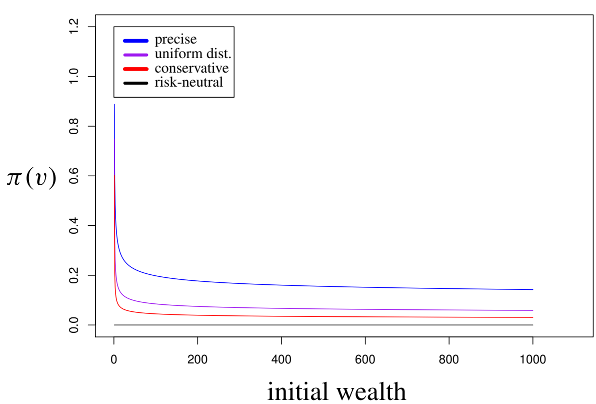

Figure 8. Proportion of value added, given $r = 3\%$, $h = 8\%$, $k = 4\%$, using the log utility. Legend labels curves in order.

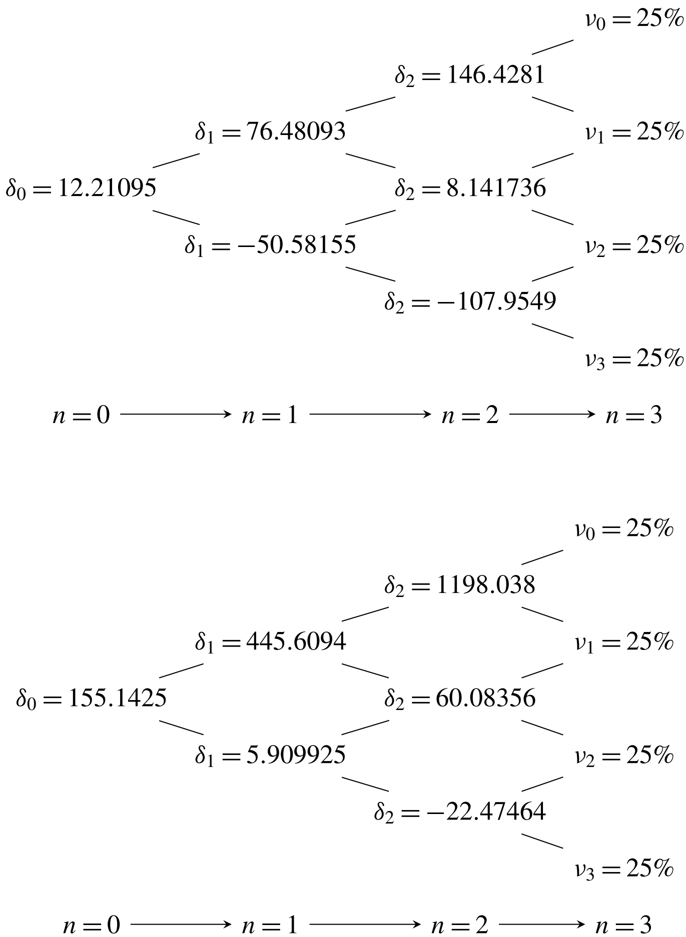

Figure 9. Two different 3-period binomial trees showing the values of δ for equal anticipations of ν using the log utility (top) and the power utility (bottom), where the constants are the same as Figure 5. In the power utility model, $\gamma = .5$.

Example 5 (power utility). Figure 9 shows the difference between the log and power utilities. As the log utility is a more relatively risk-averse utility function (for $\gamma = 0.5$), the absolute value of δ tends to be smaller when compared to the power utility function.

From Corollary 3.4 we have that the additional value for the power utility is

$$F(v, \nu) = \frac{v^\gamma (1+r)^{N\gamma}}{\gamma(\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu)^{1/(\gamma-1)}])^{\gamma-1}} \mathbb{E}^\nu\left[\left(\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\gamma/(\gamma-1)}\right] - \frac{v^\gamma (1+r)^{N\gamma}}{\gamma},$$

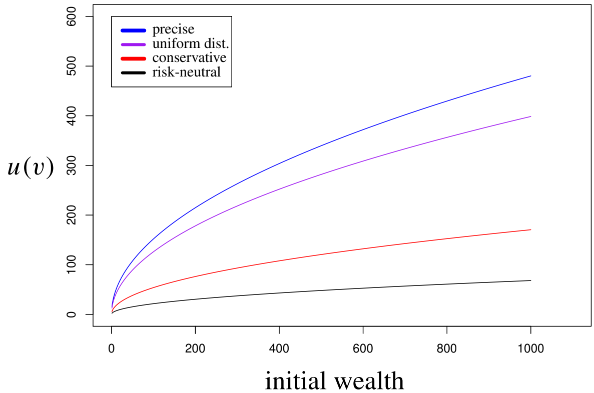

Figure 10. Value of weak information, given $r = 3\%,\ h = 8\%,\ k = 4\%$, using the power utility. Legend labels curves in order.

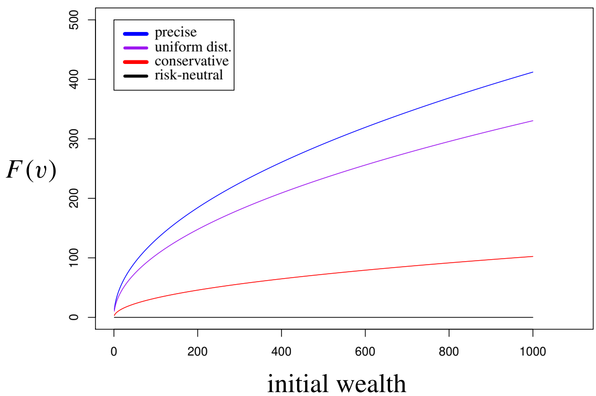

Figure 11. Additional value of weak information, given $r = 3\%,\ h = 8\%,\ k = 4\%$, using the power utility. Legend labels curves in order.

and the proportion is

$$\pi(v, \nu) = 1 - \frac{1}{\mathbb{E}^\nu[(d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu)^{\gamma/(\gamma-1)}] \cdot (\widetilde{\mathbb{E}}[(d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu)^{1/(\gamma-1)}])^{1-\gamma}}.$$

For the power utility, we have the opposite relationship for a fixed ν with the proportion remaining constant and the added value being an increasing function of initial wealth.

Example 6 (exponential utility). We can also find the financial value of weak information for exponential utility.

$$\mathbb{E}^\nu[-e^{-a\widehat{V}_N}] = e^{-v\alpha(1+r)^N - \sum_{i=0}^N \binom{N}{i} \tilde{p}^{N-i}\tilde{q}^i \ln\left(\binom{N}{i}\cdot \tilde{p}^{n-i}\tilde{q}^i/\nu_i\right)}.$$

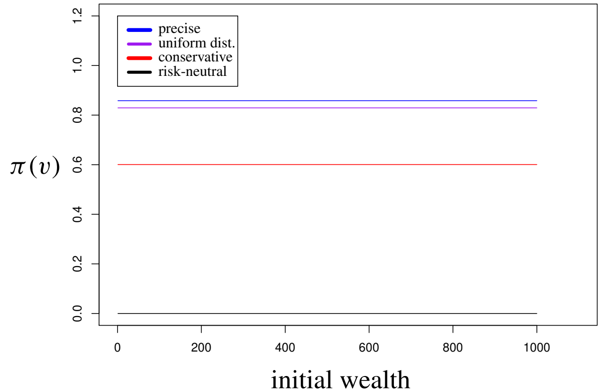

Figure 12. Proportion of value added, given $r = 3\%,\ h = 8\%,\ k = 4\%$, using the power utility. Legend labels curves in order.

We begin by solving for λ.

$$\widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v.$$

We use this equation and then plug in for I:

$$\widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\frac{-1}{\alpha}\ln\left(\frac{\lambda}{\alpha(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v.$$

We then solve for λ:

$$\lambda = \alpha(1+r)^N e^{-v\alpha(1+r)^N - \mathbb{E}^\nu[d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu \ln(d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu)]}.$$

Finally we can plug our I and our λ into our equation for the financial value of weak information to solve for the value as it specifically relates to exponential utility:

$$\begin{aligned} u(v,\nu) &= \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right] \\ &= \mathbb{E}^\nu[-e^{-a(-1/\alpha)\ln\left(\lambda/(\alpha(1+r)^N)\cdot(d\widetilde{\mathbb{P}}/d\mathbb{P}^\nu)\right)}] \\ &= e^{-v\alpha(1+r)^N - \sum_{i=0}^N \binom{N}{i}\tilde{p}^{N-i}\tilde{q}^i \ln\left(\binom{N}{i}\cdot\tilde{p}^{n-i}\tilde{q}^i/\nu_i\right)}. \end{aligned}$$

Appendix

The following is with respect to the general discrete case in a complete market. As in Section 3, we denote by Ψ^v the set of self-financing portfolios given initial wealth v.

Theorem A.1. *The discounted wealth process is a martingale under the martingale measure $\mathbb{Q}$.*

Proof. See [Runggaldier 2005]. □

Theorem A.2. *Maximizing $\mathbb{E}[U(V_N)]$ over the set of self-financing portfolios Ψ^v is equivalent to maximizing $\mathbb{E}[U(V_N)]$ subject to $\widetilde{\mathbb{E}}[U(V_N)] = v$, with $\widetilde{\mathbb{P}}$ being the unique equivalent martingale measure.*

Proof. See [Rásonyi and Stettner 2005, Lemma 4.9]. □

Theorem A.3.

$$\widehat{V}_N = I\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right).$$

More specifically, optimal terminal wealth $\widehat{V}_N$ is attained when λ satisfies

$$v = \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right].$$

Proof. See [Runggaldier 2005, p. 16]. □

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