ENTIRE AND ANCIENT SOLUTIONS OF A SUPERCRITICAL SEMILINEAR HEAT EQUATION

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ABSTRACT. We consider the semilinear heat equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$. Assuming that $N \geq 3$ and p is greater than the Sobolev critical exponent $(N+2)/(N-2)$, we examine entire solutions (classical solutions defined for all $t \in \mathbb{R}$) and ancient solutions (classical solutions defined on $(-\infty, T)$ for some $T < \infty$). We prove a new Liouville-type theorem saying that if p is greater than the Lepin exponent $p_L := 1 + 6/(N-10)$ ($p_L = \infty$ if $N \leq 10$), then all positive bounded radial entire solutions are steady states. The theorem is not valid without the assumption of radial symmetry; in other ranges of supercritical p it is known not to be valid even in the class of radial solutions. Our other results include classification theorems for nonstationary entire solutions (when they exist) and ancient solutions, as well as some applications in the theory of blowup of solutions.

1. **Introduction.** Entire and ancient solutions play an important role in studies of singularities and long-time behavior of solutions of many evolution problems. In that vein, of prominent importance are entire and ancient solutions of some specific equations which serve as scaling limits of many other equations with a given structure.

In this paper, we consider the semilinear heat equation

$$u_t = \Delta u + u^p, \tag{1.1}$$

where $u = u(x,t) > 0$, $x \in \mathbb{R}^N$, and $p > 1$. We investigate positive classical solutions of the problems

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N, \ t \in (-\infty, \infty), \tag{1.2}$$

(entire solutions of (1.1)), and

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N, \ t \in (-\infty, T), \tag{1.3}$$

where $T < \infty$ (ancient solutions of (1.1)).

Note that equation (1.1) is invariant under the scaling

$$u(x,t) \mapsto \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t).$$

With respect to the same scaling, (1.1) can be considered as the scaling limit of a large class of equation whose nonlinearities have polynomial growth, such as equations of the form

$$u_t = \Delta u + u^p + g(u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where g is a continuous function with $\lim_{u \to \infty} u^{-p} g(u) = 0$. More specifically, applying the above scaling to equation (1.4) and taking formally $\lambda \to \infty$, one obtains equation (1.1). Of course, the connection between (1.4) and (1.1) is not just formal; it is well known that with good understanding of (1.1), in particular of its entire and ancient solutions, one can draw interesting conclusions about solutions of the Cauchy problem for (1.4) (Corollary 1.2 below is an example of this).

We are mainly interested in radially symmetric solutions of (1.2) and (1.3). If no confusion seems likely, we will often consider a radial solution u as a function of $r := |x|$ and t, i.e. $u = u(r, t)$.

The simplest entire solutions are steady states. Positive steady states of (1.2) exist if and only if $p \geq p_S$, where

$$p_S := \begin{cases} \frac{N+2}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N \leq 2 \end{cases}$$

is the Sobolev exponent (see [13], [7] or [36]). If $p \geq p_S$, then radial positive steady states form a one-parameter family $\{\phi_\alpha\}_{\alpha > 0}$, where $\phi_\alpha(0) = \alpha$. These solutions are ordered—that is, $\phi_\alpha < \phi_\beta$ for $\alpha < \beta$—if and only if $p \geq p_{JL}$, where

$$p_{JL} := \begin{cases} 1 + 4 \frac{N - 4 + 2\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ +\infty & \text{if } N \leq 10, \end{cases}$$

see [39] or [36]. Ordered or not, the family $\{\phi_\alpha\}_{\alpha > 0}$ approaches as $\alpha \to \infty$ the singular steady state

$$\phi_\infty(x) := L|x|^{-2/(p-1)}, \qquad L := \left(\frac{2}{(p-1)^2} ((N-2)p - N) \right)^{\frac{1}{p-1}},$$

which has a special role in this paper. It exists whenever $p(N-2) > N$.

In regard to time-dependent entire solutions, denoting

$$p^* := \begin{cases} \frac{N(N+2)}{(N-1)^2} & \text{if } N > 2, \\ +\infty & \text{if } N \leq 2, \end{cases}$$

the following Liouville-type theorem is known (see [31, 5, 34]):

Theorem 1.1. *If $p < p_S$, then (1.2) does not possess positive radial solutions. If $p < p^*$, then (1.2) does not possess any positive solutions.*

Nonexistence of positive (non-radial) solutions of (1.2) for $p \in [p^*, p_S)$ is still an open problem. On the other hand, a nonexistence result for sign-changing radial solutions has been obtained in [3].

Theorem 1.1 has a number of interesting applications in equations (1.1), (1.4), and even more general problems [31]. As an illustration, we just state the following optimal universal estimate for positive solutions of (1.4) on any time interval (τ, T) (see [31, Theorem 3.1]).

Corollary 1.2. *Assume g is a continuous function such that $u^{-p}g(u) \to 0$ as $u \to \infty$ and let u be a positive solution of* (1.4) *on an interval (τ, T). Assume that either u is radial and $p < p_S$, or $p < p^*$. Then*

$$\|u(\cdot, t)\|_\infty \leq C(1 + (t-\tau)^{-1/(p-1)} + (T-t)^{-1/(p-1)}) \quad \text{for all} \quad t \in (\tau, T), \quad (1.5)$$

where $C = C(g, n)$ is a constant independent of u, τ, and T. If $g \equiv 0$, then the following stronger version of (1.5) *holds:*

$$\|u(\cdot, t)\|_\infty \leq C((t-\tau)^{-1/(p-1)} + (T-t)^{-1/(p-1)}) \quad \text{for all} \quad t \in (\tau, T). \quad (1.6)$$

Since C is independent of τ, taking $\tau \to -\infty$, we obtain from (1.6) the following estimates for ancient solutions of (1.1):

$$\|u(\cdot, t)\|_\infty \leq C(T-t)^{-1/(p-1)} \quad \text{for all} \quad t \in (-\infty, T). \quad (1.7)$$

For ancient solutions satisfying (1.7) the following classification theorem has been proved in [23]:

Theorem 1.3. *Let $p < p_S$ and u be a positive solution of* (1.3) *satisfying*

$$\|u(\cdot, t)\|_\infty \leq C(T-t)^{-1/(p-1)} \quad \text{as} \quad t \to -\infty. \quad (1.8)$$

Then there exists $T^* \geq T$ such that $u(x, t) = \kappa(T^* - t)^{-1/(p-1)}$, where

$$\kappa := (p-1)^{-1/(p-1)}.$$

(In this theorem and below, we use C, C_1 etc., to denote constants independent of the solution in question.) Thus, Corollary 1.2 in conjunction with Theorem 1.3 shows that the only positive radial ancient solutions are the (spatially constant) ancient solutions of the equation $\dot{\xi} = \xi^p$ (if $p < p^*$, the word "radial" can be omitted in this statement). Theorem 1.3 has other interesting and important consequences in the study of the blowup behavior of solutions of (1.1), which can be found in [23].

The above results are all concerned with the subcritical case $p < p_S$. Of course, in the critical or supercritical cases, the existence of positive radial steady states has to be taken into account in the formulation of any Liouville-type theorems or problems. A first natural question is whether there are any positive entire solutions other than the steady states. In some cases, this question has been answered in the negative, but only when rather severe extra bounds on the solutions are imposed. Namely, the following Liouville-type results are known (see [12, Theorem 2.4] and [33, Theorem 1.2]).

Theorem 1.4. *Let u be a nonnegative solution of* (1.3).
  (i) *Assume $p_S \leq p < p_{JL}$ and $u(\cdot, t) \leq \phi_\infty$ for all $t \leq T$. Then $u \equiv 0$.*
 (ii) *Assume $p > p_{JL}$ and $\phi_\alpha \leq u(\cdot, t) \leq \phi_\infty$ for some $\alpha > 0$ and all $t \leq T$. Then $u(\cdot, t) \equiv \phi_\gamma$ for some $\gamma \geq \alpha$.*

Without the extra bounds, these results are not valid, at least in the range $p_S \leq p < p_L$, where

$$p_L := \begin{cases} 1 + \frac{6}{N-10} & \text{if } N > 10, \\ +\infty & \text{if } N \leq 10 \end{cases}$$

is the critical exponent for the existence of positive bounded non-constant radial steady states of a rescaled equation (see (1.10) below). Notice that $p_L > p_{JL}$ if $N > 10$. Positive radial bounded solutions of (1.2) which do depend on time are provided by the following results of [12].

Theorem 1.5. (i) *If $p_S < p < p_L$, then there exists a positive radial bounded solution u of* (1.2) *satisfying* $\lim_{|t| \to \infty} \|u(\cdot, t)\|_\infty = 0$ *(i.e. u is a homoclinic solution to the trivial steady state). In addition, given $T \in \mathbb{R}$, u also satisfies* (1.8).

(ii) *If $p_S \le p < p_{JL}$ and ϕ is a positive radial steady state of* (1.2)*, then there exists a positive radial bounded solution of* (1.2) *satisfying*

$$\lim_{t \to -\infty} \|u(\cdot, t) - \phi\|_\infty = \lim_{t \to \infty} \|u(\cdot, t)\|_\infty = 0$$

(i.e. u connects ϕ to zero).

With the above results, the problem of the existence of positive radial entire (nonstationary) solutions is settled for all $p < p_L$. One of the primary objectives of our present study is to address the problem in the range $p > p_L$. We have the following result, the main Liouville-type theorem of this paper.

Theorem 1.6. *Assume $p > p_L$. Then any positive radial bounded solution of* (1.2) *is a steady state.*

The proof of this theorem is given in Section 3; as it is rather involved, we precede it by an informal outline.

Theorem 1.6 is not valid without the assumption of radial symmetry. Indeed, as indicated in a remark following Theorem 2.1 in [12], one can find nontrivial entire solutions by extensions of solutions in lower dimensions. To make this remark more precise, fix any $p > p_L$. Then one can always find an integer $j \in \{3, \ldots, N-1\}$ such that p is between $p_S(j)$ and $p_L(j)$, the Sobolev and Lepin exponents in dimension j. Take now an entire solution $u(\tilde{x}, t)$, $\tilde{x} \in \mathbb{R}^j$, as provided by Theorem 1.5(i). Viewing u as a function of t and x, constant in the last $N - j$ variables, we obtain a positive bounded nonstationary entire solution of (1.1).

Similarly as in the subcritical case, the Liouville theorem for $p > p_L$ has important applications. For example, we will show in Section 5 that Theorem 1.6 can be used to prove the convergence of profiles of both global and blowing-up solutions.

When nonstationary entire solutions do exist, it is still an interesting question if they can be classified in some way. Our next theorem gives a classification of entire solutions satisfying (1.8). Its conclusion is, in a sense, complementary to Theorem 1.5(i) in the case $p_S < p < p_{JL}$.

Theorem 1.7. *If $p_S < p < p_{JL}$ and u is a positive radial bounded solution of* (1.2) *satisfying* (1.8)*, then* $\lim_{t \to \infty} \|u(\cdot, t)\|_\infty = 0$ *(hence, u is a homoclinic solution to the trivial steady state).*

We believe that the same statement is valid if $p_{JL} \le p < p_L$, but presently we can only prove this under an additional condition (see Remark 1.11 below).

We now consider ancient solutions. In order to describe our results, we introduce the backward similarity variables

$$y := \frac{x}{\sqrt{T-t}}, \ \ s := -\log(T-t),$$

and the rescaled function

$$\begin{aligned}
v(y, s) &:= (T-t)^{1/(p-1)} u(x, t) \\
&= e^{-s/(p-1)} u(e^{-s/2} y, T - e^{-s}).
\end{aligned} \tag{1.9}$$

Notice that if u solves (1.3), then v is an entire solution of the equation

$$v_s = \Delta v - \frac{y}{2} \cdot \nabla v - \frac{v}{p-1} + v^p, \qquad y \in \mathbb{R}^N, \ s \in (-\infty, \infty). \tag{1.10}$$

Problem (1.10) has a positive constant steady state $v \equiv \kappa$ for all $p > 1$ and the singular steady state ϕ_∞ whenever $p(N-2) > N$. Positive bounded non-constant radial steady states of (1.10) exist if $p \in (p_S, p_L)$, while such solutions do not exist if $p > p_L$, see [18, 24] and references therein. In the case $p = p_L$, the nonexistence is stated in the main result of [25], however the proof given there contains a gap, which does not seem to have been fixed yet.

We have the following result concerning ancient solutions.

Theorem 1.8. *Let either $p_S < p < p_{JL}$ or $p > p_L$. Let u be a positive radial solution of (1.3), and let v denote the corresponding rescaled function.*

If u satisfies (1.8), then v is either a positive bounded radial steady state of (1.10) or connects a positive bounded radial steady state w of (1.10) to a nonnegative bounded radial steady state $\tilde{w} \neq w$ of (1.10):

$$\lim_{s \to -\infty} v(\cdot, s) = w, \quad \lim_{s \to \infty} v(\cdot, s) = \tilde{w}, \tag{1.11}$$

with the convergence in $C^1_{loc}(\mathbb{R}^N)$.

If (1.8) fails, then v connects the singular steady state ϕ_∞ to a nonnegative bounded radial steady state $\tilde{w}$ of (1.10), that is, (1.11) holds with $w = \phi_\infty$, where the convergence is in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$ in the case of w and in $C^1_{loc}(\mathbb{R}^N)$ in the case of $\tilde{w}$.

Thus, if $p_S < p < p_{JL}$ or $p > p_L$, the positive radial ancient solutions can be classified as heteroclinic connections in self-similar variables, possibly with the singular backward limit. This statement in the regular backward limit case (the first part of Theorem 1.8) can be viewed as a (radial) analogue of Theorem 1.3 in the given supercritical ranges of p. Indeed, using the rescaled function v, Theorem 1.3 can be formulated as follows (see [23, Corollary 1.5]):

Remark 1.9. Let $p < p_S$ and u be a positive solution of (1.3) satisfying (1.8). Then the rescaled function v is either equal to the constant κ or there exists $s_0 \in \mathbb{R}$ such that $v(y, s) = \varphi(s - s_0)$, where $\varphi(s) := \kappa(1 + e^s)^{-1/(p-1)}$ (hence v connects κ to zero).

As an application of Theorem 1.8, we now examine the character of blowup of ancient solutions. First we recall some terminology. Let u be a positive radial solution of (1.1) defined on a time interval $(0, T)$. This solution is said to blow up at $t = T$ if $\|u(\cdot, t)\|_\infty \to \infty$ as $t \to T$. The blowup is of type I if the function $(T - t)^{1/(p-1)}\|u(\cdot, t)\|_\infty$ stays bounded as $t \to T$, otherwise it is of type II. As proved in [14] (see also Corollary 1.2 above), type II blowup never occurs if $p < p_S$ (this is also true with the assumption of radial symmetry dropped). The absence of type II blowup is also known for some classes of radial solutions (for example, radially nonincreasing solutions) if $p_S \leq p < p_{JL}$ [19, 20, 27]. On the other hand, type II blowup is known to occur for some positive radial solutions if $p \geq p_{JL}$ (see [16, 26, 21, 37]). Let us now add the assumption that u is an ancient solution. Our question is whether from the fact that u has some "past" one can draw a definite conclusion about the type of its blowup. If $p_S < p < p_{JL}$ or $p > p_L$, we can give a positive answer:

Corollary 1.10. *Let either $p_S < p < p_{JL}$ or $p > p_L$. Let u be a positive radial solution of (1.3). If u blows up at $t = T$, then the blowup is of type I.*

This result follows directly from Theorem 1.8, which gives a bound on $(T - t)^{1/(p-1)}u(\cdot, t)$ in any compact set, and the universal estimate (2.3) proved in Proposition 2.1 below, which yields a bound on this function away from the origin in $\mathbb{R}^N$.

Remark 1.11. We conclude the introduction with a few remarks concerning exponents p not covered by the above results. As previously mentioned, we expect Theorem 1.7 to hold in the range $p_{JL} \leq p < p_L$ and can actually prove this (see Section 5) under an additional condition. Specifically, the condition requires that each classical positive radial steady state w of (1.10) satisfy the relation $E(w) < E(\phi_\infty)$, where E is the standard energy functional for equation (1.10) (see Subsection 2.2). In Section 5 we also give some heuristics as to why the energy condition is plausible, but it is not clear to us if it can be proved by any readily available tools. In the borderline case $p = p_L$, the statement of Theorem 1.7 is most likely void, for we do not expect any positive radial bounded solution of (1.2) to exist—$p = p_L$ is not included in Theorem 1.6 for several technical reasons. In Theorem 1.8 (and Corollary 1.10), we left out the range $p_{JL} \leq p \leq p_L$. Again, we believe that both statements of Theorem 1.8 are valid in this range as well, but can only prove it under the above energy condition (see Remark 5.3).

The rest of the paper is organized as follows. The next section contains several preliminary results concerning the energy functional for (1.10), zero number for differences of solutions of equations (1.2), (1.3) and their rescaled versions, and the α- and ω-limit sets of solutions of (1.10). In the same preliminary section, we also give universal a priori estimates on radial entire and ancient solutions, and examine the relation of two radial solutions of (1.10) for large values of $\rho = |y|$. The proof of Theorem 1.6 and its informal outline are given in Section 3. Section 4 is devoted to the proofs of Theorems 1.7, 1.8. In Section 5, we discuss some applications of our results. In particular, we state and prove there a theorem on the convergence of profiles of blowup solutions.

2. **Preliminaries.** In the rest of this paper, we consider radial solutions only, although some of the results in this preliminary section, notably those concerning the energy functional, hold for nonradial solutions. Notice that radial solutions of (1.2) or (1.3), viewed as functions of r and t, satisfy the equation

$$u_t = u_{rr} + \frac{N-1}{r}u_r + u^p \quad \text{in} \ (0, \infty) \times (-\infty, T) \tag{2.1}$$

with $T \leq \infty$, and the rescaled functions $v = v(\rho, s)$ (where $\rho := |y|$) satisfy the equation

$$v_s = v_{\rho\rho} + \frac{N-1}{\rho}v_\rho - \frac{\rho}{2}v_\rho - \frac{v}{p-1} + v^p \quad \text{in} \ (0, \infty) \times (-\infty, \infty). \tag{2.2}$$

2.1. **Universal estimates.** The following universal estimates for positive radial solutions u of (1.2), (1.3) and the corresponding rescaled functions v will play an important role in our analysis. Notice first that if v is any solution of (2.2) and u is defined by (1.9), then u is a solution of (2.1), hence any solution v of (2.2) corresponds to a solution u of (2.1).

Proposition 2.1. *Assume $p > 1$. Then there exists $C = C(N, p) > 0$ with the following properties: If $u = u(r, t)$ is a positive solution of (2.1) in $Q_T := (0, \infty) \times (-\infty, T)$ with $T \leq \infty$, then*

$$u(r, t) + |u_r(r, t)|^{2/(p+1)} + |u_{rr}(r, t)|^{1/p} \leq C(r^{-2/(p-1)} + m(t)) \ \text{in} \ Q_T, \tag{2.3}$$

where $m(t) = (T - t)^{-1/(p-1)}$ if $T < \infty$ and $m(t) = 0$ if $T = \infty$. If $T < \infty$, then the corresponding rescaled function $v = v(\rho, s)$ satisfies

$$v(\rho, s) + |v_\rho(\rho, s)|^{2/(p+1)} + |v_{\rho\rho}(\rho, s)|^{1/p} \leq C(\rho^{-2/(p-1)} + C_T) \quad in \ \ Q_\infty, \qquad (2.4)$$

where $C_T = 1$. If u is an entire solution and v is defined by (1.9) with $T < \infty$, then (2.4) is true with $C_T = 0$.

Proof. The proof is a straightforward modification of the doubling and rescaling arguments in [31] and the Liouville theorem for positive solutions of (1.2) with $N = 1$; cp. also [3]. First notice that (2.3) and (1.9) imply (2.4), hence it is sufficient to prove (2.3). In addition, (2.3) with $T = \infty$ is a consequence of (2.3) with $T < \infty$ since the constant C does not depend on T. Consequently, we may assume $T < \infty$.

Set

$$M[u](r, t) := u(r, t)^{(p-1)/2} + |u_r(r, t)|^{(p-1)/(p+1)} + |u_{rr}(r, t)|^{(p-1)/2p}.$$

Assume that (2.3) is not true. Then there exist T_k, solutions u_k of (2.1) in Q_{T_k} and points $(r_k, t_k) \in Q_{T_k}$ such that

$$M_k := M[u_k](r_k, t_k) > 2k/d_k(r_k, t_k), \quad k = 1, 2, \ldots, \qquad (2.5)$$

where $d_k(r, t) := \min(r, \sqrt{T_k - t})$ denotes the parabolic distance of (r, t) to the topological boundary of Q_{T_k}. Then [30, Lemma 5.1] guarantees that after possible modification of (r_k, t_k), (2.5) holds and, in addition, we may assume $M[u_k](r, t) \leq 2M_k$ whenever $|r - r_k| + \sqrt{|t - t_k|} < k/M_k$. Set

$$U_k(\rho, s) := \lambda_k^{2/(p-1)} u_k(r_k + \lambda_k \rho, t_k + \lambda_k^2 s),$$

where $\lambda_k := 1/M_k$. Then U_k satisfies the equation

$$U_s = U_{\rho\rho} + \frac{N-1}{r_k/\lambda_k + \rho} U_\rho + U^p,$$

$U_k, (U_k)_\rho, (U_k)_{\rho\rho}$ are bounded in $\{(\rho, s) : |\rho| + \sqrt{|s|} < k\}$ by a constant independent of k, and $U_k(0, 0) + |(U_k)_\rho(0, 0)| + |(U_k)_{\rho\rho}(0, 0)| \geq c_0 > 0$. Clearly, $r_k/\lambda_k \to \infty$. Using standard parabolic estimates, we conclude that (a suitable subsequence of) $\{U_k\}$ converges to a positive solution of (1.2) with $N = 1$. But this contradicts the corresponding Liouville theorem, see [31], for example. $\qquad\square$

2.2. Lyapunov functional.
Equation (1.10) can also be written in the form

$$v_s = \frac{1}{\varrho} \nabla \cdot (\varrho \nabla v) - \frac{v}{p-1} + v^p, \qquad y \in \mathbb{R}^N, \ s \in (-\infty, \infty), \qquad (2.6)$$

where ϱ is the Gaussian weight defined by

$$\varrho(y) := e^{-|y|^2/4}.$$

It is known that this problem possesses the Lyapunov functional

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + \frac{1}{2(p-1)} w^2 - \frac{1}{p+1} w^{p+1} \right) \varrho \, dy.$$

More precisely, we have the following proposition (see [36, Proposition 23.8] for more details; note that the assumption $v(\cdot, s_0) \in BC^1(\mathbb{R}^N)$ in [36] is satisfied for radial solutions of (2.6) due to Proposition 2.1 and the fact that we consider classical solutions).

Proposition 2.2. *Let $p > 1$ and let v be a positive radial solution of* (2.6). *Then* $E(v(\cdot, s)) \geq 0$ *and*

$$\frac{d}{ds} E(v(\cdot, s)) = - \int_{\mathbb{R}^N} v_s^2(y, s) \varrho(y) \, dy, \tag{2.7}$$

for all $s \in \mathbb{R}$.

Notice also that

$$E(w) = \Big(\frac{1}{2} - \frac{1}{p+1} \Big) \int w^{p+1} \varrho \, dy > 0 \tag{2.8}$$

for any bounded positive radial steady state w of (2.6) (or (1.10)) and this also remains true for the singular steady state ϕ_∞ if $p > p_S$ since $\phi_\infty \in H^1_{loc} \cap L^{p+1}_{loc}(\mathbb{R}^N)$ for such p.

It is known that if $p > p_S$ and w is a positive radial non-constant steady state of (2.6) or $w = \phi_\infty$, then $E(w) > E(\kappa)$, see [21, Remark 1.17]. In particular,

$$E(\phi_\infty) > E(\kappa). \tag{2.9}$$

The proof of (2.9) in [21] is quite long and involved. In the proof of the following proposition we use a simpler and more direct argument to prove (2.9) (cf. also the beginning of Subsection 3.3 in [21]). This argument enables us also to show that the ratio $E(\phi_\infty)/E(\kappa)$ is monotone with respect to p.

Proposition 2.3. *Let $N > 2$ and $F : (p_S, \infty) \to \mathbb{R}$ denote the function $p \mapsto E(\phi_\infty)/E(\kappa)$. Then F is decreasing, $\lim\limits_{p \to p_S} F(p) = \infty$ and $\lim\limits_{p \to \infty} F(p) = 1$.*

Proof. Set $\xi := (p+1)/(p-1)$. Then a straightforward calculation based on (2.8) shows $F(p) = f(\xi)$, where

$$f : (1, N/2) \to \mathbb{R} : \xi \mapsto \frac{\Gamma(N/2 - \xi)}{\Gamma(N/2)} \Big(\frac{N - (1+\xi)}{2} \Big)^\xi,$$

and Γ stands for the standard gamma function. Since $\lim\limits_{\xi \to 1} f(\xi) = 1$ and $\lim\limits_{\xi \to N/2} f(\xi) = \infty$, it is sufficient to prove $f'(\xi) > 0$ for $\xi \in (1, N/2)$. This inequality is equivalent to

$$\psi\Big(\frac{N}{2} - \xi \Big) < \log \frac{N - (1+\xi)}{2} - \frac{\xi}{N - (1+\xi)}, \tag{2.10}$$

where

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} < \log z - \frac{1}{2z} \quad \text{for} \quad z > 0,$$

see [1, 6.3.21]. Consequently, to prove (2.10) it is sufficient to show

$$\log \frac{N - 2\xi}{2} - \frac{1}{N - 2\xi} \leq \log \frac{N - (1+\xi)}{2} - \frac{\xi}{N - (1+\xi)},$$

which is equivalent to

$$\log\Big(1 + \frac{\xi - 1}{N - 2\xi}\Big) \geq \frac{\xi}{N - (1+\xi)} - \frac{1}{N - 2\xi}.$$

Setting $\eta := (\xi - 1)/(N - 2\xi)$, the last inequality is equivalent to

$$\log(1 + \eta) \geq \eta - \frac{\eta}{\eta + 1} \frac{N\eta + 1}{N - 2} \qquad (\eta > 0).$$

Using the estimate $(N\eta + 1)/(N - 2) \geq \eta$ we see that it is sufficient to show

$$\log(1 + \eta) \geq \frac{\eta}{\eta + 1} \qquad (\eta > 0).$$

The last inequality is easy to prove (consider the derivatives of the left and right hand sides, for example). $\square$

2.3. **Zero number.** Recall that radial solutions of (1.2) or (1.3) satisfy equation (2.1) with $T \leq \infty$, and the boundary condition $u_r(0,t) = 0$, and the rescaled functions $v = v(\rho, s)$ satisfy equation (2.2) and the boundary condition $v_\rho(0,s) = 0$. The singular steady state $\phi_\infty = \phi_\infty(r)$ satisfies both (2.1) and (2.2) and the boundary condition $\phi_\infty(0) = \infty$.

If u_1, u_2 are radial solutions of (1.2) or (1.3) (or u_1, u_2 are radial solutions of (1.10)), then $U := u_1 - u_2$ solves the linear equation

$$U_t = U_{rr} + \frac{N-1}{r}U_r - c\frac{r}{2}U_r + fU \quad \text{in } (0,\infty) \times (-\infty, T) \qquad (2.11)$$

and satisfies the boundary condition $U_r(0,t) = 0$, where $T \leq \infty$, $c \in \{0,1\}$ and $f = f(r,t)$ is in $L^\infty((0,\infty) \times (t_1, t_2))$ whenever $-\infty < t_1 < t_2 < T$ (the boundedness comes from Proposition 2.1 and the fact that we consider classical solutions). If $u_1 \equiv \phi_\infty$ and u_2 is as above, then U satisfies (2.11), the boundary condition $U(0,t) = \infty$, and $f \in L^\infty((\delta, \infty) \times (t_1, t_2))$ for any $\delta > 0$.

If $I \subset [0, \infty)$ is an interval and $g : I \to \mathbb{R}$ is a continuous function, we denote by $z_I(g)$ the number of zeros of g in I. We also set $z(g) = z_{(0,\infty)}g$.

The next proposition follows from zero number theorems of [8, 22].

Proposition 2.4. *Let U be as above, $U \not\equiv 0$, $t_1 < t_2 < T$. Then we have:*
  (i) *The function $t \mapsto z(U(\cdot, t))$ is nonincreasing. If $z(U(\cdot, t_1)) < \infty$ and*

$$U(r_0, t_0) = U_r(r_0, t_0) = 0 \text{ for some } r_0 \geq 0 \text{ and } t_0 \in (t_1, t_2), \qquad (2.12)$$

then

$$z(U(\cdot, t)) > z(U(\cdot, s)) \text{ for all } t_1 < t < t_0 < s < t_2. \qquad (2.13)$$

  (ii) *Assume $R > 0$, $U(R,t) \neq 0$ for all $t \in [t_1, t_2]$. Then the function $t \mapsto z_{(0,R)}(U(\cdot, t))$ is nonincreasing and finite. If (2.12) is true for some $r_0 \in [0, R)$, then (2.13) is true with z replaced by $z_{(0,R)}$.*

2.4. **Steady states and limit sets of** (2.2)**.** In what follows we assume that v is a positive solution of (2.2) and $p > p_S$. Estimate (2.4) guarantees that the Lyapunov functional $E(v(\cdot, s))$ is uniformly bounded for $s \in \mathbb{R}$ and $E(v(\cdot, t_k)) \to E(w)$ whenever $v(\cdot, t_k) \to w$ in $C^1_{loc}(0, \infty)$. Consequently, standard arguments (see Appendix G in [36], for example) show that the α- and ω-limit sets

$$\alpha(v) := \{w \in C^1(0, \infty) : (\exists t_k \to -\infty) \, v(\cdot, t_k) \to w \text{ in } C^1_{loc}(0, \infty)\},$$

$$\omega(v) := \{w \in C^1(0, \infty) : (\exists t_k \to \infty) \, v(\cdot, t_k) \to w \text{ in } C^1_{loc}(0, \infty)\},$$

are nonempty connected sets consisting of nonnegative steady states of (2.2). In addition, if v corresponds to an entire solution u (hence (2.4) is true with $C_T = 0$) and v is bounded in $(0, \infty) \times (T_1, T_2)$ for some $-\infty \leq T_1 < T_2 \leq \infty$, then the convergence $v(\cdot, t_k) \to w$ in $C^1_{loc}(0, \infty)$ with $t_k \in (T_1, T_2)$ implies the convergence $v(\cdot, t_k) \to w$ in $BC^1(0, \infty)$.

We now summarize further useful properties of $\alpha(v)$ and $\omega(v)$ reflecting the structure of steady states of the present problem. In particular, we show that $\alpha(v)$ and $\omega(v)$ are singletons.

First note that estimate (2.4) with $C_T = 0$ implies $\kappa \notin \alpha(v)$. Our assumption $p > p_S$ guarantees that ϕ_∞ is the only nonnegative steady state of (2.2) satisfying $\limsup_{\rho \to 0} w(\rho) = \infty$, see [25, Theorem 1.2] or [35]. Notice also that $0 \notin \alpha(v)$ since

$E(0) = 0 < E(w)$ for any positive steady state of (2.2) (cp. (2.8)) and $s \mapsto E(v(\cdot, s))$ is decreasing unless v is a steady state.

Any nonnegative steady state w of (2.2) satisfying $\limsup_{\rho \to 0} w(\rho) < \infty$ is uniquely determined by its value at $\rho = 0$. If w is nonconstant, then [24, Lemmas 2.2–2.3] and [4] yield the following relations

$$w(0) > \kappa, \qquad w' < 0 \text{ on } (0, \infty), \qquad z(w - \phi_\infty) \geq 2. \qquad (2.14)$$

Denote by $\mathcal{A}$ the set of $a \in [0, \infty)$ for which there exists a steady state $w_a \geq 0$ of (2.2) satisfying $w_a(0) = a$. By [22, Proposition 2.3 and the proof of Lemma 2.4], for any $a \in \mathcal{A} \setminus \{0, \kappa\}$ there exists $c_a := \lim_{\rho \to \infty} w_a(\rho) \rho^{2/(p-1)} \in (0, \infty) \setminus \{L\}$, and the mapping $a \mapsto c_a : \mathcal{A} \setminus \{0, \kappa\} \to (0, \infty)$ is injective. In particular, $z(w_a - \phi_\infty) < \infty$ for any $a \in \mathcal{A}$. Set

$$\mathcal{A}_k := \{a \in \mathcal{A} : z(w_a - \phi_\infty) = k\}, \qquad k = 0, 1, 2, \dots.$$

By (2.14), $\mathcal{A}_0 = \{0\}$ and $\mathcal{A}_1 = \{\kappa\}$. As proved in [29], the set $\mathcal{A}$ is discrete. This—in conjunction with the uniqueness of the unbounded positive steady state ϕ_∞—shows that for any positive solution v of (2.2), the sets $\alpha(v)$ and $\omega(v)$ are singletons consisting of either ϕ_∞ or w_a for some $a \in \mathcal{A}$.

As already mentioned in the introduction, if $p > p_L$, then $\mathcal{A} = \{0, \kappa\}$, i.e. $w_0 \equiv 0$ and $w_\kappa \equiv \kappa$ are the only bounded nonnegative steady states of (2.2). In this case, each of the sets $\alpha(v)$ and $\omega(v)$ has to be one of the sets $\{\phi_\infty\}$, $\{\kappa\}$, or $\{0\}$. We also know that $\alpha(v) \neq \{0\}$ (and $\alpha(v) \neq \{\kappa\}$ if v corresponds to an entire solution u). Proposition 2.3 guarantees $\omega(v) \neq \{\phi_\infty\}$.

Let now $p_S < p < p_{JL}$. Then each of the sets $\mathcal{A}_k$ is nonempty (see [38, 17, 6, 11, 28] and references therein) and bounded (this follows from the first sentence in the proof of [10, Lemma 2.2], for example), hence finite. On the other hand, an easy contradiction argument shows $\inf \mathcal{A}_k \to \infty$ as $k \to \infty$.

The arguments in the proof of [11, Proposition 2.4] show that if w_1, w_2 are two different positive steady states of (2.2) (possibly unbounded), then

$$w_1(\rho) = w_2(\rho) \text{ for some } \rho > 0 \text{ implies } w_1(\rho) \geq \kappa. \qquad (2.15)$$

Hence, w_1 and w_2 do not intersect for large values of ρ. This is also a consequence of Proposition 2.5 below, where we examine similar intersection properties for time-dependent solutions of (2.2).

2.5. Comparison arguments and intersections of solutions of (2.2) for large ρ. Let v_1, v_2 be two positive solutions of (2.2). Then $V := v_1 - v_2$ satisfies

$$V_s = V_{\rho\rho} + \frac{N-1}{r} V_\rho - \frac{\rho}{2} V_\rho + fV, \qquad (2.16)$$

where

$$f = f(\rho, s) = -\frac{1}{p-1} + \begin{cases} pv_1^{p-1} & \text{if } V(\rho, s) = 0, \\ \frac{v_1^p - v_2^p}{V} & \text{otherwise.} \end{cases}$$

By the Mean Value Theorem,

$$v_1, v_2 \leq C_v \quad \Rightarrow \quad f \leq C_0 := -\frac{1}{p-1} + pC_v^{p-1}. \qquad (2.17)$$

In particular,

$$f \leq -\delta_0 := -\frac{1}{2(p-1)} \quad \text{provided} \quad C_v \leq c_0 := \left(\frac{1}{2p(p-1)} \right)^{1/(p-1)}. \qquad (2.18)$$

Proposition 2.5. *Let $v_1, v_2, V, c_0, \delta_0$ be as above, $s_0 \in \mathbb{R}$, $\rho_0 > 0$ and $V(\rho_0, s_0) \neq 0$. Set*

$$\left.\begin{aligned}
D &:= \{(\rho, s) \in (0, \infty) \times (-\infty, s_0] : V(\rho, s) \neq 0\}, \\
D_0 &:= \text{ the connected component of } D \text{ containing } (\rho_0, s_0), \\
\Omega(s) &:= \{\rho : (\rho, s) \in D_0\}.
\end{aligned}\right\} \tag{2.19}$$

Assume

$$v_2 \leq c_0 \quad \text{in } D_0, \tag{2.20}$$

and

$$\lim_{\rho \to \infty, \ \rho \in \Omega(s)} v_2(\rho, s) = 0, \quad \text{locally uniformly in } s. \tag{2.21}$$

Then $V(\rho_0, s_0) > 0$.

In applications of this proposition, we verify condition (2.21) using an a priori bound, such as (2.4) with $C_T = 0$. By the same a priori bound, we will have (2.20) verified, provided $\rho_1(s) := \inf \Omega(s)$ is large enough for all s.

Notice that if v_1 also satisfies such an a priori bound, then v_1 and v_2 can be interchanged. In this case, Proposition 2.5 says in effect that $\rho_1(s)$ cannot be large for all s. This in particular entails statement (2.15) for steady states, as noted at the end of the previous subsection.

Proof of Proposition 2.5. Let

$$\left.\begin{aligned}
m(s) &:= \sup_{\Omega(s)} |V(\cdot, s)|, \\
S &:= \inf\{s < s_0 : \Omega(s) \neq \emptyset\},
\end{aligned}\right\} \tag{2.22}$$

The proof is by contradiction. Assume that $V(\rho_0, s_0) < 0$. Then

$$0 < v_1 < v_2 \leq c_0 \quad \text{in } D_0, \tag{2.23}$$

hence $m(s) \leq c_0$ for all $s \leq s_0$. The comparison principle used for equation (2.16) together with estimate (2.18) give

$$m(s_0) \leq e^{-\delta_0(s_0 - s)} m(s) \quad \text{for} \quad s \in (S, s_0). \tag{2.24}$$

If $S = -\infty$, then (2.24) and (2.23) yield $m(s_0) = 0$. If $S > -\infty$, then (2.21) and the continuity of V guarantee $m(s) \to 0$ as $s \to S+$, hence $m(s_0) = 0$ again. But $m(s_0) = 0$ contradicts our assumption $V(\rho_0, s_0) \neq 0$. $\qquad \square$

3. **Proof of Theorem 1.6.** The proof of Theorem 1.6 is long and rather technical at places. We first give an outline. Let $u = u(r, t)$ be a positive solution of (1.2) with $p > p_L$. Fixing any $T \in \mathbb{R}$, let v be the corresponding rescaled solution of (2.2). Using considerations in Subsection 2.4, we first show easily that $\alpha(v) = \{\phi_\infty\}$. Thus, formally, v can be viewed as a solution on the unstable manifold of the singular steady state. (The term "manifold" is used loosely here; the manifold structure of the solutions approaching ϕ_∞ backward in time is not actually established.) At the same time, as observed in [33], the solutions of (2.2) corresponding to the radial steady states of the original equation (1.2) form a one-dimensional manifold that can be considered as the principal part of the unstable manifold of ϕ_∞: As time approaches $-\infty$, these rescaled solutions approach ϕ_∞ monotonically and at an exponential rate given by the principal eigenvalue of the linearization of the right-hand side of (2.2) at ϕ_∞. Our main goal is to derive suitable estimates on $\phi_\infty - v$ in order to show that the entire solution v has to lie on the principal part of the

unstable manifold, or, in other words, u is a steady state. This is achieved by careful analysis of the abstract form of equation (2.2) and, in particular, of the remainder on the right-hand side after the linearization has been subtracted from it. This analysis, which is really the crux of our proof, is carried out in the next subsection. We remark that the proof of Theorem 1.4(ii), as given in [33], follows a similar general scenario. However, the bounds $\phi_\alpha \leq u(\cdot, t) \leq \phi_\infty$ assumed there make all the necessary estimates considerably simpler, even when nonradial solutions are allowed; those estimates from [33] are of little help in our present analysis (we make use of other technical results from [33]).

Another ingredient of the proof of Theorem 1.6 is the radial monotonicity of the entire solutions, which we prove in Subsection 3.2 for any $p > p_S$. We then complete the proof of the theorem in Subsection 3.3.

3.1. Linearization of (2.2) at ϕ_∞ and estimates of the remainder.

In this subsection, we first assume assume $p > p_{JL}$ (some abstract results that we recall are valid in this range), and then focus on the case $p > p_L$.

Set $a(\rho) := \rho^{N-1} e^{-\rho^2/4}$. We consider the weighted Lebesgue space

$$X := L^2(0, \infty; a(\rho)d\rho)$$

endowed with the scalar product

$$\langle f, g \rangle := \int_0^\infty f(\rho)g(\rho)a(\rho) \, d\rho$$

and the corresponding norm $\|f\|_X = \langle f, f \rangle^{1/2}$. Let

$$Y := \{f \in H_{loc}^1(0, \infty) : f, f' \in X\}$$

be endowed with the norm $\|f\|_Y := \|f\|_X + \|f'\|_X$. It was shown in [16, Lemma 2.3] that the operator

$$Af := f'' + \Big(\frac{N-1}{\rho} - \frac{\rho}{2}\Big)f' + \Big(\frac{pL^{p-1}}{\rho^2} - \frac{1}{p-1}\Big)f \qquad (3.1)$$

with domain

$$D(A) := \{f \in Y : Af \in X \text{ in the distributional sense}\}$$

can be extended in a unique way to a self-adjoint operator in X (still denoted by A), with the following properties:

(A1) $D(A) \subset Y$,

(A2) $(\exists c_A > 1)(\forall \phi \in D(A)) \ \langle \phi, A\phi \rangle \leq (c_A - 1)\langle \phi, \phi \rangle$,

(A3) the spectrum $\sigma(A)$ consists of a sequence of simple eigenvalues

$$\mu_j := -\Big(\frac{\beta}{2} + \frac{1}{p-1} + j\Big), \quad j = 0, 1, 2, \ldots,$$

where

$$\beta := \frac{1}{2}\Big(-(N-2) + \sqrt{(N-2)^2 - 4pL^{p-1}}\Big) < 0,$$

and the corresponding eigenfunctions (normalized in X) have the form $\vartheta_j(\rho) = \hat{c}_j \rho^\beta M_j(\rho^2/4)$, where $\hat{c}_j > 0$,

$$M_j(z) := M\Big(-j, \beta + \frac{N}{2}, z\Big)$$

and M denotes the standard Kummer function (hence M_j is a polynomial of degree j). Also, for $j = 0, 1, \ldots$, the function ϑ_j has exactly j zeros, all of them positive and simple.

The operator $-\tilde{A} := -A + c_A$ is a positive self-adjoint operator and its fractional powers $(-\tilde{A})^\alpha$ are well defined for all $\alpha \in \mathbb{R}$ (see [2, Section III.4.6]). We denote by $\{(X_\alpha, -A_\alpha) : \alpha \in [-1, 1]\}$ the corresponding fractional interpolation-extrapolation scale of spaces and operators (see [2, Section V.1] for its definition and properties); the norm in X_α will be denoted by $\|\cdot\|_\alpha$. In particular, $X_0 = X$, $A_1 = \tilde{A}$, $X_1 = D(\tilde{A})$, $X_{-1} = X_1'$ (where the duality is taken with respect to the duality pairing $\langle \cdot, \cdot \rangle$). Recall also that this scale is equivalent to the scale generated by $(X, -\tilde{A})$ and the complex interpolation functor $[\cdot, \cdot]_\theta$. The space $X_{1/2}$ is isomorphic to Y, see [16, Lemma 2.4]. By general result of [2, Section V.2], A_α generates an analytic semigroup e^{sA_α} in X_α and the following estimate is true for any $\sigma \geq 0$

$$\|e^{s(A_\gamma + \sigma)}\phi\|_\alpha \leq cs^{\gamma - \alpha} e^{\sigma s}\|\phi\|_\gamma, \qquad -1 \leq \gamma \leq \alpha \leq 1, \quad s > 0. \qquad (3.2)$$

If v is as in Proposition 2.1, $w := \phi_\infty - v$,

$$h := \phi_\infty^p - v^p - p\phi_\infty^{p-1}w,$$

and $f := v(\cdot, s)$ for some s, then estimate (2.4) and formulas [16, (2.52), (2.59)] show that (3.1) and the variation-of-constants formula

$$w(s) = e^{(s-s_0)A}w(s_0) + \int_{s_0}^s e^{(s-\tau)A}h(\cdot, \tau)\,d\tau$$

are true with A replaced by $A_{-1/2} + c_A$. Since no confusion seems likely, in what follows we set $A := A_{-1/2} + c_A$. In particular, estimate (3.2) implies

$$\|e^{sA}\phi\|_\alpha \leq cs^{\gamma - \alpha} e^{c_A s}\|\phi\|_\gamma, \qquad -1/2 \leq \gamma \leq \alpha \leq 1, \quad s > 0. \qquad (3.3)$$

Henceforth we assume that $p > p_L$.

Let $v, w, h, A, \mu_j, \vartheta_j, (X_\alpha, \|\cdot\|_\alpha)$ be as above. Crucial for our proof of Theorem 1.6 is a good understanding of the behavior of v in the following case:

$$0 \leq v(\cdot, s) < \phi_\infty \qquad \text{and} \qquad \alpha(v) = \{\phi_\infty\}. \qquad (3.4)$$

Here, the α-limit set $\alpha(v)$ is as in Subsection 2.4. In the following proposition we prove, loosely speaking, that along a sequence of times the function v approaches ϕ_∞ in the direction of the eigenfunction ϑ_0 and at the rate $\exp(\mu_0 s)$.

Proposition 3.1. *Under the above assumptions and notation, there exist a constant $c > 0$ and a sequence $s_k \to -\infty$ such that*

$$\|w(\cdot, s_k) - ce^{\mu_0 s_k}\vartheta_0\|_0 = o(e^{\mu_0 s_k}) \quad as \quad k \to \infty. \qquad (3.5)$$

Proof. Recall that $p > p_L$ implies that $\mu_0 > \mu_1 > 0 > \mu_2$ (this can be easily checked using the formulas in (A3)).

Let P be the orthogonal projection onto the orthogonal complement of $\{\vartheta_0, \vartheta_1\}$ in X. Let ξ_0, ξ_1 be defined by

$$w(\cdot, s) = \xi_0(s)\vartheta_0 + \xi_1(s)\vartheta_1 + \tilde{w}(\cdot, s), \qquad (3.6)$$

where $\tilde{w}(\cdot, s) := P(w(\cdot, s))$. Since $E(v(\cdot, s)) \to E(\phi_\infty)$ as $s \to -\infty$ (see Section 2.4), we have $w(\cdot, s) \to 0$ in X, hence

$$\|\tilde{w}(\cdot, s)\|_0 \to 0 \quad \text{and} \quad \xi_i(s) \to 0 \ (i = 0, 1) \quad \text{as} \quad s \to -\infty. \qquad (3.7)$$

Our first goal is to prove that

$$\|\tilde{w}(s)\|_0 = o(|\xi_0(s)| + |\xi_1(s)|) \quad \text{as} \quad s \to -\infty. \tag{3.8}$$

We start with some estimates of the function h.

By assumption, $h \le 0 < w$, hence $\langle h, w \rangle \le 0$. In addition,

$$0 \le -h = p(\phi_\infty^{p-1} - v_\theta^{p-1})w \le p(\phi_\infty^{p-1} - v^{p-1})w \le p(p-1)\phi_\infty^{p-2}w^2$$

for some $v_\theta \in (v, \phi_\infty)$, hence, given any $\delta \in [0,1)$,

$$-h \le f_\delta w^{1+\delta}, \tag{3.9}$$

where $f_\delta = f_\delta(\rho, s)$ is given by

$$f_\delta := [p(\phi_\infty^{p-1} - v^{p-1})]^{1-\delta}[p(p-1)\phi_\infty^{p-2}]^\delta.$$

Note that

$$f_\delta(\rho, s) \to 0 \quad \text{as} \quad s \to -\infty,$$

$$|f_\delta(\rho, s)| \le C\rho^{-\nu_\delta}, \quad \text{where} \quad \nu_\delta := 2(1-\delta) + 2\frac{p-2}{p-1}\delta \le 2. \tag{3.10}$$

Choose $\zeta \in (1 - \beta/2 - N/4, 1/2)$, $\zeta \ge 0$. We will specifically take $\zeta = 0$ when $1 - \beta/2 - N/4 < 0$, which is the case if

$$p > p_H := 1 + 4\frac{N + 2\sqrt{N} - 4}{N^2 - 12N + 16}.$$

Clearly, there is $z > 1$ such that

$$\frac{N}{z} > \frac{N}{2} - 2\zeta \quad \text{and} \quad \frac{N}{z'} > 2 - \beta,$$

where $z' := z/(z-1)$. Fixing such z, if $\delta_0 > 0$ is small enough, we have

$$\frac{N}{z(1+\delta)} > \frac{N}{2} - 2\zeta \quad \text{and} \quad \frac{N}{z'} > \nu_\delta - \beta > 2 \quad \text{for any } \delta \in [0, \delta_0]. \tag{3.11}$$

Since $|\vartheta_i(\rho)| \le C\rho^\beta(1 + \rho^{2i})$, $i = 0, 1$ (cp. (A3)), estimate (3.9) gives the following relations (omitting the argument ρ of the indicated functions)

$$|h\vartheta_i a| \le Cf_\delta\rho^\beta(1 + \rho^{2i})w^{1+\delta}a$$

$$\le \begin{cases} C\big((f_\delta\rho^\beta)^{z'}a\big)^{1/z'}\big(w^{(1+\delta)z}a\big)^{1/z}, & \rho \le 1, \\ C\big((f_\delta\rho^{\beta+2})^{2/(1-\delta)}a\big)^{(1-\delta)/2}\big(w^2 a\big)^{(1+\delta)/2}, & \rho > 1. \end{cases} \tag{3.12}$$

Next, the embedding inequalities $\|w\|_0 \le C\|w\|_\zeta$,

$$\|w|_{(0,1)}\|_{L^{z(1+\delta)}(0,1;a(\rho)d\rho)} \le C\|w\|_\zeta$$

(the latter follows from (3.11)), and the Hölder inequality imply

$$|\langle h, \vartheta_i \rangle| \le C\Big[\Big(\int_0^1 (f_\delta\rho^\beta)^{z'}a\,d\rho\Big)^{1/z'} + \Big(\int_1^\infty (f_\delta\rho^{\beta+2})^{2/(1-\delta)}a\Big)^{(1-\delta)/2}\Big]\|w\|_\zeta^{1+\delta}.$$

Now, using (3.10), (3.11), and the Lebesgue theorem, we obtain

$$|\langle h, \vartheta_i \rangle| = o(\|w\|_\zeta^{1+\delta}) \text{ as } s \to -\infty, \quad (i = 0, 1). \tag{3.13}$$

Notice also that $\vartheta_i \in X_1$, hence

$$\|w(\cdot, s) - \tilde{w}(\cdot, s)\|_\alpha \le C(|\xi_0(s)| + |\xi_1(s)|), \qquad \alpha \le 1. \tag{3.14}$$

With the above estimate of the function h at hand, we next examine differential equations for ξ_1, ξ_2, and $\|\tilde{w}\|_0$. Multiplying the equation $w_s = Aw + h$ by $\vartheta_i a$, $i = 0, 1$, and integrating over $(0, \infty)$, we obtain

$$\dot{\xi}_i = \mu_i \xi_i + \langle h, \vartheta_i \rangle, \quad i = 0, 1,$$

hence (3.13) and (3.14) imply

$$\frac{d}{ds}\xi_i^2 = 2\mu_i \xi_i^2 + g_i, \quad i = 0, 1, \tag{3.15}$$

where

$$g_i = o(\|w\|_\zeta^{1+\delta}|\xi_i|)$$
$$= o((\|\tilde{w}\|_\zeta + |\xi_0| + |\xi_1|)^{1+\delta}|\xi_i|)$$
$$= o((\|\tilde{w}\|_\zeta^2 + \xi_0^2 + \xi_1^2)^{1+\delta/2}) \quad \text{as} \quad s \to -\infty.$$

Similarly, multiplying the equation $\tilde{w}_s = A\tilde{w} + Ph$ by $2\tilde{w}$ and using $\langle A\tilde{w}, \tilde{w}\rangle \leq -c_0(\|\tilde{w}\|_{1/2}^2 + \|\tilde{w}\|_0^2)$, $\langle h, w\rangle \leq 0$, and (3.13) we obtain

$$\frac{d}{ds}\|\tilde{w}\|_0^2 = 2\langle A\tilde{w}, \tilde{w}\rangle + 2\langle h, w - \xi_0\vartheta_0 - \xi_1\vartheta_1\rangle$$
$$\leq -2c_0(\|\tilde{w}\|_{1/2}^2 + \|\tilde{w}\|_0^2) + g_2, \tag{3.16}$$

where $g_2 = o((\|\tilde{w}\|_\zeta^2 + \xi_0^2 + \xi_1^2)^{1+\delta/2})$ as $s \to -\infty$.

We are now ready to complete the proof of (3.8). Fix any

$$\varepsilon_0 \in (0, \min(1, \mu_1, c_0)/5).$$

Then there exists s_0 such that

$$|g_i| \leq \varepsilon_0^3(\|\tilde{w}\|_{1/2}^2 + \xi_0^2 + \xi_1^2) \quad \text{for} \quad s \leq s_0, \quad i = 0, 1, 2. \tag{3.17}$$

Assume for a contradiction that there exists $s_1 \leq s_0$ such that

$$\|\tilde{w}(\cdot, s_1)\|_0 \geq 2\varepsilon_0(|\xi_0(s_1)| + |\xi_1(s_1)|).$$

Then (3.16) and the convergence $\|\tilde{w}(\cdot, s)\|_0 \to 0$ as $s \to -\infty$ guarantee the existence of $s_2 < s_1$ such that

$$\|\tilde{w}(\cdot, s)\|_0 \geq \varepsilon_0(|\xi_0(s)| + |\xi_1(s)|) \quad \text{for} \quad s \in [s_2, s_1],$$
$$\|\tilde{w}(\cdot, s_2)\|_0 = \varepsilon_0(|\xi_0(s_2)| + |\xi_1(s_2)|).$$

In addition, setting $\psi := \xi_0^2 + \xi_1^2$, (3.15) implies

$$2\mu_0\psi + g_0 + g_1 \geq \frac{d}{ds}\psi \geq 2\mu_1\psi + g_0 + g_1. \tag{3.18}$$

Integrating (3.16) and the second inequality in (3.18) over (s_2, s_1) we obtain

$$4\varepsilon_0^2\psi(s_1) \leq \|\tilde{w}(\cdot, s_1)\|_0^2$$

$$\leq \|\tilde{w}(\cdot, s_2)\|_0^2 - 2c_0\int_{s_2}^{s_1}\|\tilde{w}\|_{1/2}^2\,ds + \int_{s_2}^{s_1}g_2\,ds$$

$$\leq 2\varepsilon_0^2\psi(s_2) - 2c_0\int_{s_2}^{s_1}\|\tilde{w}\|_{1/2}^2\,ds + \int_{s_2}^{s_1}g_2\,ds$$

$$\leq 2\varepsilon_0^2\psi(s_1) - 2c_0\int_{s_2}^{s_1}\|\tilde{w}\|_{1/2}^2\,ds$$

$$- 2\varepsilon_0^2\mu_1\int_{s_2}^{s_1}\psi\,ds + \int_{s_2}^{s_1}(g_2 - 2\varepsilon_0^2(g_0 + g_1))\,ds,$$

and (3.17) yields a contradiction. Thus, (3.8) is proved.

We now complete the proof of Proposition 3.1, first in the case $p > p_H$, then in the case $p \in (p_L, p_H]$.

Assume $p > p_H$ (notice that this assumption is automatically satisfied if $N \geq 16$ due to $p > p_L$). As noted above, in this case $\zeta := 0$ is our (legitimate) choice. Set

$$\xi := \xi_0^2, \qquad \eta := \xi_1^2 + \|\tilde{w}\|_0^2.$$

Then (3.15) and (3.16) imply

$$\dot{\xi} = 2\mu_0 \xi + o((\xi + \eta)^{1+\delta/2}),$$
$$\dot{\eta} \leq 2\mu_1 \eta + o((\xi + \eta)^{1+\delta/2}).$$

Such differential inequalities are considered in [33]. According to [33, Proposition 4.4(i)], as $s \to -\infty$, we have either $\eta(s) = o(\xi(s))$ or

$$\xi(s) = o(\eta(s)) \quad \text{as} \quad s \to -\infty. \tag{3.19}$$

Assume that (3.19) is true. Then (3.8) implies

$$\|\xi_0(s)\vartheta_0 + \tilde{w}(\cdot, s)\|_0 = o(\|\xi_1(s)\vartheta_1\|_0) \quad \text{as} \quad s \to -\infty. \tag{3.20}$$

Since $\vartheta_1(r)$ changes sign, (3.20) guarantees that $w(\cdot, s) = \xi_0(s)\vartheta_0 + \xi_1(s)\vartheta_1 + \tilde{w}(\cdot, s)$ changes sign for some s, which is a contradiction. Consequently, (3.19) fails and [33, Proposition 4.4(i)] implies

$$\eta(s) = o(\xi(s)) \quad \text{and} \quad \xi(s) + \eta(s) = O(e^{\mu_0 s}) \qquad \text{as} \quad s \to -\infty. \tag{3.21}$$

The previous relations and [33, Proposition 4.4(ii)] further imply

$$\xi(s) = \tilde{c}e^{2\mu_0 s} + o(e^{2\mu_0 s}) \quad \text{as} \quad s \to -\infty, \tag{3.22}$$

where $\tilde{c}$ is a constant. We have $\tilde{c} \neq 0$ due to [33, (4.13)]. Consequently, there exists a constant $c \neq 0$ such that

$$|\xi_0(s) - ce^{\mu_0 s}| + |\xi_1(s)| + \|\tilde{w}(\cdot, s)\|_0 = o(e^{\mu_0 s}) \quad \text{as} \quad s \to -\infty. \tag{3.23}$$

Since $\xi_0\vartheta_0 + \xi_1\vartheta_1 + \tilde{w} = w > 0$, we have $c > 0$ and estimate (3.23) yields

$$\|w(\cdot, s) - ce^{\mu_0 s}\vartheta_0\|_0 = o(e^{\mu_0 s}) \quad \text{as} \quad s \to -\infty.$$

This completes the proof of Proposition 3.1 in the case $p > p_H$.

Next assume $11 \leq N \leq 15$ and $p \in (p_L, p_H]$ (hence $\zeta \in (0, 1/2)$). Taking $s_2 < s_1$ with $s_1 \to -\infty$, (3.16), (3.8) imply

$$c_0 \int_{s_2}^{s_1} \|\tilde{w}\|_{1/2}^2 \, ds \leq \|\tilde{w}(s_2)\|_0^2 + o\left(\int_{s_2}^{s_1} \psi \, ds\right) = o\left(\psi(s_2) + \int_{s_2}^{s_1} \psi \, ds\right). \tag{3.24}$$

Hence, choosing any small $\varepsilon \in (0, 1)$, (3.18) guarantees that

$$(1 - \varepsilon)\psi(s_2) \leq \psi(s_1) - \mu_1 \int_{s_2}^{s_1} \psi \, ds \leq \psi(s_1),$$
$$(1 + \varepsilon)\psi(s_2) \geq \psi(s_1) - 3\mu_0 \int_{s_2}^{s_1} \psi \geq \left(1 - \frac{3\mu_0}{1 - \varepsilon}(s_1 - s_2)\right)\psi(s_1),$$

provided s_1 is negative and sufficiently large. Consequently, there exists $\tau \in (0, 1)$ (independent of s_2) such that for any sufficiently large negative s_2 we have

$$\frac{1}{2}\psi(s_2) \leq \psi(s) \leq 2\psi(s_2), \quad s \in (s_2, s_2 + \tau). \tag{3.25}$$

Now (3.24) and (3.25) imply

$$\int_s^{s+\tau} \|\tilde{w}\|_{1/2}^2 = o\left(\int_s^{s+\tau} \psi \, ds\right) \quad \text{as} \quad s \to -\infty. \tag{3.26}$$

Set

$$\xi(s) := \int_s^{s+\tau} \xi_0^2(\sigma) \, d\sigma, \qquad \eta(s) := \int_s^{s+\tau} (\xi_1^2(\sigma) + \|\tilde{w}(\cdot, \sigma)\|_0^2) \, d\sigma. \tag{3.27}$$

Notice also that for $\delta > 0$ small enough, interpolation, inequality $\|\tilde{w}\|_0 \leq C\|\tilde{w}\|_{1/2}$ and (3.8) imply

$$\|\tilde{w}\|_\zeta^{2(1+\delta/2)} \leq C\|\tilde{w}\|_0^\delta \|\tilde{w}\|_{1/2}^2 = o(\psi^{\delta/2})\|\tilde{w}\|_{1/2}^2. \tag{3.28}$$

Integrating (3.15) with $i = 0$ over the interval $(s, s + \tau)$ and using (3.25), (3.26), and (3.28), we obtain

$$\dot{\xi} = 2\mu_0 \xi + o\left(\int_s^{s+\tau} (\|\tilde{w}\|_\zeta^2 + \psi)^{1+\delta/2} \, d\sigma\right)$$
$$= 2\mu_0 \xi + o(\psi^{1+\delta/2})$$
$$= 2\mu_0 \xi + o((\xi + \eta)^{1+\delta/2}).$$

Similarly, integrating (3.15) with $i = 1$ and (3.16) we obtain

$$\dot{\eta} \leq 2\mu_1 \eta + o((\xi + \eta)^{1+\delta/2}).$$

Again, [33, Proposition 4.4] guarantees the existence of $\tilde{c} \neq 0$ such that (3.21) and (3.22) hold, this time with ξ and η as in (3.27). Consequently, there exist $c \neq 0$ and $s_k \to -\infty$ such that

$$|\xi_0(s_k) - ce^{\mu_0 s_k}| + |\xi_1(s_k)| + \|\tilde{w}(\cdot, s_k)\|_0 = o(e^{\mu_0 s_k}) \quad \text{as} \quad k \to \infty.$$

In addition, similarly as in the case $p > p_H$ we obtain $c > 0$, hence (3.5) is true. $\qquad\square$

3.2. **Radial monotonicity of entire solutions.** We next establish the radial monotonicity of positive radial entire solutions.

Proposition 3.2. *Assume $p > p_S$. Let u be a positive radial solution of* (1.2). *Then u is radially decreasing.*

Proof. Fix $T \in \mathbb{R}$ and let $v = v(\rho, s)$ be the rescaled function corresponding to u and T. It is sufficient to prove that v is radially decreasing. Due to Subsection 2.4, the α-limit set $\alpha(v)$ is a singleton $\{w\}$, where either $w = w_a$ with $a \in \mathcal{A}_k$, $k \geq 2$, or $w = \phi_\infty$. The derivative v_ρ solves a linear parabolic equation whose zero order coefficient is

$$a_0(\rho, s) := -\frac{N-1}{\rho^2} - \frac{1}{2} - \frac{1}{p-1} + pv^{p-1}(\rho, s).$$

Estimate (2.4) with $C_T = 0$ guarantees the existence of $R_0 > 0$ such that $a_0(\rho, s) < -1/2$ when $\rho \geq R_0$.

Since $w_\rho < 0$ for $\rho > 0$ (cp. (2.14)), given $\varepsilon > 0$ there exists $s(\varepsilon)$ such $v_\rho(\rho, s) < 0$ for $\rho \in [\varepsilon, R_0]$ and $s \leq s(\varepsilon)$. Assume $v_\rho(\rho_0, s_0) > 0$ for some $\rho_0 > R_0$ and $s_0 \leq s(\varepsilon)$. Set $V := v_\rho$ and let $D, D_0, \Omega(s), m(s), S$ be defined by (2.19), (2.22) and $\rho_1(s) :=$ inf $\Omega(s)$. Then $\rho_1(s) > R_0$ for $s \in (S, s_0]$ and the same arguments as in the proof of Proposition 2.5 yield $m(s_0) = 0$, which is a contradiction. Consequently, the maximum principle shows that $v(\cdot, s)$ is decreasing on $[\varepsilon, \infty)$ for any $s \leq s(\varepsilon)$.

If $w = w_a$ for some $a \in \mathcal{A}_k$ with $k \geq 2$, then $v(\cdot, s) \to w_a$ in $BC^1[0, \infty)$ as $s \to -\infty$, hence there exist $\varepsilon > 0$ and $\tilde{s}_0 \leq s(\varepsilon)$ such that $a_0(\rho, s) < -1$ if $\rho \in [0, \varepsilon]$

and $s \leq \tilde{s}_0$. If we assume $v_\rho(\rho_0, s_0) > 0$ for some $\rho_0 \in (0, \varepsilon)$ and $s_0 \leq \tilde{s}_0$, then the same arguments as above yield a contradiction. Consequently, $v(\cdot, s)$ is also nonincreasing on $[0, \varepsilon]$ for $s \leq \tilde{s}_0$, and the maximum principle guarantees that $v(\cdot, s)$ is decreasing on $[0, \infty)$ for all $s \in \mathbb{R}$.

Finally consider the case $w = \phi_\infty$ and assume on the contrary that $v(\cdot, \tilde{s}_0)$ is not decreasing for some $\tilde{s}_0$. Fix $\varepsilon \in (0, 1)$ such that

$$\frac{\kappa + 1}{p - 1} \varepsilon < \frac{C_\varepsilon}{3}, \quad \text{where} \quad C_\varepsilon := \frac{N - 1}{\varepsilon} - \frac{\varepsilon}{2}, \tag{3.29}$$

and

$$\phi_\infty(\varepsilon) > \kappa + 1. \tag{3.30}$$

Then we can find $s_0 \leq \tilde{s}_0$ such that for any $s \leq s_0$, $v(\cdot, s)$ is decreasing on $[\varepsilon, \infty)$ and $v(\cdot, s)$ attains a local minimum at some $\rho(s) \in [0, \varepsilon)$. Fix $\delta \in (0, 1/2)$. For any $s \leq s_0$, if $v(\rho(s), s) \geq \kappa + \delta$, then the relations $v_\rho(\rho(s), s) = 0 \leq v_{\rho\rho}(\rho(s), s)$ and the equation for v imply $v_s(\rho(s), s) \geq \tilde{\delta}$ for some $\tilde{\delta} > 0$ (depending only on p and δ). It follows that there exists $s_1 < s_0$ such that

$$\min_{[0,\varepsilon]} v(\cdot, s) < \kappa + \delta \quad \text{for} \quad s \leq s_1. \tag{3.31}$$

On the other hand, by (3.30) there exist $s_2 \leq s_1$ such that $v(\varepsilon, s) > \kappa + 1$ if $s \leq s_2$.

Let z be the solution of the linear equation

$$z_s = z_{\rho\rho} + \frac{N - 1}{\rho} z_\rho - \frac{\rho}{2} z_\rho - \frac{z}{p - 1}$$

in $(0, \varepsilon) \times (0, \infty)$ satisfying the boundary conditions $z_\rho(0, s) = 0$, $z(\varepsilon, s) = \kappa + 1$, and the initial condition $z(\rho, 0) = 0$. Then z is increasing in time and, since $\kappa + 1$ is a supersolution to z, we obtain $z_\rho(\varepsilon, s) \geq 0$, hence $z_\rho \geq 0$ by the maximum principle. Also, z approaches a steady state Z as $s \to \infty$ with $Z_\rho \geq 0$, $Z_\rho(0) = 0$, and $Z(\varepsilon) = \kappa + 1$. We have

$$\frac{\kappa + 1}{p - 1} \geq \frac{Z(\rho)}{p - 1} = Z_{\rho\rho} + \left(\frac{N - 1}{\rho} - \frac{\rho}{2}\right) Z_\rho \geq Z_{\rho\rho} + C_\varepsilon Z_\rho.$$

Integrating over $\rho \in (0, \varepsilon)$ we obtain

$$\frac{\kappa + 1}{p - 1} \varepsilon \geq Z_\rho(\varepsilon) - Z_\rho(0) + C_\varepsilon(Z(\varepsilon) - Z(0)) \geq C_\varepsilon(\kappa + 1 - Z(0)),$$

hence (3.29) implies $Z(0) > \kappa + 2/3$. Since $z_\rho \geq 0$ and $z(0, s) \to Z(0)$ as $s \to \infty$, we have $z(\cdot, S) > \kappa + 1/2$ for some S large enough. Since $v(\cdot, s_2 - S + s) \geq z(\cdot, s)$ on $[0, \varepsilon]$ for $s \in [0, S]$ by the comparison principle, we obtain $v(\cdot, s_2) \geq \kappa + 1/2$ on $[0, \varepsilon]$, which contradicts (3.31). $\qquad\square$

3.3. Completion of the proof of Theorem 1.6. In this subsection we assume $p > p_L$. By $\zeta(\cdot, \alpha)$ we will denote the solution of

$$\zeta_{\rho\rho} + \frac{N - 1}{r} \zeta_\rho - \frac{\rho}{2} \zeta_\rho - \frac{1}{p - 1} \zeta + \zeta^p = 0 \tag{3.32}$$

with $\zeta(0, \alpha) = \alpha$. By [24], for each $\alpha > 0$ there is ρ_α such that $\zeta(\cdot, \alpha) > 0$ on $[0, \rho_\alpha)$ and $\zeta(\rho_\alpha, \alpha) = 0$. Also, the following property is proved in [24, Lemma 2.5] (although it is not stressed in [24, Lemma 2.5], it can be checked that the constant C there is independent of ε):

(p1) For each compact interval $I \subset (0, \infty)$ one has $\zeta(\cdot, \alpha) - \phi_\infty \to 0$ in $C^1(I)$ as $\alpha \to \infty$.

We shall also need the following property of $\zeta(\cdot, \alpha)$.

Lemma 3.3. *There is $\alpha_0 > 0$ such that for each $\alpha \geq \alpha_0$ one has*

$$z_{[0,\rho_\alpha]}(\zeta(\cdot, \alpha) - \phi_\infty) \leq 2. \tag{3.33}$$

Proof. Assume that, to the contrary, there are arbitrarily large values α with $z_{[0,\rho_\alpha]}(\zeta(\cdot, \alpha) - \phi_\infty) \geq 3$. Clearly, the zeros of $\zeta(\cdot, \alpha) - \phi_\infty$ are all simple and, since $\zeta(\rho_\alpha, \alpha) = 0$, their number is even. Thus, there must be at least 4 of them. We denote by $\xi_1^\alpha < \cdots < \xi_4^\alpha$ the first four zeros of $\zeta(\cdot, \alpha) - \phi_\infty$.

Let now μ_2 be the third eigenvalue of the linearization at ϕ_∞ and ϑ_2 a corresponding eigenfunction (cp. (A3) in the Subsection 3.1). Then $\mu_2 < 0$, $z(\vartheta_2) = 2$, and both zeros of ϑ_2 are positive. Let $\eta_1 < \eta_2$ denote these zeros. As noted in [24, Lemma 2.9], a Sturm comparison argument implies that $\eta_1 \in (0, \xi_1^\alpha)$ and $\eta_2 \in (\xi_2^\alpha, \xi_3^\alpha)$. Using (p1) and taking α sufficiently large we obtain that

$$p\zeta^{p-1}(\rho, \alpha) < p\phi_\infty^{p-1}(\rho) - \mu_2 \quad (\rho \in (\eta_1, \eta_2)).$$

This relation and the fact that $\eta_1 < \xi_1^\alpha < \xi_2^\alpha < \eta_2$ make the Sturm comparison argument applicable to the interval $(\xi_1^\alpha, \xi_2^\alpha)$ as well. We conclude that this interval contains a third zero of ϑ_2, which is a contradiction. □

Although it is not needed below, we remark that (3.33) in fact holds for all $\alpha > \kappa$. This follows from the observation that the zero number in (3.33) does not change as one varies $\alpha \in (\kappa, \infty)$ (the fact that for $p > p_L$ one has $\rho_\alpha < \infty$ for all $\alpha > \kappa$ is important here). One can also turn the argument around and prove (3.33) by using the independence of the zero number of α in conjunction with the fact that the zero number is equal to 2 for $\alpha > \kappa$ sufficiently close to κ (see [24, Lemma 2.3]).

Proof of Theorem 1.6. Let $u = u(r, t)$ be a positive bounded (radial) solution of (1.2) and $p > p_L$. Fix $T \in \mathbb{R}$ and let v be the corresponding rescaled solution of (2.2). We know from Subsection 2.4 that each of the sets $\alpha(v)$ and $\omega(v)$ has to be one of the sets $\{0\}$, $\{\kappa\}$ and $\{\phi_\infty\}$, and $\alpha(v) \neq \{0\}$. Estimate (2.4) (with $C_T = 0$) guarantees $\alpha(v) \neq \{\kappa\}$. Consequently, $\alpha(v) = \{\phi_\infty\}$.

We prove that

$$z(v(\cdot, s) - \phi_\infty) \leq 1 \quad (s \in \mathbb{R}). \tag{3.34}$$

In fact, assume there is s_0 such that $z(v(\cdot, s) - \phi_\infty) \geq 2$ for $s = s_0$ (hence for all $s \leq s_0$). Making s_0 smaller if needed we may assume that the first two zeros ξ_1, ξ_2 of $v(\cdot, s_0) - \phi_\infty$ are simple. Clearly, ξ_2 being the second zero, there is $\bar{\xi} > \xi_2$ such that $v(\bar{\xi}, s_0) < \phi_\infty(\bar{\xi})$. Using (p1) and Lemma 3.3, we find α such that (3.33) holds along with the following statements

(a1) $v(\cdot, s_0) - \zeta(\cdot, \alpha)$ has zeros $\tilde{\xi}_1, \tilde{\xi}_2$ (near ξ_1, ξ_2, respectively) with $\tilde{\xi}_1 < \tilde{\xi}_2 < \bar{\xi}$,
(a2) $v(\bar{\xi}, s_0) < \zeta(\bar{\xi}, \alpha)$.

Relations $v > 0$, $\zeta(\rho_\alpha, \alpha) = 0$, and (a2) imply that $v(\cdot, s_0) - \zeta(\cdot, \alpha)$ has another zero in $(\bar{\xi}, \rho_\alpha)$. Thus, $z_{[0,\rho_\alpha]}(v(\cdot, s) - \zeta(\cdot, \alpha)) \geq 3$ for all $s \leq s_0$.

Let α be as above. Proposition 3.2 and the convergence $v(\cdot, s) \to \phi_\infty$ in $C^1_{loc}(0, \infty)$ imply that there exist $s_1 \in \mathbb{R}$ and $\delta > 0$ such that $v(\rho, s) > \zeta(\rho, \alpha)$ for all $\rho \in [0, \delta]$ and $s \in (-\infty, s_1)$. Using this relation (and the convergence again), we obtain that for all sufficiently large negative s we have

$$z_{[0,\rho_\alpha]}(v(\cdot, s) - \zeta(\cdot, \alpha)) = z_{[\delta,\rho_\alpha]}(v(\cdot, s) - \zeta(\cdot, \alpha)) \leq z_{[\delta,\rho_\alpha]}(\phi_\infty - \zeta(\cdot, \alpha)) \leq 2$$

(cp. (3.33)). This contradiction completes the proof of (3.34).

We now show that the case $z(v(\cdot, s) - \phi_\infty) = 1$ for some s is impossible. Indeed, if this holds, then $z(u(\cdot, t_0) - \phi_\infty) = 1$ for some t_0. Setting $u_0(r) := \min\{u(r, t_0), \phi_\infty(r)\}$, we have $0 < u_0 \leq \phi_\infty$ and $\phi_\infty - u_0$ has compact support. By [32], the solution of $\bar{u}_t = \Delta\bar{u} + \bar{u}^p$, $\bar{u}(\cdot, t_0) = u_0$ is unbounded (it approaches ϕ_∞), and the comparison principle then implies that the same it true of $u(\cdot, t)$, in contradiction to our assumption.

Thus $z(v(\cdot, s) - \phi_\infty) = 0$, that is, $v(\cdot, s) < \phi_\infty$, for all $s \in \mathbb{R}$.

To complete the proof, we now apply some results of [33]. Recall from Subsection 3.1 that $\vartheta_0(\rho) = \hat{c}_0\rho^\beta$, where $\hat{c}_0 > 0$, and $\|\cdot\|_0$ denotes the norm in $L^2(0, \infty; \rho^{N-1}e^{-\rho^2/4}d\rho)$ The steady states ϕ_α satisfy

$$\phi_\alpha(r) = Lr^{-2/(p-1)} - b(\alpha)r^\beta + O(r^{\beta - \varepsilon}) \quad \text{as} \quad r \to \infty,$$

where $\varepsilon > 0$, $b(\alpha) = b_1\alpha^{1+\beta(p-1)/2}$, and $b_1 > 0$ is a constant; see [15, 39]. According to [33, Lemma 2.2], the rescaled functions

$$\psi_\alpha(\rho, s) = e^{-s/(p-1)}\phi_\alpha(e^{-s/2}\rho)$$

(cf. (1.9)) satisfy

$$\left\|\phi_\infty - \psi_\alpha(\cdot, s) - \frac{b(\alpha)}{\hat{c}_0}e^{\mu_0 s}\vartheta_0\right\|_0 = o(e^{\mu_0 s}) \quad \text{as} \quad s \to -\infty. \tag{3.35}$$

Fix α such that $b(\alpha) = c\hat{c}_0$, where c is from (3.5). Then (3.35) and Proposition 3.1 imply

$$\|v(\cdot, s_k) - \psi_\alpha(\cdot, s_k)\|_0 = o(e^{\mu_0 s_k}) \quad \text{as} \quad k \to \infty.$$

As shown in [33, Lemma 4.2], this estimate guarantees that $v \equiv \psi_\alpha$, hence $u \equiv \phi_\alpha$. $\qquad\square$

4. Proofs of Theorems 1.7 and 1.8.

In the proofs of Theorems 1.7, 1.8, we will use the following result.

Proposition 4.1. *Let $p > p_S$ and u be a positive radial solution of* (1.3) *satisfying* (1.8). *Then there is a positive integer m such that $z(u(\cdot, t) - \phi_\alpha) \leq m$ for all $t < T$ and $\alpha \in (0, \infty]$.*

Proof. Fix any $\alpha \in (0, \infty]$ and set $\phi := \phi_\alpha$,

$$v(\rho, s) := e^{-s/(p-1)}u(e^{-s/2}\rho, T - e^{-s}), \quad \psi(\rho, s) := e^{-s/(p-1)}\phi(e^{-s/2}\rho) \tag{4.1}$$

(in particular, $\psi \equiv \phi_\infty$ when $\alpha = \infty$). Then v, ψ solve equation (2.2). By (1.8), there is $s_0 \in \mathbb{R}$ such that v is bounded for $s \leq s_0$.

Remarks in Subsection 2.4 show that for the α-limit set of v in $C^1_{loc}[0, \infty)$ we have either $\alpha(v) = \{w_a\}$ with $a \in \mathcal{A}_k$ for some $k > 0$ or $\alpha(v) = \{\phi_\infty\}$. The latter is ruled out by the boundedness of v for $s \leq s_0$, so we have the former. We prove that the conclusion of the proposition holds with $m = k + 1$ (which is independent of α). Note that $\alpha(\psi) = \{\phi_\infty\}$ in $C^1_{loc}(0, \infty)$.

Since $\phi(r) \leq Cr^{-2/(p-1)}$, we also have $\psi(\rho, s) \leq C\rho^{-2/(p-1)}$ for all s and we can fix $R_0 > \phi_\infty^{-1}(\kappa)$ such that

$$\psi(\rho, s) \leq c_0 \quad \text{for all} \quad s \quad \text{and} \quad \rho \geq R_0, \tag{4.2}$$

where c_0 is defined in (2.18).

Consider the function $V := v - \psi$. The k zeros of $w_a - \phi_\infty$ belong to the interval $(0, \phi_\infty^{-1}(\kappa)] \subset (0, R_0)$ (cp. (2.15)). Also, since w_a, ϕ_∞ solve the same second order

ODE, the zeros are simple. This fact and the convergence of v, ψ as $s \to -\infty$ guarantee that, decreasing s_0 if necessary, we have

$$z_{(0,R_0)}(V(\cdot, s)) = k \quad \text{for all} \quad s \le s_0. \tag{4.3}$$

Assume, for a contradiction, that $z((V(\cdot, s_1)) \ge k+2$ for some $s_1 \le s_0$. Decreasing s_0 further if needed, we may assume $s_1 = s_0$. Denoting by ρ^* the $(k+1)$-th zero of $V(\cdot, s_0)$, we can choose $\rho_0 > \rho^*$ such that $V(\rho_0, s_0) < 0$. Let $D, D_0, \Omega(s), m(s), S$ be as in (2.19), (2.22), and $\rho_1(s) := \inf \Omega(s)$. Clearly, $\rho_1(s)$ is a zero of $V(\cdot, s)$ for each $s \in (S, s_0]$, and, by the monotonicity of the zero number, $\rho_1(s)$ is at least $(k+1)$-th zero of $V(\cdot, s)$. Hence (4.3) implies $\rho_1(s) \ge R_0$ for $s \in (S, s_0]$. Now (4.2) and Proposition 2.5 give $V(\rho_0, s_0) > 0$, and we have a contradiction.

Consequently, $z((V(\cdot, s)) \le k+1$ for all $s \le s_0$ and the monotonicity of the zero number gives the same estimate for $s > s_0$. This gives the desired estimate $z(u(\cdot, t) - \phi_\alpha) \le k+1$. □

Proof of Theorem 1.7. By standard results, since the solution u is bounded, its ω-limit set in $C^1_{loc}([0,\infty))$, $\omega(u)$, is a nonempty compact set in this space and the desired conclusion $u(\cdot, t) \to 0$ is equivalent to $\omega(u) = \{0\}$. Also, $\omega(u)$ is invariant: for any $u^0_\infty \in \omega(u)$ there is a radial solution of (1.2) satisfying $u_\infty(\cdot, 0) = u^0_\infty$ and $u_\infty(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. Obviously, any such u_∞ is nonnegative and bounded.

Set

$$\ell^- := \liminf_{t \to \infty} u(0, t) \quad \text{and} \quad \ell^+ := \limsup_{t \to \infty} u(0, t).$$

By the boundedness of u, these limits are finite. We first prove that $\ell^+ = \ell^-$. Assume not and fix $\alpha \in (\ell^-, \ell^+)$. Then $u(0, t_k) = \alpha = \phi_\alpha(0)$ (and $u_r(0, t_k) = 0 = \phi'_\alpha(0)$) for an infinite sequence $t_k \to \infty$. It follows that $z(u(\cdot, t) - \phi_\alpha)$ drops at each t_k (cp. Proposition 2.4), which is a contradiction to Proposition 4.1. Thus, $\ell^+ = \ell^- =: \alpha$, which implies that $u(0, t) \to \alpha$ as $t \to \infty$.

Consequently, any element u^0_∞ of $\omega(u)$ has $u^0_\infty(0) = \alpha = \phi_\alpha(0)$. We show that actually $u^0_\infty \equiv \phi_\alpha$. Assume that, to the contrary, $u^0_\infty(r_0) \ne \phi_\alpha(r_0)$ for some $r_0 > 0$. Let u_∞ be the entire solution of (1.2) corresponding to u^0_∞, as above. Then $u(0, t) = \alpha$ (and $u_r(0, t) = 0$) for all t, and $u_\infty(r_0, t) \ne \phi_\alpha(r_0)$ for $t \approx 0$. Hence $z_{(0,r_0)}(u_\infty(\cdot, t) - \phi_\alpha)$ is finite for t near 0 and drops at any such t, which is absurd. Thus we have showed that $\omega(u) = \{\phi_\alpha\}$.

To conclude, assume $\alpha > 0$ and fix $\beta > 0$, $\beta \ne \alpha$. Then $z(u(\cdot, t) - \phi_\beta)$ is bounded by Proposition 4.1. However, in the considered range $p_S < p < p_{JL}$ we have $z(\phi_\alpha - \phi_\beta) = \infty$ (see [39, 36]) and the zeros of $\phi_\alpha - \phi_\beta$ are simple. The convergence of $u(\cdot, t)$ to ϕ_α therefore implies that $z(u(\cdot, t) - \phi_\beta) \to \infty$ as $t \to \infty$, a contradiction. Thus, $\alpha = 0$ and we have proved the desired conclusion $\omega(u) = \{0\}$. □

Proof of Theorem 1.8. Assume that v is not a steady state of (1.10). We know from Subsection 2.4 that each of the sets $\alpha(v), \omega(v)$ is a singleton consisting of either w_a for some $a \in \mathcal{A}$, or ϕ_∞. In addition, $\alpha(v) \ne \{0\}$ and monotonicity of the energy functional (cp. Proposition 2.2) gives $\omega(v) \ne \alpha(v)$. Obviously, $\alpha(v) = \{\phi_\infty\}$ if and only if (1.8) fails; if (1.8) holds, we necessarily have $\alpha(v) = \{w_a\}$ for some $a \in \mathcal{A}$.

We next prove that $\omega(v) = \{\tilde{w}\}$ where $\tilde{w} = w_a$ for some $a \in \mathcal{A}$ (possibly $a = 0$). For that, we just need to show that $\omega(v) \ne \{\phi_\infty\}$. If $p > p_L$, this follows from Proposition 2.3, as already noted in Subsection 2.4 (thus, $\omega(v) = \{0\}$ or $\omega(v) = \{\kappa\}$ in this case). If $p_S < p < p_{JL}$ and (1.8) fails, then the relation follows from $\{\phi_\infty\} = \alpha(v) \ne \omega(v)$.

If $p_S < p < p_{JL}$ and (1.8) is true, Proposition 4.1 applies. Let m be as in the proposition. Suppose $\omega(v) = \{\phi_\infty\}$. Since $z(\phi_\infty - \phi_\alpha) = \infty$ for any $\alpha > 0$ (see [39] or [36]), for all sufficiently large s, the function $v(\cdot, s) - \phi_1$ has at least $m+1$ zeros. Pick any such s and set $\alpha := e^{s/(p-1)}$. By the scaling invariance of equation (1.1), we can write $\phi_1(\rho) = \alpha^{-1}\phi_\alpha(\alpha^{-(p-1)/2}\rho)$. Using this and the relation between u and v (cp. (4.1)), we obtain, for $t = T - e^{-s}$,

$$m + 1 \leq z(v(\cdot, s) - \phi_1) = z(v(\cdot, s) - e^{-s/(p-1)}\phi_\alpha(e^{-s/2}\rho))) = z(u(\cdot, t) - \phi_\alpha),$$

and we have a contraction to Proposition 4.1.

To complete the proof of Theorem 1.8, it remains to show that $\omega(v) = \{\tilde{w}\}$ implies the convergence

$$\lim_{s\to\infty} v(\cdot, s) = \tilde{w} \tag{4.4}$$

in $C^1_{loc}(\mathbb{R}^N)$ (and not just in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$, the space used in the definition of $\omega(v)$), and that $\alpha(v) = \{w\}$ in conjunction with (1.8) implies the convergence

$$\lim_{s\to-\infty} v(\cdot, s) = w \tag{4.5}$$

in $C^1_{loc}(\mathbb{R}^N)$. The latter is a simpler: (4.5) follows from the convergence in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$, the boundedness of $v(\cdot, s)$ as $s \to -\infty$ (condition (1.8)), and parabolic estimates.

The former can be proved similarly once we show that as $s \to \infty$ the function $v(\cdot, s)$ stays bounded on a neighborhood of the origin. For this, we use a "no-needle" lemma, Lemma 2.14 of [20]. Consider the functions $v(\cdot, k+\cdot)$, $k = 1, 2, \ldots$. Since the sequence $v(\cdot, k)$ converges in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$ to $\tilde{w}$, a bounded function, [20, Lemma 2.14] yields positive constants $\delta > 0$, M_1 such that $v(\cdot, k + \delta) \leq M_1$ on $\mathbb{R}^N$ for $k = 1, 2, \ldots$. Making M_1 larger if necessary, we may also assume that $\tilde{w}(0) < M_1$. Take now any $a > M_1$ and let w_a be the solution of

$$w_{\rho\rho} + \frac{N-1}{\rho}w_\rho - \frac{\rho}{2}w_\rho - \frac{w}{p-1} + w^p = 0, \quad \rho > 0,$$
$$w(0) = a, \quad w'(0) = 0. \tag{4.6}$$

Then w_a is defined (at least) on a small interval $[0, R]$ and, making $R > 0$ smaller if needed, we have $w_a > M_1 > \tilde{w}$ on $[0, R]$. Since $v(R, s) \to \tilde{w}(R)$, $v(R, s) < w_a(R)$ for all sufficiently large s. Since also $v(r, k + \delta) \leq M_1 < w_a(r)$ for all $r \in [0, R]$ and $k = 1, 2, \ldots$, we obtain from the comparison principle that $v(r, s) < w_a(r)$ for all $r \in [0, R]$ if s is large enough. This is the desired estimate, from which (4.4) is proved easily. $\qquad\square$

We remark that the monotonicity of $s \mapsto z(v(\cdot, s) - \phi_\infty)$ implies that the steady states w and $\tilde{w}$ in (1.11) satisfy $z(w - \phi_\infty) \geq z(\tilde{w} - \phi_\infty)$.

5. **Further results and applications.** In the following theorem, we consider two classes of positive radial solutions of (1.1) for $p > p_L$. The first class consists of solutions which exhibit a type II blowup and the second class of global solutions which decay to 0 with rate slower than $t^{-1/(p-1)}$, or do not decay at all. As an application of our new Liouville theorem, Theorem 1.6, we show that at least along a sequence of times, the profiles of the solutions have a limit.

Theorem 5.1. *Let $p > p_L$ and u be a positive radial solution of* (1.1) *in* $\mathbb{R}^N \times (0, T)$. *Assume that*

$$\text{either} \ \ T < \infty, \ \ \limsup_{t \to T}(T - t)^{\frac{1}{p-1}}\|u(\cdot, t)\|_\infty = \infty \ \ \text{(type II blowup)},$$

$$\text{or} \ \ T = \infty, \ \ \limsup_{t \to T} t^{\frac{1}{p-1}}\|u(\cdot, t)\|_\infty = \infty \ \ \text{(slow or no decay)}.$$

Then there exist $t_k \to T$ such that

$$\lambda_k^{2/(p-1)}u(\lambda_k r, t_k) \to \phi_1(r), \qquad \lambda_k := \frac{1}{\|u(\cdot, t_k)\|_\infty^{(p-1)/2}}, \tag{5.1}$$

uniformly in $r \in [0, \infty)$.

In the blowup case, this theorem is known to hold for any $p \geq p_S$ under the extra assumption that $z(u_t(\cdot, t)) < \infty$: a long and technically involved proof can be found in [19]. The convergence in (5.1) plays a key role in [19] in the study of blowup rates and profiles.

Proof of Theorem 5.1. The proof is based on doubling, scaling, one-dimensional Liouville theorem, and Theorem 1.6.

Considering equation (1.1) on the time interval (δ, T) instead of $(0, T)$ (where $0 < \delta < T$) we may assume that

$$\|u(\cdot, t)\|_\infty \text{ is bounded for } t \in (0, \tau) \text{ whenever } \tau < T. \tag{5.2}$$

Set

$$M(t) := \|u(\cdot, t)\|_\infty^{(p-1)/2}.$$

Our assumptions imply that there exist $t_k \to T$ such that $M_k := M(t_k) > 2k/d(t_k)$, where $d(t) := \min(\sqrt{t}, \sqrt{T - t})$ ($d(t) = \sqrt{t}$ if $T = \infty$). The Doubling Lemma [30, Lemma 5.1] guarantees that, possibly after modifying the sequence $\{t_k\}$, the following additional condition is satisfied for $k = 1, 2, \ldots$: $M(t) \leq 2M_k$ whenever $\sqrt{|t - t_k|} < k/M_k$.

Set $\lambda_k := 1/M_k$. We claim that given any $\varepsilon_0 \in (0, 1)$ there exists $R_0 = R_0(\varepsilon_0)$ such that for a suitable subsequence of k we have $u(r, t_k) < \varepsilon_0\|u(\cdot, t_k)\|_\infty$ whenever $r/\lambda_k > R_0$. Assume that no such R_0 exist. Then we can find (a subsequence of k still denoted by k and) r_k such that $r_k/\lambda_k \to \infty$ and $u(r_k, t_k) \geq \varepsilon_0\|u(\cdot, t_k)\|_\infty$. Set

$$U_k(\rho, s) := \lambda_k^{2/(p-1)}u(r_k + \lambda_k\rho, t_k + \lambda_k^2 s).$$

Then for $k = 1, 2, \ldots$, U_k satisfies the equation

$$U_s = U_{\rho\rho} + \frac{N - 1}{r_k/\lambda_k + \rho}U_\rho + U^p,$$

U_k is bounded in $\{(\rho, s) : \sqrt{|s|} < k, \ \rho \geq -r_k/\lambda_k\}$ by a constant independent of k, and $U_k(0, 0) \geq \varepsilon_0$. Since $r_k/\lambda_k \to \infty$, (a suitable subsequence of) $\{U_k\}$ converges to a positive solution of (1.2) with $N = 1$, which contradicts the corresponding Liouville theorem (see the second part of Theorem 1.1). The claim is thus proved.

Take now a decreasing sequence $\varepsilon_j \to 0$. Using a diagonalization argument, we find a subsequence of k such that $u(r, t_k) < \varepsilon_j\|u(\cdot, t_k)\|_\infty$ whenever $r/\lambda_k > R_j := R_0(\varepsilon_j)$ and k is large enough.

Next set

$$V_k(\rho, s) := \lambda_k^{2/(p-1)}u(\lambda_k\rho, t_k + \lambda_k^2 s). \tag{5.3}$$

Then V_k satisfies the equation

$$V_s = V_{\rho\rho} + \frac{N-1}{\rho}V_\rho + V^p,$$

V_k is bounded by 2 in $\{(\rho, s) : \rho > 0, \ \sqrt{|s|} < k\}$, and $V_k(\cdot, 0)$ attains its maximum 1 in the compact set $\{\rho : |\rho| \leq R_0\}$ (since $V_k(\rho, 0) \leq \varepsilon_0$ for $\rho > R_0$). A suitable subsequence of $\{V_k\}$ converges (in C_{loc}, for example) to a positive solution of (1.2), hence to a steady state ϕ_α. Since $\max V_k(\cdot, 0) = 1$, we have $\alpha = 1$. Since $V_k(\rho, 0) \leq \varepsilon_j$ for $\rho > R_j$ and k large enough, we see that the convergence is uniform on $[0, \infty)$. $\qquad\square$

We now return to the classification problem for entire solutions satisfying (1.8) (cp. Theorem 1.7). As mentioned in the introduction, we believe that the statement of Theorem 1.7 holds also in the range $p_{JL} \leq p < p_L$. We can actually prove this, see Proposition 5.2 below, provided the following condition on the energies of steady states of (2.2) is satisfied:

$$E(w_a) < E(\phi_\infty) \quad \text{for all} \quad a \in \mathcal{A}. \tag{5.4}$$

This looks plausible, although the proof may not be easy. One way (5.4) could be verified is by proving the existence of a solution of (2.2) connecting ϕ_∞ to w_a, for any $a \in \mathcal{A}$. Then the monotonicity of the energy would give (5.4) immediately. The question whether such connections indeed exist is of independent interest. A positive answer would give an interesting information on the variety of entire solutions of (2.2). What seems to be crucial for establishing the connections is a description of the (global) bifurcation diagram for the steady states of (2.2) when p decreases from p_L down to p_{JL}. Optimally, one would prove that all classical steady states lie on bifurcation branches emanating from the singular steady state at some bifurcation values of p. If this could be proved, then there is hope that the connections can first be established locally, near bifurcation points, then globally by continuation, somewhat in the spirit of [9, Section 3].

Proposition 5.2. *Let $p_{JL} \leq p < p_L$. Assume (5.4). If u is a positive radial bounded solution of (1.2) satisfying (1.8), then $\lim_{t \to \infty} \|u(\cdot, t)\|_\infty = 0$ (i.e. u is a homoclinic solution to the trivial steady state).*

Proof. Assume $p_{JL} \leq p < p_L$. Let u be a positive radial bounded solution of (1.2) satisfying (1.8) and let C be the constant from (1.8). Set $\mathcal{A}^C := \mathcal{A} \cap [0, C]$. As proved in [29], the set $\mathcal{A}^C$ is finite. Using (5.4), we find $\varepsilon > 0$ such that

$$\varepsilon < E(\phi_\infty) - \max_{a \in \mathcal{A}^C} E(w_a). \tag{5.5}$$

Using Proposition 4.1 and the same arguments as in the proof of Theorem 1.7, one shows that $\omega(u) = \{\phi_\alpha\}$ for some $\alpha \in [0, \infty)$. We need to prove that $\alpha = 0$. Suppose for a contradiction that $\alpha > 0$. Set $u_k(r, t) := k^{2/(p-1)}u(kr, k^2t)$. Then u_k is a positive radial bounded solution of (1.2) satisfying (1.8) and $\omega(u_k) = \{\phi_{k^{2/(p-1)}\alpha}\}$. Notice that $\phi_{k^{2/(p-1)}\alpha} \nearrow \phi_\infty$ as $k \to \infty$. Therefore, we can find k and T such that $E(u_k(\cdot, T-1)) > E(\phi_\infty) - \varepsilon$. Let v_k be the rescaled function corresponding to u_k and T. Then $v_k(\cdot, 0) = u_k(\cdot, T-1)$, hence $E(v_k(\cdot, 0)) > E(\phi_\infty) - \varepsilon$. Assumption

(1.8) guarantees that

$$|v_k(0, -\log(T-t))| = (T-t)^{-1/(p-1)} k^{2/(p-1)} u(0, k^2 t)$$
$$\leq C(T-t)^{-1/(p-1)} k^{2/(p-1)} (k^2 t)^{-1/(p-1)}$$
$$\to C \quad \text{as} \quad t \to -\infty,$$

hence $\alpha(v_k) = \{w_a\}$ for some $a \in \mathcal{A}^C$, $E(w_a) > E(v_k(\cdot, 0)) > E(\phi_\infty) - \varepsilon$ and we have a contradiction to (5.5). $\square$

Remark 5.3. Condition (5.4) is also sufficient for the validity of Theorem 1.8 for $p_{JL} \leq p \leq p_L$ (cp. Remark 1.11). Indeed, the proof of Theorem 1.8 as given above applies in the case $p_{JL} \leq p \leq p_L$ with a single exception of the argument we used for proving the relation $\omega(v) \neq \{\phi_\infty\}$ in the case that $\alpha(v) = \{w_a\}$ for some $a \in \mathcal{A}$. Obviously, if (5.4) holds, then instead of that argument one can simply refer to the monotonicity of the energy functional.

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