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Stable manifolds to bounded solutions in possibly ill-posed PDEs ☆

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Abstract

We prove several results establishing existence and regularity of stable manifolds for different classes of special solutions for evolution equations (these equations may be ill-posed): a single specific solution, an invariant torus filled with quasiperiodic orbits or more general manifolds of solutions. In the later cases, which include several orbits, we also establish the invariant manifolds of an orbit depend smoothly on the orbit (analytically in the case of quasi-periodic orbits and finitely differentiably in the case of more general families).

We first establish a general abstract theorem which, under suitable (spectral, non-degeneracy, analyticity) assumptions on the linearized equation, establishes the existence of the desired manifold. Related results appear in the literature, but our results allow that the nonlinearity is unbounded and we obtain smoothness of the invariant manifolds. This makes the results in this paper applicable to some several models of current interest that could not be treated otherwise. We discuss in detail the Boussinesq equation of water waves (similar phenomena happen in other long wave approximations) and complex Ginzburg-Landau equation. More recently, we observed [11] that our results also apply to Mean Field Games.

Since the equations we consider may be ill-posed, part of the requirements for the stable manifold is that one can define the (forward) dynamics on them. Note also that the methods that are based in the existence of dynamics (such as graph transform) do not apply to ill-posed equation. We use the methods based on

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integral equations (Perron method) associated with the partial dynamics, but we need to take advantage of smoothing properties of the partial dynamics. Note that, even if the families of solutions we started with are finite dimensional, the stable manifolds may be infinite dimensional.

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1. Introduction and organization of the paper

The goal of this paper is to construct stable manifolds around some special solutions of some partial differential equations (which may be ill-posed).

The main motivation for us was the paper [26], which constructed quasi-periodic solutions in some ill-posed equations. We will take a very similar set up in which the PDEs are not necessarily well posed but we can construct some special solutions and even smooth families of such solutions. Note that even if [26] constructed only finite dimensional manifolds of solutions, this paper constructs infinite dimensional manifolds of solutions. Of course, we expect that our abstract results have other applications such as neutral delay or state dependent delay equations. The main difficulty in our study is that the equations may not be well-posed, so that some methods in finite-dimensional systems such as graph transform do not work. The study of invariant manifolds requires to impose not only invariance but that the evolution can be defined.

We will prove several variants of the results with very similar proofs. The main differences arise when we consider families of solutions, in which case, we also establish regularity with respect to the parameters in the family. The regularity with respect to parameters in the manifold is rather subtle. In the case of tori filled densely with quasi-periodic solutions, we will conclude that the dependence on the base points is analytic. For more general center manifolds, we will only obtain finite regularity of the manifolds on the base points (this is optimal even in finite dimensions as it is well known).

To motivate the results, we recall that a quasi-periodic solution is geometrically a embedding of the torus in which the motion is equivalent to a rigid rotation. The quasi-periodic solution is dense on the torus. We can consider the stable manifold of an initial condition u_0, which is the set of points whose trajectories are asymptotic to the trajectory starting in u_0. (Notice that these manifolds are not invariant. Evolving the stable manifold to u_0 by a fixed time, we obtain the stable manifold to the evolution of u_0.) We can also consider the stable manifold of the invariant torus, which is the set of initial conditions whose trajectory is asymptotic to the torus. This stable manifold for an invariant set will indeed be invariant.

Notice that, even in the case of finite dimensional systems the stable manifolds for a point and for a set are very different objects and indeed have different dimensions. As it turns out, the stable manifold of the torus will be the union of all the stable manifolds for all the initial conditions in the torus (this is very similar to the situation in normally hyperbolic manifolds [32, 61]) and moreover the stable manifold to an orbit parameterized by a point in a torus will depend analytically on the points in the torus.

We will also consider asymptotic manifolds to invariant sets that are more complicated than quasi-periodic tori and then the results of regularity will be different.

We will present three results: *a*) stable invariant manifolds for invariant tori which are the closure of a quasi-periodic trajectory. *b*) stable invariant manifolds to a bounded trajectory. *c*) stable manifolds to a general invariant set (contained in a center manifold). We will show that in the case *a*) the manifolds to the whole orbit are analytic (in the direction of the stable spaces and in the directions describing the torus). We will also show that the stable manifolds in *b*) are indeed analytic and that their dimension is equal to that of the stable space (it could well be infinite-dimensional). Finally, in case *c*) we will show that the stable manifold is analytic in the stable directions for bounded solution (but that it can be only finitely differentiable in the center directions).

The main difference among the results *a*), *b*) and *c*) mentioned above is that in *b*) we prove results for one orbit, but in *a*) and *c*) we consider families of orbits that are invariant under the evolution and we study the regularity with respect to the points in the family. In *a*), the dynamics restricted to the family is a rotation (and we obtain analytic dependence with respect to the variable in the manifold. In the case *c*) we assume more general dynamics on the invariant set and obtain only C^r dependence. In the case *c*) we can not obtain (it is false even in finite dimensions) that there is analytic dependence on the base points. We also note that our results in part c) do not assume that the center manifold is finite dimensional. Even if in the examples we present, the center manifolds are finite dimensional, the general theory works even for infinite dimensional center manifolds.

Notice that the invariant manifolds constructed here are very different from the center manifolds results. The manifolds we construct are very hyperbolic and we will obtain analyticity in many directions and the manifolds we construct will be infinite dimensional.

In the applications we consider (PDE's), the solutions we construct will be analytic functions both in space and time. They will organize themselves in manifolds of solutions. Depending on the models considered, they will be organized in analytic manifolds or in finitely differentiable manifolds. More precisely, in the most general case, we obtain finite differentiable manifolds which are obtained by joining together analytic submanifolds. This sort of regularity is optimal even in finite dimensional systems.

1.1. Some concrete examples

Two examples that serve as motivation for the abstract results in this paper and which we will discuss in detail (for previous results on these models, see [23,26]) are the Boussinesq equation

$$u_{tt} = \mu u_{xxxx} + u_{xx} + (u^2)_{xx}, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{1.1}$$

and complex Ginzburg-Landau equation

$$u_t = \nu u + (b_1 + \mathrm{i}b_2)\Delta u + |u|^2 u_x, \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \tag{1.2}$$

where $\mu, \nu, b_1, b_2 \in \mathbb{R}$ and Δ is the Laplace operator. See [19,20,13,50,36] for previous results on the complex Ginzburg-Landau equation.

The equation (1.1) appears in the theory of water waves as a long wave/small amplitude of the water wave equation. In the application to water wave the physically relevant sign for μ is $\mu > 0$. See [9]. This equation also appears as long wave, low amplitude approximation of other phenomena. In such a case, the sign of μ could be negative. In this paper we will just concentrate in the $\mu > 0$ case which makes the equation ill-posed but susceptible to our methods. The paper

[26] produced analytic quasi-periodic solutions and the paper [23] constructed $C^{r-1+Lip}$ center manifolds for any r.

The equation (1.2) is a modification of Schrödinger equation adding to it nonlinear and (damping/growth) terms. Because the coefficients of (1.2) are complex, the equation contains a term which is a heat equation when $b_1 > 0$ and a backwards heat equation when $b_1 < 0$. The limit b_1 close to zero is a very interesting singular limit [56].

Note that the linear operators in the two models above just have the discrete spectrum, we remark that this is not essential. We allow both the linear operators have continuous spectrum, for example, the complex Ginzburg-Landau equation (1.2) can be replaced by the model in [52]:

$$\begin{cases} \theta_t = B(\theta), \\ u_t = \nu u + (b_1 + \mathrm{i}b_2)\big(\Delta + N(N+1)\mathrm{sech}^2 x\big)u - R'(|u|^2)u, x \in \mathbb{T}^d, \end{cases}$$

where $N \in \mathbb{Z}^+$, $\nu < 0$, $b_1 > 0$ and $R(x) = x^2 + \text{h.o.t}$ is a real analytic function over $[0, \infty)$. The linear operator $\Delta + N(N+1)\mathrm{sech}^2 x$ is self-adjoint on the domain $H^2(\mathbb{R})$ and possesses both the continuous spectrum and the finite number of negative eigenvalues. Please refer [52] for details. We can also verify that the system above also satisfies the non-degeneracy condition of our main results Theorems 3.1-3.3.

Another set of ill-posed equations where recently there is a systematic study of some special solutions is state dependent delay equations [40]. The set up of this paper does not apply to state dependent delay equations directly since the non-linearity in state dependent delay equations is the composition operator, which is very discontinuous (even if bounded).

Other situations where ill-posed equations have appeared in the literature are elliptic equations in cylinder domains, [53,54,62,51,30], several boundary value problems in free boundaries [29] and more recently mean field games [1,2,11]. Neutral delay differential equations, which do not define evolutions and have a long history, may have center manifolds which are a tool in the study of lattice systems. (In this case, simpler proofs than those in this paper apply since the nonlinear terms are bounded [45,46,18].) In this paper, we will not discuss the application of our abstract results to the models mentioned in this paragraph.

Of course, our results apply a fortiori to equations which indeed define an evolution but these already have a very large literature [67,15–17,66,68,42,7,41]. In this paper, we allow not only that the equations are ill-posed, but also the perturbations are unbounded. Furthermore, we obtain that the stable manifolds (which may be infinite dimensional) are very smooth and that they depend somewhat smoothly in the base point.

This paper is organized as follows: In Section 2, we give a quick overview of the problem we consider and give the precise formulation of our results. The Section 3 is the main part of this paper. In this section we construct the three types of stable manifolds mentioned above and formulate the main results of this paper. In Section 4, we formulate and prove the results for the Boussinesq equation and complex Ginzburg-Landau equation.

We also present several appendices with more technical results which may be or independent interest. Notably, in Appendix A, we present the persistence under (possibly unbounded perturbation) of invariant splittings in which the solutions are smoothing.

2. Overview of the method and previous results

In this section, we present a somewhat informal overview of the assumptions and the method. A more precise presentation will be done in Section 3. The present informal presentation may highlight the aspects that need to be made precise. We also review briefly some previous results on the study of invariant objects in ill posed equations.

We consider partial differential equation (PDE)

$$\frac{d}{dt}u = \mathscr{X} \circ u, \tag{2.1}$$

where $\mathscr{X}$ is a differential and possibly non-linear operator which is defined in a domain inside a Banach space. We will not assume that it is well-posed.

We will assume that $\mathscr{X}$ has the form:

$$\mathscr{X} = \mathcal{A} + \mathcal{N}, \tag{2.2}$$

where $\mathcal{A}$ is linear, possible unbounded operator and $\mathcal{N}$ is a nonlinear and possibly unbounded operator. Moreover, $\mathcal{N}$ will be of lower order with respect to $\mathcal{A}$.

We will not assume the equation

$$u_t = \mathcal{A}u \tag{2.3}$$

defines a dynamical evolution for all initial conditions and only assume that it generates forward and backward evolutions in the center-stable and the center-unstable spaces respectively. We will also assume that the evolutions thus defined have some smoothing properties (if we take initial conditions with some smoothness, the solutions are smoother. See later for a precise formulation).

In the center space (the intersection of the center-stable and the center-unstable, the evolution is defined forwards in time and backwards in time), but in general, it may be impossible to define both the forwards and backwards evolutions outside of the center manifold. Since the equation (2.3) is a constant equation, the existence of these partial evolutions can be studied using techniques from semigroup theory [59]. We note that our results apply when the stable/unstable spaces are trivial (i.e. zero dimensional).

Even if the equation (2.2) does not admit solutions for all initial conditions, it could well admit some solutions, which may be of interest and indeed there are many cases of ill-posed equations where such solutions have been constructed, [26,23]. The goal of this paper is to take these particular solutions and construct more solutions which are asymptotic to them.

Given a solution $u(t)$ of (2.2) we consider the formal linearized equation around it. That is, we consider the formal equation

$$\frac{d}{dt}\xi(t) = [\mathcal{A} + D\mathcal{N}(u(t))]\xi(t). \tag{2.4}$$

The equation (2.4) may not admit solutions for all initial conditions (as with (2.1)), nevertheless we will show that under suitable assumptions it inherits the existence of partial dynamics from (2.3). That is there is a space (the stable space) on which we can define the evolution of (2.4) forward, another space (unstable space) in which we can define a backwards evolution. Since (2.4) is non-autonomous, we cannot use methods from semigroup theory, but under the assumptions that $\mathcal{N}$ is of lower order than $\mathcal{A}$ we will be able to construct the spaces where the partial evolutions can be defined. Since (2.4) is not autonomous, the spaces on which the partial dynamics are defined could depend on t.

The main goal of this paper is to show that, under appropriate hypotheses on the nonlinearity, there are nonlinear analogues of the above linear spaces invariant under the linearized evolution for the full equation (2.1) of (2.2). That is, given a solution, we can find smooth manifolds of initial conditions whose forward orbits can be defined and approach $u(t)$ (these are the stable manifolds). By reversing the direction of time, we can also get the unstable manifolds.

In the case that we are given a family of orbits $u_\theta(t)$, we also want to discuss the dependence of these stable/unstable manifolds on the orbits.

Of course, the above results are analogues of well known results in finite dimensional systems. They are also known for infinite dimensional equations that define an evolution [17].

In this paper, however, we will not assume that the equations define an evolution. Hence, one of the requirements that we have to impose to the initial conditions in the manifold is that they define an semi-orbit. Hence the equations we have to study are somewhat more involved than the usual equations in Perron Method. Note that some of the standard methods in invariant manifold theory (e.g. the graph transform method) rely on the existence of an evolution and, hence, cannot be applied in the present set up.

A technical tool that we have to use several times is that if we have an invariant splitting of the space and that in each component, there is a smoothing semiflow, then this structure persists under perturbations which may be unbounded, but not as severe as the smoothing properties.

To formulate a rigorous set up for all the above results, we have found very useful the *two spaces approach* of [41].

We will assume that there are two Banach spaces $X \subset Y$, roughly, X consists of smooth functions and Y of less smooth functions. The properties of the differential equation will be expressed in terms of properties of the operators with respect to the spaces.

We will assume the operator $\mathcal{N}$ is analytic (or C^r) from X to Y. The operator $\mathcal{A}$ may, in general, be unbounded from X to Y (it is higher order than $\mathcal{N}$) but we will assume that it has hyperbolic properties (in the sense of dynamical systems). That is we will assume that we can define forwards and backwards evolution in complementary spaces. We will also assume that these partial evolutions are smoothing (following [41], this is formulated as saying that these partial evolutions map Y to X with quantitative bounds).

More precisely, we will assume that there is a decomposition

$$X = X^s \oplus X^c \oplus X^u \tag{2.5}$$

(also $Y = Y^s \oplus Y^c \oplus Y^u$, with $X^\sigma \subset Y^\sigma$, $\sigma = s, c, u$), which is invariant under $\mathcal{A}$. That is, if $x \in \mathrm{Dom}(\mathcal{A}) \cap X^\sigma$, then $\mathcal{A}x \in X^\sigma$. For the results on the manifolds (the results alluded as *c*) above), we will assume that the space X^c admits a smooth bump function (i.e. a smooth real valued function identically 1 in a ball and vanishing in a larger ball. This is a non-trivial assumption in infinite dimensional Banach spaces [6,21]. It is true in Hilbert spaces or in spaces where the norm is smooth outside of the origin). We note that we do not need to assume that the spaces X^s, X^u, X admit smooth bump functions.

We will introduce the notation $\mathcal{A}^\sigma = \mathcal{A}|_{X^\sigma}$ and assume that the operator $\mathcal{A}^s$ defines a forward evolution in X^s, $\mathcal{A}^u$ defines a backwards evolution and $\mathcal{A}^c$ defines an evolution for all times.

That is, we assume that there are semigroups $\{U^s(t)\}_{t\geq 0}$, $\{U^u(t)\}_{t\leq 0}$ and a group $\{U^c(t)\}_{t\in\mathbb{R}}$. These semigroups satisfy:

$$\frac{d}{dt}U^\sigma(t) = \mathcal{A}U^\sigma(t), \quad U^\sigma(0) = Id. \tag{2.6}$$

We will assume that $\mathcal{N}$ is analytic (or C^r) as an operator from X to Y. Furthermore, we assume that the semigroups $U^{s,c,u}$ defined above are smoothing in the sense that

$$\begin{aligned} &\|U^s(t)\|_{Y^s,X^s} \le C_h e^{-\beta_1 t} t^{-\alpha_1}, \quad t>0,\\ &\|U^u(t)\|_{Y^u,X^u} \le C_h e^{-\beta_2 |t|} |t|^{-\alpha_2}, \quad t<0,\\ &\|U^c(t)\|_{Y^c,X^c} \le C_h e^{\beta_3^+ t}, \quad t\ge 0,\\ &\|U^c(t)\|_{Y^c,X^c} \le C_h e^{\beta_3^- |t|}, \quad t\le 0 \end{aligned} \tag{2.7}$$

with $\beta_1 > \beta_3^-$, $\beta_2 > \beta_3^+$ and $\alpha_1, \alpha_2 \in [0,1)$. For positive t, the fact that U^s maps Y – a space consisting of rough functions into the space X – consisting of smooth functions – can be described by saying that U^s is smoothing. Similarly for $t<0$ and U^u.

Note that the bounds (2.7) blow up as $t \to 0$ for U^s and U^u. This is natural since at $t=0$ the operators U^s and U^u are just the identity, which is an unbounded operator from Y to X.

As remarked in [41], when the spaces X, Y are spaces consisting of functions with a different number of derivatives. If we denote the difference in the number of derivatives in X and Y by p, the assumptions $\alpha_1, \alpha_2 \in [0,1)$, in (2.7), imply that the order of $\mathcal{A}$ is bigger than p. If we recall that the operator $\mathcal{N}$ was assumed to map the space X to the space Y we see that the operator $\mathcal{N}$ has order smaller than the order of $\mathcal{A}$. The borderline cases $\alpha_i = 1, i = 1,2$, appear in some applications [28], but we will not say anything about them in this paper.

We note that the assumption (2.7) is an strengthening of the usual *trichotomy* assumptions [66]. Assumption (2.7) is a trichotomy with smoothing. One of the results of [26] is that this structure of trichotomy with smoothing persists when we modify the $\mathcal{N}$ term.

In this paper we will also use the standard trichotomy assumption which was not considered in [26]

$$\begin{aligned} &\|U^s(t)\|_{X^s,X^s} \le C_h e^{-\beta_1 t}, \quad t>0,\\ &\|U^u(t)\|_{X^u,X^u} \le C_h e^{-\beta_2 |t|}, \quad t<0,\\ &\|U^c(t)\|_{X^c,X^c} \le C_h e^{\beta_3^+ t}, \quad t\ge 0,\\ &\|U^c(t)\|_{X^c,X^c} \le C_h e^{\beta_3^- |t|}, \quad t\le 0. \end{aligned} \tag{2.8}$$

We note that one of the main difficulty of the arguments in our case is that we cannot assume that the evolution is defined for general initial conditions. At all steps we have to use functional equations expressing the invariance and manipulating them so that one can reduce them to fixed point problems. One of the consequences we have to establish is the existence of the evolution.

The arguments follow the same type of strategy in all the cases:

(1) We prove that given a bounded orbit, we can find splittings of the tangent space along the orbits in which the linearized equations admit forwards and backwards evolution. (Note that this step is not needed in ODE's.)
(2) We show that the evolutions defined in these changed solution spaces satisfy the hyperbolicity and smoothing bounds (2.7), (2.8).
(3) We construct an invariant manifold for the full evolution as the graph of a function from one space in the splitting to the complementary space.

(4) In the case that we consider families of orbits (for example in the case of the stable manifolds around an invariant torus or around a center manifold) we also establish regularity with respect to the base point.

For the sake of simplicity, we will state only the results obtained for the stable and center-unstable splittings and the stable manifolds. Changing t to $-t$ (equivalently, $\mathcal{A}$ to $-\mathcal{A}$ and $\mathcal{N}$ to $-\mathcal{N}$), we obtain results for the unstable and center-stable splittings and the unstable manifolds. Of course, the center manifolds are the intersections of the center-unstable and the center-stable. This construction is, of course, very standard in the study of normally hyperbolic manifolds and we leave the details to the reader.

2.1. Previous results

The equations (2.1) subject to (2.6) and (2.7) have been studied before and shown to contain bounded solutions under some extra assumptions.

For us, some papers that served as motivation (by no means the only results in the literature) are:

(1) In [23] it was shown that (2.1) admits a $C^{r-1+Lip}$ locally invariant center manifold. In many cases, this center manifold will contain periodic, quasi-periodic solutions, horseshoes, attractors, etc.
(2) In [26] it was shown that if $\mathcal{N}$ is analytic, X^c is finite dimensional and (2.1) preserves a symplectic structure (in a very weak sense) and is exact (in a suitable weak form), then there are analytic invariant tori.
(3) In [19,20,13] the authors constructed the finite dimensional quasi-periodic solutions of (1.2) by constructing a KAM theorem under suitable assumptions.
(4) In [50,36] the authors constructed the existence and uniqueness of global solutions of (1.2).
(5) The analytic quasi-periodic solutions of a state-dependent delay differential equation with quasi-periodic forcing was constructed in [40]. It was shown that the system admits analytic quasi-periodic solutions under some hypotheses.
(6) The paper [14] which presented computer assisted proofs of period orbits of Boussinesq equation.
(7) In the papers [1] and [2], the author constructed strong solutions for time-dependent mean field games.

In this paper we will construct the infinite dimensional stable manifolds attached to the solutions in (1)-(4) and come back to (5) and (6). Note that, even if the above mentioned previous papers serve as motivation for our results, we will not use any of the above results.

2.2. Results in this paper

The goal of this paper is to show that these bounded solutions (or sets of bounded solutions) possess stable invariant manifolds. These stable manifolds are sets of initial conditions on which one can define the forward evolution and the forward orbit thus defined converges to the orbit of the bounded solutions. Note that in our set up, the existence of initial conditions for which the forward evolution can be defined is, a priori, non-trivial. These manifolds for a particular orbit will be modelled on X^s and will depend very smoothly on the stable coordinate. They will be $C^{r-2+Lip}$ when $\mathcal{N}$ is C^r from X to Y and analytic when $\mathcal{N}$ is analytic from X to Y. See Section 3.2.

When we consider families of several bounded orbits (for example in a center manifold), we will also consider the stable manifolds to the set. It will turn out that (using the hyperbolicity assumptions), this can be considered as the union of the stable manifolds to all orbits in the set. The regularity of these stable manifolds to sets involves the regularity of the manifolds of each orbit (which was studied before) as well as the dependence of the manifolds on the points. Even in finite dimensional cases is known to be less smooth than the differential equation and the dependence of stable manifold on the point in general only finitely many times differentiable. This is indeed the case when we consider regularity of invariant sets in the center manifold (see Section 3.3). When we consider the quasi-periodic solutions produced in [26], we will obtain that the manifolds are analytic (see Section 3.1).

3. Three different stable manifold results

We construct the three results on invariant manifold, stable manifolds around the whiskered tori and then we generalize to any forward bounded solution, at last we construct the stable manifold around the center manifold constructed in [23]. In each part we formulate the existence of invariant manifold into a fixed point problem and formulate the hypotheses we need, then present the main result and give the proof of the main result. For the case of stable manifold around the center manifold we need to assume that the subspace X^c admits bump functions. The main reason is the set we consider in the center manifold may be not bounded, we need the cut-off function to make the functions considered be globally bounded.

3.1. Stable manifold around a whiskered torus

In this section, we construct analytic stable manifolds for quasi-periodic solutions of (2.1). That is, we want to show that there are many solutions that converge in the future to the quasi-periodic solutions and that they organize into analytic manifolds.

This work is motivated by the paper [26] which constructed solutions of (3.3) for (2.1) under some extra assumptions, such as that (2.1) has a Hamiltonian structure. These assumptions are satisfied by several interesting equations in the literature. In the present paper, we will not use Hamiltonian structure, so we will just refer to [26] for the existence of quasi-periodic solutions. Of course, it is quite possible that one can construct solutions of (3.3) by methods different from those of [26]. Notably, the paper [62] uses a very different method to produce finitely differentiable quasi-periodic solutions (the first step of [62] is reducing to a finitely differentiable center manifold and then, verifying the hypothesis of a finite-dimensional KAM theorem with finite differentiability).

3.1.1. Description of the result

We recall that an analytic quasi-periodic solution of (2.1) is a function of the form

$$u(t) = K(\theta + \omega t), \tag{3.1}$$

where $K : \mathbb{T}^d_\rho \to X$ is an analytic map and $\omega \in \mathbb{R}^d$ and $\mathbb{T}^d_\rho$ is the standard complex strip around $\mathbb{T}^d$. More concretely, for $\rho > 0$ we define $\mathbb{T}^d_\rho$ as

$$\mathbb{T}^d_\rho = \{\theta \in \mathbb{T}^d : |Im(\theta_i)| < \rho, \quad i = 1, \cdots, d\}. \tag{3.2}$$

Note that, for any $t \in \mathbb{R}$, the map $\Phi_t : \ \theta \to \theta + \omega t$ is a diffeomorphism from $\mathbb{T}^d_\rho$ to itself. In particular, we can use either θ or $\theta + \omega t$ as a dummy variable for points in $\mathbb{T}^d_\rho$. For example, $\sup_{\theta\in\mathbb{T}^d_\rho} f(\theta) = \sup_{\theta\in\mathbb{T}^d_\rho} f(\theta + \omega t)$. See, in particular (3.9), the definitions of bundle (3.20) and (3.36).

Note that u satisfying (3.1) is a solution of (2.1) if and only if K satisfies

$$DK \cdot \omega = \mathscr{X} \circ K. \tag{3.3}$$

The solutions in the stable manifold will be solutions of the form

$$u(t) = K(\theta + \omega t) + \xi(t); \qquad \xi(t) \to 0 \tag{3.4}$$

with $\xi(t)$ going to zero fast. Substituting (3.3) and (3.4) into (2.1) one obtains that we can rewrite (2.1) as:

$$\frac{d}{dt}\xi(t) = A(\Phi_t(\theta))\xi(t) + M(\Phi_t(\theta), \xi(t)), \tag{3.5}$$

where

$$\Phi_t(\theta) = \theta + \omega t$$

is the solution of the differential equation $\frac{d}{dt}\Phi_t(\theta) = \omega$ with $\Phi_0(\theta) = \theta$ and

$$A(\Phi_t(\theta)) = \mathcal{A} + D\mathcal{N} \circ K(\Phi_t(\theta)), \tag{3.6}$$

and

$$M(\theta, \xi) = \mathcal{N}(K(\theta) + \xi)) - \mathcal{N}(K(\theta)) - D\mathcal{N}(K(\theta))\xi. \tag{3.7}$$

Due to the assumptions on $\mathcal{N}$ and K, we know that M maps $\mathbb{T}^d_\rho \times X$ to Y. Moreover,

$$M(\Phi_t(\theta), 0) = 0, \quad D_2M(\Phi_t(\theta), 0) = 0. \tag{3.8}$$

The equation (3.5) is considered as the evolution equation for ξ provided that the curve $K(\Phi_t(\theta))$ is given and fixed. Actually, (3.5) is the formal variational equation of (2.1). Note that (3.5) is non-linear and non-autonomous.

Note that we allow that (2.1) is ill-posed so we will need to carefully choose the initial conditions such that the evolution (3.5) can be defined. Furthermore, we will show that the initial conditions that we construct lie on an analytic manifold modeled on $\mathbb{T}^d_\rho \times X^s$. See Theorem 3.1 for a precise formulation.

Furthermore, it was shown in [26] that the splittings, (3.9), inherit the properties (2.7) of the spectral splittings of $\mathcal{A}$ including the geometric properties of splitting and the analytic properties of smoothing. The paper [26] showed that this splitting depends analytically on the base point. In Appendix A we will present a slightly different proof of a related result (persistence of the splitting for a simple orbit).

3.1.2. The precise set up

The set up for quasi-periodic solutions in this paper is motivated by the results of [26] but with some changes. Notably, in this paper we do not need any assumption on the symplectic character of the equations. On the other hand we need more precise notions of stability.

The first ingredient of the set-up is a formulation of the evolution equation and of the hyperbolicity/smoothing properties of the linearized equation. We have found that it is useful to use the two-space formalism of [41].

(H1): There are two Banach spaces

$$X \hookrightarrow Y$$

with continuous embedding. The space X (resp. Y) is endowed with the norm $\|\cdot\|_X$ (resp. $\|\cdot\|_Y$). Furthermore, X is dense in Y. When discussing analytic regularity (Theorem 3.1, 3.2) we will assume that the space X, Y are complex Banach spaces. In Theorem 3.3 we will consider finitely differentiable functions. The results will often include that the solutions lie in the closed (real) subspace of X consisting of functions that produce real values when given real arguments. See Remark 3.7.

For typographical reason, given a operator A from X_1 to X_2, sometimes we will write $\|A\|_{X_1,X_2}$ rather than $\|A\|_{\mathscr{L}(X_1,X_2)}$.

(H2): The function $\mathcal{N}$ in (2.2) is analytic from X to Y. Moreover,

$$\mathcal{N}(0) = 0, \quad D\mathcal{N}(0) = 0.$$

As a consequence, when we consider the linearized evolution (3.5) with $M(\theta, \xi)$, the remainder of the Taylor expansion of $\mathcal{N}$ around $K(\theta)$, we have that M is analytic from $\mathbb{T}^d_\rho \times X$ to Y. We will also assume that the functions considered here are real valued for real arguments.

$$M(\theta, 0) = 0, \quad D_2 M(\theta, 0) = 0, \quad \forall \theta \in \mathbb{T}^d_\rho.$$

Denote $B(0, \delta, X)$ as the ball of radius $\delta (< 1)$, centered at 0 in X.

We will assume that

$$\|M\|_{C^2(\mathbb{T}^d_\rho \times B(0,\delta,X),Y)},$$

(which we just write as $\|M\|_{C^2}$) is sufficiently small. The precise conditions will be expressed in the proof.

We note that this assumption can be obtained without any loss of generality simply by making a change of scales. Since we have:

$$\sup_{\theta \in \mathbb{T}^d_\rho} \sup_{\xi \in B(0,\delta,X)} \frac{\|D_\xi M(\theta, \xi)\|_{X,Y}}{\|\xi\|_{B(0,\delta,X)}} \le c \|M\|_{C^2(\mathbb{T}^d_\rho \times B(0,\delta,X),Y)}$$

and

$$\sup_{\theta\in\mathbb{T}^d_\rho}\sup_{\xi\in B(0,\delta,X)}\frac{\|M(\theta,\xi)\|_Y}{\|\xi\|^2_{B(0,\delta,X)}}\le c\|M\|_{C^2(\mathbb{T}^d_\rho\times B(0,\delta,X),Y)},$$

we are also assuming, automatically, that the left hand sides of the above equations are small.

Remark 3.1. In the results on analyticity, we will consider complex Banach spaces, which will lead quickly to results. On the other hand, in some of the problems that serve as motivation, the equations of interest are real valued.

It follows from our constructions that if $\mathcal{A}$ and $\mathcal{N}$ are such that they map real functions into real functions (as it happens in the Boussinesq equation) the solutions we construct will consist of functions that give real values when given real arguments. This follows from the fact that the space of such real functions is closed under C^0 limits and all the iterative steps in the proof of the fixed point equation giving the manifold preserve this property.

For the complex Ginzburg-Landau equation this is not the case and the solutions lying on the invariant manifolds may be complex.

Definition 3.1. We say that an embedding $K:\mathbb{T}^d_\rho\to X$ is spectrally nondegenerate if for every θ in $\mathbb{T}^d_\rho$, we can find a splitting

$$X=X^s_\theta\oplus X^c_\theta\oplus X^u_\theta \tag{3.9}$$

with associated bounded projection $\Pi^{s,c,u}_\theta\in\mathscr{L}(X,X)$ depending analytically on $\theta\in\mathbb{T}^d_\rho$ and extending continuously to the closure $\mathbb{T}^d_\rho$ and $X^{s,c,u}_\theta$ have the following properties.

(SD1) We can find families of operators

$$\begin{aligned}
&U^s_\theta(t):Y^s_\theta\to X^s_{\Phi_t(\theta)}, \quad t>0,\\
&U^u_\theta(t):Y^u_\theta\to X^u_{\Phi_t(\theta)}, \quad t<0,\\
&U^c_\theta(t):Y^c_\theta\to X^c_{\Phi_t(\theta)}, \quad t\in\mathbb{R},
\end{aligned}$$

such that:

(SD1.1) The operators $U^{s,c,u}_\theta(t)$ are cocycles over the rotation of angle ω satisfying

$$U^{s,c,u}_{\Phi_t(\theta)}(\tau)U^{s,c,u}_\theta(t)=U^{s,c,u}_\theta(\tau+t). \tag{3.10}$$

(SD1.2) The operators $U^{s,c,u}_\theta(t)$ are smoothing in the time direction where they can be defined and they satisfy assumptions in the quantitative rates. There exist constants $\beta_1,\beta_2,\beta_3^+,\beta_3^->0$ with

$$\beta_1>\beta_3^-,\quad \beta_2>\beta_3^+$$

and $C_h>1$, independent of θ, such that the evolution operators satisfy the following rate conditions:

$$\|U^s_\theta(t)\|_{\rho,Y^s_\theta,X^s_\theta}\le C_he^{-\beta_1 t}t^{-\alpha_1},\quad t>0, \tag{3.11}$$

$$\|U^u_\theta(t)\|_{\rho,Y^u_\theta,X^u_\theta}\le C_he^{-\beta_2|t|}|t|^{-\alpha_2},\quad t<0, \tag{3.12}$$

and

$$\begin{aligned} &\|U_\theta^c(t)\|_{\rho,Y_\theta^c,X_\theta^c} \le C_h e^{\beta_3^+ t}, \quad t \ge 0, \\ &\|U_\theta^c(t)\|_{\rho,Y_\theta^c,X_\theta^c} \le C_h e^{\beta_3^- |t|}, \quad t \le 0. \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\|U_\theta^s(t)\|_{\rho,X_\theta^s,X_\theta^s} \le C_h e^{-\beta_1 t}, \quad t > 0, \\ &\|U_\theta^u(t)\|_{\rho,X_\theta^u,X_\theta^u} \le C_h e^{-\beta_2 |t|}, \quad t < 0. \end{aligned} \tag{3.14}$$

(SD1.3) The operators $U_\theta^{s,c,u}(t)$ are solutions of the variational equations in the sense that

$$\begin{aligned} U_\theta^s(t) &= Id + \int_0^t A(\Phi_\tau(\theta))U_\theta^s(\tau)d\tau, \quad t > 0, \\ U_\theta^u(t) &= Id + \int_0^t A(\Phi_\tau(\theta))U_\theta^s(\tau)d\tau, \quad t < 0, \\ U_\theta^s(t) &= Id + \int_0^t A(\Phi_\tau(\theta))U_\theta^s(\tau)d\tau, \quad t \in \mathbb{R}. \end{aligned} \tag{3.15}$$

If spaces X and Y consist of function with enough derivatives, then $A^s(\theta)$ can operate on them and produce continuous function, in the sense, ($i.e.$ (3.15) is equivalent to the following differential equation)

$$\begin{aligned} \frac{d}{dt}U_\theta^s(t) &= A(\Phi_t(\theta))U_\theta^s(t), \quad t > 0, \\ \frac{d}{dt}U_\theta^u(t) &= A(\Phi_t(\theta))U_\theta^u(t), \quad t < 0, \\ \frac{d}{dt}U_\theta^c(t) &= A(\Phi_t(\theta))U_\theta^c(t), \quad t \in \mathbb{R}. \end{aligned} \tag{3.16}$$

Remark 3.2. The splitting $X = X_\theta^s \oplus X_\theta^c \oplus X_\theta^u$ in (3.9) is in more precise geometrical terms a splitting of the tangent space of the phase space at $K(\theta)$. Since we are in a Banach space, all tangent spaces at a point are just the Banach space. We use $X_\theta^\sigma, \sigma = s, c, u$ and ignore the issue of what is the base point.

Remark 3.3. The hypothesis (3.14) was not considered in [26] since it was not needed for the results in that paper. It is, however, a natural hypothesis and we will show it holds in the setups both of [26] and of this paper.

Remark 3.4. In [26] it was shown that for equations of the form (2.2) the spectral non-degeneracy (except for (3.14)) follows from spectral properties of the operator $\mathscr{A}$ (when $\mathcal{N}$ is of lower order).

For selfadjoint operators $\mathscr{A}$ (this requires that X is a Hilbert space), the spectral non-degeneracy follows from

$$\mathrm{Spec}(\mathscr{A}) \subset (-\infty, -\beta_1] \cup [-\beta_3^-, \beta_3^+] \cup [\beta_2, \infty), \ \beta_1, \beta_2, \beta_3^\pm > 0. \tag{3.17}$$

Note that verification of the spectral nondegeneracy by (3.17) does not require assumptions on the nature of the spectrum. Provided that the spectrum satisfies the inclusions in (3.17), it could be discrete or continuous spectrum.

For non-selfadjoint operators in general Banach spaces, there are characterization of spectral sets that lead to the existence of evolution operators (they also require some mild decay properties of the resolvent). See [63, Theorem X.47a]. Again, in the case that the spectrum is only eigenvalues of finite multiplicity (as it happens in the applications to CGL equation), the conditions of spectral degeneracy can be verified by elementary methods.

Remark 3.5. When we construct the stable manifold we need the condition $\beta_1 > \beta_3^-$ to control the growth of the evolution in the center space, if we construct unstable manifold, we need the condition $\beta_2 > \beta_3^+$ to control the growth of the evolution in the center space. However, when we construct the center manifold we need both conditions, $\beta_1 > \beta_3^-, \beta_2 > \beta_3^+$, as it is very standard in invariant manifold theory [35,33,34]. See [23] for the extension of these arguments to ill-posed equations.

3.1.3. Verification of the set up in some cases

The above setup was shown to indeed hold in several interesting situations and the motivation of this paper is precisely to take advantage of the previous results.

The paper [26] showed that, under the assumptions on the spectrum of $\mathcal{A}$ and the regularity of $\mathcal{N}$, then the small enough quasi-periodic solutions $u(t) = K(\theta + \omega t)$[3] of (3.3) are spectrally non degenerate, *i.e.* a modification of the splitting (3.9) is invariant under the linearized equation

$$\frac{d}{dt}\xi(t) = (D\mathscr{X})(K(\Phi_t(\theta)))\xi(t) = [\mathcal{A} + D\mathcal{N}(K(\Phi_t(\theta)))]\xi(t) \equiv A(\theta + \omega t)\xi(t). \tag{3.18}$$

In the paper [26] it was shown that the linear equation (3.18) allowed an invariant splitting into complementary spaces consisting of initial condition which lead to solution in the past or in the future. Furthermore, it was shown in [26] that these modified splittings inherit the hyperbolicity and smoothing properties (2.7) of the splitting of $\mathcal{A}$ in (2.2). The property (3.14) was not considered in [26] but we stress it in this paper and give a proof of its persistence. Note that, since now (3.18) is time-dependent, its fundamental solutions will not form a semigroup but a cocycle on $\Phi_t(\theta)$, *i.e.* U_θ^σ satisfy (3.10).

We recall that, for the fixed θ, the paper [26] established the existence of the spaces X_θ^σ by writing them as the graph of linear functions $G^\sigma(\theta) : X^\sigma \to X^\mu$, where μ denotes the complementary indices, X^σ and X^μ are the components of X in the splitting (2.5) corresponding to the dominant operator $\mathcal{A}$. That is, the spaces X_θ^σ invariant for the time-dependent evolution are given by:

$$X_\theta^\sigma = \{\xi + G^\sigma(\theta)\xi \mid \quad \xi \in X^\sigma\}. \tag{3.19}$$

[3] The paper [26] also showed that if the equations (2.1) satisfy some extra properties such as Hamiltonian structure, then such small quasiperiodic solutions exist. There can be other methods [5] to produce the quasi-periodic solutions. In this paper, we only use the hyperbolicity properties and we do not consider the problem of constructing the quasi-periodic solutions.

It was shown in [26] that the $G^\sigma(\theta)$ are analytic functions of θ for $\theta \in \mathbb{T}^d_\rho$. Actually, in (3.19) $\xi + G^\sigma(\theta)\xi$ can also be written as $(\xi, G^\sigma(\theta)\xi)$ if the second notation, $A^\sigma(\theta)\xi$, refers to the components in the decomposition X^{cu} corresponding to the operator $\mathcal{A}$. See [26] for details. Appendix A of this paper presents a similar result. Note that in the present set up one of the conditions for the invariant linear spaces is that there are only backwards or forwards evolutions.

3.1.4. The bundle language

It is natural to use the language of bundles [43,55] to describe the geometric set-up. This could be omitted in the cases of [26] since the bundles are trivial, but we think it is useful since in finite dimensions [39] finds that the bundles can be non-trivial near a resonance.

We define the set

$$\mathbf{B}^\sigma \equiv \{(\theta, \xi) | \quad \xi \in X^\sigma_\theta, \theta \in \mathbb{T}^d_\rho\}, \tag{3.20}$$

$\mathbf{B}^\sigma$ can be made into a bundle over the base $\mathbb{T}^d_\rho$ endowing it with the projection which just considers the variable θ. In such a case the fiber over θ is just X^σ_θ.

The representation (3.19) shows that, in the cases considered in [26] the bundle is trivial, but the proof of invariant manifolds in this paper does not need that. Of course, for the applications to the problems in [26] one could avoid the bundle language and just use the product using the coordinate representation given by (3.19). We will indicate along the proof how things simplify when we consider the case of trivial bundles.

An important concept in the formulation of the problem is the notion of bundle maps. We recall that a bundle map is a map from one bundle to another bundle in such a way that fibers are mapped into fibers. We say that a bundle map V covers a map Φ in the base when $Range(V|_{X_\theta}) \subset X_{\Phi(\theta)}$, where X_θ, the fiber of θ, is the domain of V and $X_{\Phi(\theta)}$, the fiber of $\Phi(\theta)$, is the range of V. In our result, the bundle map w covers the identity, *i.e.* $Range(w|_{X^s_\theta}) \subset X^{cu}_\theta$. The space of bundle maps covering a fixed map in the base is a linear space. Note that if V_1 is a bundle map covering ϕ_1 and V_2 is a bundle map covering ϕ_2, then $V_1 \circ V_1$ is a bundle map covering $\phi_1 \circ \phi_2$.

3.1.5. Formulation of the invariance equation for the stable manifolds

Because of the invariance of the splitting, (3.5) is equivalent to the following three equations

$$\frac{d}{dt}\xi^\sigma(t) = A^\sigma(\Phi_t(\theta))\xi^\sigma(t) + M^\sigma(\Phi_t(\theta), \xi(t)) \tag{3.21}$$

where the index $\sigma = c, u, s$ indicates the projections on the subspaces $X^\sigma_{\Phi_t(\theta)}$.

As usual in center manifold theory [49,17,10,68], we write the manifold as the graph of a bundle map w covering the identity from $\mathbf{B}^s$ to $\mathbf{B}^{cu}$. $w : \mathbf{B}^s \to \mathbf{B}^{cu}$. That is: $w_\theta : X^s_\theta \to X^{cu}_\theta$.

To perform analysis, we will write $w_\theta(\xi)$ also as $w(\theta, \xi)$, but keep in mind that in such a case we need to ensure that the second argument of w is in the fiber of $\mathbf{B}^s$ corresponding to the first argument and that the range of w is in the fiber of $\mathbf{B}^{cu}$ corresponding to the first argument of w.

We will need to study spaces of bundle maps later. We anticipate that when the bundles will be assumed to be analytic, we will consider spaces of mappings w that are analytic in θ, ξ. In section 3.1.6 we will give a formulation of these spaces of maps.

That is we will try to find a manifold which we will represent as a graph of a bundle map w covering the identity in the base

$$\mathscr{W} = \{\mathscr{W}_\theta = (\theta, \xi^s, w(\theta, \xi^s)) : \theta \in \mathbb{T}^d_\rho, \xi^s \in X^s_\theta, \|\xi^s\|_{X^s_\theta} \le 1\}. \tag{3.22}$$

We will also impose that for all $\theta \in \mathbb{T}^d_\rho$, w satisfies

$$\begin{aligned} &w(\theta, 0) = 0, \\ &D_2 w(\theta, 0) = 0. \end{aligned} \tag{3.23}$$

The first condition in (3.23) ensures that the given quasi-periodic orbit belongs to its stable manifold and the second one ensures that the manifold $\mathscr{W}_\theta$ is tangent to the space X^s_θ.

In terms of the original equation (2.1), we see that the invariance of the graph of w means that if we consider initial conditions of the form $u(0) = K(\theta) + \xi^s_0 + w(\theta, \xi^s_0)$ with ξ^s_0 in X^s_θ and sufficiently small, then we can find a solution $u(t)$ of (2.1) of the form

$$u(t) = K(\Phi_t(\theta)) + \xi^s(t) + w(\Phi_t(\theta), \xi^s(t))$$

with $\xi^s(t) \in X^s_{\Phi_t(\theta)}$, $\xi^s(0) = \xi^s_0 \in X^s_\theta$ and $w(\Phi_t(\theta), \xi^s(t)) \in X^{cu}_{\Phi_t(\theta)}$. Note that the fact that we can find an forward solution is a non-trivial requirement in our set-up.

Now we give the statement of the main result of this case, Theorem 3.1. The proof is based on formulating it as a fixed point problem and then applying a variant of the standard contraction argument.

Theorem 3.1. *Assume that X, Y are Banach spaces satisfying* **(H1)** *and that we have an equation* (2.1) *of the form* (2.2) *satisfying* **(H2)**. *Assume that we have an analytic K parameterizing a quasiperiodic solution of* (2.1) *(that is K satisfies* (3.3)*) which satisfies* **(SD1)**, *in particular, we can find bundles $\mathbf{B}^s$, $\mathbf{B}^{cu}$ based on the torus. (We will show that if K is small, it indeed satisfies* **(SD1)**.*)*

Then, there exists an analytic bundle map covering the identity, $w \in \mathcal{X}_1$, defined on $\boldsymbol{B}^s$ and mapping $\boldsymbol{B}^s$ to $\boldsymbol{B}^{cu}$ satisfying $w(\Phi_t(\theta), 0) = 0$, $D_2 w(\Phi_t(\theta), 0) = 0$. Furthermore, $\mathscr{W}$, the graph of w is globally forward invariant by (3.21).

Actually, w will be analytic both in the angle variable and on the fiber variable. In the proof we will present very explicit results about the domain of analyticity.

3.1.6. *The functional equations for w*

Our next goal is to derive heuristically a functional equation for w that encodes the geometric assumption that $\mathscr{W}$ is invariant under the forward evolution of the equation (3.21). It is important to note that, our procedure differs from that in [60] because we do not assume that our equations are well posed. We will need to formulate another equation to select the initial conditions that allow to construct solutions. Hence, in our case, we will have to deal with two equations, one equation which ensures that $\mathscr{W}$, the graph of w, is invariant by (3.28) and another equation which ensures that the forward semi-flow of (3.24) can be defined on it.

These two equations are coupled but they can be formulated as a fixed point equation for an operator, $\mathscr{T}$, which we will show is a contraction in appropriate spaces which encode some geometric properties and the regularity with respect to parameters. See (3.31), (3.32) and the subsequent discussions.

To derive the desired two equations for w we follow the standard procedure [60] of manipulating the solutions (assumed to exist and to lie in a manifold given by the graph) till we derive the equation for w which is formulated as the fixed point of an operator $\mathscr{T}$.

Once we have proved the existence of the fixed points for $\mathscr{T}$ and some of their properties, it will be easy to justify the manipulations and to check that the solution found corresponds to an invariant manifold.

Assume $U_\theta^\sigma(t)$ are solutions of (3.16) (in the sense of (3.15), we know that X and Y have sufficiently high derivatives, so we can define (3.16)). We separate (3.21) into two equations, $i.e.$ $\sigma = s$ and $\sigma = c, u$. Since the fundamental solutions for the s component are defined in the future and those for the c, u component are defined in the past, we impose the initial condition at $t = 0$ for ξ^s and the initial condition at $t = T$ for ξ^{cu} and by applying the Duhamel's formula to (3.21) we obtain, (recall that $U_\theta^s(t)$ is the linearized evolution and M as before is the Taylor remainder of $\mathcal{N}$ along $K(\Phi_t(\theta))$. See **(H.2)** for details),

$$\xi^s(t) = U_\theta^s(t)\xi_0 + \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)M^s(\Phi_\tau(\theta), \xi(\tau))d\tau, t \in [0, T], \tag{3.24}$$

and

$$\xi^{cu}(0) = U^{cu}_{\Phi_T(\theta)}(-T)\xi^{cu}(T) - \int_0^T U^{cu}_{\Phi_t(\theta)}(-t)M^{cu}(\Phi_t(\theta), \xi(t))dt. \tag{3.25}$$

Now, we will also impose that the solutions in (3.24) and (3.25) remain in the graph of the bundle map, $\mathcal{W}$. Since the variable θ has an evolution, then $\xi^s(t)$ should be a vector based on $X^s_{\Phi_t(\theta)}$ and $\xi^{cu}(t)$ should be a vector based on $X^{cu}_{\Phi_t(\theta)}$ and $\xi^s(t), \xi^{cu}(t)$ being in the graph of w should be

$$\xi^{cu}(t) = w(\Phi_t(\theta), \xi^s(t)). \tag{3.26}$$

Substitute (3.26) into (3.24) and (3.25) we obtain

$$\xi^s(t) = U_\theta^s(t)\xi_0 + \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta,\xi_0))])d\tau, \tag{3.27}$$

and

$$\begin{aligned} w(\theta, \xi_0) &= U^{cu}_{\Phi_T(\theta)}(-T)w(\Phi_T(\theta), \xi^s_T(\theta,\xi_0)) \\ &\quad - \int_0^T U^{cu}_{\Phi_t(\theta)}(-t)M^{cu}(\Phi_t(\theta), [\xi^s_t(\theta,\xi_0) + w(\Phi_t(\theta), \xi^s_t(\theta,\xi_0))])dt. \end{aligned} \tag{3.28}$$

We will consider the RHS of (3.28) as an operator that, given a bundle map, generates another bundle map. We will consider the operator defined in classes of functional spaces which are bounded. See (3.34) and (3.36). Therefore, we know that

$$\sup_{(\theta,\xi_0)\in \mathbf{B}^s, \|\xi_0\|_{X^s_\theta}\le\delta} \lim_{T\to\infty} \|U^{cu}_{\Phi_T(\theta)}(-T)w(\Phi_T(\theta), \xi^s_T(\theta,\xi_0))\|_{X^{cu}_{\Phi_T(\theta)}}$$
$$\le \lim_{T\to\infty} C_h e^{\beta_3^- T}(C_h e^{-\beta_1 T}\delta)^2 = 0.$$

So by taking the limit $T\to\infty$, (3.28) becomes

$$w(\theta,\xi_0) = -\int_0^\infty U^{cu}_{\Phi_t(\theta)}(-t)M^{cu}(\Phi_t(\theta), [\xi^s_t(\theta,\xi_0) + w(\Phi_t(\theta), \xi^s_t(\theta,\xi_0))])dt. \tag{3.29}$$

By combining (3.27) and (3.29) we obtain that (w, ξ^s) is the fixed point of an operator $\mathscr{T}$ (the spaces on which $\mathscr{T}$ acts will be make explicit in Section 3.1.7)

$$\begin{pmatrix}\xi^s\\ w\end{pmatrix}(t,\theta,\xi_0) = \mathscr{T}[\xi^s, w](t,\theta,\xi) = \begin{pmatrix}\mathscr{T}_s[\xi^s, w]\\ \mathscr{T}_{cu}[\xi^s, w]\end{pmatrix}(t,\theta,\xi_0) \tag{3.30}$$

with

$$\mathscr{T}_s[\xi^s, w](t,\theta,\xi_0) = U^s_\theta(t)\xi_0 + \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta,\xi_0))])d\tau, \tag{3.31}$$

and

$$\mathscr{T}_{cu}[\xi^s, w](\theta,\xi_0) = -\int_0^\infty U^{cu}_{\Phi_t(\theta)}(-t)M^{cu}(\Phi_t(\theta), [\xi^s_t(\theta,\xi_0) + w(\Phi_t(\theta), \xi^s_t(\theta,\xi_0))])dt. \tag{3.32}$$

In subsequent sections, we will specify the spaces in which $\mathscr{T}$ acts and show that it has fixed points. Once we have the fixed points that enjoy good properties it is standard [49] that the fixed points indeed give invariant manifolds. In our case, the argument requires the extra step of verifying that the evolution is defined on these initial data.

3.1.7. Function spaces for the analysis of $\mathscr{T}$ and the proof of Theorem 3.1

We produce our invariant manifold as the fixed point of the operator $\mathscr{T}$ defined in (3.30). To apply a fixed point argument it is necessary to define spaces of functions on which $\mathscr{T}$ acts.

We note that $\mathscr{T}$ has two arguments ξ^s and w. Let us try to understand what spaces do these arguments act.

For a fixed t, ξ^s_t is a bundle map covering Φ_t. More explicitly, given $(\theta,\xi_0)\in\mathbf{B}^s$ with $\xi_0\in X^s_\theta$, $\xi^s_t(\theta,\xi_0)$ will be an element in $X^s_{\Phi_t(\theta)}$. We will also impose some normalizations that indicate that the zero section of the stable bundle corresponds to the invariant torus. This is natural when we consider that the vectors in the fibers of the bundle are small perturbations.

Hence, ξ^s will be a map from $\mathbb{R}^+$ to a space of bundle maps. Equivalently, it will be a map of $\mathbb{R}^+\times\mathbf{B}^s\to\mathbf{B}^s$ in such a way that ξ^s_t is a bundle map covering Φ_t. To obtain a metric space to apply the contraction fixed point we specify spaces of maps with a certain regularity and define an appropriate norm. There are many such choices that work. We have chosen to impose analyticity in the $\mathbf{B}^s$ variables and continuity in t. Higher regularity in t can be deduced afterward using the equation. Of course, other choices or regularities are possible and indeed in other contexts we will make different choices.

In this section, we will take $\rho > 0$ as fixed because it is the analyticity domain of the quasi-periodic solution. Hence, we will omit it from some notations.

For $0 < \delta < \frac{1}{2C_h}$, we define the domain of ξ_t^s and w:

$$\mathbf{B}_\delta^s = \{ (\theta, \xi_0) \in \mathbf{B}^s : \ \theta \in \mathbb{T}_\rho^d, \ \ \|\xi_0\|_{X_\theta^s} < \delta\}. \tag{3.33}$$

Then we define

$$\begin{aligned}\mathcal{L}_\delta = \Big\{ & \xi^s : \mathbb{R}^+ \times \mathbf{B}_\delta^s \to \mathbf{B}_\delta^s | \ \forall t \in \mathbb{R}^+, (\theta, \xi_0) \in \mathbf{B}_\delta^s, \ \xi_t^s(\theta, \xi_0) \in X^s_{\Phi_t(\theta)}, \\ & \xi_0^s(\theta, \xi_0) = \xi_0, \quad \xi_t^s(\theta, 0) = 0, \ \xi^s \text{ is continuous in t and} \\ & \text{analytic in } (\theta, \xi_0) \in \mathbf{B}_\delta^s, \ \|D_2\xi_t^s(\theta, \xi_0)\|_{X_\theta^s, X^s_{\Phi_t(\theta)}} \le 2C_h e^{-\beta_1 t} \Big\}.\end{aligned} \tag{3.34}$$

Because of $\xi_t^s(\theta, 0) = 0$ and $\|D_2\xi_t^s(\theta, \xi_0)\|_{X_\theta^s, X^s_{\Phi_t(\theta)}} \le 2C_h e^{-\beta_1 t}$, then the functions ξ^s which belong to $\mathcal{L}_\delta$ satisfy

$$\|\xi_t^s(\theta, \xi_0)\|_{X^s_{\Phi_t(\theta)}} \le 2C_h e^{-\beta_1 t} \|\xi_0\|_{X_\theta^s}, \tag{3.35}$$

in particular, ξ^s is uniformly bounded.

Similarly, we argue that w should range in a space of bundle mappings covering the identity. We will impose that $w(\theta, 0) = 0$ so that graph of the mapping contains the zero section (that is, the stable manifold contains the invariant set). We will also impose that $D_2 w(\theta, 0) = 0$ so that the invariant manifold is tangent to the stable space. This is quite analogous to the normalizations in [49] in the simpler case of invariant manifolds of fixed points. As for the topology of the space we will impose analyticity in both the θ and ξ^s variables. We will consider only functions w defined on the unit ball in the fibers. This is because we will be considering only the local manifold.

We define

$$\begin{aligned}\mathcal{X}_1 = \Big\{ & w : \mathbf{B}_\delta^s \to \mathbf{B}^{cu} | \quad \forall \theta \in \mathbb{T}_\rho^d, \ \xi \in X_\theta^s, \ \|\xi\|_{X_\theta^s} < \delta, \\ & w(\theta, \xi) \in X_\theta^{cu}, w(\theta, 0) = 0, D_2 w(\theta, 0) = 0, \ w \text{ is analytic in} \\ & (\theta, \xi) \in \mathbf{B}_\delta^s \ \text{ and } \xi \in X_\theta^s, \ \|D_2 w(\theta, \xi)\|_{X_\theta^s, X_\theta^{cu}} \le 2C_h \|\xi\|_{X_\theta^s} \Big\}.\end{aligned} \tag{3.36}$$

Because of $w(\theta, 0) = 0$, $D_2 w(\theta, 0) = 0$ and $\|D_2 w(\theta, \xi)\|_{X_\theta^s, X_\theta^{cu}} \le 2C_h \|\xi\|_{X_\theta^s}$, then the functions $w \in \mathcal{X}_1$ also satisfy

$$\|w(\theta, \xi_s)\|_{X_\theta^{cu}} \le C_h \|\xi_s\|^2_{X_\theta^s}. \tag{3.37}$$

For $\beta_3^- < \beta < \beta_1$, we define the weighted norm

$$\begin{aligned} \|\xi^s\|_{C^0}^{(\beta} &\equiv \sup_{t\in\mathbb{R}^+} \sup_{(\theta,\xi_0)\in\mathbf{B}_\delta^s} \|\xi_t^s(\theta,\xi_0)\|_{X^s_{\Phi_t(\theta)}} e^{\beta t}, \\ \|w\|_{C^0} &\equiv \sup_{(\theta,\xi_0)\in\mathbf{B}_\delta^s} \frac{\|w(\theta,\xi_0)\|_{X_\theta^{cu}}}{\|\xi_0\|_{X_\theta^s}}, \\ \|(\xi^s,w)\|_{C^0} &= \max\{\|\xi^s\|_{C^0}^{(\beta}, \quad \|w\|_{C^0}\}. \end{aligned} \tag{3.38}$$

The induced metric on $\mathcal{L}_\delta \times \mathcal{X}_1$ is

$$d((\xi^s,w),(\widetilde{\xi^s},\widetilde{w})) = \|(\xi^s-\widetilde{\xi^s}, w-\widetilde{w})\|_{C^0}. \tag{3.39}$$

Note that for the functions $\xi^s \in \mathcal{L}_\delta$ and $w \in \mathcal{X}_1$, they satisfy

$$\|\xi^s\|_{C^0}^{(\beta} \le 2C_h\delta < 1, \quad \|w\|_{C^0} \le C_h\delta < 1. \tag{3.40}$$

We also note that in the above spaces, the real valued functions for real valued arguments are a closed real space and that the iterations, used to produce the fixed point, preserve this space if the original evolution equations do. Hence when the original equations preserve the real valued functions, the fixed points will also be real valued.

Proof of Theorem 3.1. The proof of the above Theorem 3.1 is based on the contraction fixed point theorem. We will check that $\mathscr{T} = (\mathscr{T}_s, \mathscr{T}_{cu})$ in (3.30) indeed defines a contraction operator in the function space $\mathcal{L}_\delta \times \mathcal{X}_1$ under the $d-$distance defined in (3.39). We separate the proof of Theorem 3.1 into two steps. First, we prove that $\mathscr{T}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset (\mathcal{L}_\delta \times \mathcal{X}_1)$ (Step 1) and then we prove that $\mathscr{T}$ is a contraction in $\mathcal{L}_\delta \times \mathcal{X}_1$ (Step 2).

Consider the integral

$$\int_0^t e^{-\beta(t-\tau)}(t-\tau)^{-\alpha}d\tau = \int_0^t e^{-\beta\tau}\tau^{-\alpha}d\tau,\ \beta>0,\ \alpha\in[0,1),\ t\ge 0,$$

first,

I: $0 \le t \le 1$.

$$\int_0^t e^{-\beta\tau}\tau^{-\alpha}d\tau \le \int_0^t \tau^{-\alpha}d\tau \le \frac{1}{1-\alpha}.$$

II: $t > 1$.

$$\int_0^t e^{-\beta\tau}\tau^{-\alpha}d\tau \le \int_0^1 e^{-\beta\tau}\tau^{-\alpha}d\tau + \int_1^t e^{-\beta\tau}\tau^{-\alpha}d\tau$$

$$
\begin{aligned}
&\leq \int_0^1 \tau^{-\alpha} d\tau + \int_1^t e^{-\beta\tau} d\tau \\
&\leq \frac{1}{1-\alpha} + \frac{1}{\beta}.
\end{aligned}
$$

The above two inequalities yield

$$
\begin{aligned}
\int_0^t e^{-\beta(t-\tau)}(t-\tau)^{-\alpha} d\tau &= \int_0^t e^{-\beta\tau}\tau^{-\alpha} d\tau \\
&\leq \frac{1}{1-\alpha} + \frac{1}{\beta}, \ \ \beta > 0, \ \alpha \in [0,1), \ t \geq 0.
\end{aligned}
$$

Moreover, by the inequality above we also have

$$
\begin{aligned}
\int_0^t e^{-\beta\tau}(t-\tau)^{-\alpha} d\tau &= \int_0^t e^{-\beta(t-\tau)}\tau^{-\alpha} d\tau \\
&= \int_0^{\frac{t}{2}} e^{-\beta(t-\tau)}\tau^{-\alpha} d\tau + \int_{\frac{t}{2}}^t e^{-\beta(t-\tau)}\tau^{-\alpha} d\tau \\
&\leq \int_0^{\frac{t}{2}} e^{-\beta\tau}\tau^{-\alpha} d\tau + \int_{\frac{t}{2}}^t e^{-\beta(t-\tau)}(t-\tau)^{-\alpha} d\tau \\
&< \int_0^t e^{-\beta\tau}\tau^{-\alpha} d\tau + \int_0^t e^{-\beta(t-\tau)}(t-\tau)^{-\alpha} d\tau \\
&= 2\int_0^t e^{-\beta\tau}\tau^{-\alpha} d\tau \\
&< 2\Big(\frac{1}{1-\alpha} + \frac{1}{\beta}\Big), \ \ \beta > 0, \ \alpha \in [0,1), \ t \geq 0.
\end{aligned}
$$

The above inequalities will be used in many places, we will omit the reference about them.

(Step 1) $\mathscr{T}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset (\mathcal{L}_\delta \times \mathcal{X}_1)$. The fact that $\mathscr{T}$ is an analytic function in θ and ξ_0 defined in the domain (3.33) is a direct consequence of the fact that the composition of analytic functions is an analytic function. Now we give the norm estimates. We estimate $\mathscr{T}_s$ first. For (3.31), obviously, $\mathscr{T}_s[\xi_t^s, w](\theta, 0) = 0$. Moreover,

$$
D_{\xi_0}\mathscr{T}_s[\xi^s, w](t,\theta,\xi_0) = U_\theta^s(t)
$$

$$+\int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)D_2M^s(\Phi_\tau(\theta),[\xi^s_\tau(\theta,\xi_0)+w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))])$$
$$\cdot\big[Id+D_2w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))\big]D_2\xi^s_\tau(\theta,\xi_0)dt.$$

Then from (2.7) and the triangle inequality we obtain

$$\begin{aligned}
\|D_{\xi_0}\mathscr{T}_s[\xi^s,w](t,\theta,\xi_0)\|_{X^s_\theta,X^s_{\Phi_t(\theta)}} &\le \|U^s_\theta(t)\|_{X^s_\theta,X^s_{\Phi_t(\theta)}}\\
&+\Big\|\int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)D_2M^s(\Phi_\tau(\theta),[\xi^s_\tau(\theta,\xi_0)+w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))])\\
&\qquad\cdot\big[Id+D_2w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))\big]D_2\xi^s_\tau(\theta,\xi_0)dt\Big\|_{X^s_\theta,X^s_{\Phi_t(\theta)}}\\
&\le C_he^{-\beta_1t}+4C_h^3\|\xi_0\|_{X^s_\theta}\|M\|_{C^2}\int_0^t e^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}e^{-2\beta_1\tau}d\tau\\
&= C_he^{-\beta_1t}+4C_h^3\|\xi_0\|_{X^s_\theta}\|M\|_{C^2}e^{-\beta_1t}\int_0^t e^{-\beta_1\tau}(t-\tau)^{-\alpha_1}d\tau\\
&\le C_he^{-\beta_1t}+8C_h^3e^{-\beta_1t}\|\xi_0\|_{X^s_\theta}\Big(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1}\Big)\|M\|_{C^2}\\
&< 2C_he^{-\beta_1t},
\end{aligned}$$

the last inequality is from the smallness of $\|M\|_{C^2}$. That is, we have established that $\mathscr{T}_s[\xi^s,w]\in\mathcal{L}_\delta$.

Now we consider $\mathscr{T}_{cu}[\xi^s,w]$. First, we verify $\mathscr{T}_{cu}[\xi^s,w](\theta,0)=0$ and $D_2\mathscr{T}_{cu}[\xi^s,w](\theta,0)=0$. For $\xi^s_t(\theta,0)=0$, $M(\theta,0)=0, D_2M(\theta,0)=0$ and $w(\theta,0)=0, D_2w(\theta,0)=0$, then $\mathscr{T}_{cu}[\xi^s,w](\theta,0)=0$. Furthermore,

$$\begin{aligned}
&D_{\xi_0}\mathscr{T}_{cu}[\xi^s,w](\theta,\xi_0)\\
&\quad=-\int_0^\infty U^{cu}_{\Phi_t(\theta)}(-t)D_2M^{cu}(\Phi_t(\theta),[\xi^s_t(\theta,\xi_0)+w(\Phi_t(\theta),\xi^s_t(\theta,\xi_0))])\\
&\qquad\big[Id+D_2w(\Phi_t(\theta),\xi^s_t(\theta,\xi_0))\big]D_2\xi^s_t(\theta,\xi_0)dt.
\end{aligned}$$

Obviously, $D_{\xi_0}\mathscr{T}_{cu}[\xi^s,w](\theta,0)=0$.

For $D_{\xi_0}\mathscr{T}_c[\xi^s, w](\theta, \xi_0)$, note that $(\xi^s, w) \in \mathcal{L}_\delta \times \mathcal{X}_1$, then we have

$$\begin{aligned}
&\|D_{\xi_0}\mathscr{T}_c[\xi^s, w](\theta, \xi_0)\|_{X^s_\theta, X^c_\theta} \\
&\qquad = \Big\| - \int_0^\infty U^c_{\Phi_t(\theta)}(-t) D_2 M^c(\Phi_t(\theta), [\xi^s_t(\theta, \xi_0) + w(\Phi_t(\theta), \xi^s_t(\theta, \xi_0))]) \\
&\qquad\qquad \big[Id + D_2 w(\Phi_t(\theta), \xi^s_t(\theta, \xi_0))\big] D_2\xi^s_t(\theta, \xi_0) dt \Big\|_{X^s_\theta, X^c_\theta} \\
&\qquad \le 4C_h^3 \|\xi_0\|_{X^s_\theta} \|M\|_{C^2} \int_0^\infty e^{\beta_3^- t} e^{-2\beta_1 t} dt \\
&\qquad \le 4C_h^3 \|\xi_0\|_{X^s_\theta} \frac{1}{2\beta_1 - \beta_3^-} \|M\|_{C^2} \\
&\qquad < 2C_h \|\xi_0\|_{X^s_\theta},
\end{aligned}$$

the last inequality is from the smallness of $\|M\|_{C^2}$. Similarly, for $D_{\xi_0}\mathscr{T}_u[\xi^s, w]$ we have

$$\begin{aligned}
\left\| D_{\xi_0}\mathscr{T}_u[\xi^s, w](\theta, \xi_0) \right\|_{X^s_\theta, X^{cu}_\theta} &\le 4C_h^3 \|\xi_0\|_{X^s_\theta} \Big(\frac{1}{2\beta_1 + \beta_2} + \frac{1}{1 - \alpha_2}\Big) \|M\|_{C^2} \\
&\le 2C_h \|\xi_0\|_{X^s_\theta}.
\end{aligned}$$

From the discussions above we know that $\mathscr{T}_{cu}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset \mathcal{X}_1$. Then together with the discussions about $\mathscr{T}_s$ we know that $\mathscr{T}(\mathcal{L}_\delta \times \mathcal{X}_1) \subset (\mathcal{L}_\delta \times \mathcal{X}_1)$.

Remark 3.6. It is amusing to note that, since the sets $\mathcal{L}_\delta$ and $\mathcal{X}_1$ are convex and compact in the C^0 topology, which makes $\mathscr{T}$ continuous, we can apply the Schauder fixed point theorem and obtain the existence (but not the uniqueness) of a fixed point at this stage.

This remark applies to many of the textbook proofs the invariant manifold theorem based on functional analysis since many of them involve some propagated bounds in the proofs in [8,10,49].

Of course, Step 2, provides uniqueness, gives a constructive algorithm to find the fixed point, allows to validate approximate calculations, gives slightly better regularity. One can argue that the contraction mapping is more elementary than Schauder fixed point theorem even if it requires more work.

(Step 2) $\mathscr{T}$ is a contraction in $\mathcal{L}_\delta \times \mathcal{X}_1$ under the $d-$distance defined in (3.39). For any $(\tilde{\xi}^s, \tilde{w}), (\xi^s, w) \in \mathcal{L}_\delta \times \mathcal{X}_1$ and $(\theta, \xi_0) \in \mathbf{B}^s$ with $\|\xi_0\|_{X^s_\theta} \le \delta$ we have

$$\begin{aligned}
&\mathscr{T}_s[\tilde{\xi}^s, \tilde{w}](t, \theta, \xi_0) - \mathscr{T}_s[\xi^s, w](t, \theta, \xi_0) \\
&\qquad = \int_0^t U^s_{\Phi_\tau(\theta)}(t - \tau) \Big[M^s(\Phi_\tau(\theta), [\tilde{\xi}^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0))])
\end{aligned}$$

$$- M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))])\Big] d\tau.$$

Note that

$$\begin{aligned}
&\Big\| w(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0)) - w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0)) \Big\|_{X^{cu}_{\Phi_\tau(\theta)}} \\
&\qquad \le \|\xi_0\|_{X^s_\theta} \|\tilde{\xi}^s_\tau(\theta, \xi_0) - \xi^s_\tau(\theta, \xi_0)\|_{X^s_{\Phi_\tau(\theta)}} \\
&\qquad \le \|\xi_0\|_{X^s_\theta} \|\tilde{\xi} - \xi\|^{(\beta}_{C^0} e^{-\beta\tau} \le \|\tilde{\xi} - \xi\|^{(\beta}_{C^0} e^{-\beta\tau}.
\end{aligned} \tag{3.41}$$

By adding and subtracting terms and triangle inequality and from (3.41) we obtain

$$\begin{aligned}
&\sup_{(\theta,\xi_0)\in \mathbf{B}^s_\delta} \Big\| \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau) \Big[M^s(\Phi_\tau(\theta), [\tilde{\xi}^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0))]) \\
&\quad - M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))])\Big] d\tau \Big\|_{X^s_{\Phi_t(\theta)}} e^{\beta t} \\
&\le \sup_{(\theta,\xi_0)\in \mathbf{B}^s_\delta} \Big\| \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau) \Big[M^s(\Phi_\tau(\theta), [\tilde{\xi}^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0))]) \\
&\quad - M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0))])\Big] d\tau \Big\|_{X^s_{\Phi_t(\theta)}} e^{\beta t} \\
&+ \sup_{(\theta,\xi_0)\in \mathbf{B}^s_\delta} \Big\| \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau) \Big[M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \tilde{\xi}^s_\tau(\theta, \xi_0))]) \\
&\quad - M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))])\Big] d\tau \Big\|_{X^s_{\Phi_t(\theta)}} e^{\beta t} \\
&+ \sup_{(\theta,\xi_0)\in \mathbf{B}^s_\delta} \Big\| \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau) \Big[M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + \tilde{w}(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))]) \\
&\quad - M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))])\Big] d\tau \Big\|_{X^s_{\Phi_t(\theta)}} e^{\beta t} \\
&\le 4C_h^2 \|M\|_{C^2} \big(2\|\tilde{\xi}^s - \xi^s\|^{(\beta}_{C^0} + \|\tilde{w} - w\|_{C^0}\big) \int_0^t e^{-(\beta_1-\beta)(t-\tau)}(t-\tau)^{-\alpha_1} d\tau \\
&\le 8C_h^2 \|M\|_{C^2} \Big(\|\tilde{\xi}^s - \xi^s\|^{(\beta}_{C^0} + \|\tilde{w} - w\|_{C^0} \Big) \Big(\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1} \Big) \\
&\le 16C_h^2 \|M\|_{C^2} \|(\xi^s - \widetilde{\xi^s}, w - \widetilde{w})\|_{C^0} \Big(\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1} \Big).
\end{aligned}$$

That is

$$\|\mathscr{T}_s[\tilde{\xi}^s, \tilde{w}] - \mathscr{T}_s[\xi^s, w]\|_{C^0}^{(\beta} \leq 16C_h^2\|M\|_{C^2}\|(\xi^s - \widetilde{\xi^s}, w - \widetilde{w})\|_{C^0}\Big(\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1}\Big).$$

In the case of center and unstable spaces we get

$$\|\mathscr{T}_c[\tilde{\xi}^s, \tilde{w}] - \mathscr{T}_c[\xi^s, w]\|_{X_\theta^c} \leq 16C_h^2\frac{\|M\|_{C^2}}{\beta - \beta_3^-}\|(\xi^s - \widetilde{\xi^s}, w - \widetilde{w})\|_{C^0},$$

and

$$\|\mathscr{T}_u[\tilde{\xi}^s, \tilde{w}] - \mathscr{T}_u[\xi^s, w]\|_{X_\theta^c} \leq 16C_h^2\|M\|_{C^2}\Big(\frac{1}{\beta_2 + \beta} + \frac{1}{1-\alpha_2}\Big)\|(\xi^s - \widetilde{\xi^s}, w - \widetilde{w})\|_{C^0}.$$

That is

$$\|\mathscr{T}_{cu}[\tilde{\xi}^s, \tilde{w}] - \mathscr{T}_{cu}[\xi^s, w]\|_{C^0} \leq 16c_1C_h^2\|M\|_{C^2}\|(\xi^s - \widetilde{\xi^s}, w - \widetilde{w})\|_{C^0}$$

with

$$c_1 = \max\left\{\frac{1}{\beta - \beta_3^-}, \frac{1}{\beta_2 + \beta} + \frac{1}{1-\alpha_2}\right\}.$$

By the discussions above we know that

$$d(\mathscr{T}[\tilde{\xi}^s, \tilde{w}], \mathscr{T}[\xi^s, w]) \leq c d((\tilde{\xi}^s, \tilde{w}), (\xi^s, w)),$$

where $c = 16C_h^2\kappa\|M\|_{C^2}$ with

$$\kappa = \max\left\{\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1}, \frac{1}{\beta - \beta_3^-}, \frac{1}{\beta_2 + \beta} + \frac{1}{1-\alpha_2}\right\}.$$

From the smallness of $\|M\|_{C^2}$ we know that $c \in (0, 1)$. That is the operator $\mathscr{T}$ is a contraction in $d-$distance defined in (3.39).

From the contraction mapping theorem we know that there is unique solution of (3.30), *i.e.* (ξ^s, w) in the C^0 closure of $\mathcal{L}_\delta \times \mathcal{X}_1$. Furthermore, since the uniform limit of the analytic functions is analytic, the fixed point is a analytic function.

Obviously, w satisfies the condition of the Theorem 3.1 and its graph, $\mathcal{W}$, is forward invariant under (3.21).

3.2. Stable manifold around a forward bounded solution

In this section, we construct analytic stable manifolds for the forward bounded solutions, $K(t)$, of (2.1), *i.e.*

$$\frac{d}{dt}K(t) = \mathscr{X} \circ K(t). \tag{3.42}$$

That is we want to show that there are many solutions that converge in the future to the bounded solutions. This work is motivated by the paper [23] which constructed center manifold for (2.1) under some extra non-degeneracy assumptions.

Since (2.1) could be ill-posed, not all the initial values can define the evolution. We will need to carefully choose the special initial values $K(0)$ so that the evolution, which starts from $K(0)$, of (2.1) can be defined. We will not consider the dependence and the regularity on the initial value $K(0)$. We just fix the bounded solution $K(t)$ of (2.1). The whiskered torus $K(\theta + \omega t)$ is an special case of bounded solution $K(t)$. So the discussions of this case are similar to the discussions of the whiskered torus case by fixing $\theta = 0$ and replacing the quasi-periodic function $K(\theta + \omega t)$ with $K(t)$.

In Appendix A we will show the invariance of the invariant splitting of the linear operator $A(t)$ under small perturbations. Concretely, we will show that if linear operator $A(t)$ admits a hyperbolic/smoothing splitting and $\widetilde{A}(t)$ is the one such that $\|A(t) - \widetilde{A}(t)\|_{X,Y}$ is small enough, then the evolution equations corresponding to $\tilde{A}(t)$ also admit a hyperbolic/smoothing splitting. Furthermore, we will also bound, in terms of $\|A(t) - \widetilde{A}(t)\|_{X,Y}$ the difference between the invariant spaces and the parameters of the splitting for $\widetilde{A}$ and those corresponding to A. See Appendix A for more precise statements and for proofs.

Theorem 3.2. *Assume that X, Y are Banach spaces satisfying* **(H1)** *and that we have an forward equation* (2.1) *of the form* (2.2) *satisfying* **(H2)**. *Assume that we have a bounded solution, $\{K(t)\}_{t\in[0,\infty)}$, of* (2.1) *(that is $K(t)$ satisfies* (3.42)*) which satisfies* **(SD1)**, *in particular, we can find bundles $\mathbf{B}^s$, $\mathbf{B}^{cu}$ based on the forward solution. (We will show that if K is forward small, it indeed satisfies* **(SD1)**).

Then, there exists a analytic bundle map covering the identity, $w \in \mathcal{P}_1$, defined on $\boldsymbol{B}^s$ and mapping $\boldsymbol{B}^s$ to $\boldsymbol{B}^{cu}$ satisfying $w(t, 0) = 0$, $D_2 w(t, 0) = 0$. Furthermore, $\mathscr{W}$, the graph of w is globally forward invariant by (3.21).

In this case, since we fixed the angle variable so w will be only analytic on the fiber variable. With the same calculations in the whiskered torus case we obtain

$$\begin{pmatrix} \xi^s \\ w \end{pmatrix}(t, \xi_0) = \mathscr{E}[\xi^s, w](t, \xi_0) = \begin{pmatrix} \mathscr{E}_s[\xi^s, w] \\ \mathscr{E}_{cu}[\xi^s, w] \end{pmatrix}(t, \xi_0) \tag{3.43}$$

with

$$\mathscr{E}_s[\xi^s, w](t, \xi_0) = U_0^s(t)\xi_0 + \int_0^t U_\tau^s(t-\tau) M^s(\tau, [\xi^s(\tau, \xi_0) + w(\tau, \xi^s(\tau, \xi_0))]) d\tau, \tag{3.44}$$

and

$$\mathscr{E}_{cu}[\xi^s, w](0, \xi_0) = -\int_0^\infty U_t^{cu}(-t) M^{cu}(t, [\xi^s(t, \xi_0) + w(t, \xi^s(t, \xi_0))])dt. \tag{3.45}$$

We will again produce our invariant manifold as the fixed point of the operator $\mathscr{E}$ defined in (3.43). We define spaces of functions on which $\mathscr{E}$ acts to apply a fixed point argument.

For $0 < \delta < 1$, we also define the domain of ξ_t^s and w as $\mathbf{B}_\delta^s$ which is similar to the one defined in (3.33). Then we define the space $\mathcal{S}_\delta$ on which $\mathscr{E}_s$ acts

$$\begin{aligned} \mathcal{S}_\delta = \Big\{ & \xi^s : \mathbb{R}^+ \times \mathbf{B}_\delta^s \to \mathbf{B}^s \,|\, \forall t \in \mathbb{R}^+,\ (0, \xi_0) \in \mathbf{B}_\delta^s,\ \xi^s(t, \xi_0) \in X_t^s, \\ & \xi^s(0, \xi_0) = \xi_0,\ \xi^s(t, 0) = 0,\ \xi^s \text{ is continuous in t} \\ & \text{and analytic in } \xi_0,\ \|D_{\xi_0}\xi^s(t, \xi_0)\|_{X_0^s, X_t^s} \le 2C_h e^{-\beta_1 t} \Big\}. \end{aligned} \tag{3.46}$$

Similarly, we also define the space $\mathcal{P}_1$ on which $\mathscr{E}_{cu}$ acts

$$\begin{aligned} \mathcal{P}_1 = \Big\{ & w : \mathbf{B}_\delta^s \to \mathbf{B}^{cu} | \quad \forall t \in \mathbb{R}^+,\ \xi \in X_t^s,\ w(t, \xi) \in X_t^{cu}, \\ & w(t, 0) = 0,\ D_2 w(t, 0) = 0,\ w \text{ is continuous t and} \\ & \text{analytic in } \xi,\ \|D_2 w(t, \xi)\|_{X_t^s, X_t^{cu}} \le 2C_h \|\xi\|_{X_t^s} \Big\}. \end{aligned} \tag{3.47}$$

For $\beta_3^- < \beta < \beta_1$, we also adopt the weighted norms $\|\cdot\|_{C^0}^{(\beta}$, $\|\cdot\|_{C^0}$ and $\|(\cdot_1, \cdot_2)\|_{C^0}$ defined in (3.38) for the functions ξ^s, w and (ξ^s, w), respectively. The induced metric on $\mathcal{S}_\delta \times \mathcal{P}_1$ is also the $d-$distance defined in (3.39). For the functions $\xi^s \in \mathcal{S}_\delta$ and $w \in \mathcal{P}_1$, (3.35) and (3.37) also hold true.

The proof of Theorem 3.2 is the same as that of Theorem 3.1 but even simpler, since we do not need to consider the regularity in θ. The proof is word by word the same except that θ is fixed in this case.

3.3. Stable manifold around an invariant set in the center manifold

We recall that the paper [23] established the existence of finite dimensional center manifolds for equation similar to (2.1) under some non-degenerate assumptions similar to the ones in Section 2.

The goal of this section is to establish the existence of solutions that converge in the future to solutions in the center manifold and also establish regularity of these stable solutions with respect to the parameters and to the initial point of the solutions in the center manifold.

As it is well known [49,10,68,53], the results about center manifolds are subtle to formulate because center manifolds are not unique and they are only locally invariant. Since the center manifold is only locally invariant, the solutions which leave the center manifold could cease to exist after a finite time.

The well known standard solution, [49,10,68,53], to these problems is to consider some "prepared" equations which agree with the original equations in a small neighborhood but which are better behaved globally. For these prepared equations there is a unique invariant center manifold.

Of course, these center manifolds for the prepared equations will be locally invariant for the original problem. The non-uniqueness of center manifolds for the problem originates from the fact that there are many ways to obtain prepared equations and each of them leads to a possibly different manifold. Nevertheless, it is worth pointing out that the solutions of the prepared equation which remain for all time in a small neighborhood have to be in any center manifold.

We remark that in our problem, the prepared equations themselves may be ill-posed. The "preparation"affects only the center direction and not the stable or unstable ones which are the ones that lead to ill-posedness. Hence, in our case, the existence of solutions will be part of the problem and we will have to choose initial conditions not only to fix the long term behavior but also to ensure existence.

For solutions that remain bounded in the future, the stable manifold will be unique. Of course, the set of such future bounded solutions will, in general, be complicated, but our results show that the stable manifolds will depend smoothly on the base points in the sense that they extend to a smooth function in a smooth submanifold (the center manifold of any of the prepared equations). The notion of regularity of functions defined in general closed sets is studied in the celebrated papers [69,70], (see also [65]), which gave an intrinsic characterization of functions which admit an extension. In our case, we show directly that the manifolds we construct for sets of bounded orbits extend to a smooth family. So, we obtain that the solutions which we construct for a closed set of bounded solutions are smooth in the sense of Whitney. In this paper, the precise meaning of C^r functions involves some uniformity assumptions, see Definition C.1 which is the customary definition in infinite-dimensional spaces.

Remark 3.7. In this section, we will assume that the spaces X, Y are real spaces. The reason is that we will assume that the nonlinearity $\|\mathscr{N}\|_{C^r}$ is bounded, but now, we need to assume bounds in the whole space X_c. If the spaces X was a complex space, by Liouville theorem and $\mathscr{N}(0) = 0$, we see that the assumptions apply only to trivial nonlinearities. The theorems we present indeed remain true for complex spaces as stated, but they are trivial.

3.3.1. The prepared equations

Under the assumption of existence of invariant splittings for $\mathscr{A}$, we see that the equation (2.1) is equivalent to a system of three equations

$$\begin{aligned}
\frac{d}{dt}u^s &= \mathscr{A}^s u^s + \mathscr{N}^s(u),\\
\frac{d}{dt}u^c &= \mathscr{A}^c u^c + \mathscr{N}^c(u),\\
\frac{d}{dt}u^u &= \mathscr{A}^u u^u + \mathscr{N}^u(u),
\end{aligned} \tag{3.48}$$

where $\mathscr{N}^\sigma$ indicates the projection over the corresponding spaces.

The prepared equations correspond to modifying (3.48) so that it is well behaved at infinity. One important role in this modification is played by cut-off functions.

Definition 3.2. Let Z be a Banach space. We say that a C^r function $\varphi : Z \to \mathbb{R}$ is a cut-off function when $\varphi(u) = 1$ when $||u||_Z \le 1$ and $\varphi(u) = 0$ when $\|u\|_Z \ge 2$ and all the derivatives of φ of order up to r are uniformly bounded.

Of course, for $r = 0$ or $r = Lip$, cut-off functions are constructed using Urysohn's lemma, but for $r \geq 2$, the existence of cut-off function is a very subtle problem.

Remark 3.8. Of course, C^r cut-off functions exist in any finite dimensional Banach space or in any Banach space with a smooth norm (in particular in Hilbert spaces. Nevertheless, it is well known that some spaces common in analysis (e.g. $C^0[0, 1]$) do not admit C^2 bump functions [6,21].

Lets, however, emphasize that the existence of bump function in X^c is a sufficient condition for our results, but it is by no means necessary. The only thing needed is that we can get the prepared equations. In many problems of interest, the nonlinearity is of the form $\Phi \circ u$ with Φ being a finite dimensional function composed on the left with unknown function u. In such case, we can prepare the equation by just cutting off the finite dimensional function Φ.

In this section, we will make the assumption:

- **(H3)**
  - **(H3.1)** In the notations of Section 2, we have $X^c = Y^c$.
  - **(H3.2)** The space X^c admits C^r cut-off functions.

Of course, assumption **(H3)** is implied by the following assumption **(H3')** which is satisfied in all the concrete applications in Section 4.

- **(H3')** The space X^c is finite dimensional.

The prepared equations [49] are obtained by scaling the variables u^σ,

$$u^\sigma = \eta \tilde{u}^\sigma, \quad \sigma = s, c, u \tag{3.49}$$

and by cutting-off the resulting equation.

$$\begin{aligned}
\frac{d}{dt}\tilde{u}^s &= \mathscr{A}^s \tilde{u}^s + \eta^{-1}\mathscr{N}^s(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u), \\
\frac{d}{dt}\tilde{u}^c &= \mathscr{A}^c \tilde{u}^c + \varphi(\tilde{u}^c)\eta^{-1}\mathscr{N}^c(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u), \\
\frac{d}{dt}\tilde{u}^u &= \mathscr{A}^u \tilde{u}^u + \eta^{-1}\mathscr{N}^u(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u).
\end{aligned} \tag{3.50}$$

Note that, if we take away the φ from (3.50), the result is equivalent to (3.48) under the change of variables (3.49). Hence, (3.50) are equivalent to the original equation when $\tilde{u}^c \in E_\eta(X^c) \equiv \{\tilde{u}^c : \ \|\tilde{u}^c\|_{X^c} < 1\}$, that is, in the original variables when $\|u^c\|_{X^c} < \eta$.

Observe that the equations (3.50) are also equations of the form (2.1) but with a modified nonlinear term $\mathcal{N}$. But the equations (3.50) are linear for $\tilde{u}^c$ large. Furthermore, the nonlinear terms in the center direction of (3.50) are small when η is small. More precisely, if we denote

$$\tilde{\mathscr{N}}(\tilde{u}^s, \tilde{u}^u, \tilde{u}^c) \equiv \eta^{-1}\big(\mathscr{N}^s(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u), \varphi(\tilde{u}^c)\mathscr{N}^c(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u), \mathscr{N}^u(\eta\tilde{u}^s, \eta\tilde{u}^c, \eta\tilde{u}^u)\big),$$

then

$$\|\tilde{\mathcal{N}}\|_{C^r(B(0,1,X),Y)} \ll 1, \quad \text{for} \;\; \eta \;\; \text{small}. \tag{3.51}$$

(We refer to [49] for the small calculation needed.) Note that considering η small corresponds to cutting off the non-linearity outside of a ball of radius η in X^c in the original coordinates. The smallness assumptions in the nonlinearity of the prepared equations correspond to considering only orbits of the original equations that lie in small neighborhoods of the origin.

Notice that the norm involved in (3.51) involves domains which are unit balls in the X^s, X^u spaces, but that in the X^c we require the whole space. The range of $\mathcal{N}$ is the space Y.

In our main results in this section, Theorem 3.3, we will assume that $||\mathcal{N}||_{C^r}$ is small. In some applications this could hold for the full system (3.48), but in general, it will hold only for the prepared equations (3.50). In such a case, our result will apply to the original equations only when the solutions remain in an small ball in the future.

The results of [23] showed that for the prepared equation, one obtains a unique center manifold which is expressed as the graph of a function from X^c to $X^s \oplus X^u$. This center manifold is invariant under the prepared equations and is tangent to the center space at the origin. Hence, the center manifold is locally invariant for the original equation in the domain where the prepared equation is equivalent to the original one.

This center manifold will, in general, depend on the cut-off function used. Nevertheless, it is known that any two manifolds produced by different cut-off functions will be very close at the origin [64]. We note that the proof of [64] even if presented only in finite dimensions, works line by line in our set up. Even if we will not use it much in this paper, we also remark that the solutions that remain bounded for all positive and negative times (e.g. periodic or quasi-periodic solutions) of the original equations have to be in any of the center manifolds whose projection to X^c include the projection of the bounded solutions considered.

The solutions of the prepared equation that remain in the set $E_\eta(X^c)$ are also solutions of the full equation. Of course, the solutions of the prepared equation that step out of this set could fail to be solutions of our original equation. Hence if we obtain bounded (with a sufficiently small bound) solutions of the prepared equation in the center manifold and consider the solutions of the prepared equations that converge to them, they will also be solutions of the original equation. Hence the stable manifolds under the prepared equations solutions that remain small in the future will also be stable manifolds under the true equations. Of course, if the solutions step out of the region in the future, then there is no unique notion of stable manifold for them.

Besides the above remarks, that are well known in center manifold theory, we note that in our case, since we are considering ill-posed equations, we also have to worry about the possible blow up of the solutions or that the dynamics cannot be defined. The construction of the stable manifolds will have to include conditions that ensure that the dynamics can be defined.

3.3.2. Description of the result

Since the center manifold $\mathcal{W}^c$ is the graph of a function $w : X^c \to X^s \oplus X^u$, it is convenient to consider the embeddings

$$K : X^c \to \mathcal{W}^c$$

given by

$$K(\theta) = (\theta, w(\theta)). \tag{3.52}$$

Note that if the manifold $\mathscr{W}^c$ is C^r close enough to X^c, then the K of the form (3.52) (and hence the w) are uniquely determined. Actually, the embedding K is a diffeomorphism between X^c and $\mathcal{W}^c$, it defines a dynamics on X^c, which we denote by $\Phi_t(\theta)$, it is constructed in [23] and was denoted by $J_t^w(\theta)$. (In the paper [23] one also find $J_t^w(\theta)$ which is the dynamics related to a not necessarily invariant graph. This was useful since it allowed to define the operator in a whole neighborhood of maps w. If w^* is the map whose graph gives the center manifold, then we set $\Phi_t(\theta) \equiv J_t^{w^*}(\theta)$.)

We assume $\Phi_t(\theta)$ is the solution of

$$\dot{\theta} = \mathcal{B}(\theta), \quad \theta \in X^c \tag{3.53}$$

and the unbounded function, $K(\Phi_t(\theta))$, which is in the center manifold, is the solution of (3.50), $i.e.$

$$\frac{d}{dt} K^\sigma(\Phi_t(\theta)) = \mathcal{A}^\sigma K^\sigma(\Phi_t(\theta)) + \mathcal{N}^\sigma(K(\Phi_t(\theta))), \quad \sigma = s, c, u, \tag{3.54}$$

where $\mathcal{B}$ is a smooth vector field on a manifold X^c. The evolution of $\Phi_t(\theta)$ satisfies, for $1 \le j \le r-1$,

$$\begin{aligned} \|D_\theta^j \Phi_t(\theta)\|_{(X^c)^{\otimes j}, X^c} &\le C\epsilon^{\frac{-j}{4r}} e^{(\beta_3^+ + \epsilon^{\frac{1}{4r}})t}, \quad t \ge 0, \\ \|D_\theta \Phi_t(\theta)\|_{X^c, X^c} &\le C\epsilon^{\frac{-j}{4r}} e^{(\beta_3^+ + \epsilon^{\frac{1}{4r}})|t|}, \quad t \le 0. \end{aligned} \tag{3.55}$$

The solutions in the stable manifold will be solutions of the form

$$u(t) = K(\Phi_t(\theta)) + \xi(t); \qquad \xi(t) \to 0 \tag{3.56}$$

with $\xi(t)$ going to zero fast enough. In a way completely analogous to the one to derive (3.5), we get the evolution equation for ξ

$$\frac{d}{dt}\xi^\sigma(t) = A^\sigma(\Phi_t(\theta))\xi^\sigma(t) + M^\sigma(\Phi_t(\theta), \xi(t)), \quad \sigma = s, c, u, \tag{3.57}$$

where A^σ is the one defined in (3.6) and $M^\sigma(\Phi_t(\theta), \xi(t))$ is the remainder of the Taylor expansion of $\mathcal{N}^\sigma$ along $K(\Phi_t(\theta))$, $\sigma = s, c, u$. Since $\mathcal{N}$ has C^r regularity, from (3.7) we know that M is just C^{r-1} regularity, note also that $\Phi_t(\theta)$ is $C^{r-1+Lip}$, so the stable manifold we obtain is just $C^{r-2+Lip}$ (see Remark 3.9).

The following result Theorem 3.3 will be our main result for the stable manifolds of solutions based on a center manifold. We will state the theorem for the prepared equations (3.50) and, hence we will assume that the nonlinearity is globally small. See the previous discussion on how to apply this result to the original equation.

Theorem 3.3. *Assume that X, Y are real Banach spaces satisfying* **(H1)** *and* **(H3)** *and that we have an equation* (2.1) *of the form* (2.2) *satisfying* **(H2)** *with $\|\mathcal{N}\|_{C^r}$ small enough. Assume furthermore that $\beta_1 > (r-1)\beta_3^+ + \beta_3^-$ and $\beta_2 > (r-1)\beta_3^- + \beta_3^+$, then we have*

- *For each* $\theta \in X^c$,

$$A(t) = \mathcal{A} + D\mathcal{N}(K(\Phi_t(\theta))),$$

the linearization of (2.1) *around the orbit* $K(\Phi_t(\theta))$, *satisfies the spectral hypothesis* **(SD1)**. *In particular, we can find a decomposition of* X *based at* $K(\Phi_0(\theta)) = K(\theta)$, $X = X^s_\theta \oplus X^c_\theta \oplus X^u_\theta$ *with* $X^c_\theta = T_{K(\theta)}\mathcal{W}^c$.
- *For each* $\theta \in X^c$, *there is a* $C^{r-2+Lip}$ *local stable manifold to* $K(\Phi_t(\theta))$ *tangent to* X^s_θ. *(This manifold will be obtained as the graph of a map* $w_\theta : X^s_\theta \to X^{cu}_\theta$; $w_\theta(0) = 0$, $Dw_\theta(0) = 0$.*)*
- *The manifolds depend smoothly on* θ *in the sense that* $w_\theta(\xi_0)$ *is* $C^{r-2+Lip}$ *in* θ *for fixed* ξ_0 *and* $C^{r-2+Lip}$ *in* ξ_0 *for fixed* θ *with uniform bounds on the derivatives.*

Remark 3.9. It is well known to experts that one can improve our regularity conclusions from $C^{r-2+Lip}$ to C^{r-1}.

One possible method [33,8,27] is to derive a functional equation for $D^{r-1}w$, (assuming it exists) which we denote as F. This functional equation can be shown to have a solution, which is the only possible candidate for $D^{r-1}w$. We denote it provisionally by $D^{r-1}w$ even if we have not yet shown it is a derivative.

To show our candidate is a derivative, we consider

$$\widetilde{w}(\theta) = w(\theta - \sigma) + Dw(\theta - \sigma)\sigma + \cdots + \frac{1}{(r-1)!} D^{r-1}w(\theta - \sigma)\sigma^{\otimes(r-1)}.$$

We show, using the equation satisfied by $D^{r-1}w$ that $\|F(\widetilde{w}) - \widetilde{w}\| \le o(|\sigma|^{r-1})$. Using that F is a uniform contraction, we obtain that $D^{r-1}w$ is a true derivative.

For convenience, we estimate the derivatives of $w_\theta(\xi_0)$ only in the θ and ξ_0 directions separately and do not study the mixed derivatives. Using regularity results on functions of several variables due to [57,58] and reproduced here as Lemma C.3, we obtain automatically the existence, continuity and boundedness of the mixed derivatives $\partial^i_\theta \partial^j_{\xi_0} w_\theta(\xi_0)$ when $i + j < r - 2 + Lip$.

That is, the function $(\theta, \xi_0) \to w_\theta(\xi_0)$ is $C^{r'}$ jointly in θ, ξ_0 for any $r' < r - 2 + Lip$. We remark that, with appropriate definitions of $C^{r'}$ for non-integer values of r', Lemma C.3 extends to non-integer values of r' and hence, our results also extend.

Other more complicated proofs (estimating also mixed derivatives as in [12]) seem to lead to sharper regularity results, but we decided not to include them here for simplicity.

The reason to estimate separately the regularity in θ and ξ_0 is that the variables θ, ξ_0 have a different role (one can think of θ as parameters and ξ_0 as the dynamical variables) and then the formulas for the iterated derivatives are simpler to understand than the formulas of mixed derivatives.

Similar with the discussions in the whiskered torus case, for (3.57) by using the Duhamel's formula we get

$$\begin{pmatrix} \xi^s \\ w \end{pmatrix}(t, \theta, \xi_0) = \mathscr{F}[\xi^s, w](t, \theta, \xi_0) = \begin{pmatrix} \mathscr{F}_s[\xi^s, w] \\ \mathscr{F}_{cu}[\xi^s, w] \end{pmatrix}(t, \theta, \xi_0) \tag{3.58}$$

with

$$\begin{aligned}\mathscr{F}_s[\xi^s, w](t,\theta,\xi_0) &= U^s_\theta(t)\xi_0\\ &\quad + \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta,\xi_0))])d\tau\end{aligned} \tag{3.59}$$

and

$$\mathscr{F}_{cu}[\xi^s, w](\theta,\xi_0) = -\int_0^\infty U^{cu}_{\Phi_t(\theta)}(-t)M^{cu}(\Phi_t(\theta), [\xi^s_t(\theta,\xi_0) + w(\Phi_t(\theta), \xi^s_t(\theta,\xi_0))])dt. \tag{3.60}$$

Recall that M is the Taylor remainder defined in (3.7). In this case we will also produce our invariant manifold as the fixed point the operator $\mathscr{F}$ defined in (3.58). We define spaces of functions on which $\mathscr{F}$ acts to apply a fixed point argument.

For $0 < \delta < 1$ we define the domain of ξ^s_t and w as $\mathbf{B}^s_\delta$ which is similar to the one defined in (3.33) and the space $\mathcal{Q}_\delta$ on which $\mathscr{F}_s$ acts. Note that in the previous cases we were considering analytic functions defined in complex Banach spaces, whereas in this case, we are considering real Banach spaces and finitely differentiable functions on them.

$$\begin{aligned}\mathcal{Q}_\delta = \Big\{ &\xi^s : \mathbb{R}^+ \times \mathbf{B}^s_\delta \to \mathbf{B}^s \mid \forall t \in \mathbb{R}^+,\ (\theta,\xi_0) \in \mathbf{B}^s_\delta,\ \xi^s_t(\theta,\xi_0) \in X^s_{\Phi_t(\theta)},\\ &\xi^s_0(\theta,\xi_0) = \xi_0,\ \xi^s_t(\theta,0) = 0,\ \xi^s \text{ is continuous in t and r} - 2\ (\text{r} - 1)\\ &\text{order differentiable in } \theta\ (\xi_0),\ \|D_{\xi_0}\xi^s_t(\theta,\xi_0)\|_{X^s_\theta, X^s_{\Phi_t(\theta)}} \le 2C_h e^{-\beta_1 t},\\ &\|D^j_{\xi_0}\xi^s_t(\theta,\xi_0)\|_{C^0} \le C_h e^{-\beta_1 t}, \quad j = 2,\cdots,r-1,\\ &\|D^j_\theta \xi^s_t(\theta,\xi_0)\|_{X_c^{\otimes j}, X^s_{\Phi_t(\theta)}} \le C_h \epsilon^{\frac{1}{2}} e^{-[\beta_1 - j(\beta_3^+ + \epsilon^{\frac{1}{4r}})]t}\|\xi_0\|_{X^s_\theta},\ j = 1\cdots,r-2,\\ &Lip_\theta\big(D^{r-2}_\theta \xi^s_t(\theta,\xi_0)\big|_{X_c^{\otimes(r-2)}, X^s_{\Phi_t(\theta)}}\big) \le C_h e^{-[\beta_1 - (r-1)(\beta_3^+ + \epsilon^{\frac{1}{4r}})]t}\|\xi_0\|_{X^s_\theta}\Big\},\end{aligned} \tag{3.61}$$

where we write X^c as X_c and $D^{r-2}_\theta \xi^s_t(\theta,\xi_0)\big|_{X_c^{\otimes(r-2)}, X^s_{\Phi_t(\vartheta)}}$ means that $D^{r-2}_\theta \xi^s_t(\theta,\xi_0) \in \mathscr{L}(X_c^{\otimes(r-2)}, X^s_{\Phi_t(\vartheta)})$. Similarly, we also define the space $\mathcal{R}_1$ on which $\mathscr{F}_{cu}$ acts

$$\begin{aligned}\mathcal{R}_1 = \Big\{ &w : \mathbf{B}^s_\delta \to \mathbf{B}^{cu} \mid\ \forall(\theta,\xi) \in \mathbf{B}^s,\ w(\theta,\xi) \in X^{cu}_\theta,\ w(\theta,0) = 0,\\ &D_2 w(\theta,0) = 0,\ w \text{ is r} - 2\ (\text{r} - 1) \text{ order differentiable in } \theta\ (\xi),\\ &\|D_\xi w(\theta,\xi)\|_{X^s_\theta, X^{cu}_\theta} \le 2C_h\|\xi\|_{X^s_\theta}, \quad \|D^i_\theta w(\theta,\xi)\|_{X_c^{\otimes j}, X^{cu}_\theta} \le \|\xi\|^2_{X^s_\theta},\\ &\|D^j_\xi w(\theta,\xi)\|_{(X^s_\theta)^{\otimes i}, X^{cu}_\theta} \le 1,\ i = 1,\cdots,r-2,\ j = 2,\cdots,r-2,\\ &Lip_\theta\big(D^{r-2}_\theta w(\theta,\xi)\big) \le 1\Big\}.\end{aligned} \tag{3.62}$$

In this case, for $\beta_3^- < \beta < \beta_1$, we also use the weighted norms $\|\cdot\|_{C^0}^{(\beta}$, $\|\cdot\|_{C^0}$ and $\|(\cdot_1, \cdot_2)\|_{C^0}$ defined in (3.38) for the functions ξ^s, w and (ξ^s, w), respectively. The induced metric on $\mathcal{Q}_\delta \times \mathcal{R}_1$ is also the $d-$distance defined in (3.39). For the functions $\xi^s \in \mathcal{Q}_\delta$ and $w \in \mathcal{R}_1$, (3.35) and (3.37) also hold.

Proof of Theorem 3.3. The first conclusion of Theorem 3.3 is proved by Lemma B.1 in Appendix B.

Similar to the proof of Theorem 3.1, the proof of the last two conclusions of Theorem 3.3 is also based on the contraction fixed point theorem. So we also separate this proof into two parts.

(Step 1): $\mathscr{F}(\mathcal{Q}_\delta \times \mathcal{R}_1) \subset (\mathcal{Q}_\delta \times \mathcal{R}_1)$. The fact that $\mathscr{F}[\xi^s, w]$ is $(r-2)$ times differentiable with respect to θ and $(r-1)$ times with respect to ξ_0 is a direct consequence of the fact that the composition of l order differentiable functions is a l order differentiable function. Now we estimate the norms.

The estimate about $D_{\xi_0}(\mathscr{F}_s[\xi^s, w](t, \theta, \xi_0))$ is the same with the one in the proof of Theorem 3.1 we omit the details.

To estimate $D^i_{\xi_0}(\mathscr{F}_s[\xi^s, w](t, \theta, \xi_0))$, for $2 \le i \le r-1$, we follow standard calculations [49] and [23]. Note that (from (3.59)), it can be written in the form

$$D^i_{\xi_0}(\mathscr{F}_s[\xi^s, w])(t, \theta, \xi_0) = \int_0^t U^s_{\Phi_\tau(\theta)}(t-\tau)\widetilde{R}_i^{\xi^s, w}(\tau, \theta, \xi_0)d\tau, \tag{3.63}$$

where

$$\begin{aligned}\widetilde{R}_i^{\xi^s, w}(\tau, \theta, \xi_0) \equiv\ & D_2 M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))]) \\ & \big[Id + D_2 w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0)\big]) D^i_2 \xi^s_\tau(\theta, \xi_0) + R_i^{\xi^s, w}(\tau, \theta, \xi_0)\end{aligned} \tag{3.64}$$

is the i order derivative of $M^s(\Phi_\tau(\theta), [\xi^s_\tau(\theta, \xi_0) + w(\Phi_\tau(\theta), \xi^s_\tau(\theta, \xi_0))])$ about ξ_0. $R_i^{\xi^s, w}(\tau, \theta, \xi_0)$ is a sum of monomials, e.g. $(D^{i_1}_{\xi_0}\xi^s)^{\otimes j_1}(D^{i_2}_{\xi_0}\xi^s)^{\otimes j_2}\cdots(D^{i_k}_{\xi_0}\xi^s)^{\otimes j_k}$ with $i_1 j_1 + i_2 j_2 + \cdots + i_k j_k = i$, whose factors are derivatives of M^s (evaluated at ξ^s, w) and of w up to order i. Actually, the monomials in $R_i^{\xi^s, w}(\tau, \theta, \xi_0)$ contain at least a factor which is a derivative of M^s of order not more than i. The last statement about the order of the derivatives of ξ^s is a consequence of the Faa di Bruno formula (C.3). It is easy to see that all the derivatives of ξ^s appearing in the derivatives of $\mathscr{F}_s$ are of order at most i and we have pulled out explicitly the terms containing derivatives of ξ^s of order i. Obviously, all the monomials in $R_i^{\xi^s, w}(\tau, \theta, \xi_0)$ contain at least one factor which is a derivative of M^s. Taking into account that $\xi^s \in \mathcal{Q}_\delta$, $w \in \mathcal{R}_1$, we can arrange that

$$\|\widetilde{R}_i^{\xi^s, w}(\tau, \theta, \xi_0)\|_{Y^s_{\Phi_\tau(\theta)}} \le c_i \|M\|_{C^{r-1}} C_h e^{-i\beta_1 \tau}, \quad 2 \le i \le r-1. \tag{3.65}$$

Hence, for (3.63), from the smallness of $\|M\|_{C^{r-1}}$ and by applying (3.11) and (3.65) we obtain

$$\begin{aligned}
&\|D^i_{\xi_0}(\mathscr{F}_s[\xi^s,w])(t,\theta,\xi_0)\|_{(X^s_\theta)^{\otimes i},X^s_{\Phi_t(\theta)}}\\
&\quad\le\int_0^t C_h e^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}c_i\|M\|_{C^{r-1}}C_h e^{-i\beta_1\tau}d\tau\\
&\quad\le C_h e^{-\beta_1 t},\quad 2\le i\le r-1.
\end{aligned}\tag{3.66}$$

Next, we estimate the derivatives in θ. For $1\le j\le r-2$, we get

$$\begin{aligned}
&D^j_\theta\mathscr{F}_s[\xi^s,w](t,\theta,\xi_0)=D^j_\theta U^s_\theta(t)\xi_0\\
&\quad+\int_0^t\Big\{[D^j_\theta U^s_{\Phi_\tau(\theta)}(t-\tau)]M^s(\Phi_\tau(\theta),[\xi^s_\tau(\theta,\xi_0)+w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))])\\
&\quad+U^s_{\Phi_\tau(\theta)}(t-\tau)D_1M^s(\Phi_\tau(\theta),[\xi^s_\tau(\theta,\xi_0)+w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))])D^j_\theta\Phi_\tau(\theta)\\
&\quad+U^s_{\Phi_\tau(\theta)}(t-\tau)D_2M^s(\Phi_\tau(\theta),[\xi^s_\tau(\theta,\xi_0)+w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))])\\
&\quad\big\{\big[Id+D_2w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))\big]D^j_\theta\xi^s_\tau(\theta,\xi_0)\\
&\quad+D_1w(\Phi_\tau(\theta),\xi^s_\tau(\theta,\xi_0))D^j_\theta\Phi_\tau(\theta)\big\}+U^s_{\Phi_\tau(\theta)}(t-\tau)Y^{\xi^s,w}_j(\tau,\theta,\xi_0)\Big\}d\tau,
\end{aligned}\tag{3.67}$$

where $Y^{\xi^s,w}_j(\tau,\theta,\xi_0)$ is a sum of monomials like the one $R^{\xi^s,w}_i(\tau,\theta,\xi_0)$ in (3.64). Then with (3.11), (B.11), (B.5) and (B.6) and the fact that $\xi^s\in\mathcal{Q}_\delta$, $w\in\mathcal{R}_1$ and the smoothing estimates, we get,

$$\begin{aligned}
&\|D^j_\theta\mathscr{F}_s[\xi^s,w](t,\theta,\xi_0)\|_{X_c^{\otimes j},X^s_{\Phi_t(\theta)}}\le C_h\epsilon e^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})]t}\|\xi_0\|_{X^s_\theta}\\
&\quad+\int_0^t\Big\{C_h e^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})](t-\tau)+j(\beta_3^++\epsilon^{\frac{1}{4r}})\tau}(t-\tau)^{-\alpha_1}4C_h^2e^{-2\beta_1\tau}\|\xi_0\|^2_{X^s_\theta}\\
&\quad+2C_h e^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}4C_h^2e^{-\beta_1\tau}\|\xi_0\|^2_{X^s_\theta}e^{j(\beta_3^++\epsilon^{\frac{1}{4r}})\tau}\\
&\quad+4C_h e^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}C_he^{-\beta_1\tau}C_he^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})]\tau}\|\xi_0\|^2_{X^s_\theta}\\
&\quad+C_{j,h}C_he^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}C_he^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})]\tau}\|\xi_0\|_{X^s_\theta}\Big\}d\tau\|M\|_{C^{r-1}}\epsilon^{\frac{-1}{4}}\\
&\quad\le C_h\epsilon e^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})]t}\|\xi_0\|_{X^s_\theta}+C(j,\beta_1,\beta_3^+,\alpha_1)\|M\|_{C^{r-1}}\epsilon^{\frac{-1}{4}}\\
&\quad\cdot e^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{4r}})]t}\|\xi_0\|_{X^s_\theta}\int_0^t e^{-j(\beta_3^++\epsilon^{\frac{1}{4r}})(t-\tau)}(t-\tau)^{-\alpha_1}d\tau\\
&\quad\le C_h[\epsilon+c(j,\beta_1,\beta_3^+,\alpha_1)\|M\|_{C^{r-1}}\epsilon^{\frac{-1}{4}}]e^{-[\beta_1-j(\beta_3^++\epsilon^{\frac{1}{2}})]t}\|\xi_0\|_{X^s_\theta}
\end{aligned}\tag{3.68}$$

$$\leq C_h \epsilon^{\frac{1}{2}} e^{-[\beta_1 - j(\beta_3^+ + \epsilon^{\frac{1}{4r}})]t} \|\xi_0\|_{X_\theta^s},$$

$C(j, \beta_1, \beta_3^+, \alpha_1)$ and $C(j, \beta_1, \beta_3^+, \alpha_1)$ are constants only depends on $j, \beta_1, \beta_3^+, \alpha_1$ and the last inequality of (3.68) is from $\|M\|_{C^{r-1}} \leq \epsilon$ and the smallness of ϵ. Moreover, by (3.67), for any θ, ϑ, we have the following

$$\begin{aligned}
&D_\theta^{r-2}\mathscr{F}_s[\xi^s, w](t,\theta,\xi_0) - D_\vartheta^{r-2}\mathscr{F}_s[\xi^s, w](t,\vartheta,\xi_0) = D_\theta^{r-2}U_\theta^s(t)\xi_0 - D_\vartheta^{r-2}U_\vartheta^s(t)\xi_0 \\
&\quad + \int_0^t \Big\{ [D_\theta^{r-2}U^s_{\Phi_\tau(\theta)}(t-\tau)]M^s(\Phi_\tau(\theta), [\xi_\tau^s(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi_\tau^s(\theta,\xi_0))]) \\
&\quad + U^s_{\Phi_\tau(\theta)}(t-\tau)D_1M^s(\Phi_\tau(\theta), [\xi_\tau^s(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi_\tau^s(\theta,\xi_0))])D_\theta^{r-2}\Phi_\tau(\theta) \\
&\quad + U^s_{\Phi_\tau(\theta)}(t-\tau)D_2M^s(\Phi_\tau(\theta), [\xi_\tau^s(\theta,\xi_0) + w(\Phi_\tau(\theta), \xi_\tau^s(\theta,\xi_0))]) \\
&\qquad \big\{\big[Id + D_2w(\Phi_\tau(\theta), \xi_\tau^s(\theta,\xi_0))\big]D_\theta^{r-2}\xi_\tau^s(\theta,\xi_0) \\
&\qquad + D_1w(\Phi_\tau(\theta), \xi_\tau^s(\theta,\xi_0))D_\theta^{r-2}\Phi_\tau(\theta)\big\} + U^s_{\Phi_\tau(\theta)}(t-\tau)Y^{\xi^s,w}_{r-2}(\tau,\theta,\xi_0)\Big\}d\tau \\
&\quad - \int_0^t \Big\{ [D_\vartheta^{r-2}U^s_{\Phi_\tau(\vartheta)}(t-\tau)]M^s(\Phi_\tau(\vartheta), [\xi_\tau^s(\vartheta,\xi_0) + w(\Phi_\tau(\vartheta), \xi_\tau^s(\vartheta,\xi_0))]) \\
&\quad + U^s_{\Phi_\tau(\vartheta)}(t-\tau)D_1M^s(\Phi_\tau(\vartheta), [\xi_\tau^s(\vartheta,\xi_0) + w(\Phi_\tau(\vartheta), \xi_\tau^s(\vartheta,\xi_0))])D_\theta^{r-2}\Phi_\tau(\theta) \\
&\quad + U^s_{\Phi_\tau(\vartheta)}(t-\tau)D_2M^s(\Phi_\tau(\vartheta), [\xi_\tau^s(\vartheta,\xi_0) + w(\Phi_\tau(\vartheta), \xi_\tau^s(\vartheta,\xi_0))]) \\
&\qquad \big\{\big[Id + D_2w(\Phi_\tau(\vartheta), \xi_\tau^s(\vartheta,\xi_0))\big]D_\vartheta^{r-2}\xi_\tau^s(\vartheta,\xi_0) \\
&\qquad + D_1w(\Phi_\tau(\vartheta), \xi_\tau^s(\vartheta,\xi_0))D_\vartheta^{r-2}\Phi_\tau(\vartheta)\big\} + U^s_{\Phi_\tau(\vartheta)}(t-\tau)Y^{\xi^s,w}_{r-2}(\tau,\vartheta,\xi_0)\Big\}d\tau.
\end{aligned} \tag{3.69}$$

Then by adding and subtracting terms and the triangle inequality and with calculations similar to those leading to (3.68) we obtain

$$Lip_\theta\big(D_\theta^{r-2}\mathscr{F}_s[\xi^s, w](t,\theta,\xi_0)\big|_{X_c^{\otimes(r-2)}, X^s_{\Phi_t(\theta)}}\big) \leq C_h e^{-[\beta_1 - (r-1)(\beta_3^+ + \epsilon^{\frac{1}{4r}})]t} \|\xi_0\|_{X_\theta^s}.$$

This finishes the verification of $\mathscr{F}_s[\mathcal{Q}_\delta \times \mathcal{R}_1] \subset \mathcal{Q}_\delta$.

Next we will prove $\mathscr{F}_{cu}[\mathcal{Q}_\delta \times \mathcal{R}_1] \subset \mathcal{R}_1$. The estimates about $\mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0)$ and $D_{\xi_0}(\mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0))$ are the same as the one in the proof of Theorem 3.1 and we omit the details.

To estimate $D^i_{\xi_0}(\mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0))$, for $2 \leq i \leq r-1$, we note that (from (3.60)), it can be written in the form

$$\begin{aligned}
&D^i_{\xi_0}(\mathscr{F}_{cu}[\xi^s, w])(\theta, \xi_0) = \\
&\quad - \int_0^\infty U^s_{\Phi_t(\theta)}(-t)\Big[D_2M^s(\Phi_t(\theta), [\xi_t^s(\theta,\xi_0), w(\Phi_t(\theta), \xi_t^s(\theta,\xi_0))])
\end{aligned} \tag{3.70}$$

$$[Id + D_2 w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0))] D_{\xi_0}^i \xi_t^s(\theta, \xi_0) + P_i^{\xi^s, w}(t, \theta, \xi_0) \Big] dt,$$

where $P_i^{\xi^s, w}(t, \theta, \xi_0)$ is a sum of monomials like the one $R_i^{\xi^s, w}(\tau, \theta, \xi_0)$ in (3.64). Then, applying arguments similar to those used to get (3.66) we obtain

$$\| D_{\xi_0}^i (\mathscr{F}_{cu}[\xi^s, w])(\theta, \xi_0) \|_{(X_\theta^s)^{\otimes i}, X_\theta^{cu}} \le 1, \quad 2 \le i \le r - 1.$$

The only thing that remains to do is to estimate the derivative of $\mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0)$ in θ. From (B.20) we get, $(1 \le j \le r - 2)$

$$\begin{aligned}
& D_\theta^j \mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0) \\
&= - \int_0^\infty \Big[[D_\theta U_{\Phi_t(\theta)}^{cu}(-t)] M^{cu}(\Phi_t(\theta), [\xi_t^s(\theta, \xi_0), w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0))]) \\
&\quad + U_{\Phi_t(\theta)}^{cu}(-t) D_1 M^{cu}(\Phi_t(\theta), [\xi_t^s(\theta, \xi_0), w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0))]) D_\theta^j \Phi_t(\theta) \\
&\quad + U_{\Phi_t(\theta)}^{cu}(-t) D_2 M^s(\Phi_t(\theta), [\xi_t^s(\theta, \xi_0), w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0))]) \\
&\quad \big\{ \big[Id + D_2 w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0)) \big] D_\theta^j \xi_t^s(\theta, \xi_0) \\
&\quad + D_1 w(\Phi_t(\theta), \xi_t^s(\theta, \xi_0)) D_\theta^j \Phi_t(\theta) \big\} + U_{\Phi_t(\theta)}^{cu}(-t) E_i^{\xi^s, w}(t, \theta, \xi_0) \Big] dt.
\end{aligned}$$

Using calculations similar to those used to obtain (3.68), we get

$$\| D_\theta^j \mathscr{F}_{cu}[\xi^s, w](\theta, \xi_0) \|_{X_c^{\otimes j}, X_\theta^{cu}} \le \| \xi_0 \|_{X_\theta^s}^2, \quad 1 \le j \le r - 2.$$

With the similar discussions we can obtain the estimate about the Lipschitz constant, we omit the details. That is, we have proved that $\mathscr{F}_{cu}[\mathcal{Q}_\delta \times \mathcal{R}_1] \subset \mathcal{R}_1$. Together with the discussions about $\mathscr{F}_s$ we know that $\mathscr{F}[\mathcal{Q}_\delta \times \mathcal{R}_1] \subset [\mathcal{Q}_\delta \times \mathcal{R}_1]$.

(Step 2) $\mathscr{F}$ is a contraction in $\mathcal{Q}_\delta \times \mathcal{R}_1$ under the $d-$distance defined in (3.39). The calculations of this part is the same as the one in the whiskered torus case since we adopt the same metric, we omit the details.

Then from the contraction fixed theorem we know that there is a unique solution of (3.58), (ξ^s, w), which is in the C^0 closure of $\mathcal{Q}_\delta \times \mathcal{R}_1$. Note that (ξ^s, w) can be written as

$$(\xi^s, w) = \lim_{n \to \infty} \mathscr{F}^{(n)}[0, 0](t, \theta, \xi_0),$$

where

$$\begin{aligned}
\mathscr{F}^{(2)}[0, 0](t, \theta, \xi_0) &= \mathscr{F}[\mathscr{F}_s[0, 0], \mathscr{F}_{cu}[0, 0]](t, \theta, \xi_0) \\
&= (\mathscr{F}_s^{(2)}[0, 0](t, \theta, \xi_0), \mathscr{F}_{cu}^{(2)}[0, 0](\theta, \xi_0)).
\end{aligned}$$

Since $\mathscr{F}_{cu}^{(n)}[0, 0](\theta, \xi_0) \in \mathcal{R}_1$, from Lemma C.1 and the definition of $\mathcal{R}_1$ we know that the function w is $C^{r-2+Lip}$ derivatives in the variable θ. Moreover, since w is in the C^0 closure of $\mathcal{R}_1$,

w is also $C^{r-2+Lip}$ derivatives in the variable ξ_0. That is w is $C^{r-2+Lip}$ derivatives in the both variables θ and ξ_0. The discussions above imply that w satisfies the second and third conclusions of the Theorem 3.3. □

4. Applications

This section is devoted to applications of Theorem 3.1 to the concrete equations: Boussinesq equation and complex Ginzburg-Landau equation.

Note that these equations are ill posed and that the non-linear terms are unbounded. We can nevertheless produce infinite dimensional families of solutions. We show that these families are smooth.

We mention without giving details, the Boussinesq system. Recently we noticed that one can apply the results of this paper to mean field games with noise. We refer to [11] for a complete treatment.

$$\begin{aligned} u_t &= -\Delta u - \mathcal{H}(t, x, Du, m), \\ m_t &= \Delta m - \operatorname{div}\big(m\mathcal{H}_p(t, x, Du, m)\big), \end{aligned} \tag{4.1}$$

where $\triangle$ is the Laplace operator and the independent variables (t, x) are taken from $\mathbb{R} \times \mathbb{T}^n$ $(n \geq 1)$. The function $\mathcal{H}$ is given and the unknowns are m, u m is a probability measure, and u is a value function.

4.1. *The Boussinesq equation*

The Boussinesq equation, introduced in [9] as model for water waves (but since considered as a model for other problems) was one of the main examples in [23,26].

$$u_{tt} = \mu u_{xxxx} + u_{xx} + (u^2)_{xx}, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}. \tag{4.2}$$

In the applications to water waves, $\mu > 0$, which makes the equation ill-posed. This is the only case we will consider in this paper.

We will consider, as in the above papers, the equation (4.2) supplemented by the periodic boundary conditions $u(t, x+1) = u(t, x)$, so that $x \in \mathbb{T}$. It is standard to write equation (4.2) as a first order system, which fits our formalism.

$$\dot{z} = \mathcal{A}_\mu z + \mathcal{N}(z) \tag{4.3}$$

with

$$\mathcal{A}_\mu = \begin{pmatrix} 0 & 1 \\ \partial_x^2 + \mu\partial_x^4 & 0 \end{pmatrix}$$

and

$$\mathcal{N}(z) = (0, \quad \partial_x^2 u^2).$$

4.1.1. Choice of spaces

We first discuss the suitable phase spaces.

For $\zeta > 0$ and $r \in \mathbb{N}$, we denote by $H^{\zeta,r}$ the analytic functions u from $\mathbb{T}_\zeta$ ($\zeta > 0$) to $\mathbb{C}$ with the Fourier expansion $u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{2\pi i k x}$ such that the norm

$$\|u\|^2_{\zeta,r} = \sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 \exp(4\pi\zeta|k|)(|k|^2+1)^r \tag{4.4}$$

is finite.

The fact that we are considering x ranging on the torus is equivalent to the fact that we are restricting ourselves to periodic functions. That is, we are supplementing (4.2) with periodic boundary conditions. It is very standard in the theory of evolutionary equations that the spatial boundary conditions are formulated as the choice of the space of functions considered.

For any $\zeta > 0$ and $r \in \mathbb{N}$ the space $\left(H^{\zeta,r}, \|\cdot\|_{\zeta,r}\right)$ is a Hilbert space.

In [26], it was shown that

$$X = H^{\zeta,r} \times H^{\zeta,r-2}, \qquad Y = H^{\zeta,r} \times H^{\zeta,r-2} \tag{4.5}$$

for $r > 5/2$ is a space in which we can verify the regularity of the non-linearity.

One can also consider the evolution defined subspaces of the above spaces consisting of functions with symmetries preserved by the equation

$$\int_0^1 z(\cdot,x)dx = 0, \quad \int_0^1 \partial_t z(\cdot,x)dx = 0, \quad z(\cdot,x) = z(\cdot,-x). \tag{4.6}$$

For the results in [26], which involve non-degeneracy conditions considering subspaces makes a big difference, but for our results it does not and our results apply for the equation (4.2) defined in the whole space of function or in the space of functions with symmetries.

An elementary calculation (detailed in [26]) shows that the operator $\mathscr{A}_\mu$ acting in X has discrete spectrum

$$\lambda_k^2 = -4\pi^2 k^2 (1 - 4\mu\pi^2 k^2) \tag{4.7}$$

and that the eigenvectors are exponentials. The spectrum has a finite number of purely imaginary eigenvalues and the other eigenvalues have multiplicity 1 in the spaces with symmetry or multiplicity 2 in spaces without the symmetry. Note also that, because of the Hamiltonian structure of the original equation the spectrum is symmetric under reflection around 0.

Hence, for any $-\beta_1 < -\beta_3^- < 0 < \beta_3^+ < \beta_2$ such that none of them is an eigenvalue, we can choose the invariant spaces corresponding to the spectral subspaces corresponding to the sets $\mathrm{Re}(\lambda_k) \in (\infty, -\beta_1]$, $\mathrm{Re}(\lambda_k) \in [-\beta_3^-, \beta_3^+]$, $\mathrm{Re}(\lambda_k) \in [\beta_2, \infty)$. The spectral non-degeneracy is easy to verify in our case since the evolution generated by $\mathcal{A}$ is diagonal in the basis of Fourier coefficients, the invariant spaces are selected by the real part of the eigenvalues.

Remark 4.1. Note that we are not assuming that the center spaces are exactly those corresponding to the purely imaginary eigenvalues. It suffices that the absolute value of the real part is smaller than that of the other eigenvalues. This includes the so-called pseudo-stable and pseudo-unstable manifolds. Since these pseudo-(un)stable manifolds are finite dimensional, it is possible to use finite dimensional arguments to obtain finite dimensional stable manifolds inside the pseudo-(un) stable manifold. The results here provide infinite dimensional stable manifolds to these finite dimensional stable manifolds inside the pseudo-stable manifold.

With these choices, it is easy to verify the spectral assumptions using that the partial evolutions are diagonal in Fourier spaces. More detailed calculations about the smoothing properties are in [26].

Remark 4.2. In the symplectic case, it is natural to choose $\beta_1 = \beta_2$, $\beta_3^+ = \beta_3^-$. To produce the quasi-periodic solutions in [26], one chooses $\beta_3^\pm$ very close to 0. In that case the center manifolds we produce are proper center manifolds.

Once we have identified the phase space, the quasiperiodic solutions will be analytic functions from $\mathbb{T}^d_\rho$ (for some $\rho > 0$) into X.

4.1.2. Verifying the smoothing properties of the partial evolutions of the linearization around bounded solution

We now come to verifying the assumptions on the evolution operators and their smoothing properties. We have:

Lemma 4.1. [26] *The operator $\mathcal{A}_\mu$ generates semi-groups of operators $U^{s,u}(t)$ in positive and negative times and a group operator $U^c(t)$ for all times. Furthermore, the following estimates hold:*

$$
\begin{aligned}
\|U^s(t)\|_{Y,X} &\le C_h \frac{e^{-\beta_1 t}}{t^{\alpha_1}}, \quad t > 0,\\
\|U^u(t)\|_{Y,X} &\le C_h \frac{e^{-\beta_2 |t|}}{|t|^{\alpha_2}}, \quad t < 0,\\
\|U^c(t)\|_{Y,X} &\le C_h e^{\beta_3^+ |t|}, \quad t \ge 0,\\
\|U^c(t)\|_{Y,X} &\le C_h e^{\beta_3^- |t|}, \quad t < 0,\\
\|U^s(t)\|_{X,X} &\le C_1 e^{-\beta_1 t}, \quad t > 0,\\
\|U^u(t)\|_{X,X} &\le C_2 e^{-\beta_2 |t|}, \quad t < 0,\\
\|U^c(t)\|_{X,X} &\le C_h e^{\beta_3^+ |t|}, \quad t \ge 0,\\
\|U^c(t)\|_{X,X} &\le C_h e^{\beta_3^- |t|}, \quad t < 0,
\end{aligned}
$$

where $\beta_1 > \beta_3^- > 0$, $\beta_2 > \beta_3^+ > 0$, $\alpha_1, \alpha_2 \in [0, 1)$ and $C_h > 1$ is a constant only depends on μ.

Since the norm in X is a combination of the modulus of the Fourier coefficients and the partial evolutions are just multiplication of the Fourier coefficients by a time dependent factor, the norm of the evolution is bounded by the suprema of the multiplication factors.

4.2. The complex Ginzburg-Landau equation

The complex Ginzburg-Landau equation:

$$\begin{aligned} u_t &= \nu u + (b_1 + \mathrm{i}b_2)\Delta u + |u|^2 u_x, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}^d \\ &= \mathcal{A}_{\nu,b} u + \mathcal{N}(u) \end{aligned} \tag{4.8}$$

where $\triangle$ is the Laplace operator and

$$\mathcal{A}_{\nu,b} = \nu + (b_1 + \mathrm{i}b_2)\triangle, \quad \mathcal{N}(u) = |u|^2 u_x.$$

4.2.1. Choice of spaces

For the system (4.8), it is natural to consider the space for $\zeta > 0$ and $r > 3/2$ we take the space

$$X = H^{\zeta,r}, \quad Y = H^{\zeta,r-1}. \tag{4.9}$$

By the Banach algebra property of the scale of spaces $H^{\zeta,r}$ when $r > 1/2$ and the particular form of the nonlinearity, we have the following proposition (see [23]).

Proposition 4.1. *The nonlinearity $\mathcal{N}$ is C^∞ from $H^{\zeta,r}$ into $H^{\zeta,r-1}$ when $r > 3/2$.*

Proof. We note that the function $u \to u$ is linear and bounded from $H^{\zeta,r}$ to $H^{\zeta,r}$, the function $u \to \bar{u}$ is bounded (anti-linear) from $H^{\zeta,r}$ to $H^{\zeta,r}$ and the function $u \to u_x$ is bounded from $H^{\zeta,r}$ to $H^{\zeta,r-1}$. Then, the nonlinearity $\mathcal{N}$: $u \mapsto u\bar{u}u_x$ is the project of the three operators. Obviously, $H^{\zeta,r-1}$ is a Banach algebra when $r > 3/2$, so the linearity $\mathcal{N}$ is C^∞ from $H^{\zeta,r}$ into $H^{\zeta,r-1}$ when $r > 3/2$.

Note that the nonlinearity is not complex analytic because of the anti-linear complex conjugate, but it is real analytic.

4.2.2. Linearization around the whiskered tori

Assume $K(\theta)$, $\theta \in \mathbb{T}^d_\rho$, is the quasiperiodic solution of (4.8), then $A(\theta) = \mathcal{A}_{\nu,b} + D\mathcal{N}(K(\theta))$.

We need to study the eigenvalue properties of $A(\theta)$. We first study the eigenvalue problem of $\mathcal{A}_{\nu,b}$ for $U \in X$, $\sigma \in \mathbb{C}$

$$\mathcal{A}_{\nu,b} U = \sigma U.$$

This leads to the following formula for eigenvalues

$$\sigma_k = \nu - (b_1 + \mathrm{i}b_2)(2\pi)^2 |k|^2, \quad k \in \mathbb{Z}^d. \tag{4.10}$$

We assume (4.8) is ill-posed, *i.e.* $b_1 < 0$. To consider the most complicated case, i.e., the operator $\mathcal{A}_{\nu,b}$ possesses both hyperbolic spectrum and center spectrum, we assume $\nu < -1$ and $(1+\nu) < \hat{b}_1 < 0$, where $\hat{b}_1 \equiv b_1(2\pi)^2$ and $[k_+] \geq 2$, with $k_+ = \frac{\nu+1}{\hat{b}_1}$ and $[k_+]$) being the integer part of k_+.

Lemma 4.2. *For $\nu < -1$, $(1+\nu) < \hat{b}_1 < 0$, and $[k_+] \geq 2$, the operator $\mathcal{A}_{\nu,b}$ has discrete spectrum in X. Furthermore, we have the following:*

The center spectrum of $\mathcal{A}_{\nu,b}$ consists of a finite number of eigenvalues and the dimension of the center subspace is even.

The hyperbolic spectrum is well separated from the center spectrum.

Proof. The real parts of eigenvalues of $\mathcal{A}_{\nu,b}$ are $Re\lambda_k = \nu - \hat{b}_1|k|^2$, $(\nu < -1$ and $(1+\nu) < \hat{b}_1 < 0)$. Denote $k_+ = \frac{\nu+1}{\hat{b}_1}$, $k_- = \frac{\nu-1}{\hat{b}_1}$ and $k_* = \frac{\nu}{\hat{b}_1}$, so (note that $|k|^2$ are integer numbers)

$$
\begin{aligned}
Re\lambda_k &= \nu - \hat{b}_1|k|^2 \leq -1 + \hat{b}_1, \quad k \in \mathcal{J}_1 \equiv \{k: \quad |k|^2 \leq [k_+] - 1\}, \\
Re\lambda_k &= \nu - \hat{b}_1|k|^2 > 1, \quad k \in \mathcal{J}_2 \equiv \{k: \quad |k|^2 \geq [k_-] + 1\}, \\
Re\lambda_k &= \nu - \hat{b}_1|k|^2 \in (0, 1], \quad k \in \mathcal{J}_3 \equiv \{k: \quad [k_*] + 1 \leq |k|^2 \leq [k_-]\},
\end{aligned}
$$

and

$$
Re\lambda_k = \nu - \hat{b}_1|k|^2 \in (-1 + \hat{b}_1, 0], \quad k \in \mathcal{J}_4 \equiv \{k: \quad [k_+] \leq |k|^2 \leq [k_*]\}.
$$

As $\mathcal{J}_3$, we have the following:

I): If $[k_*] = [k_-]$, then $\mathcal{J}_3 \equiv \emptyset$. That is there is no spectrum belongs to (0, 1].

II): If $[k_*] + 1 = [k_-]$, then $\mathcal{J}_3 \equiv \{k: \quad |k|^2 = [k_-]\}$.

III): If $[k_*] + 1 < [k_-]$, then $\mathcal{J}_3 \equiv \{k: \quad [k_*] + 1 \leq |k|^2 \leq [k_-]\}$.

Obviously, the operator $\mathcal{A}_{\nu,b}$ has discrete spectrum in X. When k belongs to $\mathcal{J}_1 \cup \mathcal{J}_2$ and $\mathcal{J}_3 \cup \mathcal{J}_4$, λ_k is the hyperbolic spectrum and center spectrum, respectively. Then center spectrum of $\mathcal{A}_{\nu,b}$ consists in a finite number of eigenvalues and the dimension of the center subspace is even, moreover, the hyperbolic spectrum is well separated from the center spectrum.

4.2.3. Verifying the smoothing properties of the partial evolutions of the linearization around bounded solution

We now come to the evolution operators and their smoothing properties. We have:

Lemma 4.3. *For $\nu < -1$ and $(1+\nu) < \hat{b}_1 < 0$, the operator $\mathcal{A}_{\nu,b}$ generates semi-group operators $U^{s,u}$ in positive and negative times and group operator U^c in $\mathbb{R}$. Furthermore, the following estimates hold*

$$
\begin{aligned}
\|U^s(t)\|_{Y,X} &\leq C_h \frac{e^{-\beta_1 t}}{t^{\alpha_1}}, \quad t > 0, \\
\|U^u(t)\|_{Y,X} &\leq C_h \frac{e^{-\beta_2 |t|}}{|t|^{\alpha_2}}, \quad t < 0, \\
\|U^c(t)\|_{Y,X} &\leq C_h e^{\beta_3^+ |t|}, \quad t \geq 0, \\
\|U^c(t)\|_{Y,X} &\leq C_h e^{\beta_3^- |t|}, \quad t \leq 0, \\
\|U^s(t)\|_{X,X} &\leq C_h e^{-\beta_1 t}, \quad t > 0, \\
\|U^u(t)\|_{X,X} &\leq C_h e^{-\beta_2 |t|}, \quad t < 0,
\end{aligned}
$$

$$\|U^c(t)\|_{X,X} \le C_h e^{\beta_3^+|t|}, \quad t \ge 0,$$
$$\|U^c(t)\|_{X,X} \le C_h e^{\beta_3^-|t|}, \quad t \le 0,$$

where

$$\begin{aligned} &\beta_1 = -\nu + (\hat{b}_1 - b_*)\{[k_+] - 1\}, \ \ \beta_3^- = -\nu + \hat{b}_1[k_+], \ \ \beta_3^+ = \nu - \hat{b}_1[k_-], \\ &\beta_2 = \nu - (\hat{b}_1 + b_{**})\{[k_-] + 1\}, \ \ \alpha_1 = \alpha_2 = \frac{1}{2}, \end{aligned} \tag{4.11}$$

$C_h > 1$ *is a constant only depends on* $\nu, \hat{b}_1$ *with*

$$0 < b_* < \hat{b}_1 + \frac{-\nu}{[k_+] - 1}, \quad 0 < b_{**} < -\hat{b}_1 + \frac{\nu}{[k_-] + 1}. \tag{4.12}$$

Proof. Assume $u \in H^{\zeta, r-1}$ with the Fourier expansion

$$u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \exp(\mathrm{i}2\pi \langle \mathrm{k}, \mathrm{x} \rangle)$$

and the norm

$$\|u\|^2_{H^{\zeta,r-1}} = \sum_{k \in \mathbb{Z}^d} |\hat{u}_k|^2 \exp(4\pi\zeta|k|)(|k|^2 + 1)^{r-1}.$$

Then

$$U^s(t)u = \sum_{k \in \mathcal{J}_1} e^{\lambda_k t} \hat{u}_k \exp(\mathrm{i}2\pi \langle \mathrm{k}, \mathrm{x} \rangle),$$

so we have

$$\begin{aligned} \|U^s(t)u\|^2_{H^{\zeta,r}} &= \sum_{k \in \mathcal{J}_1} |e^{\lambda_k t} \hat{u}_k|^2 \exp(4\pi\zeta|k|)(|k|^2+1)^r, \\ &= \sum_{k \in \mathcal{J}_1} |e^{2\lambda_k t}|(|k|^2+1)|\hat{u}_k|^2 \exp(4\pi\zeta|k|)(|k|^2+1)^{r-1} \\ &\le \sup_{k \in \mathcal{J}_1} e^{2(\nu - \hat{b}_1|k|^2)t}(|k|^2+1)\|u\|^2_{H^{\zeta,r-1}}, \quad t > 0. \end{aligned} \tag{4.13}$$

For $\sup_{k \in \mathcal{J}_1} e^{2(\nu - \hat{b}_1|k|^2)t}(|k|^2+1)$ we have the following,

$$\begin{aligned} \sup_{k \in \mathcal{J}_1} e^{(\nu - \hat{b}_1|k|^2)t}(|k|^2+1)^{\frac{1}{2}} &\le 2 \sup_{k \in \mathcal{J}_1} e^{[\nu - (\hat{b}_1 - b_*)|k|^2]t}|t|^{-\frac{1}{2}}[e^{-b_*|k|^2|t|}|k||t|^{\frac{1}{2}}] \\ &\le 2|t|^{-\frac{1}{2}} e^{-[-\nu + (\hat{b}_1 - b_*)\{[k_+]-1\}t} \sup_{z \in \mathbb{R}^+} e^{-b_* z^2} z \end{aligned}$$

$$\leq \frac{\sqrt{2}}{\sqrt{b_* e}} |t|^{-\frac{1}{2}} e^{-\beta_1 t}, \quad t > 0.$$

That is

$$\|U^s(t)u\|_{H^{\zeta,r}} \leq \frac{\sqrt{2}}{\sqrt{b_* e}} |t|^{-\frac{1}{2}} e^{-\beta_1 t} \|u\|_{H^{\zeta,r-1}}, \quad t > 0,$$

so

$$\|U^s(t)\|_{H^{\zeta,r-1},H^{\zeta,r}} \leq \frac{\sqrt{2}}{\sqrt{b_* e}} |t|^{-\frac{1}{2}} e^{-\beta_1 t}, \quad t > 0.$$

Similarly, we also have

$$\|U^u(t)\|_{H^{\zeta,r-1},H^{\zeta,r}} \leq \frac{\sqrt{2}}{\sqrt{b_{**} e}} |t|^{-\frac{1}{2}} e^{-\beta_2 |t|}, \quad t < 0.$$

For the center space which $k \in \mathcal{J}_3$, $i.e.\ Re\lambda_k \in [0, 1)$, we just consider the evolution $U^c(t)$ in the case $t \geq 0$ for the case $t \leq 0$ is dominated by this case. Since $Y^c = X_c$, we have

$$\|U^c(t)\|_{H^{\zeta,r},H^{\zeta,r}} \leq C_{\nu,b_1} e^{\beta_3^+ t}, \quad t \geq 0.$$

Similarly, for the center space which means $k \in \mathcal{J}_4$, $i.e.\ Re\lambda_k \in (-1, 0]$ we just consider the evolution $U_\theta^c(t)$ in the case $t \leq 0$. We have

$$\|U^c(t)\|_{H^{\zeta,r},H^{\zeta,r}} \leq \widehat{C}_{\nu,b_1} e^{\beta_3^- |t|}, \quad t \leq 0.$$

Easily, we obtain

$$\|U^s(t)\|_{H^{\zeta,r},H^{\zeta,r}} \leq e^{-\beta_1 t}, \quad t > 0,$$
$$\|U^u(t)\|_{H^{\zeta,r},H^{\zeta,r}} \leq e^{-\beta_2 |t|}, \quad t < 0.$$

Denote $C_h = \max\{1, C_{\nu,b_1}, \widehat{C}_{\nu,b_1}, \frac{\sqrt{2}}{\sqrt{b_* e}}, \frac{\sqrt{2}}{\sqrt{b_{**} e}}\}$, then from the discussions above we finish the proof of this Lemma. □

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Appendix A. Persistence of the splitting and smoothing properties under small perturbations

In this Appendix, we present a proof of the stability of splittings and smoothing properties **(SD1)** for a single bounded orbit. This is a concept of hyperbolicity suited to dealing with unbounded perturbations, so that we use the two-spaces approach of [41]. Hence, we need bounds on the evolution as considered as operators in several spaces. Compared with [26], we have found it useful to add bounds from X to X which were not used there.

The arguments are similar (but not identical) to those in [26] for the similar case of splittings near quasi-periodic orbits. The main difference is that [26] also had to include a discussion of the analyticity in θ, but that it did not consider the property **(SD1.2')**. At the end, we will indicate the (minor) changes needed to obtain the corresponding results for center manifolds. Roughly, we just need to obtain the regularity with respect to parameters. For a fixed orbit, the result presented here is based on a contraction argument. Once we have uniform contraction and some regularity of the problem in parameters, the result of differentiability follows by variants of the contraction mapping argument with parameters (the fiber contraction theorem).

Lemma A.1. *Assume that $K(t), t \geq 0$ is a fixed bounded solution of (2.1) and $\{A(t)\}_{t\in[0,\infty)}$ is a family of linear maps defined by (3.6) which satisfies the* **(SD1)**.

Let $\{\widetilde{A}(t)\}_{t\in[0,\infty)}$ be another family of linear maps such that $\|\widetilde{A}(t) - A(t)\|_{X,Y}$ is small enough. Then there exists a family of splittings

$$X = \widetilde{X}_t^s \oplus \widetilde{X}_t^c \oplus \widetilde{X}_t^u,$$

which is invariant under the linearized equations. That is there exist cocycles satisfying

$$\frac{d}{dt}\widetilde{U}_\tau(t) = \widetilde{A}(\tau+t)\widetilde{U}_\tau(t) \tag{A.1}$$

and

$$\widetilde{U}_\tau^{s,c,u}(t)\widetilde{X}_\tau^{s,c,u} = \widetilde{X}_{\tau+t}^{s,c,u}. \tag{A.2}$$

We denote $\widetilde{\Pi}_t^{s,c,u}$ and $\widetilde{\Pi}_t^{s,c,u}$ the projections in the X space associated to the above two splittings. Then, there exist

$$\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3^+, \widetilde{\beta}_3^- > 0, \widetilde{\alpha}_1, \widetilde{\alpha}_2 \in [0,1) \tag{A.3}$$

and $\widetilde{C}_h > 1$ independent of t satisfying

$$\widetilde{\beta}_3^- < \widetilde{\beta}_1, \quad \widetilde{\beta}_3^+ < \widetilde{\beta}_2 \tag{A.4}$$

and such that the splitting is characterized by the following rate conditions:

$$
\begin{aligned}
&\|\widetilde{U}_\tau^s(t)\|_{Y,X} \le \widetilde{C}_h \frac{e^{-\widetilde{\beta}_1 t}}{t^{\widetilde{\alpha}_1}}, \quad \tau, t \ge 0,\\
&\|\widetilde{U}_\tau^u(t)\|_{Y,X} \le \widetilde{C}_h \frac{e^{-\widetilde{\beta}_2 |t|}}{|t|^{\widetilde{\alpha}_2}}, \quad \tau \ge 0, t \le 0,\\
&\|\widetilde{U}_\tau^c(t)\|_{Y,X} \le \widetilde{C}_h e^{\widetilde{\beta}_3^+ t}, \quad \tau \le 0, t \ge 0,\\
&\|\widetilde{U}_\tau^c(t)\|_{Y,X} \le \widetilde{C}_h e^{\widetilde{\beta}_3^- |t|}, \quad \tau \ge 0, t \le 0,
\end{aligned}
\tag{A.5}
$$

and

$$
\begin{aligned}
&\|\widetilde{U}_\tau^s(t)\|_{X,X} \le \widetilde{C}_h e^{-\widetilde{\beta}_1 |t|}, \quad \tau, t \ge 0,\\
&\|\widetilde{U}_\tau^u(t)\|_{X,X} \le \widetilde{C}_h e^{-\widetilde{\beta}_2 |t|}, \quad \tau \ge 0, t \le 0,\\
&\|\widetilde{U}_\tau^c(t)\|_{X,X} \le \widetilde{C}_h e^{\widetilde{\beta}_3^+ t}, \quad \tau \le 0, t \ge 0,\\
&\|\widetilde{U}_\tau^c(t)\|_{X,X} \le \widetilde{C}_h e^{\widetilde{\beta}_3^- |t|}, \quad \tau \ge 0, t \le 0.
\end{aligned}
\tag{A.6}
$$

Furthermore, the following estimates hold

$$
\begin{aligned}
&\|\widetilde{\Pi}_t^{s,c,u} - \Pi_t^{s,c,u}\|_{Y,Y} \le C\|\widetilde{A} - A\|_{X,Y}^{\frac{1}{2}},\\
&|\widetilde{\beta}_i - \beta_i| \le C\|\widetilde{A} - A\|_{X,Y}^{\frac{1}{2}}, i = 1, 2, 3^{\pm},\\
&\widetilde{\alpha}_i = \alpha_i, i = 1, 2,\\
&|\widetilde{C}_h - C_h| \le C\|\widetilde{A} - A\|_{X,Y}^{\frac{1}{2}}.
\end{aligned}
\tag{A.7}
$$

Remark A.1. In the applications of this paper, we will just use Lemma A.1 with $A(t) = \mathcal{A}$ which admits a smoothing hyperbolic splitting and $\widetilde{A}(t) = \mathcal{A} + D\mathcal{N}(K(t))$. Similar results were obtained in [26]. The results in [26] were more precise (and had a more complicated proof) because in [26], the results were part of an iterative process and, therefore, had to be very quantitative. In our case we just need to apply the result once. Since we fixed the angle variable θ and will not need to consider the analytic regularity with respect to it.

Proof. The idea of the proof is very similar to the one in the proof of Lemma 6.1 in [26], but to make this paper easily to read we still give the details. □

A.1. Construction of the invariant splitting

We need to find the invariant subspaces for the linearized evolution equation (A.1). We just concentrate on the stable subspace since the theory for the other bundles is similar. First, we will characterize the initial conditions of the linearized evolution equation (A.1) that lead to a forward evolution which is a contraction. By formulating the new space as the graph of a bundle map M covering the identity from $\mathbf{B}^s$ to $\mathbf{B}^{cu}$ and by formulating another equation for the evolution, we get two coupled equations (one for the space and the other one for the evolution). These two equations are formulated as fixed point problem that can be solved by a variation of the contraction mapping principle.

We proceed first to formulate the fixed point equation which encodes both that the partial evolutions can be defined in the spaces and that the partial evolution keep them invariant. We start from the initial time $\tau = 0$ and the initial condition $K(0)$.

The (A.1) can be rewritten as

$$\frac{d}{dt} W_0(t) = \widetilde{A}(t) W_0(t) = A(t) W_0(t) + B(t) W_0(t) \tag{A.8}$$

with $B(t) = \widetilde{A}(t) - A(t)$. Denote $\epsilon = \|\widetilde{A} - A\|_{X,Y} = \|B\|_{X,Y}$ which we will assume to be small. Since we assume that (2.1) is ill-posed, so (A.8) may not define solutions for all initial values. We just need to construct the forward solutions. We compute the evolution of the projections of $W_0^\sigma(t)$ along the invariant bundles X_t^σ by the linearized equation when $B = 0$. For $\sigma = s, c, u$ we have:

$$\frac{d}{dt} \Pi_t^\sigma W_0(t) = A^\sigma(t) \Pi_t^\sigma W_0(t) + \Pi_t^\sigma B(t) W_0(t). \tag{A.9}$$

Our next goal is to try to find a subspace, $\widetilde{X}_t^s$, in which the solutions of (A.9) can be defined forward in time. We assume that this space where solutions can be defined is given as the graph of a linear function G_t from X_t^s to X_t^{cu}, *i.e.*

$$\widetilde{X}_t^s = \{W_0^s(t) + G_t W_0^s(t) | \quad W_0^s(t) \in X_t^s\},$$

$\Big($that is, we introduce the notation

$$W_0^{cu}(t) = \Pi_t^{cu} W_0(t), W_0^s(t) = \Pi_t^s W_0(t)\Big).$$

We will assume that the solutions of (A.9) have the form

$$W_0^{cu}(t) = G_t W_0^s(t).$$

Give any $T > 0$, for (A.9), from Duhamel's formula we get

$$\begin{aligned} W_0^s(t) &= \mathcal{E}_1(N, G)(t) \\ &= U_0^s(t) W_0^s(0) + \int_0^t U_\tau^s(t-\tau) B^s(\tau)(Id + G_\tau) W_0^s(\tau) d\tau, 0 \le t \le T, \\ M_0 W_0^s(0) &= \mathcal{E}_1(N, G)(T) = U_T^{cu}(-T) G_T W_0^s(T) \\ &\quad - \int_0^T U_t^{cu}(-t) B^{cu}(t)(Id + G_t) W_0^s(t) dt. \end{aligned}$$

Assume that $W_0^s(t) = N(t) W_0^s(0)$. Obviously, $N(t)$ is the linear operator and $N(0) = Id$. Then the above equation can be rewritten as

$$N(t)=\mathcal{E}_1[N,G](t)=U_0^s(t)+\int_0^t U_\tau^s(t-\tau)B^s(\tau)(Id+G_\tau)N(\tau)d\tau, 0\le t\le T,$$

$$G_0=\mathcal{E}_2[N,G](0)=U_T^{cu}(-T)G_TN(T)-\int_0^T U_t^{cu}(-t)B^{cu}(t)(Id+G_t)N(t)dt.$$

From the definition of $\mathcal{V}$, we take the limit $T\to\infty$ for the RHS of the second equation of the above system and obtain

$$\begin{aligned} N(t)&=\mathcal{E}_1[N,G](t)=U_0^s(t)+\int_0^t U_\tau^s(t-\tau)B^s(\tau)(Id+G_\tau)N(\tau)d\tau,\\ G_0&=\mathcal{E}_2[N,G](0)=-\int_0^\infty U_t^{cu}(-t)B^{cu}(t)(Id+G_t)N(t)dt. \end{aligned} \tag{A.10}$$

We regard (A.10) as the equations for the two unknowns G and N, where G and N are the functions of t. We will obtain G and N as the fixed point of operator, $\mathcal{E}=(\mathcal{E}_1,\mathcal{E}_2)$, defined by the RHS of (A.10). For a fixed t, $N(t)$ is a bundle map covering the advance by t on the base, *i.e.* $\tau\to\tau+t$. What we need to do is just to prove the operator $\mathcal{E}$ is a contraction, to do this we need to define the spaces on which $\mathcal{E}$ acts and the distances that make it a contraction.

A.2. *Definition of spaces*

For $\delta>2C_h$ and $\beta_3^-<\beta<\beta_1$ we denote

$$\mathcal{V}=\Big\{(N,G):\mathbb{R}^+\ni t\mapsto(N_t,G_t)\in(\mathscr{L}(X_0^s,X_t^s),\mathscr{L}(X_t^s,X_t^{cu})),\\ \|N\|_{C^0}^{(\beta}\le\delta,\quad \|G\|_{C^0}\le\epsilon^{\frac12}\Big\},$$

where

$$\begin{aligned} \|N\|_{C^0}^{(\beta}&=\sup_{t\in[0,\infty)}\|N_t\|_{X_0^s,X_t^s}e^{\beta t},\\ \|G\|_{C^0}&=\sup_{t\in[0,\infty)}\|G_t\|_{X_t^s,X_t^{cu}},\\ \|(N,G)\|_{C^0}&=\max\{\|N\|_{C^0}^{(\beta},\|G\|_{C^0}\} \end{aligned} \tag{A.11}$$

and the induced metric on $\mathcal{V}$ is also the $d-$distance defined in (3.39) with $\|(\cdot_1,\cdot_2)\|_{C^0}$ defined in (A.11) for the function $(N,G)\in\mathcal{V}$.

A.3. Elementary estimates

We also separate the proof into two steps, $\mathcal{E}\mathcal{V} \subset \mathcal{V}$ (Step 1) and then $\mathcal{E}$ is a contraction in $\mathcal{V}$ (Step 2).

(Step 1): $\mathcal{E}\mathcal{V} \subset \mathcal{V}$. From $\delta > 2C_h$ and the smallness of ϵ we obtain

$$
\begin{aligned}
&\|U_0^s(t) + \int_0^t U_\tau^s(t-\tau)B^s(\tau)(Id + G_\tau)N(\tau)d\tau\|_{\mathscr{L}(X_0^s, X_t^s)} e^{\beta t} \\
&\le C_h e^{-(\beta_1-\beta)t} + 2C_h\epsilon e^{\beta t}\int_0^t e^{-\beta_1(t-\tau)}(t-\tau)^{-\alpha_1}\|N(\tau)\|^{(\beta}_{\mathscr{L}(X_0^s, X_\tau^s)} e^{-\beta\tau}d\tau \\
&\le C_h e^{-(\beta_1-\beta)t} + 2C_h\epsilon\Big(\frac{1}{\beta_1-\beta} + \frac{1}{1-\alpha_1}\Big)\|N\|^{(\beta}_{C^0} \\
&\le C_h + 2C_h\epsilon\Big(\frac{1}{\beta_1-\beta} + \frac{1}{1-\alpha_1}\Big)\delta \\
&\le \delta,
\end{aligned}
$$

that is $\|\mathcal{E}_1[N, G]\|^{(\beta}_{C^0} \le \delta$. Similarly for $\mathcal{E}_2$, from the smallness of ϵ, we get

$$
\begin{aligned}
&\Big\| -\int_0^\infty U_t^c(-t)B^c(t)(Id + G_t)N(t)dt\Big\|_{C^0} \\
&\le 2C_h\epsilon\int_0^\infty e^{\beta_3^- t}\|N(t)\|^{(\beta}_{\mathscr{L}(X_0^s, X_t^s)} e^{-\beta t}dt \\
&\le \frac{2C_h\epsilon\delta}{\beta-\beta_3^-} \\
&\le \epsilon^{\frac{1}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\| -\int_0^\infty U_t^u(-t)B^u(t)(Id + G_t)N(t)dt\Big\|_{C^0} \\
&\le 2C_h\epsilon\int_0^\infty e^{-\beta_2 t}t^{-\alpha_2}\|N(t)\|^{(\beta}_{\mathscr{L}(X_0^s, X_t^s)} e^{-\beta t}dt \\
&\le 2C_h\epsilon\delta\Big(\frac{1}{\beta+\beta_2} + \frac{1}{1-\alpha_2}\Big) \\
&\le \epsilon^{\frac{1}{2}},
\end{aligned}
$$

that is $\|\mathcal{E}_2[N, G]\|_{C^0} \le \epsilon^{\frac{1}{2}}$. The discussions above yield $\mathcal{E}\mathcal{V} \subset \mathcal{V}$.

(Step 2): $\mathcal{E}$ is a contraction in $\mathcal{V}$. Take any (N, G), $(\widetilde{N}, \widetilde{G}) \in \mathcal{V}$ we obtain

$$\begin{aligned}&\mathcal{E}_1[N, G](t) - \mathcal{E}_1[\widetilde{N}, \widetilde{G}](t)\\&\qquad = \int_0^t U_\tau^s(t-\tau)B^s(\tau)\big[(Id + G_\tau)N(\tau) - (Id + \widetilde{G}_\tau)\widetilde{N}(\tau)\big]d\tau.\end{aligned}$$

Then from the triangle inequality we obtain

$$\begin{aligned}&\left\| \int_0^t U_\tau^s(t-\tau)B^s(\tau)\big[(Id + G_\tau)N(\tau) - (Id + \widetilde{G}_\tau)\widetilde{N}(\tau)\big]d\tau \right\|_{\mathscr{L}(X_0^s, X_t^s)} e^{\beta t}\\&\qquad \le C_h\epsilon\Big(\|N - \widetilde{N}\|_{C^0}^{(\beta}(1 + \|G\|_{C^0}) + \|\widetilde{N}\|_{C^0}^{(\beta}\|G - \widetilde{G}\|_{C^0}\Big)\\&\qquad\quad \cdot \int_0^t e^{-(\beta_1-\beta)(t-\tau)}(t-\tau)^{-\alpha_1}d\tau\\&\qquad \le 2C_h\delta\epsilon\Big(\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1}\Big)\|(N - \widetilde{N}, G - \widetilde{G})\|_{C^0},\end{aligned}$$

that is

$$\|\mathcal{E}_1[N, G] - \mathcal{E}_1[\widetilde{N}, \widetilde{G}]\|_{C^0}^{(\beta} < 2C_h\epsilon\delta\Big(\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1}\Big)\|(N - \widetilde{N}, G - \widetilde{G})\|_{C^0}.$$

Similarly, we get

$$\|\mathcal{E}_2[N, G] - \mathcal{E}_2[\widetilde{N}, \widetilde{G}]\|_{C^0} < 2C_h\delta\epsilon c_1\|(N - \widetilde{N}, G - \widetilde{G})\|_{C^0},$$

with $c_1 = \max\left\{\frac{1}{\beta - \beta_3^-}, \frac{1}{\beta_2+\beta} + \frac{1}{1-\alpha_2}\right\}$. Then we obtain

$$d\big(\mathcal{E}[N, G], \mathcal{E}[\widetilde{N}, \widetilde{G}]\big) < cd\big((N, G), (\widetilde{N}, \widetilde{G})\big),$$

where $c = 2C_h\epsilon\delta\kappa$ with

$$\kappa = \max\left\{\frac{1}{\beta_1 - \beta} + \frac{1}{1-\alpha_1}, \frac{1}{\beta - \beta_3^-}, \frac{1}{\beta_2 + \beta} + \frac{1}{1-\alpha_2}\right\}.$$

From the smallness of ϵ we know $0 < c < 1$, that is $\mathcal{E}$ is a contraction in $\mathcal{V}$. Therefore, with the above choices we can obtain the solution of (A.10), $(N(t), G_t)$, in the C^0 closure of $\mathcal{V}$ which yields a forward evolution and that the graph is invariant under this evolution.

Then we obtain the stability of the splittings $\widetilde{X}_t^s$ and $\widetilde{X}_t^{cu}$. Similarly we get the stability of the splittings $\widetilde{X}_t^u$ and $\widetilde{X}_t^{cs}$. Then $\widetilde{X}_t^c = \widetilde{X}_t^{cu} \bigcap \widetilde{X}_t^{cs}$. Above all we get the stability of the splittings $X = \widetilde{X}_t^s \oplus \widetilde{X}_t^c \oplus \widetilde{X}_t^u$.

A.4. Estimates on the projections

To get the bounds for the projections we use the same argument as in [31]. We only give the argument for the stable subspace. Let G_t^{cu} be the linear map whose graph gives $\widetilde{X}_t^{cu}$. We write

$$\Pi_t^s\xi = (\xi^s, 0), \qquad \widetilde{\Pi}_t^s\xi = (\eta^s, G_t\eta^s),$$
$$\Pi_t^{cu}\xi = (0, \xi^{cu}), \qquad \widetilde{\Pi}_t^{cu}\xi = (G_t^{cu}\eta^{cu}, \eta^{cu}),$$

then

$$\xi^s = \eta^s + G_t^{cu}\eta^{cu}, \xi^{cu} = G_t\eta^s + \eta^{cu}.$$

Since G_t and G_t^{cu} are $O(\epsilon)$ in $L(X; X)$ we can write

$$\begin{pmatrix} \eta^s \\ \eta^{cu} \end{pmatrix} = \begin{pmatrix} Id, & G_t^{cu} \\ G_t, & Id \end{pmatrix}^{-1} \begin{pmatrix} \xi^s \\ \xi^{cu} \end{pmatrix}. \tag{A.12}$$

So

$$\|\widetilde{\Pi}_t^s - \Pi_t^s\|_{X_t^s, X_t^s} = \sup_{(0,\xi)\in\mathbf{B}^s, \|\xi\|_X\le 1} \|(\widetilde{\Pi}_t^s - \Pi_t^s)\xi\|_{X_t^s} \le \|\eta^s - \xi^s, G_t\eta^s\|_{X_t^s} \le \epsilon^{\frac{1}{2}}.$$

Analogously, we obtain the estimates about the two other subspaces (center space and the unstable space), we omit the details.

A.5. Stability of the smoothing hyperbolic properties

In this step, we will prove that under the lower order perturbations the smoothing properties of the cocycles are preserved. That is, we show that if we define the evolutions in the invariant spaces constructed above, they satisfy the bounds of the form in (A.5) and (A.7) just with slightly worse parameters. The paper [26] presented an slightly different argument with more precise estimates that were needed because this would be applied infinitely many times in [26]. We just give the estimate about the stable case since the unstable case is studied in the same way, just reversing the direction of time.

From (A.10) we know that $W_0^s(t)$ satisfies

$$W_0^s(t) = U_0^s(t) + \int_0^t U_z^s(t-z)(\widetilde{A} - A)^s(z)(Id + G_z)W_0^s(z)dz.$$

Using the Grönwall's inequality [38] and noting that $(W_0^s, G) \in \mathcal{V}$, we obtain

$$\|W_0^s(t)\|_{Y,X} = \|U_0^s(t) + \int_0^t U_z^s(t-z)(\widetilde{A} - A)^s(z)(Id + G_z)W_0^s(z)dz\|_{Y,X}$$

$$\le \|U_0^s(t)\|_{Y,X} + \Big\| \int_0^t U_0^s(t-z)(\widetilde{A}-A)^s(z)(Id+G_z)W_0^s(z)dz \Big\|_{Y,X}$$

$$\le C_h e^{-\beta_1 t} t^{-\alpha_1} + 2\int_0^t C_h \epsilon e^{-\beta_1(t-z)}(t-z)^{-\alpha_1}\|W_0^s(z)\|_{Y,X} dz$$

$$\le C_h e^{-\beta_1 t} t^{-\alpha_1} + 2C_h^2\epsilon \int_0^t e^{-\beta_1 z} z^{-\alpha_1} e^{-\beta_1(t-z)}(t-z)^{-\alpha_1} e^{\beta(z)} dz,$$

where $\beta(z) = \int_z^t C_h \epsilon e^{-\beta_1(t-s)}(t-s)^{-\alpha_1} ds$, for this integral we have

$$\beta(z) = \int_z^t C_h \epsilon e^{-\beta_1(t-s)}(t-s)^{-\alpha_1} ds$$
$$= C_h \epsilon \int_0^{t-z} e^{-\beta_1 \tau}\tau^{-\alpha_1} d\tau$$
$$\le C_h \epsilon t^{1-\alpha_1}\left(\frac{1}{\beta_1} + \frac{1}{1-\alpha_1}\right).$$

Then we have the following

$$\|W_0^s(t)\|_{Y,X} \le C_h e^{-\beta_1 t} t^{-\alpha_1} + 2C_h^2 \epsilon e^{C_h \epsilon t^{1-\alpha_1}(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})} e^{-\beta_1 t}\int_0^t z^{-\alpha_1}(t-z)^{-\alpha_1} dz$$
$$\le C_h e^{-\beta_1 t} t^{-\alpha_1} + 2C_h^2 \epsilon e^{C_h \epsilon t^{1-\alpha_1}(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})} e^{-\beta_1 t}\frac{4t^{1-2\alpha_1}}{1-\alpha_1}$$
$$= C_h e^{-\beta_1 t} t^{-\alpha_1} + \frac{8C_h^2\epsilon}{1-\alpha_1} t^{1-\alpha_1} e^{C_h \epsilon t^{1-\alpha_1}(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})} e^{-\beta_1 t} t^{-\alpha_1}.$$

I: $t \in [0, 1]$, then

$$\|W_0^s(t)\|_{Y,X} \le C_h e^{-\beta_1 t} t^{-\alpha_1} + \frac{8C_h^2\epsilon}{1-\alpha_1} t^{1-\alpha_1} e^{C_h \epsilon t^{1-\alpha_1}(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})} e^{-\beta_1 t} t^{-\alpha_1}$$
$$\le C_h e^{-\beta_1 t} t^{-\alpha_1} + \frac{8C_h^2\epsilon}{1-\alpha_1} e^{C_h \epsilon (\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})} e^{-\beta_1 t} t^{-\alpha_1}$$
$$= \widehat{C}_h e^{-\widehat{\beta}_1 t} t^{-\widehat{\alpha}_1},$$

where

$$\begin{aligned}\widehat{C}_h &= C_h + \frac{8C_h^2\epsilon}{1-\alpha_1}e^{C_h\epsilon(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})},\\ \widehat{\beta}_1 &= \beta_1,\\ \widehat{\alpha}_1 &= \alpha_1.\end{aligned} \tag{A.13}$$

II: $t \in (1, \infty)$, then

$$\begin{aligned}\|W_0^s(t)\|_{Y,X} &\le C_h e^{-\beta_1 t}t^{-\alpha_1} + \frac{8C_h^2\epsilon}{1-\alpha_1}t^{1-\alpha_1}e^{C_h\epsilon t(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})}e^{-\beta_1 t}t^{-\alpha_1}\\ &= C_h e^{-\beta_1 t}t^{-\alpha_1} + \frac{8C_h^2\epsilon}{1-\alpha_1}t^{1-\alpha_1}e^{-\epsilon^{\frac{1}{2}}t}e^{C_h\epsilon t(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})}e^{-(\beta_1-\epsilon^{\frac{1}{2}})t}t^{-\alpha_1}\\ &\le C_h e^{-\beta_1 t}t^{-\alpha_1} + \frac{8C_h^2\epsilon^{\frac{1+\alpha_1}{2}}}{(1-\alpha_1)^{\alpha_1}e^{1-\alpha_1}}e^{C_h\epsilon t(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1})}e^{-(\beta_1-\epsilon^{\frac{1}{2}})t}t^{-\alpha_1}\\ &= \widehat{C}_h e^{-\widehat{\beta}_1 t}t^{-\widehat{\alpha}_1},\end{aligned}$$

(the last inequality is from

$$t^{1-\alpha_1}e^{-\epsilon^{\frac{1}{2}}t} \le \epsilon^{-\frac{1+\alpha_1}{2}}(1+\alpha_1)^{1+\alpha_1}e^{-(1+\alpha_1)} \quad),$$

where

$$\begin{aligned}\widehat{C}_h &= C_h + \frac{8C_h^2\epsilon^{\frac{1+\alpha_1}{2}}}{(1-\alpha_1)^{\alpha_1}e^{1-\alpha_1}},\\ \widehat{\beta}_1 &= \beta_1 - \epsilon^{\frac{1}{2}} - C_h\epsilon\Big(\frac{1}{\beta_1}+\frac{1}{1-\alpha_1}\Big),\\ \widehat{\alpha}_1 &= \alpha_1.\end{aligned} \tag{A.14}$$

From (A.13) and (A.14),

$$\widetilde{U}_0^s(t) = W_0^s(t) + G_t W_0^s(t)$$

and

$$\|G_t\|_{X_t^s, X_t^{cu}} \le \epsilon^{\frac{1}{2}},$$

we know that

$$\|\widetilde{U}_0^s(t)\|_{Y,X} \le \widetilde{C}_h e^{-\widetilde{\beta}_1 t}t^{-\widetilde{\alpha}_1} \tag{A.15}$$

with

$$\widetilde{C}_h \le (1+\epsilon^{\frac{1}{2}})\widehat{C}_h, \quad \widetilde{\beta}_1 = \widehat{\beta}_1, \quad \widetilde{\alpha}_1 = \widehat{\alpha}_1,$$

that is

$$|\widetilde{C}_h - C_h| \le c\epsilon^{\frac{1}{2}} = c\|\widetilde{A} - A\|_{X,Y}^{\frac{1}{2}},$$
$$|\widetilde{\beta}_1 - \beta_1| \le C\epsilon^{\frac{1}{2}} = C\|\widetilde{A} - A\|_{X,Y}^{\frac{1}{2}},$$
$$\widetilde{\alpha}_1 = \alpha_1,$$

with

$$c = \frac{8C_h^2}{(1-\alpha_1)^{\alpha_1}}\epsilon^{\frac{\alpha_1}{2}} e^{-(1-\alpha_1)} + \widehat{C}_h,$$
$$C = 1 + C_h\epsilon^{\frac{1}{2}}\left(\frac{1}{\beta_1} + \frac{1}{1-\alpha_1}\right).$$

Similarly, we also obtain

$$\|\widetilde{U}_0^s(t)\|_{X,X} \le \widetilde{C}_h e^{-\widetilde{\beta}_1 t},$$

where $\widetilde{C}_h$ and $\widetilde{\beta}_1$ are the ones in (A.15).

Appendix B. Existence of invariant hyperbolic smoothing splittings in the center manifolds and their regularity

In this appendix, we prove that there are hyperbolic and smoothing invariant splittings based on points in a center manifold and establish that they depend regularly on the point on the manifold. We also prove smooth dependence on the parameters related to the invariant splitting.

We will consider the "prepared" equations (see Section 3.3) and establish the results for the center manifold corresponding to these prepared equations. We recall that, the results may depend on the choice of the prepared equations. See the observations at the end of Section 3.3.1.

We will be considering formal evolution of equations of the form:

$$\dot{u} = \mathcal{A}u + \mathcal{N}(u),$$

where $\mathcal{A}$ is as in (2.1) and $\mathcal{N}$ is nonlinear, unbounded and is of lower order than $\mathcal{A}$.

Since we will be considering the prepared equations, we will assume that $\mathcal{N} : X \to Y$ is uniformly C^r $(r \ge 2)$ small.

In this situation [23] established the existence of a uniformly $C^{r-1+Lip}$ function w

$$w : X_c \to X^s \oplus X^u$$

with $w(0) = 0$, $Dw(0) = 0$, such a way that

$$\mathcal{W} \equiv \{(\theta, w(\theta))| \quad \theta \in X_c\}$$

is invariant under (2.1). Furthermore, the $C^{r-1+Lip}-$norm w can be as small as desired by assuming the smallness of $\|N\|_{C^r}$.

The mapping

$$K : \theta \to (\theta, w(\theta))$$

provides a $C^{r-1+Lip}$ diffeomorphism of X_c into $\mathcal{W}$. Note that we can assume that

$$\|D^j K\|_{X_c^{\otimes j}, X} \leq \eta, \quad 0 \leq j \leq r-1 \tag{B.1}$$

with η arbitrarily small. The invariance of $\mathcal{W}$ is equivalent to the existence of a $C^{r-1+Lip}$ vector field $\mathcal{B}$ on X_c such that

$$DK(\theta)\mathcal{B}(\theta) = \mathcal{A}K(\theta) + \mathcal{N}(K(\theta)).$$

If $\Phi_t(\theta)$ is the evolution of $\mathcal{B}$ in X_c, then $u(t) = K(\Phi_t(\theta)) = (\Phi_t(\theta), w(\Phi_t(\theta)))$ are solutions of the evolution equation (2.1).

The following Lemma B.1 will be the main result in this Appendix. The novelty of Lemma B.1 is that we establish that the splitting depends regularly with respect to the base point. Similar results are standard in the theory of invariant manifolds [32,33]. Nevertheless, in our set up, most of the standard proofs, based on the graph transform method do not work since the equations we consider are ill-posed and we cannot iterate forward geometric objects. Indeed, part of our problem consists precisely in choosing the initial conditions so that we can define the forward evolution. We also establish smooth dependence on the base points for the evolution operators in the splittings. We note that we obtain smooth dependence both as operators X to X and as operators Y to X.

Of course, in the case of quasi-periodic solutions, we established that the dependence on the base point was analytic (by using a very different proof). In the present case, one does not expect that the dependence will be even C^∞ since there are finite dimensional examples where the dependence is only finitely differentiable.

Lemma B.1. *Assume that $\|\mathcal{N}\|_{C^r(X,Y)} \leq \epsilon$ with ϵ small enough, and the parameters $\beta_1,\ \beta_2,\ \beta_3^{\pm}$, in (2.7), satisfy $\beta_1 > (r-1)\beta_3^+ + \beta_3^-$, $\beta_2 > (r-1)\beta_3^- + \beta_3^+$. Then for every $\theta \in X_c$ we can find a splitting*

$$\begin{aligned} X &= X_\theta^s \oplus X_\theta^c \oplus X_\theta^u, \\ Y &= Y_\theta^s \oplus Y_\theta^c \oplus Y_\theta^u \end{aligned} \tag{B.2}$$

and a family of operators $U_\theta^\sigma(t), \sigma = s, c, u$ in such a way that

(SD2)

$$U_\theta^\sigma(t) : Y_\theta^\sigma \to X_{\Phi_t(\theta)}^\sigma, \quad \sigma = s, c, u$$

such that:

(SD2.1) *The families $U_\theta^{s,c,u}(t)$ are cocycles satisfying*

$$U_{\Phi_t(\theta)}^{s,c,u}(z) U_\theta^{s,c,u}(t) = U_\theta^{s,c,u}(z+t),\ U_\theta^{s,c,u}(0) = Id.$$

(SD2.2) *The operators $U_\theta^{s,c,u}$ are smoothing in the time direction where they can be defined (they map the Y spaces into the X spaces and they satisfy quantitative estimates as indicated below).*

There exist $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in [0,1), \widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3^+, \widetilde{\beta}_3^- > 0$ with $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in [0,1), \widetilde{\beta}_1 > (r-1)\widetilde{\beta}_3^+ + \widetilde{\beta}_3^-, \widetilde{\beta}_2 > (r-1)\widetilde{\beta}_3^- + \widetilde{\beta}_3^+$ and $\widetilde{C}_h > 1$ independent of θ such that the evolution operators are characterized by the following rate conditions:

$$\begin{aligned} \|U_\theta^s(t)\|_{Y,X} &\le \widetilde{C}_h e^{-\widetilde{\beta}_1 t} t^{-\widetilde{\alpha}_1}, \quad t>0, \\ \|U_\theta^s(t)\|_{X,X} &\le \widetilde{C}_h e^{-\widetilde{\beta}_1 |t|}, \quad t>0. \end{aligned} \tag{B.3}$$

The evolution $U_\theta^{s,c,u}(t)$ operators solve the "variational" *equation*

$$\begin{aligned} \frac{d}{dt}U_\theta^\sigma(t) &= A(\Phi_t(\theta))U_\theta^\sigma(t), \\ &= [\mathcal{A} + D\mathscr{N}(K\circ\Phi_t(\theta))]U_\theta^\sigma(t), \;\; U_\theta^\sigma(0) = Id, \; \sigma = s,c,u. \end{aligned} \tag{B.4}$$

Furthermore, the mapping $\theta \to U_\theta^\sigma(t)$ is $C^{r-2+Lip}$ when we give the $U_\theta^\sigma(t)$ the topology of $\mathscr{L}(Y_\theta^s, X_{\Phi_t(\theta)}^s)$, and for $1 \le j \le r-2$ we have

$$\begin{aligned} &\|D_\theta^j U_\theta^s(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_\theta^s, X_{\Phi_t(\theta)}^s)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t}, \quad t\ge 0, \\ &\|D_\theta^j U_\theta^s(t)\|_{X_c^{\otimes j}, \mathscr{L}(Y_\theta^s, X_{\Phi_t(\theta)}^s)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t} t^{-\widetilde{\alpha}_1}, \quad t\ge 0, \\ &Lip_\theta\big(D_\theta^{r-2} U_\theta^s(t)\big|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_\theta^s, X_{\Phi_t(\theta)}^s)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t}, \quad t\ge 0, \\ &Lip_\theta\big(D_\theta^{r-2} U_\theta^s(t)\big|_{X_c^{\otimes(r-2)}, \mathscr{L}(Y_\theta^s, X_{\Phi_t(\theta)}^s)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t} t^{-\widetilde{\alpha}_1}, \; t\ge 0, \end{aligned} \tag{B.5}$$

and

$$\begin{aligned} &\|D_\theta^j U_{\Phi_\tau(\theta)}^s(t)\|_{X_c^{\otimes j}, \mathscr{L}(Y_{\Phi_\tau(\theta)}^s, X_{\Phi_{t+\tau}(\theta)}^s)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t + j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau} t^{-\widetilde{\alpha}_1}, \tau, t\ge 0, \\ &\|D_\theta^j U_{\Phi_\tau(\theta)}^s(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_{\Phi_\tau(\theta)}^s, X_{\Phi_{t+\tau}(\theta)}^s)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t + j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau}, \quad \tau, t\ge 0, \\ &Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^s(t)\big|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_\theta^s, X_{\Phi_t(\theta)}^s)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t + (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau}, \; t\ge 0, \\ &Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^s(t)\big|_{X_c^{\otimes(r-2)}, \mathscr{L}(Y_\theta^s, X_{\Phi_t(\theta)}^s)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t + (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau} t^{-\widetilde{\alpha}_1}, \\ &\quad t \ge 0, \end{aligned} \tag{B.6}$$

where $A\big|_{X_c^{\otimes(r-2)}, \mathscr{L}(Z_\vartheta^s, X_{\Phi_t(\vartheta)}^s)}$ means that operator $A \in \mathscr{L}(X_c^{\otimes(r-2)}, \mathscr{L}(Z_\vartheta^s, X_{\Phi_t(\vartheta)}^s))$ and $\vartheta = \theta, \; \Phi_\tau(\theta), \; Z = Y, X$.

Remark B.1. In the estimates for the operators we need only to estimate $U^s_{\Phi_\tau(\theta)}(t-\tau),\ 0<\tau<t$ (see (3.59)), but we have chosen to present estimates for $U^s_{\Phi_\tau(\theta)}(t)$ with $t,\ \tau$ arbitrary positive numbers and the same sign. This allows to break the estimates into two parts. In the cases of unstable and center, we will use similar arguments but then t,τ will have opposite signs (see (3.60)).

Remark B.2. As a useful mnemonic rule, we remark that $Lip_\theta D_\theta^{r-2}A(\theta)$ satisfies the same bounds as $\|D_\theta^{r-1}A(\theta)\|$ would satisfy in (B.5) and (B.6). As a matter of fact, this mnemonic rule can be justified because to derive the bounds, we only use the bounds for composition and bounded for the rules for product and sum of the derivatives, which are also true for Lipschitz constant. This remark applies in similar computations in this paper.

Before starting the proof of Lemma B.1, we recall the following technical result that will play a role in the future

Proposition B.1. *Let $\Phi_t(\theta)$ be a semi-flow defined for positive t from Banach space X to Banach space Y.*

Assume that for all $1\le j\le r-1$, we have

$$\sup_{t\in[0,1),\theta\in X}\|D_\theta^j\Phi_t(\theta)\|_{X^{\otimes j},Y}\le C_h<\infty.$$

If we assume that, for some $\beta_3^+>0$, the inequality

$$\sup_{\theta\in X}\|D_\theta\Phi_t(\theta)\|_{X,Y}\le C_h e^{\beta_3^+ t},\ \ t\in[0,\infty) \tag{B.7}$$

holds, then, we have, for all $1\le j\le r-1$,

$$\sup_{\theta\in X}\|D_\theta^j\Phi_t(\theta)\|_{X^{\otimes j},Y}\le C_h e^{j\beta_3^+ t}t^j,\ \ t\in[0,\infty). \tag{B.8}$$

If instead of assuming (B.7), we assume, $\beta>0$,

$$\sup_{\theta\in X}\|D_\theta\Phi_t(\theta)\|_{X,Y}\le C_h e^{-\beta t},\ \ t\in[0,\infty), \tag{B.9}$$

then, we have for all $1\le j\le r-1$,

$$\sup_{\theta\in X}\|D_\theta^j\Phi_t(\theta)\|_{X^{\otimes j},Y}\le C_h e^{-\beta t}t^j,\ \ t\in[0,\infty). \tag{B.10}$$

Proof. This proposition is a particular case of [27, Lemma 5.2] which is proved by an easy counting argument. □

Even if the results of Proposition B.1 are enough for our purposes, we expect that more elaborate proofs could get rid of the t^j factors in (B.8) and (B.10). For simplicity of notation we will prefer to deal only with exponentials, so we worsen slightly the exponents and add a constant. That is, we state (B.8) and (B.10) as (for the ϵ in Lemma B.1)

$$\sup_{\theta\in X} \|D^j_\theta \Phi_t(\theta)\|_{X^{\otimes j},Y} \le C_h e^{-j}\epsilon^{\frac{-j}{4r}} e^{j(\beta_3^+ + \epsilon^{\frac{1}{4r}})t},\ t\in[0,\infty),\ 1\le j\le r-1, \tag{B.11}$$

and

$$\sup_{\theta\in X} \|D^j_\theta \Phi_t(\theta)\|_{X^{\otimes j},Y} \le C_h e^{-j}\epsilon^{\frac{-j}{4r}} e^{-(\beta - \epsilon^{\frac{1}{4r}})t},\ t\in[0,\infty)\ 1\le j\le r-1. \tag{B.12}$$

Proof of Lemma B.1. We adopt the same method in Appendix A to get the results of Lemma B.1. Note that we can get (B.6) from (B.5) by applying Faa-di Bruno formula (C.3), so, it is suffices to prove (B.5).

We take the unperturbed vector field as $\mathcal{A}$ and the perturbation vector field as $\mathcal{A} + D\mathcal{N} \circ K \circ \Phi_t(\theta)$ for a fixed θ. (Recall that we are assuming that $\|D\mathcal{N}\|_{C^{r-1}}$ is uniformly small because we are dealing with the "prepared" equation, we denote $\|D\mathcal{N}\|_{C^{r-1}} \le \epsilon$.) Denote the operators generated by $\mathcal{A}$ as $U^\sigma(t), \sigma = s, u, c$, as in (2.6) (hope there is no confusion with the operators $U^\sigma_\theta, \sigma = s, u, c$, generated by $A(\theta) = [\mathcal{A} + D\mathcal{N}(K \circ \Phi_t(\theta))]$), that is

$$\frac{d}{dt}U^\sigma(t) = \mathcal{A}U^\sigma(t),\quad \sigma = s,u,c,\ \ U^\sigma(0) = I.$$

Then with the same discussions to get (A.10) in Appendix A we get

$$\begin{aligned} N_\theta(t) &= \mathcal{F}_s(N,G)(t,\theta)\\ &\equiv U^s(t) + \int_0^t U^s(t-\tau)B^s(K(\Phi_\tau(\theta)))(Id + G_\theta(\tau))N_\theta(\tau)d\tau,\\ G(\theta) &= \mathcal{F}_{cu}(N,G)(0,\theta)\\ &\equiv -\int_0^\infty U^{cu}(-t)B^{cu}(K(\Phi_\tau(\theta)))(Id + G_\theta(t))N_\theta(t)dt, \end{aligned} \tag{B.13}$$

where $B = D\mathcal{N}$ and B^s is the projection to the stable space and B^{cu} is the projection to unstable and center space.

In appendix A, we fixed θ and constructed the invariant splittings along the orbit and the linearized evolution operators U^σ_θ showing that they were obtained as the fixed point of some operators (defined by the RHS of (A.10)) that were contraction in a supremum norm along the orbit. It is important for future reference that the contraction properties of the operator depended only on norms of the perturbation and that they are uniform in θ since we are using the prepared equations, which we assumed that have a nonlinearity which is C^{r-1} small. The desired result Lemma B.1 will be obtained by considering an analogous of (A.10) that considers the dependence on θ (see (B.13)).

Assume that the parameters $\widetilde{\beta}_i,\ i = 1,2,\ \widetilde{\beta}_3^\pm$ and $\widetilde{\alpha}_i,\ i = 1,2, \widetilde{C}_h$ are the ones in Lemma A.1. That is $\widetilde{\beta}_i = \beta_i - \epsilon^{\frac{1}{2}} - C_h\epsilon(\frac{1}{\beta_i} + \frac{1}{1-\alpha_i}),\ \widetilde{\alpha}_i = \alpha_i,\ i = 1,2,\ \widetilde{\beta}_3^\pm = \beta_3^\pm + \epsilon^{\frac{1}{2}} + \frac{C_h\epsilon}{\beta_3^\pm}$ and $\widetilde{C}_h = (1 + c\epsilon^{\frac{1}{2}})C_h$. Since we also assume $\beta_1 > (r-1)\beta_3^+ + \beta_3^-$ and $\beta_1 > (r-1)\beta_3^- + \beta_3^+$, by the smallness of ϵ we know that $\widetilde{\beta}_1 > (r-1)\widetilde{\beta}_3^+ + \widetilde{\beta}_3^-$ and $\widetilde{\beta}_1 > (r-1)\widetilde{\beta}_3^- + \widetilde{\beta}_3^+$.

For the operators $U^\sigma(t),\ \sigma = s, u, c$, by (2.7) we know that

$$
\begin{aligned}
&\|U^s(t)\|_{\mathscr{L}(Y^s,X^s)} \le C_h e^{-\beta_1 t} t^{-\alpha_1} \le C_h e^{-(\widetilde{\beta}_1+\epsilon^{\frac{1}{2}})t} t^{-\widetilde{\alpha}_1},\ t \ge 0,\\
&\|U^s(t)\|_{\mathscr{L}(X^s,X^s)} \le C_h e^{-\beta_1 t} \le C_h e^{-(\widetilde{\beta}_1+\epsilon^{\frac{1}{2}})t},\ t \ge 0\\
&\|U^u(t)\|_{\mathscr{L}(Y^u,X^u)} \le C_h e^{-\beta_2 t} t^{-\alpha_2} \le C_h e^{-(\widetilde{\beta}_2+\epsilon^{\frac{1}{2}})t} t^{-\widetilde{\alpha}_2},\ t \le 0,\\
&\|U^u(t)\|_{\mathscr{L}(X^u,X^u)} \le C_h e^{-\beta_3 t} \le C_h e^{-(\widetilde{\beta}_3+\epsilon^{\frac{1}{2}})t},\ \ t \le 0,\\
&\|U^c(t)\|_{\mathscr{L}(Y^c,X^c)} \le C_h e^{\beta_3^+ t} \le C_h e^{(\widetilde{\beta}_3^+-\epsilon^{\frac{1}{2}})t},\ t \ge 0,\\
&\|U^s(t)\|_{\mathscr{L}(X^s,X^s)} \le C_h e^{\beta_3^- |t|} \le C_h e^{(\widetilde{\beta}_3^- -\epsilon^{\frac{1}{2}})|t|},\ \ t \le 0.
\end{aligned}
\tag{B.14}
$$

Define the set

$$
\begin{aligned}
\mathcal{H} = \Big\{ &N : \mathbb{R}^+ \times X_c \ni (t,\theta) \mapsto N_\theta(t) \in \mathscr{L}(X^s_\theta, X^s_{\Phi_t(\theta)}),\\
&G : \mathbb{R}^+ \times X_c \ni (t,\theta) \mapsto G_\theta(t) \in \mathscr{L}(X^s_{\Phi_t(\theta)}, X^{cu}_{\Phi_t(\theta)}),\ \|N_\theta(t)\|_{X^s_\theta, X^s_{\Phi_t(\theta)}} \le C_h e^{\widetilde{\beta}_1 t},\\
&\|D^j_\theta N_\theta(t)\|_{X_c^{\otimes j}, \mathscr{L}(X^s_\theta, X^s_{\Phi_t(\theta)})} \le C_h \epsilon e^{-[\widetilde{\beta}_1 - j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t},\ j = 1, \cdots, r-1,\\
&\|D^j_\theta G_\theta(t)\|_{X_c^{\otimes j}, \mathscr{L}(X^s_{\Phi_t(\theta)}, X^{cu}_{\Phi_t(\theta)})} \le \epsilon^{\frac{1}{2}},\ \ j = 0, 1, \cdots, r-1 \Big\}.
\end{aligned}
$$

We adopt the weighted norms $\|\cdot\|^{(\widetilde{\beta}_1}_{C^0}$, $\|\cdot\|_{C^0}$ and $\|(\cdot_1, \cdot_2)\|_{C^0}$ which are defined in (A.11) for the functions N, G and (N, G), respectively. The induced metric on $\mathcal{H}$ is also the $d-$distance defined in (3.39) with $\widetilde{\beta}_1$ in place of $\widetilde{\beta}$.

Following the standard strategy in center manifold theory, we will prove that the operator $\mathcal{F} = (\mathcal{F}_s, \mathcal{F}_{cu})$ is a contraction in $\mathcal{H}$. In this situation, we can appeal to [49, Proposition A2] (which shows that the C^0 closure of functions with uniformly bounded C^r norms is $C^{r-1+Lip}$) or to Hadamard's interpolation theorem (see Theorem C.1 in Appendix C) which shows that a C^0 contraction in spaces of uniformly bounded C^r functions also converges in C^{r-1}. That is for the fixed θ, by applying Lemma A.1 we get the existence of the splitting (B.2) and the estimates (B.3).

To finish Lemma B.1, it suffices to establish the regularity with respect θ, *i.e.* verify that the derivatives of $(N_\theta(t), G_\theta(t))$ about θ satisfy (B.5).

For $1 \le i \le r-1$, take i order derivatives of (B.13) with θ, from Faa-di Bruno formula (C.3) we get

$$
\begin{aligned}
D^i_\theta \mathcal{F}_s[N, G](t,\theta) = &\sum_{\substack{i_1+i_2+i_3=i,\\ 0\le i_k \le 1}} \frac{i!}{i_1! i_2! i_3!}\\
&\cdot \int_0^t U^s(t-z) D^{i_1}_\theta B^s(K(\Phi_z(\theta))) D^{i_2}_\theta (Id + G_\theta(z)) D^{i_3}_\theta N_\theta(z) dz.
\end{aligned}
\tag{B.15}
$$

From Faa-di Bruno formula (C.3), (B.1) and (B.11) we have

$$\begin{aligned}&\|D_\theta^{i_1} B^s(K(\Phi_z(\theta)))\|_{X_c^{\otimes i_1}, Y_\theta^s} \le C_h \epsilon e^{-i_1 \epsilon^{\frac{-i_1}{4r}}} e^{i_1(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})z},\\ &\|D_\theta^{i_2}(Id + G_\theta(z))\|_{X_c^{\otimes i_2}, \mathscr{L}(X^s_{\Phi_z(\theta)}, X^{cu}_{\Phi_z(\theta)})} \le 2.\end{aligned} \tag{B.16}$$

Then from (B.11), (B.14)-(B.16) and the smallness of ϵ we obtain

$$\begin{aligned}\|D_\theta^i \mathcal{F}_s[N,G](t,\theta)\|_{X_c^{\otimes i}, \mathscr{L}(X_\theta^s, X^s_{\Phi_t(\theta)})} &\le \sum_{\substack{i_1+i_2+i_3=i,\\ 0\le i_k \le i}} \frac{i!}{i_1!i_2!i_3!} 2C_h^3 \epsilon^2 e^{-i_1 \epsilon^{\frac{-i_1}{4r}}}\\ &\quad\cdot \int_0^t e^{-(\widetilde{\beta}_1 + \epsilon^{\frac{1}{2}})(t-z)} (t-z)^{-\alpha_1} e^{i_1(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})z} e^{-[\widetilde{\beta}_1 - i_3(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]z} dz\\ &\le C_r C_h^3 \epsilon^{2-\frac{r-1}{4r}} \int_0^t e^{-\widetilde{\beta}_1(t-z)}(t-z)^{-\alpha_1} e^{-[\widetilde{\beta}_1 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]z} dz\\ &= C_r C_h^3 \epsilon^{2-\frac{r-1}{4r}} e^{-[\widetilde{\beta}_1 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t} \int_0^t (t-z)^{-\alpha_1} e^{-i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})(t-z)} dz\\ &\le C_r C_h^3 \epsilon^{2-\frac{r-1}{4r}} e^{-[\widetilde{\beta}_1 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t} \left[\frac{1}{1-\alpha_1} + \frac{1}{i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})}\right]\\ &\le C_h \epsilon e^{-[\widetilde{\beta}_1 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t},\end{aligned} \tag{B.17}$$

where C_r is a bounded constant depending only on r, (it stands, sometimes, for different value in different place). For the derivatives about θ of $\mathcal{F}_{cu}$ we have

$$\begin{aligned}D_\theta^i \mathcal{F}_{cu}[N,G](0,\theta) = &- \sum_{\substack{i_1+i_2+i_3=i,\\ 0\le i_k \le i}} \frac{i!}{i_1!i_2!i_3!}\\ &\cdot \int_0^\infty U^{cu}(-t) D_\theta^{i_1} B^s(K(\Phi_t(\theta))) D_\theta^{i_2}(Id + G_\theta(t)) D_\theta^{i_3} N_\theta(t) dt.\end{aligned}$$

Then with the analogue calculations in (B.17) we obtain

$$\begin{aligned}\Bigg\| \sum_{\substack{i_1+i_2+i_3=i,\\ 0\le i_k \le i}} \frac{i!}{i_1!i_2!i_3!} \int_0^\infty U^c(-t) D_\theta^{i_1} B^c(K(\Phi_t(\theta)))\\ \cdot D_\theta^{i_2}(Id + G_\theta(t)) D_\theta^{i_3} N_\theta(t) dt \Bigg\|_{X_c^{\otimes i}, \mathscr{L}(X_\theta^s, X_\theta^c)}\end{aligned}$$

$$\leq \frac{C_r C_h^3 \epsilon^{2-\frac{r-1}{4r}}}{\widetilde{\beta}_1 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}}) - \widetilde{\beta}_3^-]}$$
$$\leq \epsilon^{\frac{1}{2}},$$

and

$$\Bigg\| \sum_{\substack{i_1+i_2+i_3=i,\\ 0\leq i_k\leq i}} \frac{i!}{i_1!i_2!i_3!} \int_0^\infty U^u(-t) D_\theta^{i_1} B^u(K(\Phi_t(\theta)))$$
$$\cdot D_\theta^{i_2}(Id + G_\theta(t)) D_\theta^{i_3} N_\theta(t) dt \Bigg\|_{X_c^{\otimes i}, \mathscr{L}(X_\theta^s, X_\theta^u)}$$
$$\leq C_r C_h^3 \epsilon^{2-\frac{r-1}{4r}} \Big[\frac{1}{1-\alpha_2} + \frac{1}{\widetilde{\beta}_1 + \widetilde{\beta}_2 - i(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})}\Big]$$
$$\leq \epsilon^{\frac{1}{2}}.$$

That is

$$\| D_\theta^i \mathcal{F}_{cu}[N, G](0, \theta) \|_{X_c^{\otimes i}, \mathscr{L}(X_\theta^s, X_\theta^{cu})} \leq \epsilon^{\frac{1}{2}}.$$

(Step 2): $\mathcal{F}$ is a contraction in $\mathcal{H}$. Take any $(N, G), (\widetilde{N}, \widetilde{G}) \in \mathcal{H}$, with the same calculations in Appendix A we obtain

$$d\big(\mathcal{F}[N, G], \mathcal{E}[\widetilde{N}, \widetilde{G}]\big) < c d\big((N, G), (\widetilde{N}, \widetilde{G})\big),$$

where $c = 2C_h^2 \epsilon \kappa$ with

$$\kappa = \max\Big\{\frac{1}{\widetilde{\beta}_1 + \epsilon^{\frac{1}{2}} - \widetilde{\beta}_1} + \frac{1}{1-\widetilde{\alpha}_1}, \frac{1}{\widetilde{\beta}_1 - (\widetilde{\beta}_3^- + \epsilon^{\frac{1}{2}})}, \frac{1}{\widetilde{\beta}_2 + \epsilon^{\frac{1}{2}} + \widetilde{\beta}_1} + \frac{1}{1-\widetilde{\alpha}_2}\Big\}$$
$$= \frac{1}{\epsilon^{\frac{1}{2}}} + \frac{1}{1-\alpha_1}.$$

Then by the smallness of ϵ we know that $c = 2C_h^2 \epsilon \delta \kappa < \frac{1}{2}$. That is, $\mathcal{F}$ is a C^0-contraction operator in $\mathcal{H}$, so there is a unique solution of (B.13) $(N_\theta^*(t), G_\theta^*(t))$ in the C^0 closure of $\mathcal{H}$. So $N_\theta^*(t)$ and $G_\theta^*(t)$ is $C^{r-2+Lip}$ in θ and satisfies the estimates in $\mathcal{H}$ with $j \leq r-2$. Moreover, from (B.15) we have

$$\sup_{\theta\neq\vartheta} (\theta - \vartheta)^{-1} \big(D_\theta^{r-2} N_\theta^*)(t) - D_\theta^{r-2} N_\vartheta^*(t)\big)$$
$$= \sup_{\theta\neq\vartheta} (\theta - \vartheta)^{-1} \sum_{\substack{i_1+i_2+i_3=r-2,\\ 0\leq i_k\leq 1}} \frac{i!}{i_1!i_2!i_3!}$$

$$\cdot \int_0^t U^s(t-z)\big[D_\theta^{i_1} B^s(K(\Phi_z(\theta)))D_\theta^{i_2}(Id+G_\theta^*(z))D_\theta^{i_3}N_\theta^*(z)$$
$$- D_\theta^{i_1} B^s(K(\Phi_z(\vartheta)))D_\vartheta^{i_2}(Id+G_\vartheta^*(z))D_\vartheta^{i_3}N_\vartheta^*(z)\big]dz.$$

With the similar calculations in (B.17) we can obtain

$$Lip_\theta\big(D_\theta^{r-2}N_\theta^*(t)\big|_{X_c^{\otimes(r-2)},\mathscr{L}(X_\theta^s,X_{\Phi_t(\theta)}^s)}\big) \le C_h\epsilon e^{-[\widetilde{\beta}_1-(r-1)(\widetilde{\beta}_3^+ +\epsilon^{\frac{1}{4r}})]t}$$

Similarly, we can also obtain

$$Lip_\theta\big(D_\theta^{r-2}G_\theta^*\big|_{X_c^{\otimes(r-2)},\mathscr{L}(X_\theta^s,X_\theta^{cu})}\big) \le C_h\epsilon^{\frac{1}{2}}.$$

From Appendix A we know that

$$U_\theta^s(t) = W_\theta^s(t) + G_\theta(t)W_\theta^s(t), \tag{B.18}$$

where $W_\theta^s(t) = N_\theta(t)W_\theta^s(0)$ with $\|W_\theta^s(0)\|_{X_\theta^s} \le 1$. Then from the definition of $\mathcal{H}$, Lemma C.1 and (B.18) we know that $U_\theta^s(t)$ satisfies the first inequality in (B.5).

To prove the second estimate in (B.5) we need the following inequality:

$$\begin{aligned} t^{-\alpha}\int_0^t e^{-\beta(t-z)}(t-z)^{-\alpha}z^\alpha dz &\le \int_0^t e^{-\beta(t-z)}(t-z)^{-\alpha}dz \\ &\le \big(\frac{1}{\beta}+\frac{1}{1-\alpha}\big), \quad \beta>0, \quad 0<\alpha<1. \end{aligned} \tag{B.19}$$

With (B.19) and the same tricks to obtain the first inequality in (B.5) we get the second inequality in (B.5). We omit the details.

By Faa-di Bruno formula (C.3) we know that

$$D_\theta^{i_1}\big[\Phi_z(\Phi_\tau(\theta))\big] = \sum_{k=1}^{i_1} D_\theta^k\big[\Phi_z\big]\circ\Phi_\tau(\theta)\sum_{p(i_1,k)} i_1!\Pi_{j=1}^{i_1}\frac{\big(D_\theta^j\Phi_\tau(\theta)\big)^{\lambda_j}}{(\lambda_j!)(j!)^{\lambda_j}},$$

where

$$p(i_1,k) = \Big\{(\lambda_1,\cdots,\lambda_{i_1}) : \lambda_j\in\mathbb{N}, \sum_{j=1}^{i_1}\lambda_j = k, \sum_{j=1}^{i_1} j\lambda_j = i_1\Big\}.$$

Then from (B.11) we have

$$\|D_\theta^{i_1}B^s(K(\Phi_z(\Phi_\tau(\theta)))\|_{X_c^{\otimes i_1},Y_{\Phi_\tau(\theta))}^s} \le \epsilon C_h^{i_1+1}c_{i_1}\epsilon^{\frac{-i_1}{2r}}e^{i_1(\widetilde{\beta}_3^+ +\epsilon^{\frac{1}{4r}})z}e^{i_1(\widetilde{\beta}_3^+ +\epsilon^{\frac{1}{4r}})\tau},$$

where c_{i_1} is constant depending only on i_1. With the same calculations to obtain the first estimate in (B.5) (with the above inequality in place of the first inequality in (B.16)), we obtain the two inequalities in (B.6).

For $U_\theta^{cu}(t)$ and $U_{\Phi_\tau(\theta)}^{cu}(t)$, by changing the direction of time t we have, for $j = 1, \cdots, r-2$,

$$
\begin{aligned}
&\|D_\theta^j U_\theta^u(t)\|_{X_c^{\otimes j}, \mathscr{L}(Y_\theta^u, X_{\Phi_t(\theta)}^u)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_2 - j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})]|t|} t^{-\alpha_2}, \quad t \le 0,\\
&\|D_\theta^j U_\theta^u(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_\theta^u, X_{\Phi_t(\theta)}^u)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_2 - j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})]|t|}, \quad t \le 0,\\
&\|D_\theta^j U_{\Phi_\tau(\theta)}^u(t)\|_{X_c^{\otimes j}, \mathscr{L}(Y_{\Phi_\tau(\theta)}^u, X_{\Phi_{t+\tau}(\theta)}^u)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_2 - j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})]t + j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau} t^{-\alpha_2}, \quad \tau \ge 0, t \le 0,\\
&\|D_\theta^j U_{\Phi_\tau(\theta)}^u(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_{\Phi_\tau(\theta)}^u, X_{\Phi_{t+\tau}(\theta)}^u)} \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_2 - j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}}]t + j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau}, \quad \tau \ge 0, t \le 0,\\
&\|D_\theta^j U_\theta^c(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_\theta^u, X_{\Phi_t(\theta)}^u)} \le \widetilde{C}_h \epsilon e^{j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})t}, \quad t \ge 0,\\
&\|D_\theta^j U_\theta^c(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_\theta^c, X_{\Phi_t(\theta)}^c)} \le \widetilde{C}_h \epsilon e^{j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})|t|}, \quad t \le 0,\\
&\|D_\theta^j U_{\Phi_\tau(\theta)}^c(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_{\Phi_\tau(\theta)}^c, X_{\Phi_{t+\tau}(\theta)}^c)} \le \widetilde{C}_h \epsilon e^{j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})t + j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})|\tau|}, \ \tau \le 0, t \ge 0,\\
&\|D_\theta^j U_{\Phi_\tau(\theta)}^c(t)\|_{X_c^{\otimes j}, \mathscr{L}(X_{\Phi_\tau(\theta)}^c, X_{\Phi_{t+\tau}(\theta)}^c)} \le \widetilde{C}_h \epsilon e^{j(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})|t| + j(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau}, \ \tau \ge 0, t \le 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_\theta^u(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_\theta^u, X_{\Phi_t(\theta)}^u)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]|t|}, \ t \le 0, \qquad \text{(B.20)}\\
&Lip_\theta\big(D_\theta^{r-2} U_\theta^u(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(Y_\theta^u, X_{\Phi_t(\theta)}^u)}\big) \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]|t|} |t|^{-\widetilde{\alpha}_2}, \ t \le 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^u(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_{\Phi_\tau(\theta)}^u, X_{\Phi_{t+\tau}(\theta)}^u)}\big)\\
&\qquad \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]|t| + (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau}, \ t \le 0, \tau \ge 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^s(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(Y_{\Phi_\tau(\theta)}^u, X_{\Phi_{t+\tau}(\theta)}^u)}\big)\\
&\qquad \le \widetilde{C}_h \epsilon e^{-[\widetilde{\beta}_1 - (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]|t| + (r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})\tau} |t|^{-\widetilde{\alpha}_2}, t \le 0, \tau \ge 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_\theta^c(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_\theta^c, X_{\Phi_t(\theta)}^c)}\big) \le \widetilde{C}_h \epsilon e^{(r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})]t}, \ t \ge 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_\theta^c(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_\theta^c, X_{\Phi_t(\theta)}^c)}\big) \le \widetilde{C}_h \epsilon e^{(r-1)(\widetilde{\beta}_3^- + \epsilon^{\frac{1}{4r}})]t}, \ t \le 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^c(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_{\Phi_\tau(\theta)}^c, X_{\Phi_{t+\tau}(\theta)}^c)}\big) \le \widetilde{C}_h \epsilon e^{(r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})(t+|\tau|)}, \ t \ge 0, \tau \le 0,\\
&Lip_\theta\big(D_\theta^{r-2} U_{\Phi_\tau(\theta)}^c(t)|_{X_c^{\otimes(r-2)}, \mathscr{L}(X_{\Phi_\tau(\theta)}^c, X_{\Phi_{t+\tau}(\theta)}^c)}\big) \le \widetilde{C}_h \epsilon e^{(r-1)(\widetilde{\beta}_3^+ + \epsilon^{\frac{1}{4r}})(|t|+\tau)}, \ t \le 0, \tau \ge 0. \quad \square
\end{aligned}
$$

Appendix C. Spaces of differentiable functions in Banach spaces

In this appendix, we collect some classical (but perhaps not so well known) results about spaces of differentiable functions in Banach spaces which are useful for us in establishing the contraction mapping arguments and in obtaining the results. Particular importance is given to the dependence on parameters in these spaces and on the closure properties.

C.1. Basic definitions

Definition C.1. Let X, Y be two Banach spaces. Let $O \subset X$ be an open set. We will denote by $C^r(O, Y)$ the space of all functions from X to Y which possess uniformly bounded and uniformly continuous derivatives of orders $0, 1, \cdots, r$. We endow $C^r(O, Y)$ with the norm of the supremum of all the derivatives, e.g.

$$\|f\|_{C^r(O,Y)} = \max_{0\le i\le r} \sup_{\xi\in O} |[D^i f](\xi)|_{X^{\otimes i},Y}. \tag{C.1}$$

The $|A|_{X^{\otimes i},Y} \equiv \sup_{|\xi_1|_X=1,\ldots|\xi_i|_X=1} |A(\xi_1,\ldots,\xi_i)|_Y$ is the usual norm of symmetric multilinear functions from X taking values in Y. As it is well known, the norm (C.1) makes $C^r(O, Y)$ a Banach space.

Definition C.2. We will denote by $C^{r-1+Lip}(O, Y)$ the space of functions in $C^{r-1}(O, Y)$ whose $(r-1)-th$ derivative is Lipschitz. The Lipschitz constant is

$$Lip_{O,Y} D^{r-1} f = \sup_{\xi\neq\zeta} \frac{|D^{r-1} f(\xi) - D^{r-1} f(\zeta)|_{X^{\otimes(r-1)},Y}}{\|\xi-\zeta\|_X}$$

and the norm in $C^{r-1+Lip}(O, Y)$ is the max of the C^{r-1} norm and $Lip_{O,Y} D^{r-1} f$.

Again this norm makes $C^{r-1+Lip}$ into a Banach space.

We note that since O may be not compact, this definition is different from the Whitney definition in which the topology is given by seminorms of suprema in compact sets. We will not use the Whitney definition of C^r in this paper.

Definition C.3. An open set O is called a compensated domain if there is a constant such given $x, y \in O$, there is a C^1 path γ contained in O joining x, y such that $|\gamma| \le C\|x-y\|$.

For O a compensated domain, we have the mean value theorem

$$\|f(x) - f(y)\|_Y \le C\|f\|_{C^1(O,Y)}\|x-y\|_X. \tag{C.2}$$

In particular, C^1 functions in a compensated domain are Lipschitz. It is not difficult to construct non-compensated domains with C^1 functions which are not Lipschitz.

Of course a convex set is compensated and the compensation constant is 1. In our paper, we will just be considering domains which are balls or full spaces. See [25] for the effects of the compensation constants in many problems of the function theory.

C.2. Hadamard interpolation theorem

We have the following result:

Theorem C.1. *Let O be a compensated domain. Let $f \in C^r(O, Y)$. Then if we define $\eta(r) \equiv \|f\|_{C^r(O,Y)}$, we have that $\log(\eta(r))$ is convex in r. That is for $0 \le \theta \le 1$, $0 \le a, b \le r$, we have*

$$||f||_{C^{\theta a+(1-\theta)b}(O,Y)} \le C\|f\|^{\theta}_{C^a(O,Y)}\|f\|^{1-\theta}_{C^b(O,Y)}.$$

A proof of Theorem C.1 extending for non-integer values of r for suitable definitions of C^r can be found in [25]. In finite dimensional spaces it was proved in [37]. See also [47]. We also note that the interpolation is a consequence of the existence of *Smoothing operators* [71].

For us, the following corollary will be important.

Corollary C.1. *Assume that $\{f_n\}_{n=1}^{\infty} \subset C^r(O, Y)$ is such that $\|f_n\|_{C^r(O,Y)} \le M$. Assume that $\|f_n - f_{n+1}\|_{C^0(O,Y)} \le C\kappa^n$. Then,*

$$\|f_n - f_{n+1}\|_{C^{r-1}(O,Y)} \le (2M)^{(r-1)/r} C^{1/r}\kappa^{n/r}.$$

Of course, even if the corollary is true for all values of κ, it is more interesting for $k < 1$ as it happens in contraction mapping principles.

The corollary shows that a C^0-contraction mapping a C^r-bounded set of functions to itself, is a contraction in $C^{r'}$ $(r' < r)$ norm. Of course, in particular it is a contraction in C^{r-1} norm. Using the uniform convergence of the C^{r-1} derivative and that the C^{r-1} derivative is Lipschitz, shows that the limit is in $C^{r-1+Lip}$. This gives an alternative proof of the application of Lemma C.1 to our problems.

Remark C.3. As we mentioned above, the interpolation Theorem C.1 extends for non-integer values of r' with a suitable definition of the norm. With this definition, we have Corollary C.1 for all values of $r' < r$. The same applies to the following result Corollary C.2.

A further corollary of Corollary C.1 is

Corollary C.2. *Assume that $\{f_n\}_{n=1}^{\infty} \subset C^r(O, Y)$ is such that $\|f_n\|_{C^r(O,Y)} \le M$.*
Assume that for some $f_\infty \in C^0(O, Y)$ $\|f_n - f_\infty\|_{C^0(O,Y)} \to 0$. Then, for all $r' < r$,

$$f_\infty \in C^{r'}$$

and $f_n \to f_\infty$ in $C^{r'}$.

Proof. Given a subsequence f_{n_i} of f_n we can obtain a further subsequence $f_{n_{i_j}}$ which satisfies $||f_{n_{i_j}} - f_{n_{i_{j+1}}}||^0_C \le (1/2)^j$. By Corollary C.1 we get that $f_{n_{i_j}}$ converges in $C^{r'}$. This limit obtained in $C^{r'}$ sense has to be f_∞. □

It is an exercise in metric space topology that, if for all subsequences we can obtain a subsequence that converges and all these limits are the same, then, the original sequence converges.

C.3. Lanford's closure lemma

The following result is [49, Proposition A2].

Lemma C.1. *Let O be a convex set inside of a Banach space X. Let Y be another Banach space. Denote by $\mathcal{B}$ the set of functions*

$$\mathcal{B} = \{u : O \to Y,\ \|D^j u\| \le 1,\ 0 \le j \le r-1,\ Lip(D^{r-1}u) \le 1\}.$$

Assume that $\{f_n\} \subset \mathcal{B}$ and that for each value $x \in O$, $f_n(x)$ converges weakly to $f_\infty(x)$. Then, $f_\infty \in C^{r-1+\text{Lipschitz}}$ and for $1 \le j \le r-1$, $D^j f_n$ converges uniformly to $D^j f_\infty$. As a consequence, $f_\infty \in \mathcal{B}$.

The assumption of weak pointwise converge is, of course, much weaker than the assumption of uniform convergence, which is what will appear in our applications.

The statement in [49, Proposition A2] is only for the case $O = X$, but the proof consists in studying the behavior of f_n on one dimensional segments in O, (then the result is deduced from Ascoli-Arzelá theorem). Then, one checks that the result is uniform in the segment considered. The proof works without any change for convex O. With only minor modifications the argument works also when O is a compensated domain.

C.4. Faa Di Bruno formula

Lemma C.2. *Let $g(x)$ be defined on a neighborhood of x_0 and have derivatives up to order n at x_0. Let $f(y)$ be defined on a neighborhood of $y_0 = g(x_0)$ and have derivatives up to order n at y_0. Then, the n-th derivative of the composition $h(x) = f[g(x)]$ at x_0 is given by the formula*

$$h_n = \sum_{k=1}^{n} f_k \sum_{p(n,k)} n!\Pi_{i=1}^{n} \frac{g_i^{\lambda_i}}{(\lambda_i!)(i!)^{\lambda_i}}. \tag{C.3}$$

In the above expression, we set

$$h_n = \frac{d^n}{dx^n} h(x_0), \quad f_k = \frac{d^k}{dy^k} f(y_0), \quad g_i = \frac{d^i}{dy^i} g(x_0)$$

and

$$p(n,k) = \left\{ (\lambda_1, \cdots, \lambda_n) : \lambda_i \in \mathbb{N}, \sum_{i=1}^{n} \lambda_i = k, \sum_{i=1}^{n} i\lambda_i = n \right\}.$$

The formula (C.3) without an explicit expression of the combinatorial coefficients was obtained in [4].

The explicit computation of the combinatorial coefficients is less straightforward, but can be found in [3].

C.5. Functions of several variables and partial regularity

In several applications, we have to consider functions of several variables. One can think of one as the regularity of the function and the other is the regularity with respect to parameters.

In some of our applications it is easy to estimate the regularity in each of the variables since they play a different role.

The following result shows that if we can estimate the derivatives in each of the variables, we can obtain automatically also the mixed derivatives.

Lemma C.3. *Let X_1, X_2, Y be Banach spaces. $O_1 \subset X_1$, $O_2 \subset X_2$ be convex, bounded sets.*

Let $f : O_1 \times O_2 \to Y$ be a continuous function.

Assume that for all $x_1 \in O_1, x_2 \in O_2$, $i, j \le r$, we have

$$\begin{aligned} ||\partial^i_{x_1} f(x_1, x_2)|| &\le M < \infty, \\ ||\partial^j_{x_2} f(x_1, x_2)|| &\le M < \infty. \end{aligned} \tag{C.4}$$

Then, for every n, m such that $n + m < r$, we have that the function f admits mixed partial derivatives $\partial^n_{x_1} \partial^m_{x_2} f$. Furthermore, we have

$$\sup_{x_1 \in O_1, x_2 \in O_2} \partial^n_{x_1} \partial^m_{x_2} f(x_1, x_2) \le \Gamma(M, O_1, O_2).$$

Of course, in analytic regularity, the fact that analyticity in several complementary directions is the celebrated Hartog's theorem [48]. In our case, we are assuming that the functions are bounded, but the Hartog's theorem does not need that assumption. The Hartog's theorem is much easier under the assumption that the functions are bounded.

For finite dimensional spaces X_1, X_2, this result is a classical result in the theory of Riesz potentials. A modern proof can be found in [47, Lemma 9.1]. This result is the basis of many results in the regularity theory of elliptic equations. There are also results when the number of derivatives is asymmetric and also for fractional derivatives.

Results of this type were found useful in the theory of Anosov systems when the partial derivatives along the coordinate axis are generalized to be partial derivatives along stable and unstable foliations [24, Lemma 2.5]. A more elementary and more general proof based on the theory of Morrey-Campanato spaces is in [44]. A very elementary proof using just the converse Taylor theorem and generalizing to some fractal sets is in [22]. To go from the finite dimensional proofs above to the infinite dimensional case, it suffices to take finite dimensional sections and observe that the bounds obtained are independent of the finite dimensional space considered.

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