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Semilattice sums of algebras and Mal'tsev products of varieties

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Abstract. The Mal'tsev product of two varieties of similar algebras is always a quasivariety. We consider the question of when this quasivariety is a variety. The main result asserts that if $\mathcal{V}$ is a strongly irregular variety with no nullary operations and at least one non-unary operation, and $\mathcal{S}$ is the variety, of the same type as $\mathcal{V}$, equivalent to the variety of semilattices, then the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ is a variety. It consists precisely of semilattice sums of algebras in $\mathcal{V}$. We derive an equational base for the product from an equational base for $\mathcal{V}$. However, if $\mathcal{V}$ is a regular variety, then the Mal'tsev product may not be a variety. We discuss various applications of the main result, and examine some detailed representations of algebras in $\mathcal{V} \circ \mathcal{S}$.

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1. Introduction

Let $\mathcal{K}$ be a quasivariety of Ω-algebras, and $\mathcal{Q}$ and $\mathcal{R}$, two of its subquasivarieties. Assume additionally that $\mathcal{R}$-algebras are idempotent. Then the *Mal'tsev product, $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$, of $\mathcal{Q}$ and $\mathcal{R}$ relative to $\mathcal{K}$*, consists of $\mathcal{K}$-algebras A, with a congruence θ, such that A/θ is in $\mathcal{R}$, and each θ-class a/θ is in $\mathcal{Q}$. Note that by the idempotency of $\mathcal{R}$, each θ-class is always a subalgebra of A. If $\mathcal{K}$ is the variety of all Ω-algebras, then the Mal'tsev product $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is called simply

the *Mal'tsev product* of $\mathcal{Q}$ and $\mathcal{R}$, and is denoted by $\mathcal{Q} \circ \mathcal{R}$. It follows from results of Mal'tsev [15] that the Mal'tsev product $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is a quasivariety.

In this paper we are interested in Mal'tsev products $\mathcal{V} \circ \mathcal{S}$ in which $\mathcal{V}$ is a variety of Ω-algebras and $\mathcal{S}$ is the variety, of the same type as $\mathcal{V}$, equivalent to the variety of semilattices. A member of $\mathcal{V} \circ \mathcal{S}$ is a disjoint union of $\mathcal{V}$-algebras over a homomorphic image, which has the structure of a semilattice. Such algebras are known as semilattice sums of $\mathcal{V}$-algebras. The product $\mathcal{V} \circ \mathcal{S}$ is always a quasivariety. However, until recently it was not known under what conditions it is a variety. The main result of this paper (Theorem 6.3) asserts that if $\mathcal{V}$ is a strongly irregular variety of a type with no nullary operations and at least one non-unary operation, then the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ is a variety. We also show how to build an equational base for this variety from an equational base for $\mathcal{V}$. In addition we provide an example showing that in the case that $\mathcal{V}$ is regular, $\mathcal{V} \circ \mathcal{S}$ does not need to be a variety.

Next we present examples of (relative) Mal'tsev products which exhibit different aspects of such products. In fact, two special examples inspired us to investigate semilattice sums in general: one concerning certain semilattice sums of lattices and one involving certain semilattice sums of Steiner quasigroups. (See Section 7.)

As a converse process to the decomposition of an algebra A from $\mathcal{V} \circ \mathcal{S}$ into a semilattice sum of $\mathcal{V}$-algebras, some construction techniques are available to recover an algebra A from its summands and a semilattice quotient. Such techniques include, for example, Płonka sums, and the more general Lallement sums [24, Ch. 4]. We recall some of the basic constructions of this type, and discuss their applicability for representing algebras in the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$.

In [15], Mal'tsev proved that if $\mathcal{K}$ is a congruence permutable quasivariety with a constant term whose value is always idempotent, then the relative Mal'tsev product of any two subvarieties of $\mathcal{K}$ is a variety. This result was extended by Iskander [13] to weakly congruence permutable varieties of algebras.

We begin with introductory information on semilattice sums. In Sections 3 and 4 we provide some insight into the quasivariety of semilattice sums of $\mathcal{V}$-algebras. In particular we construct, for any variety $\mathcal{V}$, a supervariety $\mathcal{V}^p$, called the *prolongation of* $\mathcal{V}$, containing $\mathcal{V} \circ \mathcal{S}$. We prove (Proposition 4.2) that free $\mathcal{V}^p$-algebras lies in $\mathcal{V} \circ \mathcal{S}$.

The main result, Theorem 6.3, asserts that if $\mathcal{V}_t$ is a variety satisfying an identity of the form $t(x, y) = x$ (i.e., a strongly irregular variety, see Definition 2.1), then $\mathcal{V}_t \circ \mathcal{S} = \mathcal{V}_t{}^p$. As a consequence, $\mathcal{V}_t \circ \mathcal{S}$ is again a variety. The proof is divided into two parts. In Section 5 we treat the special case of the variety $\mathcal{T}_t$ of Ω-algebras defined by a single identity $t(x, y) = x$. Then in Section 6 we leverage that observation to obtain the result for an arbitrary strongly irregular variety. We also illustrate the necessity of strong irregularity with an example showing that $\mathcal{S} \circ \mathcal{S}$ fails to be a variety.

Section 7 discusses a number of examples of Mal'tsev products and relative Mal'tsev products. Relative products are often more interesting, as they

may be used to represent algebras in more familiar varieties. The paper finishes with a discussion concerning the detailed representation of algebras in the Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$. Here a version of a construction of algebras called the Lallement sum [23] is used. In particular, certain very special Lallement sums provide a good detailed representation for algebras in $\mathcal{V}_t \circ \mathcal{S}$, in the case when $\mathcal{V}_t$-algebras have units.

We use notation and conventions similar to those of [23,24]. For details and further information concerning quasivarieties and Mal'tsev products of quasivarieties we refer the reader to [15] and [16], and then also to [1] and [24, Ch. 2]; for universal algebra, see [2] and [24]; in particular, for methods of constructing algebras as semilattice sums see [21] and [24, Ch. 4]; for semigroup theory, see [6,11,12]; for barycentric algebras and convex sets, we refer the reader to the monographs [23,24].

2. Background

Let $\tau \colon \Omega \to \mathbb{N}$ be a similarity type of Ω-algebras with no nullary operation symbols and containing at least one non-unary operation symbol. Such a similarity type is called *plural* [24]. For a positive integer n, let T_n be the set of all n-ary Ω-terms containing precisely n different variables. They may be denoted $x_1, x_2, \ldots x_n$ or by some other letters like y or z.

Definition 2.1. An identity is called *regular* if precisely the same variables appear on both sides. At the opposite extreme, an identity of the form $t(x, y) = x$, for some $t \in T_2$ is called *strongly irregular*. A variety is regular if it satisfies only regular identities. It is strongly irregular if it satisfies at least one strongly irregular identity.

Let $t \in T_2$. We denote by $\mathcal{T}_t$ the variety defined by the single identity $t(x, y) = x$. We use the notation $\mathcal{V}_t$ to indicate a typical subvariety of $\mathcal{T}_t$, that is, an arbitrary variety satisfying the strongly irregular identity $t(x, y) = x$. We often find it convenient to write $x \cdot y$ (or $x \cdot_t y$) in place of $t(x, y)$.

Let us note that the variety of semilattices satisfies all the regular identities of the type with one binary operation.

Definition 2.2. The variety $\mathcal{S}$ of Ω-*semilattices* is defined to be the variety of Ω-algebras satisfying all regular identities.

It is fundamental to the subject that in a plural similarity type, $\mathcal{S}$ is the unique variety that is equivalent to semilattices. Fix any $t \in T_2$ and write $x \cdot y$ in place of $t(x, y)$. Then for any $S \in \mathcal{S}$, the algebra $\langle S, \cdot \rangle$ is a semilattice. Conversely, for any basic n-ary operation symbol $\omega \in \Omega$ we have that

$$x_1 \ldots x_n \omega = x_1 \cdot \ldots \cdot x_n \tag{2.1}$$

holds in S.

Definition 2.3. Let $\mathcal{V}$ be a variety of Ω-algebras. An Ω-algebra A is a *semilattice sum* of $\mathcal{V}$-algebras if A has a congruence ϱ such that the quotient $S = A/\varrho$

is an Ω-semilattice (or briefly a semilattice) and every congruence block is a $\mathcal{V}$-algebra.

Note that since the quotient A/ϱ is a semilattice, it is idempotent. Hence the congruence blocks of ϱ are always subalgebras, and A is a disjoint sum of these subalgebras. We will denote such a sum by $\bigsqcup_{s \in S} A_s$, where the A_s's are the congruence blocks of ϱ and S is (isomorphic to) A/ϱ. The class of semilattice sums of $\mathcal{V}$-algebras forms the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ of the varieties $\mathcal{V}$ and $\mathcal{S}$, and is known to be a quasivariety. (See Mal'tsev [15,16] as well as Bergman [1] and also Romanowska, Smith [24, Ch. 3].) Whenever A is a semilattice sum of $\mathcal{V}$-algebras, it suffices to take ϱ to be the semilattice replica congruence of A, i.e., the smallest congruence of A with the corresponding quotient a semilattice. (See [1], Proposition 8.) Furthermore, in the case that $\mathcal{V}$ is a strongly irregular variety, the semilattice replica congruence of A is the unique congruence ϱ providing a decomposition of A into a semilattice sum of $\mathcal{V}$-subalgebras (see [24, Sec. 3.3]).

In this paper we are especially interested in semilattice sums of $\mathcal{T}_t$-algebras. Since $\mathcal{T}_t$ is a strongly irregular variety, each algebra A in $\mathcal{T}_t \circ \mathcal{S}$ decomposes into a semilattice sum $\bigsqcup_{s \in S} A_s$ of $\mathcal{T}_t$-algebras A_s in a unique way over the quotient S of A given by its semilattice replica congruence.

The variety $\mathcal{T}_t$ contains other irregular varieties. In particular, if $\mathcal{V}_t$ is a subvariety of $\mathcal{T}_t$, then it is known that $\mathcal{V}_t$ is defined by a set Σ_t consisting of a set Γ_t of regular identities and the single identity $t(x, y) = x$. (See e.g. [21] and [24, Ch. 4].) The Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$ consists of algebras which are semilattice sums of $\mathcal{V}_t$-subalgebras. Moreover it contains $\mathcal{V}_t$ and $\mathcal{S}$ as subvarieties, and is contained in the quasivariety $\mathcal{T}_t \circ \mathcal{S}$. The smallest variety $\mathsf{V}(\mathcal{V}_t, \mathcal{S})$ containing both $\mathcal{V}_t$ and $\mathcal{S}$ is called the *regularization* of $\mathcal{V}_t$ and is denoted by $\widetilde{\mathcal{V}}_t$. (See e.g. [24, Ch. 4].) The smallest quasivariety $\mathsf{Q}(\mathcal{V}_t, \mathcal{S})$ containing both $\mathcal{V}_t$ and $\mathcal{S}$ is called the *quasi-regularization* of $\mathcal{V}_t$ and is denoted by $\widetilde{\mathcal{V}}_t^q$. (See [4].) Note that $\mathsf{V}(\mathcal{V}_t, \mathcal{S})$ and $\mathsf{Q}(\mathcal{V}_t, \mathcal{S})$ are not necessarily equal. For each $t(x, y) \in T_2$, we have the following chain of quasivarieties:

$$\mathcal{V}_t, \mathcal{S} \subseteq \widetilde{\mathcal{V}}_t^q \subseteq \widetilde{\mathcal{V}}_t \subseteq \mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{T}_t \circ \mathcal{S}.$$

In particular, if the set Γ_t of regular identities is empty, then $\mathcal{V}_t$ coincides with $\mathcal{T}_t$.

3. Some varieties related to the quasivariety $\mathcal{T}_t \circ \mathcal{S}$

Recall that, for $x \cdot y = t(x, y) \in T_2$, the subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$ is defined by a set Σ_t consisting of the identity $x \cdot y = x$ and a set of regular identities Γ_t. The best known subvariety of $\mathcal{V}_t \circ \mathcal{S}$, different from $\mathcal{V}$ and $\mathcal{S}$, is the regularization $\widetilde{\mathcal{V}}_t$ of the variety $\mathcal{V}_t$. We will briefly recall the basic facts concerning the regularization.

The variety $\mathcal{V}_t$ can be viewed as a category in which the morphisms are simply the homomorphisms between $\mathcal{V}_t$-algebras. Any ordered set, S, can be viewed as a category whose objects are the elements of the set. If $r, s \in S$ then there is a unique morphism from r to s if $r \geq s$. Otherwise there are no

morphisms from r to s. Of course, any semilattice can be ordered by considering the basic operation to be the infimum.

Suppose that S is a semilattice and there is a covariant functor

$$F\colon S \to \mathcal{V}_t;\ (r \geq s) \mapsto (\varphi_{r,s}\colon A_r \to A_s).$$

Then the disjoint sum $\bigsqcup_{s \in S} A_s$ may be viewed as the semilattice-ordered system

$$\bigl(\{A_s \mid s \in S\};\ \{\varphi_{r,s} \mid r, s \in S, r \geq s\}\bigr), \tag{3.1}$$

where the Ω-operations on the sum are defined by

$$\omega\colon A_{s_1} \times \cdots \times A_{s_n} \to A_s;(a_{s_1}, \ldots, a_{s_n}) \mapsto a_{s_1}\varphi_{s_1,s} \cdots a_{s_n}\varphi_{s_n,s}\,\omega$$

for each (n-ary) $\omega \in \Omega$, where each $s_i \in S$ and $s = s_1 \cdot \ldots \cdot s_n$. The resulting semilattice sum is called the *Płonka sum* of the algebras A_s, and is denoted by $\sum_{s \in S} A_s$. (See [19,21], [24, Ch. 4].)

It is a remarkable fact that the class of Płonka sums of $\mathcal{V}_t$-algebras forms the *regularization* $\widetilde{\mathcal{V}}_t$ of the variety $\mathcal{V}_t$. The class $\widetilde{\mathcal{V}}_t$ is the join (in the lattice of varieties) of $\mathcal{V}_t$ and $\mathcal{S}$. Thus it satisfies precisely the regular identities true in $\mathcal{V}_t$. Its axiomatization follows from Płonka's theorem (see [21], [24, Ch. 4]). The variety $\widetilde{\mathcal{V}}_t$ is defined by the regular identities Γ_t of Σ_t and the following identities (P1) – (P5):

(P1) $x \cdot x = x$,
(P2) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
(P3) $x \cdot (y \cdot z) = x \cdot (z \cdot y)$,
(P4) $y \cdot x_1 \ldots x_n\omega = y \cdot x_1 \cdot \ldots \cdot x_n$,
(P5) $x_1 \ldots x_n\omega \cdot y = (x_1 \cdot y) \ldots (x_n \cdot y)\,\omega$,

where ω ranges through all symbols in Ω. Let us note that if Γ_t is empty, then the regularization $\widetilde{\mathcal{V}}_t$ becomes the regularization $\widetilde{\mathcal{T}}_t$, and is defined by the identities (P1) – (P5).

By results of [4] it is known that the quasi-regularization $\widetilde{\mathcal{V}}_t^q$ of $\mathcal{V}_t$ consists of Płonka sums of $\mathcal{V}_t$-algebras as in (3.1) in which the morphisms $\varphi_{r,s}$ are all injective. It coincides with the class $\mathsf{ISP}(\mathcal{V}_t \cup \mathcal{S})$ of (isomorphic copies of) subalgebras of products of algebras in $\mathcal{V}_t \cup \mathcal{S}$, and is defined, relative to $\widetilde{\mathcal{V}}_t$, by the quasi-identity

$$(x \cdot y = x\ \&\ y \cdot x = y\ \&\ x \cdot z = z \cdot x = z\ \&\ y \cdot z = z \cdot y = z) \to (x = y).$$

The congruence ϱ of a $\widetilde{\mathcal{V}}_t$-algebra A providing its decomposition into the Płonka sum of its $\mathcal{V}_t$-summands can be obtained explicitly as

$$\varrho = \{(a, b) \mid a \cdot b = a \text{ and } b \cdot a = b\}. \tag{3.2}$$

Now, let $\overline{\mathcal{T}_t}$ denote the variety defined by the identities (P1) – (P4) above. It follows immediately that $\widetilde{\mathcal{T}_t} \subseteq \overline{\mathcal{T}_t}$. For a subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$, let $\overline{\mathcal{V}_t}$ be defined by Γ_t together with (P1) – (P4). As observed by C. Bergman and D. Failing [3], $\overline{\mathcal{V}_t} \subseteq \mathcal{V}_t \circ \mathcal{S}$. That is to say, each algebra A in $\overline{\mathcal{V}_t}$ is a semilattice sum of $\mathcal{V}_t$-subalgebras. Moreover, the decomposition is given by the same formula (3.2) as is the case for the regularization. However, the $\overline{\mathcal{V}_t}$-algebras are not

necessarily Płonka sums of $\mathcal{V}_t$-algebras, though the reduct $(A, \cdot_t)$ of each $\overline{\mathcal{V}_t}$-algebra $A = \bigsqcup_{s \in S} A_s$ is a left-normal band and is the Płonka sum $\sum_{s \in S} A_s$ of the left-zero band reducts $(A_s, \cdot)$. We will call the variety $\overline{\mathcal{V}_t}$ the *pseudo-regularization* of the variety $\mathcal{V}_t$. If the operation $x \cdot y$ of an Ω-algebra (A, Ω) satisfies the conditions (P1)–(P4), then it is called a *pseudopartition operation*. If it satisfies all of (P1)–(P5) it is called a *partition operation*.

The third variety we are interested in will play an essential role in our main results.

Definition 3.1. Let σ be an identity of type τ of the form

$$y_1 \ldots y_n u = y_1 \ldots y_n v,$$

where not all the variables y_i necessarily appear in u and v.

For each $m > 0$, define

$$\sigma_m^p = \big\{\, \mathbf{x} r_1 \ldots \mathbf{x} r_n u = \mathbf{x} r_1 \ldots \mathbf{x} r_n v \mid$$
$$r_1, \ldots, r_n \in T_m, \mathbf{x} r_i = x_1 \ldots x_m r_i \,\big\}.$$

Let σ^p be the union of all σ_m^p for $m > 0$. For a set Σ of identities, define its *prolongation* to be

$$\Sigma^p = \bigcup_{\sigma \in \Sigma} \sigma^p.$$

The intent of Definition 3.1 is that, for every positive integer m, each variable y_i in σ is replaced by a term of the form $x_1 \ldots x_m r_i$. However, since $r_i \in T_m$, each term r_i must contain precisely all of the variables $x_1, \ldots, x_m$. Thus even if the identity σ is irregular, every member of σ^p will be a regular identity.

Definition 3.2. Let $\mathcal{V}$ be a variety of Ω-algebras, and let $\text{Id}(\mathcal{V})$ be the set of all equations that hold in $\mathcal{V}$. We define *the prolongation of* $\mathcal{V}$, denoted $\mathcal{V}^p$, to be the variety of all algebras satisfying $\text{Id}(\mathcal{V})^p$. Put succinctly, $\mathcal{V}^p = \text{Mod}(\text{Id}(\mathcal{V})^p)$.

In fact, in this definition, it suffices to replace $\text{Id}(\mathcal{V})$ by any equational base for $\mathcal{V}$. This will follow from Corollary 4.3.

Lemma 3.3. *Let σ be an identity that holds in a variety $\mathcal{V}$. Then the quasivariety $\mathcal{V} \circ \mathcal{S}$ satisfies the identities of σ^p.*

Proof. Assume σ is as in Definition 3.1 and let A be in $\mathcal{V} \circ \mathcal{S}$ with semilattice replica congruence ϱ. Let m be a positive integer and $r_1, \ldots, r_n \in T_m$. Choose $a_1, \ldots, a_m \in A$ and write $\mathbf{a} r_i$ as a shorthand for $a_1 \ldots a_m r_i$. We must show that $\mathbf{a} r_1 \ldots \mathbf{a} r_n u = \mathbf{a} r_1 \ldots \mathbf{a} r_n v$ holds in A.

For any $i, j \leq m$, the identity $x_1 \ldots x_m r_i = x_1 \ldots x_m r_j$ is regular. Since A/ϱ is a semilattice, it must satisfy this identity. Hence

$$(a_1 \ldots a_m r_i, \, a_1 \ldots a_m r_j) \in \varrho.$$

Thus all of the elements $\mathbf{a} r_i$, for $i = 1, \ldots, n$, lie in the same congruence block. By assumption, each congruence block is a member of $\mathcal{V}$, so it satisfies the identity σ. From this we obtain $\mathbf{a} r_1 \ldots \mathbf{a} r_n u = \mathbf{a} r_1 \ldots \mathbf{a} r_n v$ as desired. $\square$

Proposition 3.4. *For any variety* $\mathcal{V}$,

$$\mathcal{V} \circ \mathcal{S} \subseteq \mathcal{V}^p. \tag{3.3}$$

4. Free algebras in varieties $\mathcal{V}^p$

Basic properties of free algebras in sub(quasi)varieties of $\mathcal{V}^p$ follow from basic properties of free algebras in prevarieties of Ω-algebras (i.e. classes of similar algebras closed under subalgebras and direct products) as given for example in [24, §3.3]. If $\mathcal{Q}$ and $\mathcal{Q}'$ are two subquasivarieties with $\mathcal{Q}' \leq \mathcal{Q}$, then the free $\mathcal{Q}'$-algebra $X\mathcal{Q}'$ over X is isomorphic to the $\mathcal{Q}'$-replica of the algebra $X\mathcal{Q}$. (See also [2, Thm. 4.28].) In particular, if $\mathcal{V}$ is a regular variety, then the semilattice replica $X\mathcal{V}/\varrho$ of $X\mathcal{V}$ is isomorphic to the free semilattice $X\mathcal{S}$:

$$X\mathcal{V}/\varrho \cong X\mathcal{S}. \tag{4.1}$$

Recall also that the free semilattice $X\mathcal{S}$ over X is isomorphic to the semilattice $P_f(X)$ of all finite non-empty subsets of X under union.

For an Ω-term w, $\mathsf{var}(w)$ will denote the set of variables that actually appear in w. Suppose that $\mathcal{V}$ is a regular variety and let a be an element of $X\mathcal{V}$. There is a term w with $\mathsf{var}(w) = \{x_1, \ldots, x_n\} \subseteq X$ so that $a = x_1, \ldots, x_n w$. If $a = y_1, \ldots, y_m v$ for some other term v and $y_1, \ldots, y_m \in X$, then by the freeness of $X\mathcal{V}$ the identity $x_1, \ldots, x_n w = y_1, \ldots, y_m v$ holds in $\mathcal{V}$. By regularity we must have $\mathsf{var}(w) = \mathsf{var}(v)$, which we can take as a definition of $\mathsf{var}(a)$. As is customary, we often blur the distinction between an element of the free algebra and a representing term.

Lemma 4.1. *Let* $\mathcal{V}$ *be a regular variety and let* ϱ *be the semilattice replica congruence of the free algebra* $X\mathcal{V}$ *over* X. *Then for* $w, v \in X\mathcal{V}$

$$(w, v) \in \varrho \iff \mathsf{var}(w) = \mathsf{var}(v).$$

Proof. Let θ be the binary relation on $A = X\mathcal{V}$ defined by

$$(w, v) \in \theta \iff \mathsf{var}(w) = \mathsf{var}(v).$$

It is easy to see that the relation θ is a congruence relation, and is the kernel of the homomorphism

$$h \colon X\mathcal{V} \to P_f(X); \ x_1 \ldots x_n u \mapsto \mathsf{var}(u)$$

onto the semilattice of all finite subsets of X under union. Now by universality of replication (see [24, Lemma 3.3.1]), it follows that there is a unique homomorphism $\overline{h} \colon X\mathcal{V}/\varrho \to P_f(X)$ such that the composition $(\mathsf{nat}\varrho)\overline{h}$ equals h. In particular, under this composition $u \mapsto u/\varrho \mapsto \mathsf{var}(u)$. By (4.1), $\overline{h}$ is an isomorphism. Hence $\theta = \varrho$. $\qquad\square$

In particular, for any variety $\mathcal{V}$, the variety $\mathcal{V}^p$ is regular. Thus

$$X\mathcal{V}^p/\varrho \cong X\mathcal{S}. \tag{4.2}$$

Proposition 4.2. *Let $\mathcal{V}$ be a variety with equational base Σ. Let $\mathcal{W}$ be the variety defined by the prolongation Σ^p (see Definition 3.1). Then the free $\mathcal{W}$-algebra $A = X\mathcal{W}$ is a semilattice of $\mathcal{V}$-algebras. More precisely, $A = \bigsqcup_{s \in S} A_s$, where $S = X\mathcal{S}$ is the free semilattice over X and all A_s are members of $\mathcal{V}$.*

In particular, by taking Σ to be the set of all identities holding in $\mathcal{V}$,

$$A = X\mathcal{V}^p \in \mathcal{V} \circ \mathcal{S}.$$

Proof. By (4.2), we already know that A/ϱ is isomorphic to $S = X\mathcal{S}$. We complete the proof by showing that each A_s satisfies the identities of Σ. Consider any identity $y_1 \ldots y_n u = y_1 \ldots y_n v$ of Σ. Let $a_1, \ldots, a_n$ be elements of A_s. Since A_s is a ϱ-class, it follows that all $a_1, \ldots, a_n$ are ϱ-equivalent. Thus, by Lemma 4.1, $\mathsf{var}(a_1) = \cdots = \mathsf{var}(a_n) = \{x_1, \ldots, x_m\}$ for some $m > 0$. This means that each a_i may be represented as a term $x_1 \ldots x_m r_i \in T_m$. Thus

$$a_1 \ldots a_n u = (x_1 \ldots x_m r_1) \cdots (x_1 \ldots x_m r_n)u$$
$$= (x_1 \ldots x_m r_1) \cdots (x_1 \ldots x_m r_n)v = a_1 \ldots a_n v,$$

since the middle equality is an identity in Σ^p. $\qquad\square$

Corollary 4.3. *Let $\mathcal{V}$ be a variety with equational base Σ. Then Σ^p is an equational base for $\mathcal{V}^p$.*

Proof. Let $\mathcal{W}$ be the variety defined by Σ^p and let Δ be the set of all identities holding in $\mathcal{V}$. Applying Proposition 4.2 to both Σ and Δ we see that for any set X, both $X\mathcal{W}$ and $X\mathcal{V}^p$ are free algebras in the quasivariety $\mathcal{V} \circ \mathcal{S}$. But free algebras are unique and furthermore determine the variety. So $\mathcal{W} = \mathcal{V}^p$. $\qquad\square$

Corollary 4.4. *The variety $\mathcal{V}^p$ is generated by the quasivariety $\mathcal{V} \circ \mathcal{S}$.*

In the next two sections we will show that if $\mathcal{V}$ is a strongly irregular variety, then the variety $\mathcal{V}^p$ coincides with the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$.

5. The Mal'tsev product $\mathcal{T}_t \circ \mathcal{S}$

In this section we clarify the relationship between the quasivariety $\mathcal{T}_t \circ \mathcal{S}$ and the variety $\mathcal{T}_t^p$.

Following Proposition 4.2, we assume that the free $\mathcal{T}_t^p$-algebra $A = X\mathcal{T}_t^p$ over X is the semilattice sum $A = \bigsqcup_{s \in S} A_s$, where S, the semilattice replica of A, is the free semilattice over X, and the summands A_s are in $\mathcal{T}_t$. The elements of A_s will be denoted by small letters $a_s, b_s, \ldots$ with the same index as in A_s, and we will write xy for $t(x, y)$. Note that if $a_r \in A_r$ and $b_s \in A_s$ then the semilattice structure dictates that $a_r b_s \in A_{rs}$.

Lemma 5.1. *Let θ be a congruence of the free $\mathcal{T}_t^p$-algebra $A = X\mathcal{T}_t^p$. Assume that there is a pair (a_r', b_s') of elements of A, where $r, s \in S$, such that $(a_r', b_s') \in \theta$. Then for any $a_r \in A_r$ and any $b_s \in A_s$*

$$(b_s, b_s a_r) \in \theta.$$

Proof. From our assumptions, $b_s, b'_s \in A_s$, $a_r, a'_r \in A_r$ and $(b_s a'_r), (a_r b'_s) \in A_{rs}$. Each of A_r, A_s, and A_{rs} lie in $\mathcal{T}_t$, so they satisfy the identity $xy = x$. Consequently

$$b_s = b_s b'_s, \quad a_r = a_r a'_r, \quad (b_s a'_r)(a_r b'_s) = b_s a'_r.$$

Combining these equalities with the assumption that $b'_s \, \theta \, a'_r$ we obtain

$$b_s a_r = (b_s b'_s)(a_r a'_r) \, \theta \, (b_s a'_r)(a_r b'_s) = (b_s a'_r) \, \theta \, (b_s b'_s) = b_s. \qquad \square$$

Remark. Suppose $r \neq s$ in S. If $(a'_r, b'_s) \in \theta$, then each a_r and each b_s is θ-related to some element of A_{rs}. Moreover, if additionally $q < rs < r$ and also $(c'_{rs}, d'_q) \in \theta$ for some $c'_{rs} \in A_{rs}$ and $d'_q \in A_q$, then each element of A_{rs} is θ-related to some element of A_q, and in particular, $a_r \, \theta \, a_r c_{rs} \, \theta \, a_r c_{rs} \cdot d_q$ for any a_r, c_{rs}, d_q.

The next lemma follows from [17, Lemma 4.66].

Lemma 5.2. *Let A be an Ω-algebra and let α and β be congruences of A. Then the following conditions are equivalent.*

(a) $\beta \circ \alpha \circ \beta \subseteq \alpha \circ \beta \circ \alpha$,

(b) $\alpha \vee \beta = \alpha \circ \beta \circ \alpha$.

Lemma 5.3. *Let θ be a congruence of the free $\mathcal{T}_t^p$-algebra $A = X\mathcal{T}_t^p$, and let ϱ be the semilattice replica congruence of A. Then*

$$\theta \vee \varrho = \theta \circ \varrho \circ \theta. \tag{5.1}$$

Proof. Suppose that

$$a_r \, \varrho \, a'_r \, \theta \, b'_s \, \varrho \, b_s.$$

Since A_r and A_s lie in $\mathcal{T}_t$, and A/ϱ is commutative

$$a_r = a_r a'_r \, \theta \, a_r b'_s \, \varrho \, b_s a'_r \, \theta \, b_s b'_s = b_s.$$

Therefore

$$\varrho \circ \theta \circ \varrho \subseteq \theta \circ \varrho \circ \theta.$$

By Lemma 5.2, it follows that

$$\theta \vee \varrho = \theta \circ \varrho \circ \theta. \qquad \square$$

Let θ be a congruence of a $\mathcal{T}_t^p$-algebra A, and ϱ the semilattice replica congruence of A. Let ψ be the join of the congruences θ and ϱ. It is well known that $(A/\theta)/(\psi/\theta) \cong A/\psi$, and since $\psi \geq \varrho$, it follows that A/ψ is a member of the variety $\mathcal{S}$ of Ω-semilattices.

Lemma 5.4. *Let θ be a congruence of a $\mathcal{T}_t^p$-algebra A, and let $\psi = \theta \vee \varrho$. Then a congruence class $(a_r/\theta)/(\psi/\theta)$ of A/θ satisfies the identity $x \cdot y = x$ if and only if, for any $b_s \in A$*

$$(a_r, b_s) \in \psi \implies (a_r b_s, a_r) \in \theta. \tag{5.2}$$

Proof. Let a_r/θ and b_s/θ be members of A/θ. Then

$$(a_r/\theta, b_s/\theta) \in \psi/\theta \iff (a_r, b_s) \in \psi.$$

On the other hand

$$(a_r/\theta) \cdot (b_s/\theta) = a_r/\theta \iff (a_r b_s, a_r) \in \theta.$$

Hence the assertion that $(a_r/\theta)/(\psi/\theta)$ satisfies $x \cdot y = x$ is equivalent to the implication (5.2). $\qquad \square$

Proposition 5.5. *Let θ be a congruence of the free $\mathcal{T}_t{}^p$-algebra $A = X\mathcal{T}_t{}^p$ over X, and set $\psi = \theta \vee \varrho$. Then the quotient $B = A/\theta$ of A is a semilattice sum of $\mathcal{T}_t$-subalgebras. In particular, ψ/θ is the semilattice replica congruence of A/θ, and B is the semilattice sum of the ψ/θ-classes $(a_r/\theta)/(\psi/\theta)$ as a_r ranges over A.*

Proof. Since $\psi \geq \varrho$, it follows that $A/\psi \in \mathcal{S}$. Thus we only need to show that for each $a_r/\theta \in A/\theta$, the congruence class $(a_r/\theta)/(\psi/\theta)$ satisfies $x \cdot y = x$. By Lemma 5.4, this is equivalent to the condition (5.2).

So our aim now is to verify the implication (5.2). For $a_r \in A_r$ and $b_s \in A_s$, let $(a_r, b_s) \in \psi$. By Lemma 5.3, we know that

$$\psi = \theta \vee \varrho = \theta \circ \varrho \circ \theta.$$

Thus there are elements c and d in A such that $a_r \ \theta \ c \ \varrho \ d \ \theta \ b_s$. But $c \ \varrho \ d$ implies $c = cd$. Hence $a_r \ \theta \ c = cd \ \theta \ a_r b_s$. $\qquad \square$

Let us note that A/θ is isomorphic to the semilattice sum of the ψ-classes a/ψ of A over the semilattice A/ψ, where each a/ψ is the union of θ-classes b/θ such that $(a, b) \in \psi$ (or equivalently $(a/\theta, b/\theta) \in (\psi/\theta)$).

As a direct corollary of Proposition 5.5 one obtains the following theorem.

Theorem 5.6. *The quasivariety $\mathcal{T}_t \circ \mathcal{S}$ and the variety $\mathcal{T}_t{}^p$ coincide. In particular, the Mal'tsev product $\mathcal{T}_t \circ \mathcal{S}$ is a variety, and $\{t(x, y) = x\}^p$ is its equational base.*

Proof. We have already observed that $\mathcal{T}_t \circ \mathcal{S} \subseteq \mathcal{T}_t{}^p$. Conversely, if an algebra B is a member of $\mathcal{T}_t{}^p$, then there is a set X so that B is a homomorphic image of $X\mathcal{T}_t{}^p$. By Proposition 5.5, the algebra B belongs to $\mathcal{T}_t \circ \mathcal{S}$. $\qquad \square$

5.1. The most basic case of $\mathcal{T}_t$

If we take $\mathcal{G}$ to be the variety of all groupoids (magmas, binars) $(G, \cdot)$, and set $t(x, y) = x \cdot y$, then $\mathcal{G}_t$ is the variety $\mathcal{LZ}$ of left-zero semigroups and one obtains the following corollary.

Corollary 5.7. *The quasivariety $\mathcal{LZ} \circ \mathcal{S}$ is a variety. In particular $\mathcal{LZ} \circ \mathcal{S} = \mathcal{LZ}^p$.*

Note however that members of $\mathcal{LZ}^p$ are not necessarily semigroups. First observe that the equational base of $\mathcal{LZ}^p$ contains the idempotent law $x \cdot x = x$. The free $\mathcal{LZ}^p$-algebra $A = X(\mathcal{LZ}^p)$ over the two element set $X = \{x, y\}$ is the semilattice sum of three subalgebras: two one-element subalgebras $A_x = \{x\}$

and $A_y = \{y\}$ and one subalgebra $A_{x,y}$ consisting of all elements represented by terms of T_2. Obviously $xy, yx \in A_{x,y}$. To describe further elements, let us introduce the following notation. For any element a of $A_{x,y}$ and a variable z, let $aR(z) = a \cdot z$ and $aL(z) = z \cdot a$. Then let $aE(z)$ be either $aR(z)$ or $aL(z)$.

Lemma 5.8. *Each element of $A_{x,y}$ different from xy and yx may be expressed in the standard form*

$$(xy)E(z_1)\ldots E(z_k) \text{ or } (yx)E(z_1)\ldots E(z_k), \qquad (5.3)$$

where $z_i = x$ or $z_i = y$, and $k = 1, 2, \ldots$

Proof. The proof is by induction on the length of the element. The shortest elements xy and yx are already in $A_{x,y}$. Since $A_{x,y}$ is a left-zero band, it follows that for any $u, v \in A_{x,y}$, we have $u \cdot v = u$. So longer elements of $A_{x,y}$ may be obtained from elements of the form (5.3) only by multiplying them by a variable x or a variable y from the left or from the right. $\square$

Corollary 5.9. *The free algebra $A = X\mathcal{LZ}^p$ is not a semigroup.*

Proof. By Lemma 5.8, it is easy to see that the elements $x \cdot yx$ and $xy \cdot x$ are different. $\square$

On the other hand, the Mal'tsev product $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ of the varieties $\mathcal{LZ}$ and $\mathcal{S}$, but taken relative to the variety $\mathcal{SG}$ of semigroups, is a variety of semigroups, and may be described in a simple way. Note that $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ is just the intersection $\mathcal{LZ}^p \cap \mathcal{SG}$, and that the members of this variety are bands (idempotent semigroups).

Proposition 5.10. *The Mal'tsev product $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ coincides with the variety $\mathcal{LR}$ of left-regular bands defined by the identity $xyx = xy$.*

Proof. ($\Rightarrow$) Let A be a member of $\mathcal{LZ}\circ_{\mathcal{SG}}\mathcal{S}$. Then A is a band and a semilattice of left-zero bands. For $a, b \in A$, we have

$$ab/\varrho = a/\varrho \cdot b/\varrho = b/\varrho \cdot a/\varrho = ba/\varrho.$$

Hence $ab \, \varrho \, ba$, so $ab \cdot ba = ab$. On the other hand $ab \cdot ba = abba = aba$, which implies $aba = ab$.

($\Leftarrow$) Now assume that $A \in \mathcal{LR}$. By McLean's Theorem [18], each band is a semilattice of rectangular bands. Hence $A = \bigsqcup_{s \in S} A_s$, where S is the semilattice replica of A and all A_s are rectangular bands, defined by the identity $xyx = x$. This together with the identity $xyx = xy$ gives $xy = x$. Hence all A_s are left-zero bands. $\square$

5.2. Mal'tsev products of some regular varieties

We now demonstrate that the assumption of strong irregularity of the variety $\mathcal{T}_t$ in Theorem 5.6 is essential.

Consider again the type τ with one binary multiplication $\cdot$. Let $\mathcal{CG}$ be the variety of commutative groupoids.

Proposition 5.11. *The quasivariety $\mathcal{CG} \circ \mathcal{S}$ satisfies the quasi-identity*

$$(zx = x \text{ \& } zy = y \text{ \& } xz = yz) \rightarrow (xy = yx). \qquad (5.4)$$

Proof. Let $A \in \mathcal{CG} \circ \mathcal{S}$ and let ϱ be the semilattice replica congruence of A. Assume that $a, b, c \in A$ with $ca = a$, $cb = b$, and $ac = bc$. We will show that $ab = ba$. Since A/ϱ is a semilattice, it is commutative. Hence $ac \varrho ca$ and $bc \varrho cb$. Consequently

$$a = ca \varrho ac = bc \varrho cb = b.$$

This implies

$$ab = ba,$$

as desired. $\qquad\square$

Example 5.12. *Let A be the groupoid whose multiplication table is given below.*

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	0	2	2	0	0	2
3	0	0	2	3	0	1	2
4	0	0	0	0	4	4	4
5	0	0	0	0	4	5	4
6	0	0	2	3	4	5	6

Let us note that the elements $0, 1, 2, 3, 4$ form a subsemilattice of A (that means a subgroupoid of A which is a semilattice). Similarly the elements $0, 1, 2, 4, 5$ form a subsemilattice, and the elements $0, 1, 2, 4, 6$ form a subsemilattice. However $3 \cdot 5 = 1 \neq 0 = 5 \cdot 3$, whence A itself is not a semilattice. The equivalence ϱ of A with the 2-element classes $\{0, 1\}$, $\{2, 3\}$, and $\{4, 5\}$, and one element class $\{6\}$, is a congruence of A, each ϱ-class is a semilattice and A/ϱ is a semilattice. Thus $A \in \mathcal{S} \circ \mathcal{S}$, and hence $A \in \mathcal{CG} \circ \mathcal{S}$. Moreover, ϱ is the semilattice replica congruence of A. Indeed, if A/σ is a semilattice for some congruence σ of A, then for all $a, b \in A$ we must have $(a \cdot b, b \cdot a) \in \sigma$. Hence

$$(0, 1) = (5 \cdot 3, 3 \cdot 5) \in \sigma,$$
$$(2, 3) = (3 \cdot 6, 6 \cdot 3) \in \sigma,$$
$$(4, 5) = (5 \cdot 6, 6 \cdot 5) \in \sigma.$$

Thus ϱ is the semilattice replica congruence.

Now let θ be the congruence on A generated by 2 and 4. One easily checks that the congruence θ has one 3-element class $\{0, 2, 4\}$ and each of the remaining classes consists of one element. Let $B = A/\theta$. By taking $x = 3/\theta = \{3\}$, $y = 5/\theta = \{5\}$ and $z = 6/\theta = \{6\}$ we see that B fails the quasi-identity (5.4). By Proposition 5.11, $B \notin \mathcal{CG} \circ \mathcal{S}$, and hence $B \notin \mathcal{S} \circ \mathcal{S}$. Consequently, neither $\mathcal{S} \circ \mathcal{S}$ nor $\mathcal{CG} \circ \mathcal{S}$ is a variety.

From Proposition 5.11, Example 5.12, and the fact that $\mathcal{S}$ is the smallest regular variety, we have the following.

Corollary 5.13. *For no regular subvariety $\mathcal{V}$ of $\mathcal{CG}$ is $\mathcal{V} \circ \mathcal{S}$ a variety.*

By Corollary 4.3, the prolongation $\mathcal{CG}^p$ is defined by the set of identities Σ^p consisting of all identities of the form

$$(x_1 \ldots x_m r_1) \cdot (x_1 \ldots x_m r_2) = (x_1 \ldots x_m r_2) \cdot (x_1 \ldots x_m r_1), \qquad (5.5)$$

in which $m > 0$ and r_1, r_2 are terms from T_m. Our discussion above suggests the following problem.

Problem 5.14. Is the quasivariety $\mathcal{CG} \circ \mathcal{S}$ axiomatized by the identities in (5.5) together with the single quasi-identity (5.4) of Proposition 5.11?

Presumably, there is nothing special about commutativity here. Thus we are motivated to pose the following problem.

Problem 5.15. Let $\mathcal{V}$ be a proper regular variety of groupoids. Is it true that $\mathcal{V} \circ \mathcal{S}$ fails to be a variety?

With regard to this latter problem, let us note that, with $\mathcal{G}$ denoting the variety of all groupoids, $\mathcal{G} \circ \mathcal{S} = \mathcal{G}$, which is obviously a variety. Thus the requirement that $\mathcal{V}$ be a proper subvariety is necessary.

6. The Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$

Now let $\mathcal{V}_t$ be a (strongly irregular) subvariety of the variety $\mathcal{T}_t$ of Ω-algebras defined by the identity $t(x, y) = x$ for some $t \in T_2$ and a set Γ_t of regular identities. Recall from Section 3 that the regularization $\widetilde{\mathcal{V}}_t$ of $\mathcal{V}_t$ is defined by Γ_t and the identities (P1) – (P5), with $x \cdot y = t(x, y)$, and consists of Płonka sums of $\mathcal{V}_t$-algebras. Moreover

$$\mathcal{T}_t \wedge \widetilde{\mathcal{V}}_t = \mathcal{V}_t \text{ and } \mathcal{T}_t \vee \widetilde{\mathcal{V}}_t = \widetilde{\mathcal{T}}_t.$$

Since every Płonka sum is a semilattice sum, $\widetilde{\mathcal{V}}_t \subseteq \mathcal{V}_t \circ \mathcal{S}$. On the other hand, $\mathcal{V}_t \circ \mathcal{S}$ is obviously contained in $\mathcal{T}_t \circ \mathcal{S}$. And by Proposition 3.4, $\mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{V}_t^p$. Finally, by Theorem 5.6, $\mathcal{T}_t \circ \mathcal{S} = \mathcal{T}_t^p$. Summarizing

$$\mathcal{V}_t \subseteq \widetilde{\mathcal{V}}_t \subseteq \mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{V}_t^p \subseteq \mathcal{T}_t \circ \mathcal{S} = \mathcal{T}_t^p.$$

And by Corollary 4.4, the variety $\mathcal{V}_t^p$ is the smallest variety containing the quasivariety $\mathcal{V}_t \circ \mathcal{S}$.

Recall also that, by (4.2),

$$X\mathcal{V}_t^p / \varrho \cong X\mathcal{S},$$

and for any two Ω-terms w, v

$$(w, v) \in \varrho \iff \mathsf{var}(w) = \mathsf{var}(v).$$

As a direct consequence of Proposition 4.2 one obtains the following corollary.

Corollary 6.1. *For any subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$, the free $\mathcal{V}_t^p$-algebra $B = X\mathcal{V}_t^p$ is a semilattice of $\mathcal{V}_t$-algebras. In particular, $B = \bigsqcup_{s \in S} B_s$, where $S = X\mathcal{S}$ is the free semilattice over X and all B_s are members of $\mathcal{V}_t$. Hence*

$$B = X\mathcal{V}_t^p \in \mathcal{V}_t \circ \mathcal{S}.$$

We intend to show that the classes $\mathcal{V}_t \circ \mathcal{S}$ and $\mathcal{V}_t^p$ coincide.

Proposition 6.2. *Let δ be a congruence of the free $\mathcal{V}_t^p$-algebra $B = X\mathcal{V}_t^p$ over X. Then the quotient $C = B/\delta$ is a semilattice sum of $\mathcal{V}_t$-algebras.*

Proof. We first use the fact that C is the quotient B/δ of B to describe the decomposition of C into a semilattice sum of $\mathcal{T}_t$-algebras in terms of the algebra B. Let ϱ_B be the semilattice replica congruence of the algebra B, and let $\psi = \delta \vee \varrho_B$. Since $\psi \geq \varrho_B$ it follows that B/ψ is a semilattice. On the other hand, the minimality of ϱ_B guarantees that ψ is the smallest congruence above δ such that $B/\psi \in \mathcal{S}$. It follows that ψ/δ is the semilattice replica congruence of B/δ. Since each element of C is of the form b/δ for some $b \in B$, Theorem 5.6 implies that

$$C = \bigsqcup_{b/\delta \in B/\delta} (b/\delta)/(\psi/\delta)$$

in which each summand $(b/\delta)/(\psi/\delta) \in \mathcal{T}_t$.

On the other hand, by Corollary 6.1, $B = \bigsqcup_{s \in S} B_s$, in which S is a semilattice and all B_s are members of $\mathcal{V}_t$.

It remains to show that the (ψ/δ)-delta classes, $(b/\delta)/(\psi/\delta)$, satisfy the identities of Γ_t. Let

$$x_1 \ldots x_k u = x_1 \ldots x_k v \tag{6.1}$$

be a typical member of Γ_t. We want to show that (6.1) holds in each ψ/δ-class. This means that for any $c^1 = b^1/\delta, \ldots, c^k = b^k/\delta \in (b/\delta)/(\psi/\delta)$, with $b^i \in B_{s_i}$ and $b \in B$, we have

$$c^1 \ldots c^k u = c^1 \ldots c^k v, \tag{6.2}$$

which is equivalent to

$$(b^1 \ldots b^k u, b^1 \ldots b^k v) \in \delta. \tag{6.3}$$

Note that $b^i/\delta, b^j/\delta \in (b/\delta)/(\psi/\delta)$ if and only if $(b^i, b^j) \in \psi$. Since each $(b^i/\delta)/(\psi/\delta)$ satisfies the identity $x \cdot y = x$, we may use Lemma 5.4 repeatedly to obtain the following:

$$b^i \; \delta \; b^i \cdot b^1 \; \delta \; (b^i \cdot b^1) \cdot b^2 \; \delta \; ((b^i \cdot b^1) \cdot b^2) \cdot b^3 \; \delta \ldots$$
$$\delta \; (\ldots ((b^i \cdot b^1) \cdot b^2) \cdot \ldots) \cdot b^k \; =: d^i.$$

Thus $b^i \; \delta \; d^i$ for each $1 \leq i \leq k$, and the elements $d^1, d^2, \ldots, d^k$ all belong to the same congruence class $B_{s_1 \cdots s_k}$ of ϱ_B. Since ϱ_B-classes satisfy the identities of Γ_t, it follows that

$$d^1 \ldots d^k u = d^1 \ldots d^k v.$$

Finally note that

$$(b^1 \ldots b^k u, d^1 \ldots d^k u) \in \delta \quad \text{and} \quad (b^1 \ldots b^k v, d^1 \ldots d^k v) \in \delta,$$

which implies

$$(b^1 \ldots b^k u, b^1 \ldots b^k v) \in \delta.$$

This verifies (6.3). □

Let us note that in particular, ψ/δ is the semilattice replica congruence of B/δ and the quotient B/δ is the semilattice sum of ψ/δ-classes $(b/\delta)/(\psi/\delta)$.

Theorem 6.3. *Let $\mathcal{V}_t$ be a variety satisfying the identity $t(x, y) = x$ and with equational base Σ. Then the Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$ is a variety, having Σ_t^p as an equational base.*

Proof. We have already observed that $\mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{V}_t{}^p$. Conversely, if an algebra C is a member of $\mathcal{V}_t{}^p$, then there is a set X so that C is a homomorphic image of $X\mathcal{V}_t{}^p$. Then by Proposition 6.2, the algebra C belongs to $\mathcal{V}_t \circ \mathcal{S}$. Thus the quasivariety $\mathcal{V}_t \circ \mathcal{S}$ and the variety $\mathcal{V}_t{}^p$ coincide. $\square$

Corollary 6.4. *Let $\mathcal{W}$ be any variety of Ω-algebras. Then the Mal'tsev product $\mathcal{V}_t \circ_{\mathcal{W}} \mathcal{S}$ of $\mathcal{V}_t$ and $\mathcal{S}$ relative to $\mathcal{W}$ is a variety.*

Problem 6.5. Does Theorem 6.3 hold for an irregular variety which is not strongly irregular?

7. Examples and counterexamples

First consider the type τ with one binary multiplication $\cdot$, as in Sections 5.1 and 5.2. We could observe that the Mal'tsev products $\mathcal{LZ} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{S}$ considered in these examples do not satisfy the associative law. This is however an instance of a more general phenomenon. Consider a variety of plural type τ, and assume that the equational base Σ of a variety $\mathcal{V}$ contains a (non-trivial) linear identity σ. (An identity is *linear* if the multiplicities of each variable on each side are at most 1. This is not to be confused with the notion of linear identity as it appears in [10].) Then the identities of σ^p obtained from σ are never linear, since each variable of σ was replaced in σ^p by a term $r_i \in T_m$ with $m \geq 2$, and all of these terms contain the same sets of variables. Moreover, consequences of the identities Σ^p are never linear.

This can be summarised as follows.

Lemma 7.1. *For any non-trivial variety $\mathcal{V}$, the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ does not satisfy any (non-trivial) linear identity.*

In particular, for no variety $\mathcal{V}$ of semigroups is the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ a class of semigroups. A similar observation can be made for many familiar varieties, for example groups, rings, lattices, semilattices etc.

Frequently more interesting are proper subvarieties of $\mathcal{V} \circ \mathcal{S}$.

Example 7.2 (Bands). As already mentioned in Section 5.1, each band is a semilattice sum of rectangular bands. (See [18,11,12].) Let $\mathcal{SG}$ be the variety of semigroups, $\mathcal{B}$ the variety of bands, and $\mathcal{RB}$ the variety of rectangular bands. Then, by Corollary 6.4,

$$\mathcal{B} = \mathcal{RB} \circ_{\mathcal{SG}} \mathcal{S} = \mathcal{SG} \cap (\mathcal{RB} \circ \mathcal{S}) = \mathcal{SG} \cap \mathcal{RB}^p.$$

Note that $\mathcal{RB}^p$ is a variety of groupoids, but is not a variety of semigroups.

In the following examples, the (pseudo-)partition operation $x \cdot y$ will be denoted by $t(x, y)$ to avoid conflict with the basic operation of multiplication in the algebras considered.

Example 7.3 (Birkhoff systems). Lattices may be defined as algebras $(A, +, \cdot)$ with two semilattice reducts, the (join-)semilattice $(A, +)$ and the (meet-)semilattice $(A, \cdot)$, satisfying the *Birkhoff identity* $x + xy = x(x + y)$ and the

absorption law $t(x, y) = x + xy = x$. They form the variety $\mathcal{L}$ of lattices. By dropping the absorption law one obtains the definition of the more general variety $\mathcal{BS}$ of *Birkhoff systems*. (See [8,9].) An essential role in this variety is played by two 3-element bi-chains $\mathbf{3}_m$ and $\mathbf{3}_j$ built on the set $\{0, 1, 2\}$ by defining the meet-semilattice order as $0 <. 1 <. 2$ and the join-semilattice order as $0 <_+ 2 <_+ 1$ in $\mathbf{3}_m$ and as $1 <_+ 0 <_+ 2$ in $\mathbf{3}_j$. Let $\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j)$ denote the class of Birkhoff systems which do not contain either $\mathbf{3}_m$ or $\mathbf{3}_j$ as subalgebras. It was shown in [9] that the class $\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j)$ is a variety, and is characterized as precisely the class of Birkhoff systems which are semilattice sums of lattices. It follows that

$$\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j) = \mathcal{L} \circ_{\mathcal{BS}} \mathcal{S} = \mathcal{BS} \cap (\mathcal{L} \circ \mathcal{S}) = \mathcal{BS} \cap \mathcal{L}^p.$$

Note that none of the basic operations of $\mathcal{L}^p$-algebras are associative or commutative.

As already mentioned, for a strongly irregular variety $\mathcal{V}_t$, the best known and understood subvariety of $\mathcal{V}_t{}^p$ (different from $\mathcal{V}_t$ and $\mathcal{S}$) is the regularization, $\widetilde{\mathcal{V}}_t$, of $\mathcal{V}_t$. Much less is known about the pseudo-regularization. It is an easy exercise to show that the regularization and the pseudo-regularization of the variety $\mathcal{LZ}$ of left-zero bands coincide. However it is possible for the regularization of a variety to be distinct from its pseudo-regularization. The first example was discovered in [3], and investigated in connection with the constraint satisfaction problem.

Example 7.4 (Steiner quasigroups). A *Steiner quasigroup* or *squag* is a commutative idempotent groupoid satisfying the identity $t(x, y) = xy \cdot y = x$. The variety formed by these groupoids is denoted by $\mathcal{SQ}$. As shown in [3], the regularization $\widehat{\mathcal{SQ}}$ of $\mathcal{SQ}$ is the variety of commutative and idempotent groupoids satisfying the identity $x(x \cdot yz) = (x \cdot xy)z$. The pseudo-regularization $\overline{\mathcal{SQ}}$ of $\mathcal{SQ}$ is defined to be the variety of commutative idempotent groupoids satisfying the identities (P1) – (P4) from Section 2, where $\cdot$ is replaced by $t(x, y) = xy \cdot y$. Groupoids in the regularization are Płonka sums of squags. Groupoids in the pseudo-regularizations are semilattice sums of squags. As was shown in [3], the variety of commutative idempotent groupoids satisfying the identity $x(y \cdot yz) = (xy \cdot y)z$ (called $\mathcal{T}_2$ in that paper) is contained in the variety $\overline{\mathcal{SQ}}$, and is different from $\widetilde{\mathcal{SQ}}$. It follows that the three varieties $\mathcal{SQ}$, $\widetilde{\mathcal{SQ}}$ and $\overline{\mathcal{SQ}}$ are distinct:

$$\mathcal{SQ} \subset \widetilde{\mathcal{SQ}} \subset \overline{\mathcal{SQ}}.$$

However we do not know if the varieties $\mathcal{T}_2$ and $\overline{\mathcal{SQ}}$ coincide, though we think it is not likely. The Mal'tsev product $\mathcal{SQ} \circ \mathcal{S}$ coincides with the variety $\mathcal{SQ}^p$. Since commutativity is a linear identity, members of the variety $\mathcal{SQ}^p$ are not commutative. The pseudo-regularization $\overline{\mathcal{SQ}}$ is obviously contained in but not equal to $\mathcal{SQ}^p$.

In fact, Examples 7.3 and 7.4 were the inspiration for undertaking investigations of the most general class of algebras representable as semilattice sums of their subalgebras.

Examples of other varieties of algebras with distinct regularization and pseudo-regularization are provided by bisemilattices, algebras with two semilattice operations (one interpreted as a join and one as a meet, similarly as in the case of Birkhoff systems.)

Example 7.5 (Semilattice sums of lattices). First recall that members of the class $\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j)$ of Birkhoff systems considered above are all semilattice sums of lattices. However, it is not difficult to check that this class is not the pseudo-regularization of the variety $\mathcal{L}$ of lattices. The partition operation $t(x, y) = x + xy$ of the regularization $\widetilde{\mathcal{L}}$ is not a pseudo-partition operation for $\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j)$. Indeed, an easy exercise shows that the requirements for a pseudo-partition operation are not satisfied. A witness is a certain 3-element member of $\mathcal{BS}^-(\mathbf{3}_m, \mathbf{3}_j)$. This is the bichain $\mathbf{3}_n$ built on the set $\{0, 1, 2\}$ with the meet-semilattice order defined as for $\mathbf{3}_m$ and $\mathbf{3}_j$ and the join-semilattice order defined by $2 <_+ 0 <_+ 1$. Note that $\mathbf{3}_n$ is the semilattice sum of the 2-element lattice $\{0, 1\}$ and the 1-element lattice $\{2\}$, but it is not a Płonka sum of these lattices. (See [8,9].)

Now let A be a subdirectly irreducible lattice and set $A_1 = A$. Let A_0 be a 1-element lattice $\{\infty\}$ disjoint from A_1. Finally let A^∞ be the Płonka sum of A_1 and A_0 over the 2-element meet semilattice $\{0, 1\}$ with $0 < 1$. Then the bisemilattice A^∞ generates the regularization of the variety $\mathsf{V}(A)$ generated by A. (See e.g. [24, Ch. 4].)

There is another interesting family of semilattice sums obtained from the lattice A in the case when A is a bounded lattice. Once again set $A_1 = A$ and denote the bounds of A by 0_A and 1_A. We define the bisemilattice $A_\infty = A \cup \{\infty\}$ as follows. For an $a \in A$, $a \cdot \infty = 0_A$ and $a + \infty = 1_A$. It is easy to see that this yields a bisemilattice, and is a semilattice sum of the lattices A_1 and A_0 over the meet semilattice $\{0, 1\}$ with $1 < 0$. A somewhat more complicated exercise shows that the operation $t(x, y) = x + xy$ is a pseudo-partition operation on the algebra A_∞, but it is not a partition operation. It follows that each variety $\mathsf{V}(A_\infty)$ is contained in the pseudo-regularization of $\mathcal{L}$ but is different from the regularization of the variety $\mathsf{V}(A)$. We do not know if the varieties $\mathsf{V}(A_\infty)$ and $\overline{\mathsf{V}(A)}$ coincide.

Example 7.6 (Groups). Groups may be defined as algebras $(G, \cdot, ^{-1})$ satisfying the identities $xy \cdot z = x \cdot yz$, $y^{-1}y \cdot x = x \cdot yy^{-1}$, and $x \cdot yy^{-1} = x$. They form a strongly irregular variety with $t(x, y) = x \cdot yy^{-1}$, which will be denoted by $\mathcal{GP}$. The regularization $\widetilde{\mathcal{GP}}$ of $\mathcal{GP}$ may be defined by the first two identities defining $\mathcal{GP}$ and the identities $(xy)^{-1} = y^{-1}x^{-1}$, $(x^{-1})^{-1} = x$ and $xxx^{-1} = x$. (See e.g. [20,21].) The variety $\widetilde{\mathcal{GP}}$ consists of Płonka sums of groups. In semigroup theory, Płonka sums of groups are called *strong semilattices of groups*. By results of Clifford [5], it is known that the class of Płonka sums of groups coincides with the class of semigroups which are semilattices of groups and with the class of so-called *Clifford semigroups*, semigroups with a unary operation $^{-1}$ satisfying the identities $(x^{-1})^{-1} = x, xx^{-1}x = x, xx^{-1} = x^{-1}x$ and $(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$. (See also [6,11,12].) The class $\widetilde{\mathcal{GP}}$ is a subvariety of the variety $\mathcal{IS}$ of inverse semigroups, semigroups with one unary

operation $^{-1}$ satisfying the last three identities defining $\widetilde{\mathcal{GP}}$. Members of $\widetilde{\mathcal{GP}}$ are precisely inverse semigroups which are Płonka sums of groups (or as semigroup theorists would say, strong semilattices of groups). (See e.g. [21] and references there.) In fact,

$$\widetilde{\mathcal{GP}} = \mathcal{IS} \cap (\mathcal{GP} \circ \mathcal{S}).$$

Groups defined as algebras with two basic operations $\cdot$ and $^{-1}$ as above do not contain any nullary operations in the similarity type. Hence Theorem 6.3 applies, and the Mal'tsev product $\mathcal{GP} \circ \mathcal{S}$ is the variety $\mathcal{GP}^p$. Note that $\mathcal{GP}^p$ is a variety of groupoids with one unary operation, but it is not a variety of semigroups. The pseudo-regularization of $\mathcal{GP}$ has not yet been investigated.

The assumption that the type of a variety $\mathcal{V}$, in a Mal'tsev product $\mathcal{V} \circ \mathcal{S}$, has no nullary operation symbols is due to the fact that semilattices have no nullary operations. However, if $\mathcal{V}$ is a variety with constant (nullary) operations, then the latter assumption may easily be omitted, as it was done in the example above. Instead of a constant operation one may take a unary operation defining this constant, and consider an equivalent variety $\mathcal{V}'$ with such a unary operation replacing each constant operation. Then if $\mathcal{V}$ is strongly irregular, Theorem 6.3 applies, and the Mal'tsev product $\mathcal{V}' \circ \mathcal{S}$ is a variety.

If $\mathcal{V}$ is an irregular, but not strongly irregular variety, then the class of Płonka sums of $\mathcal{V}$-algebras does not necessarily coincide with the regularization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$. (See [24, Ch. 4] and [21] for more information and some references.) The following example concerns the case of semigroups.

Example 7.7 (Semigroups). It was shown by Saliĭ [25, 26] that each semigroup in the regularization $\widetilde{\mathcal{V}}$ of an irregular variety $\mathcal{V}$ of semigroups embeds into a Płonka sum of $\mathcal{V}$-algebras. (This however is not true for general algebras, see [7, 22].) Hence in the case of semigroups the regularization $\widetilde{\mathcal{V}}$ is contained in the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$. However not all of its members are Płonka sums of $\mathcal{V}$-algebras.

Example 7.8. This example comes from the geometry of affine spaces and convex sets. (See [24] and references there.) Affine spaces over a subfield F of the field $\mathbb{R}$ of reals (affine F-spaces) may be defined as algebras $(A, \underline{F})$, where $\underline{F}$ is the set of binary operations $xy\underline{p} = \underline{p}(x, y) = x(1 - p) + yp$, for all $p \in F$. Affine F-spaces defined in this way form a variety. Convex subsets of affine F-spaces (convex F-sets) may be defined as algebras $(C, \underline{I}^o)$, where $\underline{I}^o = \{\underline{p} \mid p \in I^o = \,]0, 1[\}$ and $]0, 1[$ is the open unit interval of F. Fix a subfield F of $\mathbb{R}$. Then the convex F-subsets of affine F-spaces generate the variety $\mathcal{BA}$ of *barycentric algebras*. Convex F-sets form a subquasivariety $\mathcal{C}$ of $\mathcal{BA}$. The variety $\mathcal{BA}$ is defined by the following identities

$$xx\underline{p} = x,$$
$$xy\,\underline{p} = yx\,\underline{1 - p},$$
$$xy\,\underline{p}\,z\,\underline{q} = x\,\,yz\,\underline{q/(p \circ q)}\,\,\underline{p \circ q}$$

for all p, q in I^o. Here $p \circ q = p + q - pq$. (See [24, §5.8].) Its subquasivariety $\mathcal{C}$ is defined by the cancellation laws

$$(xy\underline{p} = xz\underline{p}) \to (y = z), \tag{7.1}$$

which hold for all $p \in I^o$. It is known [24, §7.5] that each barycentric algebra is a semilattice sum of (open) convex sets. This shows that $\mathcal{BA}$ is contained in the Mal'tsev product $\mathcal{C} \circ \mathcal{S}$:

$$\mathcal{BA} \subset \mathcal{C} \circ \mathcal{S} \subset \mathcal{BA}^p.$$

Since the quasivariety $\mathcal{C}$ satisfies linear identities, the varieties $\mathcal{BA}$ and $\mathcal{BA}^p$ are distinct.

In fact, a stronger result was proved, which we provide here in a somewhat reformulated way.

Theorem 7.9 [24, Sec. 7.5] *Each barycentric algebra is a semilattice sum of open convex sets over its semilattice replica, and is a subalgebra of a Płonka sum of convex sets over its semilattice replica.*

The summands of the Płonka sum are certain canonical extensions of the summands of the semilattice sum.

8. Concluding remarks and the problem of representation of algebras in Mal'tsev products

The fact that an algebra A is a semilattice sum of $\mathcal{V}_t$-subalgebras says very little about its detailed structure. An exception is of course given by algebras represented as Płonka sums. (The second best representation is given by subalgebras of Płonka sums.) So it is natural to ask how the summands of a semilattice sum are put together or how to reconstruct the algebraic structure of a semilattice sum $A = \bigsqcup_{s \in S} A_s$ from its $\mathcal{V}_t$-summands A_s and the quotient semilattice S. Such a construction exists and was introduced for general algebras in [23, §6.2], under the name of a Lallement sum, as a generalization of a Płonka sum. (See also [14] for a basic construction introduced in the case of semigroups, and [22, 21] and [24, Ch. 4] for similar constructions in the case of general algebras.) The primary idea was to relax the requirement of functoriality in the definition of Płonka sums. There are several types of such constructions and the definition may be formulated for algebras of any plural type. However, to avoid excessive notation we will consider here only plural type with operations of arity two, and will limit ourself to the following definition.

Definition 8.1. Let $(S, \cdot)$ be a (meet) semilattice. (Here all operations of Ω are equal to $\cdot$). For each $s \in S$, let a $\mathcal{V}_t$-algebra A_s of a plural binary type $\tau \colon \Omega \to \{2\}$ be given, and for each operation $\star \in \Omega$ an extension $(E_s^\star, \star)$ of $(A_s, \star)$. For each pair $t \leq s$ of S, let

$$\varphi_{s,t}^\star \colon (A_s, \star) \to (E_t^\star, \star)$$

be a $\star$-homomorphism such that the following three conditions are satisfied:

(1) $\varphi_{s,s}^\star$ is the embedding of A_s into $E_s^\star$;

(2) $\varphi_{s,s\cdot t}^\star(A_s) \star \varphi_{t,s\cdot t}^\star(A_t) \subseteq A_{s\cdot t}$;

(3) for each $u \le s \cdot t$ in S and $a_s \in A_s$, $b_t \in A_t$

$$\varphi_{s\cdot t,u}^\star(\varphi_{s,s\cdot t}^\star(a_s) \star \varphi_{t,s\cdot t}^\star(b_t)) = \varphi_{s,u}^\star(a_s) \star \varphi_{t,u}^\star(b_t).$$

An Ω-algebra structure on the disjoint sum A of all A_s is given by defining the operations $\star$ in Ω as follows:

$$a_s \star b_t = \varphi_{s,s\cdot t}^\star(a_s) \star \varphi_{t,s\cdot t}^\star(b_t).$$

Then all A_s are subalgebras of A, and S is a quotient of A. The semilattice sum A of A_s is said to be the *semilattice sum of A_s by the mappings $\varphi_{s,t}^\star$*. If additionally, for each $t \in S$, one has $E_t^\star = \{\varphi_{s,t}^\star(a_s) \mid s \ge t, a_s \in A_s\}$, and all $E_s^\star$ are certain canonical extensions of A_s (see [23, § 6.1]), then this semilattice sum is called a *Lallement sum*.

By [23, Th. 624] each semilattice sum of $\mathcal{V}_t$-algebras can be reconstructed as a Lallement sum of these algebras. The usefulness of Lallement sums depends on properties of the available extensions $E_s^\star$. In particular, a nice situation appears if for each $s \in S$, all extensions $E_s^\star$ coincide with the summand A_s. Such Lallement sums are called *strict*.

We next consider a special case of strict Lallement sums of $\mathcal{V}_t$-algebras, which extends a construction described for Birkhoff systems in [9]. This construction works nicely for $\mathcal{V}_t$-algebras where each operation $\star \in \Omega$ has a one-sided unit. In what follows we assume that all one-sided units are right-sided and call them simply units. We will call such algebras briefly *$\mathcal{V}_t$-algebras with units*.

Let A be a semilattice sum $\bigsqcup_{s \in S} A_s$ of $\mathcal{V}_t$-algebras A_s with units, and let $e_s^\star$ be the (right-)unit element of $\star$ in A_s. Additionally assume that for all $s, t, u \in S$ with $u \le s, t$

$$(a_s \star b_t) \star e_u^\star = (a_s \star e_u^\star) \star (b_t \star e_u^\star), \tag{8.1}$$

and for $t \le s$ in S define the maps

$$\varphi_{s,t}^\star \colon A_s \to A_t; \; a_s \mapsto a_s \star e_t^\star.$$

Note that $t \le s$ in S means $s \cdot t = t$, so these maps are well defined. It is easily seen that each $\varphi_{s,t}^\star$ is a $\star$-homomorphism from A_s into A_t. Moreover, by (8.1),

$$a_s \star b_t = (a_s \star b_t) \star e_{st}^\star = (a_s \star e_t^\star) \star (b_s \star e_t^\star) = \varphi_{s,s\cdot t}^\star(a_s) \star \varphi_{t,s\cdot t}^\star(b_t).$$

Then for each $u \le s \cdot t$ in S

$$\begin{aligned}
\varphi_{s\cdot t,u}^\star(\varphi_{s,s\cdot t}^\star(a_s) \star \varphi_{t,s\cdot t}^\star(b_t)) &= \varphi_{s\cdot t,u}^\star(a_s \star b_t) \\
&= (a_s \star b_t) \star e_u^\star = (a_s \star e_u^\star) \star (b_t \star e_u^\star) = \varphi_{s,u}^\star(a_s) \star \varphi_{t,u}^\star(b_t).
\end{aligned}$$

It follows that the maps $\varphi_{s,t}^\star$ satisfy the requirements of a strict Lallement sum. One obtains the following theorem, a corollary of Theorem 624 of [23], and an extension of Theorem 4.15 of [9].

Theorem 8.2. *Let A be a semilattice sum of $\mathcal{V}_t$-algebras A_s with units of a plural binary type. Then $A = \bigsqcup_{s \in S} A_s$ satisfies the condition* (8.1) *if and only if it is a strict Lallement sum of the subalgebras A_s over the semilattice S given by the homomorphisms $\varphi^\star_{s,t}$ described above.*

Examples are provided by Birkhoff systems which are semilattices of bounded lattices, and more general bisemillatices which are semilattice sums of bounded lattices. In particular, members of the pseudo-regularizations of the variety $\mathcal{L}$ of lattices provided above are of this type.

Note also that the condition (8.1) holds for all algebras A of plural binary type with self-entropic operations, i.e., satisfying $(x \star y) \star (z \star t) = (x \star z) \star (y \star t)$, for each $\star \in \Omega$.

One more final remark concerns the assumption about strong irregularity of the summands A_s. In this paper, we were interested in semilattices sums of algebras in a strongly irregular variety. However, both Definition 8.1 and Theorem 8.2 may be extended easily to semilattice sums of algebras in any variety.

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